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REPUBLIQUE ALGERIENNE DEMOCRATIQUE ET POPULAIRE

وزارة الـتـعـلـيـم الـعـالـي و الـبـحـث الـعـلـمـي

MINISTERE DE L'ENSEIGNEMENT SUPERIEUR ET DE LA RECHERCHE SCIENTIFIQUE

UNIVERSITÉ DE MUSTAFA STAMBOULI

MASCARA

FACULTÉ DES SCIENCES EXACTES

DEPARTEMENT DE MATHEMATIQUES

جامعة مصطفى اسطمبولي

مـعـسـكـر

كـلـيـة الـعـلـوم الـدقـيـقـة

قسم الرياضيات

Thèse de doctorat

Spécialité : *Mathématiques*

Option : *Analyse Mathématique*

Sujet de la thèse :

Study of the global existence and stabilization of some evolutions problems

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La soutenance sera prévue le: 05/06/2023

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L'Année Universitaire: 2022-2023

Etude de l'existence globale et de la stabilisation de quelques problèmes d'évolution

Study of the global existence and stabilization of some evolutions problems

Figure 1: Université Mustapha Stambouli de Mascara. Faculté des Sciences Exactes

Figure 2: Laboratoire d'Analyse et Contrôle des Equation aux Dérévées Partielles .Université Djillali Liabes , Sidi Bel Abbes

Remerciements

Je tiens tout d'abord à exprimer ma reconnaissance à mon directrice de recherche, **Mdm.ABDELLI Mama**. d'avoir dirigé cette thèse. Grâce à ses conseils, ses orientations et son humanisme.
Je la remercie aussi pour ses conseils, ses remarques et sa très grande disponibilité.
Mes remerciements vont profondment mon co-directeur de thèse, **Mr. BAHLIL Mounir** , pour toutes ses aides et son soutien continu pendant mon doctorat.

Je tiens vraiment à remercier **Mr.Ahmed MOHAMMED CHERIF** qui me fait un grand honneur en tant que Président du jury de cette thèse.
Je remercie également **Mr.Abbes BENAISSA** d'avoir accepté d'evaluer ce travail et de faire partie de mon jury.
Je remercie chaleureusement **Mr.Ammar KHEMMOUDJ** et **Mr.Ahmed BCHATNIA** qui me font l'honneur d'être dans le jury de ma soutenance.

Je tiens également remercier tous mes collègues du laboratoire d'Analyse et Contrôle des EDP. Université Djillali Liabes, Sidi Bel Abbes.

Enfin, j'adresse mes remerciements et ma gratitude à ma famille : mes parents, ma soeur et mes frères.

Contents

ملخص

تركز هذه الأطروحة على دراسة الوجود العام والسلوك في وقت طويل لحلول بعض معادلات التطور ...
يتكون هذا العمل من أربعة فصول في الفصل الأول ، قدمنابعض المفاهيم الأساسية حول فضاءات سوبولاف و .بعض النظريات الرئيسية في التحليل الدالي في الفصل الثاني ، نقوم بدراسة معادلة الصفيحة اللاخطية في وجود معامل اضمحلال لزج مطاطي و معامل اضمحلال آخر قوي يتسبب في وجود تبديد لطاقة المرفقة للمعادلة بشكل عام وبالاعتماد على طريقة فايدو غلاركين لقد حصلنا على الوجود للحلول العامة في الفضاءات الدالية سوبولاف بالإضافة ،إلى ، إعطاء الاستقرار العام الحلول بواسطة طريقة ليابونوف ،بالاعتماد وجود دالة الإسترخاء . مع بعض خصائص الدوال محدبة في الفصل الثالثة ، درسنا الوجود العام وانتظام الحل والاستقرار الداخلي .من معادلة بتروفسكي غير الخطية عن طريق التخميد القوي الموزع محل ً يا قد برهنا أنه ، في ظل ظروف معينة على معامل التخميد الذي يسمح بالانعدام الاضمحلال تقبل هذه المعادلة حل وحيدا بالاعتماد على نظريات نصف الزمرة ،اما بالاستخدام طريقة فايدة غ ، فيΩ مجال فرعي لـ لاركين توصلنا الى أن الحل يكون اكثر انتظاما . باستخدام طريقة المضاعفاا متعددة القطع ، أوضحنا أن الطاقة .تتناقص بشكل سريع جدا)اسي (ومتعددة الحدود نحو الصفر ، في ظل ظروف هندسية معينة أخي ً را ، في الفصل الرابع ، درسنا نظا ً ما مزدو ً جا يتكون من معادلة كيرشوف ومعادلة الحرارة في مجال محدود. لقد برهنا وجود حل عام وحيد ،بالاعتمادا دائما على طريقة فايدو غلاركين التقريبية. و استخدمنا طريقة المضاعفات لإيجاد استقرار عام

Résumé

Cette thèse porte essentiellement sur l'étude d'existence globale et du comportement en temps long des solutions de certaines équations d'évolution. Ce travail se compose de quatre chapitres. Dans le premier chapitre, on a introduit quelques notions de base sur les espaces de Sobolev et quelques théormes principaux en analyse fonctionnelle. Dans le deuxime chapitre, on a considéré l'équation de plaque non linéaire en présence de termes dissipatifs: un terme dissipatif viscoélastique et un terme dissipatif fort et de forme générale. Par la méthode de Faedo-Galerkin on a obtenu l'existence globale des solutions dans des espaces de Sobolev. De plus, sous des conditions sur la fonction de relaxation la stabilité générale est donnée par la méthode de Lyapunov combinée avec certaines propriétés des fonctions convexes. Dans le troisième chapitre, on a étudié l'existence globale, la régularité de la solution et la stabilisation interne de l'équation de Petrovsky non linéaire par une force d'amortissement localement distribué. On a montré que, sous certaines conditions sur le coefficient $a(x)$ qui permettent à celui-ci d'être nul sur un sous-domaine de Ω, ce problème admet une unique solution par la théorie des semi-groupes et par la méthode de Faedo-Galerkin on a trouvé que la solution est rgulière. A l'aide d'une méthode de multiplicateus par morceaux on a prouvé que l'énergie de la solution décroît exponentillement et polynomiallement vers zéro, sous des conditions géométriques. Finalement, dans le quatrième chapitre, on a considéré un système couplé constitué d'une équation de Kirchhoff et d'une équation de la chaleur dans un domaine bornée. On a montré l'existence et l'unicité d'une solution globale en se basant sur les approximations de Faedo-Galarkin. Et on a utilisé la méthode des multiplicateurs pour trouver une stabilité générale.

Mots clés: Equation de Petrovsky, equation de Kirchhoff, equation de la chaleur, Terme d'amortissement non-linéaire fort localement distibué, terme viscoélastique, existence globale décroissance exponentiel, décroissance polynomiale, décroissance générale, méthode de Faedo-Galerkin, théories des semi-groupes, la méthode de Lyapunov, la métode de multiplicateurs,

Abstract

This thesis focuses on the study of global existence and long-time behavior of the solutions of certain evolution equations. This work consists of four chapters. In the first chapter, we introduced some basic notions on Sobolev spaces and some main theorems in functional analysis. In the second chapter, we considered the nonlinear plate equation in the presence of dissipative terms: a viscoelastic dissipative term and a strong dissipative term of general form. By the Faedo-Galerkin method we have obtained the global existence of solutions in Sobolev spaces. Moreover, under conditions on the relaxation function the general stability is given by Lyapunov's method combined with some properties of convex functions. In the third chapter, we studied the global existence, the regularity of the solution and the internal stabilization of the nonlinear Petrovsky equation by a strong locally distributed damping. We showed that, under certain conditions on the damping term $a(x)$ which allow it to be zero on a subdomain of Ω, this problem admits a unique solution by the theory of semigroup and by the Faedo-Galerkin method we find that the solution is regular. Using a piecewise multiplier method, we proved the energy decreases exponentially and polynomially towards zero, under geometric conditions. Finally, in the fourth chapter, we considered a coupled system consisting of the Kirchhoff equation and the heat equation in a bounded domain. We showed the existence and uniqueness of a global solution based on the Faedo-Galarkin approximations. And we used the method of multipliers to find a general stability.

Keywords: Petrovsky equation, Kirchhoff equation, Heat equation, nonlinear localized strong damping, viscoelastic term, global existence, exponential stability, polynomial stabilization, general decay, Faedo-Galerkin method, semi-groups theory, Lyapunov method, multiplier method.

Publications

1. H. M. Braiki, M. Abdelli and Kh. Zennir, *Well-posedness and stability for a Petrovsky equation with properties of nonlinear localized for strong damping,* Mathematical Methods in the Applied Sciences, **44**(5), 3568-3587, (2021).

2. H. M. Braiki. M. Abdelli. N, Louhibi and A. Hakem, *Well-posedness and general decay of solutions for a Petrovsky equation with a memory term,* Analele Universitătii Oradea, Issue No.1, 31-45, (2022).

3. H. M. Braiki, M. Abdelli and S. Mansouri, *Well-posedness and exponential stability of coupled non-degenrate Kirchhoff equation and the heat equation,* . Applicable Analysis, 1-15,(2023).

List of symbols

Ω: Bounded domain in $\mathbb{R}^n$.
Γ: Topological boundary of Ω.
$x = (x_1, x_1, ..., x_n)$:Generic point of $\mathbb{R}^n$.
$d\,x = d\,x_1 d\,x_2 ... d\,x_n$: Lebesgue measuring on Ω.
$\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_n}\right)$: Gradient of u.
$\Delta u = \sum_{i=1}^{i=n} \frac{\partial^2 u}{\partial x_i^2}$: Laplacien of u.
a.e: Almost everywhere.
$p^{'}$: Conjugate of p, i.e $\frac{1}{p} + \frac{1}{p'} = 1$.
$\mathcal{C}(\Omega)$: Space of real continuous functions on Ω.
$\mathcal{C}^k(\Omega), k \in \mathbb{N}$: Space of k times continuously differentiable functions on Ω.
$\mathcal{C}_0^\infty(\Omega) = \mathcal{D}(\Omega)$: Space of differentiable functions with compact support on Ω.
$\mathcal{D}'(\Omega)$: Distribution space on Ω.
$D^\alpha u = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \ldots \partial_n^{\alpha_n} u = \frac{\partial^{\alpha_1 + \ldots + \alpha_n} u}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$
$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n, |\alpha| = \alpha_1 + \ldots + \alpha_n$

General introduction

This thesis presents a wide-ranging survey of many important topics in partial differential equations theory, in particular we study the well-posedness and stability of some evolutions problems, we discuss the well-posedness and regularity of the solutions of the system by using the nonlinear semigroup theory and the Faedo Galerkin scheme, we show the exponential and polynomial stabilities by multiplied method and Lyapunov function.

0.1 Time-Delay

Time-delay (memory term) often appears in many real-world engineering systems either in the state, the control input, or the measurements. Delays are strongly involved in challenging areas of communication and information technologies: in stabilization of networked controlled systems and in high-speed communication networks. Time-delay is, in many cases, a source of instability. However, for some systems, the presence of delay can have a stabilizing effect. The stability analysis and robust control of timedelay systems (TDSs) are, therefore, of theoretical and practical importance.
Time-Delay Systems (TDSs) are also called systems with aftereffect or dead-time, hereditary systems, equations with deviating argument, or differential-difference equations. They belong to the class of functional differential equations which are infinite-dimensional, as opposed to ordinary differential equations (ODEs). The simplest example of such a system is

$$\dot{u}(t) = -u(t-s) \quad u(t) \in \mathbb{R}$$

where $s > 0$ is the time-delay. Time-delay often appears in many control systems (such as aircraft, chemical or process control systems, and communication networks), either in the state, the control input, or the measurements. There can be transport, communication, or measurement delays.

