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Contents

Acknowledgements **5**

Introduction **7**

1 Elementary Differential Geometry **11**
1.1 Differentiable Manifold and Submanifold 11
1.1.1 The Definition of a Differentiable Manifold 11
1.1.2 Differentiable Function and Mapping 12
1.1.3 Rank of Mapping and Immersion 13
1.1.4 Submanifold 14
1.2 Vector Fields on a Differential Manifolds 16
1.2.1 The Tangent Space at Point of a Differential Manifold 16
1.2.2 Vector Field 17
1.3 Riemannian Manifolds 19
1.3.1 Tangent Covectors 19
1.3.2 Riemannian Metric 20
1.3.3 Orientable Manifold 20
1.3.4 Integration on Manifold 22
1.3.5 Integral of a Differential Form over a Manifold 22
1.3.6 Manifolds with Boundary 23
1.4 The Connection on Riemannian Manifolds 23
1.4.1 Linear Connection 23
1.4.2 Torsion Tensor 24
1.4.3 Levi-Civita Connection 24
1.5 Induced Connection on the Tangent Bundle 25
1.6 Second Fundamental Form 27
1.7 Curvature 28
1.7.1 Curvature Tensor 28
1.7.2 Sectional Curvature 29
1.7.3 Ricci Curvature 29
1.8 Operators On Riemannian Manifolds 30
1.8.1 Gradient Operator 30

1.8.2 Hessian Operator 31
1.8.3 Divergence Operator 32
1.8.4 Laplacian Operator 33
1.8.5 Divergence Theorem 33
1.8.6 Green Theorem 34
1.9 Conformally Metric 34

2 Geometry of Submanifolds **36**
2.1 Second Fundamental Form 36
2.2 The Curvature of Riemannian Submanifold 38
2.3 Shape operator 40

3 Harmonic and Biharmonic mappings **43**
3.1 Harmonic maps 43
3.1.1 First variation of energy 45
3.1.2 Second variation of energy 47
3.2 Biharmonic maps 51
3.2.1 First variation of bi-energy 51
3.3 Stress energy tensor 54
3.4 Stress bi-energy tensor 55

4 p-Harmonic and p-Biharmonic mappings **60**
4.1 The p-Biharmonic maps 60
4.1.1 First variation of p-energy 60
4.1.2 Second Variation of p-energy Functional 62
4.2 p-biharmonic map 65
4.2.1 First variation of p-bienergy 65
4.3 Stress p-energy tensor 67
4.4 Stress p-bienergy tensors 68

5 On the p-biharmonic Submanifold **76**
5.1 p-Biharmonic Submanifold in Space Form 76
5.2 Proper p-Biharmonic Hypersurface in Space Form 81
5.3 p-Biharmonic Hypersurface in Riemannian Manifold 84
5.4 p-biharmonic hypersurface in conformally flat space 85

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I dedicate this work

To

My parents, Fatima and Ali Benaoumeur.

My brothers.

My sisters .

All my family.

My close friends .

My lovely niece Fatima Zahraa, and my new nephew Islem Abdelrahmen.

For their love, endless supports and encouragements.

KHADIDJA

Abstract

Harmonic maps are mappings between Riemannian manifolds which extremize a natural energy functional. They include geodesics, minimal surfaces.
p-harmonic maps with ($p \geq 2$) defined as critical points of the p-energy functional.
The p-biharmonic maps are generalization of the notion of p-harmonic.
In this work, we study p-biharmonic submanifolds.
The main result are

- The definition of p-biharmonic submanifold;
- The necessary conditions for submanifold to be p-biharmonic submanifold in space form;
- Some properties for p-biharmonic hypersurfaces in Riemannian submanifolds in an Einstein space;
- the construction of new examples of proper p-biharmonic hypersurfaces.

Introduction

Denote by $C^\infty(M,N)$ the space of smooth maps $\varphi:(M,g)\longrightarrow(N,h)$ between two Riemannian manifolds, for any compact domain D of M the energy functional of φ is defined by

$$E(\varphi;D)=\frac{1}{2}\int_D |d\varphi|^2 v^g, \tag{1}$$

where $|d\varphi|$ is the Hilbert-Schmidt norm of the differential $d\varphi$ and v^g is the volume element on (M,g). A map φ is called harmonic if it is a critical point of the energy functional over any compact subset D in M, let $\{\varphi_t\}_{t\in(-\epsilon,\epsilon)}$

$$\frac{d}{dt}E(\varphi_t;D)\Big|_{t=0}=-\int_D h(v,\tau(\varphi))\,v^g, \tag{2}$$

where $v=\dfrac{\partial\varphi_t}{\partial t}\Big|_{t=0}$ denotes the variation vector field of φ, and $\tau(\varphi)$ is the tension field of φ given by $\tau(\varphi)=trace_g\nabla d\varphi$.
A natural generalization of harmonic maps is given by integrating the square of the functional of φ is defined by

$$E_2(\varphi;D)=\frac{1}{2}\int_D |\tau(\varphi)|^2 v^g \tag{3}$$

we say that φ id biharmonic map if it is a critical point of the bienergy functional, that is to say, if it satisfies the Euler-Lagrange of the functional (3), that is

$$\tau_2(\varphi)=-\Delta^\varphi\tau(\varphi)-trace_g R^n(d\varphi,\tau(\varphi))d\varphi$$

were studied by J.Eells, J.H.Sampson, L.Lemaire [10, 9], A .Lichnerwicz [17] and G.Y.Jiang [14]
A Submanifold in a Riemannian manifold is called harmonic (or minimal) if the isometric immersion defining the submanifold is a harmonic map, and it's called biharmonic if the isometric immersion defining the submanifold is a biharmonic map

We will call proper biharmonic submanifold a biharmonic submanifold with is non minimal (see[3])

The variational problem associated by considering for a fixed map, the energy functional defined on the set of Riemannian metric on the compact domain gave rise to the stress-energy tensor .

The p-harmonic map is a critical point of the p-energy functional

$$E_p(\varphi;D)=\frac{1}{p}\int_D |d\varphi|^p v_g, \tag{4}$$

over any compact subset D of M. Let $\{\varphi_t\}_{t\in(-\epsilon,\epsilon)}$ be a smooth variation of φ supported in D. Then

$$\frac{d}{dt}E_p(\varphi_t;D)\Big|_{t=0} = -\int_D h(\tau_p(\varphi),v)v_g, \tag{5}$$

where $v = \frac{\partial\varphi_t}{\partial t}\Big|_{t=0}$ denotes the variation vector field of φ,

$$\tau_p(\varphi) = \operatorname{div}^M(|d\varphi|^{p-2}d\varphi). \tag{6}$$

Let $\tau(\varphi)$ the tension field of φ defined by

$$\tau(\varphi) = \operatorname{trace}_g \nabla d\varphi = \sum_{i=1}^m \left\{\nabla^\varphi_{e_i} d\varphi(e_i) - d\varphi(\nabla^M_{e_i}e_i)\right\}. \tag{7}$$

(see [2]), where $\{e_1,...,e_m\}$ is an orthonormal frame on (M,g), $m = \dim M$, ∇^M is the Levi-Civita connection of (M,g), and ∇^φ denote the pull-back connection on $\varphi^{-1}TN$. If $|d\varphi|_x \neq 0$ for all $x \in M$, the map φ is p-harmonic if and only if (see [1, 4, 11])

$$|d\varphi|^{p-2}\tau(\varphi) + (p-2)|d\varphi|^{p-3}d\varphi(\operatorname{grad}^M|d\varphi|) = 0. \tag{8}$$

A natural generalization of p-harmonic maps is given by integrating the square of the norm of $\tau_p(\varphi)$. More precisely, the p-bienergy functional of φ is defined by

$$E_{2,p}(\varphi;D) = \frac{1}{2}\int_D |\tau_p(\varphi)|^2 v^g. \tag{9}$$

We say that φ is a p-biharmonic map if it is a critical point of the p-bienergy functional, that is to say, if it satisfies the Euler-Lagrange equation of the functional (9), that is (see [20])

$$\begin{aligned}\tau_{2,p}(\varphi) \;=\; & -|d\varphi|^{p-2}\operatorname{trace}_g R^N(\tau_p(\varphi),d\varphi)d\varphi - \operatorname{trace}_g \nabla^\varphi|d\varphi|^{p-2}\nabla^\varphi\tau_p(\varphi)\\ & -(p-2)\operatorname{trace}_g \nabla < \nabla^\varphi\tau_p(\varphi), d\varphi > |d\varphi|^{p-4}d\varphi = 0.\end{aligned} \tag{10}$$

Let $\{e_1,...,e_m\}$ be an orthonormal frame on (M,g), we have

$$\operatorname{trace}_g R^N(\tau_p(\varphi),d\varphi)d\varphi = \sum_{i=1}^m R^N(\tau_p(\varphi),d\varphi(e_i))d\varphi(e_i),$$

$$\operatorname{trace}_g \nabla^\varphi|d\varphi|^{p-2}\nabla^\varphi\tau_p(\varphi) = \sum_{i=1}^m \left(\nabla^\varphi_{e_i}|d\varphi|^{p-2}\nabla^\varphi_{e_i}\tau_p(\varphi) - |d\varphi|^{p-2}\nabla^\varphi_{\nabla^M_{e_i}e_i}\tau_p(\varphi)\right),$$

$$< \nabla^\varphi\tau_p(\varphi), d\varphi >= \sum_{i=1}^m h\left(\nabla^\varphi_{e_i}\tau_p(\varphi), d\varphi(e_i)\right),$$

$$\begin{aligned}\operatorname{trace}_g \nabla < \nabla^\varphi \tau_p(\varphi), d\varphi > |d\varphi|^{p-4} d\varphi &= \sum_{i=1}^m \Big(\nabla^\varphi_{e_i} < \nabla^\varphi \tau_p(\varphi), d\varphi > |d\varphi|^{p-4} d\varphi(e_i) \\ &\quad - < \nabla^\varphi \tau_p(\varphi), d\varphi > |d\varphi|^{p-4} d\varphi(\nabla^M_{e_i} e_i) \Big).\end{aligned}$$

The p-energy functional (resp. p-bienergy functional) includes as a special case ($p = 2$) the energy functional (resp. bi-energy functional), whose critical points are the usual harmonic maps (resp. bi-harmonic maps),
p-harmonic maps are always p-biharmonic maps by definition. In particular, if (M, g) is a compact orientable Riemannian manifold without boundary, and (N, h) is a Riemannian manifold with non-positive sectional curvature. Then, every p-biharmonic map from (M, g) to (N, h) is p-harmonic.

A submanifold in a Riemannian manifold is called a p-biharmonic submanifold if the isometric immersion defining the submanifold is a p-biharmonic map. We will call proper p-biharmonic submanifolds a p-biharmonic submanifols which is non p-harmonic.

The present thesis mainly deals with: the study of p-biharmonic submanifold in space form and p-Biharmonic hypersurfaces in Einstein space and conformally flat space.

The first chapter is intended to establish the notations and recall some basic of Riemannian manifolds, witch will be used throughout the entire theses,

In the second chapter we introduce the notion of Riemannian submanifold,

In the third chapter, we shall introduce the theory of harmonic and biharmonic maps,

In the fourth chapter, we extend the definition of p-harmonic and p-biharmonic maps between two Riemannian manifolds and we present some new properties of stress p-bienergy tensor (see [23]),

The fifth chapter contains new methods for constructing proper p-biharmonic hypersurfaces in Einstein space and conformally flat space, and we present fundamental results and examples of proper p-biharmonic submanifold.

Chapter 1

Elementary Differential Geometry

In this chapter we recall the fundamental notation of Differential and a Riemannian geometry, as differentiable manifolds and submanifolds, Riemannian manifolds, connection, curvatures and operators on Riemannian manifolds,...ect.
Which will be used throughout the entire thesis.

1.1 Differentiable Manifold and Submanifold

References [5], [16], [26], [27], [28].

1.1.1 The Definition of a Differentiable Manifold

Definition 1.1.1. (*Chart*) *A topological space M is locally Euclidean of dimension n if every point p in M has a neighborhood U such that there is a homeomorphism φ from U onto an open subset of $\mathbb{R}^n$. We call the pair (U, φ) a chart.*

Definition 1.1.2. (*Topological Manifold*) *A topological manifold of dimension n is a Hausdorff, locally Euclidean space of dimension n and has a countable basis of open sets.*

Example 1.1.1. *[28] (A Cusp) The graph of $y = x^{\frac{2}{3}}$ in $\mathbb{R}^2$ is a topological manifold (Figure 1.1.1(a)). By virtue of being a subspace of $\mathbb{R}^2$, it is Hausdorff and second countable. It is locally Euclidean, because it is homeomorphic to $\mathbb{R}$ via $(x, x^{\frac{2}{3}}) \mapsto x$.*

Example 1.1.2. *[28] (A cross) The cross in $\mathbb{R}^2$ in (Figure 1.1.1(b)) with the subspace topology is not locally Euclidean at the intersection p, and so cannot be a topological manifold.*

Definition 1.1.3. (*Compatible Charts*) *Two charts $(U, \varphi : U \to \mathbb{R}^n)$, $(V, \psi : V \to \mathbb{R}^n)$ of a topological are C^∞-compatible if the two maps*
$\varphi \circ \psi^{-1} : \psi(U \cap V) \longrightarrow \varphi(U \cap V)$, $\psi \circ \varphi^{-1} : \varphi(U \cap V) \longrightarrow \psi(U \cap V)$ are C^∞ these two maps are called the transition map between the charts with $U \cap V \neq \emptyset$.

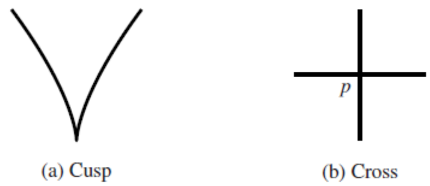


Figure 1.1:

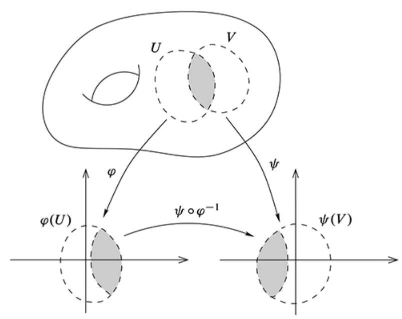


Figure 1.2: The transition map $\psi \circ \varphi^{-1}$ is define on $\varphi(U \cap V)$

Definition 1.1.4. (*Atlas*) *A C^∞ atlas on a locally Euclidean space M is a collection $\mathcal{A} = (U_\alpha, \varphi_\alpha)$ of pairwise C^∞-compatible charts that cover M, i.e., such that $M = \bigcup_\alpha U_\alpha$*

Definition 1.1.5. (*Smooth Manifold*) *A C^∞-Manifold or differentiable manifold is a topological manifold together with a C^∞-atlas.*

Theorem 1.1.1. [5] *Let M and N be C^∞-manifold of dimension m and n. the $M \times N$ is a C^∞-manifold of dimension $m+n$ with C^∞-atlas $\mathcal{A} = \{(U_\alpha \times V_\beta, \varphi_\alpha \times \psi_\beta)\}$ where $(U_\alpha, \varphi_\alpha)$ and (V_β, ψ_β) are a charts on M and N respectively, and $\varphi \times \psi$ is defined by $(\varphi \times \psi)(p, q) = (\varphi(p), \psi(q))$ in $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$.*

1.1.2 Differentiable Function and Mapping

Let f be a real-valued function defined on an open set W_f of C^∞-manifold M of dimension n, possibly all of M ; in brief $f : W_f \longrightarrow \mathbb{R}$. If (U, φ) is a chart such that $W_f \cap U \neq \emptyset$ and if $x^1, \ldots, x^n$ denotes the local coordinates, then f corresponds to a function $\tilde{f}(x^1, \ldots, x^n)$ on $\varphi(W_f \cap U)$ defined by $\tilde{f}(x^1(p), \ldots, x^n(p)) = \tilde{f}(\varphi(p))$ for all $p \in W_f \cap U$.

Definition 1.1.6. (*Differentiable Function*) *$f : W_f \longrightarrow \mathbb{R}$ is a C^∞ function if each $p \in W_f$ lies in a coordinate neighborhood (U, φ) such that $f \circ \varphi^{-1}(x^1, \ldots, x^n) = \tilde{f}(x^1, \ldots, x^n)$ is C^∞ on $\varphi(W_f \cap U)$.*

Definition 1.1.7. (*Differentiable Mapping*) *Let M and N be two differential manifolds. F is a smooth mapping of M into N if for every $p \in M$ there exist a coordinated neighborhood (U, φ) of p and (V, ψ) of $F(p)$ with $F(U) \subset V$ such that $\psi \circ F \circ \varphi^{-1} : \varphi(U) \longrightarrow \psi(V)$ is smooth.*

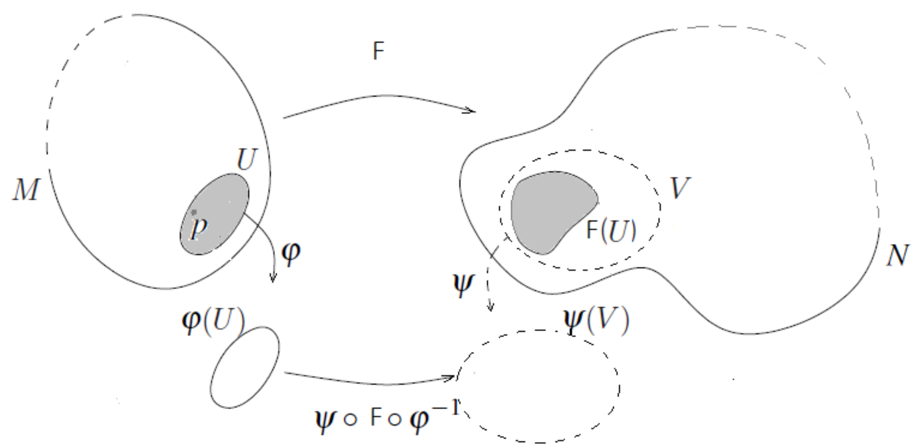


Figure 1.3: Differentiable Mapping between two manifolds

Definition 1.1.8. *A differentiable mapping $F : M \longrightarrow N$ between C^∞-manifolds is a diffeomorphism if it is homeomorphism and F^{-1} is C^∞, M and N are diffeomorphic if there exists a diffomorphism $F : M \longrightarrow N$.*

1.1.3 Rank of Mapping and Immersion

Let $F : M^m \longrightarrow N^n$ be a differentiable mapping of C^∞-manifolds and let $p \in M$, If (U, φ) and (V, ψ) are coordinate neighborhoods of p and $F(p)$, respectively and $F(U) \subset V$, then we have a corresponding expression for F in local coordinates, namely

$$\widetilde{F} = \psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(V)$$

Definition 1.1.9. *The rank of F at p is defined to be the rank of $\widetilde{F}$ at $\varphi(p)$. Thus the rank at p is the rank at $a = \varphi(p)$ of the Jacobian matrix*

$$\begin{pmatrix} \frac{\partial f^1}{\partial x^1} & \cdots & \frac{\partial f^1}{\partial x^m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f^n}{\partial x^1} & \cdots & \frac{\partial f^n}{\partial x^m} \end{pmatrix}_a$$

of the mapping $\widetilde{F}(x^1, \ldots, x^m) = (f^1(x^1, \ldots, x^m), \ldots, f^n(x^1, \ldots, x^m))$

Definition 1.1.10. *$F : M^m \longrightarrow N^n$ is said to be an immersion (resp. submersion) if rank $F = m = dimM$ (resp. $= n = dimN$) everywhere if F is an injective (one-to-one) immersion, then F establishes a one-to-one correspondence to endow $\widetilde{M}$ with a topology and C^∞ structure, then $\widetilde{M}$ will be called a submanifold (or immersed submanifold) and $F : M \longrightarrow \widetilde{M}$ is a diffeomorphism.*

We note that rank $F \leq \max(m, n)$ at every point it follows that if F is an immersion (resp. submersion), then $m \leq n$ (respectively $n \leq m$). In general the topology and C^∞ structure of an immersed submanifold $\widetilde{M} \subset N$ depend on F as well as M i.e, $\widetilde{M}$ is not a subspace of N.

Definition 1.1.11. *An imbedding is a one to one immersion $F : M \longrightarrow N$ with is a homeomorphism of M into N, that is, F is a homeomorphism of M onto its image, $\widetilde{M} = F(M)$ with its topology as a subspace of N. The image of an imbedding is called an imbedded submanifold.*

Theorem 1.1.2. [5] *Let $F : M \longrightarrow N$ be an immersion. Then each $p \in M$ has a neighborhood U such that $F|U$ is an imbedding of U in N.*

1.1.4 Submanifold

Definition 1.1.12. *A subset M of a C^∞ manifold N is said to have the m-submanifold property if each $p \in M$ has a coordinate neighborhood (U, φ) on N with local coordinates $x^1, \ldots, x^m$ such that*

(i) $\varphi(p) = (0, \ldots, 0)$,

(ii) $\varphi(U) = C^n_\epsilon(0) = \{x \in \mathbb{R}^n / |x^i| < \epsilon, i = 1 \ldots n, \}$

(iii) $\varphi(U \cap M) = \{x \in C^n_\epsilon(0) / x^{m+1} = \ldots = x^n = 0\}$.

ifM has this property, coordinate neighborhoods of this type are called preferred coordinates (relative to M) ($Figure$ 1.1.4) show such a subset M in $N = \mathbb{R}^3$
($m = 2$ and $n = 3$).

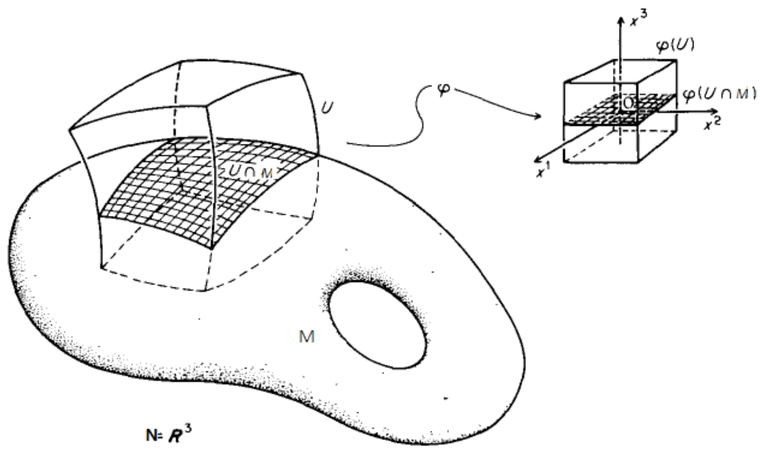


Figure 1.4:

Lemma 1.1.1. [5] *Let $M \subset N$ have the m-submanifold property then M with the relative topology is a topological m-manifold and each preferred coordinate system (U, φ) of N (relative to M) defines a local coordinates on M are C^∞-comapatible wherever they overlap and determine a C^∞ structure on M relative to which the inclusion $\mathbf{i}$: $M \to N$ is an imbedding.*

Definition 1.1.13. *A regular submanifold of a C^∞ manifold N is any sub-space M with submanifold property and with C^∞ structure that the corresponding preferred coordinate neighborhoods determine on it.*

Theorem 1.1.3. [5] *Let $F : M' \to N$ be an imbedding of C^∞-manifold M of dimension m in C^∞-manifold N of dimension n then $M = F(M')$ has a m-submanifold property and thus M is a regular submanifold. As such it is diffeomorphic to M' with respect to the mapping $F : M' \to M$.*

Theorem 1.1.4. [5] *If $F : M \to N$ is a one-to-one immersion and M is compact, then F is imbedding and $\widetilde{M} = F(M)$ a regular submanifold. Thus a submanifold of N, if compact in N is regular.*

Theorem 1.1.5. [5] *Let M be C^∞-manifold of dimension m, N be C^∞-manifold of dimension n, and $F : M \to N$ be a C^∞ mapping suppose that F has constant rank k on M and that $q \in F(M)$. Then $F^{-1}(q)$ is a closed, regular submanifold of M of dimension $m - k$.*

Corollary 1.1.1. *If $F : M \to N$ is a C^∞ mapping of manifolds, dim $N = n \leq m =$ dim$M$, and if the rank of $F = n$ at every of $A = F^{-1}(q)$, then A is closed, regular submanifold of M.*

Example 1.1.3. *The map*

$$\begin{array}{rcl} F:\mathbb{R}^n & \longrightarrow & \mathbb{R}, \\ x=(x_1\dots x_n) & \longmapsto & \sum_{i=1}^{n} x_i^2 \end{array}$$

F has a rank 1 on $\mathbb{R}^n\backslash\{0\}$, which contains $F^{-1}(1)=S^{n-1}$, thus S^{n-1} is an $(n-1)$-dimensional submanifold by corollary 1.1.1.

Example 1.1.4. *The map*

$$\begin{array}{rcl} F:\mathbb{R}^3 & \longrightarrow & \mathbb{R}, \\ (x,y,z) & \longmapsto & (a-(x^2+y^2)^{\frac{1}{2}})^2+z^2 \end{array}$$

has rank 1 at each point of $F^{-1}(b^2)$,$a>b>0$ thus locus $F^{-1}(b^2)$, the torus in $\mathbb{R}^3$ is a submanifold of dimension 2.

1.2 Vector Fields on a Differential Manifolds

1.2.1 The Tangent Space at Point of a Differential Manifold

Let M denote a C^∞ manifold of dimension n. Just as for $\mathbb{R}^n$, we define a germ of a C^∞ function at p in M to be an equivalence class of C^∞ functions defined in a neighborhood of p in M, two such functions being equivalent if they agree on some, possibly smaller, neighborhood of p. The set of germs of C^∞ real-valued functions at p in M is denoted by $C_p^\infty(M)$. The addition and multiplication of functions make $C_p^\infty(M)$ into a ring, with scalar multiplication by real numbers, $C_p^\infty(M)$ becomes an algebra over $\mathbb{R}$. Choosing an arbitrary (U,φ) about p it is easily verified that $\varphi^*: C^\infty_{\varphi(p)}(M)\to C_p^\infty(M)$ given by $\varphi^*(f)=f\circ\varphi$ is an isomorphism of the algebra of germs of C^∞ function at $\varphi(p)\in\mathbb{R}^n$ onto the algebra $C_p^\infty(M)$.

Definition 1.2.1. *We define the tangent space $T_p(M)$ to M at p to be a set of all mappings $X_p: C_p^\infty(M)\to R$ satisfying for all α, $\beta\in\mathbb{R}$ and $f,g\in C_p^\infty(M)$ the two conditions*

(i) $X_p(\alpha f+\beta g)=\alpha X_p(f)+\beta X_p(g)$ *(linearity)*

(ii) $X_p(fg)=(X_pf)g(p)+f(p)(X_pf)$ *(Leibniz rule)*

with the vector space operations in $T_p(M)$ defined by

$$(X_p+Y_p)f=X_pf+Y_pf$$

$$(\alpha X_p)f=\alpha(X_pf)$$

A tangent vector to M at p is any $X_p\in T_p(M)$.

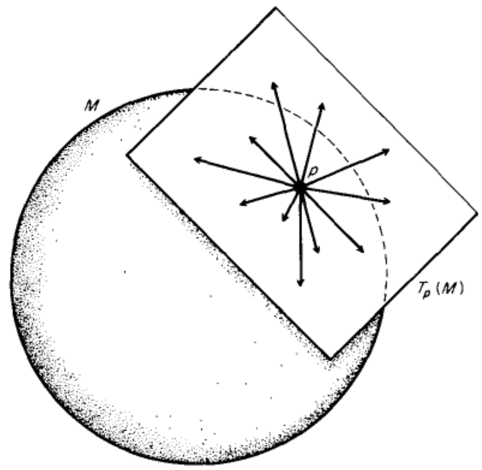


Figure 1.5: Tangent space on M at p

Lemma 1.2.1. [26] *Let* $X_p \in T_p(M)$

1. *If* $f, g \in C^\infty(M)$ *are equal on a neighborhood of* p *, then* $X_p(f) = X_p(g)$.
2. *If* $h \in C^\infty(M)$ *is a constant on a neighborhood of* p *then* $X_p(h) = 0$

Definition 1.2.2. *Let* $\varphi = (x^1 \ldots, x^n)$ *be a coordinate system in* M *at* p *if* $f \in C^\infty(M)$, *let*

$$\partial_i|p(f) = \frac{\partial f}{\partial x^i}(p) = \frac{\partial(f \circ \varphi)}{\partial u^i}(\varphi(p)) \quad (1 \leq i \leq n)$$

where $u^1 \ldots, u^n$ *are the standard coordinate on* $\mathbb{R}^n$.

Theorem 1.2.1. [26] *If* $\varphi = (x^1, \ldots, x^n)$ *is a coordinate system in* M *at* p, *then its coordinate vectors* $\partial_1|p, \ldots, \partial_n|p$ *form a basis for the tangent space* $T_p(M)$ *and*

$$X_p = \sum_{i=1}^{n} \partial_i|p \,\forall\, X_p \in T_p(M).$$

Definition 1.2.3. (*Differential of a Map*) *Let* $\phi : M \to N$ *be a smooth mapping for each* $p \in M$ *the function* $d_p\phi : T_p(M) \to T_{F(p)}N$ *sending* X_p *to* $X_{\phi(p)}$ *is called the differential map of* ϕ *at* p.
Thus $d_p\phi$ *is characterized by the equation* $d_p\phi(X_p)(g) = X_p(g \circ \phi)$ *for all* $X_p \in T_p(M)$ *and* $g \in C^\infty(N)$ *it follows that differential map is linear.*

1.2.2 Vector Field

Definition 1.2.4. (*Tangent Bundle*) *Let* M *be a smooth manifold and* $T_p(M)$ *be a tangent space at* $p \in M$. *The tangent bundle of* M *is the union of all the tangent spaces* *of M :*

$$TM = \bigcup_{p \in M} T_p(M)$$

TM is a smooth manifold and dim $TM = 2$ dim M (see [28]).

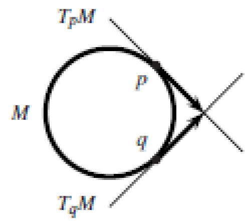


Figure 1.6: Tangent space to a circle .

Definition 1.2.5. (*Vector Bundle*)

- ⋄ *Let E, M be smooth manifolds and let $\pi : E \longrightarrow M$ be a smooth surjective map we call that priem$_\pi(p) = E_p$ the fibre at p for each $p \in M$.*

- ⋄ *On the tangent bundle TM of a smooth manifold M, the natural projection map $\pi : TM \longrightarrow M$, $\pi(p, v) = p$ makes TM into a smooth vector bundle over M.*

Definition 1.2.6. (*Vector Field*)

- ⋄ *A section of a vector bundle $\pi : E \longrightarrow M$ is a map $s : M \longrightarrow E$ such that $\pi \circ s = Id_M$*

- ⋄ *A vector field X on a manifold M is a function that assigns a tangent vector $X_p \in T_pM$ to each point $p \in M$. In terms of the tangent bundle, a vector field on M is simply a section of the tangent bundle $\pi : TM \longrightarrow M$ and the vector field is smooth if it is smooth as a map from M to TM.*

- ⋄ *The set of vector field $X : M \longrightarrow TM$ is denoted by $\Gamma(TM)$.*

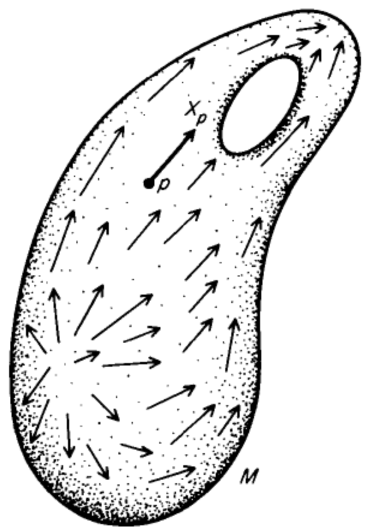


Figure 1.7: Vector field X on M .

