الجمهوريــة الجزائريــة الديمقراطيــة الشعبيــة

République Algérienne Démocratique et Populaire

و البـحـث الـعلـمــي وزارة الـتعلـيــم الـعـالي

Ministère de l'Enseignement Supérieur et de la Recherche Scientifique

Université MUSTAPHA Stambouli

Mascara

جامعة مصطفى اسطمبولي

معسكر

Faculté des sciences exactes

Département de Mathématiques

MEMOIRE de MAGISTER

En Mathématiques

Option : Géométrie des sous variétés, Géométrie non commutative et applications

Intitulée

Sur la géométrie des surfaces dans les espaces H^2xR et Nil3

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Le 04/07/2023

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Année Universitaire : 2022-2023

Acknowledgement

In particular, I would like to thank the research supervisor Pr. Belarbi Lakehal who agreed to lead this work and who introduced to a wide range of methods and concepts in the field of differential geometry. I am also very grateful to him for acquiring new mathematical concepts in this research.

I would like to thank Pr. Ahmed Mohamed Cheriif who honored me with chairing this jury.

I would also like to thank the members of the jury Pr. Elhendi Hichem, Pr. Gherici Beldjilali and Pr. Bouzir Habib for having honored me with their presence and for the time they gave to participating in this thesis committee.

Finally, I sincerely thank all my family and friends who have always supported and encouraged me in setting up this memorial.

Abstract

The initial subject of research was the study of minimal translation surfaces in the product space $\mathbb{H}^2 \times \mathbb{R}$ (the product space of the half Poincaré plane $\mathbb{H}^2$ and the real space $\mathbb{R}$). On the other hand the study of surfaces of constant extrinsic Gaussian curvature in the Heisenberg space noted Nil_3, this homogeneous space interests geometers by the fact that it admits a fairly large mobility (because its group of isometries is dimension four) and it admits a simply transitive subgroup of isometries which is nilpotent.

Keywords: Minimal surfaces, Flat surfaces, Homogenous spaces.

Résumé

Le sujet initial de recherche a été l'étude des surfaces de translation minimales dans l'espace produit $\mathbb{H}^2 \times \mathbb{R}$ (l'espace produit du demi plan de poincaré $\mathbb{H}^2$ et l'espace réelle $\mathbb{R}$). D'autre part l'étude des surfaces de courbure de Gauss extrinsèque constante dans l'espace de Heisenberg noté Nil_3, cet espace homogène intéresse les géomètres par le fait qu'il admet une assez grande mobilité (car son groupe d'isométries est de dimension quatre) et il admet un sous groupe simplement transitif d'isométries qui est nilpotent.

Mot clés: Surfaces minimales, Surfaces plates, Espaces homogènes.

ملخص

تتناول هذه الأطروحة في بدايتها دراسة الأسطح المتعدية الأصغرية في الفضاء $\mathbb{H}^2 \times \mathbb{R}$ (فضاء جداء نصف المستوي لـ poincaré $\mathbb{H}^2$ و فضاء الأعداد الحقيقة).
وبعد ذالك تم التطرق الى دراسة الأسطح التي لديها الانحناء الجاوسي الخارجي ثابتا في فضاء Heisenberg الذي يرمزله Nil_3 ، هذا الفضاء المتجانس هو محل اهتمام المتخصصين في الهندسة نظرا لأهميته (لأن زمرة تقايساته من الدرجة الرابعة) كما أنه يقبل زمرة جزئية بسيطة متعدية لتقايسات متلاشية.

كلمات مفتاحية: الأسطح الأصغرية ، الأسطح المستوية ، الفضاءات المتجانسة

Contents

INTRODUCTION

In mathematics and in physics, a minimal surface is a surface minimizes its area while achieving certain conditions on board.

In elemental differentiel geometry, a minimal surface is a closed and bounded surface of a real Euclidean space of dimension 3 with regular board minimizing the total area with fixed contour.

In 1744, Leonhard Euler posed and solved the first minimal surface problem: finding between all surfaces passing through two parallel circles, the one with the smallest surface. In particular, as the study of minimum surfaces, L.Euler found that the only minimum surfaces of revolution are planes and catenoids.

In 1760, Lagrange generalised Euler's results for calculating variations for integrals to one variable in the case of two variables. He sought to solve the following problem: "given a closed curve of E^3, to determine a minimum area having this curve as a boundary " such a surface is called a minimum area.

In 1776, Meusnier showed that the differential equation obtained by Lagrange being equivalent to a condition on the mean curvature: "an area is minimal if and only if its mean curvature at any point is zero".

We have eight homogeneous spaces of dimension 3: $E^3, H^3, S^3, S^2 \times \mathbb{R}, H^2 \times \mathbb{R}, SL(2, \mathbb{R}), Nil_3$ and Sol_3. In particular, our study will be space $H^2 \times \mathbb{R}$.

In this brief we have made it possible to obtain classification results concerning the minimum translation areas of two properly prolonged types in the $H^2 \times \mathbb{R}$ space. From D. W. Yoon's article , we will address the following information:

Let H^2 be represented by the upper half-plane model $\{(x, y) \in \mathbb{R}^2 | y > 0\}$ equipped with the metric $g_H = (dx^2 + dy^2) \diagup y^2$. The space H^2, with the group structure derived by the composition of proper affine maps, is a Lie group and the metric g_H is left invariant. Therefore the Riemannian product space $H^2 \times \mathbb{R}$ is a Lie group with respect to the operetion

$$(x, y, z) * (\bar{x}, \bar{y}, \bar{z}) = (\bar{x}y + x, y\bar{y}, z + \bar{z})$$

and the left invariant product metric

$$g = \frac{dx^2 + dy^2}{y^2} + dz^2.$$

During the recent years, there has been a rapidly growing interest in the geometry of surfaces in three homogenous spaces focusing on flat and constant Gaussian curvature surfaces. Many works are studying the geometry of surfaces in homogeneous 3-manifolds. See for example [3],[4],[19],[20],[22],[47], [30],[24],[31],[15] and [32].

The concept of translation surfaces in $\mathbb{R}^3$ can be generalized the surfaces in the three dimensional Lie group, in particular, homogeneous manifolds. In Euclidean 3−space, every cylinder is flat. Conversely, complete flat surfaces in $\mathbb{E}^3$ are cylinders over complete curves.

See [30]. L'opez and Munteanu [26] studied invariant surfaces with constant mean curvature and constant Gaussian curvature in Sol_3 space. Yoon and Lee [46] studied translation surfaces in Heisenberg group $\mathbb{H}_3$ whose position vector x satisfies the equation $\Delta x = Ax$, where Δ is the Laplacian operator of the surface and A is a $3 \times 3-$real matrix.

Flat G_4-invariant surfaces are nothing but surfaces invariant under $SO(2)-$action, *i.e.* rotational surfaces. Flat rotational surfaces are classified by Caddeo, Piu and Ratto in [14].

In [20], J.I.Inoguchi give a classification of intrinsically flat G_1-invariant translation surfaces in Heisenberg group $\mathbb{H}_3$. Let M be a surface invariant under G_3, then M is locally expressed as

$$X(u,v) = (0,0,v).(x(u),y(u),0) = (x(u),y(u),v),\ u \in I,\ v \in \mathbb{R}.$$

Here I is an open interval and u is the arclength parameter. Note that $(x,y,0)$ and $(0,0,v)$ commute. Then the sectional curvature $K(X_x \wedge X_y) = \frac{1}{4}$ and the extrinsically Gaussian curvature $K_{ext} = -\frac{1}{4}$. Direct computation show that M is flat.(cf.[19],[20],[21],[37]).

My work is divided into three chapters:

In the first chapter we recall a number of definitions of a differential manifolds, map, atlas, ect. We also report the definition of a group and Lie algebra, tanget space and vector fields, rct. In section 1.6 we introduce the notion of a Riemannian manifolds and the connection of Levi-Civita. We also write the curvature of Gauss and that of the mean curvature, ect.

In the second chapter we present the result concerning the classification of minimum areas of type I and II in the $H^2 \times \mathbb{R}$ space, according to the article by D.W.Yoon. We begin with the study of the metric g and we calculate the symbols of Chtistoffel Γ_{ij}^k and the connecting forms $\tilde{\nabla}$, the first and second fundamental form, ect.

In the third chapter we classify G_2-invariant surfaces of the Heisenberg group $\mathbb{H}_3$ with constant extrinsically Gaussian curvature K_{ext}, including extrinsically flat G_2-invariant surfaces.

Chapter 1

Riemannian manifold

In this chapter we present the basic concepts of the theory of differential geometry. We first define topological and abstract manifold, differential maps (section 1.1.3). Next, we define and give example of submanifolds of $\mathbb{R}^n$ (section 1.1.4). Moreover, the notions of tangent space, vector fields, brackets, Lie group and Lie algebra are definies.

In section 1.6 we present the definitions of Riemannian manifolds, Riemannian metric. In section 1.6.1 we introduce the concept of isometry, the first and second fundamental form, Christoffel symbols Γ_{ij}^k. In addition, we need to define what the cannonical connection, and in section 1.6.5 we define the curvature average.

1.1 The notion of manifolds

Differential manifolds constitute the basic framework of differential topology and differential geometry. The notion of differentiable manifold generalizes the differential and integral calculus that we know how to define on a euclidean space of dimension n ($\mathbb{R}^n$).

1.1.1 Differentiable manifolds

Let $\mathcal{M}$ be a paracompact topological space i.e $\mathcal{M}$ is separated and such that any open covering admits a finer and locally finite open covering .

Definition 1. *We say that $\mathcal{M}$ is a topological manifold of dimension $n \in \mathbb{N}$ if any point $x \in \mathcal{M}$ has an open neighborhood U homeomorphic to $\mathbb{R}^n$ i.e there exists a one-to-one map $\phi\colon \mathbb{R}^n \to U$ such that ϕ and its inverse ϕ^{-1} be continuous.*

Example 1. *$\mathbb{R}^n$ is trivially a topological manifold of dimension n.*

Definition 2. *We say that the topological manifold $\mathcal{M}$ is of dimension "n" if and only if $\forall U \subset \mathcal{M}$open set of $\mathcal{M}$ there exists an open set $O \subset \mathbb{R}^n$ of $\mathbb{R}^n$ such that : U and O are homeomorphie (i.e : $\exists f\colon U \subset \mathcal{M} \to O \subset \mathbb{R}^n$ homeomorphism) .*

And $(x_1, \ldots, x_n) = \phi^{-1}(x)$ will be the coordinates of x. If (U, φ) and (V, ψ) are two local maps such that the intersection $U \cap V$ is non-empty then a point $x \in U \cap V$ will be identified by its coordinates $(x_1, \ldots, x_n)$ in U and its coordinates $(x'_1, \ldots, x'_n)$ in V.

Can we have

$$(x'_1, \ldots, x'_n) = \psi^{-1} \circ \varphi (x_1, \ldots, x_n). \tag{1.1.1}$$

The application $\psi^{-1} \circ \varphi$ is called changing the coordinates of the map (U, φ) to the map (V, ψ).

Definition 3. *A map in a topological manifold $\mathcal{M}$ is a pair (U, φ) such that :*

1) $U \subset \mathcal{M}$ is an open set of $\mathcal{M}$.

2) $\varphi \colon U \subset \mathcal{M} \to \varphi(U) \subset \mathbb{R}^n$ is a homeomorohism.

1.1.2 Abstract Manifolds

Definition 4. *Let$\mathcal{A} = \{(U_i, \phi_i)\}_{i \in \mathcal{A}}$ be a collection of $\mathbb{R}^n$-valued charts on a set $\mathcal{M}$. We call $\mathcal{A}$ an $\mathbb{R}^n$-valued atlas of class C^p if the following conditions are satisfied:*

(i) $\bigcup_{i \in \mathcal{A}} U_i = \mathcal{M}$.

(ii) The sets of the form $\phi_i(U_i \cap U_j)$ for $i, j \in \mathcal{A}$ are all open in $\mathbb{R}^n$.

(iii) Whenever $U_i \cap U_j$ is not emply, the map

$$\phi_j \cap \phi_i^{-1} \colon \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$$

is a C^p diffeomorphism $(p \geq 1)$.

Definition 5. *The pairs (U_i, ϕ_i) are called the charts of the atlas $\{(U_i, \phi_i)\}$. A chart at or around $x \in X$ is one whose domain contains x, and a chart centered at x is one mapping x to the origin in $\mathbb{R}^d$. The local coordinates associated with a chart (U_i, ϕ_i) are the functions $\phi_{i,k} \colon U_i \to \mathbb{R}\,(1 \leq k \leq d)$ such that $\phi_i(x) = (\phi_{i,1}(x), \ldots, \phi_{i,d}(x))$.*

Definition 6. *Let $\{(U_i, \phi_i)\}_{i \in I}$ be an atlas on $\mathcal{M}$, let U be a subset of $\mathcal{M}$ and $\phi \colon U \to \mathbb{R}^d$ a bijection onto an open subset of $\mathbb{R}^d$. The pair (U, ϕ) is said to be a chart compatible with the atlas $\{(U_i, \phi_i)\}_{i \in I}$ if the union $\{(U, \phi)\} \cup \{(U_i, \phi_i)\}_{i \in I}$ is still an atlas. Two atlases (of same dimension and differentiability class) are compatible if their union is still an atlas.*

In order for (U, ϕ) to be compatible with an atlas $\{(U_i, \phi_i)\}_{i \in I}$ it is necessary that each $\phi(U \cap U_i)$ and $\phi_i(U \cap U_i)$ be an open subset of $\mathbb{R}^d$ and that the maps $\phi \circ \phi_i^{-1}$ and $\phi^{-1} \circ \phi_i$ be of class C^p on their domains of definition.

Definition 7. *A differntiable manifold is a pair $(\mathcal{M}, \mathcal{A})$ where M is a topological manifold, and $\mathcal{A}$ a differentiable atlas on M .*

Example 2. *The sphere $S^n = \{x \in \mathbb{R}^{n+1} |\ |x| = 1\}$ is an n-manifold.*

We construct an atlas $\{(U_1,\phi_1),(U_2,\phi_2)\}$ with the aid of a standard well-known map called stereographic projection. Let $U_1 = S^n \setminus \{(0,\dots,0,1)\}$ and $U_2 = S^n \setminus \{(0,\dots,0,-1)\}$.

Note that $U_1 \cup U_2 = S^n$. Let $\phi_1(x_1,x_2,\dots,x_{n+1}) = \left(\frac{x_1}{1-x_{n+1}},\dots,\frac{x_n}{1-x_{n+1}}\right)$ and

$$\phi_2(x_1,x_2,\dots,x_{n+1}) = \left(\frac{x_1}{1+x_{n+1}},\dots,\frac{x_n}{1+x_{n+1}}\right).$$

Then map $\phi_1\colon U_1 \to \mathbb{R}^n$ is called stereographic projection. The inverse map $\phi_1^{-1}\colon \mathbb{R}^n \to U_1$ is defined by

$$\phi_1^{-1}(y_1,\dots,y_n) = \left(\frac{2y_1}{\sum\limits_{i=1}^n y_i^2+1},\frac{2y_2}{\sum\limits_{i=1}^n y_i^2+1},\dots,\frac{2y_n}{\sum\limits_{i=1}^n y_i^2+1},1-\frac{2}{\sum\limits_{i=1}^n y_i^2+1}\right).$$

Both ϕ_1 and ϕ_1^{-1} are continuous and hence ϕ_1 is a homeomorphism.

The second coordinate chart (U_2,ϕ_2), stereographic projection from the south pole, is given by $\phi_2 = -\phi_1 \circ (-I_{S^n})$ where $(-I_{S^n})$ is multiplication by $-I_{S^n}$ on the sphere. Since multiplication by -1 is a homeomorphism of the sphere to itself (its inverse is itself), the map $\phi_2\colon U_2 \to \mathbb{R}^n$ is a homeomorphism.

Checking the compatibility conditions, we have

$$\phi_2 \circ \phi_1^{-1}(y_1,\dots,y_n) = \frac{1}{\sum\limits_{i=1}^n y_i^2}(y_1,\dots,y_n)$$

and $\phi_2 \circ \phi_1^{-1} = \phi_1 \circ \phi_2^{-1}$. Hence, S^n is shown to be an n-manifold.

Compatibility is an equivalence relation. Thus we arrive at the definition of a manifold:

Definition 8. *A C^p differentiable structure $(p \geq 1)$ on a set $\mathcal{M}$ is an equivalence class of d-dimensional atlases of class C^p on $\mathcal{M}$. A d-dimensional manifold of class C^p is a set $\mathcal{M}$ endowed with a C^p differentiable structure. A chart on $\mathcal{M}$ is any chart belonging to any atlas in the differentiable structure of $\mathcal{M}$.*

1.1.3 Differentiable Maps

Definition 9. *Let X and Y be manifolds, of dimension d and e and class C^q and C^r, respectively. Let $p \leq \inf(q,r)$. We say that a continuous map $f\colon X \to Y$ is of class C^p, or C^p differentiable, or a C^p morphism, if for every chart (U,ϕ) at $x \in X$ and every chart (V,ψ) at $f(x) \in Y$, the map $\psi \circ f \circ \phi^{-1}\colon \phi(U \cap f^{-1}(V)) \to \mathbb{R}^e$ is of class C^p. We will denote be $C^p(X,Y)$ the set of C^p differentiable maps from X into Y.*

This definition, involving as it does all possible charts at x and $f(x)$, is not always convenient to use. The next theorem helps:

Theorem 1. *Let X and Y be manifolds of dimension d and e, respectively, and class $\geq p$. Let $f\colon X \to Y$ be a continuous map. The following conditions are equivalent:*

(i) f is C^p differentiable;

(ii) for every $x \in X$, every chart (U, ϕ) at x and every chart (V, ψ) at $f(x)$ such that $f(U) \subset V$, the composition $\psi \circ f \circ \phi^{-1} \colon \phi(U) \to \mathbb{R}^e$ is of class C^p;

(iii) for every $x \in X$, there exists a chart (U, ϕ) at x and a chart (V, ψ) at $f(x)$ such that $f(U) \subset V$ and $\psi \circ f \circ \phi^{-1} \subset C^p(\phi(U), \mathbb{R}^e)$.

Proof. (i) $\Rightarrow$ (ii) is immediate from the definition, just notice that $f(U) \subset V$ implies $U \cap f^{-1}(V) = U$.

(ii) $\Rightarrow$ (iii). Let (V, ψ) be chart at $f(x)$. Since f is continuous, $f^{-1}(V)$ is open in X and contains x, by the definition of canonical topology there exists a chart (U, ϕ) at x such that $U \subset f^{-1}(V)$, whence $f(U) \subset V$. If (ii) is true it follows that $\psi \circ f \circ \phi^{-1}$ is of class C^p from $\phi(U)$ into $\mathbb{R}^e$.

(iii) $\Rightarrow$ (i). Let (S, α) be a chart at $x \in X$ and (T, β) one at $f(x) \in Y$. We must show that the map $\beta \circ f \circ \alpha^{-1}$, from the open subset $\alpha(S \cap f^{-1}(T))$ of $\mathbb{R}^d$ into $\mathbb{R}^e$, is of class C^p. It is enough to show that it is C^p on a neighborhood of each point of its domain.

Take $u \in \alpha(S \cap f^{-1}(T))$ and $x' = \alpha^{-1}(u) \in S$. Property (iii), applied to x', gives a chart (U, ϕ) at x' and a chart (V, ψ) at $f(x')$ such that $f(U) \subset V$ and that $\psi \circ f \circ \phi^{-1}$ is of class C^p on $\phi(U)$. Now we can write

$$\beta \circ f \circ \alpha^{-1} = \left(\beta \circ \psi^{-1}\right) \circ \left(\psi \circ f \circ \phi^{-1}\right) \circ \left(\phi \circ \alpha^{-1}\right),$$

with the understanding that this only makes sense if each step in the composition is defined. If we can prove that each step is defined and C^p on a neighborhood of the image of u by the previous steps, we will have shown that $\beta \circ f \circ \alpha^{-1}$ is C^p on a neighborhood of u, and we'll be done.

The coordinate change $\phi \circ \alpha^{-1} \colon \alpha(S \cap U) \to \phi(S \cap U)$ is of class C^p, and its domain contains $u = \alpha(x')$. Next, $\psi \circ f \circ \phi^{-1}$ is of class C^p on $\phi(U)$, and its domain contains $\phi(x')$, the image of u under $\phi \circ \alpha^{-1}$, by the very choice of U, so $\psi \circ f \circ \phi^{-1}$ is of class C^p on a neighborhood of $\phi(x')$.