An example of time-delay system

A simple example of TDS is described as follows. Imagine a showering person wishing to achieve the desired value T_d of water temperature by rotating the mixer handle for cold and hot water [35]. Let $u(t)$ denote the water temperature in the mixer output and let s be the constant time needed by the water to go from the mixer output to the person's head (see Fig.3). Assume that the change of the temperature is proportional to the angle of rotation of the handle, whereas the rate of rotation of the handle is proportional to $T(t) - T_d$. At time t the person feels the water temperature leaving the mixer at time $t - s$, which results in the following equation with the constant delay h:

$$\dot{u}(t) = -k[u(t-s) - T_d], \quad k \in \mathbb{R}.$$

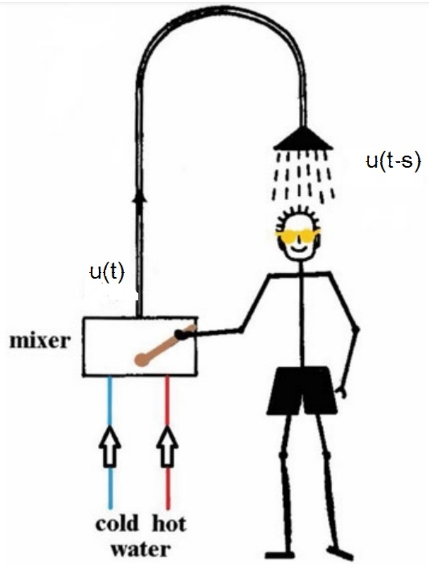

Figure 3:

0.2 The Kirchhoff equation

The mathematical description of transversal small vibrations of elastic string, fixed at the ends, is an old question. The first investigations on this problem were done by d'Alembertt $(1717-1793)$ and Euler (1707-1783). We consider an orthogonal Cartesian coordinate system (x, u) in $\mathbb{R}^2$. Suppose that the string, in the rest position, is on the x axis with fixed ends at the points M and N. If $u(x,t)$ is the vertical displacement of a point X of the string, with coordinate x, at time t, the mathematical model proposed by d'Alembert, in the modern notation, is:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$

where $c^2 = \dfrac{P_0}{\rho}$, with P_0 the initial tension and ρ the mass of the string MN.
D'Alembert observed that the configurations of the displacement of the string are given by:

$$u(x,t) = \Phi(x+ct) + \Psi(x-ct),$$

where Φ and Ψ , after d'Alembert, are arbitrary functions.
To obtain the d'Alembert model we impose many restrictions on the physical problem. Another model for the same physical problem of the vertical displacement of the elastic strings was proposed by Kirchhoff [31] and Carrier [17], which we will find, as a particular case of moving ends, in the next section. If P_0 is the initial tension, that is, the tension at the rest position, the Kirchhoff-Carrier model for small vertical vibration of elastic string, with fixed ends, is:

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{\rho\kappa} + \frac{E\kappa}{2L\rho} \int_0^L \left| \frac{\partial u}{\partial x}(x.t) \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0 \tag{0.1}$$

where $0 \leq x \leq L$ and $t > 0$, represent the string in repose, $u(x,t)$ is the vertical displacement of the point x at the instant t, ρ the mass density, κ is the area of the cross section of the

string, L is the lenght of the string, P_0 the initial tension on the string and E the Young's modulus of the material.
The natural generalization of the model (0.1) is given by the following nonlinear mixed problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - M\left(\displaystyle\sum_{i=1}^{n}\int_\Omega \left|\frac{\partial u}{\partial x}(x.t)\right|^2 dx\right)\Delta u = f & on \;\; \Omega\times(0,T) \\ u = 0 & on\; \mathit{\Gamma} \;\times\; (0,T) \\ u(x,0) = \phi_0(x) & on \;\; \Omega, \\ \dfrac{\partial u}{\partial t}(x,0) = \phi_1(x) & on \;\; \Omega, \end{cases} \tag{0.2}$$

where Ω is a bounded open set of $\mathbb{R}^n$ with smooth boundary Γ, $M : [0;\infty) \to \mathbb{R}$ is a positive real function and $\Delta = \sum_{i=1}^{n} \dfrac{\partial^2}{\partial x_i^2}$ is the Laplace operator. We say that a problem is :

1) Coercive if $M(r) \geq \nu > 0$ for each $r \geq 0$.
2) Coercive at ∞ if $M(r) \geq 0$ for each $r \geq 0$, and $\int_0^\infty M(r)dr = \infty$.
3) Mildly degenerate if $M(\|\nabla\phi_0(x)\|_2^2) > 0$.
4) Really degenerate if $M(\|\nabla\phi_0(x)\|_2^2) = 0$.

In the Kirchhoff-Carrier model (0.1) , $M : [0;\infty) \to \mathbb{R}$ *is* $M(\lambda) = \dfrac{P_0}{\rho\kappa} + \dfrac{E}{2L\rho\kappa}\lambda$. Several authors have investigated the nonlinear problem (0.2). When $n = 1$ *and* $\Omega = (0;L)$, it was studied by Dickey [23] and Bernstein [10] whom considered ϕ_0 and ϕ_1 analytic functions with some growth conditions. Assuming Ω bounded open set of $\mathbb{R}^n$, ϕ_0 and ϕ_1 analytic functions, Pohozaev[60] obtained existence and uniqueness of global solutions for the mixed problem (0.2). In Lions [39], he formulated the Pohozaevs results in an abstract context obtaining better results and presenting a collection of problems. One of the problems proposed by Lions [39] was the study of the problem (0.2) with $M : \Omega \times [0;\infty) \to \mathbb{R}$, i.e., the problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - M\left(\displaystyle\sum_{i=1}^{n}\int_\Omega \left|\frac{\partial u}{\partial x}(x.t)\right|^2 dx\right)\Delta u = f & on \;\; \Omega\times(0,T) \\ u = 0 & on\;\; \mathit{\Gamma} \;\times\; (0,T) \\ u(x,0) = \phi_0(x) & on \;\; \Omega, \\ \dfrac{\partial u}{\partial t}(x,0) = \phi_1(x) & on \;\; \Omega, \end{cases} \tag{0.3}$$

that is, for nonhomogeneous materials. This case has its origin in the model (0.1) when the physic elements ρ, κ and E are not constants, but depends on the point x in the string. In Rivera Rodrigues [62] the author proved the existence and uniqueness of local solutions for the problem (0.3).
In a more general context it is correct to consider ρ, h and E changing not only with the point x in the string but with the instant t too, i.e., $\rho = \rho(x;t)$, $\kappa = \kappa(x;t)$ and $E = E(x;t)$. In this case, we have the problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - M\left(x, t, \displaystyle\sum_{i=1}^{n} \int_\Omega \left|\frac{\partial u}{\partial x}(x.t)\right|^2 dx\right) \Delta u = f & on \;\; \Omega \times (0,T) \\ u = 0 & on \;\; \Gamma \;\; \times \;\; (0,T) \\ u(x,0) = \phi_0(x) & on \;\; \Omega, \\ \dfrac{\partial u}{\partial t}(x,0) = \phi_1(x) & on \;\; \Omega, \end{cases} \tag{0.4}$$

where $M : [0;\infty) \times [0,T] \times [0,\infty) \to \mathbb{R}$. The problem (0.4) were treated by Cicero Lopes Frota they making use of the same technique used by Rivera Rodrigues [62], they proved that if ϕ_0, ϕ_1, f *and* $\dfrac{\partial M}{\partial t}$ are small in some sense, then exist one, and only one, nonlocal solution for the problem (0.4). It's important to observe that it's a good assumption to consider $\dfrac{\partial M}{\partial t}$ small, because in normal conditions ρ, κ and E have a small variation with the time. For the study of problem (0.2) with dissipative terms we have, for instance, Brito [16] and Medeiros-Milla Miranda [51]. The problem (0.2) in the degenerate case can be find in Arosio-Spagnolo [6], Ebihara-Medeiros-Milla Miranda [24], Arosio-Garavaldi [5], Crippa [21], Yamada [70], Nishihara-Yamada [55] and Nishihara [56].

0.3 Thesis overview

This thesis is divided into four chapters.

Chapter 1: Notations and Preliminaries

In the first chapter, we collect some notions and results of functional analysis as well as some technical methods used to establish either existence or stability of some nonlinear evolution problems. These results are needed to develop further arguments.

Chapter 2: Well-posedness and general decay of solutions for a Petrovsky equation with a memory term

Let Ω is a bounded domain in $\mathbb{R}^n$, having a boundary $\Gamma = \partial\Omega$. Now consider a viscoelastic Petrovsky equation in a bounded domain with a nonlinear strong damping

$$\begin{cases} u_{tt} + \Delta^2 u - \displaystyle\int_0^t h(t-s)\Delta^2 u(s)ds - g(\Delta u_t) = 0, & x \in \Omega \times [0, +\infty[, \\ u = \Delta u = 0, & x \in \Gamma \times [0, \infty[, \\ u(x,0) = u_0(x), \,, \; u_t(x,0) = u_1(x) & x \in \Omega \times [0, +\infty[. \end{cases} \tag{0.5}$$

In the absence of the viscoelastic term (i.e. if $h = 0$), problem (0.5) has been investigated in [**32**] by Komornik, he showed that the well-posedness by the semigroup method. Then, using the multiplier technique, he directly proved exponential and polynomial decay estimates for the associated energy. When the damping term is general and without the memory term, Lakroumbe et al. [**44**] showed the global existence of weak solutions using the Faedo-Galerkin method and obtained general stability estimates by introducing Lyapunov method combined with some properties of convex functions.

In this chapter, we prove a global existence result using the energy method combined with the Faedo- Galerkin. Meanwhile, under suitable conditions on relaxation function $h(.)$, we study the asymptotic behaviour of solutions using a perturbed energy method and some properties of the convex functions, the general Young inequality and Jensen's inequality. We make the following hypotheses on the relaxation function and the damping function

(A1) Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a $\mathcal{C}^2$ real function such that $h(0) = h_0 > 0$ and

$$l = \int_0^\infty h(s)\ ds < 1$$

There exists a non-increasing differentiable function $\nu : \mathbb{R}_+ \to \mathbb{R}_+$ such that $h'(s) \leq -\nu(s)h(s), \forall s \geq 0$ and

$$\int_0^\infty \nu(s)\ ds = +\infty.$$

(A2) Consider $g : \mathbb{R} \to \mathbb{R}$ a non-decreasing $\mathcal{C}^1(\mathbb{R})$ function such that

$$g(v)v > 0, \text{ for all } v \neq 0,$$

and there exist constants $\varepsilon, c_1, c_2 > 0$ and a convex increasing function $G : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ of class $\mathcal{C}^1(\mathbb{R}_+) \cap \mathcal{C}^2(\mathbb{R}_+^*)$ satisfying G linear on $[0, \varepsilon]$ or ($G'(0) = 0$ and $G'' > 0$ on $]0, \varepsilon]$, such that

$$c_1 |s| \leqslant |g(s)| \leqslant c_2 |s|, \text{ if } |s| > \varepsilon,$$
$$|s|^2 + |g(s)|^2 \leqslant G^{-1}(sg(s)), \text{ if } |s| \leqslant \varepsilon.$$

Let us introduce for brevity the Hilbert spaces

$$\mathcal{H} = H_0^1(\Omega), \quad V = \{v \in H^3(\Omega) | v = \Delta v = 0 \text{ on } \Gamma\},$$

and

$$W = \{v \in H^5(\Omega) | v = \Delta v = \Delta^2 v = 0 \text{ on } \Gamma\}.$$

Introduce the energy

$$E(t) = \frac{1}{2}\|\nabla u_t\|^2 + \frac{1}{2}(h \circ \nabla\Delta u)(t) + \frac{1}{2}\Big(1 - \int_0^t h(s)ds\Big)\|\nabla\Delta u\|^2,$$

then E(t) is a nonincreasing function for $t > 0$ and

$$E'(t) = -\frac{1}{2}h(t)\|\nabla\Delta u\|^2 + \frac{1}{2}h' \circ \nabla\Delta u(t) - \int_\Omega g(\Delta u_t)\Delta u_t\ dx \leq 0.$$

where

$$(h \circ v)(t) = \int_0^t h(t-s)\|v(t) - v(s)\|^2\, ds.$$

We are now in the position to state our results:

Theorem 0.3.1. *(Well-posedness) Assume that*

$$(u_0, u_1) \in W \times V,$$

then the solution of the problem (0.5) *satisfies*

$$u_t \in L^\infty(0, T; V) \ ; \ u_{tt} \in L^\infty(0, T; \mathcal{H})$$

and

$$u \in L^\infty(0, T; H^4(\Omega) \cap V),$$

such that for any $T > 0$.