1.3 Riemannian Manifolds

1.3.1 Tangent Covectors

Definition 1.3.1. (*Covector on Manifold*) *Let M be a smooth manifold and assume $p \in M$. We denote by $T_p^*(M)$ (where $T_p^*(M)$ the dual space to $T_p(M)$, thus $\sigma_p \in T_p^* M$ is a linear mapping $\sigma_p : T_pM \to \mathbb{R}$ and its value on $X_p \in T_p(M)$ is denote by $\sigma_p(X_p)$ or $< X_p, \sigma_p >$. Given a basis $e_{1p}, \ldots, e_{np}$ of $T_p(M)$, there is a a uniquely determined dual basis $\omega_p^1, \ldots, \omega_p^n$ satisfying, by definition, $\omega_p^i(e_{jp}) = \delta_j^i$ so that*

$$\sigma_p = \sum_{i=1}^{n} \sigma_p(e_{ip})\omega_p^i.$$

Definition 1.3.2. (*C^∞-covector field*) *A C^∞-covector field σ on M is a function which assigns to each $p \in M$ a covector $\sigma_p \in T_p^* M$ in such a manner that for any chart (U, φ) with coordinate frame $e_1, \ldots, e_n$, the functions $\sigma(e_i)$, $i = 1 \ldots n$ are of class C^∞ on U.*

Remarks 1.3.1.

- *The cotangent bundle of M is denoted by $T^* M$*

$$T^* M = \bigcup_{p \in M} T_p^* M$$

- *The map σ is $C^\infty(M)$-linear*

- *Let* (U, φ) *be a chart on* M *,dim* $M = n$ *and* $U' = \varphi(U) \subset \mathbb{R}^n$ *. Then* $dx_1, \ldots, dx_n$ *on* U' *are dual basis to* $\partial x_1 \ldots, \partial x_n$*.*

Definition 1.3.3. (*Tensor Field*) *Let* M *be a smooth manifold and for any* $p \in M$*. We define a vectorial space*

$$T_x^{(r,s)}M = \underbrace{T_pM \otimes \ldots \otimes T_pM}_{r-once} \otimes \underbrace{T_p^*M \otimes \ldots \otimes T_p^*M}_{s-once} := \{T \ \setminus T\, is\, (r,s) - tensor\}$$

Then :

1. *The tensor bundle denote by* $T^{(r,s)}M = \bigcup_{p\in M} T_p^{(r,s)}M$
2. *A* (r, s)*-tensor field on a manifold* M *is a function that assigns a section of the tensor bundle*

Example 1.3.1.

i) *A function on a manifold* M *is a* $(0, 0)$*-tensor.*

ii) *A vector field* X *is a* $(1, 0)$*-tensor.*

iii) *A differential 1-form* ω *on a manifold* M *is a* $(0, 1)$*-tensor.*

1.3.2 Riemannian Metric

Definition 1.3.4. *A Riemannian metric* g *on a smooth manifold* M *is a* C^∞*-bilinear, symmetric, positive definite* $(0, 2)$*-tensor field on* M *smoothly assigns to each point* $p \in M$ *a scalar product* g_p *on the tangent space* T_pM *where,*

$$\begin{array}{rcl} g_p = g\big|_{T_pM\otimes T_pM} : T_pM \times T_pM & \longrightarrow & \mathbb{R} \\ X_p\ ,\ Y_p & \longmapsto & g(X,Y)_p = g_p(X_p, Y_p) \end{array}$$

The pair (M, g) *is called Riemannian manifold.*

1.3.3 Orientable Manifold

Definition 1.3.5. *An oriented vector space in a vector plus an equivalence of allowable bases: all those bases with the same orientation as chosen on; they will be called oriented or positively oriented bases or frames.*

Lemma 1.3.1. *Let* $\Omega \neq 0$ *be a alternating covariant tensor on* $\mathcal{V}$ *of order* $n = dim\mathcal{V}$ *and let* $e_1, \ldots, e_n$ *be a basis of* $\mathcal{V}$*. then for any set of vectors* $v_1, \ldots, v_n$ *with* $v_i = \sum_{j=1}^{n} \alpha_i^j e_j$*, we have*

$$\Omega(v_1, \ldots, v_n) = det(\alpha_i^j)\Omega(e_1, \ldots, e_n)$$

Corollary 1.3.1. *A non vanishing $\Omega \in \Lambda^n(v)$ has the same sing (or opposite sing) on two bases if they have the same (respectively , opposite) orientation: thus choice of an $\Omega \neq 0$ determines an orientation of $\mathcal{V}$. Two such forms Ω_1, Ω_2 determine the same orientation if and only if $\Omega_1 = \lambda_2$, where λ is a positive real number.*

Definition 1.3.6. *We shall say that M is orientable if it is possible to define C^∞ n-form $\Omega = \lambda$ where $\lambda > 0$ is C^∞ function would give M the same orientation.*

Theorem 1.3.1. *A manifold M is orientable if and if only it has covering $\{U_\alpha, \varphi_\alpha\}$ of coherently oriented coordinate neighborhoods.*

Theorem 1.3.2. *Let M be an orientable Riemannian manifold with Riemannian metric g, corresponding to an orientation to an orientation of M there is an uniquely determined n-form Ω which gives the orientation and which has the value $+1$ on every oriented orthonormal frame*

Example 1.3.2. *The sphere $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \,|\, \|x\| = 1\}$ is an orientable manifold. Endeed; $\mathbb{S}^n$ is a Hausdorff topological space, where $\mathcal{T}_{\mathbb{S}^n}$ is the topology induced by $\mathbb{R}^{n+1}$ (it is the topology whose open sets are of the form $U = \Omega \cap \mathbb{S}^n$ with Ω is open of $\mathbb{R}^{n+1}$). We consider the stereographic projection*

$$\begin{aligned} \varphi_N : U_N = \mathbb{S}^n - \{N\} &\longrightarrow \mathbb{R}^n; \\ (u_1, ..., u_{n+1}) &\longmapsto \left(\frac{u_1}{1-u_{n+1}}, ..., \frac{u_n}{1-u_{n+1}}\right) \end{aligned}$$

$$\begin{aligned} \varphi_S : U_S = \mathbb{S}^n - \{S\} &\longrightarrow \mathbb{R}^n, \\ (u_1, ..., u_{n+1}) &\longmapsto \left(\frac{u_1}{1+u_{n+1}}, ..., \frac{u_n}{1+u_{n+1}}\right) \end{aligned}$$

where $N = (0, .., 0, 1)$ denote the "north pole", and $S = (0, ..., 0, -1)$ denote the "south pole".
Using $u_1^2 + ... + u_{n+1}^2 = 1$, we get

$$\begin{aligned} \varphi_N^{-1} : \mathbb{R}^n &\longrightarrow U_N; \\ (x_1, ..., x_n) &\longmapsto \left(\frac{2x_1}{\|x\|^2+1}, ..., \frac{2x_n}{\|x\|^2+1}, \frac{\|x\|^2-1}{\|x\|^2+1}\right) \end{aligned}$$

$$\begin{aligned} \varphi_S^{-1} : \mathbb{R}^n &\longrightarrow U_S. \\ (y_1, ..., y_n) &\longmapsto \left(\frac{2y_1}{\|y\|^2+1}, ..., \frac{2y_n}{\|y\|^2+1}, -\frac{\|y\|^2-1}{\|y\|^2+1}\right) \end{aligned}$$

So, the maps $\varphi_N : U_N \longrightarrow \mathbb{R}^n$ and $\varphi_S : U_S \longrightarrow \mathbb{R}^n$ are homeomorphismes. The smooth transition maps between charts are given by,

$$\varphi_S \circ \varphi_N^{-1}(x) = \frac{x}{\|x\|^2}, \quad \varphi_N \circ \varphi_S^{-1}(y) = \frac{y}{\|y\|^2}, \quad \forall x, y \in \mathbb{R}^n - \{0\}.$$

Thus, $\mathcal{A}_{\mathbb{S}^n} = \{(U_N, \varphi_N), (U_S, \varphi_S)\}$ form a differentiable atlas on $\mathbb{S}^n$. So, $\mathbb{S}^n$ is a differential manifold of dimension n. with, $U_N \cap U_S = \mathbb{S}^n - \{N, S\}$ are connexe, then $\mathbb{S}^n$ is orientable.

1.3.4 Integration on Manifold

1.3.5 Integral of a Differential Form over a Manifold

Integration of an n-form on $\mathbb{R}^n$ is not so different from integration of a function . Our approach to integration over a general manifold has several distinguishing features

- The manifold must be oriented
- On a manifold of dimension n, one can integrate only n-forms, not function
- The n-forms must have compact support

Let M be an oriented manifold of dimension n, with an oriented atlas $(U_\alpha, \varphi_\alpha)$ giving the orientation of M. Denote by $\Omega_c^k(M)$ the vector space of C^∞ k-forms with compact support on M. Suppose (U, φ) is a chart in this atlas. If $\omega \in \Omega_c^n(U)$ n-form with compact support on U, then because $\varphi : U \to \varphi(U)$ is a diffeomorphism, $(\varphi^{-1})^*\omega$is an n-form with compact support on the open subset $\varphi(U) \in \mathbb{R}^n$. We define the integral of ω on U to be

$$\int_U \omega := \int_{\varphi(U)} (\varphi^{-1})^*\omega. \tag{1.1}$$

If (U, ψ) is another chart in the oriented atlas with the same U, then $\varphi \circ \psi^{-1} : \psi(U) \to \varphi(U)$ is an orientation-preserving diffeomorphism, and so

$$\int_{\varphi(U)} (\varphi^{-1})^*\omega = \int_{\psi(U)} (\varphi \circ \psi^{-1})^*(\varphi^{-1})^*\omega = \int_{\psi(U)} (\psi^{-1})^*\omega$$

Thus, the integral $\int_U \omega$ on a chart U of of the atlas well define, independent of the choice of coordinates on U, by the the linearity of the integral on $\mathbb{R}^n$, if $\omega, \tau \in \Omega_c^n(U)$ then

$$\int_U (\omega + \tau) = \int_U \omega + \int_U \tau$$

Now let $\omega \in \Omega_c^n(M)$. Choose a partition of unity $\{\rho_\alpha\}$ subordinate to the open cover $\{U_\alpha\}$. Because ω has compact support and a partition of unity has locally finite supports, all except finitely many $\rho_\alpha\omega$ are identically zero, In particular,

$$\omega = \sum_\alpha \rho_\alpha \omega$$

is a finite su. Since

$$\text{supp}(\rho_\alpha\omega) \subset \text{supp}\rho_\alpha \cap \text{supp}\omega$$

supp($\rho_\alpha \omega$) is a closed subset of the compact set supp ω . Hence, supp($\rho_\alpha\omega$) is compact. Since $\rho_\alpha\omega$ is an n-form with compact support in the chart U_α, its integral $\int_{U_\alpha} \rho_\alpha\omega$ is defined. Therefore, we can define the integral of ω over M to be the finite sum

$$\int_M \omega := \sum_\alpha \int_{U_\alpha} \rho_\alpha \omega$$

1.3.6 Manifolds with Boundary

Definition 1.3.7. *A smooth manifold with boundary is a Hausdorff space M with a countable basis of open sets and a differentiable structure $\mathcal{A} = \{U_\alpha, \varphi_\alpha\}$ where $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha) \subset \mathbb{H}^n$ is homeomorphism, such that*

- $\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \;/\; x_1 \geq 0\}$. (*half-space*)
- *the union of U_α cover M*
- *If $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are two elements of $\mathcal{A}$ the $\varphi_\beta \circ \varphi_\alpha^{-1}$ and $\varphi_\alpha \circ \varphi_\beta^{-1}$ are diffeomorphisms of $\varphi_\alpha(U_\alpha \cap U_\beta)$ and $\varphi_\beta(U_\alpha \cap U_\beta)$, open subsets of $\mathbb{H}^n$*
- *$\mathcal{A}$ is maximal with respect first and third properties*

Remarks 1.3.2.

- *The boundary of the half-space noted $\partial\mathbb{H}^n$ is given by: $\partial\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \;/\; x_1 = 0\}$.*
- *Let (φ, U) be a chart on M at p if $\varphi(p) \in \partial\mathbb{H}^n$ in one coordinate system, then this holds for all coordinate systems the collection of such points is called boundary of M, denoted by ∂M.*
- *$M \backslash \partial M$ is a manifold which we denote by $Int(M)$.*
- *If $\partial M = \emptyset$, then M is called manifold without boundary.*

1.4 The Connection on Riemannian Manifolds

1.4.1 Linear Connection

Definition 1.4.1. *A linear connection on a smooth Riemannian manifold M is a map:*

$$\begin{aligned} \nabla : \Gamma(TM) \times \Gamma(TM) &\to \Gamma(TM), \\ (X, Y) &\mapsto \nabla_X Y \end{aligned}$$

such that:

1. $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$,

2. $\nabla_X(fY) = f\nabla_X Y + X(f)Y$,

3. $\nabla_{X+fY}(Z) = \nabla_X Z + f\nabla_Y Z$,

for all $X, Y, Z \in \Gamma(TM)$ *and* $f \in C^\infty(M)$. *We say that* $\nabla_X Y$ *is the covariant derivative of* Y *with the direction of* X.

Definition 1.4.2. *A section* $Y \in \Gamma(TM)$ *is said to be parallel with respect to the connection* ∇ *if*

$$\nabla_X Y = 0, \ \forall \ X \in \Gamma(TM).$$

Definition 1.4.3. *If* g *is a Riemannian metric on* M *then a connection* ∇ *is said to be metric or compatible with* g *if,*

$$\nabla g = 0 \ \textit{ i.e } \ (\nabla_X g)(Y, Z) = 0,$$

that is:

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \quad \forall \ X, Y, Z \in \Gamma(TM).$$

1.4.2 Torsion Tensor

Definition 1.4.4. *Let* M *be a smooth manifold, and* ∇ *be a connection on the tangent bundle* TM, *then the torsion of* ∇ *is a tensor field of type* $(1,2)$ *defined by*

$$\begin{array}{rcl} T : \Gamma(TM) \times \Gamma(TM) & \longrightarrow & \Gamma(TM) \\ (X, \ Y) & \longmapsto & \nabla_X Y - \nabla_Y X - [X, Y], \end{array}$$

where $[\,,\,]$ *is the Lie bracket on* $\Gamma(TM)$. *The connection* ∇ *on the tangent bundle* TM *is said to be torsion-free if the corresponding torsion* T *vanishes i.e.*

$$[X, Y] = \nabla_X Y - \nabla_Y X \quad \forall \ \ X, Y \in \Gamma(TM).$$

Remark 1.4.1. $T(X,Y) = -T(Y,X)$, *foll all* $X, Y \in \Gamma(TM)$ *(*T *is an antisymmetric).*

1.4.3 Levi-Civita Connection

Definition 1.4.5. *Let* (M, g) *be a Riemannian manifold then the map*

$$\nabla : \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM)$$

defined by the Koszul formula:

$$\begin{array}{rcl} 2g(\nabla_X Y, Z) & = & X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) \\ & & +g(Z,[X,Y]) + g(Y,[Z,X]) - g(X,[Y,Z]), \end{array} \tag{1.2}$$

is called the Levi-Civita connection of (M, g).

Theorem 1.4.1. *Let (M, g) be a Riemannian manifold. Then the Levi-Civita connection is an unique linear connection compatible with g and torsion free.*

Proposition 1.4.1. *Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇. Further let (U, φ) be a local coordinate on M and put $\partial_i = \frac{\partial}{\partial x_i} \in \Gamma(TU)$. Then $\{\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_m}\}$ is a local frame of TM on U. We define the Christoffel symbols $\Gamma_{ij}^k : U \longrightarrow \mathbb{R}$ of the connection ∇ with respect to (U, φ) by*

$$\Gamma_{ij}^k = \frac{1}{2}\sum_{l=1}^m g^{kl}\{\frac{\partial g_{jl}}{\partial x_i} + \frac{\partial g_{il}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l}\},$$

where $g_{ij} = g(e_i, e_j) = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ are the components of g, and $(g^{ij}) = (g_{ij})^{-1}$ is the inverse matrix.

1.5 Induced Connection on the Tangent Bundle

Definition 1.5.1. *Let $\varphi : M \longrightarrow N$ be a smooth map between two differentiable manifolds M and N and let ∇^N be a linear connection on N, then the Pull-back connection on the tangent bundle $\varphi^{-1}TN$ is defined by:*

$$\begin{aligned} \nabla^\varphi : \Gamma(TM) \times \Gamma(\varphi^{-1}TN) &\longrightarrow \Gamma(\varphi^{-1}TN), \\ (X, V) &\longrightarrow \nabla_X^\varphi V = \nabla_{d\varphi(X)}^N \widetilde{V} \end{aligned} \tag{1.3}$$

where $\widetilde{V} \in \Gamma(TN)$ such that $\widetilde{V} \circ \varphi = V$.

Locally

$$\begin{aligned} \nabla_X^\varphi V &= \nabla^\varphi_{X^i \frac{\partial}{\partial x_i}} V^\alpha (\frac{\partial}{\partial y_\alpha} \circ \varphi) \\ &= X^i \left\{ \frac{\partial V^\alpha}{\partial x_i} (\frac{\partial}{\partial y_\alpha} \circ \varphi) + V^\alpha \nabla^\varphi_{\frac{\partial}{\partial x_i}} (\frac{\partial}{\partial y_\alpha} \circ \varphi) \right\} \end{aligned}$$

Note that

$$\begin{aligned} \nabla^\varphi_{\frac{\partial}{\partial x_i}} (\frac{\partial}{\partial y_\alpha} \circ \varphi) &= \nabla^N_{d\varphi(\frac{\partial}{\partial x_i})} \frac{\partial}{\partial y_\alpha} \\ &= \frac{\partial \varphi_\beta}{\partial x_i} \left(\nabla^N_{\frac{\partial}{\partial y_\beta}} \frac{\partial}{\partial y_\alpha} \right) \circ \varphi \\ &= \frac{\partial \varphi_\beta}{\partial x_i} \left(\Gamma^\gamma_{\alpha\beta} \frac{\partial}{\partial y_\gamma} \right) \circ \varphi \end{aligned}$$

So that

$$\nabla^{\varphi}_{X}V = X^{i}\left\{\frac{\partial V^{\gamma}}{\partial x_i} + V^{\alpha}\frac{\partial \varphi_{\beta}}{\partial x_i}\left(\Gamma^{\gamma}_{\alpha\beta}\circ\varphi\right)\right\}\left(\frac{\partial}{\partial y_{\gamma}}\circ\varphi\right)$$

Then the relation (1.3) is independent of the choice of $\widetilde{V}$ i.e. this connection is well defined.

Definition 1.5.2. *If $\varphi : M \longrightarrow N$ is a map between differeniable manifolds, then two vector fields $X \in \Gamma(TM)$, $\widetilde{X} \in \Gamma(TN)$ are said to be φ-related if*

$$d\varphi_p(X) = \widetilde{X}_{\varphi(p)} \ \forall \ p \in M.$$

In that case we write $\widetilde{X} = d\varphi(X)$.

Proposition 1.5.1. *Let $\varphi : M \longrightarrow N$ be a smooth map and let ∇^{N} be a linear connection compatible with the Riemaniann metric h on N, then the linear connection ∇^{φ} is compatible with the induce Riemannian metric on $\varphi^{-1}TN$, that is*

$$X(h(V,W)) = h(\nabla^{\varphi}_{X}V, W) + h(V, \nabla^{\varphi}_{X}W),$$

for all $X \in \Gamma(TM)$ and V, $W \in \Gamma(\varphi^{-1}TN)$.

Proof. Let $X \in \Gamma(TM), V, W \in \Gamma(\varphi^{-1}TN)$ and $\widetilde{X}, \widetilde{V}, \widetilde{W} \in \Gamma(TN)$, such that

$$d\varphi(X) = \widetilde{X}\circ\varphi, \widetilde{V}\circ\varphi = V \text{ and } \widetilde{W}\circ\varphi = W$$

Then:

$$\begin{aligned}
X(h(V,W)) &= X(h(\widetilde{V}\circ\varphi, \widetilde{W}\circ\varphi)) \\
&= X(h(\widetilde{V}, \widetilde{W})\circ\varphi) \\
&= d(h(\widetilde{V}, \widetilde{W})\circ\varphi)(X) \\
&= dh(\widetilde{V}, \widetilde{W})(d\varphi(X)) \\
&= d\varphi(X)(h(\widetilde{V}, \widetilde{W})) \\
&= \widetilde{X}(h(\widetilde{V}, \widetilde{W}))\circ\varphi \\
&= h(\nabla^{N}_{\widetilde{X}}\widetilde{V}, \widetilde{W})\circ\varphi + h(\widetilde{V}, \nabla^{N}_{\widetilde{X}}\widetilde{W})\circ\varphi \\
&= h(\nabla^{N}_{\widetilde{X}\circ\varphi}\widetilde{V}, \widetilde{W}\circ\varphi) + h(\widetilde{V}\circ\varphi, \nabla^{N}_{\widetilde{X}\circ\varphi}\widetilde{W}) \\
&= h(\nabla^{\varphi}_{X}V, W) + h(V, \nabla^{\varphi}_{X}W).
\end{aligned}$$

□

Proposition 1.5.2. *Let ∇^{N} be a torsion free connection on N, then*

$$\nabla^{\varphi}_{X}d\varphi(Y) = \nabla^{\varphi}_{Y}d\varphi(X) + d\varphi([X,Y]),$$

For all $X, Y \in \Gamma(TM)$.

Proof. Let $V, W \in \Gamma(TN)$ be a φ-related with X and Y respectively, then:

$$\begin{aligned} [V,W]\circ\varphi &= d\varphi\circ[X,Y] \\ \nabla^N_V W &= \nabla^N_W V + [V,W]. \end{aligned}$$

From where:

$$\begin{aligned} \nabla^\varphi_X d\varphi(Y) &= \nabla^\varphi_X W\circ\varphi \\ &= \nabla^N_{d\varphi(X)} W \\ &= \big(\nabla^N_V W\big)\circ\varphi \\ &= \big(\nabla^N_W V + [V,W]\big)\circ\varphi \\ &= \nabla^\varphi_Y d\varphi(X) + d\varphi([X,Y]). \end{aligned}$$

□

1.6 Second Fundamental Form

Definition 1.6.1. *Let $\varphi : (M,g) \to (N,h)$ be a smooth map between two Riemannian manifolds. The second fundamental form of φ is the covariant derivative of vectorial 1-form $d\varphi$, defined by:*

$$\nabla d\varphi(X,Y) = \nabla^\varphi_X d\varphi(Y) - d\varphi(\nabla^M_X Y)$$

For all $X, Y \in \Gamma(TM)$.

Definition 1.6.2. *A map $\varphi : (M,g) \longrightarrow (N,h)$ is said to be totally geodesic if its second fundamental form vanishes.*

Property 1.6.1. *Let $\varphi : (M,g) \longrightarrow (N,h)$ be a smooth map between two Riemannian manifolds, the second fundamental form of φ is a vectorial 1-form $C^\infty(M)$-bilinear symmetric. i.e.*

$$\nabla d\varphi(f_1.X, f_2.Y) = f_1 f_2 \nabla d\varphi(Y,X),$$

for all X, $Y \in \Gamma(TM)$, and $f_1, f_2 \in C^\infty(M)$.

Proposition 1.6.1. *Let $\varphi : M \longrightarrow N$ and $\psi : N \longrightarrow P$ be a two smooth maps , then*

$$\nabla d(\psi\circ\varphi) = d\psi(\nabla d\varphi) + \nabla d\psi(d\varphi, d\varphi).$$

Proof. Let $X, Y \in \Gamma(TM)$, then

$$\begin{aligned} \nabla d(\psi\circ\varphi)(X,Y) &= \nabla^{\psi\circ\varphi}_X d(\psi\circ\varphi)(Y) - d(\psi\circ\varphi)(\nabla^M_X Y) \\ &= \nabla^{\psi\circ\varphi}_X d\psi(d\varphi(Y)) - d\psi(d\varphi(\nabla^M_X Y)) \end{aligned}$$

$$
\begin{aligned}
&= \nabla^P_{d\psi(d\varphi(X))} d\psi(d\varphi(Y)) - d\psi(d\varphi(\nabla^M_X Y)) \\
&= \nabla^\psi_{d\varphi(X)} d\psi(d\varphi(Y)) - d\psi(d\varphi(\nabla^M_X Y)) \\
&= \nabla d\psi(d\varphi(X), d\varphi(Y)) + d\psi(\nabla^N_{d\varphi(X)} d\varphi(Y)) - d\psi(d\varphi(\nabla^M_X Y)) \\
&= \nabla d\psi(d\varphi(X), d\varphi(Y)) + d\psi(\nabla d\varphi(X, Y)).
\end{aligned}
$$

□

Definition 1.6.3. *Let (M, g) be an m-dimensional Riemannian manifold, the frame $\{e_i\}_{i=1}^m$ is said geodesic frame at $x \in M$, if it is orthonormal that is $g(e_i, e_j) = \delta_{ij}$ on $U \subset M$, and $(\nabla_{e_i} e_j)|_x = 0, \quad \forall i, j = 1...m$.*

1.7 Curvature

1.7.1 Curvature Tensor

Definition 1.7.1. *The curvature tensor R is a tensor field of type $(1, 3)$ defined by*

$$
\begin{aligned}
R(X, Y)Z &:= [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z \\
&= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z, \qquad \forall X, Y, Z \in \Gamma(TM).
\end{aligned}
$$

The curvature tensor of type $(0, 4)$ is given by

$$R(X, Y, Z, W) = g(R(X, Y)Z, W).$$

Proposition 1.7.1. *Let (M, g) be a smooth Riemannian manifold. For vector fields X, Y, Z, W on M we have*

1. *$R(X, Y)Z = -R(Y, X)Z$ (antisymmetric).*
2. *$g(R(X, Y)Z, W) = g(R(Z, W)X, Y)$.*
3. *$g(R(X, Y)Z, Z) = 0$.*
4. *R verified Bianchi's identity algebraic:*

$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0.$$

5. *R verified Bianchi's identity differential:*

$$(\nabla_X R)(Y, Z) + (\nabla_Y R)(Z, X) + (\nabla_Z R)(X, Y) = 0.$$

1.7.2 Sectional Curvature

Definition 1.7.2. *For a point $p \in M$ the function*

$$\begin{aligned} K_p : \Gamma(TM) \times \Gamma(TM) &\to \mathbb{R} \\ (X, Y) &\mapsto \frac{g(R(X,Y)Y, X)}{g(X,X)g(Y,Y) - g(X,Y)^2}. \end{aligned}$$

is called the sectional curvature at p.
The Riemannian manifold M is said to be of constant curvature if there exists $k \in \mathbb{R}$ such that $K(X, Y) = k$.

Definition 1.7.3. *Let (M, g) be a smooth Riemannian manifold. We define the smooth tensor field $R_1 : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ of type $(3, 1)$ by*

$$R_1(X, Y)Z = g(Y, Z)X - g(X, Z)Y.$$

for all $X, Y, Z \in \Gamma(TM)$.

Corollary 1.7.1. *Let $(M^m, g)(m \geq 2)$ be a Riemannian manifold of constant curvature k. Then the curvature tensor R is given by*

$$R(X, Y)Z = k[R_1(X, Y)Z].$$

for all $X, Y, Z \in \Gamma(TM)$.

1.7.3 Ricci Curvature

Definition 1.7.4. *Let (M^m, g) be a Riemannian manifold, $p \in M$ and $\{e_1, ...e_m\}$ be an orthonormal frame of T_pM. Then*

1. *the Ricci tensor at p is defined by*

$$\text{Ricci}(X) = \sum_{i=1}^{m} R(X, e_i)e_i, \qquad \forall\ X \in T_pM.$$

2. *the Ricci curvature at p is defined by*

$$\text{Ric}(X, Y) = \sum_{i=1}^{m} g(R(X, e_i)e_i, Y), \qquad \forall\, X,\, Y \in T_pM.$$

3. *the scalar curvature S is defined by*

$$\begin{aligned} S &= \text{trace}_g \text{Ric} \\ &= \sum_{i,j=1}^{m} g(R(e_i, e_j)e_j, e_i) \end{aligned}$$

Remark 1.7.1. *For all $X, Y \in \Gamma(TM)$ we have*

$$\mathrm{Ric}(X, Y) = g(\mathrm{Ricci}(X), Y)$$

Corollary 1.7.2. *Let (M^m, g) be a Riemannian manifold of constant curvature k, then*

1. $\mathrm{Ricci}(X) = (m-1)kX$.
2. $\mathrm{Ric}(X, Y) = (m-1)kg(X, Y)$.
3. $S = m(m-1)k$.

Example 1.7.1.

1. *The sphere $\mathbb{S}^n$ has constant sectional curvature $+1$.*
2. *The space $\mathbb{R}^n$ has curvature 0 .*
3. $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 \ y > 0\}$ *The hyperbolic space with the metric $g = \frac{dx^2+dy^2}{y^2}$, has constant sectional curvature -1.*

1.8 Operators On Riemannian Manifolds

1.8.1 Gradient Operator

Let (M, g) be a Riemannian manifold,

$$\begin{aligned} \sharp : \Gamma(T^*M) &\to \Gamma(TM) \\ \omega &\mapsto \omega^\sharp \end{aligned}$$

be a isomorphism map between the cotangent bundle and the tangent bundle given by

$$\forall X \in \Gamma(TM), \ g(\omega^\sharp, X) = \omega(X).$$

Definition 1.8.1. *Let (M, g) be a Riemannian manifold, the gradient operator is given by*

$$\begin{aligned} \mathrm{grad} : C^\infty(M) &\longrightarrow \Gamma(TM). \\ f &\mapsto \mathrm{grad}\, f = (df)^\sharp \end{aligned}$$

So that for all $X \in \Gamma(TM)$ we have

$$g(\mathrm{grad}\, f, X) = X(f) = df(X).$$

Locally:

$$\operatorname{grad} f = \sum_{i=1}^{m} g^{ij} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j},$$

where $(\frac{\partial}{\partial x_i})_{i=1,\dots m}$ is a local coordinate. Let $\{e_i\}_{i=1,\dots m}$ be an orthonormal frame on (M, g). Then

$$\operatorname{grad} f = \sum_{i=1}^{m} e_i(f) e_i.$$

Proposition 1.8.1. *Let (M, g) be a Riemannian manifold, then*

1. $\operatorname{grad}(f + h) = \operatorname{grad} f + \operatorname{grad} h$;
2. $\operatorname{grad}(fh) = h \operatorname{grad} f + f \operatorname{grad} h$;
3. $(\operatorname{grad} f)(h) = (\operatorname{grad} h)(f)$.
4. $g(\nabla_X \operatorname{grad} f, Y) = g(\nabla_Y \operatorname{grad} f, X)$,

where f, $h \in \mathbf{C}^\infty(M)$ and $X, Y \in \Gamma(TM)$.