Finally, $\beta \circ \psi^{-1}$ is a C^p diffeomorphism between $\psi(T \cap V)$ and $\beta(T \cap V)$. Its domain $\psi(T \cap V)$ contains the image $\psi(f(x'))$ of u under the composition so far, since $f(x') \in V$ by our choice of V and $x' \in f^{-1}(T)$ as the image of $u \in \alpha(S \cap f^{-1}(T))$ under α^{-1}. Thus $\beta \circ \psi^{-1}$ is C^p on a neighborhood of $\psi(f(x'))$, concluding the proof that $\beta \circ f \circ \alpha^{-1}$ is C^p on a neighborhood of u.

Proposition 1. *Let X and Y be C^p manifolds of dimension d and e and having atlases $(U_i, \phi_i)_{i \in I}$ and $(V_j, \psi_j)_{j \in J}$, respectively. The atlas $(U_i \times V_j, \phi_i \times \psi_j)_{(i,j) \in I \in J}$, where*

$$\phi_i \times \psi_j \colon (x, y) \longmapsto (\phi_i(x), \psi_j(y)) \in \mathbb{R}^d \times \mathbb{R}^e = \mathbb{R}^{d+e},$$

makes $X \times Y$ into $a\,(d+e)$ dimensional C^p manifolds.

Examples of differentiable maps

Proposition 2. *Let X and Y be manifolds. The canonical projections $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ are differentiable.*

Proof. We prove the result for p. By Theorem 1. (iii), it suffices to show that, for every $(x, y) \in X \times Y$, there exists a chart $(U \times V, \phi \times \psi)$ at (x, y) and a chart (W, θ) at $x \in X$ such that $p(U \times V) \subset W$ and $\theta \circ p \circ (\phi \circ \psi)^{-1} : (\phi \times \psi)(U \times V) \to \mathbb{R}^d$ (where d is the dimension of X) is of class C^∞.

Let $(U \times V, \phi \times \psi)$ be a product of charts, as in 1.1.1, at the point (x, y). For (W, θ) we take the chart (U, ϕ) at x. We have $p(U \times V) = U$, and the map $\phi \circ p \circ (\phi \times \psi)^{-1}$ is defined on $(\phi \times \psi)(U \times V)$ by

$$(s,t) \longmapsto \underbrace{\left(\phi^{-1}(s), \psi^{-1}(t)\right)}_{\in U \times V} \overset{p}{\longmapsto} \phi^{-1}(s) \overset{\phi}{\longmapsto} s,$$

which is of class C^∞.

1.1.4 Submanifolds of $\mathbb{R}^n$

For $d \leq n$ the canonical inclusion $\mathbb{R}^d \subset \mathbb{R}^n$ is defined as the map

$$i\colon (x_1, \ldots, x_d) \mapsto (x_1, \ldots, x_d, 0, \ldots, 0).$$

Similarly, the canonical isomorphism is $\mathbb{R}^n = \mathbb{R}^d \times \mathbb{R}^{n-d}$.

Definition 10. *Let V be a subset of $\mathbb{R}^n$. We say that V is a d-dimensional C^p submanifold of $\mathbb{R}^n$ if, for every $x \in V$, there exists an open neighborhood $U \subset \mathbb{R}^n$ of x and a map $f\colon U \to \mathbb{R}^n$ such that $f(U) \subset \mathbb{R}^n$ is open, f is a C^pdiffeomorphism onto its image and $f(U \cap V) = f(U) \cap \mathbb{R}^n$. The codimension of V is $n - d$.*

Example 3. *[5]The sphere*

The sphere $S^d = \left\{x \in \mathbb{R}^{d+1}\colon \|x\| = 1\right\}$ is a compact, d-dimensional, C^∞ submanifold of $\mathbb{R}^{d+1}$. (We call S^1 a circle; S^0 is equal to two points).

To see this, write

$$S^d = \left\{x = (\xi_1, \ldots, \xi_{d+1}) : \xi_1^2 + \cdots + \xi_{d+1}^2 - 1 = 0\right\}.$$

Thus S^d is the zero-set of the map $f(\xi_1, \ldots, \xi_{d+1}) = \xi_1^2 + \cdots + \xi_{d+1}^2 - 1$, which is C^∞; furthermore, since

$$f'(x) = (2\xi_1, \ldots, 2\xi_{d+1}),$$

f has non-zero derivative whenever $x = (\xi_1, \ldots, \xi_{d+1})$ is on S^d.

1.2 Tangent Spaces

Before introducing tangent spaces to abstract manifolds, we study the case of submanifolds of $\mathbb{R}^n$.

Definition 11. *Let V be a submanifold of $\mathbb{R}^n$. A vector $z \in \mathbb{R}^n$ is said to be tangent to V at x if there exists a C^1 curve $\alpha\colon I \to V$ (where $I \subset \mathbb{R}$ is an interval containing 0) such that $\alpha(0) = x$ and $\alpha'(0) = z$.*

Remark 1. *Strictly speaking, $\alpha'(0)$ is a linear map from $\mathbb{R}$ into $\mathbb{R}^n$, but we have identified it with the vector $\alpha'(0).1 \in \mathbb{R}^n$.*

The condition $0 \in I$ just lightens the notation somewhat, but we could allow the curve to be defined on an interval I containing some t_0 such that $\alpha(t_0) = x$ and $\alpha'(t_0) = z$.

Definition 12. *Let X be a manifold and $x \in X$ a point. A tangent vector to X at x is a $\sim$-equivalence class of triples (U, ϕ, u). The set of tangent vectors to X at x will be denoted by T_xX.*

Remark 2. *A chart (U, ϕ) at x determines an associated isomorphism*

$$\theta_x\colon T_xX \to \mathbb{R}^d,$$

which takes $z \in T_xX$ to the unique vector $u \in \mathbb{R}^d$ such that $(U, \phi, u) \in z$. Bijectivity follows because the vector $u \in \mathbb{R}^d$ in (U, ϕ, u) is arbitrary.

1.3 Vector fields; brackets

Definition 13. *A vector field X on a differentiable manifold M is a correspondence that associates to each point $p \in M$ a vector $X(p) \in T_pM$. In terms of mappings, X is a mapping of M into the tangent bundle TM. The field is differentiable if the mapping $X\colon M \to TM$ is differentiable.*

Considering a parametrization $x\colon U \subset \mathbb{R}^n \to M$ we can write

$$X(p) = \sum_{i=1}^{n} a_i(p) \frac{\partial}{\partial x_i} \tag{1.3.1}$$

where each $a_i\colon U \to \mathbb{R}$ is a function on U and $\left\{\frac{\partial}{\partial x_i}\right\}$ is the basis associated to x, $i = 1, \ldots, n$. It is clear that X is differentiable if and only if the functions a_i are differentiable for some (and, therefore, for any) parametrization.

Occasionally, it is convenient to use the idea suggested by (1.3.1) and think of a vector field as a mapping $X\colon \mathcal{D} \to \mathcal{F}$ from the set $\mathcal{D}$ of differentiable functions on M to the set $\mathcal{F}$ of functions on M, defined in the following way

$$X(f)(p) = \sum_{i} a_i(p) \frac{\partial f}{\partial x_i}(p), \tag{1.3.2}$$

where f denotes, by abuse of notation, the expression of f in the parametrization x. Indeed, this idea of a vector as a directional derivative was precisely what was used to define the notion of tangent vector. It is easy to check that the function Xf obtained in (1.3.2) does not depend on the choice of parametrization x. In this context, it is immediate that X is differentiable if and only if $X\colon \mathcal{D} \to \mathcal{D}$, that is, $Xf \in \mathcal{D}$ for all $f \in \mathcal{D}$.

Observe that if $\varphi\colon M \to M$ is a diffeomorphism, $v \in T_pM$ and f is a differentiable function in a neighborhood of $\varphi(p)$, we have

$$(d\varphi(v) f)\varphi(p) = v(f \circ \varphi)(p).$$

Indeed, let $\alpha\colon (-\varepsilon, \varepsilon) \to M$ be a differentiable curve with $\alpha'(0) = v$, $\alpha(0) = p$. Then

$$(d\varphi(v) f)\varphi(p) = \frac{d}{dt}(f \circ \varphi \circ \alpha)\,|_{t=0} = v(f \circ \varphi)(p).$$

The interpretation of X as an operator on $\mathcal{D}$ permits us to consider the iterates of X. For example, if X and Y are differentiable fields on M and $f\colon M \to \mathbb{R}$ is a differentiable function, we can consider the functions $X(Yf)$ and $Y(Xf)$. In general, such operations do not lead to vector fields, because they involve derivatives of order higher than one. Nevertheless, we can affirm the following.

Lemme 1. *Let X and Y be differentiable vector fields on a differentiable manifold M. Then there exists a unique vector field Z such that, for all $f \in \mathcal{D}$,*

$$Zf = (XY - YX)f.$$

Proof. First, we prove that if Z exists, then it is unique. Assume, therefore, the existence of such a Z. Let $p \in M$ and let $x\colon U \to M$ be a parametrization at p, and let

$$X = \sum_i a_i \frac{\partial}{\partial x_i}, \quad Y = \sum_j b_j \frac{\partial}{\partial x_j}$$

be the expressions for X and Y in these parameterizations. Then for all $f \in \mathcal{D}$,

$$\begin{aligned} XYf &= X\left(\sum_i b_j \frac{\partial f}{\partial x_j}\right) \\ &= \sum_{i,j} a_i \frac{\partial b_j}{\partial x_i}\frac{\partial f}{\partial x_j} + \sum_{i.j} a_i b_j \frac{\partial^2 f}{\partial x_i \partial x_j}, \end{aligned}$$

$$\begin{aligned} YXf &= Y\left(\sum_i a_i \frac{\partial f}{\partial x_i}\right) \\ &= \sum_{i,j} b_j \frac{\partial a_i}{\partial x_j}\frac{\partial f}{\partial x_i} + \sum_{i.j} a_i b_j \frac{\partial^2 f}{\partial x_i \partial x_j}. \end{aligned}$$

Therefore, Z is given, in the parametrization x, by

$$\begin{aligned} Zf &= XYf - YXf \\ &= \sum_{i,j}\left(a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i}\right)\frac{\partial f}{\partial x_j} \end{aligned}$$

which proves the uniqueness of Z.

To show existence, define Z_α in each coordinate neighborhood $x_\alpha(U_\alpha)$ of a differentiable structure $\{(U_\alpha, x_\alpha)\}$ on M by the previous expression. By uniqueness, $Z_\alpha = Z_\beta$ on $x_\alpha(U_\alpha) \cap x_\beta(U_\beta) \neq \emptyset$, which allows us to define Z over the entire manifold M.

The vector field Z given by Lemma (1) is called the bracket $[X,Y] = XY - YX$ of X and Y; Z is obviously differentiable.

The bracket operation has the following properties:

Proposition 3. *If X, Y and Z are differentiable vector fields on M, a, b are real numbers, and f, g are differentiable functions, then:*

(a) $[X,Y] = -[Y,X]$ (anticommutativity),

(b) $[aX + bY, Z] = a[X,Z] + b[Y,Z]$ (linearity),

(c) $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$ (Jacobi identity),

(d) $[fX, gY] = fg[X,Y] + fX(g)Y - gY(f)X$.

Proof. (a) and (b) are immediate. In order to prove (c), it suffices to observe that, on the one hand,

$$\begin{aligned} [[[X,Y],Z]] &= [XY - YX, Z] \\ &= XYZ - YXZ - ZXY + ZYX \end{aligned}$$

while, on the other hand,

$$[X,[Y,Z]] + [Y,[Z,X]]$$
$$= XYZ - XZY - YZX + ZYX + YZX - YXZ - ZXY + XZY.$$

Because the second members of the expressions above are equal, (c) follows using (a).

Finally, to prove (d), calculate

$$\begin{aligned} [fX, gY] &= fX(gY) - gY(fX) \\ &= fgXY + fX(g)Y - gfYX - gY(f)X \\ &= fg[X,Y] + fX(g)X - gY(f)X. \end{aligned}$$

1.4 Lie groups

[8]The space $\mathbb{R}^n$ is a C^∞ manifold and at the same time an Abelean group with group operation given by componentwise addition. Moreover the algebraic and differentiable structures are related: $(x,y) \to x + y$ is a C^∞ mapping of the product manifold $\mathbb{R}^n \times \mathbb{R}^n$ onto $\mathbb{R}^n$, that is, the group operation is differentiable. We also see that the mapping of $\mathbb{R}^n$ onto $\mathbb{R}^n$ given by taking each element x to its inverse $-x$ is differentiable.

Now let G be a group which is at the same time a differentiable manifold. For $x, y \in G$ let xy denote their product and x^{-1} the inverse of x.

Definition 14. *G is a Lie group provided that the mapping of $G \times G \to G$ defined by $(x, y) \to xy$ and the mapping of $G \to G$ defined by $x \to x^{-1}$ are both C^{∞} mappings.*

Example 4. *$\mathbb{R}$ is a one-dimensional (Abelean) Lie group, where the group multiplication is the usual addition $+$. Similarly, any real or complex vector space is a Lie group under vector addition.*

1.5 Lie algebra

Definition 15. *We denote by $\mathfrak{X}(M)$ the set of all $C^{\infty}-$vector fields defined on $C^{\infty}-$manifold M.*

We shall say that a vector space $\mathfrak{X}(M)$ over $\mathbb{R}$ is a (real) Lie algebra if in addition to its vector space structure it possesses a product, that is, a map $\mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, taking the pair (X, Y) to the element $[X, Y]$ of $\mathfrak{X}(M)$, which has the following properties:

(1) it is bilinear over $\mathbb{R}$:

$$\begin{aligned} [\alpha_1 X_1 + \alpha_2 X_2, Y] &= \alpha_1 [X_1, Y] + \alpha_2 [X_2, Y], \\ [X, \alpha_1 Y_1 + \alpha_2 Y_2] &= \alpha_1 [X, Y_1] + \alpha_2 [X, Y_2], \end{aligned}$$

(2) it is skew commutative:

$$[X, Y] = -[Y, X],$$

(3) it satisfies the Jacobi identity:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

Theorem 2. *$\mathfrak{X}(M)$ with the product $[X, Y]$ is a Lie algebra.*

Proof. If $\alpha, \beta \in \mathbb{R}$ and X_1, X_2, Y are C^{∞}-vector fields, then it is straightforward to verify that

$$[\alpha X_1 + \beta X_2, Y] f = \alpha [X_1, Y] f + \beta [X_2, Y] f.$$

Thus $[X, Y]$ is linear in the first variable. Since the skew commutativity $[X, Y] = -[Y, X]$ is immediate from the definition, we see that linearity in the first variable implies linearity in the second. Therefore $[X, Y]$ is bilinear and skew-commutative. There remains the Jacobi identity which follows immediately if we evaluate

$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]]$ applied to a C^{∞}-function f. Using the definition, we obtain

$$\begin{aligned} [X, [Y, Z]] f &= X (([Y, Z]) f) - [Y, Z] (Xf) \\ &= X (Y (Zf)) - X (Z (Yf)) - Y (Z (Xf)) + Z (Y (Xf)). \end{aligned}$$

Permuting cyclically and adding establishes the identity.

1.6 Riemannian manifolds

The space

$$L^2(T_mM,\mathbb{R}) = \{\alpha\colon T_mM \times T_mM \to \mathbb{R} / \alpha \text{ is bilinear}\}$$

has a basis where the

$$\{dx_i \otimes dx_j / i,j = 1,\ldots,n\}$$

where the dx_i form the dual basis of the dual space

$$(T_mM)^* = L(T_mM,\mathbb{R}) = \{w\colon T_mM \to \mathbb{R} / \text{ linear form}\}$$

defined as follows :

$$dx_i\left(\frac{\partial}{\partial x_j}\right) = \delta_{ij} = \begin{cases} 1 \text{ if } i=j \\ 0 \text{ if } i \neq j \end{cases}$$

Bilinear forms $dx_i \otimes dx_j$ are defined in terms of their action based on :

$$(dx_i \otimes dx_j)\left(\frac{\partial}{\partial x_k},\frac{\partial}{\partial xl}\right) = \delta_{ik}\delta_{jl} = \begin{cases} 1 \text{ if } i=k \text{ and } j=l \\ 0 \text{ otherwise} \end{cases}$$

By inserting the base, for the coefficients of the representation

$$\alpha = \sum_{i,j} \alpha_{ij} dx_i \otimes dx_j$$

we get the expression

$$\alpha_{ij} = \alpha\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right).$$

Definition 16. *A Riemannian metric (or Riemannian structure) on a differentiable manifold M is a correspondence which associates to each point p of M an inner product $\langle\ ,\ \rangle_p$ (that is, a symmetric, bilinear, positive-definite form) on the tangent space T_pM, which varies differentiably in the following sense: If $x\colon U \subset \mathbb{R}^n \to M$ is a system of coordinates around p, with $x(x_1,x_2,\ldots,x_n) = q \in x(U)$ and $\frac{\partial}{\partial x_i}(q) = dx_q(0,\ldots,1,\ldots,0)$, then $\left\langle \frac{\partial}{\partial x_i}(q), \frac{\partial}{\partial x_j}(q)\right\rangle_q = g_{ij}(x_1,\ldots x_n)$ is a differentiable function on U.*

Remark 3. *A Riemannian metric g on M is a map $m \mapsto g_m \in L^2(T_mM,\mathbb{R})$ such that the following conditions hold :*

1. *$g_m(X,Y) = g_m(Y,X)$ for everything X,Y.*
2. *$g_m(X,X) > 0$ for everything $X \neq 0$.*
3. *The coefficients g_{ij} in each local representation (i.e , in any map)*

$$g_m = \sum_{i,j} g_{ij}(m)\, dx_i \otimes dx_j$$

are differentiable functions .

(M,g) is then called Riemannian manifold.

Example 5. *In $\mathbb{R}^3$, the Euclidian metric $g_0 = dx^2 + dy^2 + dz^2$ is a Riemannian metric .*

1.6.1 Isometry

Definition 17. *$f\colon (M,g) \to (N,h)$ an isometry ((M,g) and (N,h) are two Riemannian manifolds) if f is a diffeomorphism such that*

$h\left(D_m f(X), D_m f(X)\right) = g(X,Y)$ at any point $m \in M$ and for all vectors X and Y tangent in m to M.

1.6.2 The first and second fundamental form

Definition 18. *Given a surface X, for any point $p = X(u,v)$ on X, and letting*

$$E = \langle X_u, X_u \rangle, \quad F = \langle X_u, X_v \rangle, \quad G = \langle X_v, X_v \rangle.$$

The positive definite quadratic form $(x,y) \to Ex^2 + 2Fxy + Gy^2$ is called the first fundamental form of X at p. It is often denoted as I_p and in matrix form, we have

$$I_p(x,y) = (x,y) \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Since the map $(x,y) \to Ex^2 + 2Fxy + Gy^2$ is a positive definite quadratic form, we must have $E \neq 0$ and $G \neq 0$.

Then, we can write

$$Ex^2 + 2Fxy + Gy^2 = E\left(x + \frac{F}{E}y\right)^2 + \frac{EG - F^2}{E}y^2.$$

Since this quantity must be positive, we must have $E > 0$, $G > 0$, and also $EG - F^2 > 0$.

Definition 19. *Given a surface X, for any point $p = X(u,v)$ on X, and letting*

$$l = \langle X_{\mathrm{uu}}, N \rangle, \quad m = \langle X_{uv}, N \rangle, \quad n = \langle X_{vv}, N \rangle,$$

where N is the unit normal vector such that

$$N = \frac{X_u \times X_v}{\|X_u \times X_v\|}.$$

The quadratic form $(x,y) \to lx^2 + 2mxy + ny^2$ is called the second fundamental form of X at p. It is often denoted as II_p and in matrix form, we have

$$I_p(x,y) = (x,y) \begin{pmatrix} l & m \\ m & n \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

1.6.3 Christoffel symbols

Definition 20. *Let $g\colon U \to \mathbb{R}^{2\times 2}$ be a metric tensor of class C^2. The Christoffel symbols of the first kind of this metric tensor are the 2^3 functions.*

$$\Gamma_{ijk}\colon = \frac{1}{2}\left(\partial_i g_{jk} + \partial_j g_{ki} + \partial_k g_{ij}\right) : U \to \mathbb{R}$$

$(1 \leq i,j,k \leq 2)$ *and the Christoffel symbols of the second kind of this metric tensor are the* 2^3 *functions.*

$$\Gamma_{ij}^{k}\colon = \sum_{\alpha} g^{\alpha k}\Gamma_{ij\alpha}\colon U \to \mathbb{R}.$$

$(1 \leq i,j,k \leq 2)$

Where $\left(g^{\alpha k}\right)$ *is the inverse matrix of* (g_{ij}).

1.6.4 The canonical connection

Definition 21. *An affine connection* ∇ *on a differentiable manifold* M *is a mapping*

$$\nabla\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

which is denoted by $(X,Y) \overset{\nabla}{\to} \nabla_X Y$ *and which satisfies the following properties :*

i) $\nabla_{fX+gY}Z = f\nabla_X Z + g\nabla_Y Z.$

ii) $\nabla_X (Y+Z) = \nabla_X Y + \nabla_X Z.$

iii) $\nabla_X (fY) = f\nabla_X Y + X(f)Y,$

in which $X,Y,Z \in \mathfrak{X}(M)$ and $f,g \in \mathcal{D}(M)$.