Theorem 0.3.2. *(Stabilization) Assume that (A1) and (A2) hold. Then there exist positive constants* k_0 *and* k_1 *such that the solution of the problem* (0.5) *satisfies*

$$E(t) \leq k_0 G_1^{-1} \left(k_1 \int_0^t \nu(s)\, ds \right), \quad \forall\, t \in \mathbb{R}_+,$$

where

$$G_1(t) = \int_t^1 \frac{1}{G_2(s)}\ ds$$

and

$$G_2(t) = \begin{cases} t, & \textit{if } G \textit{ is linear on } [0, \varepsilon] \\ tG'(\varepsilon_0 t), & \textit{if } G'(0) = 0 \textit{ and } G'' > 0 \textit{ on }]0, \varepsilon] \end{cases}$$

Chapter 3: Well-posedness and stability for a Petrovsky equation with properties of nonlinear localized for strong damping

We consider a locally damped Petrovsky equation in a bounded domain. The damping is nonlinear, and is localized in a suitable open subset of the domain under consideration.

$$\begin{cases} u_{tt} + \Delta^2 u - a(x) g(\Delta u_t) = 0, & (x,t) \in \Omega \times [0, +\infty[\\ u = \Delta u = 0, & (x,t) \in \Gamma \times [0, \infty[\\ u(x,0) = u^0(x), \ u_t(x,0) = u^1(x), & x \in \Omega, \end{cases} \tag{0.6}$$

where Ω is a bounded domain of $\mathbb{R}^n$ with boundary Γ which assumed to be regular.

L. Tebou [65] considered a wave equation with a nonlinear strong damping term localized in a neighborhood of a suitable subset of the domain under consideration, he proved the well-posedness and regularity of the solutions of the system by using a combination of the nonlinear semigroup theory and the Faedo-Galerkin scheme. Then, using the energy method combined with the piecewise multipliers method, he investigated the exponential decay of the energy when the nonlinear damping grows linearly. When $g(\Delta u_t) = |\Delta u_t|^{p-2} \Delta u_t$ the problem (0.6) was treated by L. Tebou [67]. The author proved the existence and uniqueness of global solution u for (0.6). Then, using an appropriate perturbed energy combined with multiplier technique, he directly proved exponential and polynomial decay estimates for the associated energy.

In this chapter, the well-posedness and regularity of solution is discussed owing to the nonlinear semigroup theory together with the Faedo-Galerkin approach. By energy method combined with the piecewise multiplied method and relying on the localized smoothing property, we show the exponential and polynomial stabilities by discussing with respect to the parameter p. We assume that $a(x)$ and $g(s)$ satisfies the following hypotheses:

(H1) The function $a : \Omega \to \mathbb{R}$ is a nonnegative and bounded such that

$$\exists a_0 > 0, \quad a(x) \geq a_0 > 0, \quad \text{a.e} \quad \text{in} \quad \omega.$$
$$a(x) \in W^{1,\infty}(\Omega).$$

(H2) $g \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ is nondecreasing function with $g(0) = 0$, and globally Lipschitz. Suppose that there exist $c_i > 0$, $i = 1, 2, 3, 4$ and $p \geq 1$ such that

$$c_1|s|^p \leq g(s) \leq c_2|s|^{\frac{1}{p}}, \quad \text{if} \quad |s| \leq 1$$

$$c_3|s| \leq g(s) \leq c_4|s|, \quad \text{if} \quad |s| > 1.$$

Set

$$V = H_0^1(\Omega), \quad W = \{u \in H^3(\Omega) \cap H_0^1(\Omega), \;\; \Delta u = 0 \;\text{ on }\; \Gamma\},$$

and

$$\widetilde{W} = \{u \in H^5(\Omega) \cap H_0^1(\Omega), \;\; \Delta u = \Delta^2 u = 0 \;\text{ on }\; \Gamma\}.$$

We introduce the functional energy

$$E(t) = \frac{1}{2}\|\nabla u_t(t)\|^2 + \frac{1}{2}\|\nabla \Delta u(t)\|^2.$$

The energy E is a nonincreasing function of the time variable t and we have for almost every $t \geq 0$

$$E'(t) = -\int_\Omega a(x)\Delta u_t g(\Delta u_t)\, dx.$$

Theorem 0.3.3. *(Well-posedness) Let $(u_0, u_1) \in W \times V$ and suppose that (H1) and (H2) hold. Then, there exists a solution for system* (0.6) *satisfies*

$$u \in \mathcal{C}([0, \infty), W) \cap \mathcal{C}^1([0, \infty), V)$$

Theorem 0.3.4. *(Regular solutions) Let $(u_0, u_1) \in \widetilde{W} \times W$ and suppose that (H1) and (H2) hold. Then, there exists a solution of system* (0.6) *that satisfies*

$$u \in L^\infty([0, \infty), \widetilde{W}) \cap L^\infty([0, \infty), W).$$

We now turn to the statements of our stabilization result. Before stating it, we now introduce a geometric constraint (GC) on the subset ω where the dissipation is effective. Let $x^0 \in \mathbb{R}^n$ be an arbitrary point of $\mathbb{R}^n$, we set

$$\Gamma(x^0) = \Big\{x \in \Gamma; \quad m(x).\nu(x) > 0\Big\},$$

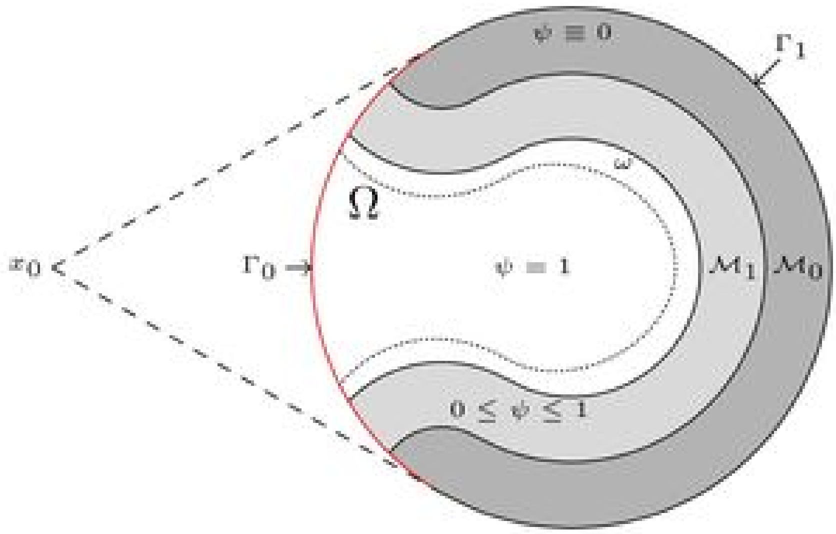

Figure 4: geometric contraint

where ν represents the unit normal vector pointing towards the exterior of Ω and

$$m(x) = x - x^0.$$

Let ω be a neighborhood of $\Gamma(x^0)$ in Ω and consider δ sufficiently small such that

$$\mathcal{M}_0 = \left\{ x \in \Omega; d(x, \Gamma(x^0)) < \delta \right\} \subset \omega,$$

$$\mathcal{M}_1 = \left\{ x \in \Omega; d(x, \Gamma(x^0)) < 2\delta \right\} \subset \omega.$$

If $A \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, we have

$$d(x; A) = \inf_{y \in A}(|x - y|),$$

then $\mathcal{M}_0 \subset \mathcal{M}_1 \subset \omega$.

Theorem 0.3.5. *(Stabilization) Let $(u_0, u_1) \in \widetilde{W} \times W$ and suppose that (H1) and (H2) hold. Then, any weak solution of* (0.6) *satisfies the estimate*

$$E(t) \le CE(0)e^{-kt} \quad \forall t > 0, \quad \textit{and} \quad p = 1$$

and

$$E(t) \le Ct^{-2/(p-1)} \quad \forall t > 0, \quad \textit{and} \quad p > 1$$

Chapter 4 : Well-posedness and general stability for coupled non-degenerate Kirchhoff equation and the heat equation

Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth enough boundary. Let α and β be two nonzero real numbers with the same sign. Consider the coupled wave/heat system

$$\begin{cases} y_{tt} - \gamma\Delta y_{tt} - \phi(\|\nabla y(t)\|^2)\Delta y + \alpha\Delta\theta = 0, & \text{in } \Omega \times (0, +\infty) \\ \theta_t - \sigma\Delta\theta - \beta\Delta y_t = 0, & \text{in } \Omega \times (0, +\infty) \\ y = \theta = 0, & \text{on } \partial\Omega \times (0, +\infty) \\ y(\cdot, 0) = y_0,\ y_t(\cdot, 0) = y_1,\ \theta(\cdot, 0) = \theta_0, & \text{in } \Omega \end{cases} \tag{0.7}$$

where γ and σ are positive physical constants representing respectively, the rotational force constant, thermal conductivity, and ϕ is given function. The functions (y_0, y_1, θ_0) are the given initial data.

When $\gamma = 0$ and $\phi(s) = m_0 + m_1 s$, with $m_0 > 0$ and $m_1 > 0$, Ben Aissa [8] has studied the global existence for small data and the uniform exponential decay rate of the energy.

Moulay Khatir and Shel [58] studied the thermoelastic system with delay

$$\begin{cases} u_{tt}(x,t) - \alpha u_{xx}(x, t-\tau) + \gamma\theta_x(x,t) = 0, & \text{in } (0,l) \times (0, +\infty) \\ \theta_t(x,t) - \kappa\theta_{xx}(x,t) - \gamma u_{xt} = 0, & \text{in } (0,l) \times (0, +\infty) \\ u(0,t) = u(l,t) = \theta_x(0,t) = \theta_x(l,t) = 0, & \text{on } t \geq 0 \\ u(\cdot, 0) = u_0,\ u_t(\cdot, 0) = u_1,\ \theta(\cdot, 0) = \theta_0, & \text{in } \Omega \end{cases}$$

where α, γ, κ and l are some positive constants. To avoid this problem, we added to the system, at the delayed equation, a Kelvin-Voigt damping. They proved the well-posedness of the system by the semigroup theory. Under appropriate assumptions, they obtained the exponential stability of the system by introducing a suitable Lyapunov functional.

Mansouri et al. [57] considered a coupled system consisting of a Kirchhoff thermoelastic plate and an undamped wave equation

$$\begin{cases} y_{tt} - \gamma\Delta y_{tt} + a\Delta^2 y + \alpha\Delta\theta + \mu z = 0, & \text{in } \Omega \times (0, +\infty) \\ \theta_t - \gamma\Delta\theta - \beta\Delta y_t = 0, & \text{in } \Omega \times (0, +\infty) \\ z_{tt} - \mu\Delta z + \mu y = 0, & \text{in } \Omega \times (0, +\infty) \\ y = \partial_\nu y = 0, \quad z = \theta = 0, & \text{on } \partial\Omega \times (0, +\infty) \\ y(\cdot, 0) = y_0,\ y_t(\cdot, 0) = y_1,\ \theta(\cdot, 0) = \theta_0, & \text{in } \Omega, \\ z(\cdot, 0) = z_0,\ z_t(\cdot, 0) = z_1 & \text{in } \Omega \end{cases}$$

They showed that the coupled system is not exponentially stable. Afterwards, they proved that the coupled system is polynomially stable, and provided an explicit polynomial decay rate of the associated semigroup.

In this chapter, we establish the well-posedness result of the solutions of the system by using the Faedo-Galerkin scheme. By energy method combined with the multiplied method, we show the exponential stability.