1.8.2 Hessian Operator

Definition 1.8.2. *Let f be a differentiable function on (M^m, g), then*

$$\begin{array}{rcl} \operatorname{Hess} f : \Gamma(TM) \times \Gamma(TM) & \longrightarrow & C^\infty(M) \\ (X, \quad Y) & \mapsto & (\operatorname{Hess} f)(X, Y) = g(\nabla_X \operatorname{grad} f, Y) \end{array}$$

we have

1. $\operatorname{Hess} f$ *be a tensor of type* $(0, 2)$.
2. $\operatorname{Hess} f$ *is symmetric.*

Locally:

$$\operatorname{Hess} f = \sum_{i,j=1}^{m} (\operatorname{Hess} f)_{ij}\, dx_i \otimes dx_j,$$

where

$$\begin{aligned} (\operatorname{Hess} f)_{ij} &= g(\nabla_{\partial_i} \operatorname{grad} f, \partial_j) \\ &= \frac{\partial^2 f}{\partial x_i \partial x_j} - \sum_{k=1}^{m} \Gamma_{ij}^k \frac{\partial f}{\partial x_k}. \end{aligned}$$

1.8.3 Divergence Operator

Let X be a vector field on (M, g), then

$$\begin{array}{rcl} \nabla X : \Gamma(TM) & \longrightarrow & \Gamma(TM) \\ Z & \mapsto & \nabla_Z X \end{array}$$

is a smooth linear mapping.

Definition 1.8.3. *The divergence of the vector field* $X \in \Gamma(TM)$*, denoted* $\operatorname{div} X$ *is defined by*

$$\operatorname{div} X = \operatorname{trace}\nabla X.$$

Let $\{e_i\}_{i=1,\dots m}$ be an orthonormal frame on M, then

$$\operatorname{div} X = \sum_{i=1}^{m} g(\nabla_{e_i} X, e_i).$$

Properties 1.8.1. *Let* (M, g) *be a Riemannian manifold, then*

1. $\operatorname{div}(X+Y) = \operatorname{div} X + \operatorname{div} Y$;
2. $\operatorname{div}(fX) = f\operatorname{div} X + X(f)$,

for all $X, Y \in \Gamma(TM)$ *and* $f \in C^\infty(M)$.

Example 1.8.1. *Let* $(\mathbb{R}^n, dx_1^2 + dx_2^2 + \dots + dx_n^2)$*, et* $P = \sum_{i=1}^{n} x_i \partial_i$ *the postion field in* $\mathbb{R}^n$*. for all* $Y = \sum_{j=1}^{n} Y^j \partial_j$*,we have :*

$$\begin{aligned} \nabla_Y P &= \sum_{i,j=1}^{n} \nabla_{Y^j \partial_j} x_i \partial_i \\ &= \sum_{i,j=1}^{n} Y^j \left[\frac{\partial x_i}{\partial x_j} \partial_i + x_i \underbrace{\nabla_{\partial j} \partial_i}_{=0} \right] \\ &= \sum_{i,j=1}^{n} Y^j \delta_{ij} \partial_i \\ &= \sum_{j=1}^{n} Y^j \partial_j. \end{aligned}$$

Then, $\nabla_Y P = Y$. Thus

$$\begin{aligned}\operatorname{div}^{\mathbb{R}^n} P &= \sum_{i=1}^{n} g(\nabla_{e_i} P, e_i) \\ &= \sum_{i=1}^{n} g(e_i, e_i) = n.\end{aligned}$$

Definition 1.8.4. *The divergence of 1-form $\omega \in \Gamma(T^*M)$ is defined by*

$$\operatorname{div}^M \omega = \sum_{i=1}^{m} (e_i(\omega(e_i)) - \omega(\nabla^M_{e_i} e_i))$$

Proposition 1.8.2. *Let $\omega, \eta \in \Gamma(T^*M)$ and $f \in \mathbf{C}^\infty(M)$, then*

1. *$div(\omega + \eta) = div\, \omega + div\, \eta$.*
2. *$div(f\,\omega) = f div\, \omega + \omega(\text{grad } f)$.*

1.8.4 Laplacian Operator

Definition 1.8.5. *Let (M^m, g) be a Riemannian manifold, the Laplacian operator noted $\triangle$, on M is defined by*

$$\begin{aligned}\triangle : C^\infty(M) &\longrightarrow C^\infty(M) \\ f &\mapsto \triangle(f) = \operatorname{div}(\operatorname{grad} f)\end{aligned}$$

Properties 1.8.2. *Let (M^m, g) be a Riemannian manifold, then*

1. *$\triangle(f + h) = \triangle(f) + \triangle(h)$;*
2. *$\triangle(fh) = h \triangle (f) + f \triangle (h) + 2g(\operatorname{grad} f, \operatorname{grad} h)$,*

for all $f, h \in C^\infty(M)$.

1.8.5 Divergence Theorem

Proposition 1.8.3. *Let D a compact domain on board in a Riemannian manifold (M, g). Let ω a 1-forme differential and X a field of vectors defined on a neighborhood included in D. Then :*

$$\int_D (\operatorname{div}^M \omega) v^D = \int_{\partial D} \omega(\mathfrak{n}) v^{\partial D} \qquad et \qquad \int_D (\operatorname{div}^M \omega) v^D = \int_{\partial D} g(X, \mathfrak{n}) v^{\partial D},$$

where ∂D is a board of D, and $\mathfrak{n} = \mathfrak{n}(x)$ is the unit vector normal to ∂D.

Corollary 1.8.1. *for all ω a 1-form differential and X a compact supported vector field in a domain D,then :*

$$\int_D (\operatorname{div} \omega) v^D = 0 \qquad and \qquad \int_D (\operatorname{div} X) v^D = 0.$$

1.8.6 Green Theorem

Theorem 1.8.1. *Let (M, g) a compact orientable and without boundary Riemannian manifold $(\partial M = \emptyset)$. Then, $\forall X \in \Gamma(TM)$, $\forall w \in \Gamma(T^*M)$, we have :*

$$\int_M (\operatorname{div}^M X)v^g = 0, \quad \int_M (\operatorname{div}^M w)v^g = 0,$$

where $v^g = \sqrt{det(g_{ij})}dx^1 \wedge ... \wedge dx^m$.

1.9 Conformally Metric

Definition 1.9.1. *A smooth map φ from a Riemannian manifold (M, g) to (M', g') is called conformal if there is function f such that:*

$$\varphi^* g' = fg.$$

Proposition 1.9.1. *[29] Let (M, g) be a Riemannian manifold of a dimension m if we deform the metric g by $\tilde{g} = e^{2\gamma} g$ where $\gamma \in C^\infty(M)$. Then:*

$$\tilde{\nabla}_X Y = \nabla_X Y + X(\gamma)Y + Y(\gamma)X - g(X, Y) \operatorname{grad} \gamma \tag{1.4}$$

for all $X, Y, Z \in \Gamma(TM)$

Proof. Let $\{e_i\}_{i=1...m}$ be an orthonormal frame on (M, g) such that $\nabla_{e_i} e_j = 0$ at $x \in M$ for all $i, j = 1 \dots m$, so at $x \in M$, we have

$$\begin{aligned}
2\tilde{g}(\tilde{\nabla}_X Y, Z) &= X\tilde{g}(Y, Z) + Y\tilde{g}(Z, X) - Z\tilde{g}(X, Y) \\
&= X(e^{2\gamma})g(Y, Z) + e^{2\gamma} Xg(Y, Z) + Y(e^{2\gamma})g(X, Z) + e^{2\gamma} Y g(X, Z) \\
&\quad - Z(e^{2\gamma})g(X, Y) + e^{2\gamma} Zg(X, Y) \\
&= 2e^{2\gamma} X(\gamma)g(Y, Z) \\
&\quad + 2e^{2\gamma} Y(\gamma)g(X, Z) - 2e^{2\gamma} Z(\gamma)g(X, Y) \\
&\quad + e^{2\gamma}(Xg(Y, Z) + Y g(X, Z) - Zg(X, Y)) \\
&= 2e^{2\gamma} g(X(\gamma)Y, Z) + 2e^{2\gamma} g(Y(\gamma)X, Z) - 2e^{2\gamma} \sum_{i=1}^{m} g(Z, e_i)e_i(\gamma)g(X, Y) + 2\tilde{g}(\nabla_X Y, Z) \\
&= 2\tilde{g}(\nabla_X Y + X(\gamma)Y + Y(\gamma)X - g(X, Y) \operatorname{grad} \gamma, Z).
\end{aligned}$$

□

By using the definition of curvature tensor, we have the following

Proposition 1.9.2. *[29] Let (M, g) be a Riemannian manifold of a dimension m if we deform the metric g by $\tilde{g} = e^{2\gamma} g$ where $\gamma \in C^\infty(M)$. The curvature tensors of $\tilde{\nabla}$ and ∇ satisfy*

$$\begin{aligned}\widetilde{R}(X,Y)Z &= R(X,Y)Z + X(Y(\gamma))Z - Y(Y(\gamma))X \\ &+Y(\gamma)Z(\gamma)X - X(\gamma)Z(\gamma)Y - Y(\gamma)g(X,Z)\operatorname{grad}\gamma \\ &+X(\gamma)g(Y,Z)\operatorname{grad}\gamma - g(Y,Z)\nabla_X \operatorname{grad}\gamma \\ &+g(X,Z)\nabla_Y \operatorname{grad}\gamma - g(Y,Z)|\operatorname{grad}\gamma|^2 X \\ &+g(X,Z)|\operatorname{grad}\gamma|^2 Y,\end{aligned}$$

where $X, Y, Z \in \Gamma(TM)$.

Proposition 1.9.3. *[29] Let* (M, g) *be a Riemannian manifold of a dimension* m *if we deform the metric* g *by* $\tilde{g} = e^{2\gamma} g$ *where* $\gamma \in C^\infty(M)$. *The Ricci tensors of* $\tilde{\nabla}$ *and* ∇ *satisfy :*

$$\widetilde{\mathrm{Ricci}}(X) = e^{-2\gamma}(Ricci(X) - \Delta(\gamma)X + (2-m)(|\operatorname{grad}\gamma|^2 X + \nabla_X \operatorname{grad}\gamma - X(\gamma)\operatorname{grad}\gamma)),$$

where $X \in \Gamma(TM)$.

Chapter 2

Geometry of Submanifolds

In this chapter we define the fundamental notations of Riemannian submanifolds, connection, curvature, second fundamental form ... References [15], [20].

2.1 Second Fundamental Form

Let M be an m-dimensional submanifold of n-dimensional Riemannian manifold (N, h) and $\mathbf{i} : M \hookrightarrow N$ the canonical inclusion, let g be a Riemannian metric defined by $g = \mathbf{i}^* h$. g is called the induced metric on M.

Theorem 2.1.1. *[15] The Levi-Civita connection ∇^M of (M, h) from that on (N, h), ∇^N is given by*

$$\nabla^M_X Y = (\nabla^N_X Y)^\top \text{ for all } X, Y \in \Gamma(TM).$$

where $\top : T_x N \to T_x M$ for $x \in M$ denote the orthogonal projection.

Proof. Let $X, X_1, X_2, Y \in \Gamma(TM)$ and $f \in C^\infty(M)$

1. ∇^M is linear connection

   (a)

$$\begin{aligned} \nabla^M_{fX_1+X_2} Y &= (\nabla^N_{fX_1+X_2} Y)^\top \\ &= (f\nabla^N_{X_1} Y + \nabla^N_{X_2} Y)^\top \\ &= f(\nabla^N_{X_1} Y)^\top + (\nabla^N_{X_2} Y)^\top \\ &= f\nabla^M_{X_1} Y + \nabla^M_{X_2} Y, \quad \forall X_1, X_2, Y \in \Gamma(TM), \forall f \in C^\infty(M); \end{aligned}$$

   (b)

$$\begin{aligned} \nabla^M_X (fY_1 + Y_2) &= (\nabla^N_X (fY_1 + Y_2))^\top \\ &= (X(f)Y_1 + f\nabla^N_X Y_1 + \nabla^N_X Y_2)^\top \\ &= X(f)Y_1 + f(\nabla^N_X Y_1)^\top + (\nabla^N_X Y_2)^\top \\ &= X(f)Y_1 + f\nabla^M_X Y_1 + \nabla^M_X Y_2, \quad \forall X, Y_1, Y_2 \in \Gamma(TM), \forall f \in C^\infty(M). \end{aligned}$$

2. ∇^M is compatible with the Riemmanian metric g

$$\begin{aligned} Xg(Y,Z) &= Xh(Y,Z) \\ &= h(\nabla^N_X Y, Z) + h(Y, \nabla^N_X Z) \\ &= h((\nabla^N_X Y)^\top, Z) + h(Y, (\nabla^N_X Z)^\top) \\ &= h(\nabla^M_X Y, Z) + h(Y, \nabla^M_X Z), \quad \forall X, Y \in \Gamma(TM). \end{aligned}$$

3. ∇^M is a torsion free connection

$$\begin{aligned} \nabla^M_X Y - \nabla^M_Y X &= (\nabla^N_X Y)^\top - (\nabla^N_Y X)^\top \\ &= (\nabla^N_X Y - \nabla^N_Y X)^\top \\ &= [X,Y]^\top \\ &= [X,Y], \quad \forall X, Y \in \Gamma(TM). \end{aligned}$$

□

Definition 2.1.1. *The normal bundle of (M,g) in (N,h) as the bundle $TM^\perp$ at $x \in M$ where $T_xM^\perp$ is orthogonal complement of T_xM in T_xN thus mean*

$$T_xM^\perp = \{v \in T_xN / h(v,w) = 0, \forall w \in T_xM\}.$$

Proposition 2.1.1. *[20] Let (M,g) be a Riemannian submanifold of a Riemannian manifold (N,h). Then*

$$\forall x \in M, \quad T_xN = T_xM \oplus T_xM^\perp,$$

that is,for all $x \in M$, we have

$$\forall v \in T_xN, \quad \exists! v^\top \in T_xM, \quad \exists! v^\perp \in T_xM^\perp, \quad v = v^\top + v^\perp.$$

Lemma 2.1.1. *Let (M,g) be a Riemannian submanifold of a Riemannian manifold (N,h) and let $B : \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM)^\perp, \quad B(X,Y) = (\nabla^N_X Y)^\perp$, then*

1. *B is symmetric;*

2. *B is $C^\infty(M)$-bilinear.*

Proof.

1.

$$\begin{aligned} B(X,Y) - B(Y,X) &= (\nabla^N_X Y)^\perp - (\nabla^N_Y X)^\perp \\ &= (\nabla^N_X Y - \nabla^N_Y X)^\perp \\ &= [X,Y]^\perp = 0, \quad \forall X, Y \in \Gamma(TM); \end{aligned}$$

2.

$$\begin{aligned} B(fX,Y) &= (\nabla^N_{fX}Y)^\perp \\ &= (f\nabla^N_X Y)^\perp \\ &= f(\nabla^N_X Y)^\perp \\ &= fB(X,Y), \quad \forall X,Y \in \Gamma(TM), \forall f \in C^\infty(M). \end{aligned}$$

Thus, $B(X,fY) = B(fY,X) = fB(Y,X) = fB(X,Y)$, by the same method, we find $B(X_1+X_2,Y) = B(X_1,Y) + B(X_2,Y)$ and $B(X,Y_1+Y_2) = B(X,Y_1) + B(X,Y_2), \quad \forall X,Y,X_1,X_2,Y_1,Y_2 \in \Gamma(TM), \forall f \in C^\infty(M)$.

□

Lemma 2.1.1 makes the following definition possible

Definition 2.1.2. *B is called the second fundamental form of (M,g).*

2.2 The Curvature of Riemannian Submanifold

Definition 2.2.1. *Let (M,g) be a Riemannian submanifold of a Riemannian manifold (N,h) and B the second fundamental form (M,g), the mean curvature of (M,g) is*

$$H = \frac{1}{m}\operatorname{trace}_g B = \frac{1}{m}\sum_{i=1}^m B(e_i,e_i)$$

for an orthonormal basis $\{e_1,\ldots,e_m\}$ of T_xM.

Definition 2.2.2. *A Riemannian submanifold M of the Riemannian manifold N is called minimal if its mean curvature H vanishes.*

Proposition 2.2.1. *Let (M,g) be a submanifold of (N,h), R^M(resp R^N) curvature tensor of M (resp N) then*

$$g(R^M(V,W)X,Y) = h(R^N(V,W)X,Y) - h(B(V,X),B(W,Y)) + h(B(V,Y),B(W,X)),$$

where $X,Y,V,W \in \Gamma(TM)$.

Proof. We calculate

$$R^M(V,W)X = \nabla^M_V\nabla^M_W X - \nabla^M_W\nabla^M_V X - \nabla^M_{[V,W]}X;$$

$$g(R^M(V,W)X,Y) = g(\nabla^M_V\nabla^M_W X,Y) - g(\nabla^M_W\nabla^M_V X,Y) - g(\nabla^M_{[V,W]}X,Y).$$

For the term $g(\nabla^M_V\nabla^M_W X,Y)$, we have

$$g(\nabla^M_V\nabla^M_W X,Y) = g((\nabla^N_V\nabla^M_W X)^\top,Y)$$

$$\begin{aligned}
&= h(\nabla^N_V \nabla^M_W X - (\nabla^N_V \nabla^M_W X)^\perp, Y) \\
&= h(\nabla^N_V \nabla^M_W X, Y) \\
&= h(\nabla^N_V (\nabla^N_W X - (\nabla^N_W X)^\perp), Y) \\
&= h(\nabla^N_V \nabla^N_W X, Y) - h(\nabla^N_V B(W,X), Y) \\
&= h(\nabla^N_V \nabla^N_W X, Y) - \underbrace{V h(B(W,X), Y)}_{0} + h(B(W,X), \nabla^N_V Y) \\
&= h(\nabla^N_V \nabla^N_W X, Y) - \underbrace{V h(B(W,X), Y)}_{0} + h(B(W,X), (\nabla^N_V Y)^\perp) \\
&= h(\nabla^N_V \nabla^N_W X, Y) + h(B(W,X), B(V,Y)).
\end{aligned}$$

by the same method, we find that

$$g(\nabla^M_W \nabla^M_V X, Y) = h(\nabla^N_W \nabla^N_V X, Y) + h(B(V,X), B(W,Y));$$

$$g(\nabla^M_{[V,W]} X, Y) = h(\nabla^N_{[V,W]} X, Y).$$

finely

$$g(R^M(V,W)X, Y) = h(R^N(V,W)X, Y) - h(B(V,X), B(W,Y)) + h(B(V,Y), B(W,X)).$$

□

Corollary 2.2.1. *Let (M,g) be a Riemannian submanifold of (N,h), K^M (resp. K^N) the sectional curvature of (M,g) (resp. (N,h)). Then*

$$K^M(v,w) = K^N(v,w) + \frac{h(B(v,v), B(w,w)) - h(B(v,w), B(v,w))}{g(v,v)g(w,w) - g(v,w)^2}$$

where $\{v,w\}$ is a basis of $\pi \subset T_xM$, with $x \in M$.

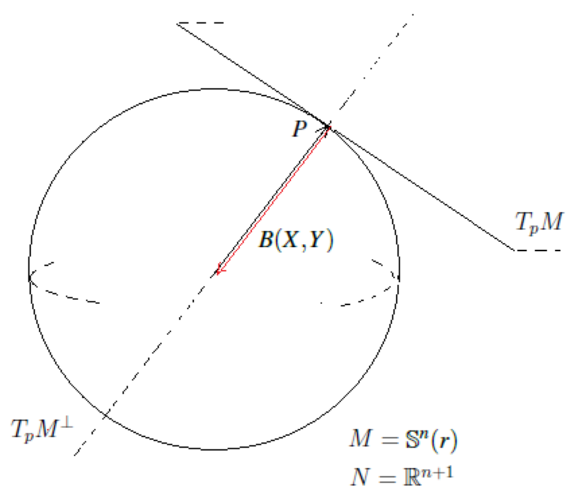
Example 2.2.1. *Let $M = \mathbb{S}^n(r) = \{x \in \mathbb{R}^{n+1} \mid \|x\| = r\}$, and $N = \mathbb{R}^{n+1}$ equipped with inner product $h = \langle\, , \rangle$. We have, $\nabla^N_X P = X$, for all $X \in \Gamma(TN)$, where $P = \sum_{x=1}^{n+1} x_i \partial_i$ (see example 1.8.1), and g the induced metric on M for h, as P is a normal field of M, we get $B(X,Y) = \alpha P$,for all $X, Y \in \Gamma(TM)$, where $\alpha \in C^\infty(M)$.*

We have, $\langle B(X,Y), P\rangle = \alpha\langle P, P\rangle$ i.e $\langle B(X,Y), P\rangle = \alpha|P|^2 = \alpha r^2$, as result

$$\begin{aligned}
\alpha &= \frac{1}{r^2}\langle(\nabla^N_X Y)^\perp, P\rangle \\
&= \frac{1}{r^2}\langle \nabla^N_X Y, P\rangle \\
&= \frac{1}{r^2}(X\langle Y, P\rangle - \langle Y, \nabla_X P\rangle) \\
&= -\frac{1}{r^2}\langle Y, X\rangle.
\end{aligned}$$

thus, $B(X,Y) = -\frac{1}{r^2}\langle X, Y\rangle P$. *Now we can calculate the sectional curvature of* $(\mathbb{S}^n, g)$:

$$\begin{aligned} K^{\mathbb{S}^n}(v,w) &= K^{\mathbb{R}^{n+1}}(v,w) + \frac{\langle B(v,v), B(w,w)\rangle - \langle B(v,w), B(v,w)\rangle}{\langle v,v\rangle\langle w,w\rangle - \langle v,w\rangle^2} \\ &= \frac{\frac{1}{r^2}\frac{1}{r^2}\langle v,v\rangle\langle w,w\rangle r^2 - \frac{1}{r^2}\frac{1}{r^2}\langle v,w\rangle\langle v,w\rangle r^2}{\langle v,v\rangle\langle w,w\rangle - \langle v,w\rangle^2} \\ &= \frac{1}{r^2}. \end{aligned}$$



2.3 Shape operator

Definition 2.3.1. *Let* (M,g) *be a Riemannian submanifold of* (N,h), U *be a normal vector field on* M. *The shape operator (or Weingarten map or second fundamental tensor) of* M *is defined by*

$$A_U : \Gamma(TM) \longrightarrow \Gamma(TM), \quad A_U X = -(\nabla_X^N U)^\top.$$

Proposition 2.3.1. *Let* (M,g) *be a Riemannian submanifold of* (N,h), U *a normal vector field on* M. *Then :*

$$g(A_U X, Y) = h(B(X,Y), U), \quad \forall X, Y \in \Gamma(TM).$$

Proof. Let $X, Y \in \Gamma(TM)$. We compute

$$\begin{aligned} g(A_U X, Y) &= -h((\nabla_X^N U)^\top, Y) \\ &= -h(\nabla_X^N U, Y) \end{aligned}$$

$$
\begin{aligned}
&= -X\underbrace{h(U,Y)}_{0}+h(U,\nabla^N_X Y)\\
&= h(U,(\nabla^N_X Y)^\perp)\\
&= h(U,B(X,Y)).
\end{aligned}
$$

□

Example 2.3.1. *The shape operator of cylinder* $C=\{(x,y,z)\in\mathbb{R}^3\,|\,x^2+y^2=r^2,\ -1<z<1\}$ *is given by*

$$A_\eta=\begin{pmatrix}\frac{1}{r} & 0\\ 0 & 0\end{pmatrix},$$

where η *is the unit normal vector field on* C *with* $r>0$*. Indeed;* $(\theta,z)\longmapsto(r\cos\theta,r\sin\theta,z)$ *is a* C *parametrization, then* $\partial_\theta=(-r\sin\theta,r\cos\theta,0)$*,* $\partial_z=(0,0,1)$*, and* $\eta=(-\cos\theta,-\sin\theta,0)$*. We have*

$$
\begin{aligned}
A_\eta(\partial_\theta) &= -(\nabla^{\mathbb{R}^3}_{\partial_\theta}\eta)^\top\\
&= -\nabla^{\mathbb{R}^3}_{\partial_\theta}\eta\\
&= \nabla^{\mathbb{R}^3}_{\partial_\theta}(\cos\theta\partial_x+\sin\theta\partial_y)\\
&= -\sin\theta\partial_x+\cos\theta\nabla^{\mathbb{R}^3}_{\partial_\theta}\partial_x+\cos\theta\partial_y+\sin\theta\nabla^{\mathbb{R}^3}_{\partial_\theta}\partial_y\\
&= -\sin\theta\partial_x+\cos\theta\nabla^{\mathbb{R}^3}_{-r\sin\theta\partial_x+r\cos\theta\partial_y}\partial_x+\cos\theta\partial_y\\
&\quad +\sin\theta\nabla^{\mathbb{R}^3}_{-r\sin\theta\partial_x+r\cos\theta\partial_y}\partial_y\\
&= -\sin\theta\partial_x-r\cos\theta\sin\theta\nabla^{\mathbb{R}^3}_{\partial_x}\partial_x+r\cos^2\theta\nabla^{\mathbb{R}^3}_{\partial_y}\partial_x\\
&\quad +\cos\theta\partial_y-r\sin^2\theta\nabla^{\mathbb{R}^3}_{\partial_x}\partial_y+r\cos\theta\sin\theta\nabla^{\mathbb{R}^3}_{\partial_y}\partial_y.
\end{aligned}
$$

As $\nabla^{\mathbb{R}^3}_{\partial_x}\partial_x=\nabla^{\mathbb{R}^3}_{\partial_x}\partial_y=\nabla^{\mathbb{R}^3}_{\partial_y}\partial_x=\nabla^{\mathbb{R}^3}_{\partial_y}\partial_y=0$*, we find* $A_\eta(\partial_\theta)=(-\sin\theta,\cos\theta,0)$*, i.e* $A_\eta(\partial_\theta)=\frac{1}{r}\partial_\theta$*, by the same way, we find that* $A_\eta(\partial_z)=0$*.*

Example 2.3.2. *Let* U *a normal vector field on* $\mathbb{S}^n(r)$ *in* $\mathbb{R}^{n+1}$*. The shape operator of* $(\mathbb{S}^n(r),g)$ *in* $(\mathbb{R}^{n+1},\langle\,,\,\rangle)$ *relative to* U *is given by :*

$$
\begin{aligned}
g(A_UX,Y) &= \langle B(X,Y),U\rangle\\
&= -\frac{1}{r^2}g(X,Y)\langle P,U\rangle,
\end{aligned}
$$

(see l'example 2.2.1). Therefore, $A_UX=-\frac{1}{r^2}\langle P,U\rangle X$ *for all* $X\in\Gamma(T\mathbb{S}^n(r))$*.*

Definition 2.3.2. *Let* (M,g) *be a Riemannian submanifold of* (N,h)*.*

$$\nabla^\perp:\Gamma(TM)\times\Gamma(TM)^\perp\longrightarrow\Gamma(TM)^\perp,\quad (X,Y)\longmapsto\nabla^\perp_XY=(\nabla^N_XY)^\perp$$

is called the normal connection of (M,g) *in* (N,h)*.*

Properties 2.3.1. *Let (M, g) be a Riemannian submanifold of (N, h).*

1. *$\nabla^{\perp}_X Y$ is $C^{\infty}(M)$-linear with respect to X and $\mathbb{R}$-linear with respect to Y;*
2. *$\nabla^{\perp}_X fY = X(f)Y + f\nabla^{\perp}_X Y, \quad \forall X \in \Gamma(TM), \forall Y \in \Gamma(TM)^{\perp}, \forall f \in C^{\infty}(M)$;*
3. *$X(h(Y, Z)) = h(\nabla^{\perp}_X Y, Z) + h(Y, \nabla^{\perp}_X Z), \quad \forall X \in \Gamma(TM), \forall Y, Z \in \Gamma(TM)^{\perp}$.*

Proof. Using the definition of $\nabla^{\perp}$ and the properties of ∇^{N}. □

Chapter 3

Harmonic and Biharmonic mappings

In this chapter we give the definition of energy functional, the necessary and sufficient conditions for a map between two Riemannian manifolds to be harmonic.

3.1 Harmonic maps

Definition 3.1.1. *Let $(M^m, g), (N^n, h)$ two Riemannian manifolds, and $\varphi \in C^\infty(M, N)$. The energy of φ is defined by*

$$E(\varphi; D) = \frac{1}{2}\int_D |d\varphi|^2\, v^g, \tag{3.1}$$

where D is a compact domain in M, $|d\varphi|$ the Hilbert-Schmidt norm of the differential $d\varphi$, v^g the volume element on (M^m, g) given by $v^g = \sqrt{det(g-ij)}dx_1 \dots dx_m$.