Corollary 1. *A connection* ∇ *on a Riemannian manifold* M *is compatible with the metric if and only if*

$$X\langle Y,Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle, \qquad X,Y,Z \in \mathfrak{X}(M). \tag{1.6.1}$$

Proof. Suppose that ∇ is compatible with the metric. Let $p \in M$ and let $c\colon I \to M$ be a differentiable curve with $c(t_0) = p$, $t_0 \in I$, and with $\frac{dc}{dt}\mid_{t=t_0} = X(p)$. Then

$$\begin{aligned} X(p)\langle Y,Z\rangle &= \frac{d}{dt}\langle Y,Z\rangle \mid_{t=t_0} \\ &= \left\langle \nabla_{X(p)}Y, Z\right\rangle_p + \left\langle Y, \nabla_{X(p)}Z\right\rangle_p. \end{aligned}$$

Since p is arbitrary, (1.6.1) follows. The converse is obvious.

Definition 22. *An affine connection* ∇ *on a smooth manifold* M *is said to be symmetric when*

$$\nabla_X Y - \nabla_Y X = [X,Y] \quad \text{for all} \quad X,Y \in \mathfrak{X}(M). \tag{1.6.2}$$

Remark 4. *In a coordinate system* (U,x), *the fact that* ∇ *is symmetric implies that for all* $i,j = 1,\ldots,n,$

$$\nabla_{X_i}X_j - \nabla_{X_j}X_i = [X_i,X_j] = 0, \quad X_i = \frac{\partial}{\partial x_i}, \tag{1.6.3}$$

which justifies the terminology (observe that (1.12.3) is equivalent to the fact that $\Gamma_{ij}^{k} = \Gamma_{ji}^{k}$).

Theorem 3. *(Levi-Civita). Given a Riemannian manifold M, there exists a unique affine connection ∇ on M satisfying the conditions:*

a. ∇ is symmetric.

b. ∇ is compatible with the Riemannian metric.

Proof. Suppose initially the existence of such a ∇. Then

$$X\langle Y.Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle, \tag{1.6.4}$$

$$Y\langle Z, X\rangle = \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle, \tag{1.6.5}$$

$$Z\langle X, Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle. \tag{1.6.6}$$

Adding (1.6.4) and (1.6.5) and subtracting (1.6.6), we have, using the symmetry of ∇, that

$$\begin{aligned} & X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ = \; & \langle [X, Z], Y\rangle + \langle [Y, Z], X\rangle + \langle [X, Y], Z\rangle + 2\langle Z, \nabla_Y X\rangle. \end{aligned}$$

Therefore

$$\langle Z, \nabla_Y X\rangle = \frac{1}{2}\left\{ \begin{array}{c} X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle - \langle [X, Z], Y\rangle \\ - \langle [Y, Z], X\rangle - \langle [X, Y], Z\rangle \end{array} \right\} \tag{1.6.7}$$

The expression (1.6.7) shows that ∇ is uniquely determined from the metric $\langle\ ,\ \rangle$. Hence, if it exists, it will be unique.

To prove existence, define ∇ by (1.6.7). It is easy to verify that ∇ is well-defined and that it satisfies the desired conditions.

Definition 23. *The curvature R of a Riemannien manifold M is a correspondence that associates to every pair $X, Y \in \mathfrak{X}(M)$ a mapping*

$R(X, Y) : \mathfrak{X}(M) \to \mathfrak{X}(M)$ given by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \quad Z \in \mathfrak{X}(M),$$

where ∇ is the Riemannian connection of M.

Observe that if $M = \mathbb{R}^n$, then $R(X, Y)Z = 0$ for all $X, Y, Z \in \mathfrak{X}(\mathbb{R}^n)$. In fact, if the vector field Z is given by $Z = (z_1, \ldots, z_n)$, with the components of Z coming from the natural coordinates of $\mathbb{R}^n$, we obtain

$$\nabla_X Z = (Xz_1, \ldots, Xz_n),$$

hence

$$\nabla_Y \nabla_X Z = (YXz_1, \ldots, YXz_n),$$

which implies that

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z = 0,$$

as was stated. We are able, therefore, to think of R as a way of measuring how much M deviates from being Euclidean.

Another way of viewing definition (24) is to consider a system of coordinates $\{x_i\}$ around $p \in M$. Since $\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0$, we obtain

$$R\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_k} = \left(\nabla_{\frac{\partial}{\partial x_i}}\nabla_{\frac{\partial}{\partial x_j}} - \nabla_{\frac{\partial}{\partial x_j}}\nabla_{\frac{\partial}{\partial x_i}}\right)\frac{\partial}{\partial x_k},$$

that is, the curvature measures the non-commutativity of the covariant derivative.

Proposition 4. *The curvature R of a Riemannian manifold has the following properties:*

(i) R is bilinear in $\mathfrak{X}(M) \times \mathfrak{X}(M)$, that is,

$$\begin{aligned} R(fX_1 + gX_2, Y_1) &= fR(X_1, Y_1) + gR(X_2, Y_1), \\ R(X_1, fY_1 + gY_2) &= fR(X_1, Y_1) + gR(X_1, Y_2), \end{aligned}$$

$f, g \in \mathcal{D}(M)$, $X_1, X_2, Y_1, Y_2 \in \mathfrak{X}(M)$.

(ii) For any $X, Y \in \mathfrak{X}(M)$, the curvature operator $R(X,Y) : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is linear, that is,

$$\begin{aligned} R(X,Y)(Z + W) &= R(X,Y)Z + R(X,Y)W, \\ R(X,Y)fZ &= fR(X,Y)Z, \end{aligned}$$

$f \in \mathcal{D}(M)$, $Z, W \in \mathfrak{X}(M)$.

Proof. Let us verify (ii) only. The first part of (ii) is obvious. As for the second, we have

$$\begin{aligned} \nabla_Y \nabla_X (fZ) &= \nabla_Y (f\nabla_X Z + (Xf)Z) \\ &= f\nabla_Y\nabla_X Z + (Yf)(\nabla_X Z) + (Xf)(\nabla_Y Z) + (Y(Xf))Z. \end{aligned}$$

Therefore,

$$\nabla_Y\nabla_X(fZ) - \nabla_X\nabla_Y(fZ) = f(\nabla_Y\nabla_X - \nabla_X\nabla_Y)Z + ((YX - XY)f)Z,$$

hence

$$\begin{aligned} R(X,Y)fZ &= f\nabla_Y\nabla_X Z - f\nabla_X\nabla_Y Z + ([Y,X]f)Z + f\nabla_{[X,Y]}Z + ([X,Y]f)Z \\ &= fR(X,Y)Z. \end{aligned}$$

Proposition 5. *(Bianchi Identity)*

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0.$$

Proof. From the symmetry of the Riemannian connection, we have,

$$\begin{aligned}
R(X,Y)Z + R(Y,Z)X + R(Z,X)Y &= \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z + \nabla_{[X,Y]}Z \\
&\quad +\nabla_Z\nabla_Y X - \nabla_Y\nabla_Z X + \nabla_{[Y,Z]}X \\
&\quad +\nabla_X\nabla_Z Y - \nabla_Z\nabla_X Y + \nabla_{[Z,X]}Y \\
&= \nabla_Y[X,Z] + \nabla_Z[Y,X] + \nabla_X[Z,Y] \\
&\quad -\nabla_{[X,Z]}Y - \nabla_{[Y,X]}Z - \nabla_{[Z,Y]}X \\
&= [Y,[X,Z]] + [Z,[Y,X]] + [X,[Z,Y]] \\
&= 0,
\end{aligned}$$

where the last equality follows from the Jacobi identity for vector fields.

From now on, we shall write $\langle R(X,Y)Z,T\rangle = (X,Y,Z,T)$.

Proposition 6. *(a)* $(X,Y,Z,T) + (Y,Z,X,T) + (Z,X,Y,T) = 0$,

(b) $(X,Y,Z,T) = -(Y,X,Z,T)$,

(c) $(X,Y,Z,T) = -(X,Y,T,Z)$,

(d) $(X,Y,Z,T) = (Z,T,X,Y)$.

Proof. (a) is just the Bianchi identity again;

(b) follows directly from Definition (curvature);

(c) is equivalent to $(X,Y,Z,Z) = 0$, whose proof follows:

$$(X,Y,Z,Z) = \left\langle \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z + \nabla_{[X.Y]}Z, Z\right\rangle.$$

But

$$\langle\nabla_Y\nabla_X Z, Z\rangle = Y\langle\nabla_X Z, Z\rangle - \langle\nabla_X Z, \nabla_Y Z\rangle,$$

and

$$\left\langle\nabla_{[X,Y]}Z, Z\right\rangle = \frac{1}{2}[X,Y]\langle Z,Z\rangle.$$

Hence

$$\begin{aligned}
(X,Y,Z,Z) &= Y\langle\nabla_X Z, Z\rangle - X\langle\nabla_Y Z, Z\rangle + \frac{1}{2}[X,Y]\langle Z,Z\rangle \\
&= \frac{1}{2}Y(X\langle Z,Z\rangle) - \frac{1}{2}X(Y\langle Z,Z\rangle) + \frac{1}{2}[X,Y]\langle Z,Z\rangle \\
&= 0,
\end{aligned}$$

which proves(c).

In order to prove (d),we use (a), and write:

$$(X,Y,Z,T) + (Y,Z,X,T) + (Z,X,Y,T) = 0,$$

$$(Y,Z,T,X) + (Z,T,Y,X) + (T,Y,Z,X) = 0,$$

$$(Z,T,X,Y) + (T,X,Z,Y) + (X,Z,T,Y) = 0,$$

$$(T, X, Y, Z) + (X, Y, T, Z) + (Y, T, X, Z) = 0.$$

Summing the equations above, we obtain

$$2\,(Z, X, Y, T) + 2\,(T, Y, Z, X) = 0$$

and, therefore,

$$(Z, X, Y, T) = (Y, T, Z, X)\,.$$

1.6.5 The curvature average

Definition 24.

$$H = \frac{lG + nE - 2mF}{2\,(EG - F^2)}.$$

If $H = 0$, we say that (S) is minimal. Where the coefficients E, F, G, l, n and m are here the coefficients of the first and second fundamental forms.

Chapter 2

Minimal translation surfaces in $H^2 \times \mathbb{R}$

The name minimal surfaces has been applied to surfaces of vanishing mean curvature, because the condition $H = 0$ will necessarily be satisfied by surfaces which minimize area within a given boundary configuration [1]. So, in the chapter we define the minimal surface (section 2.1). Then, we define the Lie group $H^2 \times \mathbb{R}$ (section 2.2), and in section 2.3 we classify the minimal translation surface of type 1. At the end, in section 2.4 we classify the minimal translation surface of type 2.

2.1 Minimal surface

Definition 25. *A minimal surface is a closed and bounded surface of a real Euclidean affine space of dimension 3 with regular boundary minimizing the total area with fixed contour. In other words, a minimal surface in a given Riemannian manifold is the embedding of a compact manifold with boundary minimizing the Riemannian volume with fixed boundary .*

Definition 26. *In the space $H^2 \times \mathbb{R}$, the surfaces which locally minimize the areas are called minimal surfaces, they satisfy the condition $H = 0$, where H is the mean curvature given by the formula*:

$$H = \frac{lG + nE - 2Fm}{2\,(EG - F^2)}.$$

2.2 The Lie group $H^2 \times \mathbb{R}$

$H^2 \times \mathbb{R}$ a Riemannian manifold endowed with a left invariant metric:

$$g_{H^2 \times \mathbb{R}} = \frac{1}{y^2}\left(dx^2 + dy^2\right) + dz^2$$

The Riemannian product space $H^2 \times \mathbb{R}$ is a Lie group with respect to the operation :

$$(x, y, z) * (\bar{x}, \bar{y}, \bar{z}) = (\bar{x}y + x, y\bar{y}, z + \bar{z})$$

An orthonormal basis of left invariant vector fields $\{E_1, E_2, E_3\}$ on $H^2 \times \mathbb{R}$ is given by

$$E_1 = y\frac{\partial}{\partial x};\ E_2 = y\frac{\partial}{\partial y};\quad E_3 = \frac{\partial}{\partial z}.$$

The Levi-Civita connection of the $H^2 \times \mathbb{R}$ space with respect to this base is

$$\begin{aligned}
\tilde{\nabla}_{E_1} E_1 &= E_2, \quad \tilde{\nabla}_{E_1} E_2 = -E_1, \quad \tilde{\nabla}_{E_1} E_3 = 0, \\
\tilde{\nabla}_{E_2} E_1 &= 0, \quad \tilde{\nabla}_{E_2} E_2 = 0, \quad \tilde{\nabla}_{E_2} E_3 = 0, \\
\tilde{\nabla}_{E_3} E_1 &= 0, \quad \tilde{\nabla}_{E_3} E_2 = 0, \quad \tilde{\nabla}_{E_3} E_3 = 0.
\end{aligned}$$

where (x, y, z) are usual coordinates of $\mathbb{R}^3$.

On the only hand, for any vectors $X = x_1 E_1 + y_1 E_2 + z_1 E_3$ and $Y = x_2 E_1 + y_2 E_2 + z_2 E_3$ in $H^2 \times \mathbb{R}$ the cross product $\times$ is defined by:

$$\begin{aligned}
X \times Y &= \begin{vmatrix} E_1 & E_2 & E_3 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} \\
&= \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} E_1 - \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} E_2 + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} E_3 \\
&= (y_1 z_2 - y_2 z_1) E_1 + (x_2 z_1 - x_1 z_2) E_2 + (x_1 y_2 - x_2 y_1) E_3 \\
&= (y_1 z_2 - y_2 z_1, x_2 z_1 - x_1 z_2, x_1 y_2 - x_2 y_1).
\end{aligned}$$

Lie brackets are :

$$[E_1, E_2] = -E_1; \quad [E_2, E_3] = 0; \quad [E_3, E_1] = 0.$$

As well as

$$\begin{aligned}
g(E_1, E_1) &= g(E_2, E_2) = g(E_3, E_3) = 1, \\
g(E_1, E_2) &= g(E_2, E_3) = g(E_1, E_3) = 0.
\end{aligned}$$

Thus we have directly the fundamental tensor of g (i.e: the matrix g_{ij}) associated with the metric, and its inverse g^{ij}. The associated matrices are :

$$(g_{ij})_{1 \le i,j \le 3} = \begin{pmatrix} \frac{1}{y^2} & 0 & 0 \\ 0 & \frac{1}{y^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (g^{ij})_{1 \le i,j \le 3} = \begin{pmatrix} y^2 & 0 & 0 \\ 0 & y^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with $\det(g_{ij}) = \frac{1}{y^4}$.

The Christoffel symbols as well as the Levi-Civita connecting forms in (x, y, z) coordinates for the metric g are :

$$\begin{aligned}
\Gamma_{ij}^k &= \frac{1}{2}\left[g^{k1}\left(\frac{\partial g_{i1}}{\partial x_j} + \frac{\partial g_{j1}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x}\right)\right] + \frac{1}{2}\left[g^{k2}\left(\frac{\partial g_{i2}}{\partial x_j} + \frac{\partial g_{j2}}{\partial x_i} - \frac{\partial g_{ij}}{\partial y}\right)\right] \\
&+ \frac{1}{2}\left[g^{k3}\left(\frac{\partial g_{i3}}{\partial x_j} + \frac{\partial g_{j3}}{\partial x_i} - \frac{\partial g_{ij}}{\partial z}\right)\right]
\end{aligned}$$

$i, j, k = 1, 2, 3$ with $x_1 = x$, $x_2 = y$, $x_3 = z$.

So we get :

$$\begin{cases}
\Gamma_{11}^2 = \frac{1}{y}, \ \Gamma_{12}^1 = \Gamma_{21}^1 = -\frac{1}{y}, \ \Gamma_{23}^1 = \Gamma_{32}^1 = 0, \ \Gamma_{22}^3 = 0, \\
\Gamma_{22}^2 = -\frac{1}{y}, \ \Gamma_{13}^2 = \Gamma_{31}^2 = 0, \ \Gamma_{33}^1 = 0, \ \Gamma_{11}^3 = 0, \ \Gamma_{23}^3 = \Gamma_{32}^3 = 0, \\
\Gamma_{11}^1 = 0, \ \Gamma_{12}^2 = \Gamma_{21}^2 = 0, \ \Gamma_{33}^2 = 0, \ \Gamma_{12}^3 = \Gamma_{21}^3 = 0, \\
\Gamma_{22}^1 = 0, \ \Gamma_{13}^1 = \Gamma_{31}^1 = 0, \ \Gamma_{33}^3 = 0, \ \Gamma_{13}^3 = \Gamma_{31}^3 = 0, \ \Gamma_{23}^2 = \Gamma_{23}^2 = 0.
\end{cases}$$

Definition 27. *A translation surfaces $\sum(\alpha,\beta)$ in $H^2\times\mathbb{R}$ is a surface parameterized by:*

$$x\colon \sum \to H^2\times\mathbb{R}, \qquad x(s,t)=\alpha(s)*\beta(t),$$

where α and β are any generating planar curves lying in orthogonal planes of $\mathbb{R}^3$.

- *We emphasize that the group operation $*$ on the space $H^2\times\mathbb{R}$ is not commutative, we have two translation surfaces, namely $\sum(\alpha,\beta)$ and $\sum(\beta,\alpha)$ which are different. According to planar curves α and β, we distinguish two*

types as follows:

We assume that $\alpha(s)$ and $\beta(t)$ lie in the yz-plane and xy-plane of $\mathbb{R}^3$, respectively. That is

$$\begin{aligned}\alpha(s) &= (0,s,f(s)),\\ \beta(t) &= (g(t),t,0),\end{aligned}$$

where $f(s)$ and $g(t)$ are smooth functions and $s,t>0$.

In this case, we have two translation surfaces $\sum_1(\alpha,\beta)$ and $\sum_2(\beta,\alpha)$ parameterized by:

$$\begin{aligned}x(s,t) &= \alpha(s)*\beta(t)\\ &= (0,s,f(s))*(g(t),t,0)\\ &= (sg(t),st,f(s)),\end{aligned}$$

and

$$\begin{aligned}x(s,t) &= \beta(t)*\alpha(s)\\ &= (g(t),t,0)*(0,s,f(s))\\ &= (g(t),st,f(s)),\end{aligned}$$

which are called the translation surfaces of type 1 and 2, respectively.

Remark 5. *1) If one curve lies in the xz-plane,then the translation surface is a part of xz-plane .*

2) The translation surfaces generated by $\alpha(s)=(0,c_1,s)$ and $\beta(t)=(t,c_2,0)$ $(c_1,\ c_2\in\mathbb{R}^+)$ are planes. So, translation surfaces except for Remarks 1) and 2) are meaningful for our study, because planes are trivial minimal surfaces.

2.3 Classification of type 1 minimal translation surface

Let $\sum_1$ be a translation surface of type 1 in Riemannian product space $H^2\times\mathbb{R}$. Then , $\sum_1$ is parameterized by :

$$x(s,t)=(sg(t),st,f(s)) \tag{2.3.1}$$

for all $s>0$ and $t>0$.