Let ϕ is a $\mathcal{C}^1$-class function on $\mathbb{R}_+$ and bijective. Assume that there exist $m_0,\ m_1 > 0$ such that and satisfies

$\phi(s) \geq m_0, \quad \text{and} \quad s\phi(s) \geq m_1\widetilde{\phi}(s), \ \ \forall s \geq 0, \quad \text{where} \ \ \widetilde{\phi}(s) = \int_0^s \phi(r)\,dr.$ (0.8)

Introduce the energy

$E(t) = \frac{1}{2}\int_\Omega |y_t(t)|^2\,dx + \frac{1}{2}\widetilde{\phi}(\|\nabla y(t)\|^2) + \frac{\gamma}{2}\int_\Omega |\nabla y_t(t)|^2\,dx + \frac{\alpha}{2\beta}\int_\Omega |\theta(t)|^2\,dx, \quad \forall t \geq 0.$ (0.9)

Then, the energy functional defined by (0.7) satisfies

$$E'(t) = -\sigma\frac{\alpha}{\beta}\int_\Omega |\nabla\theta(t)|^2\,dx \leq 0, \quad \forall t \geq 0.$$

Theorem 0.3.6. *(Well-posedness) Let $\phi : [0, +\infty[\to [0, +\infty[$ be a locally Lipschitz continuous function and $(y_0, y_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H^2(\Omega) \cap H_0^1(\Omega), \quad \theta_0 \in H^2(\Omega) \cap H_0^1(\Omega)$. Assume that $\{y_0, y_1, \theta_0\}$ are small and*

$$\max_{0 \leq s \leq E(0)} |\phi'(s)| \leq m_0.$$

Then the problem (0.7) has a unique weak solution (y, θ) such that for any $T > 0$, we have

$$y \in L^\infty(0, T; H^2(\Omega) \cap H_0^1(\Omega))$$

$$y_t \in L^\infty(0, T; H^2(\Omega) \cap H_0^1(\Omega)), \quad y_{tt} \in L^\infty(0, T; H_0^1(\Omega))$$

$$\theta \in L^\infty(0, T; H_0^1(\Omega)) \cap L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega)).$$

$$\theta_t \in L^\infty(0, T; L^2(\Omega))$$

Theorem 0.3.7. *(Stabilization) Let $(y_0, y_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H^2(\Omega) \cap H_0^1(\Omega), \quad \theta_0 \in H^2(\Omega) \cap H_0^1(\Omega)$. Assume that ϕ satisfies (0.8) and $\beta < \gamma$. The energy of the unique solution of system (0.7), given by (0.9), decays exponentially to zero, there exist positive constants M and λ, independent of the initial data, with*

$$E(t) \leq M\exp(-\lambda t)E(0).$$

Chapter 1

Notations and Preliminaries

1.1 Functional spaces

1.1.1 Sobolev Spaces

The spaces L^p, $1 \leq p \leq \infty$, of p-integrable functions were useful tools for the study of differential equations. In the papers by S. L. Sobolev published between 1935 and 1938, new spaces were introduced which are nowadays called the classical Sobolev spaces $W^{m,p}, 1 < p < \infty$, $m = 0, 1, 2, ...$ the calculus of distributions and embedding theorems were used successfully for the further development of the theory of linear partial differential equations and boundary value problems. Let Ω be open set in $\mathbb{R}^n$, we define the sobolev space $W^{m,p}(\Omega)$

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) \ \ such\ that \ \ \forall \alpha \in \mathbb{N} \ \ with \ \ |\alpha| \ \leq m \ \ D^\alpha u \in L^p(\Omega)\}$$

$W^{m,p}(\Omega)$ is Banach space with norme

$$\|u\|_{W^{m,p}(\Omega)} = \Big(\sum_{|\alpha| \leq m} \|D^\alpha u\|_{L^p(\Omega)} \Big)^{1/p}$$

If $p = 2$ we denot

$$W^{m,p}(\Omega) := H^m(\Omega)$$

Remark 1.1.1. $H^m(\Omega)$ *is a Hilbert space*

Embedding.

Theorem 1.1.1. *(Sobolev Embedding Theorem) Let Ω a bounded domain in $\mathbb{R}^n$, $(n \geq 1)$ of $\mathcal{C}^1$ class with smooth boundary $\partial\Omega$, and $1 \leq p < \infty$.*

$$W^{1,p}(\Omega) \subset \begin{cases} L^{p^*}(\Omega) \quad \text{where } \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}, \quad p < n \\ \\ L^q(\Omega) \qquad \forall\, q \in [p, \infty), \quad p = n \\ \\ L^\infty(\Omega), \quad p > n. \end{cases}$$

Furthermore, those embeddings are continuous in the following sense: there exists $C(n, p, \Omega)$ such that for $u \in W_0^{1,p}(\Omega)$

$$\|u\|_{L^{p^*}(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}, \qquad \forall p < n$$

$$\sup_\Omega |u| \leq C'.Vol(\Omega)^{\frac{1}{n}-\frac{1}{p}}.\|Du\|_{L^p(\Omega)}, \quad \forall p > n.$$

Theorem 1.1.2. *Let Ω a bounded domain in $\mathbb{R}^n$, $(n \geq 1)$ of $\mathcal{C}^1$ class with smooth boundary $\partial\Omega$, and $1 \leq p \leq \infty$.*

$$W^{1,p}(\Omega) \subset \begin{cases} L^{p^*}(\Omega) \quad \forall q \in [1, p^*[\ \text{ where } \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}, \quad p < n \\ \\ L^q(\Omega) \qquad \forall\, q \in [1, \infty), \quad p = n \\ \\ \mathcal{C}(\overline{\Omega}), \qquad p > n. \end{cases}$$

with compact imbedding.

Some inequalities.

Proposition 1.1.3. *For $u \in W(a, b, V, V')$ et $v \in V$, we have:*

$$\left\langle \frac{du}{dt}(.), v \right\rangle_{V \times V'} = \frac{d}{dt}(u(.), v), \ in \quad D'(]a, b[).$$

Young inequality : For all $a, b \in \mathbb{R}$, $(or\ \mathbb{C})$ and for all $p, q \in [1, +\infty[$ with $\frac{1}{q} + \frac{1}{p} = 1$, we have :

$$|ab| \leqslant \frac{1}{p}|a|^p + \frac{1}{q}|b|^q.$$

Hölder inequality : Let $1 < p,\ q < +\infty$, with $\frac{1}{p} + \frac{1}{q} = 1$. Let f a function of $L^p(\Omega)$ and g a function of $L^q(\Omega)$. Then Hölder l'inequality writes:

$$\|fg\|_{L^1(\Omega)} = \|f\|_{L^p(\Omega)}\ \|g\|_{L^q(\Omega)}.$$

a.e

$$\begin{cases} \int_\Omega |f(x)g(x)|\,dx \leqslant \left(\int_\Omega |f(x)^p|\,dx\right)^{\frac{1}{p}} \left(\int_\Omega |g(x)^q|\,dx\right)^{\frac{1}{q}}, & \text{if } \ p,q \in [1,+\infty[, \\ \\ \int_\Omega |f(x)g(x)|dx \leqslant \|g\|_{L^\infty} \int_\Omega |f(x)|\,dx, & \text{if } \ p=1, \quad \text{and } \ q=+\infty. \end{cases}$$

Green Formula: Let Ω an open bounded of frontiers regulars $\partial\Omega$ and $v(x)$ the normal exteriors the point x. Let u a function of $H^2(\Omega)$ and v a function of $H^1(\Omega)$, then the Green formula write :

$$\int_\Omega (\Delta u)v\,dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v\,ds - \int_\Omega \nabla u\ \nabla v\,dx,$$

and

$$\int_\Omega (u\Delta v - v\Delta u)\,dx = \int_{\partial\Omega} \left(u\frac{\partial u}{\partial \nu} - v\frac{\partial u}{\partial \nu}\right).$$

1.1.2 Weak, Weak star and strong convergence

Definition 1.1.1. *:(**Weak convergence in** E).*
Let $x \in E$ and let $\{x_n\} \subset E$. We say that $\{x_n\}$ weakly converges to x in E, and we write $x_n \rightharpoonup x$ in E, if

$$\langle f, x_n\rangle \longrightarrow \langle f, x\rangle \quad \text{for all } f \in E^{'}.$$

Definition 1.1.2. *:(**Weak Convergence in** $E^{'}$).*
Let $f \in E^{'}$ and let $\{f_n\} \subset E^{'}$. We say that $\{f_n\}$ weakly converges to f in $E^{'}$, and we write $f_n \rightharpoonup f$ in $E^{'}$, if

$$\langle f_n, x_n\rangle \longrightarrow \langle f, x\rangle \quad \text{for all } x \in E.$$

Definition 1.1.3. *:(**Weak star Convergence**).*
Let $f \in E^{'}$ and let $\{f_n\} \subset E^{'}$. We say that $\{f_n\}$ weakly star converges to f in $E^{'}$, and we write $f_n \rightharpoonup^* f$ in $E^{'}$, if

$$\langle f_n, x_n\rangle \longrightarrow \langle f, x\rangle \quad \text{for all } x \in E.$$

Definition 1.1.4. *:(**Strong Convergence**).*
Let $x \in E$ (resp. $f \in E^{'}$) and let $\{x_n\} \subset E$ (resp. $\{f_n\} \subset E^{'}$. We say that $\{x_n\}$ (resp. $\{f_n\}$) strong converge to x (resp. f), and we write $x_n \to x$ in E (resp. $f_n \to f$ in $E^{'}$), if

$$\lim_{n\to\infty} \|x_n - x\|_E = 0, (resp. \lim_{n\to\infty} \|f_n - f\|_{E'} = 0).$$

Theorem 1.1.4. *(**Bolzano- Weierstrass**).*
If $dimE < \infty$ and if $\{x_n\} \subset E$ is bounded, then there existe $x \in E$ and a subsequence $\{x_{n_k}\}$ strongly converges to x

Theorem 1.1.5. ***(Weak star Compactness, Banach-Alaoglu-Bourbaki).***
Assum that E is separable and consider $\{f_n\} \subset E^{'}$. If $\{x_n\}$ is bounded, then there existe $f \in E^{'}$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$such that $\{f_{n_k}\}$ weakly star converges to f in $E^{'}$.

1.2 Existence and uniqueness of solution

1.2.1 Semigroups

We start by introducing some basic concepts concerning the semigroups. The vast majority of the evolution equations can be reduced to the form

$$\begin{cases} U_t = AU, \qquad t > 0, \\ U(0) = U_0, \end{cases} \tag{1.1}$$

where A is the infinitesimal generator of a C_0-semigroup $S(t)$ over a Hilbert space H. Lets start by basic definitions and theorems.
Let $(X; \|.\|_X)$ be a Banach spaces and H be a Hilbert space equipped with the inner product $\langle ., . \rangle_H$ and the induced norm $\|.\|_H$.

Definition 1.2.1. *A one parameter family $(S(t))_{t\geq 0}$ of bounded linear operators from X into H is a semigroup of bounded linear operator on X if*

- *$S(0) = I$ (I is the identity operator on X)*
- *$S(t+s) = S(t).S(s)$ for every $t, s \geq 0$.*
- *For each $u \in H,\ S(t)u$ is continous in t on $[0, +\infty[$.*

Definition 1.2.2. *A semigroup is said to be uniformly continuous with respect to operator norm $\|.\|$ associated with X,*

$$\lim_{t\to 0^+} \|S(t) - I\| = 0.$$

Definition 1.2.3. *A semigroup $(S(t))_{t\geq 0}$ of bounded linear operators is a strongly continuous semigroup (or a C_0-semigroup) if*

$$\lim_{t\to 0^+} S(t) = u.$$

Definition 1.2.4. *A strongly continuous contraction semigroup $(S(t))_{t\geq 0}$ on X is a strongly continuous semigroup on X such that*

$$\|S(t) - I\|_{\mathcal{L}(X)} \leq 1 \quad \forall t \geq 0.$$

Definition 1.2.5. *For a semigroup $(S(t))_{t\geq 0}$, we define an linear operator A with domain $D(A)$ consisting of points u such that the limit.The linear operator A defined by*

$$Au = \lim_{t\to 0^+} \frac{S(t)u - u}{t}, \quad \forall u \in D(A)$$

where

$$D(A) = \left\{ u \in X; \lim_{t\to 0^+} \frac{S(t)u - u}{t} \quad exists \right\}$$

is the infinitesimal generator of the semigroup $(S(t))_{t\geq 0}$.