Remark 3.1.1. *The Hilbert Schmidt norm of differential φ is given by*

$$|d\varphi|^2 = \sum_{i=1}^{m} h(d\varphi(e_i), d\varphi(e_i)),$$

with $\{e_1, \dots, e_m\}$ be an orthonormal frame on (M, g), and the local expression of the Hilbert Schmidt norm is

$$\begin{aligned}
|d\varphi|^2 &= \sum_{i=1}^{m} h(d\varphi(e_i), d\varphi(e_i)) \\
&= \sum_{i,a,b=1}^{m} h(d\varphi(e_i^a \frac{\partial}{\partial x_a}), d\varphi(e_i^b \frac{\partial}{\partial x_b})) \\
&= \sum_{a,b=1}^{m} g^{ab} \sum_{\alpha,\beta=1}^{n} h\left(\frac{\partial \varphi^\alpha}{\partial x_a}(\frac{\partial}{\partial y_\alpha}\circ\varphi), \frac{\partial \varphi^\beta}{\partial x_b}(\frac{\partial}{\partial y_\beta}\circ\varphi)\right)
\end{aligned}$$

$$= \sum_{a,b=1}^{m} g^{ab} \sum_{\alpha,\beta=1}^{n} \frac{\partial\varphi^\alpha}{\partial x_a}\frac{\partial\varphi^\beta}{\partial x_b} h\left(\frac{\partial}{\partial y_\alpha}, \frac{\partial}{\partial y_\beta}\right) \circ \varphi$$
$$= \sum_{i,j=1}^{m}\sum_{\alpha,\beta=1}^{n} g^{ij}\frac{\partial\varphi^\alpha}{\partial x_i}\frac{\partial\varphi^\beta}{\partial x_j}(h_{\alpha\beta}\circ\varphi).$$

Example 3.1.1. *Let the mapping*

$$\begin{aligned} \varphi : \mathbb{R}^n - \{0\} &\longrightarrow \mathbb{R}^n - \{0\}. \\ x &\longmapsto \frac{x}{\|x\|^l}. \end{aligned}$$

Then

$$d\varphi(\frac{\partial}{\partial x_i}) = \|x\|^{-l}(\frac{\partial}{\partial y_i}\circ\varphi) - lx_i\|x\|^{-l-2}\sum_{\alpha=1}^{n} x_\alpha \frac{\partial}{\partial y_\alpha}\circ\varphi;$$

$$|d\varphi|^2 = \sum_{i=1}^{n} < d\varphi(\frac{\partial}{\partial x_i}), d\varphi(\frac{\partial}{\partial x_i}) >$$

$$= \sum_{i=1}^{n} < \|x\|^{-l}(\frac{\partial}{\partial y_i}\circ\varphi) - lx_i\|x\|^{-l-2}\sum_{\alpha=1}^{n} x_\alpha\frac{\partial}{\partial y_\alpha}\circ\varphi,$$
$$\|x\|^{-l}(\frac{\partial}{\partial y_i}\circ\varphi) - lx_i\|x\|^{-l-2}\sum_{\beta=1}^{n} x_\beta\frac{\partial}{\partial y_\beta}\circ\varphi >$$

$$= \|x\|^{-2l}\sum_{i=1}^{n}\delta_{ii} - l\|x\|^{-2l-2}\sum_{i=1}^{n}\sum_{\beta=1}^{n} x_i x_\beta \delta_{i\beta} - l\|x\|^{-2l-2}\sum_{i=1}^{n}\sum_{\alpha=1}^{n} x_i x_\alpha\delta_{i\alpha}$$
$$+ l^2\|x\|^{-2l-4}\sum_{i=1}^{n}\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} x_i^2 x_\alpha x_\beta \delta_{\alpha\beta}$$

$$= n\|x\|^{-2l} - l\|x\|^{-2l} - l\|x\|^{-2l} + l^2\|x\|^{-2l}$$

$$= (n - 2l + l^2)\|x\|^{-2l}.$$

Definition 3.1.2. *A variation of $\varphi \in C^\infty(M,N)$ to support in a compact domain $D \subset M$, is a smooth family maps $(\varphi_t)_{t\in(-\epsilon,\epsilon)} : M \longrightarrow N$, such that $\varphi_0 = \varphi$ and $\varphi_t = \varphi$ on $M \setminus \mathrm{int}(D)$.*

Definition 3.1.3. *A map is called harmonic if it is a critical point of the energy functional over any compact subset D of M. i.e*

$$\frac{d}{dt}E(\varphi_t; D)\Big|_{t=0} = 0.$$

3.1.1 First variation of energy

Definition 3.1.4. *Let $\varphi : (M, g) \longrightarrow (N, h)$ be a smooth map between two Riemannian manifolds. The trace of second fundamental form of φ is called tension field of φ, noted by*

$$\tau(\varphi) = \operatorname{trace}_g \nabla d\varphi. \tag{3.2}$$

Local expression of tension field

Let a smooth map $\varphi : (M, g) \longrightarrow (N, h)$, we have

$$\begin{aligned}
\tau(\varphi) &= \sum_{i,j=1}^{m} g^{ij}(\nabla d\varphi)(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) \\
&= \sum_{i,j=1}^{m}\sum_{\gamma=1}^{n}\left(\frac{\partial^2\varphi_\gamma}{\partial x_i \partial x_j} + \sum_{\alpha,\beta=1}^{n}\frac{\partial\varphi_\alpha}{\partial x_i}\,\frac{\partial\varphi_\beta}{\partial x_j}\;{}^N\Gamma^\gamma_{\alpha\beta}\circ\varphi - \sum_{k=1}^{m}\frac{\partial\varphi_\gamma}{\partial x_k}\;{}^M\Gamma^k_{ij}\right)\frac{\partial}{\partial y_\gamma}\circ\varphi.
\end{aligned}$$

$(\frac{\partial}{\partial x_i})$ (resp. $(\frac{\partial}{\partial y_\alpha})$) is a local frame of vector fields on M (resp. on N)

Theorem 3.1.1. *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map and let $(\varphi_t)_{t\in(-\epsilon,\epsilon)}$ be a smooth variation of φ supported in D. Then*

$$\frac{d}{dt}E(\varphi_t; D)\Big|_{t=0} = -\int_D h(v, \tau(\varphi))\, v^g,$$

where $v = \frac{d\varphi_t}{dt}\Big|_{t=0}$ denotes the variation vector field of $(\varphi_t)_{t\in(-\epsilon,\epsilon)}$ v^g the volume element on (M^m, g)

Proof. Let $\{e_1, \dots e_m\}$ be an orthonormal frame on (M, g) and $\frac{d}{dt}$ a frame of vector field on $]-\varepsilon, \varepsilon[$.Thus, $\{(e_i, 0), (0, \frac{d}{dt})\}_{i=1}^m$ becomes an orthonormal frame for the diagonal metric $g + dt^2$ on the product manifold $M\times]-\varepsilon, \varepsilon[$. We have

$$\left[(e_i, 0), (0, \frac{d}{dt})\right] = 0, \quad \forall i \in \{1, \dots, m\}.$$

defined

$$\phi : M\times]-\varepsilon, \varepsilon[\;\longrightarrow\; N.$$

$$(x,t) \longmapsto \phi(x,t) = \phi_t(x)$$

According to Leibniz's formula, and

$$\begin{aligned} \phi_x :]-\varepsilon, \varepsilon[&\longrightarrow N; \\ t &\longmapsto \phi_x(t) = \phi(x,t) = \varphi_t(x) \end{aligned}$$

$$\begin{aligned} \phi_t : M &\longrightarrow N, \\ x &\longmapsto \phi_t(x) = \phi(x,t) = \varphi_t(x) \end{aligned}$$

we find that

$$\begin{aligned} d\phi(e_i, 0)_{(x,0)} &= d_x\phi_0(e_i|_x) + d_0\phi_x(0) \\ &= d_x\phi_0(e_i|_x) \\ &= d_x\varphi(e_i|_x); \end{aligned}$$

$$\begin{aligned} d\phi(0, \frac{d}{dt})_{(x,0)} &= d_x\phi_0(0) + d_0\phi_x(\frac{d}{dt}|_{t=0}) \\ &= d\phi_x(\frac{d}{dt}|_{t=0}) \\ &= v(x). \end{aligned}$$

Therefore

$$d\phi(e_i, 0) = d\varphi(e_i) \quad \text{et} \quad d\phi(0, \frac{d}{dt}) = v \quad \text{en} \quad t = 0.$$

Let ∇^ϕ the Pull-Back connection associated with the map ϕ we calculate

$$\begin{aligned} \frac{d}{dt} E(\varphi_t; D)\Big|_{t=0} &= \frac{1}{2}\frac{d}{dt}\int_D |d\varphi_t|^2 v^g\Big|_{t=0} \\ &= \frac{1}{2}\int_D \frac{\partial}{\partial t}|d\varphi_t|^2\Big|_{t=0} v^g \\ &= \frac{1}{2}\int_D \frac{\partial}{\partial t}\sum_{i=1}^m h(d\varphi_t(e_i), d\varphi_t(e_i))\Big|_{t=0} v^g \\ &= \int_D \sum_{i=1}^m h(\nabla^\phi_{(0,\frac{d}{dt})} d\phi(e_i,0), d\phi(e_i,0))\Big|_{t=0} v^g \\ &= \int_D \sum_{i=1}^m h(\nabla^\phi_{(0,e_i)} d\phi(0,\frac{d}{dt}), d\phi(e_i,0))\Big|_{t=0} v^g \end{aligned}$$

$$= \int_D \sum_{i=1}^m h(\nabla^N_{d\varphi(e_i)} v, d\varphi(e_i)) v^g$$
$$= \int_D \sum_{i=1}^m h(\nabla^\varphi_{e_i} v, d\varphi(e_i))\, v^g$$
$$= \int_D \sum_{i=1}^m \left[e_i\, h(v, d\varphi(e_i)) - h(v, \nabla^\varphi_{e_i}\, d\varphi(e_i))\right] v^g. \qquad (3.3)$$

Define a 1-form ω to support in D by

$$\omega(X) = h(v, d\varphi(X)), \quad \forall X \in \Gamma(TM).$$

we have

$$\begin{aligned} \operatorname{div}^M \omega &= \sum_{i=1}^m (\nabla_{e_i}\omega)(e_i) \\ &= \sum_{i=1}^m \{e_i(\omega(e_i)) - \omega(\nabla^M_{e_i} e_i)\} \\ &= \sum_{i=1}^m \{e_i h(v, d\varphi(e_i)) - h(v, d\varphi(\nabla^M_{e_i} e_i))\}. \qquad (3.4) \end{aligned}$$

from formulas (3.3), (3.4) we obtain

$$\frac{d}{dt} E(\varphi_t, D)\Big|_{t=0} = \int_D (\operatorname{div}\omega)\, v^g - \int_D h(v, \tau(\varphi)).$$

By divergence theorem, we have

$$\frac{d}{dt} E(\varphi_t; D)\Big|_{t=0} = -\int_D h(v, \tau(\varphi)) v^g.$$

□

Theorem 3.1.2. *The map $\varphi \in C^\infty(M, N)$ between two Riemannian manifolds is harmonic if and only if $\tau(\varphi) = \operatorname{trace}\nabla d\varphi = 0$.*

3.1.2 Second variation of energy

Theorem 3.1.3. *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a harmonic map and D a compact domain of M, if $\{\varphi_{t,s}\}$ is a variation of φ with two parameters with compact support in D, then*

$$\frac{\partial^2}{\partial t \partial s} E(\varphi_{t,s}; D)\Big|_{(t,s)=(0,0)} = \int_D h(-\operatorname{trace}\, R^N(V, d\varphi)d\varphi - \operatorname{trace}(\nabla^\varphi)^2 V, W)\, v^g,$$

where $V = \dfrac{\partial \varphi_{t,s}}{\partial t}\Big|_{(t,s)=(0,0)}$ *and* $W = \dfrac{\partial \varphi_{t,s}}{\partial s}\Big|_{(t,s)=(0,0)}$ *denotes variation vector fields.*

Proof. Let $\{e_1, \ldots, e_m\}$ be an orthonormal frame on (M^m, g). We set

$$\begin{array}{rcccc} \phi & : & M \times (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) & \longrightarrow & N, \\ & & (x, t, s) & \longmapsto & \varphi_{t,s}(x) \end{array}$$

$$E_i = (e_i, 0, 0),\ \frac{\partial}{\partial t} = (0, \frac{d}{dt}, 0)\ et\ \frac{\partial}{\partial s} = (0, 0, \frac{d}{ds}).$$

Then

$$\frac{\partial^2}{\partial t \partial s} E(\varphi_{t,s}; D)\Big|_{(t,s)=(0,0)} = \frac{1}{2}\int_D \sum_{i=1}^m \frac{\partial^2}{\partial t \partial s} h(d\phi(E_i), d\phi(E_i))\, v^g \Big|_{(t,s)=(0,0)}, \quad (3.5)$$

$$\begin{aligned} \frac{1}{2}\frac{\partial^2}{\partial t \partial s} h(d\phi(E_i), d\phi(E_i)) &= \frac{\partial}{\partial t} h(\nabla^\phi_{\frac{\partial}{\partial s}} d\phi(E_i), d\phi(E_i)) \\ &= h(\nabla^\phi_{\frac{\partial}{\partial t}} \nabla^\phi_{\frac{\partial}{\partial s}} d\phi(E_i), d\phi(E_i)) \\ &\quad + h(\nabla^\phi_{\frac{\partial}{\partial s}} d\phi(E_i), \nabla^\phi_{\frac{\partial}{\partial t}} d\phi(E_i)), \end{aligned} \quad (3.6)$$

and

$$\begin{aligned} h(\nabla^\phi_{\frac{\partial}{\partial t}} \nabla^\phi_{\frac{\partial}{\partial s}} d\phi(E_i), d\phi(E_i)) &= h(\nabla^\phi_{\frac{\partial}{\partial t}} \nabla^\phi_{E_i} d\phi(\frac{\partial}{\partial s}), d\phi(E_i)) \\ &= h(R^N(d\phi(\frac{\partial}{\partial t}), d\phi(E_i)) d\phi(\frac{\partial}{\partial s}), d\phi(E_i)) \\ &\quad h(\nabla^\phi_{E_i} \nabla^\phi_{\frac{\partial}{\partial t}} d\phi(\frac{\partial}{\partial s}), d\phi(E_i)) \\ &\quad + h(\nabla^\phi_{[\frac{\partial}{\partial t}, E_i]} d\phi(\frac{\partial}{\partial s}), d\phi(E_i)). \end{aligned} \quad (3.7)$$

Define an $1-$form ω, on M by

$$\omega(X) = h(\nabla^\phi_{\frac{\partial}{\partial t}} d\phi(\frac{\partial}{\partial s})\Big|_{(t,s)=(0,0)}, d\varphi(X)), \quad X \in \Gamma(TM).$$

We use that φ is harmonic map, so

$$\begin{aligned} \operatorname{div}^M \omega &= \sum_{i=1}^m \{e_i(\omega(e_i)) - \omega(\nabla^M_{e_i} e_i)\} \\ &= \sum_{i=1}^m \{e_i(h(\nabla^\phi_{\frac{\partial}{\partial t}} d\phi(\frac{\partial}{\partial s})\Big|_{(t,s)=(0,0)}, d\varphi(e_i))) - h(\nabla^\phi_{\frac{\partial}{\partial t}} d\phi(\frac{\partial}{\partial s})\Big|_{(t,s)=(0,0)}, d\varphi(\nabla^M_{e_i} e_i))\} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{m}\{h(\nabla^{\phi}_{E_i}\nabla^{\phi}_{\frac{\partial}{\partial t}}d\phi(\frac{\partial}{\partial s})\Big|_{(t,s)=(0,0)}, d\varphi(e_i)) \\
&\quad +h(\nabla^{\phi}_{\frac{\partial}{\partial t}}d\phi(\frac{\partial}{\partial s})\Big|_{(t,s)=(0,0)}, \nabla^{\varphi}_{e_i}d\varphi(e_i)) - h(\nabla^{\phi}_{\frac{\partial}{\partial t}}d\phi(\frac{\partial}{\partial s})\Big|_{(t,s)=(0,0)}, d\varphi(\nabla^{M}_{e_i}e_i))\} \\
&= \sum_{i=1}^{m}\{h(\nabla^{\phi}_{E_i}\nabla^{\phi}_{\frac{\partial}{\partial t}}d\phi(\frac{\partial}{\partial s})\Big|_{(t,s)=(0,0)}, d\varphi(e_i)) + h(\nabla^{\phi}_{\frac{\partial}{\partial t}}d\phi(\frac{\partial}{\partial s})\Big|_{(t,s)=(0,0)}, \tau(\varphi)) \\
&= \sum_{i=1}^{m}\{h(\nabla^{\phi}_{E_i}\nabla^{\phi}_{\frac{\partial}{\partial t}}d\phi(\frac{\partial}{\partial s})\Big|_{(t,s)=(0,0)}, d\varphi(e_i)) \qquad (3.8)
\end{aligned}
$$

by (3.7) and (3.8), and since $\left[\frac{\partial}{\partial t}, e_i\right] = 0$, we have :

$$
h(\nabla^{\phi}_{\frac{\partial}{\partial t}}\nabla^{\phi}_{\frac{\partial}{\partial s}}d\phi(E_i), d\phi(E_i))\Big|_{(t,s)=(0,0)} = \sum_{i=1}^{m} h(R^N(V, d\varphi(e_i))W, d\varphi(e_i)) + \mathrm{div}^M\omega. \qquad (3.9)
$$

The second term to the right of the equality (3.6) is given by

$$
\begin{aligned}
h(\nabla^{\phi}_{\frac{\partial}{\partial s}}d\phi(E_i), \nabla^{\phi}_{\frac{\partial}{\partial t}}d\phi(E_i)) &= h(\nabla^{\phi}_{E_i}d\phi(\frac{\partial}{\partial s}), \nabla^{\phi}_{E_i}d\phi(\frac{\partial}{\partial t})) \\
&= e_i\left(h(d\phi(\frac{\partial}{\partial s}), d\phi(\frac{\partial}{\partial t}))\right) \\
&\quad -h(d\phi(\frac{\partial}{\partial s}), \nabla^{\phi}_{E_i}\nabla^{\phi}_{E_i}d\phi(\frac{\partial}{\partial t})). \qquad (3.10)
\end{aligned}
$$

If η is an 1-form, on M defined by

$$
\eta(X) = h(W, \nabla^{\varphi}_{X}V), \quad X \in \Gamma(TM).
$$

$$
\begin{aligned}
\Longrightarrow \mathrm{div}^M\eta &= \sum_{i=1}^{m}\{e_i(\eta(e_i)) - \eta(\nabla^{M}_{e_i}e_i)\} \\
&= \sum_{i=1}^{m}\{e_i(h(W, \nabla^{\varphi}_{e_i}V)) - h(W, \nabla^{\varphi}_{\nabla^{M}_{e_i}e_i}V)\}. \qquad (3.11)
\end{aligned}
$$

We use (3.10) and (3.11), we obtain

$$
\begin{aligned}
\sum_{i=1}^{m} h(\nabla^{\phi}_{\frac{\partial}{\partial t}}d\phi(E_i), \nabla^{\phi}_{\frac{\partial}{\partial s}}d\phi(E_i))\Big|_{(t,s)=(0,0)} &= \mathrm{div}^M\eta + \sum_{i=1}^{m} h(W, \nabla^{\varphi}_{\nabla^{M}_{e_i}e_i}V) \\
&\quad -\sum_{i=1}^{m} h(W, \nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}V). \qquad (3.12)
\end{aligned}
$$

From the equations (3.5), (3.6), (3.9), (3.17), and divergence theorem we have,

$$\begin{aligned}\frac{\partial^2}{\partial t\partial s}E(\varphi_{t,s};D)\bigg|_{(t,s)=(0,0)} &= \int_D \sum_{i=1}^m \big\{-h(R^N(V,d\varphi(e_i))d\varphi(e_i),W) \\ &\quad + h(W,\nabla^\varphi_{\nabla^M_{e_i}e_i}V) - h(W,\nabla^\varphi_{e_i}\nabla^\varphi_{e_i}V)\Big\}\, v^g.\end{aligned}$$

□

Definition 3.1.5. *Let $\varphi:(M^m,g)\longrightarrow(N^n,h)$ be a smooth map between two Riemannian manifolds, we define the Jacobi operator of φ, noted J_φ, by*

$$\begin{array}{rccc} J_\varphi & : & \Gamma(\varphi^{-1}TN) & \longrightarrow & \Gamma(\varphi^{-1}TN) \\ & & v & \longmapsto & J_\varphi(v)\end{array}$$

$$\begin{aligned} J_\varphi(v) &= -\operatorname{trace} R^N(v,d\varphi)d\varphi - \operatorname{trace}(\nabla^\varphi)^2 v \\ &= -\sum_{i=1}^m R^N(v,d\varphi(e_i))d\varphi(e_i) - \sum_{i=1}^m \Big[\nabla^\varphi_{e_i}\nabla^\varphi_{e_i}v - \nabla^\varphi_{\nabla^M_{e_i}e_i}v\Big].\end{aligned}$$

Properties 3.1.1. *Let $\varphi:(M^m,g)\longrightarrow(N^n,h)$ be a smooth map between two Riemannian manifolds. Then*

1. *Jacobi's operator J_φ is $\mathbb{R}-$linear,*
2. $J_\varphi(fv) = fJ_\varphi(v) - \big[\triangle(f)v + 2\nabla^\varphi_{\operatorname{grad} f}v\big]$, $\forall v\in\Gamma(\varphi^{-1}TN)$ *et* $\forall f\in C^\infty(M)$.

Proof.

1. Let $v,w\in\Gamma(\varphi^{-1}TN)$, then

$$\begin{aligned} J_\varphi(v+w) &= -\sum_{i=1}^m \Big\{R^N(v+w,d\varphi(e_i))d\varphi(e_i) + [\nabla^\varphi_{e_i}\nabla^\varphi_{e_i}v+w - \nabla^\varphi_{\nabla^M_{e_i}e_i}v+w]\Big\} \\ &= -\sum_{i=1}^m \big\{R^N(v,d\varphi(e_i))d\varphi(e_i) - R^N(w,d\varphi(e_i))d\varphi(e_i) \\ &\quad + [\nabla^\varphi_{e_i}\nabla^\varphi_{e_i}v + \nabla^\varphi_{e_i}\nabla^\varphi_{e_i}w - \nabla^\varphi_{\nabla^M_{e_i}e_i}v - \nabla^\varphi_{\nabla^M_{e_i}e_i}w]\Big\} \\ &= -\sum_{i=1}^m \Big\{R^N(v,d\varphi(e_i))d\varphi(e_i) + [\nabla^\varphi_{e_i}\nabla^\varphi_{e_i}v - \nabla^\varphi_{\nabla^M_{e_i}e_i}v]\Big\} \\ &\quad -\sum_{i=1}^m \Big\{R^N(w,d\varphi(e_i))d\varphi(e_i) + [\nabla^\varphi_{e_i}\nabla^\varphi_{e_i}w - \nabla^\varphi_{\nabla^M_{e_i}e_i}w]\Big\} \\ &= J_\varphi(v) + J_\varphi(w).\end{aligned}$$

Using the same method, we get $J_\varphi(\lambda v) = \lambda J_\varphi(v)$, $\forall\lambda\in\mathbb{R}$.

2. Let $v \in \Gamma(\varphi^{-1}TN)$ et $f \in C^\infty(M)$, then

$$\begin{aligned}
J_\varphi(fv) &= -\sum_{i=1}^{m}\Big\{R^N(fv, d\varphi(e_i))d\varphi(e_i) + [\nabla^\varphi_{e_i}\nabla^\varphi_{e_i} fv - \nabla^\varphi_{\nabla^M_{e_i}e_i} fv]\Big\} \\
&= -f\sum_{i=1}^{m} R^N(v, d\varphi(e_i))d\varphi(e_i) - \sum_{i=1}^{m}\big[\nabla^\varphi_{e_i} e_i(f)v + \nabla^\varphi_{e_i} f\nabla^\varphi_{e_i} v \\
&\quad -(\nabla^\varphi_{\nabla^M_{e_i}e_i})(f)v - f\nabla^\varphi_{\nabla^M_{e_i}e_i} v\big] \\
&= -f\sum_{i=1}^{m} R^N(v, d\varphi(e_i))d\varphi(e_i) - \sum_{i=1}^{m}\big[e_i(e_i(f))v + e_i(f)\nabla^\varphi_{e_i} v + e_i(f)\nabla^\varphi_{e_i} v \\
&\quad + f\nabla^\varphi_{e_i}(\nabla^\varphi_{e_i} v) - (\nabla^M_{e_i} e_i)(f)v - f\nabla^\varphi_{\nabla^M_{e_i}e_i} v\big] \\
&= fJ_\varphi(v) - \big[\triangle(f)v + 2\nabla^\varphi_{\mathrm{grad}\ f} v\big].
\end{aligned}$$

□

3.2 Biharmonic maps

Definition 3.2.1. *Let $\varphi : (M, g) \longrightarrow (N, h)$ be a smooth map between two Riemannian manifolds, and D compact domain in M. The bi-energy of φ on D is defined by*

$$\begin{array}{rcccc}
E_2 & : & C^\infty(M,N) & \longrightarrow & \mathbb{R}_+, \\
 & & \varphi & \longmapsto & E_2(\varphi; D) = \dfrac{1}{2}\displaystyle\int_D |\tau(\varphi)|^2 v^g
\end{array}$$

where $|\tau(\varphi)|^2 = h(\tau(\varphi), \tau(\varphi))$, and $\tau(\varphi)$ is the tension field of map φ.

Definition 3.2.2. *The smooth map $\varphi : (M^m, g) \longrightarrow (N^n, h)$ between two Riemannian manifold is called biharmonic map if it is a critical point of the bi-energy functional over any compact subset D of M, i.e.*

$$\frac{d}{dt}E_2(\varphi_t; D)\Big|_{t=0} = 0, \tag{3.13}$$

$(\varphi_t)_{t\in(-\epsilon,\epsilon)}$ is a variation of φ in compact support D.

3.2.1 First variation of bi-energy

Theorem 3.2.1. *Let a smooth map $\varphi : (M^m, g) \longrightarrow (N^n, h)$ between two Riemannian manifolds, $(\varphi_t)_{t\in(-\epsilon,\epsilon)}$ a smooth variation of φ with support in D. Then*

$$\frac{d}{dt}E_2(\varphi_t; D)\Big|_{t=0} = -\int_D h(v, \tau_2(\varphi))v^g,$$

where $v = \frac{d\varphi_t}{dt}\Big|_{t=0}$ *is the vector field of variation associated with* $(\varphi_t)_{t\in(-\epsilon,\epsilon)}$, $\tau_2(\varphi) \in \Gamma(\varphi^{-1}TN)$ *is a Pull-Back field defined relatively to an orthonormal frame* $\{e_i\}_{i=1}^m$ *on* (M^m, g) *by*

$$\begin{aligned}\tau_2(\varphi) &= -\operatorname{trace}_g R^N(\tau(\varphi), d\varphi)d\varphi - \operatorname{trace}_g(\nabla^\varphi)^2\tau(\varphi)\\ &= -\sum_{i=1}^m R^N(\tau(\varphi), d\varphi(e_i))d\varphi(e_i) - \sum_{i=1}^m\{\nabla^\varphi_{e_i}\nabla^\varphi_{e_i}\tau(\varphi) - \nabla^\varphi_{\nabla^M_{e_i}e_i}\tau(\varphi)\}.\end{aligned}$$

Proof. Let a map $\phi : M\times(-\epsilon,\epsilon) \longrightarrow N$ defined by $\phi(x,t) = \varphi_t(x)$.
Then

$$\frac{d}{dt}E_2(\varphi_t; D)\big|_{t=0} = \int_D \sum_{i,j=1}^m h(\nabla^\phi_{(0,\frac{d}{dt})}\nabla d\phi((e_i,0),(e_i,0)), \nabla d\phi((e_j,0),(e_j,0)))v^g\Big|_{t=0}. \tag{3.14}$$

We have

$$\nabla^\phi_{(0,\frac{d}{dt})}d\phi(e_i,0) = \nabla^\phi_{(e_i,0)}d\phi(0,\frac{d}{dt}), \tag{3.15}$$

Because,

$$[(0,\frac{d}{dt}),(e_i,0)] = 0.$$

Also

$$\nabla^\phi_{(0,\frac{d}{dt})}d\phi(\nabla^M_{e_i}e_i,0) = \nabla^\phi_{(\nabla^M_{e_i}e_i,0)}d\phi(0,\frac{d}{dt}). \tag{3.16}$$

Hence

$$\begin{aligned}\nabla^\phi_{(0,\frac{d}{dt})}\nabla d\phi((e_i,0),(e_i,0)) &= \nabla^\phi_{(0,\frac{d}{dt})}\{\nabla^\phi_{(e_i,0)}d\phi(e_i,0) - d\phi(\nabla^{M\times(-\epsilon,\epsilon)}_{(e_i,0)}(e_i,0))\}\\ &= \nabla^\phi_{(0,\frac{d}{dt})}\nabla^\phi_{(e_i,0)}d\phi(e_i,0) - \nabla^\phi_{(0,\frac{d}{dt})}d\phi(\nabla^{M\times(-\epsilon,\epsilon)}_{(e_i,0)}(e_i,0))\\ &= R^N(d\phi(0,\frac{d}{dt}), d\phi(e_i,0))d\phi(e_i,0) + \nabla^\phi_{(e_i,0)}\nabla^\phi_{(0,\frac{d}{dt})}d\phi(e_i,0)\\ &\quad +\nabla^\phi_{[(0,\frac{d}{dt}),(e_i,0)]}d\phi(e_i,0) - \nabla^\phi_{(0,\frac{d}{dt})}d\phi(\nabla^M_{e_i}e_i,0)\\ &= R^N(d\phi(0,\frac{d}{dt}), d\phi(e_i,0))d\phi(e_i,0) + \nabla^\phi_{(e_i,0)}\nabla^\phi_{(e_i,0)}d\phi(0,\frac{d}{dt})\\ &\quad -\nabla^\phi_{(\nabla^M_{e_i}e_i,0)}d\phi(0,\frac{d}{dt}).\end{aligned}$$

Therefor

$$\begin{aligned}\sum_{i,j=1}^m h(\nabla^\phi_{(0,\frac{d}{dt})}\nabla d\phi((e_i,0),(e_i,0)), \nabla d\phi((e_j,0),(e_j,0)))\Big|_{t=0} &= \sum_{i=1}^m h(R^N(v, d\varphi(e_i))d\varphi(e_i), \tau(\varphi))\\ &+\sum_{i=1}^m h(\nabla^\varphi_{e_i}\nabla^\varphi_{e_i}v, \tau(\varphi)) - \sum_{i=1}^m h(\nabla^\varphi_{\nabla^M_{e_i}e_i}v, \tau(\varphi)).\end{aligned} \tag{3.17}$$

Let $w \in \Gamma(T^*M)$ be an 1-form to support in D defined by

$$w(X) = h(\nabla^{\varphi}_X v, \tau(\varphi)), \forall X \in \Gamma(TM).$$

$$\begin{aligned}
\Longrightarrow \operatorname{div}^M w &= \sum_{i=1}^{m}\{e_i(w(e_i)) - w(\nabla^M_{e_i} e_i)\} \\
&= \sum_{i=1}^{m}\{e_i(h(\nabla^{\varphi}_{e_i} v, \tau(\varphi))) - h(\nabla^{\varphi}_{\nabla^M_{e_i} e_i} v, \tau(\varphi))\} \qquad (3.18)\\
&= \sum_{i=1}^{m}\{h(\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i} v, \tau(\varphi)) + h(\nabla^{\varphi}_{e_i} v, \nabla^{\varphi}_{e_i}\tau(\varphi)) - h(\nabla^{\varphi}_{\nabla^M_{e_i} e_i} v, \tau(\varphi))\}.
\end{aligned}$$

From (3.17) and (3.18), we obtain

$$\begin{aligned}
\sum_{i,j=1}^{m} h(\nabla^{\phi}_{(0,\frac{d}{dt})} \nabla d\phi((e_i,0),(e_i,0)), \nabla d\phi((e_j,0),(e_j,0)))v^g\Big|_{t=0} &= \sum_{i=1}^{m} h(R^N(v, d\varphi(e_i))d\varphi(e_i), \tau(\varphi)) \\
&\quad + \operatorname{div}^M w - \sum_{i=1}^{m} h(\nabla^{\varphi}_{e_i} v, \nabla^{\varphi}_{e_i}\tau(\varphi)).
\end{aligned} \qquad (3.19)$$

Also let $\eta \in \Gamma(T^*M)$ be an 1-form to support in D defined by

$$\eta(X) = h(v, \nabla^{\varphi}_X \tau(\varphi)),\ \forall X \in \Gamma(TM).$$

$$\begin{aligned}
\Longrightarrow \operatorname{div}^M \eta &= \sum_{i=1}^{m}\{e_i(\eta(e_i)) - \eta(\nabla^M_{e_i} e_i)\} \\
&= \sum_{i=1}^{m}\{e_i(h(v, \nabla^{\varphi}_{e_i}\tau(\varphi)) - h(v, \nabla^{\varphi}_{\nabla^M_{e_i} e_i}\tau(\varphi)))\} \qquad (3.20)\\
&= \sum_{i=1}^{m}\{h(\nabla^{\varphi}_{e_i} v, \nabla^{\varphi}_{e_i}\tau(\varphi)) + h(v, \nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}\tau(\varphi)) - h(v, \nabla^{\varphi}_{\nabla^M_{e_i} e_i}\tau(\varphi))\}.
\end{aligned}$$

Substituting (3.20) in (3.19), we have

$$\begin{aligned}
\sum_{i,j=1}^{m} h(\nabla^{\phi}_{(0,\frac{d}{dt})} \nabla d\phi((e_i,0),(e_i,0)), \nabla d\phi((e_j,0),(e_j,0)))v^g\Big|_{t=0} &= \sum_{i=1}^{m} h(R^N\tau(\varphi), d\varphi(e_i))d\varphi(e_i), v) \\
&\quad + \operatorname{div}^M w - \operatorname{div}^M \eta + h(v, \nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}\tau(\varphi)) \\
&\quad - h(v, \nabla^{\varphi}_{\nabla^M_{e_i} e_i}\tau(\varphi).
\end{aligned} \qquad (3.21)$$

From (3.14), (3.21), and divergence theorem, we obtain

$$\frac{d}{dt}E_2(\varphi_t;D)\Big|_{t=0} = -\int_D \sum_{i=1}^{m} h(-R^N(\tau(\varphi),d\varphi(e_i))d\varphi(e_i) - \nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}\tau(\varphi) + \nabla^{\varphi}_{\nabla^M_{e_i}e_i}\tau(\varphi), v)v^g.$$

□

Theorem 3.2.2. *The map $\varphi \in C^\infty(M,N)$ between two Riemannian manifolds is biharmonic if and only if*

$$\begin{aligned}\tau_2(\varphi) &= -\operatorname{trace}_g R^N(\tau(\varphi),d\varphi)d\varphi \\ &\quad -\operatorname{trace}_g(\nabla^{\varphi})^2\tau(\varphi) = 0. \end{aligned} \tag{3.22}$$

Remarks 3.2.1. *1) The equation (3.22) is called the Euler-Lagrange associated with the bi-energy functional.*
2)Let $\varphi : (M^m, g) \to (N^n, h)$ a smooth map between two Riemannian manifolds

$$\tau_2(\varphi) = J_\varphi(\tau(\varphi)).$$

3.3 Stress energy tensor

Proposition 3.3.1. *Let $\varphi : (M^m, g) \to (N^n, h)$ be a smooth map between two Riemannian manifold and D a compact domain in M, let $(g_t)_{t\in(-\epsilon,\epsilon)}$ a variation to support in D of the metric g where $g_0 = g$, then*

$$\delta g = \frac{\partial}{\partial t}g_t\Big|_{t=0} \in \Gamma(T^*M \otimes T^*M)$$

Locally $g_t = g_{ij}(t,x)dx^i \otimes dx^j$, $g_0 = g_{ij}(x)dx^i \otimes dx^j$, $\delta g = \frac{\partial}{\partial t}g_{ij}\Big|_{t=0} dx^i \otimes dx^j$.