We have

$$\begin{aligned}
\frac{\partial x}{\partial s}: &= x_s = \frac{D}{Ds}x(s,t)\\
&= (g(t), t, f'(s))\\
&= \frac{g(t)}{y}\cdot y\frac{\partial}{\partial x} + \frac{t}{y}\cdot y\frac{\partial}{\partial y} + f'(s)\cdot\frac{\partial}{\partial z} \text{ with in this case } y = st\\
&= \frac{g(t)}{st}E_1 + \frac{1}{s}E_2 + f'(s)E_3.\\
\frac{\partial x}{\partial t}: &= x_t = \frac{D}{Dt}x(s,t)\\
&= (sg'(t), s, 0)\\
&= \frac{sg'(t)}{y}\cdot y\frac{\partial}{\partial x} + \frac{s}{y}\cdot y\frac{\partial}{\partial y} \text{ with in this case } y = st\\
&= \frac{g'(t)}{t}E_1 + \frac{1}{t}E_2.
\end{aligned}$$

The coefficients of the first fundamental form of $\sum_1$ are given by:

$$\begin{aligned}
E &= \langle x_s, x_s\rangle\\
&= \left(\frac{g(t)}{st}, \frac{1}{s}, f'(s)\right)\begin{pmatrix}\frac{g(t)}{st}\\ \frac{1}{s}\\ f'(s)\end{pmatrix}\\
&= \left(\frac{g(t)}{st}\right)^2 + \frac{1}{s^2} + (f'(s))^2,
\end{aligned}$$

$$\begin{aligned}
F &= \langle x_s, x_t\rangle\\
&= \left(\frac{g(t)}{st}, \frac{1}{s}, f'(s)\right)\begin{pmatrix}\frac{g'(t)}{t}\\ \frac{1}{t}\\ 0\end{pmatrix}\\
&= \frac{g(t)g'(t)}{st^2} + \frac{1}{st},
\end{aligned}$$

$$\begin{aligned}
G &= \langle x_t, x_t\rangle\\
&= \left(\frac{g'(t)}{t}, \frac{1}{t}, 0\right)\begin{pmatrix}\frac{g'(t)}{t}\\ \frac{1}{t}\\ 0\end{pmatrix}\\
&= \left(\frac{g'(t)}{t}\right)^2 + \frac{1}{t^2}.
\end{aligned}$$

The unit normal vector field U of $\sum_1$ is given by:

$$U = \frac{x_s \times x_t}{\|x_s \times x_t\|} = -\frac{f'(s)}{wt}E_1 + \frac{f'(s)g'(t)}{wt}E_2 + \left(\frac{g(t) - tg'(t)}{wst^2}\right)E_3,$$

where $w = \|x_s \times x_t\|$ and because

$$
\begin{aligned}
x_s \wedge x_t &= \left(\frac{g(t)}{st}, \frac{1}{s}, f^{'}(s) \right) \wedge \left(\frac{g^{'}(t)}{t}, \frac{1}{t}, 0 \right) \\
&= \left(-\frac{f^{'}(s)}{t}, \frac{f^{'}(s)\, g^{'}(t)}{t}, \frac{g(t)}{st^2} - \frac{g^{'}(t)}{st} \right).
\end{aligned}
$$

To compute the second fundamental form of $\sum_1$, we have to calculate the following:

$$
\begin{aligned}
\frac{D}{Ds}E_1 &= \tilde{\bigtriangledown}_{x_s} E_1 \\
&= \tilde{\bigtriangledown}_{\frac{g(t)}{st}E_1 + \frac{1}{s}E_2 + f^{'}(s)E_3} E_1 \\
&= \frac{g(t)}{st} \tilde{\bigtriangledown}_{E_1} E_1 + \frac{1}{s} \tilde{\bigtriangledown}_{E_2} E_1 + f^{'}(s)\, \tilde{\bigtriangledown}_{E_3} E_1 \\
&= \frac{g(t)}{st} E_2,
\end{aligned}
$$

$$
\begin{aligned}
\frac{D}{Ds}E_2 &= \tilde{\bigtriangledown}_{x_s} E_2 \\
&= \frac{g(t)}{st} \tilde{\bigtriangledown}_{E_1} E_2 + \frac{1}{s} \tilde{\bigtriangledown}_{E_2} E_2 + f^{'}(s)\, \tilde{\bigtriangledown}_{E_3} E_2 \\
&= -\frac{g(t)}{st} E_1,
\end{aligned}
$$

$$
\begin{aligned}
\frac{D}{Ds}E_3 &= \tilde{\bigtriangledown}_{x_s} E_3 \\
&= \frac{g(t)}{st} \tilde{\bigtriangledown}_{E_1} E_3 + \frac{1}{s} \tilde{\bigtriangledown}_{E_2} E_3 + f^{'}(s)\, \tilde{\bigtriangledown}_{E_3} E_3 \\
&= 0.
\end{aligned}
$$

$$
\begin{aligned}
\frac{D}{Dt}E_1 &= \tilde{\bigtriangledown}_{x_t} E_1 \\
&= \tilde{\bigtriangledown}_{\frac{g^{'}(t)}{t}E_1 + \frac{1}{t}E_2} E_1 \\
&= \frac{g^{'}(t)}{t} \tilde{\bigtriangledown}_{E_1} E_1 + \frac{1}{t} \tilde{\bigtriangledown}_{E_2} E_1 \\
&= \frac{g^{'}(t)}{t} E_2,
\end{aligned}
$$

$$
\begin{aligned}
\frac{D}{Dt}E_2 &= \tilde{\bigtriangledown}_{x_t} E_2 \\
&= \frac{g^{'}(t)}{t} \tilde{\bigtriangledown}_{E_1} E_2 + \frac{1}{t} \tilde{\bigtriangledown}_{E_2} E_2 \\
&= -\frac{g^{'}(t)}{t} E_1,
\end{aligned}
$$

$$\begin{aligned}
\frac{D}{Dt}E_3 &= \tilde{\nabla}_{x_t}E_3 \\
&= \frac{g'(t)}{t}\tilde{\nabla}_{E_1}E_3 + \frac{1}{t}\tilde{\nabla}_{E_2}E_3 \\
&= 0.
\end{aligned}$$

So, the covariant derivatives are :

$$\begin{aligned}
\tilde{\nabla}_{x_s}x_s &= \frac{D}{Ds}\left(\frac{g(t)}{st}E_1 + \frac{1}{s}E_2 + f'(s)E_3\right) \\
&= \frac{g(t)}{t}\left[-\frac{1}{s^2}E_1 + \frac{1}{s}\frac{D}{Ds}E_1\right] + \left(-\frac{1}{s^2}E_2 + \frac{1}{s}\frac{D}{Ds}E_2\right) + f''(s)E_3 + f'(s)\frac{D}{Ds}E_3 \\
&= -\frac{g(t)}{s^2t}E_1 + \frac{g^2(t)}{s^2t^2}E_2 - \frac{1}{s^2}E_2 - \frac{g(t)}{s^2t}E_1 + f''(s)E_3 \\
&= -\frac{2g(t)}{s^2t}E_1 + \left(\frac{g(t)^2}{s^2t^2} - \frac{1}{s^2}\right)E_2 + f''(s)E_3,
\end{aligned}$$

$$\begin{aligned}
\tilde{\nabla}_{x_s}x_t &= \frac{D}{Ds}\left(\frac{g'(t)}{t}E_1 + \frac{1}{t}E_2\right) \\
&= \frac{g'(t)}{t}\frac{D}{Ds}E_1 + \frac{1}{t}\frac{D}{Ds}E_2 \\
&= -\frac{g(t)}{st^2}E_1 + \left(\frac{g(t)g'(t)}{st^2}\right)E_2,
\end{aligned}$$

$$\begin{aligned}
\tilde{\nabla}_{x_t}x_t &= \frac{D}{Dt}\left(\frac{g'(t)}{t}E_1 + \frac{1}{t}E_2\right) \\
&= \frac{tg''(t) - g'(t)}{t^2}E_1 + \frac{g'(t)}{t}\frac{D}{Dt}E_1 - \frac{1}{t^2}E_2 + \frac{1}{t}\frac{D}{Dt}E_2 \\
&= \frac{tg''(t) - g'(t)}{t^2}E_1 + \frac{g'(t)^2}{t^2}E_2 - \frac{1}{t^2}E_2 - \frac{g'(t)}{t}E_1 \\
&= \left(\frac{tg''(t) - 2g'(t)}{t^2}\right)E_1 + \left(\frac{g'(t)^2 - 1}{t^2}\right)E_2.
\end{aligned}$$

We have

$$U = \frac{1}{w}\left(-\frac{f'(s)}{t}, \frac{f'(s)g'(t)}{t}, \frac{g(t) - tg'(t)}{st^2}\right)$$

and

$$\begin{aligned}
\tilde{\nabla}_{x_s}x_s &= \left(-\frac{2g(t)}{s^2t}, \frac{g(t)^2}{s^2t^2} - \frac{1}{s^2}, f''(s)\right), \\
\tilde{\nabla}_{x_s}x_t &= \left(-\frac{g(t)}{st^2}, \frac{g(t)g'(t)}{st^2}, 0\right), \\
\tilde{\nabla}_{x_t}x_t &= \left(\frac{tg''(t) - 2g'(t)}{t^2}, \frac{g'(t)^2 - 1}{t^2}, 0\right).
\end{aligned}$$

which imply the coefficients of the second fundamental form of $\sum_1$ are given by :

$$
\begin{aligned}
l &= \left\langle \tilde{\nabla}_{x_s} x_s, U \right\rangle \\
&= \left(-\frac{2g(t)}{s^2 t}, \frac{g(t)^2}{s^2t^2} - \frac{1}{s^2}, f''(s) \right) \begin{pmatrix} -\frac{f'(s)}{wt} \\ \frac{f'(s)g'(t)}{wt} \\ \frac{g(t)-tg'(t)}{wst^2} \end{pmatrix} \\
&= \frac{1}{w}\left[\frac{2f'(s)g(t)}{s^2t^2} + \frac{f'(s)g'(t)g(t)^2}{s^2t^3} - \frac{f'(s)g'(t)}{s^2 t} + \frac{g(t)f''(s)}{st^2} - \frac{tg'(t)f''(s)}{st^2} \right] \\
&= \frac{1}{ws^2t^3}\left(\begin{array}{c} 2tf'(s)g(t) + f'(s)g(t)^2 g'(t) - t^2 f'(s)g'(t) + stf''(s)g(t) \\ -st^2 f''(s)g'(t) \end{array} \right),
\end{aligned}
$$

$$
\begin{aligned}
m &= \left\langle \tilde{\nabla}_{x_s} x_t, U \right\rangle \\
&= \left(-\frac{g(t)}{st^2}, \frac{g(t)g'(t)}{st^2}, 0 \right) \begin{pmatrix} -\frac{f'(s)}{wt} \\ \frac{f'(s)g'(t)}{wt} \\ \frac{g(t)-tg'(t)}{wst^2} \end{pmatrix} \\
&= \frac{1}{w}\left[-\frac{f'(s)(-g(t))}{st^3} + \frac{f'(s)g(t)g'(t)^2}{st^3} \right] \\
&= \frac{1}{wst^3}\left(f'(s)g(t) + f'(s)g(t)g'(t)^2 \right),
\end{aligned}
$$

$$
\begin{aligned}
n &= \left\langle \tilde{\nabla}_{x_t} x_t, U \right\rangle \\
&= \left(\frac{tg''(t) - 2g'(t)}{t^2}, \frac{g'(t)^2 - 1}{t^2}, 0 \right) \begin{pmatrix} -\frac{f'(s)}{wt} \\ \frac{f'(s)g'(t)}{wt} \\ \frac{g(t)-tg'(t)}{wst^2} \end{pmatrix} \\
&= \frac{1}{w}\left[\frac{-tf'(s)g''(t)}{t^3} + \frac{2f'(s)g'(t)}{t^3} + \frac{f'(s)g'(t)^3}{t^3} - \frac{f'(s)g'(t)}{t^3} \right] \\
&= \frac{1}{wt^3}\left(-tf'(s)g''(t) + f'(s)g'(t) + f'(s)g'(t)^3 \right).
\end{aligned}
$$

The translation surface $\sum_1$ of type 1 is minimal if and only if:

$$H = \frac{lG - 2mF + nE}{2(EG - F^2)} = 0 \Leftrightarrow lG - 2mF + nE = 0$$

First let's calculate lG, mF and nE:

$$lG = \frac{1}{ws^2t^3}[2tf'(s)g(t) + f'(s)g(t)^2g'(t) - t^2f'(s)g'(t) + stf''(s)g(t) - st^2f''(s)g'(t)]$$
$$[\frac{g'(t)^2}{t^2} + \frac{1}{t^2}]$$
$$= \frac{1}{ws^2t^3}[\frac{2f'(s)g(t)g'(t)^2}{t} + \frac{f'(s)g(t)^2g'(t)^3}{t^2} - f'(s)g'(t)^3 + \frac{sf''(s)g(t)g'(t)^2}{t} - s''(s)g'(t)^3 + \frac{2f'(s)g(t)}{t} + \frac{f'(s)g(t)^2g'(t)}{t^2} - f'(s)g'(t) + \frac{sf''(s)g(t)}{t} - sf''(s)g'(t)],$$

$$mF = \frac{1}{wst^3}[f'(s)g(t) + f'(s)g(t)g'(t)^2][\frac{g(t)g'(t)}{st^2} + \frac{1}{st}]$$
$$= \frac{1}{wst^3}[\frac{f'(s)g(t)^2g'(t)}{st^2} + \frac{f'(s)g(t)^2g'(t)^3}{st^2} + \frac{f'(s)g(t)}{st} + \frac{f'(s)g(t)g'(t)^2}{st}]$$

$$nE = \frac{1}{wt^3}[-tf'(s)g''(t) + f'(s)g'(t) + f'(s)g'(t)^3][\frac{g(t)^2}{s^2t^2} + \frac{1}{s^2} + f'(s)^2]$$
$$= \frac{1}{wt^3}[\frac{-f'(s)g''(t)g(t)^2}{s^2t} + \frac{f'(s)g'(t)g(t)^2}{s^2t^2} + \frac{f'(s)g'(t)^3g(t)^2}{s^2t^2} - \frac{tf'(s)g''(t)}{s^2} + \frac{f'(s)g'(t)}{s^2} + \frac{f'(s)g'(t)^3}{s^2} + -tf'(s)^3g''(t) + f'(s)^3g'(t) + f'(s)^3g'(t)^3]$$

Then we obtain:

$$
\begin{aligned}
H=0 \Leftrightarrow & \frac{1}{w}[\frac{2f'(s)g(t)g'(t)^2}{s^2t^4}+\frac{f'(s)g(t)^2g'(t)^3}{s^2t^5}-\frac{f'(s)g'(t)^3}{s^2t^3}+\frac{f''(s)g(t)g'(t)^2}{st^4} \\
& -\frac{f''(s)g'(t)^3}{st^3}+\frac{2f'(s)g(t)}{s^2t^4}+\frac{f'(s)g(t)^2g'(t)}{s^2t^5}-\frac{f'(s)g'(t)}{s^2t^3} \\
& -\frac{f''(s)g'(t)}{st^3}-\frac{2f'(s)g(t)^2g'(t)}{s^2t^5}-\frac{2f'(s)g(t)^2g'(t)^3}{s^2t^5}-\frac{2f'(s)g(t)}{s^2t^4}+\frac{f''(s)g(t)}{st^4} \\
& -\frac{2f'(s)g(t)g'(t)^2}{s^2t^4}-\frac{f'(s)g''(t)g(t)^2}{s^2t^4}+\frac{f'(s)g(t)^2g'(t)}{s^2t^5} \\
& +\frac{f'(s)g(t)^2g'(t)^3}{s^2t^5}-\frac{f'(s)g''(t)}{s^2t^2}+\frac{f'(s)g'(t)}{s^2t^3}+\frac{f'(s)g'(t)^3}{s^2t^3} \\
& -\frac{f'(s)^3g''(t)}{t^2}+\frac{f'(s)^3g'(t)}{t^3}+\frac{f'(s)^3g'(t)^3}{t^3}]=0 \\
\Leftrightarrow & [\frac{f''(s)g(t)g'(t)^2}{st^4}-\frac{f''(s)g'(t)^3}{st^3}+\frac{f''(s)g(t)}{st^4}-\frac{f''(s)g'(t)}{st^3} \\
& -\frac{f'(s)g''(t)g(t)^2}{s^2t^4}-\frac{f'(s)g''(t)}{s^2t^2}-\frac{f'(s)^3g''(t)}{t^2}+\frac{f'(s)^3g'(t)}{t^3} \\
& +\frac{f'(s)^3g'(t)^3}{t^3}]=0 \\
\Leftrightarrow & \frac{1}{s^2t^4}[sf''(s)g(t)g'(t)^2-stf''(s)g'(t)^3+sf''(s)g(t) \\
& -stf''(s)g'(t)-f'(s)g''(t)g(t)^2-t^2f'(s)g''(t) \\
& -s^2t^2f'(s)^3g''(t)+s^2tf'(s)^3g'(t)+s^2tf'(s)^3g'(t)^3]=0 \\
\Leftrightarrow & [sf''(s)[g(t)g'(t)^2-tg'(t)^3+g(t)-tg'(t)] \\
& +f'(s)[-g''(t)g(t)^2-t^2g''(t)]+s^2f'(s)^3[tg'(t)-t^2g''(t)+tg'(t)^3]]=0 \qquad (1)
\end{aligned}
$$

We multiply (1) by (-1) , we find :

$$
\left[\begin{array}{c} s^2f'(s)^3\left[t^2g''(t)-tg'(t)-tg'(t)^3\right] \\ +sf''(s)\left[tg'(t)^3+tg'(t)-g(t)g'(t)^2-g(t)\right] \\ +f'(s)\left[g''(t)g(t)^2+t^2g''(t)\right] \end{array}\right]=0 \qquad (2.3.2)
$$

We start to study equation (2.3.2) in following cases :
If $f'(s)=0$, that is, $f(s)=k$ $(k\in\mathbb{R})$, the surface $\sum_1$ is parameterized by :

$$
x(s,t)=(sg(t),st,k),
$$

where $g(t)$ is an arbitrary function.

Now, we assume that $f'(s)\neq 0$ on an open interval. Since $s>0$, divide (2.3.2) by $s^2f'(s)^3$ we obtain:

$$
\left[\begin{array}{c} \left[t^2g''(t)-tg'(t)-tg'(t)^3\right]+\frac{f''(s)}{sf'(s)^3}\left[tg'(t)^3+tg'(t)-g(t)g'(t)^2-g(t)\right] \\ +\frac{1}{s^2f'(s)^2}\left[g''(t)g(t)^2+t^2g''(t)\right] \end{array}\right]=0
$$

and take derivative with respect to s:

$$\frac{d}{ds}\left(\frac{f''(s)}{sf'(s)^3}\right)\left[tg'(t)^3+t'g(t)-g'(t)^2g(t)-g(t)\right]$$
$$+\frac{d}{ds}\left(\frac{1}{s^2f'(s)^2}\right)\left[g(t)^2g''(t)+t^2g''(t)\right]=0.$$

Hence, we deduce the existence of a real number $a\in\mathbb{R}$ such that

$$\begin{aligned}\frac{d}{ds}\left(\frac{f''(s)}{sf'(s)^3}\right)&=-a\frac{d}{ds}\left(\frac{1}{s^2f'(s)^2}\right),\\ g(t)^2g''(t)+t^2g''(t)&=a\left[tg'(t)^3+tg'(t)-g'(t)^2g(t)-g(t)\right].\end{aligned}\tag{2.3.3}$$

Let us distinguish the following cases:

1 If $a=0$ i.e $\frac{d}{ds}\left(\frac{f''(s)}{sf'(s)^3}\right)=0$, then $\frac{f''(s)}{sf'(s)^3}=b$ and

$$\begin{aligned}g(t)^2g''(t)+t^2g''(t)=0\Leftrightarrow& g''(t)\left[g(t)^2+t^2\right]=0\\ \Rightarrow& g''(t)=0,\end{aligned}$$

that is $g(t)=c_1t+c_2\,(b,\ c_1,\ c_2\in\mathbb{R})$.

(i) Let $b=0$ i.e $\frac{f''(s)}{sf'(s)^3}=0\Leftrightarrow f''(s)=0$. Then $f(s)=d_1s+d_2\,(d_1\in\mathbb{R}^*,\ d_2\in\mathbb{R})$. In this case, equation (2.3.2) becomes

$$\begin{aligned}s^2f'(s)^3\left[t^2g''(t)-tg'(t)-tg'(t)^3\right]=0\Rightarrow& s^2d_1^3\left[-tc_1-tc_1^3\right]=0\\ \Rightarrow& -s^2d_1^3tc_1\left(1+c_1^2\right)=0\\ \Rightarrow& c_1\left(1+c_1^2\right)s^2d_1^3t=0\\ \Rightarrow& c_1=0\,(s>0,\ t>0,\ d_1\neq 0).\end{aligned}$$

Thus, the surface can be parameterize as

$$x(s,t)=(c_2s,st,d_1s+d_2).$$

(ii) If $b=-k^2\neq 0$, then $f''(s)=-k^2sf'(s)^3$ and the general solution of the ODE is given by:

$$f(s)=\frac{1}{k}\ln\left(s+\sqrt{s^2+\frac{2d_1}{k^2}}\right)+d_2,\tag{2.3.4}$$

Substituting (2.3.4) into (2.3.2), we easily obtain $c_1=c_2=0$. Thus, $g(t)=0$.
Where d_1 and d_2 are constants of integration.

(iii) If $b = k^2 \neq 0 \Rightarrow f''(s) = k^2 s f'(s)^3$, then the general solution of the ODE $f''(s) = k^2 s f'(s)^3$ is given by:

$$f(s) = \frac{1}{k}\sin^{-1}\frac{ks}{\sqrt{2}d_1} + d_2 \neq 0,$$

because we have

$$\begin{aligned}
\frac{f''(s)}{f'(s)^3} = k^2 s &\Longleftrightarrow -\frac{1}{2}\cdot\frac{1}{f'^2} = \frac{k^2}{2}s^2 + k_1 \\
&\Longleftrightarrow \frac{1}{f'^2} = -k^2 s^2 - 2k_1 \\
&\Longleftrightarrow f'^2 = \frac{1}{-k^2 s^2 - 2k_1} = \frac{1}{k^2\left(-s^2 - \frac{2k_1}{k^2}\right)} \\
&\Longleftrightarrow f' = \frac{1}{k\sqrt{d_1 - s^2}} \quad \text{with } d_1 = \frac{-2k_1}{k^2},\ k_1 \in \mathbb{R}^- \quad \text{so, } d_1 > 0 \\
&\Longleftrightarrow f = \int \frac{ds}{k\sqrt{1 - \left(\frac{s}{\sqrt{d_1}}\right)^2}}
\end{aligned}$$

which implies from (2.3.2) we can also obtain $c_1 = c_2 = 0$, that is $g(t) = 0$.