Definition 1.2.6. *An unbounded linear operator $(A, D(A))$ on H, is said to be dissipative if*

$$\Re\langle Au, u\rangle \geq 0, \quad \forall u \in D(A).$$

Definition 1.2.7. *An unbounded linear operator $(A, D(A))$ on X, is said to be m-dissipative if*

- *A is a dissipative operator*
- *$\exists\ \lambda_0 > 0$, such that $\Re(\lambda_0 I - A) = X$*

Theorem 1.2.1. *(Hille-Yosida's Theorem in Banach spaces) An unbounded linear operator $(A, D(A))$ in X is the infinitesimal generator of a semigroup of contractions on X if and only if the following conditions are satisfied*

- *A is a closed operator*
- *$D(A)$ is dense in X*
- *For all $\lambda > 0$, $(\lambda I - A)$ is a bijective mapping from $D(A)$ to X, its inverse $(\lambda I - A)^{-1}$ is a bounded operator on X obeying*

$$\|(\lambda I - A)^{-1}\| \leq \frac{1}{\lambda}$$

Theorem 1.2.2. *(Hille-Yosida's Theorem in Hilbert spaces Phillips Theorem) An unbounded linear operator $(A, D(A))$ in X is the infinitesimal generator of a semigroup of contractions on X if and only if A is m-dissipative in X.*

Theorem 1.2.3. *(Hille-Yosida Theorem: Lumer-Phillips from in Hilbert spaces)*
Let $A : D(A) \subset H \to H$ be a linear operator. Then A is maximal monoton if and only if -A is the infinitesimal generator of a C_0 semigroup of contraction on H.

Theorem 1.2.4. *(Lumer-Phillips)*
Let $(A, D(A))$ be an unbounded linear operator on X, with dense domain $D(A)$ in X. A is the infinitesimal generator of a C_0-semigroup of contractions if and only if it is a m-dissipative operator.

Definition 1.2.8. *An unbounded linear operator $A : D(A) \subset E \to F$ is said to be monotone (or accretive) if it satisfies*

$$(Av, v) \geq 0 \quad \forall v \in D(A).$$

Remark 1.2.1. *A is a monotone operator $\Leftrightarrow$ -A is a dissipative operator*

Definition 1.2.9. *An unbounded linear operator $A : D(A) \subset E \to F$ is said to be maximal monotone if*

- *A is a monotone operator.*
- *$\forall f \in H \ \exists u \in D(A)$ such that $u + Au = f$.*

The first properties of maximal monotone operators are given in the result below.

Proposition 1.2.5. *Let A be a maximal monotone operator. Then*

- *D(A) is dense in H,*
- *A is a closed operator,*
- *For every $\lambda > 0$, $(I + \lambda A)$ is bijective from D(A) onto H, $(I + \lambda A)^{-1}$ is a bounded operator, and*

$$\|(I + \lambda A)^{-1}\|_{\mathcal{L}(H)} \leq 1.$$

Theorem 1.2.6. *(Browder-Minty) Let's E be a reflexive Hilbert space. Let A nonlinear operators such as*

$$\langle Au - Av, u - v \rangle \geq 0 \quad \forall v, u \in E$$

$\lim \frac{\langle Au,u \rangle}{\|v\|_E} \to \infty$ *as* $\|v\|_E \to \infty$, *so A is coercive. Then A is surjective in E' e.i (a operator $A : E \to E'$ is surjective if for each $f \in E'$, there exists $u \in E$, such that $Au = f$).*

1.2.2 The Faedo-Galerkin method

The method is based on three steps :
(i) Choose certain basis of functions in an appropriate Sobolev space, and solve the approximate problems in any finite dimensional space spanned by finite basis functions. This often turns out to be an initial value problem for nonlinear ordinary differential equations. By the well-known local existence theorem for ordinary differential equations, local existence of solution to the approximate problem follows.
(ii) Obtain the compactness estimates for the solution of the approximate problem. It also turns out that the solution to the approximate problem globally exists.

(iii) Further use of the obtained compactness estimates allows one to choose a subsequence of solutions of the approximate problem obtained in the second step, converging to a solution of the original problem; uniqueness of solution for the original problem has to be proved separately, but the compactness estimates obtained in the second step are still very useful for this purpose.

1.3 Stabilty Methods

The purpose of stabilization is to attenuate the vibrations by feedback, therefore it is to ensure the decay of the energy solutions to 0 more or less quickly by a dissipation mechanism. More precisely, the stabilization problem in which we are interested amounts to determining the asymptotic behavior of the energy that we denote by $E(t)$ (this is the norm of solutions in the state space), to study its limit in order to determine if this limit is zero or not, and, if this limit is zero, to give an estimate of the decay rate of energy to zero. They are several type of stabilization :

1) Strong stabilization:

$$\lim_{t \to +\infty} E(t) = 0.$$

2) Exponential stabilization:

$$E(t) \leq Ce^{-\delta t} \quad \forall t > 0.$$

3) Polynomial stabilization:

$$E(t) \leq \frac{C}{t^{\alpha}} \quad \forall t > 0.$$

where C, δ, and α are positive constans and C which depends on the initial data.

1.3.1 Lyapunov's method

Lyapunov design has been a primary tool for nonlinear control system design, stability and performance analysis since its introduction in 1982. The basic idea is to design a feedback control law that renders the derivative of a specified Lyapunov function candidate negative definite or negative semi-definite. Lyapunovs direct method is a mathematical interpretation of the physical property that if a systems total energy is dissipating, then the states of the system will ultimately reach an equilibrium point. The basic idea behind the method is that, if there exist a kind of continuous scalar energy functions such that this energy diminishes along the systems trajectory, then the system is said to be asymptotically stable. Since there is no need to solve the solution of the differential equations governing the system in determining its stability, it is usually referred to as the direct method.

Although Lyapunovs direct method is efficient for stability analysis, its applicability is restricted due to the difficulty in selecting a Lyapunov function. The situation is different when facing the controller design problem, where the control has not been specified, and the system under consideration is undetermined. Lyapunov functions have been effectively utilized in the synthesis of control systems. The basic idea is that, by first choosing a Lyapunov function candidate and then the feedback control law can be specified such that it renders the derivative of the specified Lyapunov function candidate negative definite, or negative semi-definite when invariance principle can be used to prove asymptotic stability. This way of designing control is called Lyapunov design. Lyapunov design depends on the selection of Lyapunov function candidates. Though the result is sufficient, it is difficult to find a Lyapunov function (LF) satisfying the requirements of Lyapunov design. Fortunately, during the past several decades, many effective control design approaches have been developed for different classes of linear and nonlinear systems based on the basic ideas of Lyapunov design. Lyapunov functions are additive, like energy, i.e., Lyapunov functions for combinations of subsystems may be derived by adding the Lyapunov functions of the subsystems.

1.3.2 The multiplier method

We use this method to get a better estimate of the decay rate, A. Haraux and V. Komornik have improved and generalized this method. They introduced integral inequalities which make it possible to obtain very efficiently and very good decay estimates for many linear or nonlinear problems.
We will use these integral inequalities to study the decay rate of the energy of a nonlinear dissipative problems.

Chapter 2

Well-posedness and general decay of solutions for a Petrovsky equation with a memory term

2.1 Introduction

In this chapter we consider the existence and decay properties of global solutions for the initial boundary value problem of non-linear Petrovsky equation with a strong damping and a memory term

$$\begin{cases} u_{tt} + \Delta^2 u - \int_0^t h(t-s)\Delta^2 u(s)ds - g(\Delta u_t) = 0, & x \in \Omega \times [0, +\infty[, \\ u(x,t) = \Delta u(x,t) = 0, & x \in \Gamma \times [0, \infty[, \\ u(x,0) = u_0(x), \; , \; u_t(x,0) = u_1(x) & x \in \Omega \times [0, +\infty[, \end{cases} \tag{2.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, Γ is a smooth boundary, (u_0, u_1) are the initial data in a suitable function space, h and g are real functions.

The study of viscoelastic problems has attracted the attention of many authors and several decay and blow up results have been established. In [18] Cavalcanti et al. considered the equation

$$u_{tt} - \Delta u - \int_0^t h(t-s)\Delta u(s)ds - a(x)u_t + u|u|^{p-2} = 0, \text{ in } \Omega \times [0, +\infty[$$

where $a : \Omega \to \mathbb{R}_+$ is a function which may vanish on a part of the domain Ω but satisfies $a(x) \geq a_0 > 0$ on $\omega \subset \Omega$ and h satisfies, for two positive constants ξ_1 and ξ_2

$$\xi_1 h(t) \leq h'(t) \leq \xi_2 h(t), \;\; \forall t \geq 0.$$

They established an exponential decay result under some restrictions on ω. Berrimi and Messaoudi [11] established the result of [18], under weaker conditions on both a and h to a problem where a

source term is competing with the damping term.
Belhannache et al. [7] considered the following problem

$$u_{tt} - \Delta u - \int_0^t h(t-s)\Delta u(s)ds - a(x)u_t + |u|^{p-2}u = 0, \text{ in } \Omega \times [0, +\infty[$$

with a

$$h'(t) \leq -\xi(t)H(h(t)),$$

they showed the global existence and obtained a general stability result.
Mustafa and Messaoudi [53] established an explicit and general decay rate for relaxation function satisfying

$$h'(t) \leq H(h(t))$$

where $H \in \mathcal{C}^1(\mathbb{R})$, with $H(0) = 0$ and H is linear or strictly increasing and strictly convex function $\mathcal{C}^2$ near the origin.
Park and Kang [59] studied the following nonlinear viscoelastic problem with damping

$$|u_t|^l u_{tt} + \Delta^2 u - \Delta u_{tt} - M(\|\nabla u\|_2^2)\Delta u + \int_0^t h(t-s)\Delta u(s)\, ds + u_t = 0, \quad x \in \Omega,\ t > 0.$$

Santos et al. [63] considered the existence and uniform decay for the following nonlinear beam equation in a non-cylindrical domain:

$$u_{tt} + \Delta^2 u - M(\|\nabla u\|_2^2)\Delta u + \int_0^t h(t-s)\Delta u(s)\, ds + \alpha u_t = 0, \qquad in\ \widehat{Q},$$

where $\widehat{Q} = \cup_{0\leq t\leq\infty}\Omega_t \times \{t\}$. Yaojun [71] proved the existence of global solution, as well as, under suitable conditions on relaxation function $h(.)$ and the positive initial energy as well as non-positive initial energy, it is proved that the solution blows up in the finite time and the lifespan estimates of solutions are also givenresult for the equation

$$u_{tt} + (-\Delta)^m u + \int_0^t h(t-s)(-\Delta)^m u(s)\, ds = |u|^{p-2}u. \tag{2.2}$$

When $m = 2$ F. Tahamatani and M. Shahrouzi [69] prove the existence of weak solutions of Eq. (2.2) with initial-boundary value conditions. Meanwhile, they show that there are solutions under some conditions on initial data which blow up in finite time with non-positive initial energy as well as positive initial energy and give the lifespan estimates of solutions. In the absence of nonlinear source term, Munoz Rivera, Lapa and Baretto [54] considered Eq. (2.2) in a bounded domain $\Omega \subset \mathbb{R}^n$ and showed that the energy of solution decays exponentially provided the relaxation function $h(.)$ also decays exponentially.
Komornik [**32**] studied the following nonlinear Petrovsky system with a strong damping

$$\begin{cases} u_{tt}(x,t) + \Delta^2 u(x,t) - g(\Delta u_t) = 0, & x \in \Omega \times [0, +\infty[, \\ u(0,t) = \Delta u = 0, & x \in \Gamma \times [0, \infty[, \\ u(x,0) = u_0(x), \ , \ u_t(x,0) = u_1(x) & x \in \Omega \times [0, +\infty[. \end{cases}$$

He used semigroup approach for sitting the well possedness and he studied the strong stability of this system by introducing a multiplier method combined with a nonlinear integral inequalities given by Martinez [50].