Definition 3.3.1. *Let $\varphi : (M^m, g) \to (N^n, h)$ a smooth map between two Riemannian. The stress energy tensor is defined by*

$$S(\varphi) = e(\varphi)g - \varphi^* h. \tag{3.23}$$

For all $X, Y \in \Gamma(TM)$, we have

$$S(\varphi)(X,Y) = e(\varphi)g(X,Y) - h(d\varphi(X), d\varphi(Y)), \tag{3.24}$$

where, $e(\varphi) = \frac{1}{2}|d\varphi|^2$ is the energy density of φ.

Proposition 3.3.2. *[19] Let $\varphi : (M^m, g) \to (N^n, h)$ be a C^∞-map between two Riemannian manifolds, D a compact domain in M and $(g_t)_{t\in(-\epsilon,\epsilon)}$ a smooth variation of the metric g. Then*

$$\frac{d}{dt}E(\varphi;D)\Big|_{t=0} = -\int_D h(S(\varphi),\delta g)\, v^g, \tag{3.25}$$

Theorem 3.3.1. *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map between two manifolds, then*

$$\operatorname{div}^M S(\varphi) = -h(\tau(\varphi), d\varphi) \tag{3.26}$$

Proof. Let $\{e_1, \ldots, e_m\}$ be an orthonormal frame such that $\nabla^M_{e_i} e_j = 0$ at $x \in M$ for all $i, j = 1 \ldots m$ and $X \in \{e_i\}_{i=1}^m$.
At $x \in M$ we have

$$\begin{aligned}
(\operatorname{div}^M S(\varphi))(x) &= \sum_{i=1}^m e_i(S(\varphi), (e_i, X) \\
&= \frac{1}{2}\sum_{i=1}^m e_i|d\varphi|^2 g_{ij} - \sum_{i=1}^m e_i h(d\varphi(e_i), d\varphi(e_j)) \quad \text{we put } X = e_j \quad \text{for} j \in \{1 \cdots m\} \\
&= \frac{1}{2}\sum_{i=1}^m e_i|d\varphi|^2 \delta_{ij} - \sum_{i=1}^m e_i h(d\varphi(e_i), d\varphi(e_j)) \\
&= \sum_{k=1}^m h(\nabla^\varphi_{e_j} d\varphi(e_k), d\varphi(e_k)) - h(\sum_{i=1}^m \nabla^\varphi_{e_i} d\varphi(e_i), d\varphi(e_j)) - \sum_{i=1}^m h(d\varphi(e_i), \nabla^\varphi_{e_i} d\varphi(e_j)) \\
&= \sum_{i=1}^m h(\nabla^\varphi_{e_i} d\varphi(e_j), d\varphi(e_i)) - h(\tau(\varphi), d\varphi(e_j)) - \sum_{i=1}^m h(d\varphi(e_i), \nabla^\varphi_{e_i} d\varphi(e_j)) \\
&= -h(\tau(\varphi), d\varphi(X))
\end{aligned}$$

□

Corollary 3.3.1. *Let $\varphi : (M^m, g) \to (N^n, h)$ a C^∞-map between two Riemannian manifolds, then from the above theorem, we have*

- *If φ is harmonic then $\operatorname{div}^M S(\varphi)$ is vanishing.*
- *If φ is submersion and if $\operatorname{div}^M S(\varphi) = 0$ then φ is harmonic.*

3.4 Stress bi-energy tensor

Definition 3.4.1. *Let $\varphi : (M^m, g) \to (N^n, h)$ be a C^∞-map between two Riemannian manifolds. The stress bi-energy tensor $S_2(\varphi) \in \Gamma(T^*M \odot T^*M)$ associated to φ is defined for all $X, Y \in \Gamma(TM)$ by*

$$\begin{aligned}
S_2(\varphi) &= -\frac{1}{2}|\tau(\varphi)|^2 g(X,Y) - < d\varphi, \nabla^\varphi \tau(\varphi) > g(X,Y) \\
&\quad + h(d\varphi(X), \nabla^\varphi_Y \tau(\varphi) + h(d\varphi(Y), \nabla^\varphi_X \tau(\varphi)),
\end{aligned} \tag{3.27}$$

where

$$< d\varphi, \nabla^\varphi \tau(\varphi) >= \sum_{i=1}^m h(d\varphi(e_i), \nabla^\varphi_{e_i} \tau(\varphi))$$

relative to an orthonormal frame $\{e_1, \ldots, e_m\}$ on (M^m, g).

Proposition 3.4.1. *Let $\varphi : (M^m, g) \to (N^n, h)$ be a C^∞-map between two Riemannian manifolds, D a compact domain in M and $(g_t)_{t\in(-\epsilon,\epsilon)}$ a smooth variation of the metric g. Then*

$$\frac{d}{dt}E_2(\varphi; D)\Big|_{t=0} = -\int_D h(S_2(\varphi), \delta g)\, v^g, \tag{3.28}$$

To proof the Proposition 3.4.1, we need the following Lemmas

Lemma 3.4.1. *Let $\varphi : (M^m, g) \to (N^n, h)$ a smooth map between two Riemannian manifolds M and $\{g_t\}$ a C^∞-variation of the metric g.The vector field $\xi = (\mathrm{div}^M(\delta g))^\sharp - \frac{1}{2}\,\mathrm{grad}^M(trace_g(\delta g))$ satisfied :*

$$\delta(|\tau(\varphi)|^2) = -2 < h(\nabla d\varphi, \tau(\varphi)), \delta g > -2h(d\varphi(\xi), \tau(\varphi)). \tag{3.29}$$

Proof. Locally, we have

$$\begin{aligned}
\delta(|\tau(\varphi)|^\alpha) &= -g^{ai}g^{bj}\delta(g_{ab})(\nabla d\varphi)^\alpha_{ij} - \xi^k\varphi^\alpha_k && (3.30)\\
< h(\nabla d\varphi, \tau(\varphi)), \delta g > &= g^{ai}g^{bj}\delta(g_{ab})(\nabla d\varphi)^\alpha_{ij}\tau(\varphi)^\beta h_{\alpha\beta} && (3.31)\\
h(d\varphi(\xi), \tau(\varphi)) &= \xi^k\varphi^\alpha_k\tau(\varphi)^\beta h_{\alpha\beta} && (3.32)\\
\delta(|\tau(\varphi)|^2) &= \delta(\tau(\varphi)^\alpha\tau(\varphi)^\beta h_{\alpha\beta})\\
&= 2\delta(\tau(\varphi)^\alpha)\tau(\varphi)^\beta h_{\alpha\beta}\\
&= -2g^{ai}g^{bj}\delta(g_{ab})(\nabla d\varphi)^\alpha_{ij}\tau(\varphi)^\beta h_{\alpha\beta}\\
&\quad -2\xi^k\varphi^\alpha_k\tau(\varphi)^\beta h_{\alpha\beta} && (3.33)
\end{aligned}$$

Subsisting the formulas (3.31) and (3.32) in (3.33) we obtain the formula (3.29) □

Lemma 3.4.2. *Let $\varphi : (M^m, g) \to (N^n, h)$ be a smooth map between two Riemannian manifolds, D a compact domain in M and $(g_t)_{t\in(-\epsilon,\epsilon)}$ a C^∞-variation of the metric g. we set*

$$\xi = (\mathrm{div}^M(\delta g))^\sharp - \frac{1}{2}\,\mathrm{grad}^M(trace_g(\delta g)).$$

Then

$$\begin{aligned}
\int_D h(d\varphi(\xi), \tau(\varphi))v^g &= \int_D < -sym(\nabla h(d\varphi, \tau(\varphi)))\\
&\quad +\frac{1}{2}\,\mathrm{div}^M(h(d\varphi, \tau(\varphi))^\sharp)g, \delta g > v^g && (3.34)
\end{aligned}$$

Proof. Let $\omega = h(d\varphi, \tau(\varphi))$, then

$$\int_D \omega(\xi)v^g = \int_D \omega((\mathrm{div}^M(\delta g))^\sharp)v^g - \frac{1}{2}\int_D \omega(\mathrm{grad}^M(trace_g(\delta g)))v^g. \tag{3.35}$$

The first term on the right of the equality (3.35) is given by

$$\begin{aligned}\int_D \omega((\operatorname{div}^M(\delta g))^\sharp)v^g &= \int_D g(\omega,(\operatorname{div}^M(\delta g))^\sharp)v^g \\ &= \int_D g^*(\omega,\operatorname{div}^M(\delta g))v^g,\end{aligned}$$

where g^* is the induced Riemannian metric on T^*M. In the other hand, if for $\sigma \in \Gamma(T^*M\otimes T^*M)$, we pose $C(\omega,\sigma)=\omega^i\sigma_{ij}dx^j$, we obtain:

$$g^*(\omega,\operatorname{div}^M\sigma) = \operatorname{div}^M(C(\omega,\sigma)^\sharp) - < \operatorname{sym}(\nabla\omega),\sigma > . \tag{3.36}$$

for $\sigma=\delta g$, from the formula (3.36), we find

$$\int_D \omega((\operatorname{div}^M(\delta g))^\sharp)v^g = -\int_D < \operatorname{sym}(\nabla\omega),\delta g > \tag{3.37}$$

Remarking for $\lambda\in C^\infty(M)$, we have

$$\omega(\operatorname{grad}^M\lambda) = g^*(\omega,d\lambda). \tag{3.38}$$

For $\lambda=\operatorname{trace}(\delta g)$, from the formula (3.38) we obtain

$$\begin{aligned}-\frac{1}{2}\int_D \omega(\operatorname{grad}^M(\operatorname{trace}(\delta g)))v^g &= -\frac{1}{2}\int_D g^*(\omega,d(trace(\delta g)))v^g \\ &= -\frac{1}{2}\int_D g(\omega^\sharp,\operatorname{grad}^M(\operatorname{trace}(\delta g)))v^g \\ &= \frac{1}{2}\int_D \operatorname{trace}(\delta g)\operatorname{div}^M(\omega^\sharp)v^g \\ &= \frac{1}{2}\int_D < \operatorname{div}^M(\omega^\sharp)g,\delta g > v^g. \end{aligned}\tag{3.39}$$

Substituting the formulas (3.37) and (3.39) in (3.35) we find (3.34). □

Proof of Prorosition (3.4.1)**.** From the formula (3.29) and the Lemma 3.4.1, we have

$$\begin{aligned}\frac{d}{dt}E_2(\varphi;D)\Big|_{t=0} &= \frac{1}{2}\int_D \delta(|\tau(\varphi)|^2)v^g + \frac{1}{2}\int_D |\tau(\varphi)|^2\delta(v_{g_t}) \\ &= \frac{1}{2}\int_D (-2 < h(\nabla d\varphi,\tau(\varphi)),\delta g > -2h(d\varphi(\xi),\tau(\varphi)))v^g \\ &\quad +\frac{1}{2}\int_D < \frac{1}{2}|\tau(\varphi)|^2 g,\delta g > v^g,\end{aligned}$$

and from the Lemma 3.4.2, we obtain

$$\frac{d}{dt}E_2(\varphi;D)\Big|_{t=0} = \frac{1}{2}\int_D (-2) < h(\nabla d\varphi,\tau(\varphi)),\delta g > v^g + \frac{1}{2}\int_D < \frac{1}{2}|\tau(\varphi)|^2 g,\delta g > v^g$$

$$+\frac{1}{2}\int_D < 2\text{sym}(\nabla h(d\varphi,\tau(\varphi))) - \text{div}^M(h(d\varphi,\tau(\varphi))^\sharp)g, \delta g > v^g. \tag{3.40}$$

We have

$$\begin{aligned} S_2(\varphi) &= -2h(\nabla d\varphi,\tau(\varphi)) + 2\text{sym}(\nabla h(d\varphi,\tau(\varphi))) \\ &\quad - \text{div}^M(h(d\varphi,\tau(\varphi))^\sharp)g + \frac{1}{2}|\tau(\varphi)|^2 g. \end{aligned} \tag{3.41}$$

Let $\{e_1,\ldots,e_m\}$ be an orthonormal frame on (M^m,g) defined in a neighborhood of point $x\in M$ such that $\nabla_{e_i}e_j=0$ at point $x\in M$, for all $i,j\in\{1\ldots m\}$.
At $x\in M$, we have

$$\begin{aligned} 2\text{sym}(\nabla h(d\varphi,\tau(\varphi)))(e_i,e_j) &= \nabla^\varphi_{e_i}h(d\varphi(e_j),\tau(\varphi)) + \nabla^\varphi_{e_j}h(d\varphi(e_i),\tau(\varphi)) \\ &= 2h(\nabla d\varphi(e_i,e_j),\tau(\varphi)) + h(d\varphi(e_i),\nabla^\varphi_{e_j}\tau(\varphi)) \\ &\quad + h(d\varphi(e_j),\nabla^\varphi_{e_i}\tau(\varphi)) \end{aligned} \tag{3.42}$$

and,

$$\begin{aligned} \text{div}^M(h(d\varphi,\tau(\varphi))^\sharp) &= \sum_{i=1}^m e_i(g(h(d\varphi,\tau(\varphi))^\sharp,e_i)) \\ &= \sum_{i=1}^m e_i(h(d\varphi(e_i),\tau(\varphi))) \\ &= \sum_{i=1}^m \{h(\nabla^\varphi_{e_i}d\varphi(e_i),\tau(\varphi)) + h(d\varphi(e_i),\nabla^\varphi_{e_i}\tau(\varphi))\} \\ &= |\tau(\varphi)|^2 + < d\varphi, \nabla^\varphi\tau(\varphi) > \end{aligned} \tag{3.43}$$

Substituting the formulas (3.42) and (3.43) in (3.41) after that in (3.40), we obtain (3.28)

Theorem 3.4.1. *Let $\varphi:(M^m,g)\to(N^n,h)$ be a smooth map between two manifolds, then :*

$$\text{div}^M S_2(\varphi) = -h(\tau_2(\varphi),d\varphi) \tag{3.44}$$

Proof. Let $\{e_1,\ldots,e_m\}$ be an orthonormal frame on (M^m,g) such that $\nabla^M_{e_i}e_j=0$ at $x\in M$ for all $i,j=1\ldots m$ and $X\in\{e_i\}_{i=1}^m$.
At $x\in M$ we have

$$(\text{div}^M S_2(\varphi))(e_j) = \sum_{i=1}^m e_i(S_2(\varphi)(e_i,e_j))$$

$$\begin{aligned}
&= \sum_{i=1}^{m} e_i(-\frac{1}{2}|\tau(\varphi)|^2\delta_{ij} - < d\varphi, \nabla^{\varphi}\tau(\varphi) > \delta_{ij}) \\
&\quad + \sum_{i=1}^{m} e_i(h(d\varphi(e_i), \nabla^{\varphi}_{e_j}\tau(\varphi)) + h(d\varphi(e_j), \nabla^{\varphi}_{e_i}\tau(\varphi))), \\
&= -h(\nabla^{\varphi}_{e_j}\tau(\varphi), \tau(\varphi)) - e_j(< d\varphi, \nabla^{\varphi}\tau(\varphi) >) \\
&\quad + \sum_{i=1}^{m} h(\nabla^{\varphi}_{e_i} d\varphi(e_i), \nabla^{\varphi}_{e_j}\tau(\varphi)) + \sum_{i=1}^{m} h(d\varphi(e_i), \nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_j}\tau(\varphi)) \\
&= -h(\nabla^{\varphi}_{e_j}\tau(\varphi), \tau(\varphi)) - \sum_{i=1}^{m} h(\nabla^{\varphi}_{e_j} d\varphi(e_i), \nabla^{\varphi}_{e_j}\tau(\varphi)) - \sum_{i=1}^{m} h(d\varphi(e_i), \nabla^{\varphi}_{e_j}\nabla^{\varphi}_{e_j}\tau(\varphi)) \\
&\quad + \sum_{i=1}^{m} h(\nabla^{\varphi}_{e_i} d\varphi(e_i), \nabla^{\varphi}_{e_j}\tau(\varphi)) + \sum_{i=1}^{m} h(d\varphi(e_i), \nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_j}\tau(\varphi))
\end{aligned}$$

At point x we have

$$\tau(\varphi) = \sum_{i=1}^{m} \nabla^{\varphi}_{e_i} d\varphi(e_i) \quad \text{and} \quad \nabla^{\varphi}_{e_i} d\varphi(e_j) = \nabla^{\varphi}_{e_j} d\varphi(e_i)$$

$$\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_j}\tau(\varphi) - \nabla^{\varphi}_{e_j}\nabla^{\varphi}_{e_i}\tau(\varphi) = R^N(d\varphi(e_i), d\varphi(e_j))\tau(\varphi)$$

A straightforward computation yields

$$\operatorname{div}^M S_2(\varphi) = -h(\tau_2(\varphi), d\varphi) \tag{3.45}$$

□

From theorem 3.4.1, we deduce

Corollary 3.4.1. *Let $\varphi : (M^m, g) \to (N^n, h)$ a C^∞-map between two Riemannian manifolds, then*

- *If φ is biharmonic then* $\operatorname{div}^M S_2(\varphi) = 0$.
- *If φ is submersion and if* $\operatorname{div}^M S_2(\varphi) = 0$ *then φ is biharmonic.*

Chapter 4

p-Harmonic and p-Biharmonic mappings

4.1 The p-Biharmonic maps

Definition 4.1.1. *Let $\varphi : (M, g) \longrightarrow (N, h)$ be a smooth map between two Riemannian manifolds, and D compact domain in M. The p-energy functional of φ on D is defined by*

$$\begin{array}{rcccc} E_p & : & C^\infty(M,N) & \longrightarrow & \mathbb{R}_+. \\ & & \varphi & \longmapsto & E_p(\varphi; D) = \dfrac{1}{p}\displaystyle\int_D |d\varphi|^p v^g \end{array}$$

Definition 4.1.2. *The smooth map $\varphi : (M^m, g) \longrightarrow (N^n, h)$ between two Riemannian manifold is called p-harmonic map if it is a critical point of the p-energy functional over any compact subset D of M, i.e.*

$$\frac{d}{dt}E_p(\varphi_t; D)\Big|_{t=0} = 0.$$

4.1.1 First variation of p-energy

Theorem 4.1.1. *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map and let $(\varphi_t)_{t\in(-\epsilon,\epsilon)}$ be a smooth variation of φ supported in D. Then*

$$\frac{d}{dt}E_p(\varphi_t; D)\Big|_{t=0} = -\int_D h(\tau_p(\varphi), v)v^g,$$

with $v = \frac{\partial\varphi_t}{\partial t}\big|_{t=0}$ is the field of variation associated with $\{\varphi_t\}_{-\epsilon<t<\epsilon}$, and $\tau_p(\varphi) \in \Gamma(\varphi^{-1}TN)$ defined by

$$\tau_p(\varphi) = \operatorname{trace}\nabla|d\varphi|^{p-2}d\varphi = \sum_{i=1}^{m}\left[\nabla^\varphi_{e_i}|d\varphi|^{p-2}d\varphi(e_i) - |d\varphi|^{p-2}d\varphi(\nabla^M_{e_i}e_i)\right],$$

where $\{e_i\}_{i=1}^m$ *is an orthonormal frame on* (M^m, g).

Proof. Let $\{e_i\}_{i=1}^m$ an orthonormal frame on (M^m, g), and $\{\frac{d}{dt}\}$ a frame on $(-\epsilon, \epsilon)$,then $\left\{(e_i, 0), (0, \frac{d}{dt})\right\}$ is an local orthonormal frame for diagonal metric on the product manifold $M \times (-\epsilon, \epsilon)$, and we have the Lie crochet $[(e_i, 0), (0, \frac{d}{dt})] = 0$, for all $i = 1, \dots, m$. Let $\phi : M \times (-\epsilon, \epsilon) \longrightarrow N$ a map defined by $\phi(x, t) = \varphi_t(x)$. We have :

$$d\phi(e_i, 0)|_{t=0} = d\varphi(e_i) \quad \text{et} \quad d\phi(0, \tfrac{d}{dt})|_{t=0} = v.$$

The Hilbert Schmidt definition of $d\varphi_t$ gives the following formula

$$|d\varphi_t|^p = (|d\varphi_t|^2)^{\frac{p}{2}}.$$

Therefor

$$\begin{aligned}
\partial_t |d\varphi_t|^p &= \frac{p}{2}(|d\varphi_t|^2)^{\frac{p}{2}-1} \sum_{i=1}^m \partial_t h(d\varphi_t(e_i), d\varphi_t(e_i)) \\
&= p(|d\varphi_t|^2)^{\frac{p-2}{2}} \sum_{i=1}^m h(\nabla^\phi_{\partial_t} d\varphi_t(e_i), d\varphi_t(e_i)) \\
&= p|d\varphi_t|^{p-2} \sum_{i=1}^m h(\nabla^\phi_{\partial_t} d\phi(e_i, 0), d\phi(e_i, 0)) \\
&= p|d\varphi_t|^{p-2} \sum_{i=1}^m h(\nabla^\phi_{(e_i,0)} d\phi(\partial_t) + d\phi([\partial_t, (e_i, 0)]), d\phi(e_i, 0)) \\
&= p|d\varphi_t|^{p-2} \sum_{i=1}^m h(\nabla^\phi_{(e_i,0)} d\phi(\partial_t), d\phi(e_i, 0)).
\end{aligned}$$

If, $\partial_t = (0, \frac{d}{dt})$. Then

$$\begin{aligned}
\partial_t |d\varphi_t|^p|_{t=0} &= p|d\varphi|^{p-2} \sum_{i=1}^m h(\nabla^\varphi_{e_i} v, d\varphi(e_i)) \\
&= p \sum_{i=1}^m h(\nabla^\varphi_{e_i} v, |d\varphi|^{p-2} d\varphi(e_i)) \\
&= p \sum_{i=1}^m \left[e_i \left(h(v, |d\varphi|^{p-2} d\varphi(e_i)) \right) - h(v, \nabla^\varphi_{e_i} |d\varphi|^{p-2} d\varphi(e_i)) \right].
\end{aligned}$$

We set $w(X) = h(v, |d\varphi|^{p-2} d\varphi(X))$ pour tout $X \in \Gamma(TM)$. The divergence of an 1-form w is given by

$$\operatorname{div} w = \sum_{i=1}^m (\nabla_{e_i} w)(e_i)$$

$$
\begin{aligned}
&= \sum_{i=1}^{m}\left\{e_i(w(e_i)) - w(\nabla^M_{e_i}e_i)\right\}\\
&= \sum_{i=1}^{m}\left\{e_i(h(v,|d\varphi|^{p-2}d\varphi(e_i)) - h(v,|d\varphi|^{p-2}d\varphi(\nabla^M_{e_i}e_i))\right\},
\end{aligned}
$$

implies that

$$
\begin{aligned}
\partial_t|d\varphi_t|^p\big|_{t=0} &= p\,\mathrm{div}w + p\sum_{i=1}^{m}\left[h(v,|d\varphi|^{p-2}d\varphi(\nabla^M_{e_i}e_i)) - h(v,\nabla^\varphi_{e_i}|d\varphi|^{p-2}d\varphi(e_i))\right]\\
&= p\,\mathrm{div}w - p\sum_{i=1}^{m}h\left(v,\nabla^\varphi_{e_i}|d\varphi|^{p-2}d\varphi(e_i) - |d\varphi|^{p-2}d\varphi(\nabla^M_{e_i}e_i)\right).
\end{aligned}
$$

Finally, from the divergence theorem, we get

$$
\frac{1}{p}\int_M \partial_t|d\varphi_t|^p\big|_{t=0}\,v^g = -\int_M\sum_{i=1}^{m}h\left(v,\nabla^\varphi_{e_i}|d\varphi|^{p-2}d\varphi(e_i) - |d\varphi|^{p-2}d\varphi(\nabla^M_{e_i}e_i)\right)v^g.
$$

□

Remark 4.1.1. *[1]*
Let $\varphi:(M^m,g)\longrightarrow(N^n,h)$ be a smooth map between two Riemannian manifolds. If $|d\varphi|_x\neq 0$ for all $x\in M$, then φ is p-harmonic if and only if

$$
|d\varphi|^{p-2}\tau(\varphi) + (p-2)|d\varphi|^{p-3}d\varphi(\mathrm{grad}^M|d\varphi|) = 0. \tag{4.1}
$$

Indeed; On an orthonormal frame $\{e_i\}_{i=1}^m$ on (M^m,g), we have

$$
\begin{aligned}
\tau_p(\varphi) &= \sum_{i=1}^{m}\left[\nabla^\varphi_{e_i}|d\varphi|^{p-2}d\varphi(e_i) - |d\varphi|^{p-2}d\varphi(\nabla^M_{e_i}e_i)\right]\\
&= \sum_{i=1}^{m}\left[(p-2)|d\varphi|^{p-3}e_i(|d\varphi|)\,d\varphi(e_i) + |d\varphi|^{p-2}\nabla^\varphi_{e_i}d\varphi(e_i)\right.\\
&\quad\left.-|d\varphi|^{p-2}d\varphi(\nabla^M_{e_i}e_i)\right]\\
&= |d\varphi|^{p-2}\tau(\varphi) + (p-2)|d\varphi|^{p-3}d\varphi(\mathrm{grad}^M|d\varphi|).
\end{aligned}
$$

4.1.2 Second Variation of p-energy Functional

Theorem 4.1.2. *Let $\varphi:(M^m,g)\longrightarrow(N^n,h)$ be a p-harmonic map and D a compact domain of M, if $\{\varphi_{t,s}\}$ is a variation of φ with two parameters with compact support in D, then*

$$
\frac{\partial^2}{\partial t\partial s}E_p(\varphi_{t,s};D)\Big|_{(t,s)=(0,0)} = \int_D h(J^\varphi_p(v),w)v^g,
$$

where J_p^φ is Jacobian operator associated to E_p defined by

$$\begin{aligned} J_p^\varphi(v) &= -|d\varphi|^{p-2}\operatorname{trace}_g R^N(v,d\varphi)d\varphi - \operatorname{trace}_g \nabla^\varphi|d\varphi|^{p-2}\nabla^\varphi v \\ & -(p-2)\operatorname{trace}_g \nabla < \nabla^\varphi v, d\varphi > |d\varphi|^{p-4}d\varphi, \end{aligned} \tag{4.2}$$

and $v = \frac{\partial\varphi_{t,s}}{\partial t}\Big|_{t=s=0}$, $w = \frac{\partial\varphi_{t,s}}{\partial s}\Big|_{t=s=0}$.