2 Suppose now $a \neq 0$. From the first equation in (2.3.3), we obtain

$$\begin{aligned}
\frac{f''(s)}{sf'(s)^3} = -a\frac{1}{s^2 f'(s)^2} + c_1 &\Leftrightarrow f''(s) + \frac{a}{s}f'(s) = c_1 s f'(s)^3 \\
&\Leftrightarrow f''(s) = -\frac{a}{s}f'(s) + c_1 s f'(s)^3 \\
&\Leftrightarrow f''(s) + \frac{a}{s}f'(s) = c_1 s f'(s)^3\,(c_1 \in \mathbb{R}), \qquad (2.3.5)
\end{aligned}$$

where c_1 is a constant of integration . We put $f'(s) = p(s)$. Then we find the Bernoulli's equation as follows :

$$\frac{dp}{ds} + \frac{a}{s}p = c_1 s p^3.$$

We divide by p^3, we obtain:

$$\frac{dp}{ds}p^{-3} + \frac{a}{s}p^{-2} = c_1 s \qquad (2)$$

To solve (2) we go through 2 stapes:
Step 1: homogenous first-order ODE

$$\begin{aligned} p^{\prime}p^{-3}+\frac{a}{s}p^{-2}=0 &\Longleftrightarrow p^{\prime}p^{-1}=-\frac{a}{s}\\ &\Longleftrightarrow \int\frac{p^{\prime}}{p}ds=-a\int\frac{1}{s}ds\\ &\Longleftrightarrow \ln|p(s)|=-a\ln s+h\\ &\Longleftrightarrow p(s)=\exp(-a\ln s+h).\end{aligned}$$

So,
$p=s^{-a}\exp h(s) \quad\Longrightarrow\quad p^{\prime}=-as^{-a-1}\exp h(s)+h^{\prime}(s)s^{-a}\exp h(s).$
Step 2: ODE of order 1 with second member

$$\begin{aligned}(2) &\Longleftrightarrow \left[-as^{-a-1}\exp h(s)+h^{\prime}(s)s^{-a}\exp h(s)\right]s^{3a}\exp(-3h(s))+\frac{a}{s}s^{2a}\exp(-2h(s))=c_1s\\ &\Longleftrightarrow -as^{2a-1}\exp(-2h(s))+h^{\prime}(s)s^{2a}\exp(-2h(s))+as^{2a-1}\exp(-2h(s))=c_1s\\ &\Longleftrightarrow h^{\prime}(s)s^{2a}\exp(-2h(s))=c_1s\\ &\Longleftrightarrow -\frac{1}{2}\int -2h^{\prime}(s)\exp(-2h(s))\,ds=\int c_1s^{1-2a}ds\\ &\Longleftrightarrow \exp(-2h(s))=\int -2c_1s^{-2a+1}ds+c_2\\ &\Longleftrightarrow h \quad=-\frac{1}{2}\ln\left(\int -2c_1s^{-2a+1}ds+c_2\right)=-\frac{1}{2}\ln\left(\frac{-2c_1}{2-2a}s^{2-2a}+c_2\right)\end{aligned}$$

So,

$$p=s^{-a}\left(\int -2c_1s^{-2a+1}ds+c_2\right)^{-\frac{1}{2}}$$

Then

$$p^{-2}=s^{2a}\left(\int -2c_1s^{-2a+1}ds+c_2\right), \tag{2.3.6}$$

where c_2 is a constant of integration.

(i) Let $a=1$. Then from (2.3.6) we have

$$p^{-2}=s^{2a}\left(-2c_1\ln s+c_2\right)$$

So

$$p=\frac{1}{s}\sqrt{c_2-2c_1\ln s}$$

We put $f^{\prime}(s)=p(s)$, then

$$f(s)=\int p(s)\,ds=\int\frac{1}{s}\sqrt{c_2-2c_1\ln s}\,ds=-\frac{1}{c_1}\int\frac{-2c_1\cdot\frac{1}{s}}{2\sqrt{c_2-2c_1\ln s}}ds. \tag{2.3.7}$$

$$
\begin{aligned}
(2.3.7) \Longrightarrow & f(s) = -\frac{1}{c_1}\sqrt{c_2 - 2c_1 \ln s} + c_3, \quad \text{where } c_3 \in \mathbb{R} \text{ and } c_1 = 0. \\
\Longrightarrow & f'(s) = \frac{1}{s\sqrt{c_2 - 2c_1 \ln s}} \\
\Longrightarrow & f''(s) = \frac{-\sqrt{c_2 - 2c_1 \ln s} - s \cdot \frac{-2c_1 \frac{1}{s}}{2\sqrt{c_2 - 2c_1 \ln s}}}{s^2 (c_2 - 2c_1 \ln s)} = \frac{-c_2 + c_1 (2\ln s + 1)}{s^2 (c_2 - 2c_1 \ln s)^{\frac{3}{2}}}.
\end{aligned}
$$

Then

$$
\begin{aligned}
(2.3.2) \Longleftrightarrow & \frac{s^2}{s^3 (c_2 - 2c_1 \ln s)^{\frac{3}{2}}} \underbrace{\left[t^2 g''(t) - t g'(t) - t g'(t)^3\right]}_{G_1(t)} \\
& + \frac{s(-c_2 + c_1 (2\ln s + 1))}{s^2 (c_2 - 2c_1 \ln s)^{\frac{3}{2}}} \underbrace{\left[t g'(t)^3 + t g'(t) - g(t) g'(t)^2 - g(t)\right]}_{G_2(t)} \\
& + \frac{1}{s\sqrt{c_2 - 2c_1 \ln s}} \underbrace{\left[g(t)^2 g''(t) + t^2 g''(t)\right]}_{G_3(t)} = 0 \\
\Longleftrightarrow & \frac{1}{s (c_2 - 2c_1 \ln s)^{\frac{3}{2}}} \left[G_1(t) + (c_1 (2\ln s + 1) - c_2) G_2(t) + (c_2 - 2c_1 \ln s) G_3(t)\right] = 0.
\end{aligned}
\tag{I}
$$

We have

$$G_2(t) = G_3(t) \quad \text{according to (2.3.3)}$$

So

$$
\begin{aligned}
(I) \Longleftrightarrow & G_1(t) + 2c_1 \ln s G_2(t) + c_1 G_2(t) - c_2 G_2(t) + c_2 G_3(t) - 2c_1 \ln s G_3(t) = 0 \\
\Longleftrightarrow & G_1(t) + 2c_1 \ln s G_3(t) + c_1 G_3(t) - c_2 G_3(t) + c_2 G_3(t) - 2c_1 \ln s G_3(t) = 0 \\
\Longleftrightarrow & G_1(t) + c_1 G_3(t) = 0 \\
\Longleftrightarrow & t^2 g''(t) - t g'(t) - t g'(t)^3 + c_1 \left(g(t)^2 g''(t) + t^2 g''(t)\right) = 0 \\
\Longleftrightarrow & (1 + c_1) t^2 g''(t) + c_1 g(t)^2 g''(t) = t g'(t) \left[1 + g'(t)^2\right].
\end{aligned}
$$

Then

$$\left[(1 + c_1) t^2 + c_1 g(t)^2\right] g''(t) = t g'(t) \left[1 + g'(t)^2\right]. \tag{2.3.8}$$

1. If $c_1 = 0$, then equation (2.3.8) becomes

$$a t^2 g''(t) = t g'(t) + t g'(t)^3 \iff g''(t) - \tfrac{1}{t} g'(t) - \tfrac{1}{t} g'(t)^3 = 0,$$

We put $g'(t) = w(t)$. Then we can obtain the Bernoulli's equation as follows:

$$\frac{dw}{dt} - \frac{1}{t} w = \frac{1}{t} w^3$$

We divide by w^3, we obtain:

$$\frac{dw}{dt}w^{-3} - \frac{1}{t}w^{-2} = \frac{1}{t} \tag{3}$$

To solve (3) we go through 2 stapes:
Step 1: homogenous first-order ODE

$$\begin{aligned} w'w^{-3} - \frac{1}{t}w^{-2} = 0 &\Longleftrightarrow w'w^{-1} = \frac{1}{t} \\ &\Longleftrightarrow \int \frac{w'}{w}dt = \int \frac{1}{t}dt \\ &\Longleftrightarrow \ln|w(t)| = \ln t + v \\ &\Longleftrightarrow w(t) = t\exp v(t) \end{aligned}$$

So

$$w = t\exp v(t) \Longrightarrow w' = \exp v(t) + tv'(t)\exp v(t).$$

Step 2: ODE of order 1 with second member

$$\begin{aligned} (3) &\Longleftrightarrow [\exp v(t) + tv'(t)\exp v(t)]\, t^{-3}\exp(-3v(t)) - \frac{t^{-2}}{t}\exp(-2v(t)) = \frac{1}{t} \\ &\Longleftrightarrow t^{-2}v'(t)\exp(-2v(t)) + t^{-3}\exp(-2v(t)) - t^{-3}\exp(-2v(t)) = \frac{1}{t} \\ &\Longleftrightarrow v'(t)\exp(-2v(t)) = \frac{1}{t^{-1}} \\ &\Longleftrightarrow -\frac{1}{2}\int -2v'(t)\exp(-2v(t))\,dt = \int tdt \\ &\Longleftrightarrow \exp(-2v(t)) = -2\left[\frac{t^2}{2}\right] + d_1 \\ &\Longleftrightarrow v = -\frac{1}{2}\ln\left(d_1 - t^2\right) \end{aligned}$$

So

$$w = t\exp\left(-\frac{1}{2}\ln\left(d_1 - t^2\right)\right) = t\left(d_1 - t^2\right)^{-\frac{1}{2}} = g'(t).$$

Then

$$g(t) = -\sqrt{d_1 - t^2} \quad (d_1 \in \mathbb{R}).$$

And from (3) give

$$\begin{aligned} f''(s) + \frac{1}{s}f'(s) = 0 &\Longleftrightarrow \frac{f''(s)}{f'(s)} = -\frac{1}{s} \\ &\Longleftrightarrow \ln|f'(s)| = -\ln s + c_2 \\ &\Longleftrightarrow f'(s) = d_2 s^{-1} \\ &\Longleftrightarrow f(s) = d_2\ln s + d_3 \quad (d_2, d_3 \in \mathbb{R}). \end{aligned}$$

(ii) Let $a \neq 1$. In this case, the function $f(s)$ satisfying equation (2.3.5) appears in the from

$$f(s) = \frac{1}{\sqrt{|c_2|}} \int \frac{1}{s\sqrt{s^{2(a-1)} + \frac{c_1}{c_2(a-1)}}} ds \tag{2.3.9}$$

because we have

$$p^{-2} = s^{2a}\left(\frac{c_1}{a-1}s^{2(a-1)} + c_2\right) = \left[\frac{c_1}{a-1}s^2 + c_2 s^{2a}\right] \Longleftrightarrow p = \frac{1}{\sqrt{\frac{c_1}{a-1}s^2 + c_2 s^{2a}}}$$

and we put $f'(s) = p(s)$, then

$$f'(s) = \frac{1}{\sqrt{\frac{c_1}{a-1}s^2 + c_2 s^{2a}}} = \frac{1}{s\sqrt{\frac{c_1}{a-1} + c_2 s^{2(a-1)}}}$$

So

$$f(s) \frac{1}{\sqrt{|c_2|}} \int \frac{1}{s\sqrt{s^{2(a-1)} + \frac{c_1}{c_2(a-1)}}} ds \Longrightarrow f'(s) = \frac{1}{\sqrt{|c_2|}} \cdot \frac{1}{s\sqrt{s^{2(a-1)} + \frac{c_1}{c_2(a-1)}}}$$

$$\Longrightarrow f''(s) = \frac{1}{\sqrt{|c_2|}} \cdot \frac{-as^{2(a-1)} - \frac{c_1}{c_2(a-1)}}{s^2\left(s^{2(a-1)} + \frac{c_1}{c_2(a-1)}\right)^{\frac{3}{2}}}$$

So,

$$
\begin{aligned}
(2.3.2) \Longleftrightarrow & \frac{1}{s\sqrt{|c_2|}^3\sqrt{s^{2(a-1)}+\frac{c_1}{c_2(a-1)}}^3}\underbrace{\left[t^2g''(t)-tg'(t)-tg'(t)^3\right]}_{G_1(t)} \\
&+\frac{-as^{2(a-1)}-\frac{c_1}{c_2(a-1)}}{s^2\sqrt{|c_2|}\left(s^{2(a-1)}+\frac{c_1}{c_2(a-1)}\right)^{\frac{3}{2}}}\underbrace{\left[tg'(t)^3+tg'(t)-g(t)g'(t)^2-g(t)\right]}_{G_2(t)} \\
&+\frac{1}{s\sqrt{|c_2|}\sqrt{s^{2(a-1)}+\frac{c_1}{c_2(a-1)}}}\underbrace{\left[g(t)^2g''(t)+t^2g''(t)\right]}_{G_3(t)}=0 \\
\Longleftrightarrow & \frac{1}{\sqrt{|c_2|}^3}\cdot\frac{1}{s\sqrt{s^{2(a-1)}+\frac{c_1}{c_2(a-1)}}^3}\left[G_1(t)\right]+\frac{1}{\sqrt{|c_2|}}\cdot\frac{-as^{2(a-1)}-\frac{c_1}{c_2(a-1)}}{s\left(s^{2(a-1)}+\frac{c_1}{c_2(a-1)}\right)^{\frac{3}{2}}}\left[G_2(t)\right] \\
&+\frac{1}{\sqrt{|c_2|}}\cdot\frac{1}{s\sqrt{s^{2(a-1)}+\frac{c_1}{c_2(a-1)}}}\left[G_3(t)\right]=0 \\
\Longleftrightarrow & \frac{1}{\sqrt{|c_2|}^3}\cdot\frac{1}{s\sqrt{s^{2(a-1)}+\frac{c_1}{c_2(a-1)}}^3}\frac{1}{c_2}G_1(t)+\left(-as^{2(a-1)}-\frac{c_1}{c_2(a-1)}\right)G_2(t) \\
&+\left(s^{2(a-1)}+\frac{c_1}{c_2(a-1)}\right)G_3(t)=0 \\
\Longleftrightarrow & \frac{1}{c^2}G_1(t)+\left(-as^{2(a-1)}-\frac{c_1}{c_2(a-1)}\right)G_2(t)+\left(s^{2(a-1)}+\frac{c_1}{c_2(a-1)}\right)G_3(t)=0.
\end{aligned}
\tag{II}
$$

In addition, we have

$$g(t)^2g''(t)+t^2g''(t)=a\left[tg'(t)^3+tg'(t)-g(t)g'(t)^2-g(t)\right] \iff G_3(t)=aG_2(t)$$

So

$$
\begin{aligned}
(II) \Longleftrightarrow & \frac{1}{c^2}G_1(t)+s^{2(a-1)}\left[-aG_2(t)+G_3(t)\right]+\frac{c_1}{c_2(a-1)}\left[-G_2(t)+G_3(t)\right]=0 \\
\Longleftrightarrow & \frac{1}{c^2}G_1(t)+s^{2(a-1)}\left[-G_3(t)+G_3(t)\right]+\frac{c_1}{c_2(a-1)}\left[-\frac{1}{a}G_3(t)+G_3(t)\right]=0 \\
\Longleftrightarrow & \frac{1}{c^2}G_1(t)+c_1G_3(t)\left[\frac{-1}{ac_2(a-1)}+\frac{1}{c_2(a-1)}\right]=0 \\
\Longleftrightarrow & \frac{a}{c^2}G_1(t)+c_1G_3(t)\left[\frac{-1}{c_2(a-1)}+\frac{a}{c_2(a-1)}\right]=0 \\
\Longleftrightarrow & \frac{a}{c^2}G_1(t)+c_1G_3(t)\left[\frac{a-1}{c_2(a-1)}\right]=0 \\
\Longleftrightarrow & aG_1(t)+c_1G_3(t)=0 \\
\Longleftrightarrow & a\left[t^2g''(t)-tg'(t)-tg'(t)^3\right]+c_1\left[g(t)^2g''(t)+t^2g''(t)\right]=0
\end{aligned}
$$

which implies

$$\left[at^2g^{''}(t)+c_1g(t)^2g^{''}(t)+c_1t^2g^{''}(t)\right]=atg^{'}(t)+atg^{'}(t)^3, \tag{4}$$

$$(4)\iff\left[(a+c_1)t^2+c_1g(t)^2\right]g^{''}(t)=atg^{'}(t)\left[1+g^{'}(t)^2\right]. \tag{2.3.10}$$

1. If $c_1=0$, then the general solution of (2.3.10) is given by $g(t)=-\sqrt{d_1-t^2}$. As the solution of equation ()and equation (2.3.5) gives:

$$\begin{aligned}f^{''}(s)+\frac{a}{s}f^{'}(s)=0 &\iff \frac{f^{''}(s)}{f^{'}(s)}=-\frac{a}{s}\\ &\iff \ln|f^{'}(s)|=-a\ln s+c_2\\ &\iff f^{'}(s)=\underbrace{\exp c_2}_{=d_2}s^{-a}\\ &\iff f(s)=d_2\cdot\frac{1}{-a+1}s^{-a+1}+d_3\end{aligned}$$

So

$$f(s)=\frac{d_2}{1-a}s^{1-a}+d_3\,(d_1,d_2,d_3\in\mathbb{R}).$$

We conclude with the following :

Theorem 4. *Let $\sum_1$ be a translation surface of type 1 in $H^2\times\mathbb{R}$. If $\sum_1$ is minimal surface, then $\sum_1$ is a plane parameterized as*

$$x(s,t)=(sg(t),st,f(s)),$$

where

(1) either $f(s)=c_1s+c_2$ and $g(t)=c_3$ or

(2) $f(s)=c_1\ln s+c_2$ and $g(t)=-\sqrt{c_3-t^2}$ or

(3) $f(s)=\frac{c_1}{1-a}s^{1-a}+c_2$ and $g(t)=-\sqrt{c_3-t^2}$ or

(4) $f(s)=\frac{-1}{c_1}\sqrt{c_2-2c_1\ln s}+c_3$ and $g(t)$ is the function satisfying equation (2.3.8) or

(5) $f(s)=\frac{1}{\sqrt{|c_2|}}\int\frac{1}{s\sqrt{s^{2(a-1)}+\frac{c_1}{c_2(a-1)}}}ds$ and $g(t)$ is the function satisfying equation (2.3.10).

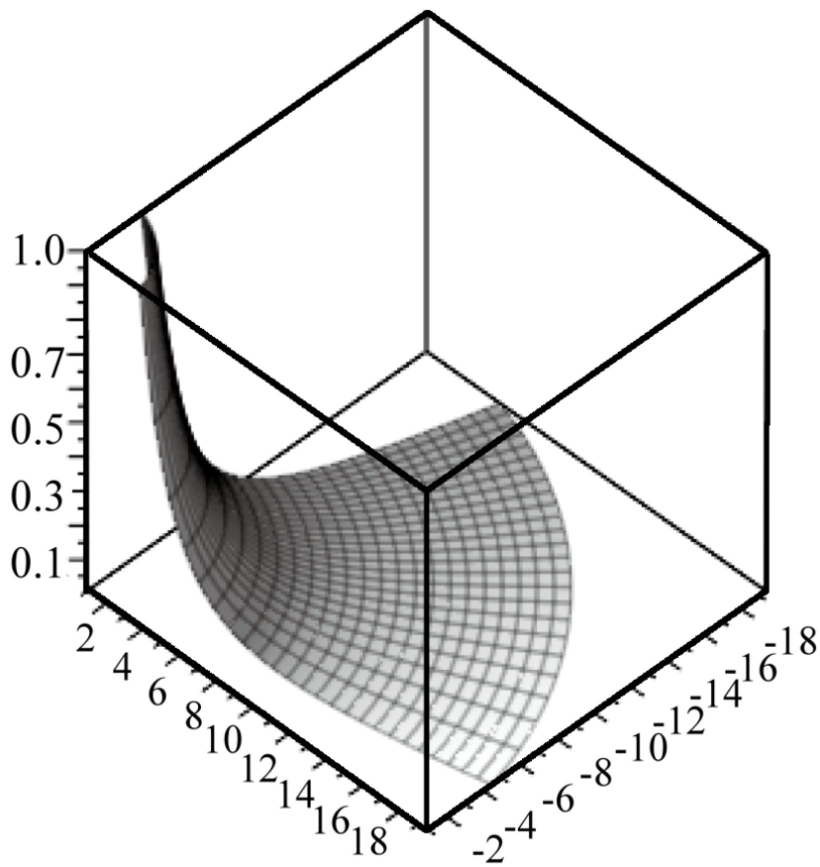

Figure 2.1: Minimal translation surface in $\mathbb{H}^2 \times \mathbb{R}$ of type 1 .