Kouémou-Patcheu [33] studied the Kirchhoff equation with a nonlinear source term

$$u_{tt} + A^2u + M(\|A^{\frac{1}{2}}u\|_H^2)Au - g(u_t) = 0,$$

where A is a linear operator in a Hilbert space H and M and g are real functions. She proved the global existence of solutions by the Faedo-Galerkin method and she used a new method recently introduced by Martinez [50] to study the decay rate of solution.
In this paper, we prove the global existence of weak solutions of the problem (2.1) by using the Galerkin method (see Lions [41]). Meanwhile, under suitable conditions on $g(.)$ and we use some techniques using Liapunov functions and some properties of convex functions. These arguments of convexity were introduced and developed by Cavalcanti et al. [19], Daoulatli et al. [22], Lasiecka and Doundykov [38] and Lasiecka and Tataru [42], and used by Liu and Zuazua [47], Eller et al. [25] and Alabau-Boussouira [3].

This paper is organized as follows. In Section 2, we present some notations and material needed for our work. In Section 3, we establish the global existence of the solution of the problem. Some technical lemmas and the decay results are presented in Sections 4.

2.2 Notation and Preliminaries

We begin by introducing some notation that will be used throughout this work.
Let us introduce three real Hilbert spaces $\mathcal{H}$, V and W by setting

$$\mathcal{H} = H_0^1(\Omega), \quad \|v\|_{\mathcal{H}}^2 = \int_\Omega |\nabla v|^2 dx$$

and

$$V = \{v \in H^3(\Omega) | v = \Delta v = 0 \text{ on } \Gamma\}, \quad \|v\|_V^2 = \int_\Omega |\nabla \Delta v|^2 dx$$

$$W = \{v \in H^5(\Omega) | v = \Delta v = \Delta^2 v = 0 \text{ on } \Gamma\}, \quad \|v\|_W^2 = \int_\Omega |\nabla \Delta^2 v|^2\, dx.$$

Identifying H with its dual H' we have

$$W \subset V \subset \mathcal{H} \subset V' \subset W',$$

with dense and compact imbedings.
If $v \in L^2(\Omega)$, we denote by $\|v\|^2_{L^2(\Omega)} = \|v\|^2$, the $H^k(\Omega)$ and $H^1_0(\Omega)$ are the Sobolev spaces.
Next, we give the precise assumptions on the functions $h(.)$ and $g(.)$.

(A1) Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a $\mathcal{C}^2$ real function such that $h(0) = h_0 > 0$ and

$$l = \int_0^\infty h(s)\ ds < 1.$$

There exists a non-increasing differentiable function $\nu : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$h'(s) \leq -\nu(s)h(s), \forall s \geq 0 \text{ and } \int_0^\infty \nu(s)\ ds = +\infty.$$

(A2) Consider $g : \mathbb{R} \to \mathbb{R}$ a non-decreasing $\mathcal{C}^1(\mathbb{R})$ function such that

$$g(v)v > 0, \ \text{ for all } \ v \neq 0, \tag{2.3}$$

and there exist constants $\varepsilon, c_1, c_2, \tau > 0$ and a convex increasing function $G : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ of class $\mathcal{C}^1(\mathbb{R}_+) \cap \mathcal{C}^2(\mathbb{R}^*_+)$ satisfying G linear on $[0, \varepsilon]$ or
($G^{'}(0) = 0$ and $G'' > 0$ on $]0, \varepsilon]$), such that

$$c_1 |s| \leqslant |g(s)| \leqslant c_2 |s|, \quad if \quad |s| > \varepsilon, \tag{2.4}$$

$$|s|^2 + |g(s)|^2 \leqslant G^{-1}(sg(s)), \quad if \quad |s| \leqslant \varepsilon, \tag{2.5}$$

$$|g'(s)| \leq \tau. \tag{2.6}$$

Remark 2.2.1. *Let us denote by G^* the conjugate function of the differentiable convex function G, i.e.,*

$$G^*(s) = \sup_{t \in \mathbb{R}_+} (st - G(t)).$$

Then G^* is the Legendre transform of G, which is given by (see Arnold [4, p. 61-62])

$$G^*(s) = s(G^{'})^{-1}(s) - G\left((G^{'})^{-1}(s)\right) \quad if s \in \left]0, G^{'}(r)\right],$$

and G^* satisfies the generalized Young inequality

$$ST \leq G^*(S) + G(T), \quad if S \in \left]0, G^{'}(r)\right], T \in \left]0, r\right]. \tag{2.7}$$

The relation (2.7) and the fact that $G(0) = 0$ and $(G')^{-1}$, G are increasing functions yield

$$G^*(s) \leq s(G^{-1})(s), \quad \forall s \geq 0 \tag{2.8}$$

2.3 Well posedeness and regularity

Theorem 2.3.1. *Assume that*

$$(u_0, u_1) \in W \times V,$$

then the solution of the problem (2.1) satisfies

$$u_t \in L^\infty(0,T;V) \;\; ; \;\; u_{tt} \in L^\infty(0,T;\mathcal{H})$$

and

$$u \in L^\infty(0,T;H^4(\Omega) \cap V),$$

such that for any $T > 0$

$$\int_0^T \left(\int_\Omega u_{tt}(x,t) + \Delta^2 u(x,t) - \int_0^t h(t-s)\Delta^2 u(s)ds - g(\Delta u_t) \; dx \right) dt = 0, \quad in \quad L^\infty(0,T;L^2(\Omega)),$$

$$u(0) = u_0, \quad u_t(0) = u_1, \quad in \quad \Omega.$$

i) *Approximate solutions:*

We will use the Faedo-Galerkin method to prove the existence of a global solution. Let $T > 0$ be fixed and let $\{w_j\}, j \in \mathbb{N}$ be a basis of $\mathcal{H}$, V and W, i.e. the space generated by $\mathcal{B}_k = \{w_1, w_2, \ldots, w_k\}$ is dense in $\mathcal{H}$, V and W.

We construct approximate solutions u^k, $k = 1, 2, 3, \ldots$, in the form

$$u^k(t) = \sum_{j=1}^{k} c_{jk}(t) w_j(x),$$

where c_{jk} is determined by the ordinary differential equations.

For any v in $\mathcal{B}_k$, $u^k(t)$ satisfies the approximate equation

$$\int_\Omega \left(u^k_{tt}(x,t) + \Delta^2 u^k(x,t) - \int_0^t h(t-s)\Delta^2 u^k(s) \; ds - g(\Delta u^k_t) \right) v \; dx = 0, \tag{2.9}$$

with initial conditions

$$u^k(0) = u^k_0 = \sum_{j=1}^{k} \langle u_0, w_j \rangle w_j \to u_0, \quad \in \quad W \quad as \quad k \to +\infty, \tag{2.10}$$

and

$$u^k_t(0) = u^k_1 = \sum_{j=1}^{k} \langle u_1, w_j \rangle w_j \to u_1, \quad in \quad V \quad as \quad k \to +\infty. \tag{2.11}$$

The standard theory of ODE guarantees that the system (2.9)-(2.11) has an unique solution in $[0, t_k)$, with $0 < t_k < T$, by Zorn lemma since the nonlinear terms in (2.9) are locally Lipschitz continuous. Note that $u^k(t)$ is of class $\mathcal{C}^2$.

In the next step, we obtain a priori estimates for the solution of the system (2.9)-(2.11), so that it can be extended outside $[0, t_k)$ to obtain one solution defined for all $T > 0$, using a standard compactness argument for the limiting procedure.

ii) *A priori estimates:*

The first estimate. Setting $v = -2\Delta u_t^k$ in (2.9) and denoting by

$$h \circ u(t) = \int_0^t h(t-s)\|u(t) - u(s)\|^2 \, ds,$$

we have

$$\frac{d}{dt}\Big[\|\nabla u_t^k(t)\|^2 + \|\nabla \Delta u^{(k)}(t)\|^2\Big(1 - \int_0^t h(s)\, ds\Big) + h \circ \nabla \Delta u^k(t)\Big]$$
$$= h(t)\|\nabla \Delta u^k(t)\|^2 + h' \circ \nabla \Delta u^k(t) - 2\int_\Omega \Delta u_t^k(t) g(\Delta u_t^k(t))\, dx.$$

Let

$$E^k(t) = \|\nabla u_t^k(t)\|^2 + \|\nabla \Delta u^k(t)\|^2\Big(1 - \int_0^t h(s)\, ds\Big) + h \circ \nabla \Delta u^k(t).$$

Integrating in $[0, t]$, $t < t_k$; using (2.10) and (2.11), we obtain

$$E^k(t) + \int_0^t \int_\Omega \Delta u_t^k(t) g(\Delta u_t^k(t))\, dx \leq E^k(0) \leq C_0, \tag{2.12}$$

for some $C_0 > 0$ independent of k.

This estimate imply that the solution $u^k(t)$ exists globally in $[0, +\infty)$. Estimate (2.12) yields

$$u^k \text{ is bounded in } L^\infty(0, T; V), \tag{2.13}$$

$$u_t^k \text{ is bounded in } L^\infty(0, T; \mathcal{H}), \tag{2.14}$$

$$\Delta u_t^k g(\Delta u_t^k) \text{ is bounded in } L^1(\Omega \times (0, T)), \tag{2.15}$$

The second estimate. Differentiating (2.9) with respect to x, taking $v = \nabla u_{tt}^k(t)$ and choosing $t = 0$, we obtain that

$$\|\nabla u_{tt}^k(0)\|^2 + \int_\Omega \nabla \Delta^2 u^k(0) \nabla u_{tt}^k(0) - \int_\Omega \nabla g(\Delta u_t^k)(0) \nabla u_{tt}^k(0)\, dx = 0.$$

Using Cauchy-Schwarz inequality and (2.6) we obtain

$$\|\nabla u_{tt}^k(0)\| \leq \|\nabla \Delta^2 u^k(0)\| + \|g'(\Delta u_t^k)(0)\|$$
$$\leq \|\nabla \Delta^2 u^k(0)\| + \tau \|\nabla \Delta u_t^k)(0)\|.$$

Taking (2.10) and (2.11), we obtain

$$u_{tt}^k(0) \text{ is bounded in } \mathcal{H}. \tag{2.16}$$

The third estimate. Differentiating (2.9) with respect to t gives

$$u_{ttt}^k(t) + \Delta^2 u_t^k - \int_0^t h'(t-s)\Delta^2 u^k(s)\, ds - h_0 \Delta^2 u^k(t) - \Delta u_{tt}^k g'(\Delta u_t^k) = 0. \tag{2.17}$$

Multiplying (2.17) by $v = -2\Delta u_{tt}^k$, integrating over Ω and applying the Green formula, we obtain

$$\frac{d}{dt}[\|\nabla u_{tt}^k\|^2 + \|\nabla u_{tt}^k\|^2] - \int_0^t h'(t-s)\Delta^2 u^k(s)\, ds - h_0 \Delta^2 u^k(t) - \Delta u_{tt}^k g'(\Delta u_t^k) = 0. \tag{2.18}$$