Remark 4.1.2. *Let $\{e_i\}_{i=1}^m$ be an orthonormal frame on (M^m, g), then*

$$\begin{aligned} \operatorname{trace}_g R^N(v,d\varphi)d\varphi &= R^N(v,d\varphi(e_i))d\varphi(e_i), \\ \operatorname{trace}_g \nabla^\varphi|d\varphi|^{p-2}\nabla^\varphi v &= \nabla^\varphi_{e_i}|d\varphi|^{p-2}\nabla^\varphi_{e_i} v - |d\varphi|^{p-2}\nabla^\varphi_{\nabla^M_{e_i}e_i} v, \\ \operatorname{trace}_g \nabla < \nabla^\varphi v, d\varphi > |d\varphi|^{p-4}d\varphi &= \nabla^\varphi_{e_i} < \nabla^\varphi v, d\varphi > |d\varphi|^{p-4}d\varphi(e_i) \\ & \quad - < \nabla^\varphi v, d\varphi > |d\varphi|^{p-4}d\varphi(\nabla^M_{e_i}e_i). \end{aligned}$$

Proof. We define $\phi : M \times (-\epsilon,\epsilon) \times (-\epsilon,\epsilon) \to N$ by

$$\phi(x,t,s) = \varphi_{t,s}(x), \quad (x,t,s) \in M \times (-\epsilon,\epsilon) \times (-\epsilon,\epsilon). \tag{4.3}$$

Let ∇^ϕ the pull-Back connection on $\phi^{-1}TN$, for all vectors field X on M, we have

$$[\partial_t, X] = 0, \quad [\partial_s, X] = 0, \quad [\partial_t, \partial_s] = 0. \tag{4.4}$$

According to the definition of p-energy functional, we have

$$\frac{\partial^2}{\partial t\partial s}E_p(\varphi_{t,s}; D)\Big|_{t=s=0} = \frac{1}{p}\int_D \frac{\partial^2}{\partial t\partial s}\left(|d\varphi_{t,s}|^2\right)^{\frac{p}{2}}\Big|_{t=s=0} v_g. \tag{4.5}$$

Let $\{e_i\}_{i=1}^m$ be a geodesic frame at $x \in M$, as $d\varphi_{t,s}(e_i) = d\phi(e_i)$. We obtain

$$\begin{aligned} \frac{1}{p}\frac{\partial^2}{\partial t\partial s}\left(|d\varphi_{t,s}|^2\right)^{\frac{p}{2}}\Big|_{(t,s)=(0,0)} &= \frac{1}{p}\partial_t\left[\partial_s\left(|d\varphi_{t,s}|^2\right)^{\frac{p}{2}}\right]\Big|_{(t,s)=(0,0)} \\ &= \frac{1}{p}\partial_t\left[\frac{p}{2}\partial_s\left(|d\varphi_{t,s}|^2\right)\left(|d\varphi_{t,s}|^2\right)^{\frac{p}{2}-1}\right]\Big|_{(t,s)=(0,0)} \\ &= \partial_t\Big[h\big(\nabla^\phi_{\partial_s}d\phi(e_i), d\phi(e_i)\big)h\big(d\phi(e_j), d\phi(e_j)\big)^{\frac{p-2}{2}}\Big]_{(t,s)=(0,0)} \\ &= h\big(\nabla^\phi_{\partial_t}\nabla^\phi_{\partial_s}d\phi(e_i)\Big|_{(t,s)=(0,0)}, d\varphi(e_i)\big)|d\varphi|^{p-2} \\ &\quad +h\big(\nabla^\phi_{\partial_s}d\phi(e_i), \nabla^\phi_{\partial_t}d\phi(e_i)\big)\Big|_{(t,s)=(0,0)}|d\varphi|^{p-2} \\ &\quad +(p-2)h\big(\nabla^\phi_{\partial_s}d\phi(e_i)\Big|_{(t,s)=(0,0)}, d\varphi(e_i)\big) \\ &\quad h\big(\nabla^\phi_{\partial_t}d\phi(e_j)\Big|_{(t,s)=(0,0)}, d\varphi(e_j)\big)|d\varphi|^{p-4}, \end{aligned} \tag{4.6}$$

The first term of (4.6) is given by

$$\begin{aligned} h\big(\nabla^{\phi}_{\partial_t}\nabla^{\phi}_{\partial_s} d\phi(e_i)\Big|_{(t,s)=(0,0)}, d\varphi(e_i)\big)|d\varphi|^{p-2} &= h\big(\nabla^{\phi}_{\partial_t}\nabla^{\phi}_{e_i} d\phi(\partial_s)\Big|_{(t,s)=(0,0)}, d\varphi(e_i)\big)|d\varphi|^{p-2} \\ &= h\big(R^N(d\phi(\partial_t), d\phi(e_i))d\phi(\partial_s)\Big|_{(t,s)=(0,0)}, d\varphi(e_i)\big)|d\varphi|^{p-2} \\ &\quad + h\big(\nabla^{\phi}_{e_i}\nabla^{\phi}_{\partial_t} d\phi(\partial_s)\Big|_{(t,s)=(0,0)}, d\varphi(e_i)\big)|d\varphi|^{p-2} \\ &\quad + h\big(\nabla^{\phi}_{[\partial_t, e_i]} d\phi(\partial_s)\Big|_{(t,s)=(0,0)}, d\varphi(e_i)\big)|d\varphi|^{p-2}. \qquad (4.7) \end{aligned}$$

We define a 1-form ω on M by

$$\omega(X) = h\big(\nabla^{\phi}_{\partial_t} d\phi(\partial_s)\Big|_{(t,s)=(0,0)}, |d\varphi|^{p-2} d\varphi(X)\big), \quad X \in \Gamma(TM).$$

We find that

$$\begin{aligned} \operatorname{div}^M \omega &= e_i\Big[h\big(\nabla^{\phi}_{\partial_t} d\phi(\partial_s)\Big|_{(t,s)=(0,0)}, |d\varphi|^{p-2} d\varphi(e_i)\big)\Big] \\ &= h\big(\nabla^{\phi}_{e_i}\nabla^{\phi}_{\partial_t} d\phi(\partial_s)\Big|_{(t,s)=(0,0)}, d\varphi(e_i)\big)|d\varphi|^{p-2} \\ &\quad + h\big(\nabla^{\phi}_{\partial_t} d\phi(\partial_s)\Big|_{(t,s)=(0,0)}, \nabla^{\varphi}_{e_i}|d\varphi|^{p-2} d\varphi(e_i)\big), \qquad (4.8) \end{aligned}$$

From (4.4), (4.7) and (4.8), we have

$$\begin{aligned} h\big(\nabla^{\phi}_{\partial_t}\nabla^{\phi}_{\partial_s} d\phi(e_i)\Big|_{(t,s)=(0,0)}, d\varphi(e_i)\big)|d\varphi|^{p-2} &= h\big(R^N(v, d\varphi(e_i))w, d\varphi(e_i)\big)|d\varphi|^{p-2} \\ &\quad + \operatorname{div}^M \omega - h\big(\nabla^{\phi}_{\partial_t} d\phi(\partial_s)\Big|_{(t,s)=(0,0)}, \tau_p(\varphi)\big). \end{aligned} \qquad (4.9)$$

The second term of (4.6) is given by

$$\begin{aligned} h\big(\nabla^{\phi}_{\partial_s} d\phi(e_i), \nabla^{\phi}_{\partial_t} d\phi(e_i)\big)\Big|_{(t,s)=(0,0)}|d\varphi|^{p-2} &= h\big(\nabla^{\phi}_{e_i} d\phi(\partial_s), \nabla^{\phi}_{e_i} d\phi(\partial_t)\big)\Big|_{(t,s)=(0,0)}|d\varphi|^{p-2} \\ &= h\big(\nabla^{\varphi}_{e_i} w, |d\varphi|^{p-2}\nabla^{\phi}_{e_i} v)\big). \qquad (4.10) \end{aligned}$$

Also, we define a 1-form η on M by

$$\eta(X) = h(w, |d\varphi|^{p-2}\nabla^{\varphi}_{X} v), \quad X \in \Gamma(TM).$$

From (4.10), we obtain

$$h\big(\nabla^{\phi}_{\partial_s} d\phi(e_i), \nabla^{\phi}_{\partial_t} d\phi(e_i)\big)\Big|_{(t,s)=(0,0)}|d\varphi|^{p-2} = \operatorname{div}^M \eta - h\big(w, \nabla^{\varphi}_{e_i}|d\varphi|^{p-2}\nabla^{\varphi}_{e_i} v\big). \quad (4.11)$$

Substituting (4.9), (4.11), and the following equation:

$$\begin{aligned}(p-2)h\big(\nabla^{\phi}_{\partial_s} d\phi(e_i)\Big|_{(t,s)=(0,0)} &, \quad d\varphi(e_i)\big)h\big(\nabla^{\phi}_{\partial_t} d\phi(e_j)\Big|_{(t,s)=(0,0)}, d\varphi(e_j)\big)|d\varphi|^{p-4} \\ &= (p-2)\operatorname{div}^M \theta - (p-2)h\big(w, \nabla^{\varphi}_{e_i} h(\nabla^{\varphi}_{e_j} v, d\varphi(e_j))|d\varphi|^{p-4} d\varphi(e_i)\big), \\ \theta(X) &= h\big(w, h(\nabla^{\varphi}_{e_j} v, d\varphi(e_j))|d\varphi|^{p-4} d\varphi(X)\big), \quad X \in \Gamma(TM),\end{aligned}$$

in (4.6), we find the equation

$$\begin{aligned}\frac{1}{p}\frac{\partial^2}{\partial t \partial s} h(\varphi_{t,s}(e_i), \varphi_{t,s}(e_i))^{\frac{p}{2}}\Big|_{(t,s)=(0,0)} &= h\big(R^N(v, d\varphi(e_i))w, d\varphi(e_i)\big)|d\varphi|^{p-2} \\ &\quad + \operatorname{div}^M \omega - h\big(\nabla^{\phi}_{\partial_t} d\phi(\partial_s)\Big|_{(t,s)=(0,0)}, \tau_p(\varphi)\big) \\ &\quad + \operatorname{div}^M \eta - h\big(w, \nabla^{\varphi}_{e_i} |d\varphi|^{p-2} \nabla^{\varphi}_{e_i} v\big) + (p-2)\operatorname{div}^M \theta \\ &\quad -(p-2)h\big(w, \nabla^{\varphi}_{e_i} h(\nabla^{\varphi}_{e_j} v, d\varphi(e_j))|d\varphi|^{p-4} d\varphi(e_i)\big). \quad (4.12)\end{aligned}$$

According to the equations (4.5), (4.12), divergence theorem, and the p-harmony of φ, we find the result of Theorem 5.4.1. □

4.2 p-biharmonic map

Definition 4.2.1. *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map between two Riemannian manifolds, and D compact domain in M. The p-bienergy functional of φ on D is defined by*

$$E_{2,p}(\varphi; D) = \frac{1}{2}\int_D |\tau_p(\varphi)|^2 v^g,$$

where $|\tau_p(\varphi)|^2 = h(\tau_p(\varphi), \tau_p(\varphi))$, and $\tau_p(\varphi)$ is the p-tension field of φ.

Definition 4.2.2. *Let (M^m, g), (N^n, h) two Riemannian manifolds, and $\varphi : (M^m, g) \longrightarrow (N^n, h)$ a smooth map. φ is said p-biharmonic if it critical point of the functional p-bienergy $E_{2,p}$ for all D compact of M, that is*

$$\frac{d}{dt} E_{2,p}(\varphi_t, D)\Big|_{t=0} = 0.$$

$\{\varphi_t\}$ being a variation of φ to support in D.

4.2.1 First variation of p-bienergy

Theorem 4.2.1. *[21] Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map and let $(\varphi_t)_{t\in(-\epsilon,\epsilon)}$ be a smooth variation of φ supported in D, $p \geq 2$. Then*

$$\frac{d}{dt} E_{2,p}(\varphi_t, D)\Big|_{t=0} = -\int_D h(\tau_{2,p}(\varphi), v) v^g,$$

where $\tau_{2,p}(\varphi) \in \Gamma(\varphi^{-1}TN)$ is given by :

$$\tau_{2,p}(\varphi) = -|d\varphi|^{p-2}\,\text{trace}_g\, R^N(\tau_p(\varphi), d\varphi)d\varphi - \text{trace}_g\, \nabla^\varphi |d\varphi|^{p-2}\nabla^\varphi \tau_p(\varphi)$$
$$-(p-2)\,\text{trace}_g\, \nabla\langle \nabla^\varphi \tau_p(\varphi), d\varphi\rangle |d\varphi|^{p-4} d\varphi,$$

and $v = \frac{d\varphi_t}{dt}\Big|_{t=0}$ *is the associated variation field to* $\{\varphi_t\}_{-\epsilon<t<\epsilon}$.

Proof. Consider the map $\phi : M \times (-\epsilon, \epsilon) \to N$ by $\phi(x, t) = \varphi_t(x)$, we have

$$\frac{d}{dt}E_{2,p}(\varphi_t; D) = \int_D h(\nabla^\phi_{\partial_t}\tau_p(\varphi_t), \tau_p(\varphi_t))\, v_g. \tag{4.13}$$

Let $\{e_i\}_{i=1}^m$ be a geodesic frame of (M^m, g) , at $x \in M$, we have

$$\nabla^\phi_{\partial_t}\tau_p(\varphi_t) = \nabla^\phi_{\partial_t}\nabla^\phi_{e_i}|d\varphi_t|^{p-2}d\varphi_t(e_i), \tag{4.14}$$

According to the definition of the Riemannian curvature of (N^n, h), we find that

$$\begin{aligned}\nabla^\phi_{\partial_t}\nabla^\phi_{e_i}|d\varphi_t|^{p-2}d\varphi_t(e_i) &= |d\varphi_t|^{p-2}R^N(d\phi(\partial_t), d\varphi_t(e_i))d\varphi_t(e_i)\\ &\quad+\nabla^\phi_{e_i}\nabla^\phi_{\partial_t}|d\varphi_t|^{p-2}d\varphi_t(e_i),\end{aligned} \tag{4.15}$$

by the compatibility of ∇^ϕ with h, we find

$$\begin{aligned}h(\nabla^\phi_{e_i}\nabla^\phi_{\partial_t}|d\varphi_t|^{p-2}d\varphi_t(e_i), \tau_p(\varphi_t)) &= e_i\big(h(\nabla^\phi_{\partial_t}|d\varphi_t|^{p-2}d\varphi_t(e_i), \tau_p(\varphi_t))\big)\\ &\quad-h(\nabla^\phi_{\partial_t}|d\varphi_t|^{p-2}d\varphi_t(e_i), \nabla^\phi_{e_i}\tau_p(\varphi_t)),\end{aligned} \tag{4.16}$$

and from property

$$\nabla^\phi_X d\phi(Y) = \nabla^\phi_Y d\phi(X) + d\phi([X, Y]),$$

with $X = \partial_t$ and $Y = |d\varphi_t|^{p-2}e_i$, we have

$$\nabla^\phi_{\partial_t}|d\varphi_t|^{p-2}d\varphi_t(e_i)\Big|_{t=0} = |d\varphi|^{p-2}\nabla^\varphi_{e_i}v + (p-2)h(\nabla^\varphi_{e_j}v, d\varphi(e_j))|d\varphi|^{p-4}d\varphi(e_i), \tag{4.17}$$

So that, according to the equations(4.16), (4.17), and the divergence theorem, we conclude that

$$\begin{aligned}\int_D h(\nabla^\phi_{e_i}\nabla^\phi_{\partial_t}|d\varphi_t|^{p-2}d\varphi_t(e_i) \quad &, \quad \tau_p(\varphi_t))\Big|_{t=0} v^g\\ &= \int_D \Big[h\big(v, \nabla^\varphi_{e_i}|d\varphi|^{p-2}\nabla^\varphi_{e_i}\tau_p(\varphi)\big)\\ &\quad+(p-2)h\big(v, \nabla^\varphi_{e_j}|d\varphi|^{p-4} < \nabla^\varphi\tau_p(\varphi), d\varphi > d\varphi(e_j)\big)\Big]v^g.\end{aligned} \tag{4.18}$$

From (4.15),and (4.18), $v = d\phi(\partial_t)$ at $t = 0$, the theorem 4.2.1 was proofed . □

From theorem 4.2.1, we find the following results

Theorem 4.2.2. *Let $\varphi : (M^m, g) \to (N^n, h)$ a smooth map between two Riemannian manifold, then φ is p-biharmonic map if and only if*

$$\begin{aligned}\tau_{2,p}(\varphi) &= -|d\varphi|^{p-2}\operatorname{trace}_g R^N(\tau_p(\varphi), d\varphi)d\varphi - \operatorname{trace}_g \nabla^\varphi |d\varphi|^{p-2}\nabla^\varphi \tau_p(\varphi) \\ &\quad -(p-2)\operatorname{trace}_g \nabla < \nabla^\varphi \tau_p(\varphi), d\varphi > |d\varphi|^{p-4} d\varphi = 0.\end{aligned}$$

Remark 4.2.1. *[21] Let $\varphi : (M^m, g) \to (N^n, h)$ be a smooth map between two Riemannian manifold, then*

$$\tau_{2,p}(\varphi) = J_p^\varphi(\tau_p(\varphi)).$$

4.3 Stress p-energy tensor

Proposition 4.3.1. *Let $\varphi : (M^m, g) \to (N^n, h)$ be a smooth map such that $|d\varphi|_x \neq 0$ for all $x \in M$, and let $(g_t)_{t\in(-\epsilon,\epsilon)}$ a one parameter variation of g. Then*

$$\frac{d}{dt}E_p(\varphi; D)\Big|_{t=0} = \frac{1}{2}\int_D < S_p(\varphi), \delta g > \; v_g,$$

where $S_p(\varphi) \in \Gamma(\odot^2 T^*M)$ is given by

$$S_p(\varphi)(X, Y) = \frac{1}{P}|d\varphi|^p g(X, Y) - |d\varphi|^{p-2} h(d\varphi(X), d\varphi(Y)). \tag{4.19}$$

$S_p(\varphi)$ is called the stress penergy tensor of φ.

Theorem 4.3.1. *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map between two manifolds, then*

$$\operatorname{div}^M S_p(\varphi) = -h(\tau_p(\varphi), d\varphi) \tag{4.20}$$

Proof. Let $\{e_1, \ldots, e_m\}$ be an orthonormal frame such that $\nabla^M_{e_i} e_j = 0$ at $x \in M$ for all $i, j = 1 \ldots m$ and $X \in \{e_i\}_{i=1}^m$.
At $x \in M$ we have

$$\begin{aligned}(\operatorname{div}^M S(\varphi))(x) &= \sum_{i=1}^m e_i(S_p(\varphi), (e_i, X) \\ &= \frac{1}{p}\sum_{i=1}^m e_i|d\varphi|^p g(X, e_i) - \sum_{i=1}^m e_i |d\varphi|^{p-2} h(d\varphi(X), d\varphi(e_i)) \\ &\quad -\sum_{i=1}^m |d\varphi|^{p-2} e_i h(d\varphi(X), d\varphi(e_i)) \\ &= \frac{1}{p} g(X, grad|d\varphi|^p) - h(d\varphi(X), d\varphi(\operatorname{grad}|d\varphi|^{p-2})) \\ &\quad -|d\varphi|^{p-2} h(\nabla_X{}^\varphi d\varphi(e_i), d\varphi(e_i)) - |d\varphi|^{p-2} h(d\varphi(X), \tau(\varphi))\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{p}g(X, grad|d\varphi|^p) \\
&\quad -h(d\varphi(X), d\varphi(\text{grad}\,|d\varphi|^{p-2}) + |d\varphi|^{p-2}\tau(\varphi)) \\
&\quad -\frac{1}{2}|d\varphi|^{p-2}X|d\varphi|^2 \\
&= \frac{1}{p}g(X, grad|d\varphi|^p) \\
&\quad -h(d\varphi(X), d\varphi(\text{grad}\,|d\varphi|^{p-2}) + |d\varphi|^{p-2}\tau(\varphi)) \\
&\quad -\frac{1}{2}|d\varphi|^{p-2}\frac{2}{p|d\varphi|^{p-2}}X|d\varphi|^p \\
&= \frac{1}{p}g(X, grad|d\varphi|^p) \\
&\quad -h(d\varphi(X), d\varphi(\text{grad}\,|d\varphi|^{p-2}) + |d\varphi|^{p-2}\tau(\varphi)) \\
&\quad -\frac{1}{p}g(X, grad|d\varphi|^p) \\
&= -h(\tau_p(\varphi), d\varphi(X))
\end{aligned}$$

□

Corollary 4.3.1. *Let $\varphi : (M^m, g) \to (N^n, h)$ a C^∞-map between two Riemannian manifolds, then from the above theorem, we have*

- *If φ is harmonic then $\text{div}^M S_p(\varphi)$ is vanishing.*
- *If φ is submersion and if $\text{div}^M S_p(\varphi) = 0$ then φ is harmonic.*

4.4 Stress p-bienergy tensors

[23] Let $\varphi : (M, g) \to (N, h)$ be a smooth map between two Riemannian manifolds and $p \geq 2$. Consider a smooth one-parameter variation of the metric g, i.e. a smooth family of metrics (g_t) $(-\epsilon < t < \epsilon)$ such that $g_0 = g$, write $\delta = \frac{\partial}{\partial t}\Big|_{t=0}$, then $\delta g \in \Gamma(\odot^2 T^* M)$ is a symmetric 2-covariant tensor field on M (see [2]). Take local coordinates (x^i) on M, and write the metric on M in the usual way as $g_t = g_{ij}(t, x)\, dx^i\, dx^j$, we now compute

$$\frac{d}{dt}E_{2,p}(\varphi; D)\Big|_{t=0} = \frac{1}{2}\int_D \delta(|\tau_p(\varphi)|^2)v_g + \frac{1}{2}\int_D |\tau_p(\varphi)|^2\delta(v_{g_t}). \tag{4.21}$$

The calculation of the first term breaks down in three lemmas.

Lemma 4.4.1. *The vector field $\xi = (\text{div}^M \delta g)^\sharp - \frac{1}{2}\,\text{grad}^M(\text{trace}\,\delta g)$ satisfies*

$$\begin{aligned}
\delta(|\tau_p(\varphi)|^2) &= -(p-2)|d\varphi|^{p-4} < \varphi^* h, \delta g > h(\tau(\varphi), \tau_p(\varphi)) \\
&\quad -2|d\varphi|^{p-2} < h(\nabla d\varphi, \tau_p(\varphi)), \delta g > -2|d\varphi|^{p-2}h(d\varphi(\xi), \tau_p(\varphi))
\end{aligned}$$

$$-(p-2)(p-4)|d\varphi|^{p-5}\langle\varphi^*h,\delta g\rangle h(d\varphi(\mathrm{grad}^M|d\varphi|),\tau_p(\varphi))$$
$$-2(p-2)|d\varphi|^{p-3} < d|d\varphi|\odot h(d\varphi,\tau_p(\varphi)),\delta g >$$
$$-(p-2)|d\varphi|^{p-4}h(d\varphi(\mathrm{grad}^M\langle\varphi^*h,\delta g\rangle),\tau_p(\varphi)),$$

where φ^*h is the pull-back of the metric h, and $\langle\, ,\,\rangle$ is the induced Riemannian metric on $\otimes^2T^*M$.

Proof. In local coordinates (x^i) on M and (y^α) on N, we have

$$\delta(|\tau_p(\varphi)|^2)=\delta\big(\tau_p(\varphi)^\alpha\tau_p(\varphi)^\beta h_{\alpha\beta}\big)=2\delta(\tau_p(\varphi)^\alpha)\tau_p(\varphi)^\beta h_{\alpha\beta}. \tag{4.22}$$

By the definition of $\tau_p(\varphi)$ we get

$$\begin{aligned}\delta(\tau_p(\varphi)^\alpha) &= \delta\big(|d\varphi|^{p-2}\tau(\varphi)^\alpha+\theta^\alpha\big)\\ &= \delta(|d\varphi|^{p-2})\tau(\varphi)^\alpha+|d\varphi|^{p-2}\delta(\tau(\varphi)^\alpha)+\delta(\theta^\alpha).\end{aligned} \tag{4.23}$$

where $\tau(\varphi)^\alpha=g^{ij}\big(\varphi^\alpha_{i,j}+^N\Gamma^\alpha_{\mu\sigma}\varphi^\mu_i\varphi^\sigma_j-^M\Gamma^k_{ij}\varphi^\alpha_k\big)$ is the component of the tension field $\tau(\varphi)$, and $\theta^\alpha=(p-2)|d\varphi|^{p-3}g^{ij}|d\varphi|_i\varphi^\alpha_j$.
The first term in the right-hand side of (5.27) is given by

$$\begin{aligned}\delta(|d\varphi|^{p-2})\,\tau(\varphi)^\alpha &= (p-2)|d\varphi|^{p-4}\delta(\frac{|d\varphi|^2}{2})\tau(\varphi)^\alpha\\ &= -\frac{p-2}{2}|d\varphi|^{p-4}<\varphi^*h,\delta g>\tau(\varphi)^\alpha.\end{aligned} \tag{4.24}$$

The second term on the right-hand side of (5.27) is (see [18])

$$|d\varphi|^{p-2}\delta(\tau(\varphi)^\alpha)=-|d\varphi|^{p-2}g^{ai}g^{bj}\delta(g_{ab})(\nabla d\varphi)^\alpha_{ij}-|d\varphi|^{p-2}\xi^k\varphi^\alpha_k, \tag{4.25}$$

Now, we compute the third term on the right-hand side of (5.27)

$$\begin{aligned}\delta(\theta^\alpha) &= (p-2)(p-3)|d\varphi|^{p-5}\delta(\frac{|d\varphi|^2}{2})g^{ij}|d\varphi|_i\varphi^\alpha_j\\ &\quad+(p-2)|d\varphi|^{p-3}\delta(g^{ij})|d\varphi|_i\varphi^\alpha_j\\ &\quad+(p-2)|d\varphi|^{p-3}g^{ij}\delta(|d\varphi|_i)\varphi^\alpha_j.\end{aligned} \tag{4.26}$$

By using $\delta(\frac{|d\varphi|^2}{2})=-\frac{1}{2}\langle\varphi^*h,\delta g\rangle$ with $\delta(|d\varphi|_i)=(\delta(|d\varphi|))_i$, the equation (4.26) becomes

$$\begin{aligned}\delta(\theta^\alpha) &= -\frac{(p-2)(p-3)}{2}|d\varphi|^{p-5}\langle\varphi^*h,\delta g\rangle g^{ij}|d\varphi|_i\varphi^\alpha_j\\ &\quad+(p-2)|d\varphi|^{p-3}\delta(g^{ij})|d\varphi|_i\varphi^\alpha_j\\ &\quad-\frac{p-2}{2}|d\varphi|^{p-4}g^{ij}\langle\varphi^*h,\delta g\rangle_i\varphi^\alpha_j\\ &\quad+\frac{p-2}{2}|d\varphi|^{p-5}g^{ij}|d\varphi|_i\langle\varphi^*h,\delta g\rangle\varphi^\alpha_j.\end{aligned} \tag{4.27}$$

Note that

$$\begin{aligned} 2\delta(|d\varphi|^{p-2})\tau(\varphi)^\alpha\tau_p(\varphi)^\beta h_{\alpha\beta} &= -(p-2)|d\varphi|^{p-4} < \varphi^*h, \delta g > \tau(\varphi)^\alpha\tau_p(\varphi)^\beta h_{\alpha\beta} \\ &= -(p-2)|d\varphi|^{p-4} < \varphi^*h, \delta g > h(\tau(\varphi), \tau_p(\varphi)), \end{aligned} \tag{4.28}$$

$$\begin{aligned} 2|d\varphi|^{p-2}\delta(\tau(\varphi)^\alpha)\tau_p(\varphi)^\beta h_{\alpha\beta} &= -2|d\varphi|^{p-2}g^{ai}g^{bj}\delta(g_{ab})(\nabla d\varphi)^\alpha_{ij}\tau_p(\varphi)^\beta h_{\alpha\beta} \\ &\quad -2|d\varphi|^{p-2}\xi^k\varphi^\alpha_k\tau_p(\varphi)^\beta h_{\alpha\beta} \\ &= -2|d\varphi|^{p-2} < h(\nabla d\varphi, \tau_p(\varphi)), \delta g > \\ &\quad -2|d\varphi|^{p-2}h(d\varphi(\xi), \tau_p(\varphi)), \end{aligned} \tag{4.29}$$

and the following

$$\begin{aligned} 2\delta(\theta^\alpha)\tau_p(\varphi)^\beta h_{\alpha\beta} &= -(p-2)(p-3)|d\varphi|^{p-5}\langle\varphi^*h, \delta g\rangle g^{ij}|d\varphi|_i\varphi^\alpha_j\tau_p(\varphi)^\beta h_{\alpha\beta} \\ &\quad +2(p-2)|d\varphi|^{p-3}\delta(g^{ij})|d\varphi|_i\varphi^\alpha_j\tau_p(\varphi)^\beta h_{\alpha\beta} \\ &\quad -(p-2)|d\varphi|^{p-4}g^{ij}\langle\varphi^*h, \delta g\rangle_i\varphi^\alpha_j\tau_p(\varphi)^\beta h_{\alpha\beta} \\ &\quad +(p-2)|d\varphi|^{p-5}g^{ij}|d\varphi|_i\langle\varphi^*h, \delta g\rangle\varphi^\alpha_j\tau_p(\varphi)^\beta h_{\alpha\beta} \\ &= -(p-2)(p-3)|d\varphi|^{p-5}\langle\varphi^*h, \delta g\rangle h(d\varphi(\operatorname{grad}^M|d\varphi|), \tau_p(\varphi)) \\ &\quad -2(p-2)|d\varphi|^{p-3} < d|d\varphi| \odot h(d\varphi, \tau_p(\varphi)), \delta g > \\ &\quad -(p-2)|d\varphi|^{p-4}h(d\varphi(\operatorname{grad}^M\langle\varphi^*h, \delta g\rangle), \tau_p(\varphi)) \\ &\quad +(p-2)|d\varphi|^{p-5}\langle\varphi^*h, \delta g\rangle h(d\varphi(\operatorname{grad}^M|d\varphi|), \tau_p(\varphi)). \end{aligned} \tag{4.30}$$

Substituting (5.27), (4.28), (4.29) and (4.30) in (5.26), the Lemma 4.4.1 follows. □

Lemma 4.4.2. *([7]) Let D be a compact domain of M. Then*

$$\begin{aligned} \int_D |d\varphi|^{p-2}h(d\varphi(\xi), \tau_p(\varphi))v_g &= \int_D \Big\langle -\operatorname{sym}\big(\nabla|d\varphi|^{p-2}h(d\varphi, \tau_p(\varphi))\big) \\ &\quad +\frac{1}{2}\operatorname{div}^M\big(|d\varphi|^{p-2}h(d\varphi, \tau_p(\varphi))^\sharp\big)g, \delta g\Big\rangle v_g. \end{aligned}$$

Lemma 4.4.3. *We set $\omega = |d\varphi|^{p-4}h(d\varphi, \tau_p(\varphi))$. Then*

$$-\int_D |d\varphi|^{p-4}h(d\varphi(\operatorname{grad}^M < \varphi^*h, \delta g >), \tau_p(\varphi))\, v_g = \int_D < \varphi^*h, \delta g > \operatorname{div}\omega\, v_g.$$

Proof. Note that

$$\operatorname{div}(< \varphi^*h, \delta g > \omega) =< \varphi^*h, \delta g > \operatorname{div}\omega + \omega(\operatorname{grad}^M < \varphi^*h, \delta g >),$$

and consider the divergence Theorem, the Lemma 4.4.3 follows. □

Theorem 4.4.1. *Let $\varphi : (M^m, g) \to (N^n, h)$ be a smooth map such that $|d\varphi|_x \neq 0$ for all $x \in M$, and let $(g_t)_{t\in(-\epsilon,\epsilon)}$ a one parameter variation of g. Then*

$$\frac{d}{dt}E_{2,p}(\varphi; D)\Big|_{t=0} = \frac{1}{2}\int_D < S_{2,p}(\varphi), \delta g > \, v_g,$$

where $S_{2,p}(\varphi) \in \Gamma(\odot^2 T^*M)$ is given by

$$\begin{aligned} S_{2,p}(\varphi)(X,Y) &= -\frac{1}{2}|\tau_p(\varphi)|^2 g(X,Y) - |d\varphi|^{p-2} < d\varphi, \nabla^\varphi \tau_p(\varphi) > g(X,Y) \\ &\quad +|d\varphi|^{p-2} h(d\varphi(X), \nabla^\varphi_Y \tau_p(\varphi)) + |d\varphi|^{p-2} h(d\varphi(Y), \nabla^\varphi_X \tau_p(\varphi)) \\ &\quad +(p-2)|d\varphi|^{p-4} < d\varphi, \nabla^\varphi \tau_p(\varphi) > h(d\varphi(X), d\varphi(Y)). \end{aligned}$$

$S_{2,p}(\varphi)$ is called the stress p-bienergy tensor of φ.