2.4 Classification of type 2 minimal translation surface

Let $\sum_2$ be a translation surface of type 2 in Riemannian product space $H^2 \times \mathbb{R}$. Then , $\sum_2$ is parameterized by :

$$x(s,t) = (g(t), st, f(s)). \tag{2.4.1}$$

for all $s > 0$ and $t > 0$.

We have

$$\begin{aligned}
x_s &= \frac{D}{Ds} x(s,t) \\
&= (0, t, f'(s)) \\
&= \frac{t}{y} \cdot y \frac{\partial}{\partial y} + f'(s) \frac{\partial}{\partial z} \text{with in this case } y = st \\
&= \frac{1}{s} E_2 + f'(s) E_3, \\
x_t &= \frac{D}{Dt} x(s,t) \\
&= (g'(t), s, 0) \\
&= \frac{g'(t)}{y} \cdot y \frac{\partial}{\partial y} + \frac{s}{y} \cdot y \frac{\partial}{\partial y} \text{with in this case } y = st \\
&= \frac{g'(t)}{st} E_1 + \frac{1}{t} E_2,
\end{aligned}$$

The coefficients of the first fundamental form of $\sum_2$ are given by:

$$\begin{aligned}
E &= \left(0, \frac{1}{s}, f'(s)\right) \begin{pmatrix} 0 \\ \frac{1}{s} \\ f'(s) \end{pmatrix} \\
&= \frac{1}{s^2} + f'(s)^2, \\
F &= \left(0, \frac{1}{s}, f'(s)\right) \begin{pmatrix} \frac{g'(t)}{st} \\ \frac{1}{t} \\ 0 \end{pmatrix} \\
&= \frac{1}{st}, \\
G &= \left(\frac{g'(t)}{st}, \frac{1}{t}, 0\right) \begin{pmatrix} \frac{g'(t)}{st} \\ \frac{1}{t} \\ 0 \end{pmatrix} \\
&= \frac{g'(t)^2}{s^2 t^2} + \frac{1}{t^2}.
\end{aligned}$$

The unit normal vector field U of $\sum_2$ is given by

$$U = -\frac{f'(s)}{wt} E_1 + \frac{f'(s) g'(t)}{wst} E_2 - \frac{g'(t)}{ws^2 t} E_3,$$

where $w = \|x_s \times x_t\|$ and because

$$\begin{aligned} x_s \wedge x_t &= \left(0, \frac{1}{s}, f^{\shortmid}(s)\right) \wedge \left(\frac{g^{\shortmid}(t)}{st}, \frac{1}{t}, 0\right) \\ &= \left(-\frac{f^{\shortmid}(s)}{t}, \frac{f^{\shortmid}(s)\, g^{\shortmid}(t)}{st}, -\frac{g^{\shortmid}(t)}{s^2 t}\right) \end{aligned}$$

To compute the second fundamental form of $\sum_2$, we have to calculate the following:

$$\begin{aligned} \frac{D}{Ds}E_1 &= \tilde{\nabla}_{x_s} E_1 \\ &= \tilde{\nabla}_{\frac{1}{s}E_2 + f^{\shortmid}(s)E_3} E_1 \\ &= \frac{1}{s}\tilde{\nabla}_{E_2} E_1 + f^{\shortmid}(s)\, \tilde{\nabla}_{E_3} E_1 \\ &= 0, \\ \frac{D}{Ds}E_2 &= \tilde{\nabla}_{x_s} E_2 \\ &= \frac{1}{s}\tilde{\nabla}_{E_2} E_2 + f^{\shortmid}(s)\, \tilde{\nabla}_{E_3} E_2 \\ &= 0, \\ \frac{D}{Ds}E_3 &= \tilde{\nabla}_{x_s} E_3 \\ &= \frac{1}{s}\tilde{\nabla}_{E_2} E_3 + f^{\shortmid}(s)\, \tilde{\nabla}_{E_3} E_3 \\ &= 0. \end{aligned}$$

$$\begin{aligned} \frac{D}{Dt}E_1 &= \tilde{\nabla}_{x_t} E_1 \\ &= \frac{g^{\shortmid}(t)}{st}\tilde{\nabla}_{E_1} E_1 + \frac{1}{t}\tilde{\nabla}_{E_2} E_1 \\ &= \frac{g^{\shortmid}(t)}{st} E_2, \end{aligned}$$

$$\begin{aligned} \frac{D}{Dt}E_2 &= \tilde{\nabla}_{x_t} E_2 \\ &= \frac{g^{\shortmid}(t)}{st}\tilde{\nabla}_{E_1} E_2 + \frac{1}{t}\tilde{\nabla}_{E_2} E_2 \\ &= -\frac{g^{\shortmid}(t)}{st} E_1, \\ \frac{D}{Dt}E_3 &= \tilde{\nabla}_{x_t} E_3 \\ &= \frac{g^{\shortmid}(t)}{st}\tilde{\nabla}_{E_1} E_3 + \frac{1}{t}\tilde{\nabla}_{E_2} E_3 \\ &= 0. \end{aligned}$$

So, the covariant derivatives are :

$$\begin{aligned}
\tilde{\nabla}_{x_s} x_s =& \frac{D}{Ds}\left(\frac{1}{s}E_2 + f'(s)\, E_3\right) \\
=& -\frac{1}{s^2}E_2 + \frac{1}{s}\frac{D}{Ds}E_2 + f''(s)\, E_3 + f'(s)\frac{D}{Ds}E_3 \\
=& -\frac{1}{s^2}E_2 + f''(s)\, E_3, \\
\tilde{\nabla}_{x_s} x_t =& \frac{D}{Ds}\left(\frac{g'(t)}{st}E_1 + \frac{1}{t}E_2\right) \\
=& \left(-\frac{g'(t)}{s^2 t}E_1 + \frac{g'(t)}{st}\frac{D}{Ds}E_1 + \frac{1}{t}\frac{D}{Ds}E_2\right) \\
=& -\frac{g'(t)}{s^2 t}E_1, \\
\tilde{\nabla}_{x_t} x_t =& \frac{D}{Dt}\left(\frac{g'(t)}{st}E_1 + \frac{1}{t}E_2\right) \\
=& \left(\frac{tg''(t) - 2g'(t)}{st^2}\right)E_1 + \frac{g'(t)}{st}\frac{D}{Dt}E_1 - \frac{1}{t^2}E_2 + \frac{1}{t}\frac{D}{Dt}E_2 \\
=& \left(\frac{tg''(t) - g'(t)}{st^2}\right)E_1 + \frac{g'(t)^2}{s^2t^2}E_2 - \frac{1}{t^2}E_2 - \frac{g'(t)}{st^2}E_1 \\
=& \left(\frac{tg''(t) - 2g'(t)}{st^2}\right)E_1 + \left(\frac{g'(t)^2 - s^2}{s^2t^2}\right)E_2,
\end{aligned}$$

which imply the coefficients of the second fundamental form of $\sum_2$ are given by :

$$\begin{aligned}
l &= \left\langle \tilde{\nabla}_{x_s} x_s, U \right\rangle \\
&= \left(0, -\frac{1}{s^2}, f''(s)\right)\begin{pmatrix} -\frac{f'(s)}{wt} \\ \frac{f'(s)g'(t)}{wst} \\ -\frac{g'(t)}{ws^2t} \end{pmatrix} \\
&= \frac{1}{w}\left[\frac{-f'(s)\, g'(t)}{s^3 t} - \frac{f''(s)\, g'(t)}{s^2 t}\right] \\
&= -\frac{g'(t)}{ws^3t}\left(f'(s) + sf''(s)\right),
\end{aligned}$$

$$\begin{aligned}
m &= \left\langle \tilde{\nabla}_{x_s} x_t, U \right\rangle \\
&= \left(-\frac{g'(t)}{s^2 t}, 0, 0\right)\begin{pmatrix} -\frac{f'(s)}{wt} \\ \frac{f'(s)g'(t)}{wst} \\ -\frac{g'(t)}{ws^2t} \end{pmatrix} \\
&= \frac{1}{ws^2t^2}\left(f'(s)\, g'(t)\right),
\end{aligned}$$

$$\begin{aligned}
n &= \left\langle \tilde{\nabla}_{x_t} x_t, U \right\rangle \\
&= \left(\left(\frac{tg''(t) - 2g'(t)}{st^2} \right), \left(\frac{g'(t)^2 - s^2}{s^2t^2} \right), 0 \right) \begin{pmatrix} -\frac{f'(s)}{wt} \\ \frac{f'(s)g'(t)}{wst} \\ -\frac{g'(t)}{ws^2t} \end{pmatrix} \\
&= \frac{1}{w} \left[\frac{-f'(s)\, g''(t)}{st^3} + \frac{2f'(s)\, g'(t)}{st^3} + \frac{f'(s)\, g'(t)^3}{s^3t^3} - \frac{s^2 f'(s)\, g'(t)}{s^3t^3} \right] \\
&= \frac{1}{ws^3t^3} \left(f'(s)\, g'(t) \left(g'(t) - s^2 \right) - s^2 f'(s) \left(tg''(t) - 2g'(t) \right) \right).
\end{aligned}$$

We suppose that the translation surface $\sum_2$ of type 2 is minimal if and only if

$$H = 0 \Longleftrightarrow lG - 2mF + nE = 0$$

First let's calculate lG, mF and nE:

$$\begin{aligned}
lG &= \frac{1}{ws^3t} \left[-g'(t)\, f'(s) - g'(t)\, sf''(s) \right] \left[\frac{g'(t)^2}{s^2t^2} + \frac{1}{t^2} \right] \\
&= \frac{1}{ws^3t} \left[\frac{-g'(t)^3 f'(s)}{s^2t^2} - \frac{sg'(t)^3 f''(s)}{s^2t^2} - \frac{g'(t)\, f'(s)}{t^2} - \frac{sg'(t)\, f''(s)}{t^2} \right], \\
mF &= \frac{1}{ws^2t^2} \left[f'(s)\, g'(t) \right] \left[\frac{1}{st} \right] \\
&= \frac{1}{ws^2t^2} \left[\frac{f'(s)\, g'(t)}{st} \right], \\
nE &= \frac{1}{ws^3t^3} \left[f'(s)\, g'(t) \left(g'(t)^2 - s^2 \right) - s^2 f'(s) \left(tg''(t) - 2g'(t) \right) \right] \left[\frac{1}{s^2} + f'(s)^2 \right] \\
&= \frac{1}{ws^3t^3} \left[\begin{array}{c} \frac{f'(s)g'(t)}{s^2} \left(g'(t)^2 - s^2 \right) - f'(s) \left(tg''(t) - 2g'(t) \right) \\ + f'(s)^3 g'(t) \left(g'(t)^2 - s^2 \right) - s^2 f'(s)^3 \left(tg''(t) - 2g'(t) \right) \end{array} \right].
\end{aligned}$$

Then we obtain:

$$H=0 \Longleftrightarrow \frac{1}{w}\left[\begin{array}{c} \frac{-g'(t)^3 f'(s)}{s^5t^3}-\frac{g'(t)^3 f''(s)}{s^4t^3}-\frac{g'(t)f'(s)}{s^3t^3}-\frac{g'(t)f''(s)}{s^2t^3}-\frac{2f'(s)g'(t)}{s^3t^3} \\ +\frac{f'(s)g'(t)^3}{s^5t^3}-\frac{f'(s)g'(t)}{s^3t^3}-\frac{f'(s)g''(t)}{s^3t^2}+\frac{2f'(s)g'(t)}{s^3t^3}+\frac{f'(s)^3g'(t)^3}{s^3t^3} \\ -\frac{f'(s)^3g'(t)}{st^3}-\frac{f'(s)^3g''(t)}{st^2}+\frac{2f'(s)^3g'(t)}{st^3} \end{array}\right]=0$$

$$\Longleftrightarrow -\frac{f''(s)\,g'(t)^3}{s^4t^3}-\frac{2f'(s)\,g'(t)}{s^3t^3}-\frac{g'(t)\,f''(s)}{s^2t^3}-\frac{f'(s)\,g''(t)}{s^3t^2}+\frac{f'(s)^3\,g'(t)^3}{s^3t^3}$$

$$-\frac{f'(s)^3\,g''(t)}{st^2}+\frac{f'(s)^3\,g'(t)}{st^3}=0$$

$$\Longleftrightarrow \frac{1}{s^4t^3}\left[\begin{array}{c} -f''(s)\,g'(t)^3-2sf'(s)\,g'(t)-s^2g'(t)\,f''(s) \\ -stf'(s)\,g''(t)+sf'(s)^3\,g'(t)^3-s^3tf'(s)^3\,g''(t) \\ +s^3f'(s)^3\,g'(t) \end{array}\right]=0$$

$$\Longleftrightarrow g'(t)^3\left[-f''(s)+sf'(s)^3\right]+g'(t)\left[-2sf'(s)-s^2f''(s)+s^3f'(s)^3\right]$$

$$+\,tg''(t)\left[-sf'(s)-s^3f'(s)^3\right]=0$$

We multiply this by (-1), we find:

$$tg''(t)\left[sf'(s)+s^3f'(s)^3\right]+g'(t)\left[2sf'(s)+s^2f''(s)-s^3f'(s)^3\right] \tag{2.4.2}$$

$$+g'(t)^3\left[f''(s)-sf'(s)^3\right]=0$$

If $g'(t)=0$, that is $g(t)=c$ $(c\in\mathbb{R})$, the surface $\sum_2$ is parameterized by:

$$x(s,t)=(c,st,f(s)),$$

where $f(s)$ is an arbitrary function.

If $g'(t)\neq 0$, then we can divide (2.4.2) by $g'(t)$

$$\frac{tg''(t)}{g'(t)}\left[sf'(s)+s^3f'(s)^3\right]+\left[2sf'(s)+s^2f''(s)-s^3f'(s)^3\right]+g'(t)^2\left[f''(s)-sf'(s)^3\right]=0$$

then, we derive that with respect to t

$$\frac{d}{dt}\left(\begin{array}{c} \frac{tg''(t)}{g'(t)}\left[sf'(s)+s^3f'(s)^3\right]+\left[2sf'(s)+s^2f''(s)-s^3f'(s)^3\right] \\ +g'(t)^2\left[f''(s)-sf'(s)^3\right] \end{array}\right)=0 \tag{4}$$

$$(4)\Longleftrightarrow \frac{d}{dt}\left(\frac{tg''(t)}{g'(t)}\right)\underbrace{\left(sf'(s)+s^3f'(s)^3\right)}_{F_1(s)}+\frac{d}{dt}\left(g'(t)^2\right)\underbrace{\left(f''(s)-sf'(s)^3\right)}_{F_3(s)}=0$$

$$\Longleftrightarrow F_1(s)\frac{d}{dt}\left(\frac{tg''(t)}{g'(t)}\right)+F_3(s)\frac{d}{dt}\left(g'(t)^2\right)=0$$

So, There is a real number $a\in\mathbb{R}$ such that

$$\frac{d}{dt}\left(\frac{tg''(t)}{g'(t)}\right)=-\,a\frac{d}{dt}\left(g'(t)^2\right),$$

$$f''(s)-sf'(s)^3=a\left(sf'(s)+s^3f'(s)^3\right).$$

Let us distinguish the following cases:

(1) Suppose that $a = 0$. Then the first equation of (2.4.3) leads to

$$\begin{aligned} tg''(t) = bg'(t)\,(b \in \mathbb{R}) &\Longleftrightarrow \int \frac{g''(t)}{g'(t)}dt = b\int \frac{1}{t}dt \\ &\Longleftrightarrow \ln|g'(t)| = b\ln|t| + k\,(k \in \mathbb{R}) \\ &\Longleftrightarrow \exp(\ln|g'(t)|) = \exp(b\ln|t| + k) = \exp(k)\cdot t^b \\ &\Longleftrightarrow g'(t) = c_1 \cdot t^b, \end{aligned}$$

where c_1 is a constant of integration .
If $b \neq -1$, then

$$\int g'(t)\,dt = c_1 \int t^b dt \Longleftrightarrow g(t) = \frac{c_1}{b+1}t^{b+1} + c_2\,(c_1,\ c_2 \in \mathbb{R})$$

and if $b = -1$, then
$\int g'(t)\,dt = c_1 \int \frac{1}{t}dt \quad \Longleftrightarrow \quad g(t) = c_1 \ln t + c_2 \ \ (t > 0)$.
From the second equation of (2.4.3), we have the ordinary differential equation

$$f''(s) - sf'(s)^3 = 0 \Longleftrightarrow f''(s) = sf'(s)^3$$

So

$$\begin{aligned} \int \frac{f''(s)}{sf'(s)^3}ds = \int sds &\Longleftrightarrow -\frac{1}{2}\cdot\frac{1}{f'^2} = \frac{s^2}{2} + k_1 \\ &\Longleftrightarrow \frac{1}{f'^2} = -s^2 - 2k_1 \\ &\Longleftrightarrow f'^2 = \frac{1}{-s^2 - 2k_1} = \frac{1}{k_2 - s^2} \quad \text{with } k_2 = -2k_1 \quad \left(k_1 \in \mathbb{R}^-\right) \\ &\Longleftrightarrow f' = \frac{1}{\sqrt{k_2 - s^2}} = \frac{1}{\sqrt{1 - \left(\frac{s}{\sqrt{k_2}}\right)^2}},\ c_3 = \sqrt{k_2}. \end{aligned}$$

Then the general solution is given by $f(s) = \text{constant}$ or $f(s) = \sin^{-1}\frac{s}{c_3} + c_4\,(c_3 \neq 0,\ c_4 \in \mathbb{R})$.