Integrating by parts, we have

$$\begin{aligned}
2\int_\Omega \int_0^t h'(t-s)\Delta^2 u^k(s)\Delta u_{tt}^k(t)\, dx &= -2\int_0^t h'(t-s)\int_\Omega \nabla\Delta u^k(s)\nabla\Delta u_{tt}^k(t)\, dx\, ds \\
&= -2\frac{d}{dt}\int_0^t h'(t-s)\int_\Omega \nabla\Delta u^k(s)\nabla\Delta u_t^k(t)\, dx\, ds \\
&+ 2h'(0)\int_\Omega \nabla\Delta u^k(s)\nabla\Delta u_t^k(t)\, dx \\
&+ 2\int_0^t h''(t-s)\nabla\Delta u^k(s)\nabla\Delta u_t^k(t)\, dx
\end{aligned}$$

and

$$\begin{aligned}
2\int_\Omega h_0\Delta^2 u^k(t)\Delta u_{tt}^k(t)\, dx &= -2h_0\int_\Omega \nabla\Delta u^k(t)\nabla\Delta u_{tt}^k(t)\, dx \\
&= 2h_0\|\nabla\Delta u_t^k(t)\|^2 - 2h_0\frac{d}{dt}\int_\Omega \nabla\Delta u^k(t)\nabla\Delta u_t^k(t)\, dx. \\
&- 2h_0\int_\Omega \nabla\Delta u^k(t)\nabla\Delta u_t^k(t)\, dx.
\end{aligned}$$

Inserting the above two equalities into (2.18), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[\|\nabla u_{tt}^k\|^2 + \|\nabla\Delta u_t^k\|^2 - 2\int_0^t h'(t-s)\int_\Omega \nabla\Delta u^k(s)\nabla\Delta u_t^k(t)\, dx\, ds\right] \\
&= -h'(0)\int_\Omega \nabla\Delta u^k(s)\nabla\Delta u_t^k(t)\, dx - \int_0^t h''(t-s)\int_\Omega \nabla\Delta u^k(s)\nabla\Delta u_t^k(t)\, dx\, ds \\
&- 2\int_\Omega g'(\Delta u_t^k(t))(\Delta u_t^k(t))^2\, dx - 2h_0\|\nabla\Delta u_t^k(t)\|^2.
\end{aligned} \tag{2.19}$$

Using Cauchy-Schwarz and Young inequalities; integrating (2.19) over $(0,t)$, yields

$$\begin{aligned}
&\|\nabla u_{tt}^k\|^2 + \|\nabla\Delta u_t^k\|^2 + 2\int_0^t\int_\Omega g'(\Delta u_t^k(s)(\Delta u_t^k(s))^2\, dx\, ds \\
&\le \|\nabla u_{tt}^k(0)\|^2 + \|\nabla\Delta u_t^k(0)\|^2 + 2\int_0^t h'(t-s)\int_\Omega \nabla\Delta u^k(s)\nabla\Delta u_t^k(t)\, dx\, ds \\
&+ 2h_0\int_\Omega \nabla\Delta u^k(t)\nabla\Delta u_t^k(t)\, dx + 2h_0\int_\Omega \nabla\Delta u^k(0)\nabla\Delta u_t^k(0)\, dx \\
&+ \left(\epsilon + \epsilon\|h''\|_{L^1}^2\right)\int_0^t \|\nabla\Delta u^k(s)\|^2\, ds + \left(\frac{h'(0)^2}{4\epsilon} + \frac{1}{4\epsilon}\right)\int_0^t \|\nabla\Delta u_t^k(s)\|^2\, ds,
\end{aligned} \tag{2.20}$$

where

$$\int_0^t h'(t-s)\int_\Omega \nabla\Delta u^k(s)\nabla\Delta u_t^k(t)\,dx\;ds \le \varepsilon\|\nabla\Delta u_t^k(t)\|^2 + \|h\|_1\|h\|_\infty\frac{\nu(0)}{4\varepsilon}\int_0^t \|\nabla\Delta u^k(s)\|^2\;ds$$

and

$$h_0\int_\Omega \nabla\Delta u^k(t)\nabla\Delta u_t^k(t)\;dx \le \varepsilon\|\nabla\Delta u_t^k\|^2 + \frac{h_0^2}{4\varepsilon}\|\nabla\Delta u^k\|^2$$

we deduce from (2.10), (2.11), (2.16), (2.20), choosing ϵ small enough and using Gronwall lemma, we obtain

$$\|\nabla u_{tt}^k\|^2 + \|\nabla\Delta u_t^k\|^2 + 2\int_0^t\int_\Omega g'(\Delta u_t^k(s)(\Delta u_t^k(s))^2\,dx\;ds \le C_1,$$

where C_1 is a positive constant independent of k. Therefore, we conclude that

$$u_t^k \quad \text{is bounded in} \qquad L^\infty(0,T;V) \tag{2.21}$$

and

$$u_{tt}^k \quad \text{is bounded in} \qquad L^\infty(0,T;\mathcal{H}). \tag{2.22}$$

By (2.21) we deduce that

$$u_t^k \quad \text{is bounded in} \qquad L^2(0,T;V)$$

Applying Rellich compactenes theorem given in [41], we deduce that

$$u_t^k \quad \text{is precompact in} \qquad L^2(0,T;L^2(\Omega)). \tag{2.23}$$

The fourth estimate. Setting $v = \Delta^2 u_t^k$ in (2.9), we have

$$\begin{aligned}\int_\Omega u_{tt}^k\Delta^2 u_t^k\,dx + \int_\Omega \Delta^2 u^k(t)\Delta^2 u_t^k\,dx - \int_\Omega\int_0^t h(t-s)\Delta^2 u^k(s)\Delta^2 u_t^k\,dx\,ds\\ - \int_\Omega g(\Delta u_t^k)\Delta^2 u_t^k\,dx = 0,\end{aligned} \tag{2.24}$$

where

$$\begin{aligned}\int_\Omega\int_0^t h(t-s)\Delta^2 u^k(s)\Delta^2 u_t^k\,dx\,ds = -\frac{1}{2}h(t)\|\|\Delta^2 u^k\|^2 + \frac{1}{2}h'\circ\Delta^2 u^k(t)\\ + \frac{1}{2}\frac{d}{dt}\Big\{-h\circ\Delta^2 u^k(t) + \Big(\int_0^t h(s)\,ds\Big)\Delta^2 u^k\Big\}.\end{aligned} \tag{2.25}$$

From (2.24) and (2.25), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big\{\|\Delta u_t^k\|^2 + \|h\circ\Delta^2 u(t) + \Big(1-\int_0^t h(s)\,ds\Big)\|\Delta^2 u^k\|^2\Big\}\\ = \int_\Omega g(\Delta u_t^k)\Delta^2 u_t^k\,dx + \frac{1}{2}h(t)\|\Delta^2 u^k\|^2 - \frac{1}{2}h'\circ\Delta^2 u^k(t).\end{aligned} \tag{2.26}$$

Taking in a acount that

$$\int_0^t\int_\Omega g(\Delta u_t^k)\Delta^2 u_t^k\;dx\;ds = -\int_0^t\int_\Omega g'(\Delta u_t^k)(\nabla\Delta u_t^k)^2\;dx\;ds$$

and integrating (2.26) over $(0,t)$, we obtain under $(A1)$ and (2.6)

$$\|\Delta u_t^k(t)\|^2 + \Big(1 - \int_0^t h(s)\,ds\Big)\|\Delta^2 u^k(t)\|^2 + h \circ \Delta^2 u^k(t) + \tau \int_0^t \int_\Omega (\nabla \Delta u_t^k)^2 \,dx\,ds$$
$$\leq \|\Delta u_t^k(0)\|^2 + \|\Delta^2 u^k(0)\|^2 + \int_0^t h(s)\|\Delta^2 u^k(s)\|^2\,ds.$$

Using Gronwall Lemma, we deduce that

$$\Delta^2 u^k \quad \text{is bounded in} \qquad L^\infty(0,T;L^2(\Omega)) \tag{2.27}$$

and

$$\Delta u_t^k \quad \text{is bounded in} \qquad L^\infty(0,T;L^2(\Omega)). \tag{2.28}$$

iii) *Passing to the limit:*

Applying Dunford-Petit theorem we conclude from (2.13), (2.22), (2.27) and (2.28), replacing the sequence u^k, with a subsequence if needed, that

$$u^k \rightharpoonup u, \ \text{weak-star in } L^\infty(0,T;V \cap H^4(\Omega)) \tag{2.29}$$

$$u_t^k \rightharpoonup u_t, \ \text{weak-star in } L^\infty(0,T;V) \tag{2.30}$$

$$u_{tt}^k \rightharpoonup u_{tt}, \ \text{weak-star in } L^\infty(0,T;\mathcal{H}) \tag{2.31}$$

$$u_t^k \longrightarrow u_t, \ \text{almost everywhere in } \Omega \times [0,+\infty) \tag{2.32}$$

$$g(\Delta u_t^k) \rightharpoonup \phi, \ \text{weak-star in } L^2([0,T]\times\Omega) \tag{2.33}$$

$$\Delta^2 u^k \rightharpoonup \psi, \ \text{weak-star in } L^\infty(0,T;L^2(\Omega)), \tag{2.34}$$

where $\psi = \Delta^2 u$.

As $(u^k)_{k\in\mathbb{N}}$ is bounded in $L^\infty(0,T;V)$ (by (2.13))) and the injection of V in $\mathcal{H}$ is compact, we have

$$u^k \longrightarrow u, \ \text{strong in } L^2(0,T;\mathcal{H}). \tag{2.35}$$

In the other hand, using (2.29), (2.31) and (2.35), we have

$$\begin{aligned}&\int_0^T \int_\Omega \Big(u_{tt}^k(x,t) + \Delta^2 u^k(x,t)) - \int_0^t h(t-s)\Delta^2 u^k(s)\,ds\Big) v\,dx\,dt \longrightarrow \\ &\int_0^T \int_\Omega \Big(u_{tt}(x,t) + \Delta^2 u(x,t)) - \int_0^t h(t-s)\Delta^2 u(s)\,ds\Big) v\,dx\,dt,\end{aligned} \tag{2.36}$$

for all $v \in L^2(0,T;L^2(\Omega))$.

It remains to show

$$\int_0^T \int_\Omega g(\Delta u_t^{(k)})\ vdx\ dt \longrightarrow \int_0^T \int_\Omega g(\Delta u_t)\ vdx\ dt,$$

when $k \to +\infty$. We claim that

$$g(\Delta u_t) \in L^1([0,T] \times \Omega).$$

Indeed, since g is continuous, we deduce from (2.32)

$$g(\Delta u_t^k) \longrightarrow g(\Delta u_t) \quad \text{almost everywhere in} \quad (0,T) \times \Omega. \tag{2.37}$$

Using (2.15) and Fatou's lemma, we deduce that

$$\Delta u_t g(\Delta u_t) \in L^1([0,T] \times \Omega). \tag{2.38}$$

set $E \subset [0,T] \times \Omega$ and

$$E_1 = \Big\{(t,x) \in (0,T) \times \Omega : |g(\Delta u_t)| \leq |E|^{-1/2}\Big\}; \text{ and } E_2 = E \setminus E_1$$

Let $J(r) = \inf \Big\{|s| : \ s \in \mathbb{R},\ |g(s)| \geq r\Big\}$, then

$$\int_E g(\Delta u_t^k)\ dx\ dt = \int_{E_1} g(\Delta u_t^k)\, dx\, dt + \int_{E_2} g(\Delta u_t^k)\, dx\, dt$$

By Cauchy-Schwarz inequality, we have

$$\int_0^T \int_\Omega |\Delta g(u_t^k)|\, dx\, dt \leq c|E|^{1/2}\Big(\int_0^T \int_\Omega |\Delta g(u_t^k)|^2\, dx\, dt\Big)^{1/2}.$$

Using (2.4), (2.5) and (2.38), we obtain

$$\begin{aligned}
\int_0^T \int_\Omega |\Delta g(u_t^k)|^2\, dx\, dt &\leq \int_0^T \int_{|\Delta u_t^k| > \varepsilon} \Delta u_t^k g(\Delta u_t^k)\, dx\, dt + \int_0^T \int_{|\Delta u_t^k| \leq \varepsilon} G^{-1}(\Delta u_t^k g(\Delta u_t^k))\, dx\, dt \\
&\leq c \int_0^T \int_\Omega \Delta u_t^k g(\Delta u_t^k)\, dx\, dt + cG^{-1}\Big(\int_E \Delta u_t^k g(\Delta u_t^k)\, dx\, dt\Big) \\
&\leq c \int_0^T \int_\Omega \Delta u_t^k g(\Delta u_t^k)\, dx\, dt + c' G^*(1) + c'' \int_\Omega \Delta u_t^k g(\Delta u_t^k)\, dx\, dt \\
&\leq cK_1 + c' G^*(1), \qquad \text{for } T > 0.
\end{aligned}$$