Proof. By using $\delta(v_{g_t}) = \frac{1}{2}\langle g, \delta g\rangle v_g$ (see [2]). Lemmas 4.4.1, 4.4.2, and 4.4.3, the equation (4.21) becomes

$$\begin{aligned} S_{2,p}(\varphi) &= -(p-2)|d\varphi|^{p-4} h(\tau(\varphi), \tau_p(\varphi))\varphi^* h \\ &\quad -2|d\varphi|^{p-2} h(\nabla d\varphi, \tau_p(\varphi)) + 2\,\mathrm{sym}\left(\nabla |d\varphi|^{p-2} h(d\varphi, \tau_p(\varphi))\right) \\ &\quad -\mathrm{div}^M\left(|d\varphi|^{p-2} h(d\varphi, \tau_p(\varphi))^\sharp\right) g \\ &\quad -(p-2)(p-4)|d\varphi|^{p-5} h(d\varphi(\mathrm{grad}^M |d\varphi|), \tau_p(\varphi))\varphi^* h \\ &\quad -2(p-2)|d\varphi|^{p-3} d|d\varphi| \odot h(d\varphi, \tau_p(\varphi)) \\ &\quad +(p-2)\,\mathrm{div}^M\left[|d\varphi|^{p-4} h(d\varphi, \tau_p(\varphi))\right]\varphi^* h + \frac{1}{2}|\tau_p(\varphi)|^2 g. \end{aligned} \tag{4.31}$$

Note that, for all $X, Y \in \Gamma(TM)$, we have

$$\begin{aligned} 2\,\mathrm{sym}\left(\nabla |d\varphi|^{p-2} h(d\varphi, \tau_p(\varphi))\right)(X,Y) &= 2|d\varphi|^{p-2} h(\nabla d\varphi(X,Y), \tau_p(\varphi)) \\ &\quad +|d\varphi|^{p-2} h(d\varphi(X), \nabla^\varphi_Y \tau_p(\varphi)) \\ &\quad +|d\varphi|^{p-2} h(d\varphi(Y), \nabla^\varphi_X \tau_p(\varphi)) \\ &\quad +X(|d\varphi|^{p-2}) h(d\varphi(Y), \tau_p(\varphi)) \\ &\quad +Y(|d\varphi|^{p-2}) h(d\varphi(X), \tau_p(\varphi)), \end{aligned} \tag{4.32}$$

and the following formula

$$\begin{aligned} -2d|d\varphi| \odot h(d\varphi, \tau_p(\varphi))(X,Y) &= -X(|d\varphi|) h(d\varphi(Y), \tau_p(\varphi)) \\ &\quad -Y(|d\varphi|) h(d\varphi(X), \tau_p(\varphi)). \end{aligned} \tag{4.33}$$

Calculating in a normal frame at x, we have

$$\mathrm{div}^M\left(|d\varphi|^{p-2} h(d\varphi, \tau_p(\varphi))^\sharp\right) = \sum_{i=1}^m e_i(g(|d\varphi|^{p-2} h(d\varphi, \tau_p(\varphi))^\sharp, e_i))$$

$$\begin{aligned}
&= \sum_{i=1}^{m} e_i(|d\varphi|^{p-2}h(d\varphi(e_i),\tau_p(\varphi)))\\
&= \sum_{i=1}^{m} e_i(|d\varphi|^{p-2})h(d\varphi(e_i),\tau_p(\varphi))\\
&\quad+\sum_{i=1}^{m}|d\varphi|^{p-2}h(\nabla^{\varphi}_{e_i}d\varphi(e_i),\tau_p(\varphi))\\
&\quad+\sum_{i=1}^{m}|d\varphi|^{p-2}h(d\varphi(e_i),\nabla^{\varphi}_{e_i}\tau_p(\varphi))\\
&= (p-2)|d\varphi|^{p-3}h(d\varphi(\operatorname{grad}^M|d\varphi|),\tau_p(\varphi))\\
&\quad+|d\varphi|^{p-2}h(\tau(\varphi),\tau_p(\varphi))\\
&\quad+|d\varphi|^{p-2}<d\varphi,\nabla^{\varphi}\tau_p(\varphi)>. \qquad (4.34)
\end{aligned}$$

From the definition of $\tau_p(\varphi)$, and equation (4.34), we get

$$\operatorname{div}^M\left(|d\varphi|^{p-2}h(d\varphi,\tau_p(\varphi))^{\sharp}\right)=|\tau_p(\varphi)|^2+|d\varphi|^{p-2}<d\varphi,\nabla^{\varphi}\tau_p(\varphi)>. \qquad (4.35)$$

With the same method of (4.34), we find that

$$\begin{aligned}
\operatorname{div}^M\left(|d\varphi|^{p-4}h(d\varphi,\tau_p(\varphi))\right) &= (p-4)|d\varphi|^{p-5}h(d\varphi(\operatorname{grad}^M|d\varphi|),\tau_p(\varphi))\\
&\quad+|d\varphi|^{p-4}h(\tau(\varphi),\tau_p(\varphi))\\
&\quad+|d\varphi|^{p-4}<d\varphi,\nabla^{\varphi}\tau_p(\varphi)>. \qquad (4.36)
\end{aligned}$$

Substituting (4.32), (4.33), (4.35) and (4.36) in (4.31), the Theorem 4.4.1 follows. □

By using the definition of divergence for symmetric $(0,2)$-tensors (see [2, 7]) we have the following result.

Theorem 4.4.2. *Let $\varphi:(M,g)\to(N,h)$ be a smooth map such that $|d\varphi|_x\neq 0$ for all $x\in M$. Then*

$$\operatorname{div}^M S_{2,p}(\varphi)(X)=-h(\tau_{2,p}(\varphi),d\varphi(X)),\quad \forall X\in\Gamma(TM). \qquad (4.37)$$

Proof. First, the p-bitension field of φ is given by

$$\begin{aligned}
\tau_{2,p}(\varphi) &= -|d\varphi|^{p-2}\operatorname{trace}_g R^N(\tau_p(\varphi),d\varphi)d\varphi-\operatorname{trace}_g\nabla^{\varphi}|d\varphi|^{p-2}\nabla^{\varphi}\tau_p(\varphi)\\
&\quad-(p-2)\operatorname{trace}_g\nabla<\nabla^{\varphi}\tau_p(\varphi),d\varphi>|d\varphi|^{p-4}d\varphi. \qquad (4.38)
\end{aligned}$$

and the p-tension field of φ is given by

$$\tau_p(\varphi)=\operatorname{div}^M(|d\varphi|^{p-2}d\varphi). \qquad (4.39)$$

when the tension field is defined by

$$\tau(\varphi) = \operatorname{trace}_g \nabla d\varphi \tag{4.40}$$

Let $\{e_i, \ldots, e_m\}$ be an orthonormal frame such that $\nabla^M_{e_i} e_j = 0$ at $x \in M$ for all $i, j = 1 \ldots m$ and $X \in \{e_i\}_{i=1}^m$.
At $x \in M$ we have

$$\begin{aligned}
\operatorname{div}^M S_{2,p}(\varphi)(X) &= \sum_{i=1}^m e_i\big[-\frac{1}{2}|\tau_p(\varphi)|^2 g(X,e_i) - |d\varphi|^{p-2} < d\varphi, \nabla^\varphi \tau_p(\varphi) > g(X,e_i) \\
&\quad + |d\varphi|^{p-2} h(d\varphi(X), \nabla^\varphi_{e_i}\tau_p(\varphi)) + |d\varphi|^{p-2} h(d\varphi(e_i), \nabla^\varphi_X \tau_p(\varphi)) \\
&\quad + (p-2)|d\varphi|^{p-4} < d\varphi, \nabla^\varphi \tau_p(\varphi) > h(d\varphi(X), d\varphi(e_i))\big] \\
&= -h(\nabla^\varphi_X \tau_p(X), \tau_p(X)) - \nabla^\varphi_X |d\varphi|^{p-2} < d\varphi, \nabla^\varphi \tau_p(\varphi) > \\
&\quad + \sum_{i=1}^m \nabla^\varphi_{e_i} |d\varphi|^{p-2} h(d\varphi(X), \nabla^\varphi_{e_i}\tau_p(\varphi)) + \sum_{i=1}^m \nabla^\varphi_{e_i}|d\varphi|^{p-2} h(d\varphi(e_i), \nabla^\varphi_X \tau_p(\varphi)) \\
&\quad + (p-2) \sum_{i=1}^m \nabla^\varphi_{e_i} |d\varphi|^{p-4} < d\varphi, \nabla^\varphi \tau_p(\varphi) > h(d\varphi(X), d\varphi(e_i)) \\
&\quad + (p-2)|d\varphi|^{p-4} < d\varphi, \nabla^\varphi \tau_p(\varphi) > \sum_{i=1}^m e_i h(d\varphi(X), d\varphi(e_i)) \\
&= -h(\nabla^\varphi_X \tau_p(X), \tau_p(X)) - X|d\varphi|^{p-2} < d\varphi, \nabla^\varphi \tau_p(\varphi) > \\
&\quad - |d\varphi|^{p-2} \sum_{i=1}^m h(\nabla^\varphi_X d\varphi(e_i), \nabla^\varphi_{e_i} \tau_p(\varphi)) - |d\varphi|^{p-2} \sum_{i=1}^m h(d\varphi(e_i), \nabla^\varphi_X \nabla^\varphi_{e_i} \tau_p(\varphi)) \\
&\quad + \sum_{i=1}^m e_i |d\varphi|^{p-2} h(d\varphi(X), \nabla^\varphi_{e_i}\tau_p(\varphi)) + |d\varphi|^{p-2} \sum_{i=1}^m h(\nabla^\varphi_{e_i} d\varphi(X), \nabla^\varphi_{e_i}\tau_p(\varphi)) \\
&\quad + |d\varphi|^{p-2} \sum_{i=1}^m h(d\varphi(X), \nabla^\varphi_{e_i}\nabla^\varphi_{e_i}\tau_p(\varphi)) + \sum_{i=1}^m e_i |d\varphi|^{p-2} h(d\varphi(e_i), \nabla^\varphi_X \tau_p(\varphi)) \\
&\quad + |d\varphi|^{p-2} \sum_{i=1}^m h(\nabla^\varphi_{e_i} d\varphi(e_i), \nabla^\varphi_X \tau_p(\varphi)) + |d\varphi|^{p-2} \sum_{i=1}^m h(d\varphi(e_i), \nabla^\varphi_{e_i}\nabla^\varphi_X \tau_p(\varphi)) \\
&\quad + (p-2) \sum_{i=1}^m \nabla^\varphi_{e_i} |d\varphi|^{p-4} < d\varphi, \nabla^\varphi \tau_p(\varphi) > h(d\varphi(X), d\varphi(e_i)) \\
&\quad + (p-2)|d\varphi|^{p-4} < d\varphi, \nabla^\varphi \tau_p(\varphi) > \sum_{i=1}^m h(\nabla^\varphi_{e_i} d\varphi(X), d\varphi(e_i)) \\
&\quad + (p-2)|d\varphi|^{p-4} < d\varphi, \nabla^\varphi \tau_p(\varphi) > \sum_{i=1}^m h(d\varphi(X), \nabla^\varphi_{e_i} d\varphi(e_i)).
\end{aligned} \tag{4.41}$$

From the equation (4.39), we have

$$h(\nabla^{\varphi}_X\tau_p(X), \sum_{i=1}^{m} e_i|d\varphi|^{p-2}d\varphi(e_i) + |d\varphi|^{p-2}\sum_{i=1}^{m}\nabla^{\varphi}_{e_i}d\varphi(e_i)) = h(\nabla^{\varphi}_X\tau_p(X), \operatorname{div}^M(|d\varphi|^{p-2}d\varphi)), \tag{4.42}$$

$$\begin{aligned}(p-2)|d\varphi|^{p-4}\sum_{i=1}^{m}h(\nabla^{\varphi}_{e_i}d\varphi(X), d\varphi(e_i)) &= (p-2)|d\varphi|^{p-4}\sum_{i=1}^{m}h(\nabla^{\varphi}_X d\varphi(e_i), d\varphi(e_i))\\ &= \frac{p-2}{2}|d\varphi|^{p-4}\sum_{i=1}^{m}Xh(d\varphi(e_i), d\varphi(e_i))\\ &= \frac{p-2}{2}|d\varphi|^{p-4}X|d\varphi|^2\\ &= (p-2)|d\varphi|^{p-3}X|d\varphi|\\ &= X|d\varphi|^{p-2}. \end{aligned} \tag{4.43}$$

By using the proprieties the curvature operator R^N we obtain

$$\begin{aligned}\sum_{i=1}^{m}h(d\varphi(e_i), \nabla^{\varphi}_X\nabla^{\varphi}_{e_i}\tau_p(\varphi) - \nabla^{\varphi}_{e_i}\nabla^{\varphi}_X\tau_p(\varphi)) &= \sum_{i=1}^{m}h(d\varphi(e_i), R^N(d\varphi(X), d\varphi(e_i))\tau_p(\varphi))\\ &= \sum_{i=1}^{m}h(d\varphi(X), R^N(\tau_p(\varphi), d\varphi(e_i))d\varphi(e_i))\\ &= h(d\varphi(X), \operatorname{trace}_g R^N(\tau_p(\varphi), d\varphi)d\varphi)\end{aligned} \tag{4.44}$$

$$\sum_{i=1}^{m}(e_i|d\varphi|^{p-2}\nabla^{\varphi}_{e_i}\tau_p(\varphi) + |d\varphi|^{p-2}\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}\tau_p(\varphi)) = \operatorname{trace}_g \nabla^{\varphi}|d\varphi|^{p-2}\nabla^{\varphi}\tau_p(\varphi) \tag{4.45}$$

$$\begin{aligned}\sum_{i=1}^{m}(\nabla^{\varphi}_{e_i}|d\varphi|^{p-4} < d\varphi, \nabla^{\varphi}\tau_p(\varphi) > d\varphi(e_i) \quad &+ \quad |d\varphi|^{p-4} < d\varphi, \nabla^{\varphi}\tau_p(\varphi) > \nabla^{\varphi}_{e_i}d\varphi(e_i))\\ &= \operatorname{trace}_g \nabla < \nabla^{\varphi}\tau_p(\varphi), d\varphi > |d\varphi|^{p-4}d\varphi\end{aligned} \tag{4.46}$$

Substituting (4.42), (4.43), (4.44), (4.45) and (4.46) in (4.41) and using equation (4.38) we obtain equation (4.37) which completes the proof of the theorem

□

Remark 4.4.1. *When $p = 2$, we have $S_{2,p}(\varphi) = S_2(\varphi)$, where $S_2(\varphi)$ is stress bienergy tensor in [18].*

Corollary 4.4.1. *Let $\varphi : (M, g) \to (N, h)$ be a smooth map. (1) Then $S_{2,m}(\varphi) = 0$ implies φ is m-harmonic, where $m = \dim M$. (2) If M is compact without boundary, and $p \neq \frac{m}{2}$. Then $S_{2,p}(\varphi) = 0$ implies φ is p-harmonic.*

Proof. Let $\{e_i\}$ be an orthonormal frame on (M, g). (1) We have

$$0 = \sum_{i=1}^{m} S_{2,p}(\varphi)(e_i, e_i) \quad = \quad -\frac{m}{2}|\tau_p(\varphi)|^2 + (p - m)|d\varphi|^{p-2} < d\varphi, \nabla^{\varphi}\tau_p(\varphi) > .$$

For $p = m$, the last equation becomes $-\frac{m}{2}|\tau_m(\varphi)|^2 = 0$. So φ is m-harmonic map. (2) We set $\theta(X) = h(|d\varphi|^{p-2}d\varphi(X), \tau_p(\varphi))$, for all $X \in \Gamma(TM)$. The trace of $S_{2,p}(\varphi)$ gives the equality

$$0 = \sum_{i=1}^{m} S_{2,p}(\varphi)(e_i, e_i) \quad = \quad (\frac{m}{2} - p)|\tau_p(\varphi)|^2 + (p - m)\operatorname{div}^M \theta.$$

By using the Green Theorem, we get

$$(\frac{m}{2} - p)\int_M |\tau_p(\varphi)|^2 v^g = 0.$$

Since $p \neq \frac{m}{2}$, we obtain $|\tau_p(\varphi)|^2 = 0$, that is φ is p-harmonic map. $\square$

Chapter 5

On the p-biharmonic Submanifold

In this chapter we give new methods for constructing proper p-biharmonic submanifold in space form and p-biharmonic hypersurfaces in Einstein space and conformally space form. These new results are contained in [23, 24].

5.1 p-Biharmonic Submanifold in Space Form

Theorem 5.1.1. *The canonical inclusion* $\mathbf{i}$ *is p-biharmonic if and only if*

$$\begin{cases} -\Delta^{\perp} H + \operatorname{trace}_g B(\cdot, A_H(\cdot)) - m\,(c - (p-2)|H|^2)\,H &= 0; \\ \\ 2\operatorname{trace}_g A_{\nabla^{\perp}_{\cdot} H}(\cdot) + \left(p - 2 + \frac{m}{2}\right)\operatorname{grad}^M |H|^2 &= 0, \end{cases} \tag{5.1}$$

Where $\Delta^{\perp}$ *is the Laplacian in the normal bundle of* (M, g).

Proof. First, the p-tension field of $\mathbf{i}$ is given by

$$\tau_p(\mathbf{i}) \;=\; |d\mathbf{i}|^{p-2}\tau(\mathbf{i}) + (p-2)|d\mathbf{i}|^{p-3}d\mathbf{i}(\operatorname{grad}^M |d\mathbf{i}|),$$

since $\tau(\mathbf{i}) = mH$ (see [1, 2]), and $|d\mathbf{i}|^2 = m$, we get $\tau_p(\mathbf{i}) = m^{\frac{p}{2}}H$. Let $\{e_i, \ldots, e_m\}$ be an orthonormal frame such that $\nabla^M_{e_i} e_j = 0$ at $x \in M$ for all $i, j = 1, \ldots, m$, then calculating at x

$$\begin{aligned} \operatorname{trace}_g R^N(\tau_p(\mathbf{i}), d\mathbf{i})d\mathbf{i} &= \sum_{i=1}^{m} R^N(\tau_p(\mathbf{i}), d\mathbf{i}(e_i))d\mathbf{i}(e_i) \\ &= m^{\frac{p}{2}} \sum_{i=1}^{m} R^N(H, e_i)e_i. \end{aligned}$$

By the following equation $R^N(X, Y)Z = c\,(\langle Y, Z\rangle X - \langle X, Z\rangle Y)$, with $\langle H, e_i\rangle = 0$, for all $X, Y, Z \in \Gamma(TN(c))$ and $i = 1, \ldots, m$, the last equation becomes

$$\operatorname{trace}_g R^N(\tau_p(\mathbf{i}), d\mathbf{i})d\mathbf{i} = m^{\frac{p+2}{2}} cH. \tag{5.2}$$

We compute the term $\operatorname{trace}_g(\nabla^{\mathbf{i}})^2\tau_p(\mathbf{i})$ at x

$$\begin{aligned}\sum_{i=1}^m \nabla^{\mathbf{i}}_{e_i}\nabla^{\mathbf{i}}_{e_i}H &= \sum_{i=1}^m \nabla^{\mathbf{i}}_{e_i}\big(-A_H(e_i)+(\nabla^{\mathbf{i}}_{e_i}H)^\perp\big)\\ &= -\sum_{i=1}^m \nabla^M_{e_i}A_H(e_i)-\sum_{i=1}^m B(e_i,A_H(e_i))\\ &\quad -\sum_{i=1}^m A_{(\nabla^{\mathbf{i}}_{e_i}H)^\perp}(e_i)x+\sum_{i=1}^m \big(\nabla^{\mathbf{i}}_{e_i}(\nabla^{\mathbf{i}}_{e_i}H)^\perp\big)^\perp, \end{aligned}\tag{5.3}$$

since $< A_H(X),Y >=< B(X,Y),H >$ for all $X,Y\in\Gamma(TM)$, we get

$$\begin{aligned}\sum_{i=1}^m \nabla^M_{e_i}A_H(e_i) &= \sum_{i,j=1}^m \big\langle \nabla^M_{e_i}A_H(e_i),e_j\big\rangle e_j\\ &= \sum_{i,j=1}^m e_i\big(\big\langle A_H(e_i),e_j\big\rangle\big)e_j\\ &= \sum_{i,j=1}^m e_i\big(\big\langle B(e_i,e_j),H\big\rangle\big)e_j\\ &= \sum_{i,j=1}^m e_i\big(\big\langle \nabla^N_{e_j}e_i,H\big\rangle\big)e_j,\end{aligned}$$

since $\nabla^N_X\nabla^N_Y Z = R^N(X,Y)Z+\nabla^N_Y\nabla^N_X Z+\nabla^N_{[X,Y]}Z$, for all $X,Y,Z\in\Gamma(TN(c))$, we conclude

$$\begin{aligned}\sum_{i=1}^m \nabla^M_{e_i}A_H(e_i) &= \sum_{i,j=1}^m \big\langle \nabla^N_{e_i}\nabla^N_{e_j}e_i,H\big\rangle e_j+\sum_{i,j=1}^m \big\langle \nabla^N_{e_j}e_i,\nabla^{\mathbf{i}}_{e_i}H\big\rangle e_j\\ &= \sum_{i,j=1}^m \big\langle R^N(e_i,e_j)e_i,H\big\rangle e_j+\sum_{i,j=1}^m \big\langle \nabla^N_{e_j}\nabla^N_{e_i}e_i,H\big\rangle e_j\\ &\quad +\sum_{i,j=1}^m \big\langle B(e_i,e_j),(\nabla^{\mathbf{i}}_{e_i}H)^\perp\big\rangle e_j,\end{aligned}$$

since $R^N(X,Y)Z = c\,(\langle Y,Z\rangle X-\langle X,Z\rangle Y)$, for all $X,Y,Z\in\Gamma(TN(c))$, we have

$$\begin{aligned}\sum_{i=1}^m \nabla^M_{e_i}A_H(e_i) &= \sum_{i,j=1}^m e_j\big(\big\langle \nabla^N_{e_i}e_i,H\big\rangle\big)e_j-\sum_{i,j=1}^m \big\langle \nabla^N_{e_i}e_i,\nabla^{\mathbf{i}}_{e_j}H\big\rangle e_j\\ &\quad +\sum_{i,j=1}^m \big\langle A_{(\nabla^{\mathbf{i}}_{e_i}H)^\perp}(e_i),e_j\big\rangle e_j\end{aligned}$$

$$\begin{aligned} &= m\sum_{j=1}^{m} e_j\big(\langle H,H\rangle\big)\,e_j - m\sum_{j=1}^{m}\langle H,\nabla^{\mathbf{i}}_{e_j}H\rangle\,e_j \\ &\quad+\sum_{i=1}^{m} A_{(\nabla^{\mathbf{i}}_{e_i}H)^{\perp}}(e_i) \\ &= \frac{m}{2}\sum_{j=1}^{m} e_j\big(\langle H,H\rangle\big)\,e_j + \sum_{i=1}^{m} A_{(\nabla^{\mathbf{i}}_{e_i}H)^{\perp}}(e_i). \qquad (5.4)\end{aligned}$$

From equations (5.3) and (5.13), we obtain

$$\begin{aligned}\operatorname{trace}_g(\nabla^{\mathbf{i}})^2\tau_p(\mathbf{i}) &= -\frac{m^{\frac{p+2}{2}}}{2}\operatorname{grad}^M|H|^2 - 2m^{\frac{p}{2}}\operatorname{trace}_g A_{(\nabla^{\perp}_{\cdot}H)}(\cdot) \\ &\quad -m^{\frac{p}{2}}\operatorname{trace}_g B(\cdot,A_H(\cdot)) + m^{\frac{p}{2}}\Delta^{\perp}H. \qquad (5.5)\end{aligned}$$

Now, we compute the term $\operatorname{trace}_g\nabla < \nabla^{\mathbf{i}}\tau_p(\mathbf{i}), d\mathbf{i} > d\mathbf{i}$ at x

$$\sum_{i,j=1}^{m}\nabla^{\mathbf{i}}_{e_i} < \nabla^{\mathbf{i}}_{e_j}\tau_p(\mathbf{i}), d\mathbf{i}(e_j) > d\mathbf{i}(e_i) = m^{\frac{p}{2}}\sum_{i,j=1}^{m}\nabla^{\mathbf{i}}_{e_i} < \nabla^{\mathbf{i}}_{e_j}H, e_j > e_i,$$

by the compatibility of pull-back connection $\nabla^{\mathbf{i}}$ with the Riemannian metric of $N(c)$, and the definition of the mean curvature vector field H of (M,g), we have

$$\begin{aligned}\sum_{j=1}^{m}\langle\nabla^{\mathbf{i}}_{e_j}H\,,e_j\rangle &= \sum_{j=1}^{m}\big\{e_j\langle H,e_j\rangle - \langle H,\nabla^{\mathbf{i}}_{e_j}e_j\rangle\big\} \\ &= -\sum_{j=1}^{m}\langle H,B(e_j,e_j)\rangle \\ &= -m|H|^2,\end{aligned}$$

by the last two equations, we have the following

$$\operatorname{trace}_g\nabla < \nabla^{\mathbf{i}}\tau_p(\mathbf{i}), d\mathbf{i} > d\mathbf{i} = -m^{\frac{p+2}{2}}\operatorname{grad}^M|H|^2 - m^{\frac{p+4}{2}}|H|^2H. \qquad (5.6)$$

The Theorem 2.1 follows by (10), (5.2), (5.14), and (5.6). □

If $p=2$ and $N=\mathbb{S}^n$, we arrive at the following Corollary.

Corollary 5.1.1. *Let M be a submanifold of sphere $\mathbb{S}^n$ of dimension m, then the canonical inclusion $\mathbf{i}: M\hookrightarrow\mathbb{S}^n$ is biharmonic if and only if*

$$\begin{cases}\frac{m}{2}\operatorname{grad}^M|H|^2 + 2\operatorname{trace}_g A_{(\nabla^{\perp}_{\cdot}H)}(\cdot) = 0,\\[2ex] -m\,H + \operatorname{trace}_g B(\cdot,A_H(\cdot)) - \Delta^{\perp}H = 0.\end{cases}$$

This result was deduced by B-Y. Chen and C. Oniciuc [6, 25].

Theorem 5.1.2. *If M is a hypersurface with nowhere zero mean curvature of $N^{m+1}(c)$, then M is p-biharmonic if only if*

$$\begin{cases} -\Delta^{\perp} H + \big(|A|^2 + m(p-2)|H|^2 - mc\big)H = 0; \\ 2A(\operatorname{grad}^M |H|) + \big(2(p-2)+m\big)|H| \operatorname{grad}^M |H| = 0. \end{cases} \tag{5.7}$$

Proof. Consider $\{e_1, ..., e_m\}$ to be a local orthonormal frame field on (M, g), and let η the unit normal vector field at (M, g) in $N^{m+1}(c)$. We have

$$\begin{aligned} H &= < H, \eta > \eta \\ &= \frac{1}{m}\sum_{i=1}^{m} < B(e_i, e_i), \eta > \eta \\ &= \frac{1}{m}\sum_{i=1}^{m} g(A(e_i), e_i)\eta \\ &= \frac{1}{m}(\operatorname{trace}_g A)\eta. \end{aligned}$$

Let $i = 1, \ldots, m$, we compute

$$\begin{aligned} A_H(e_i) &= \sum_{j=1}^{m} g(A_H(e_i), e_j)e_j \\ &= -\sum_{j=1}^{m} < \nabla^N_{e_i} H, e_j > e_j \\ &= -\sum_{j=1}^{m} e_i < H, e_j > e_j + \sum_{j=1}^{m} < H, B(e_i, e_j) > e_j \\ &= < H, \eta > \sum_{j=1}^{m} < \eta, B(e_i, e_j) > e_j, \end{aligned}$$

by the last equation and the formula $< \eta, B(e_i, e_j) >= g(Ae_i, e_j)$, we obtain the following equation $A_H(e_i) =< H, \eta > A(e_i)$. So that

$$\begin{aligned} \sum_{i=1}^{m} B(e_i, A_H(e_i)) &= \sum_{i=1}^{m} B(e_i, < H, \eta > A(e_i)) \\ &= < H, \eta > \sum_{i=1}^{m} B(e_i, A(e_i)) \\ &= < H, \eta > \sum_{i=1}^{m} g(A(e_i), A(e_i))\eta \end{aligned}$$

$$= \quad |A|^2 H. \tag{5.8}$$

In the same way, with $\eta = H/|H|$, we find that

$$\begin{aligned}
\sum_{i=1}^{m} A_{\nabla^{\perp}_{e_i} H}(e_i) &= \sum_{i,j=1}^{m} < A_{\nabla^{\perp}_{e_i} H}(e_i), e_j > e_j \\
&= -\sum_{i,j=1}^{m} < \nabla^N_{e_i} \nabla^{\perp}_{e_i} H, e_j > e_j \\
&= -\sum_{i,j=1}^{m} < e_i < H, \eta > \nabla^N_{e_i} \eta, e_j > e_j \\
&= A(\operatorname{grad}^M |H|).
\end{aligned} \tag{5.9}$$

The Theorem 2.3 follows by equations (5.15), (5.16), and Theorem 2.1. □

Corollary 5.1.2. *(i) A submanifold M with parallel mean curvature vector field in $N^n(c)$ is p-biharmonic if and only if*

$$\operatorname{trace}_g B(\cdot, A_H(\cdot)) = m\big(c - (p-2)|H|^2\big)H, \tag{5.10}$$

(ii) A hypersurface M of constant non-zero mean curvature in $N^{m+1}(c)$ is proper p-biharmonic if and only if

$$|A|^2 = mc - m(p-2)|H|^2. \tag{5.11}$$

Example 5.1.1. *We consider the hypersurface*

$$\mathbb{S}^m(a) = \Big\{(x^1, \cdots, x^m, x^{m+1}, b) \in \mathbb{R}^{m+2} \,:\, \sum_{i=1}^{m+1} (x^i)^2 = a^2\Big\} \subset \mathbb{S}^{m+1},$$

where $a^2 + b^2 = 1$. We have

$$\eta = \frac{1}{r}(x^1, \cdots, x^{m+1}, -\frac{a^2}{b}),$$

with $r^2 = \frac{a^2}{b^2}$ $(r > 0)$, is a unit section in the normal bundle of $\mathbb{S}^m(a)$ in $\mathbb{S}^{m+1}$. Let $X \in \Gamma(T\mathbb{S}^m(a))$, we compute

$$\nabla^{\mathbb{S}^{m+1}}_X \eta = \frac{1}{r} \nabla^{\mathbb{R}^{m+2}}_X (x^1, \cdots, x^{m+1}, -\frac{a^2}{b}) = \frac{1}{r} X.$$

Thus, $\nabla^{\perp}\eta = 0$ and $A = -\frac{1}{r} Id$. This implies that $H = -\frac{1}{r}\eta$, and so $\mathbb{S}^m(a)$ has constant mean curvature $|H| = \frac{1}{r}$ in $\mathbb{S}^{m+1}$. Since $|A|^2 = \frac{m}{r^2}$, according to Corollary 2.4. we conclude that $\mathbb{S}^m(a)$ is proper p-biharmonic in $\mathbb{S}^{m+1}$ if and only if $p = 1/b^2$.

5.2 Proper p-Biharmonic Hypersurface in Space Form

Theorem 5.2.1. *The hypersurface (M^m, g) with the mean curvature vector $H = f\eta$ is p-bihamronic if and only if*

$$\begin{cases} -\Delta^M(f) + f|A|^2 - f\,\mathrm{Ric}^N(\eta,\eta) + m(p-2)f^3 & = \ 0; \\ 2A(\mathrm{grad}^M f) - 2f(\mathrm{Ricci}^N \eta)^\top + (p-2+\dfrac{m}{2})\,\mathrm{grad}^M f^2 & = \ 0, \end{cases} \tag{5.12}$$

where Ric^N *(resp.* Ricci^N*) is the Ricci curvature (resp. Ricci tensor) of* $(N^{m+1}, \langle,\rangle)$.