(2) If $a \neq 0$, then the first equation of (2.4.3) writes as

$$g''(t) - \frac{b}{t}g'(t) = -\frac{a}{t}g'(t)^3, \tag{2.4.3}$$

where b is a constant of integration. We put $g'(t) = q(t)$. Then we can obtain the Bernoulli's equation as follows:

$$\frac{dq}{dt} - \frac{b}{t}q = -\frac{a}{t}q^3$$

For his resolution, we put

$$h(t) = q^{-2}(t) \Longrightarrow h^{'}(t) = -2q^{'}(t)\,q^{-3}(t)$$

Thus

$$\begin{aligned}\frac{dq}{dt} - \frac{b}{t}q = -\frac{a}{t}q^3 &\Longleftrightarrow \frac{dq}{dt}q^{-3} - \frac{b}{t}q^{-2} = -\frac{a}{t}\\ &\Longleftrightarrow -\frac{1}{2}h^{'} - \frac{b}{t}h = -\frac{a}{t}\end{aligned}$$

We obtain a linear ODE of order 1 with second member.
To solve we go through 2 stapes:
Step 1: homogenous first-order ODE

$$\begin{aligned}-\frac{1}{2}h^{'} - \frac{b}{t}h = 0 &\Longleftrightarrow -\frac{1}{2}h^{'} = \frac{b}{t}h\\ &\Longleftrightarrow \int \frac{h^{'}}{h}dt = -2b\int \frac{dt}{t}\\ &\Longleftrightarrow \ln|h(t)| = -2b\ln t + k_1\\ &\Longleftrightarrow h(t) = t^{-2b}\exp k_1\end{aligned}$$

Hence, the general solution of the ODE without second member is :

$$h(t) = t^{-2b}\exp k_1.$$

Step 2: ODE of order 1 with second member
We have

$$h(t) = t^{-2b}\exp k_1 \Longrightarrow h^{'}(t) = \left(k_1^{'}(t)\,t^{-2b}\exp k_1\right) + \left(-2bt^{-2b-1}\exp k_1\right)$$

By replacing h and $h^{'}$in the ODE, we have

$$-\frac{1}{2}h^{'} - \frac{b}{t}h = -\frac{a}{t} \tag{5}$$

$$
\begin{aligned}
(5) \Longleftrightarrow & -\frac{1}{2}\left(\left(k_1^{\prime}(t)\, t^{-2b} \exp k_1\right)+\left(-2bt^{-2b-1} \exp k_1\right)\right)-\frac{b}{t}\left(t^{-2b} \exp k_1\right)=-\frac{a}{t} \\
\Longleftrightarrow & -\frac{1}{2} k_1^{\prime}(t)\, t^{-2b} \exp k_1(t)+\frac{b}{t} t^{-2b} \exp k_1(t)-\frac{b}{t} t^{-2b} \exp k_1(t)=-\frac{a}{t} \\
\Longleftrightarrow & -\frac{1}{2} k_1^{\prime}(t)\, t^{-2b} \exp k_1(t)=-\frac{a}{t} \\
\Longleftrightarrow & k_1^{\prime}(t) \exp k_1(t)=\frac{2a}{t} t^{2b} \\
\Longleftrightarrow & \int k_1^{\prime}(t) \exp k_1(t)\, dt=2a \int t^{2b-1} dt \\
\Longleftrightarrow & \exp k_1(t)=\int 2at^{2b-1} dt \\
\Longleftrightarrow & \exp k_1(t)=\frac{2a}{2b} t^{2b}+c \quad (c \in \mathbb{R}) \\
\Longleftrightarrow & k_1(t)=\ln \left(\frac{a}{b} t^{2b}+c\right)
\end{aligned}
$$

So, the general solution in the ODE is:

$$
\begin{aligned}
h_g(t) & =\exp \left(\ln \left(\frac{a}{b} t^{2b}+c\right)\right) t^{-2b} \\
& =\frac{1}{t^{2b}}\left(\frac{a}{b} t^{2b}\right) t^{-2b} \exp c
\end{aligned}
$$

or

$$
h_g(t)=\frac{1}{t^{2b}} \int 2at^{2b-1} dt .
$$

We have $h(t)=q^{-2}(t)$. Then la solution general in the equation $\frac{dq}{dt}-\frac{b}{t} q=-\frac{a}{t} q^3$ is:

$$
\begin{aligned}
q_g(t) & =\left(\frac{1}{t^{2b}} \int 2at^{2b-1} dt\right)^{-\frac{1}{2}} \\
& =\left(\frac{a}{b}+t^{-2b} c_1\right)^{-\frac{1}{2}} \quad (c_1=\exp c) \\
& =\frac{1}{\sqrt{\frac{a}{b}+t^{-2b} c_1}}\ (c_1 \in \mathbb{R})
\end{aligned}
$$

So

$$
q^{-2}=\frac{1}{t^{2b}} \int 2at^{2b-1} dt . \tag{2.4.4}
$$

(i) If $b=0$, then the general solution of (2.4.4) appears in the form

$$
g(t)=\int \frac{1}{\sqrt{2a \ln t-d_1}} dt . \tag{2.4.5}
$$

$$
\begin{aligned}
(2.4.6) \Longrightarrow g'(t) &= \frac{1}{\sqrt{2a\ln t - d_1}} \\
\Longrightarrow g''(t) &= \frac{\frac{-2a\cdot\frac{1}{t}}{2\sqrt{2a\ln t - d_1}}}{(2a\ln t - d_1)} = \frac{-a}{t}\cdot\frac{1}{(2a\ln t - d_1)^{\frac{3}{2}}}
\end{aligned}
$$

So

$$
\begin{aligned}
(2.4.2) \Longleftrightarrow & \frac{-at}{t\,(2a\ln t - d_1)^{\frac{3}{2}}}\underbrace{\left[sf'(s) + s^3 f'(s)^3\right]}_{F_1(s)} + \frac{1}{\sqrt{2a\ln t - d_1}}\underbrace{\left[2sf'(s) - s^3 f'(s)^3 + s^2 f''(s)\right]}_{F_2(s)} \\
& + \frac{1}{(2a\ln t - d_1)^{\frac{3}{2}}}\underbrace{\left[f''(s) - sf'(s)^3\right]}_{F_3(s)} = 0 \\
\Longleftrightarrow & \frac{1}{(2a\ln t - d_1)^{\frac{3}{2}}}\left[-aF_1(s) + (2a\ln t - d_1)\,F_2(s) + F_3(s)\right] = 0 \qquad (*)
\end{aligned}
$$

In addition, we have
$f''(s) - sf'(s)^3 = a\left(sf'(s) + s^3 f'(s)^3\right) \iff F_3(s) = aF_1(s)$
So

$$
\begin{aligned}
(*) \Longleftrightarrow & - aF_1(s) + aF_1(s) + (2a\ln t - d_1)\,F_2(s) = 0 \\
\Longleftrightarrow & (2a\ln t - d_1)\,F_2(s) = 0 \\
\Longleftrightarrow & (2a\ln t - d_1)\left[2sf'(s) - s^3 f'(s)^3 + s^2 f''(s)\right] = 0
\end{aligned}
$$

$$
(2a\ln t - 2d_1)\left[2sf'(s) - s^3 f'(s)^3 + s^2 f''(s)\right] = 0. \qquad (2.4.6)
$$

From this, we obtain $2sf'(s) - s^3 f'(s)^3 + s^2 f''(s) = 0$, and it's solution is

$$
f(s) = \pm\ln\left(\frac{1+\sqrt{1+d_2 s^2}}{s}\right) + d_3\,(d_2,\ d_3 \in \mathbb{R})\,.
$$

(ii) If $b = 1$, then from (2.4.5) the function $g(t)$ is given by

$$
g(t) = \frac{1}{a}\sqrt{c_1 + at^2} + c_2\,(c_2 \in \mathbb{R})
$$

because equation (2.4.5) became

$$
q^{-2} = \frac{1}{t^2}\int 2atdt \Longleftrightarrow q = \frac{1}{t^{-1}}\left(\int 2atdt\right)^{-\frac{1}{2}} = t\left(\frac{2a}{2}t^2 + c_1\right)^{-\frac{1}{2}} = t\left(at^2 + c_1\right)^{-\frac{1}{2}},
$$

we have

$$g^{'}(t) = q(t) \Longrightarrow g(t) = \int q(t)\,dt = \int \frac{t}{\sqrt{at^2+c_1}}dt = \frac{1}{a}\int \frac{2at}{2\sqrt{at^2+c_1}}dt$$
$$= \frac{1}{a}\sqrt{c_1+at^2} + c_2\,(c_2 \in \mathbb{R}).$$

In this case , the left hand side of equation (2.4.2) is polynomial in t with functions of s as the coefficients. Therefore, the leading coefficient must vanish.

In addition, we have

$$g^{''}(t) = \frac{\sqrt{c_1+at^2} - t\cdot\frac{2at}{2\sqrt{c_1+at^2}}}{(c_1+at^2)} = \frac{c_1}{(c_1+at^2)^{\frac{3}{2}}}.$$

So

$$\begin{aligned}
(2.4.2) &\Longleftrightarrow \frac{c_1}{(c_1+at^2)^{\frac{3}{2}}}F_1(s) + \frac{t}{(c_1+at^2)^{\frac{1}{2}}}F_2(s) + \frac{t^3}{(c_1+at^2)^{\frac{3}{2}}}F_3(s) = 0 \\
&\Longleftrightarrow \frac{t}{(c_1+at^2)^{\frac{3}{2}}}\left[c_1F_1(s) + \left(c_1+at^2\right)F_2(s) + t^2F_3(s)\right] = 0 \\
&\Longleftrightarrow c_1F_1(s) + \left(c_1+at^2\right)F_2(s) + t^2F_3(s) = 0 \qquad (**)
\end{aligned}$$

$$\begin{aligned}
(**) &\Longleftrightarrow c_1F_1(s) + at^2F_1(s) + \left(c_1+at^2\right)F_2(s) = 0 \\
&\Longleftrightarrow \left(c_1+at^2\right)[F_1(s)+F_2(s)] = 0 \\
&\Longleftrightarrow F_1(s)+F_2(s) = 0 \\
&\Longleftrightarrow sf^{'}(s) + s^3f^{'}(s)^3 + 2sf^{'}(s) - s^3f^{'}(s)^3 + s^2f^{''}(s) = 0 \\
&\Longleftrightarrow s^2f^{''}(s) + 3sf^{'}(s) = 0
\end{aligned}$$

We solve this equation

$$\begin{aligned}
s^2f^{''}(s) + 3sf^{'}(s) = 0 &\Longleftrightarrow f^{''}(s) + \frac{3}{s}f^{'}(s) = 0 \\
&\Longleftrightarrow \frac{f^{''}(s)}{f^{'}(s)} = -\frac{3}{s} \\
&\Longleftrightarrow \int \frac{f^{''}(s)}{f^{'}(s)}ds = -3\int \frac{ds}{s} \\
&\Longleftrightarrow \ln|f^{'}(s)| = -3\ln s + k_1 \\
&\Longleftrightarrow f^{'}(s) = d_1s^{-3} \\
&\Longleftrightarrow f(s) = d_1\int s^{-3}ds \\
&\Longleftrightarrow f(s) = -\frac{d_1}{2}s^{-2} + d_2
\end{aligned}$$

So, $f(s) = -\frac{d_1}{2s^2} + d_2 \, (d_1, \ d_2 \in \mathbb{R})$.

(iii) If $b \notin \mathbb{R} - \{0, 1\}$, then (2.4.4) becomes:

$$q^{-2} = \frac{1}{t^{2b}} \int 2at^{2b-1} dt = \frac{1}{t^{2b}} \left[\frac{a}{b} t^{2b} + c_1 \right] \Longrightarrow q = t^b \left(\frac{a}{b} t^{2b} + c_1 \right)^{-\frac{1}{2}},$$

then the general solution of (2.4.4) is:

$$g(t) = \int q(t) \, dt = \int \frac{t^b}{\sqrt{\frac{a}{b} t^{2b} + c_1}} dt = \frac{\sqrt{|b|}}{\sqrt{|b|}} \int \frac{t^b}{\sqrt{\frac{a}{b} t^{2b} + c_1}} dt = \sqrt{|b|} \int \frac{t^b}{\sqrt{at^{2b} + bc_1}} dt.$$

So, we have:

$$\begin{aligned} g(t) = \sqrt{|b|} \int \frac{t^b}{\sqrt{at^{2b} + bc_1}} dt \Longrightarrow g'(t) &= \sqrt{|b|} \cdot \frac{t^b}{\sqrt{at^{2b} + bc_1}} \\ \Longrightarrow g''(t) &= \sqrt{|b|} \left[\frac{bt^{b-1} \sqrt{at^{2b} + bc_1} - t^b \cdot \frac{2abt^{2b-1}}{2\sqrt{at^{2b} + bc_1}}}{at^{2b} + bc_1} \right] \\ &= \sqrt{|b|} \left[\frac{bt^{b-1} \sqrt{at^{2b} + bc_1} - \frac{abt^{3b-1}}{\sqrt{at^{2b} + bc_1}}}{at^{2b} + bc_1} \right] \\ &= \sqrt{|b|} \left(\frac{bt^{b-1} \left(at^{2b} + bc_1 \right) - abt^{3b-1}}{\left(at^{2b} + bc_1 \right)^{\frac{3}{2}}} \right) \\ &= \sqrt{|b|} \left(\frac{b^2 c_1 t^{b-1}}{\left(at^{2b} + bc_1 \right)^{\frac{3}{2}}} \right). \end{aligned}$$

Then

$$
\begin{aligned}
(2.4.2) \Longleftrightarrow & \frac{\sqrt{|b|}\left(b^2 c_1 t^b\right)}{\left(at^{2b}+bc_1\right)^{\frac{3}{2}}}\left[F_1(s)\right]+\frac{\sqrt{|b|}t^b}{\left(at^{2b}+bc_1\right)^{\frac{1}{2}}}\left[F_2(s)\right]+\frac{\left(|b|\right)^{\frac{3}{2}}t^{3b}}{\left(at^{2b}+bc_1\right)^{\frac{3}{2}}}\left[F_3(s)\right]=0\\
\Longleftrightarrow & \frac{\sqrt{|b|}t^b}{\left(at^{2b}+bc_1\right)^{\frac{3}{2}}}\left[b^2c_1F_1(s)+\left(at^{2b}+bc_1\right)F_2(s)+bt^{2b}F_3(s)\right]=0\\
\Longleftrightarrow & b^2c_1F_1(s)+\left(at^{2b}+bc_1\right)F_2(s)+bt^{2b}F_3(s)=0\\
\Longleftrightarrow & b^2c_1F_1(s)+abt^{2b}F_1(s)+\left(at^{2b}+bc_1\right)F_2(s)=0\\
\Longleftrightarrow & \left(at^{2b}+bc_1\right)\left[F_2(s)+bF_1(s)\right]=0\\
\Longleftrightarrow & F_2(s)+bF_1(s)=0\\
\Longleftrightarrow & 2sf'(s)-s^3f'(s)^3+s^2f''(s)+bsf'(s)+bs^3f'(s)^3=0\\
\Longleftrightarrow & (b+2)\,sf'(s)+(b-1)\,s^3f'(s)^3+s^2f''(s)=0\\
\Longleftrightarrow & f''(s)+(b+2)\frac{1}{s}f'(s)+s(b-1)\,f'(s)^3=0\\
\Longleftrightarrow & f''(s)+(b+2)\frac{1}{s}f'(s)=s(1-b)\,f'(s)^3
\end{aligned}
$$

We pose $f'(s)=p(s)$ and we find a Bernoulli equation:

$$\frac{dp}{ds}+(b+2)\frac{1}{s}p=s(1-b)\,p^3$$

We divide by p^3, we obtain:

$$\frac{dp}{ds}p^{-3}+(b+2)\frac{1}{s}p^{-2}=s(1-b) \tag{7}$$

To solve (7) we go through 2 staps:
Step 1: homogenous first-order ODE

$$
\begin{aligned}
p'p^{-3}+(b+2)\frac{1}{s}p^{-2}=0 \Longleftrightarrow & p'p^{-1}=-\frac{1}{s}(2+b)\\
\Longleftrightarrow & \int\frac{p'}{p}ds=-(b+2)\int\frac{ds}{s}\\
\Longleftrightarrow & \ln|p(s)|=-(b+2)\ln s+k_2\\
\Longleftrightarrow & p(s)=s^{-(b+2)}\exp k_2(s)
\end{aligned}
$$

Then

$$p'(s)=-(b+2)\,s^{-(b+3)}\exp k_2(s)+s^{-(b+2)}k_2'(s)\exp k_2(s).$$

Step 2: ODE of order 1 with second member

$$\begin{aligned}
(7) \Longleftrightarrow & \left[-(b+2)\, s^{-(b+3)} \exp k_2(s) + s^{-(b+2)} k_2^{\prime}(s) \exp k_2(s)\right] s^{3(b+2)} \exp(-3k_2(s)) \\
& + (b+2)\frac{1}{s} s^{2b+3} \exp(-2k_2(s)) = s(1-b) \\
\Longleftrightarrow & -(b+2)\, s^{2b+3} \exp(-2k_2(s)) + s^{2b+4} k_2^{\prime}(s) \exp(-2k_2(s)) \\
& + (b+2)\, s^{2b+3} \exp(-2k_2(s)) = s(1-b) \\
\Longleftrightarrow & s^{2b+4} k_2^{\prime}(s) \exp(-2k_2(s)) = s(1-b) \\
\Longleftrightarrow & k_2^{\prime}(s) \exp(-2k_2(s)) = (1-b)\, s^{-2b-3} \\
\Longleftrightarrow & -\frac{1}{2} \int -2k_2^{\prime}(s) \exp(-2k_2(s))\, ds = (1-b) \int s^{-2b-3} ds \\
\Longleftrightarrow & \exp(-2k_2(s)) \quad = -2(1-b)\left(\frac{1}{-2b-2} s^{-2b-2}\right) + d_1 \\
& = -2(1-b)\left(\frac{1}{-2(b+1)} s^{-2(b+1)}\right) + d_1 = \frac{1-b}{b+1} s^{-2(b+1)} + d_1 \\
\Longrightarrow & k_2 = -\frac{1}{2} \ln\left(\frac{1-b}{b+1} s^{-2(b+1)} + d_1\right) (d_1 \in \mathbb{R}).
\end{aligned}$$

So

$$p^{-2} \;=\; s^{2(b+2)}\left(\frac{1-b}{b+1} s^{-2(b+1)} + d_1\right) = \frac{1-b}{b+1} s^2 + d_1 s^{2(b+2)}.$$

Then

$$p = \left(\frac{1-b}{b+1} s^2 + d_1 s^{2(b+2)}\right)^{-\frac{1}{2}}$$

We have

$$f^{\prime}(s) = p(s) \Longrightarrow f(s) = \int p(s)\, ds = \int \frac{1}{\sqrt{\frac{1-b}{b+1} s^2 + d_1 s^{2(b+2)}}} ds = \int \frac{1}{s\sqrt{\frac{1-b}{b+1} + d_1 s^{2(b+1)}}} ds,$$

where $d_1 \in \mathbb{R}$.

Thus, we have the following:

Theorem 5. *Let $\sum_2$ be a translation surface of type 2 in $H^2 \times \mathbb{R}$. If $\sum_2$ is minimal surface, then $\sum_2$ is a plane or parameterized as*

$$x(s,t) = (g(t), st, f(s)),$$

where

(1) either $f(s) = \sin^{-1} \frac{s}{c_3} + c_4$ and $g(t) = c_1 \ln t + c_2$ or

(2) $f(s) = \sin^{-1} \frac{s}{c_3} + c_4$ and $g(t) = \frac{c_1}{b+1} t^{b+1} + c_2$ or

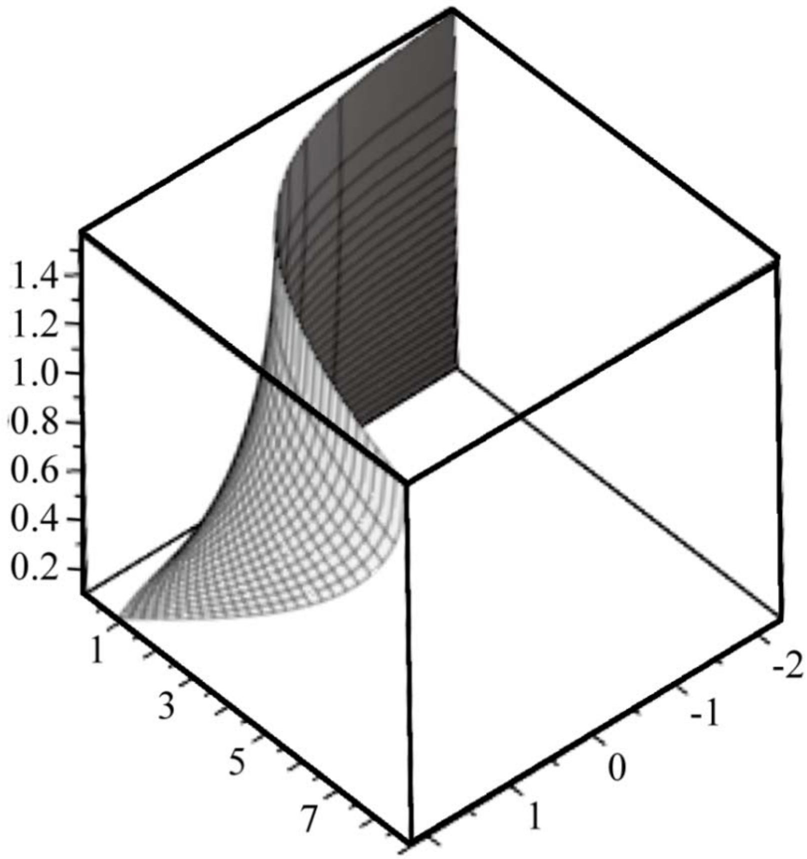

Figure 2.2: Minimal translation surface in $\mathbb{H}^2 \times \mathbb{R}$ of type 2 .

(3) $f(s) = \pm \ln\left(\frac{1+\sqrt{1+d_2 s^2}}{s}\right) + d_3$ and $g(t) = \int \frac{1}{\sqrt{2a \ln t - d_1}} dt$ or

(4) $f(s) = -\frac{d_1}{2s^2} + d_2$ and $g(t) = \frac{1}{a}\sqrt{c_1 + at^2} + c_2$ or

(5) $f(s) = \int \frac{1}{\sqrt[s]{d_1 s^{2(b+1)} - \frac{b-1}{b+1}}} ds$ and $g(t) = \sqrt{|b|} \int \frac{t^b}{\sqrt{at^{2b} + bc_1}} dt$.

Chapter 3

Surfaces with Constant Extrinsically Gaussian Curvature in the Heisenberg Group

The 3-dimensional Heisenberg group $\mathbb{H}_3$ is the simply connected and connected 2−step nilpotent Lie group.Which has the following standard representation in $GL(3,\mathbb{R})$

$$\begin{pmatrix} 1 & r & t \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix} \tag{3.0.1}$$

with $r, s, t \in \mathbb{R}$.
The Lie algebra $\mathfrak{h}_3$ of $\mathbb{H}_3$ is given by the matrices

$$A = \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \tag{3.0.2}$$

with $x, y, z \in \mathbb{R}$.The exponential map $exp : \mathfrak{h}_3 \to \mathbb{H}_3$ is a global diffeomorphism,and is given by

$$exp(A) = I + A + \frac{A^2}{2} = \begin{pmatrix} 1 & x & z + \frac{xy}{2} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.0.3}$$

The Heisenberg group $\mathbb{H}_3$ is represented as the cartesian 3−space $\mathbb{R}^3(x, y, z)$ with group structure:

$$(x_1, y_1, z_1).(x_2, y_2, z_2) := \left(x_1 + x_2, y_1 + y_2, z_1 + z_2 + \frac{1}{2}x_1 y_2 - \frac{1}{2}x_2 y_1 \right). \tag{3.0.4}$$

We equip $\mathbb{H}_3$ with the following left invariant Riemannian metric

$$g := dx^2 + dy^2 + \left(dz + \frac{1}{2}(ydx - xdy) \right)^2. \tag{3.0.5}$$

The identity component $I^{\circ}(\mathbb{H}_3)$ of the full isometry group of $(\mathbb{H}_3, g)$ is the semi-direct product $SO(2) \ltimes \mathbb{H}_3$. The action of $SO(2) \ltimes \mathbb{H}_3$ is given explicitly by

$$\begin{aligned} A &= \left(\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} . \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right) . \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ \frac{1}{2}(a\sin\theta - b\cos\theta) & \frac{1}{2}(a\cos\theta + b\sin\theta) & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} a \\ b \\ c \end{bmatrix} \end{aligned}$$

In particular, rotational around the $z-$axis and translations:

$$(x, y, z) \to (x, y, z + a), a \in \mathbb{R}$$

along the $z-$axis are isometries of $\mathbb{H}_3$.
The Lie algebra $\mathfrak{h}_3$ of $I^{\circ}(\mathbb{H}_3)$ is generated by the following Killing vector fields:

$$F_1 = \frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial z}, \;\; F_2 = \frac{\partial}{\partial y} - \frac{x}{2}\frac{\partial}{\partial z},$$

$$F_3 = \frac{\partial}{\partial z}, \;\; F_4 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

One can check that F_1, F_2, F_3 are infinitesimal transformations of the $1-$parameter groups of isometries defined by

$$G_1 = \{(t, 0, 0) | t \in \mathbb{R}\}, \;\; G_2 = \{(0, t, 0) | t \in \mathbb{R}\}, \;\; G_3 = \{(0, 0, t) | t \in \mathbb{R}\},$$

respectively.Here this groups acts on $\mathbb{H}_3$ by the left translation.The vector field F_4 generates the group of rotations around the $z-$axis. Thus G_4 is identified with $SO(2)$.

Definition 28. *A surface Σ in the Heisenberg space $\mathbb{H}_3$ is said to be invariant surface if it is invariant under the action of the $1-$parameter subgroups of isometries $\{G_i\}$, with $i \in \{1, 2, 3, 4\}$.*

The Lie algebra $\mathfrak{h}_3$ of $\mathbb{H}_3$ has an orthonormal basis $\{E_1, E_2, E_3\}$ defined by

$$E_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \;\; E_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}, \;\; E_3 = \frac{\partial}{\partial z}. \tag{3.0.6}$$

The Levi-Civita connection ∇ of g,in terms of the basis $\{E_i\}_{i=1.2.3}$,is explicitly given as follows

$$\begin{cases} \nabla_{E_1} E_1 = 0, \nabla_{E_1} E_2 = \frac{1}{2} E_3, \nabla_{E_1} E_3 = -\frac{1}{2} E_2 \\ \nabla_{E_2} E_1 = -\frac{1}{2} E_3, \nabla_{E_2} E_2 = 0, \nabla_{E_2} E_3 = \frac{1}{2} E_1 \\ \nabla_{E_3} E_1 = -\frac{1}{2} E_2, \nabla_{E_3} E_2 = \frac{1}{2} E_1, \nabla_{E_3} E_3 = 0 \end{cases} \tag{3.0.7}$$

The Riemannian curvature tensor R is a tensor field on $\mathbb{H}_3$ defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{3.0.8}$$

The components $\{R^l_{ijk}\}$ are computed as

$$R^1_{212} = -\frac{3}{4},\ R^1_{313} = \frac{1}{4},\ R^2_{323} = \frac{1}{4}. \tag{3.0.9}$$

Let us denote $K_{ij} = K(E_i, E_j)$ the sectional curvature of the plane spanned by E_i and E_j.Then we get easily the following:

$$K_{12} = -\frac{3}{4},\ K_{13} = -\frac{1}{4},\ K_{23} = -\frac{1}{4}. \tag{3.0.10}$$

The Ricci curvature Ric is defined by

$$Ric(X, Y) = trace\{Z \to R(Z, X)Y\}. \tag{3.0.11}$$

The components $\{R_{ij}\}$ of Ric are defined by

$$Ric(E_i, E_j) = R_{ij} = \sum_{k=1}^{3} \langle R(E_i, E_k)E_k, E_j\rangle . \tag{3.0.12}$$

The components $\{R_{ij}\}$ are computed as

$$R_{11} = -\frac{1}{2},\ R_{12} = R_{13} = R_{23} = 0,\ \ R_{22} = -\frac{1}{2},\ R_{33} = \frac{1}{2}. \tag{3.0.13}$$

The scalar curvature S of $\mathbb{H}_3$ is constant and we have

$$S = trRic = \sum_{i=1}^{3} Ric(E_i, E_i) = -\frac{1}{2}. \tag{3.0.14}$$

3.1 Constant Extrinsically Gaussian Curvature G_1-Invariant Translation Surfaces in Heisenberg group $\mathbb{H}_3$

3.1.1

In this subsection we study complete extrinsically flat translation surfaces Σ in Heisenberg group $\mathbb{H}_3$ which are invariant under the one parameter subgroup G_1. Clearly, such a surface is generated by a curve γ in the totally geodesic plane $\{x = 0\}$. Discarding the trivial case of a vertical plane $\{y = y_0\}$. Thus γ is given by $\gamma(y) = (0, y, v(y))$. Therefore the generated surface is parameterized by

$$X(x, y) = (x, 0, 0).(0, y, v(y)) = (x, y, v(y) + \frac{xy}{2}),\ (x, y) \in \mathbb{R}^2.$$

We have an orthogonal pair of vector fields on (Σ), namely,

$$e_1 := X_x = (1, 0, \frac{y}{2}) = E_1 + yE_3.$$

and

$$e_2 := X_y = (0, 1, v' + \frac{x}{2}) = E_2 + v'E_3.$$

The coefficients of the first fundamental form are:

$$E =< e_1, e_1 >= 1 + y^2, \; F =< e_1, e_2 >= yv', \; G =< e_2, e >= 1 + v'^2.$$

As a unit normal field we can take

$$N = \frac{-y}{\sqrt{1 + y^2 + v'^2}} E_1 - \frac{v'}{\sqrt{1 + y^2 + v'^2}} E_2 + \frac{1}{\sqrt{1 + y^2 + v'^2}} E_3$$

The covariant derivatives are

$$\widetilde{\nabla}_{e_1} e_1 = -yE_2$$
$$\widetilde{\nabla}_{e_1} e_2 = \frac{y}{2} E_1 - \frac{v'}{2} E_2 + \frac{1}{2} E_3$$
$$\widetilde{\nabla}_{e_2} e_2 = v'E_1 + v''E_3.$$

The coefficients of the second fundamental form are

$$l =< \widetilde{\nabla}_{e_1} e_1, N >= \frac{yv'}{\sqrt{1 + y^2 + v'^2}}$$

$$m =< \widetilde{\nabla}_{e_1} e_2, N >= \frac{-\frac{y^2}{2} + \frac{v'^2}{2} + \frac{1}{2}}{\sqrt{1 + y^2 + v'^2}}$$

$$n =< \widetilde{\nabla}_{e_2} e_2, N >= \frac{-yv' + v''}{\sqrt{1 + y^2 + v'^2}}.$$

Let K_{ext} be the extrinsic Gauss curvature of Σ,

$$K_{ext} = \frac{ln - m^2}{EG - F^2} = \frac{-y^2 v'^2 + yv'v'' - (-\frac{y^2}{2} + \frac{v'^2}{2} + \frac{1}{2})^2}{(1 + y^2 + v'^2)^2}. \tag{3.1.1}$$

Thus Σ is extrinsically flat invariant surface in Heisenberg group $\mathbb{H}_3$ if and only if

$$K_{ext} = 0,$$

that is, if and only if

$$-y^2 v'^2 + yv'v'' - \left(-\frac{y^2}{2} + \frac{v'^2}{2} + \frac{1}{2}\right)^2 = 0 \tag{3.1.2}$$

to classify extrinsically flat invariant surfaces must solve the equation (3.1.2)
We can writes equation (3.1.2) as

$$y^2 + yv'v'' - \left(\frac{y^2}{2} + \frac{v'^2}{2} + \frac{1}{2}\right)^2 = 0 \tag{3.1.3}$$

we assume that $z = \frac{y^2}{2} + \frac{v'^2}{2} + \frac{1}{2}$. Then

$$\begin{cases} z' = y + v'v'' \\ v'v'' = z' - y \\ v'^2 = 2z - y^2 - 1. \end{cases} \tag{3.1.4}$$

Therefor equation (3.1.3) becomes

$$yz' - z^2 = 0. \tag{3.1.5}$$

equation (3.1.5) implies that

$$-\frac{z'}{z^2} = -\frac{1}{y}. \tag{3.1.6}$$

and equation (3.1.6) implies that

$$z = \frac{1}{-\ln(y) + \alpha}. \tag{3.1.7}$$

where $\alpha \in \mathbb{R}$, and if $y \neq e^{\alpha}$.

From 3.1.4 and 3.1.7 , we have

$$\begin{aligned} v'^2 &= 2z - y^2 - 1 \\ &= \frac{2}{-\ln(y)+\alpha} - y^2 - 1. \end{aligned}$$

Thus

$$v' = \sqrt{\frac{2}{-\ln(y) + \alpha} - y^2 - 1}.$$

As conclusion, we have

Theorem 6. *•The only non-extendable extrinsically flat translation surfaces in the $3-$dimensional Heisenberg group $\mathbb{H}_3$ invariant under the $1-$parameter subgroup $G_1 = \{(t, 0, 0) \in \mathbb{H}_3/t \in \mathbb{R}\}$, are the surfaces whose parametrization is $X(x, y) = \left(x, y, v(y) + \frac{xy}{2}\right)$ where y and v satisfy*

$$v(y) = \int \sqrt{\frac{2}{-\ln(y) + \alpha} - y^2 - 1dy}.$$

where $\alpha \in \mathbb{R}$, and $y \neq e^{\alpha}$.

•There are no complete extrinsically flat translation surfaces in the $3-$dimensional Heisenberg group $\mathbb{H}_3$ invariant under the $1-$parameter subgroup $G_1 = \{(t, 0, 0) \in \mathbb{H}_3/t \in \mathbb{R}\}$.

Remark 6. *Let Σ be a G_1-invariant translation surfaces in the $3-$dimensional Heisenberg space. Then Σ is locally expressed as*

$$X(x, y) = (0, y, v(y)) \,.\, (x, 0, 0) = \left(x, y, v(y) - \frac{xy}{2}\right).$$

Then the extrinsically Gaussian curvature K_{ext} of Σ is computed as

$$K_{ext} = \frac{\left((v' - x)^2 - 1\right)^2}{4\left(1 + (v' - x)^2\right)^2}.$$

Thus Σ can not be of constant extrinsically Gaussian curvature.

3.1.2

In this subsection we study complete constant extrinsically Gaussian curvature translation surfaces Σ in Heisenberg group $\mathbb{H}_3$ which are invariant under the one parameter subgroup G_1. Clearly, such a surface is generated by a curve γ in the totally geodesic plane $\{x = 0\}$. Discarding the trivial case of a vertical plane $\{y = y_0\}$. Thus γ is given by $\gamma(y) = (0, y, v(y))$. Therefore the generated surface is parameterized by

$$X(x, y) = (x, 0, 0).(0, y, v(y)) = (x, y, v(y) + \frac{xy}{2}),\ (x, y) \in \mathbb{R}^2.$$

Theorem 7. • *The G_1-invariant constant extrinsically Gaussian curvature translation surfaces in the $3-$dimensional Heisenberg group $\mathbb{H}_3$, are:*

1. $K_{ext} = -\frac{1}{4}$.
   The surfaces of equation
   $$z = v(y) + \frac{xy}{2} = \frac{xy}{2} + \frac{1}{2}y\sqrt{2\beta - y^2} + \arctan\left(\frac{y}{\sqrt{\beta - y^2}}\right),$$
   where $\beta \in \mathbb{R}$.
2. $K_{ext} \neq -\frac{1}{4}$.
   Then y and v satisfy
   $$v(y) = \int \sqrt{\frac{1}{-2(K_{ext} + \frac{1}{4})\ln(y) + \gamma} - y^2 - 1 dy}.$$
   where $\gamma \in \mathbb{R}$, and $y \neq e^{\frac{\gamma}{2(K_{ext}+\frac{1}{4})}}$.

• *There are no complete constant extrinsically Gaussian curvature translation surfaces in the $3-$dimensional Heisenberg group $\mathbb{H}_3$ invariant under the $1-$parameter subgroup G_1.*

Proof. From (3.2.1) and (3.1.3) we have

$$K_{ext} = \frac{ln - m^2}{EG - F^2} = \frac{y^2 + yv'v'' - \frac{1}{4}(1 + y^2 + v'^2)^2}{(1 + y^2 + v'^2)^2}. \tag{3.1.8}$$

1. If $K_{ext} = -\frac{1}{4}$. Then equation (3.1.8) becomes
   $$y^2 + yv'v'' = 0 \tag{3.1.9}$$
   We note that y equal zero is solution of the equation(3.1.9).
   If y is different to zero ($y \neq 0$), equation (3.1.9) becomes
   $$v'v'' = -y. \tag{3.1.10}$$
   Integration gives us
   $$v(y) = \frac{1}{2}y\sqrt{2\beta - y^2} + \arctan\left(\frac{y}{\sqrt{\beta - y^2}}\right),$$
   where $\beta \in \mathbb{R}$.

2. If $K_{ext} \neq -\frac{1}{4}$. Then equation (3.1.8) becomes

$$y^2 + yv'v'' = (K_{ext} + \frac{1}{4})(1 + y^2 + v'^2)^2. \tag{3.1.11}$$

In fact, put $z = 1 + y^2 + v'^2$. Then z satisfies

$$\frac{1}{2}yz' = (K_{ext} + \frac{1}{4})z^2. \tag{3.1.12}$$

Hence we have

$$z = \frac{1}{-2(K_{ext} + \frac{1}{4})y + \gamma}, \tag{3.1.13}$$

where $\gamma \in \mathbb{R}$, and $y \neq e^{\overline{2(K_{ext}+\frac{1}{4})}}$. Using the equation $z = 1 + y^2 + v'^2$, we get

$$v'^2 = \frac{1}{-2(K_{ext} + \frac{1}{4})y + \gamma} - y^2 - 1.$$

3.2 Constant Extrinsically Gaussian Curvature G_2-Invariant Translation Surfaces in Heisenberg group $\mathbb{H}_3$

In this section we study constant complete extrinsically flat translation surfaces Σ in Heisenberg group $\mathbb{H}_3$ which are invariant under the one parameter subgroup G_2. Clearly, such a surface is generated by a curve γ in the totally geodesic plane $\{y = 0\}$. Discarding the trivial case of a vertical plane $\{x = x_0\}$. Thus γ is given by $\gamma(x) = (x, 0, f(x))$. Therefore the generated surface is parameterized by

$$X(x, y) = (0, y, 0).(x, 0, f(x)) = (x, y, f(x) - \frac{xy}{2}), \ (x, y) \in \mathbb{R}^2.$$

We have an orthogonal pair of vector fields on (Σ), namely,

$$e_1 := X_x = (1, 0, f' - \frac{y}{2}) = E_1 + f'E_3.$$

and

$$e_2 := X_y = (0, 1, -\frac{x}{2}) = E_2 - xE_3.$$

The coefficients of the first fundamental form are:

$$E =< e_1, e_1 >= 1 + f'^2, \ F =< e_1, e_2 >= -xf', \ G =< e_2, e >= 1 + x^2.$$

As a unit normal field we can take

$$N = \frac{-f'}{\sqrt{1 + x^2 + f'^2}}E_1 + \frac{x}{\sqrt{1 + x^2 + f'^2}}E_2 + \frac{1}{\sqrt{1 + x^2 + f'^2}}E_3$$

The covariant derivatives are

$$\widetilde{\nabla}_{e_1}e_1 = -f'E_2 + f''E_3$$
$$\widetilde{\nabla}_{e_1}e_2 = \frac{f'}{2}E_1 + \frac{x}{2}E_2 - \frac{1}{2}E_3$$
$$\widetilde{\nabla}_{e_2}e_2 = -xE_1.$$

The coefficients of the second fundamental form are

$$l =< \widetilde{\nabla}_{e_1}e_1, N >= \frac{-xf' + f''}{\sqrt{1 + x^2 + f'^2}}$$

$$m =< \widetilde{\nabla}_{e_1}e_2, N >= \frac{-\frac{f'^2}{2} + \frac{x^2}{2} - \frac{1}{2}}{\sqrt{1 + x^2 + f'^2}}$$

$$n =< \widetilde{\nabla}_{e_2}e_2, N >= \frac{-yv' + v''}{\sqrt{1 + y^2 + v'^2}}.$$

Let K_{ext} be the extrinsic Gauss curvature of Σ,

$$K_{ext} = \frac{ln - m^2}{EG - F^2} = \frac{x^2 + xf'f'' - \frac{1}{4}(x^2 + f'^2 + 1)^2}{(1 + x^2 + f'^2)^2}. \tag{3.2.1}$$

Thus Σ is extrinsically flat invariant surface in Heisenberg group $\mathbb{H}_3$ if and only if

$$K_{ext} = 0,$$

that is, if and only if

$$x^2 + xf'f'' - \frac{1}{4}(x^2 + f'^2 + 1)^2 = 0. \tag{3.2.2}$$

to classify extrinsically flat invariant surfaces must solve the equation (3.2.2)
We remark that the equation (3.2.2) is similarly to the equation (3.1.2), It is sufficient to change y by x and v by f.
As conclusion, we have

Theorem 8. $\bullet$*The only non-extendable extrinsically flat translation surfaces in the* $3-$*dimensional Heisenberg group* $\mathbb{H}_3$ *invariant under the* $2-$*parameter subgroup* $G_2 = \{(0, t, 0) \in \mathbb{H}_3/t \in \mathbb{R}\}$, *are the surfaces whose parametrization is* $X(x, y) = \left(x, y, f(x) - \frac{xy}{2}\right)$ *where* x *and* f *satisfy*

$$f(x) = \int \sqrt{\frac{2}{-\ln(x) + \alpha} - x^2 - 1dy}.$$

where $\alpha \in \mathbb{R}$, *and* $x \neq e^\alpha$.

$\bullet$*There are no complete extrinsically flat translation surfaces in the* $3-$*dimensional Heisenberg group* $\mathbb{H}_3$ *invariant under the* $1-$*parameter subgroup* $G_2 = \{(0, t, 0) \in \mathbb{H}_3/t \in \mathbb{R}\}$.

Remark 7. *Let Σ be a G_2-invariant translation surfaces in the $3-$dimensional Heisenberg space. Then Σ is locally expressed as*

$$X(x,y) = (x,0,f(x)) \,.\, (0,y,0) = \left(x, y, f(x) + \frac{xy}{2}\right).$$

Then the extrinsically Gaussian curvature K_{ext} of Σ is computed as

$$K_{ext} = -\frac{\left((f'+y)^2 - 1\right)^2}{4\left(1+(v'-x)^2\right)^2}.$$

Thus Σ can not be of constant extrinsically Gaussian curvature.

Theorem 9. • *The G_2-invariant constant extrinsically Gaussian curvature translation surfaces in the $3-$dimensional Heisenberg group $\mathbb{H}_3$, are:*

1. $K_{ext} = -\frac{1}{4}$.
   The surfaces of equation

$$z = f(x) - \frac{xy}{2} = -\frac{xy}{2} + \frac{1}{2}x\sqrt{2\beta - x^2} + \arctan\left(\frac{x}{\sqrt{\beta - x^2}}\right),$$

   where $\beta \in \mathbb{R}$.

2. $K_{ext} \neq -\frac{1}{4}$.
   Then x and f satisfy

$$f(x) = \int \sqrt{\frac{1}{-2(K_{ext}+\frac{1}{4})\ln(x)+\gamma} - x^2 - 1dy}.$$

   where $\gamma \in \mathbb{R}$, and $x \neq e^{\frac{\gamma}{2(K_{ext}+\frac{1}{4})}}$.

• *There are no complete constant extrinsically Gaussian curvature translation surfaces in the $3-$dimensional Heisenberg group $\mathbb{H}_3$ invariant under the $1-$parameter subgroup G_2.*

CONCLUSION

In this magister thesis we gives a classification of minimal translation surfaces in product Riemannian space $\mathbb{H}^2 \times \mathbb{R}$ and surfaces with constant extrinsically Gaussian curvature in the Heisenberg group .

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