Proof. Choose a normal orthonormal frame $\{e_i\}_{i=1,\dots,m}$ on (M^m, g) at x, so that $\{e_i, \eta\}_{i=1,\dots,m}$ is an orthonormal frame on the ambient space $(N^{m+1}, \langle,\rangle)$. Note that, $d\mathbf{i}(X) = X$, $\nabla^{\mathbf{i}}_X Y = \nabla^N_X Y$, and the p-tension field of $\mathbf{i}$ is given by $\tau_p(\mathbf{i}) = m^{\frac{p}{2}} f\eta$. We compute the p-bitension field of $\mathbf{i}$

$$\begin{aligned} \tau_{2,p}(\mathbf{i}) \ = \ & -|d\mathbf{i}|^{p-2}\,\mathrm{trace}_g\, R^N(\tau_p(\mathbf{i}), d\mathbf{i})d\mathbf{i} \\ & -(p-2)\,\mathrm{trace}_g\, \nabla\langle\nabla^{\mathbf{i}}\tau_p(\mathbf{i}), d\mathbf{i}\rangle|d\mathbf{i}|^{p-4}d\mathbf{i} \\ & -\mathrm{trace}_g\, \nabla^{\mathbf{i}}|d\mathbf{i}|^{p-2}\nabla^{\mathbf{i}}\tau_p(\mathbf{i}). \end{aligned} \tag{5.13}$$

The first term of (5.13) is given by

$$\begin{aligned} -|d\mathbf{i}|^{p-2}\,\mathrm{trace}_g\, R^N(\tau_p(\mathbf{i}), d\mathbf{i})d\mathbf{i} \ &= \ -|d\mathbf{i}|^{p-2}\sum_{i=1}^m R^N(\tau_p(\mathbf{i}), d\mathbf{i}(e_i))d\mathbf{i}(e_i) \\ &= \ -m^{p-1} f \sum_{i=1}^m R^N(\eta, e_i)e_i \\ &= \ -m^{p-1} f\,\mathrm{Ricci}^N \eta \\ &= \ -m^{p-1} f\left[(\mathrm{Ricci}^N \eta)^\perp + (\mathrm{Ricci}^N \eta)^\top\right]. \end{aligned} \tag{5.14}$$

We compute the second term of (5.13)

$$-(p-2)\,\mathrm{trace}_g\, \nabla\langle\nabla^{\mathbf{i}}\tau_p(\mathbf{i}), d\mathbf{i}\rangle|d\mathbf{i}|^{p-4}d\mathbf{i} \ = \ -(p-2)m^{p-2}\sum_{i,j=1}^m \nabla^N_{e_j}\langle\nabla^N_{e_i} f\eta, e_i\rangle e_j,$$

$$\begin{aligned} \sum_{i=1}^m \langle\nabla^N_{e_i} f\eta, e_i\rangle \ &= \ \sum_{i=1}^m \left[\langle e_i(f)\eta, e_i\rangle + f\langle\nabla^N_{e_i}\eta, e_i\rangle\right] \\ &= \ -f\sum_{i=1}^m \langle\eta, B(e_i, e_i)\rangle \end{aligned}$$

$$= \quad -mf^2.$$

By the last two equations, we have the following

$$-(p-2)\,\mathrm{trace}_g\,\nabla\langle\nabla^{\mathbf{i}}\tau_p(\mathbf{i}), d\mathbf{i}\rangle|d\mathbf{i}|^{p-4}d\mathbf{i} = m^{p-1}(p-2)\left(\mathrm{grad}^M f^2 + mf^3\eta\right). \quad (5.15)$$

The third term of (5.13) is given by

$$\begin{aligned} -\,\mathrm{trace}_g\,\nabla^{\mathbf{i}}|d\mathbf{i}|^{p-2}\nabla^{\mathbf{i}}\tau_p(\mathbf{i}) &= -m^{p-1}\sum_{i=1}^{m}\nabla^N_{e_i}\nabla^N_{e_i}f\eta \\ &= -m^{p-1}\sum_{i=1}^{m}\nabla^N_{e_i}[e_i(f)\eta + f\nabla^N_{e_i}\eta] \\ &= -m^{p-1}\left[\Delta^M(f)\eta + 2\nabla^N_{\mathrm{grad}^M f}\eta + f\sum_{i=1}^{m}\nabla^N_{e_i}\nabla^N_{e_i}\eta\right]. \end{aligned} \quad (5.16)$$

Thus, at x, we obtain

$$\begin{aligned} \sum_{i=1}^{m}\nabla^N_{e_i}\nabla^N_{e_i}\eta &= \sum_{i=1}^{m}\nabla^N_{e_i}\left[(\nabla^N_{e_i}\eta)^{\perp} + (\nabla^N_{e_i}\eta)^{\top}\right] \\ &= -\sum_{i=1}^{m}\nabla^N_{e_i}A(e_i) \\ &= -\sum_{i=1}^{m}\nabla^M_{e_i}A(e_i) - \sum_{i=1}^{m}B(e_i, A(e_i)). \end{aligned} \quad (5.17)$$

Since $\langle A(X), Y\rangle = \langle B(X,Y), \eta\rangle$ for all $X, Y \in \Gamma(TM)$, we get

$$\begin{aligned} \sum_{i=1}^{m}\nabla^M_{e_i}A(e_i) &= \sum_{i,j=1}^{m}\langle\nabla^M_{e_i}A(e_i), e_j\rangle e_j \\ &= \sum_{i,j=1}^{m}\left[e_i\langle A(e_i), e_j\rangle e_j - \langle A(e_i), \nabla^M_{e_i}e_j\rangle e_j\right] \\ &= \sum_{i,j=1}^{m}e_i\langle B(e_i, e_j), \eta\rangle e_j \\ &= \sum_{i,j=1}^{m}e_i\langle\nabla^N_{e_j}e_i, \eta\rangle e_j \\ &= \sum_{i,j=1}^{m}\langle\nabla^N_{e_i}\nabla^N_{e_j}e_i, \eta\rangle e_j. \end{aligned} \quad (5.18)$$

By using the definition of curvature tensor of $(N^{m+1}, \langle , \rangle)$, we conclude

$$\begin{aligned}
\sum_{i=1}^{m} \nabla^M_{e_i} A(e_i) &= \sum_{i,j=1}^{m} \Big[\langle R^N(e_i,e_j)e_i, \eta\rangle e_j + \langle \nabla^N_{e_j}\nabla^N_{e_i} e_i, \eta\rangle e_j \Big] \\
&= \sum_{i,j=1}^{m} \Big[-\langle R^N(\eta,e_i)e_i, e_j\rangle e_j + \langle \nabla^N_{e_j}\nabla^N_{e_i} e_i, \eta\rangle e_j \Big] \\
&= -\sum_{j=1}^{m} \langle \operatorname{Ricci}^N \eta, e_j\rangle e_j + \sum_{i,j=1}^{m} e_j \langle \nabla^N_{e_i} e_i, \eta\rangle e_j - \sum_{i,j=1}^{m} \langle \nabla^N_{e_i} e_i, \nabla^N_{e_i}\eta\rangle e_j \\
&= -(\operatorname{Ricci}^N \eta)^\top + m \operatorname{grad}^M f. \qquad (5.19)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\sum_{i=1}^{m} B(e_i, A(e_i)) &= \sum_{i=1}^{m} \langle B(e_i, A(e_i)), \eta\rangle \eta \\
&= \sum_{i=1}^{m} \langle A(e_i), A(e_i)\rangle \eta \\
&= |A|^2 \eta. \qquad (5.20)
\end{aligned}$$

Substituting (5.17), (5.19) and (5.20) in (5.16), we obtain

$$\begin{aligned}
-\operatorname{trace}_g \nabla^{\mathbf{i}} |d\mathbf{i}|^{p-2} \nabla^{\mathbf{i}} \tau_p(\mathbf{i}) &= -m^{p-1} \big[\Delta^M(f)\eta - 2A(\operatorname{grad}^M f) + f(\operatorname{Ricci}^N \eta)^\top \\
&\quad - \frac{m}{2} \operatorname{grad}^M f^2 - f|A|^2\eta \big]. \qquad (5.21)
\end{aligned}$$

The Theorem 5.2.1 follows by (5.13)-(5.15), and (5.21). □

As an immediate consequence of Theorem 5.2.1 we have.

Corollary 5.2.1. *A hypersurface (M^m, g) in an Einstein space $(N^{m+1}, \langle , \rangle)$ is p-biharmonic if and only if it's mean curvature function f is a solution of the following PDEs*

$$\begin{cases}
-\Delta^M(f) + f|A|^2 + m(p-2)f^3 - \frac{S}{m+1} f = 0; \\
2A(\operatorname{grad}^M f) + (p - 2 + \frac{m}{2}) \operatorname{grad}^M f^2 = 0,
\end{cases} \qquad (5.22)$$

where S is the scalar curvature of the ambient space.

Proof. It is well known that if $(N^{m+1}, \langle , \rangle)$ is an Einstein manifold then $\operatorname{Ric}^N(X,Y) = \lambda\langle X, Y\rangle$ for some constant λ, for any $X, Y \in \Gamma(TN)$. So that

$$S = \operatorname{trace}_{\langle , \rangle} \operatorname{Ric}^N$$

$$
\begin{aligned}
&= \sum_{i=1}^{m} \mathrm{Ric}^N(e_i, e_i) + \mathrm{Ric}^N(\eta, \eta) \\
&= \lambda(m+1),
\end{aligned}
$$

where $\{e_i\}_{i=1,\dots,m}$ is a normal orthonormal frame on (M^m, g) at x. Since $\mathrm{Ric}^N(\eta, \eta) = \lambda$, on conclude that

$$\mathrm{Ric}^N(\eta, \eta) = \frac{S}{m+1}.$$

On the other hand, we have

$$
\begin{aligned}
(\mathrm{Ricci}^N \eta)^\top &= \sum_{i=1}^{m} \langle \mathrm{Ricci}^N \eta, e_i \rangle e_i \\
&= \sum_{i=1}^{m} \mathrm{Ric}^N(\eta, e_i) e_i \\
&= \sum_{i=1}^{m} \lambda \langle \eta, e_i \rangle e_i \\
&= 0.
\end{aligned}
$$

The Corollary 5.2.1 follows by Theorem 5.2.1. □

5.3 p-Biharmonic Hypersurface in Riemannian Manifold

Theorem 5.3.1. *A totally umbilical hypersurface (M^m, g) in an Einstein space $(N^{m+1}, \langle , \rangle)$ with non-positive scalar curvature is p-biharmonic if and only if it is minimal.*

Proof. Take an orthonormal frame $\{e_i, \eta\}_{i=1,\dots,m}$ on the ambient space $(N^{m+1}, \langle , \rangle)$ such that $\{e_i\}_{i=1,\dots,m}$ is an orthonormal frame on (M^m, g). We have

$$
\begin{aligned}
f &= \langle H, \eta \rangle \\
&= \frac{1}{m} \sum_{i=1}^{m} \langle B(e_i, e_i), \eta \rangle \\
&= \frac{1}{m} \sum_{i=1}^{m} \langle g(e_i, e_i)\beta\eta, \eta \rangle \\
&= \beta,
\end{aligned}
$$

where $\beta \in C^\infty(M)$. The p-biharmonic hypersurface equation (5.22) becomes

$$
\begin{cases}
-\Delta^M(\beta) + m(p-1)\beta^3 - \frac{S}{m+1}\beta = 0; \\
\\
(p-1+\frac{m}{2})\beta \,\mathrm{grad}^M \beta = 0,
\end{cases}
\tag{5.23}
$$

Solving the last system, we have $\beta = 0$ and hence $f = 0$, or

$$\beta = \pm\sqrt{\frac{S}{m(m+1)(p-1)}},$$

it's constant and this happens only if $S \geq 0$. The proof is complete. □

5.4 p-biharmonic hypersurface in conformally flat space

Let $\mathbf{i} : M^m \hookrightarrow \mathbb{R}^{m+1}$ be a minimal hypersurface with the unit normal vector field η, $\widetilde{\mathbf{i}} : (M^m, \widetilde{g}) \hookrightarrow (\mathbb{R}^{m+1}, \widetilde{h} = e^{2\gamma}h)$, $x \longmapsto \widetilde{\mathbf{i}}(x) = \mathbf{i}(x) = x$, where $\gamma \in C^\infty(\mathbb{R}^{m+1})$, $h = \langle , \rangle_{\mathbb{R}^{m+1}}$, and $\widetilde{g}$ is the induced metric by $\widetilde{h}$, that is

$$\widetilde{g}(X,Y) = e^{2\gamma}g(X,Y) = e^{2\gamma}\langle X, Y\rangle_{\mathbb{R}^{m+1}},$$

where g is the induced metric by h. Let $\{e_i, \eta\}_{i=1,\dots,m}$ be an orthonormal frame adapted to the p-harmonic hypersurface on $(\mathbb{R}^{m+1}, h)$, thus $\{\widetilde{e}_i, \widetilde{\eta}\}_{i=1,\dots,m}$ becomes an orthonormal frame on $(\mathbb{R}^{m+1}, \widetilde{h})$, where $\widetilde{e}_i = e^{-\gamma}e_i$ for all $i = 1, ..., m$, and $\widetilde{\eta} = e^{-\gamma}\eta$.

Theorem 5.4.1. *The hypersurface $(M^m, \widetilde{g})$ in the conformally flat space $(\mathbb{R}^{m+1}, \widetilde{h})$ is p-biharmonic if and only if*

$$\begin{cases} \eta(\gamma)e^{-\gamma}\big[-\Delta^M(\gamma) - m\operatorname{Hess}_\gamma^{\mathbb{R}^{m+1}}(\eta,\eta) + (1-m)|\operatorname{grad}^M \gamma|^2 \\ -|A|^2 + m(1-p)\eta(\gamma)^2\big] + \Delta^M(\eta(\gamma)e^{-\gamma}) + (m-2)(\operatorname{grad}^M \gamma)(\eta(\gamma)e^{-\gamma}) = 0; \\ -2A(\operatorname{grad}^M(\eta(\gamma)e^{-\gamma})) + 2(1-m)\eta(\gamma)e^{-\gamma}A(\operatorname{grad}^M \gamma) \\ +(2p-m)\eta(\gamma)\operatorname{grad}^M(\eta(\gamma)e^{-\gamma}) = 0, \end{cases} \tag{5.24}$$

where $\operatorname{Hess}_\gamma^{\mathbb{R}^{m+1}}$ *is the Hessian of the smooth function γ in $(\mathbb{R}^{m+1}, h)$.*

Proof. By using the Kozul's formula, we have

$$\begin{cases} \widetilde{\nabla}^M_X Y = \nabla^M_X Y + X(\gamma)Y + Y(\gamma)X - g(X,Y)\operatorname{grad}^M \gamma; \\ \widetilde{\nabla}^{\mathbb{R}^{m+1}}_U V = \nabla^{\mathbb{R}^{m+1}}_U V + U(\gamma)V + V(\gamma)U - h(U,V)\operatorname{grad}^{\mathbb{R}^{m+1}} \gamma, \end{cases}$$

for all $X, Y \in \Gamma(TM)$, and $U, V \in \Gamma(T\mathbb{R}^{m+1})$. Consequently

$$\nabla^{\widetilde{\mathbf{i}}}_X d\widetilde{\mathbf{i}}(Y) = \nabla^{\widetilde{\mathbf{i}}}_X Y$$

$$\begin{aligned} &= \widetilde{\nabla}^{\mathbb{R}^{m+1}}_{d\mathbf{i}(X)} Y \\ &= \widetilde{\nabla}^{\mathbb{R}^{m+1}}_{X} Y \\ &= \nabla^{\mathbb{R}^{m+1}}_{X} Y + X(\gamma)Y + Y(\gamma)X - h(X,Y)\operatorname{grad}^{\mathbb{R}^{m+1}}\gamma, \end{aligned} \tag{5.25}$$

and the following

$$\begin{aligned} d\widetilde{\mathbf{i}}(\widetilde{\nabla}^M_X Y) &= d\mathbf{i}(\nabla^M_X Y) + X(\gamma)d\mathbf{i}(Y) + Y(\gamma)d\mathbf{i}(X) - g(X,Y)d\mathbf{i}(\operatorname{grad}^M\gamma) \\ &= \nabla^M_X Y + X(\gamma)Y + Y(\gamma)X - g(X,Y)\operatorname{grad}^M\gamma. \end{aligned} \tag{5.26}$$

From equations (5.25) and (5.26), we get

$$\begin{aligned} (\nabla d\widetilde{\mathbf{i}})(X,Y) &= \nabla^{\widetilde{\mathbf{i}}}_X d\widetilde{\mathbf{i}}(Y) - d\widetilde{\mathbf{i}}(\widetilde{\nabla}^M_X Y) \\ &= (\nabla d\mathbf{i})(X,Y) + g(X,Y)[\operatorname{grad}^M\gamma - \operatorname{grad}^{\mathbb{R}^{m+1}}\gamma] \\ &= B(X,Y) - g(X,Y)\eta(\gamma)\eta. \end{aligned} \tag{5.27}$$

So that, the mean curvature function $\widetilde{f}$ of $(M^m,\widetilde{g})$ in $(\mathbb{R}^{m+1},\widetilde{h})$ is given by $\widetilde{f} = -\eta(\gamma)e^{-\gamma}$. Indeed, by taking traces in (5.27), we obtain

$$e^{2\gamma}\widetilde{H} = H - \eta(\gamma)\eta.$$

Since (M^m,g) is minimal in $(\mathbb{R}^{m+1},h)$, we find that $\widetilde{H} = -e^{-2\gamma}\eta(\gamma)\eta$, that is $\widetilde{H} = -e^{-\gamma}\eta(\gamma)\widetilde{\eta}$.

With the new notations the equation (5.12) for p-biharmonic hypersurface in the conformally flat space becomes

$$\begin{cases} -\widetilde{\Delta}(\widetilde{f}) + \widetilde{f}|\widetilde{A}|^2_{\widetilde{g}} - \widetilde{f}\,\widetilde{\operatorname{Ric}}^{\mathbb{R}^{m+1}}(\widetilde{\eta},\widetilde{\eta}) + m(p-2)\widetilde{f}^3 &= 0; \\ 2\widetilde{A}(\widetilde{\operatorname{grad}}^M\widetilde{f}) - 2\widetilde{f}(\widetilde{\operatorname{Ricci}}^{\mathbb{R}^{m+1}}\widetilde{\eta})^\top + (p-2+\dfrac{m}{2})\widetilde{\operatorname{grad}}^M\widetilde{f}^2 &= 0, \end{cases} \tag{5.28}$$

A straightforward computation yields

$$\begin{aligned} \widetilde{\operatorname{Ricci}}^{\mathbb{R}^{m+1}}\eta &= e^{-2\gamma}\big[\operatorname{Ricci}^{\mathbb{R}^{m+1}}\eta - \Delta^{\mathbb{R}^{m+1}}(\gamma)\eta + (1-m)\nabla^{\mathbb{R}^{m+1}}_\eta\operatorname{grad}^{\mathbb{R}^{m+1}}\gamma \\ &\quad +(1-m)|\operatorname{grad}^{\mathbb{R}^{m+1}}\gamma|^2\eta - (1-m)\eta(\gamma)\operatorname{grad}^{\mathbb{R}^{m+1}}\gamma\big]; \end{aligned}$$

$$\begin{aligned} \widetilde{\operatorname{Ric}}^{\mathbb{R}^{m+1}}(\widetilde{\eta},\widetilde{\eta}) &= \widetilde{h}(\widetilde{\operatorname{Ricci}}^{\mathbb{R}^{m+1}}\widetilde{\eta},\widetilde{\eta}) \\ &= h(\widetilde{\operatorname{Ricci}}^{\mathbb{R}^{m+1}}\eta,\eta) \\ &= e^{-2\gamma}h\big(\operatorname{Ricci}^{\mathbb{R}^{m+1}}\eta - \Delta^{\mathbb{R}^{m+1}}(\gamma)\eta + (1-m)\nabla^{\mathbb{R}^{m+1}}_\eta\operatorname{grad}^{\mathbb{R}^{m+1}}\gamma \\ &\quad +(1-m)|\operatorname{grad}^{\mathbb{R}^{m+1}}\gamma|^2\eta - (1-m)\eta(\gamma)\operatorname{grad}^{\mathbb{R}^{m+1}}\gamma,\eta\big) \end{aligned}$$

$$= e^{-2\gamma}\big[-\Delta^{\mathbb{R}^{m+1}}(\gamma)+(1-m)\operatorname{Hess}_{\gamma}^{\mathbb{R}^{m+1}}(\eta,\eta)+(1-m)|\operatorname{grad}^{\mathbb{R}^{m+1}}\gamma|^2 -(1-m)\eta(\gamma)^2\big]; \quad (5.29)$$

$$\begin{aligned}
(\widetilde{\operatorname{Ricci}}^{\mathbb{R}^{m+1}}\widetilde{\eta})^{\top} &= \sum_{i=1}^{m} h(\widetilde{\operatorname{Ricci}}^{\mathbb{R}^{m+1}}\widetilde{\eta},e_i)e_i\\
&= (1-m)e^{-3\gamma}\sum_{i=1}^{m}\Big[h(\nabla_{\eta}^{\mathbb{R}^{m+1}}\operatorname{grad}^{\mathbb{R}^{m+1}}\gamma,e_i)e_i-\eta(\gamma)h(\operatorname{grad}^{\mathbb{R}^{m+1}}\gamma,e_i)e_i\Big]\\
&= (1-m)e^{-3\gamma}\Big[\sum_{i=1}^{m}h(\nabla_{e_i}^{\mathbb{R}^{m+1}}\operatorname{grad}^{\mathbb{R}^{m+1}}\gamma,\eta)e_i-\eta(\gamma)\operatorname{grad}^{M}\gamma\Big]\\
&= (1-m)e^{-3\gamma}\Big[\sum_{i=1}^{m}e_ih(\operatorname{grad}^{\mathbb{R}^{m+1}}\gamma,\eta)e_i-\sum_{i=1}^{m}h(\operatorname{grad}^{\mathbb{R}^{m+1}}\gamma,\nabla_{e_i}^{\mathbb{R}^{m+1}}\eta)e_i\\
&\quad -\eta(\gamma)\operatorname{grad}^{M}\gamma\Big]\\
&= (1-m)e^{-3\gamma}\big[\operatorname{grad}^{M}\eta(\gamma)+\sum_{i=1}^{m}h(\operatorname{grad}^{\mathbb{R}^{m+1}}\gamma,Ae_i)e_i-\eta(\gamma)\operatorname{grad}^{M}\gamma\big]\\
&= (1-m)e^{-3\gamma}\big[\operatorname{grad}^{M}\eta(\gamma)+A(\operatorname{grad}^{M}\gamma)-\eta(\gamma)\operatorname{grad}^{M}\gamma\big]; \quad (5.30)
\end{aligned}$$

$$\begin{aligned}
\widetilde{\Delta}(\widetilde{f}) &= e^{-2\gamma}[\Delta(\widetilde{f})+(m-2)d\widetilde{f}(\operatorname{grad}^{M}\gamma)]\\
&= e^{-2\gamma}[-\Delta(\eta(\gamma)e^{-\gamma})-(m-2)(\operatorname{grad}^{M}\gamma)(\eta(\gamma)e^{-\gamma})]; \quad (5.31)
\end{aligned}$$

$$\begin{aligned}
|\widetilde{A}|_{\widetilde{g}}^2 &= \sum_{i=1}^{m}\widetilde{g}(\widetilde{A}\widetilde{e}_i,\widetilde{A}\widetilde{e}_i)\\
&= \sum_{i=1}^{m}g(\widetilde{A}e_i,\widetilde{A}e_i)\\
&= \sum_{i=1}^{m}h(\widetilde{\nabla}_{e_i}^{\mathbb{R}^{m+1}}\widetilde{\eta},\widetilde{\nabla}_{e_i}^{\mathbb{R}^{m+1}}\widetilde{\eta})\\
&= \sum_{i=1}^{m}h(\nabla_{e_i}^{\mathbb{R}^{m+1}}\widetilde{\eta}+e_i(\gamma)\widetilde{\eta}+\widetilde{\eta}(\gamma)e_i,\nabla_{e_i}^{\mathbb{R}^{m+1}}\widetilde{\eta}+e_i(\gamma)\widetilde{\eta}+\widetilde{\eta}(\gamma)e_i)\\
&= \sum_{i=1}^{m}\big[h(\nabla_{e_i}^{\mathbb{R}^{m+1}}\widetilde{\eta},\nabla_{e_i}^{\mathbb{R}^{m+1}}\widetilde{\eta})+2\widetilde{\eta}(\gamma)h(\nabla_{e_i}^{\mathbb{R}^{m+1}}\widetilde{\eta},e_i)+e_i(\gamma)^2e^{-2\gamma}\\
&\quad +2e_i(\gamma)h(\nabla_{e_i}^{\mathbb{R}^{m+1}}\widetilde{\eta},\widetilde{\eta})\big]+m\widetilde{\eta}(\gamma)^2. \quad (5.32)
\end{aligned}$$

The first term of (5.32) is given by

$$\sum_{i=1}^{m}h(\nabla_{e_i}^{\mathbb{R}^{m+1}}e^{-\gamma}\eta,\nabla_{e_i}^{\mathbb{R}^{m+1}}e^{-\gamma}\eta) = \sum_{i=1}^{m}h(-e^{-\gamma}e_i(\gamma)\eta+e^{-\gamma}\nabla_{e_i}^{\mathbb{R}^{m+1}}\eta,-e^{-\gamma}e_i(\gamma)\eta+e^{-\gamma}\nabla_{e_i}^{\mathbb{R}^{m+1}}\eta)$$

$$\begin{aligned} &= \sum_{i=1}^{m}[e^{-2\gamma}e_i(\gamma)^2 + e^{-2\gamma}h(\nabla^{\mathbb{R}^{m+1}}_{e_i}\eta, \nabla^{\mathbb{R}^{m+1}}_{e_i}\eta)] \\ &= e^{-2\gamma}|\operatorname{grad}^M \gamma|^2 + e^{-2\gamma}|A|^2. \end{aligned}$$

The second term of (5.32) is given by

$$\begin{aligned} 2\widetilde{\eta}(\gamma)\sum_{i=1}^{m} h(\nabla^{\mathbb{R}^{m+1}}_{e_i}\widetilde{\eta}, e_i) &= -2e^{-\gamma}\eta(\gamma)\sum_{i=1}^{m} h(e^{-\gamma}\eta, \nabla^{\mathbb{R}^{m+1}}_{e_i} e_i) \\ &= -2me^{-2\gamma}\eta(\gamma)h(\eta, H) \\ &= 0. \end{aligned}$$

Here $H = 0$. We have also

$$\begin{aligned} 2\sum_{i=1}^{m} e_i(\gamma)h(\nabla^{\mathbb{R}^{m+1}}_{e_i}\widetilde{\eta}, \widetilde{\eta}) &= \sum_{i=1}^{m} e_i(\gamma)e_i h(\widetilde{\eta}, \widetilde{\eta}) \\ &= \sum_{i=1}^{m} e_i(\gamma)e_i(e^{-2\gamma}) \\ &= -2e^{-2\gamma}\sum_{i=1}^{m} e_i(\gamma)^2 \\ &= -2e^{-2\gamma}|\operatorname{grad}^M \gamma|^2. \end{aligned}$$

Thus

$$|\widetilde{A}|^2_{\widetilde{h}} = e^{-2\gamma}|A|^2 + me^{-2\gamma}\eta(\gamma)^2. \tag{5.33}$$

We compute

$$\begin{aligned} \widetilde{\operatorname{grad}}^M \widetilde{f} &= e^{-2\gamma}\sum_{i=1}^{m} e_i(\widetilde{f})e_i \\ &= -e^{-2\gamma}\operatorname{grad}^M(\eta(\gamma)e^{-\gamma}); \end{aligned} \tag{5.34}$$

and the following

$$\begin{aligned} \widetilde{A}(\widetilde{\operatorname{grad}}^M \widetilde{f}) &= -\widetilde{\nabla}^{\mathbb{R}^{m+1}}_{\widetilde{\operatorname{grad}}^M \widetilde{f}}\widetilde{\eta} \\ &= -\widetilde{\nabla}^{\mathbb{R}^{m+1}}_{\widetilde{\operatorname{grad}}^M \widetilde{f}} e^{-\gamma}\eta \\ &= e^{-\gamma}(\widetilde{\operatorname{grad}}^M \widetilde{f})(\gamma)\eta - e^{-\gamma}\widetilde{\nabla}^{\mathbb{R}^{m+1}}_{\widetilde{\operatorname{grad}}^M \widetilde{f}}\eta \\ &= -e^{-3\gamma}\operatorname{grad}^M(\eta(\gamma)e^{-\gamma})(\gamma)\eta + e^{-3\gamma}\widetilde{\nabla}^{\mathbb{R}^{m+1}}_{\operatorname{grad}^M(\eta(\gamma)e^{-\gamma})}\eta \\ &= -e^{-3\gamma}\operatorname{grad}^M(\eta(\gamma)e^{-\gamma})(\gamma)\eta + e^{-3\gamma}\eta(\gamma)\operatorname{grad}^M(\eta(\gamma)e^{-\gamma}) \\ &\quad + e^{-3\gamma}\operatorname{grad}^M(\eta(\gamma)e^{-\gamma})(\gamma)\eta + e^{-3\gamma}\nabla^{\mathbb{R}^{m+1}}_{\operatorname{grad}^M(\eta(\gamma)e^{-\gamma})}\eta \end{aligned}$$

$$= \quad e^{-3\gamma}\eta(\gamma)\,\mathrm{grad}^M(\eta(\gamma)e^{-\gamma}) - e^{-3\gamma}A(\mathrm{grad}^M\,\eta(\gamma)e^{-\gamma}). \qquad (5.35)$$

Substituting (5.29) – (5.35) in (5.28), and by simplifying the resulting equation we obtain the system (5.24). □

Remark 5.4.1.

1. *Using Theorem 5.4.1, we can construct many examples for proper p-biharmonic hypersurfaces in the conformally flat space.*
2. *If the functions γ and $\eta(\gamma)$ are non-zero constants on M, then according to Theorem 5.4.1, the hypersurface $(M^m, \widetilde{g})$ is p-biharmonic in $(\mathbb{R}^{m+1}, \widetilde{h})$ if and only if*
$$|A|^2 = m(1-p)\eta(\gamma)^2 - m\eta(\eta(\gamma)).$$

Example 5.4.1. *The hyperplane $\mathbf{i} : \mathbb{R}^m \hookrightarrow (\mathbb{R}^{m+1}, e^{2\gamma(z)}h)$, $x \longmapsto (x, c)$, where $\gamma \in C^\infty(\mathbb{R})$, $h = \sum_{i=1}^m dx_i^2 + dz^2$, and $c \in \mathbb{R}$, is proper p-biharmonic if and only if $(1-p)\gamma'(c)^2 - \gamma''(c) = 0$. Note that, the smooth function*

$$\gamma(z) = \frac{\ln\left(c_1(p-1)z + c_2(p-1)\right)}{p-1}, \quad c_1, c_2 \in \mathbb{R},$$

is a solution of the previous differential equation (for all c).

Example 5.4.2. *Let M be a surface of revolution in $\{(x,y,z) \in \mathbb{R}^3 \,|\, z > 0\}$. If M is part of a plane orthogonal to the axis of revolution, so that M is parametrized by*

$$(x_1, x_2) \longmapsto (f(x_2)\cos(x_1), f(x_2)\sin(x_1), c),$$

for some constant $c > 0$. Here $f(x_2) > 0$. Then, M is minimal, and according to Theorem 5.4.1, the surface M is proper p-biharmonic in 3-dimensional hyperbolic space $(\mathbb{H}^3, z^{\frac{2}{p-1}}h)$, where $h = dx^2 + dy^2 + dz^2$.

Open Problems.

1. If M is a minimal surface of revolution contained in a catenoid, that is M is parametrized by
$$(x_1, x_2) \longmapsto \left(a\cosh\left(\frac{x_2}{a} + b\right)\cos(x_1), a\cosh\left(\frac{x_2}{a} + b\right)\sin(x_1), x_2\right),$$
where $a \neq 0$ and b are constants. Is there $p \geq 2$ and $\gamma \in C^\infty(\mathbb{R}^3)$ such that M is proper p-biharmonic in $(\mathbb{R}^3, e^{2\gamma}(dx^2 + dy^2 + dz^2))$?
2. Is there a proper p-biharmonic submanifolds in Euclidean space $(\mathbb{R}^n, dx_1^2 + ... + dx_n^2)$?

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