Then

$$\int_0^T \int_E |\Delta g(u_t^k)|\, dx\, dt \leq K, \qquad \text{for } T > 0.$$

$$g(\Delta u_t^k) \longrightarrow g(\Delta u_t) \text{ in } L^1([0,T] \times \Omega),$$

then (2.33) implies that

$$g(\Delta u_t^k)) \rightharpoonup g(\Delta u_t),\ \text{weak-star in } L^2((0,T)\times\Omega).$$

We deduce, for all $v \in L^2(0,T)\times L^2(\Omega)$, that

$$\int_0^T \int_\Omega g(\Delta u_t^k))v\,dx\,dt \longrightarrow \int_0^T \int_\Omega g(\Delta u_t)v\,dx\,dt.$$

Finally we have shown that, for all $v \in L^2((0,T)\times L^2(\Omega))$:

$$\lim_k \int_0^T \int_\Omega \Big(u_{tt}^k(x,t) + \Delta^2 u^k(x,t) - \int_0^t h(t-s)\Delta^2 u^k(s)\ ds - g(\Delta u_t^k)\Big)\ v\ dx\ dt =$$

$$\int_0^T \int_\Omega \Big(u_{tt}(x,t) + \Delta^2 u(x,t) - \int_0^t h(t-s)\Delta^2 u(s)\ ds - g(\Delta u_t)\Big)\ v\ dx\ dt = 0$$

Therefore, u is a solution for the problem (2.1). The proof of Theorem 2.3.1 is now completed.

2.4 Assymptotic behavior

Introduce the energy associeted to the system (2.1) such that

$$E(t) = \frac{1}{2}\|\nabla u_t\|^2 + \frac{1}{2}(h\circ\nabla\Delta u)(t) + \frac{1}{2}\Big(1 - \int_0^t h(s)ds\Big)\|\nabla\Delta u\|^2. \tag{2.39}$$

Remark 2.4.1. *By multiplying equation (2.1) by $-\Delta u_t$, integrating over Ω and using Green formula and the boundary conditions we get*

$$E'(t) = -\frac{1}{2}h(t)\|\nabla\Delta u\|^2 + \frac{1}{2}h'\circ\nabla\Delta u(t) - \int_\Omega g(\Delta u_t)\Delta u_t\ dx \le 0. \tag{2.40}$$

Theorem 2.4.1. *Assume that (A1) and (A2) hold. Then there exist positive constants k_0 and k_1 such that the solution of the problem (2.1) satisfies*

$$E(t) \le k_0 G_1^{-1}\left(k_1 \int_0^t \nu(s)\,ds\right), \quad \forall\ \ t\in\mathbb{R}_+, \tag{2.41}$$

where

$$G_1(t) = \int_t^1 \frac{1}{G_2(s)}\ ds \tag{2.42}$$

and

$$G_2(t) = \begin{cases} t, & \text{if } G \text{ is linear on } [0,\varepsilon] \\ tG'(\varepsilon_0 t), & \text{if } G'(0)=0 \text{ and } G''>0 \text{ on }]0,\varepsilon] \end{cases}$$

For study the stability of a system (2.1), we use the Liapounov function giving some estimates.

For $M > 0$ and $\varepsilon_1 > 0$, we define a perturbed modified energy by

$$L(t) = ME(t) + \varepsilon_1 \Psi(t) + \chi(t), \tag{2.43}$$

where

$$\Psi(t) = -\int_\Omega u_t \Delta u \; dx,$$

$$\chi(t) = -\int_\Omega u_t \int_0^t h(t-s)(\Delta u(t) - \Delta u(s)) \; ds \; dx.$$

Lemma 2.4.2. *There exist two positive constants α_1 and α_2 depending on ε_1 and M such that for all $t > 0$,*

$$\alpha_1 E(t) \leq L(t) \leq \alpha_2 E(t). \tag{2.44}$$

Proof. Using Cauchy-Schwarz, Sobolev -Poincare inequalities and (A_1), we have

$$|\Psi(t)| \leq C_s \|\nabla u_t\| \|\nabla \Delta u\| \leq \frac{C_s}{2}\Big(\|\nabla u_t\|^2 + \|\nabla \Delta u\|^2\Big) \leq CE(t),$$

and

$$\begin{aligned} |\chi(t)| &\leq \frac{C_s}{2}\left(\|\nabla u_t\|^2 + \int_\Omega \left| \int_0^t h(t-s)(\nabla\Delta u(t) - \nabla \Delta u(s)) \; ds \right|^2\right) dx \\ &\leq \frac{C_s}{2}\|\nabla u_t\|^2 + \frac{C_s^2 l}{2} \; (h \circ \nabla \Delta u)(t) \leq CE(t). \end{aligned}$$

Choosing M large enough, we obtain estimate (2.44). □

Proof of Theorem 2.4.1. For each $t_0 > 0$, sufficiently large $M > 0$ and suitably small $\varepsilon_1 > 0$, there exist positive constants C_1, C_2, and C_3, such that

$$\frac{d}{dt}L(t) \leq -C_1 E(t) + C_2 (h \circ \nabla \Delta u)(t) + C_3 \|g(\Delta u_t)\|^2, \quad \forall t_0 \geq t. \tag{2.45}$$

The proof of this theorem will be carried out throughout the following two lemmas

Lemma 2.4.3. *For any $\eta > 0$, the functional $\Psi(t)$ satisfy*

$$\Psi'(t) \leq -(1 - l - \eta - C_s^2 \eta)\|\nabla \Delta u(t)\|^2 + \frac{l}{4\eta}(h \circ \nabla \Delta u)(t) + \|\nabla u_t\|^2 + \frac{1}{4\eta}\int_\Omega |g(\Delta u_t)|^2 \; dx. \tag{2.46}$$

Proof. Taking the derivative of $\Psi(t)$ with respect to t, using the first equation in the system (2.1), we obtain

$$\begin{aligned} \Psi'(t) &= -\int_\Omega u_t \Delta u_t \; dx + \int_\Omega \Big(\Delta^2 u - \int_0^t h(t-s)\Delta^2 u(s)\, ds - g(\Delta u_t)\Big)\Delta u \, dx \\ &= \int_\Omega |\nabla u_t|^2 \; dx - \int_\Omega |\nabla \Delta u|^2 dx + \int_\Omega \Big(\int_0^t h(t-s)\nabla \Delta u(s)\, ds\Big)\nabla \Delta u(t) dx \\ &+ \int_\Omega g(\Delta u_t)\Delta u \; dx. \end{aligned} \tag{2.47}$$

Now, the third term in the right-hand side of (2.47) can be estimated as follows:

$$\begin{aligned}
&\int_\Omega \Big(\int_0^t h(t-s)\nabla\Delta u(s)\,ds\Big)\nabla\Delta u(t)\,dx \\
&= \int_\Omega \Big(\int_0^t h(t-s)[\nabla\Delta u(s)-\nabla\Delta u(t)]\,ds\Big)\nabla\Delta u(t)\,dx + \int_\Omega \Big(\int_0^t h(t-s)[\nabla\Delta u(t)]^2\,ds\Big)\,dx \\
&\le \|\nabla\Delta u(t)\|\Big(\int_\Omega \Big|\int_0^t h(t-s)[\nabla\Delta u(s)-\nabla\Delta u(t)]ds\Big|^2\,dx\Big)^{1/2} + l\,\|\nabla\Delta u(t)\|^2 \\
&\le l^{1/2}\|\nabla\Delta u(t)\|(h\circ\nabla\Delta u)^{1/2}(t) + l\,\|\nabla\Delta u(t)\|^2 \\
&\le (l+\eta)\|\nabla\Delta u(t)\|^2 + \frac{l}{4\eta}(h\circ\nabla\Delta u)(t).
\end{aligned}$$

Then, we conclude

$$\Psi'(t) \le (l+\eta-1)\|\nabla\Delta u(t)\|^2 + \frac{l}{4\eta}(h\circ\nabla\Delta u)(t) + \|\nabla u_t\|^2 + \int_\Omega |g(\Delta u_t)||\Delta u|\,dx. \tag{2.48}$$

Since

$$\int_\Omega |g(\Delta u_t)||\Delta u|\,dx \le C_s^2\eta\|\nabla\Delta u(t)\|^2 + \frac{1}{4\eta}\int_\Omega |g(\Delta u_t)|^2\,dx. \tag{2.49}$$

By using (2.48) and (2.49), we obtain (2.46). □

Lemma 2.4.4. *For any $\eta > 0$, the functional $\chi(t)$ satisfy*

$$\begin{aligned}
\chi'(t) \le{}& \eta(1+l)\|\nabla\Delta u\|^2 + \Big(l + \frac{l}{4\eta} + \frac{lC_s^2}{4\eta} + \frac{l^2}{4\eta}\Big)(h\circ\nabla\Delta u)(t) \\
&- \Big(\Big(\int_0^t h(s)\,ds\Big) - C_s^2\eta\Big)\|\nabla u_t\|^2 + \eta\|g(\Delta u_t)\|^2 - \frac{h_0C_s^2}{4\eta}(h'\circ\nabla\Delta u)(t).
\end{aligned} \tag{2.50}$$

Proof. By differentiating χ, then exploiting the first equation in the system (2.1), and integrating by parts, we obtain

$$
\begin{aligned}
\chi'(t) &= -\int_\Omega u_{tt} \int_0^t h(t-s)(\Delta u(t) - \Delta u(s))\ ds\ dx - \int_\Omega u_t \int_0^t h(t-s)\Delta u_t(t)\ ds\ dx \\
&- \int_\Omega u_t \int_0^t h'(t-s)(\Delta u(t) - \Delta u(s))\ ds\ dx \\
&= -\int_\Omega \Big[\Big(-\Delta^2 u + \int_0^t h(t-s)\Delta^2 u(s) ds + g(\Delta u_t)\Big) \int_0^t h(t-s)(\Delta u(t) - \Delta u(s))\ ds\Big]\ dx \\
&- \int_0^t h(s)\, ds \|\nabla u_t(t)\|^2 - \int_\Omega u_t \int_0^t h'(t-s)(\Delta u(t) - \Delta u(s))\ ds\ dx \\
&- \int_\Omega \nabla\Delta u(t) \int_0^t h(t-s)(\nabla\Delta u(t) - \nabla\Delta u(s))\ ds\ dx \qquad (2.51) \\
&= -\int_\Omega \nabla\Delta u(t) \int_0^t h(t-s)(\nabla\Delta u(t) - \nabla\Delta u(s))\ ds\ dx \\
&+ \int_\Omega \Big[\int_0^t h(t-s)\nabla\Delta u(s)\, ds \int_0^t h(t-s)(\nabla\Delta u(t) - \nabla\Delta u(s))\ ds\Big]\ dx \\
&- \int_\Omega g(\Delta u_t) \int_0^t h(t-s)(\Delta u(t) - \Delta u(s))\ ds\ dx \\
&- \int_0^t h(s)\, ds \|\nabla u_t(t)\|^2 - \int_\Omega u_t \int_0^t h'(t-s)(\Delta u(t) - \Delta u(s))\ ds\ dx.
\end{aligned}
$$

Using Young's, Sobolev-Poincaré and Cauchy-Schwarz inequalities, we infer

$$
\begin{aligned}
-\int_\Omega u_t \int_0^t h'(t-s)(\Delta u(t) - \Delta u(s))\ ds\ dx &\leq C_s^2 \eta \|\nabla u_t\|^2 + \frac{C_s^2}{4\eta}\Big(\int_0^t h'(s) ds\Big)(h' \circ \nabla\Delta u)(t) \\
&\leq C_s^2 \eta \|\nabla u_t\|^2 - \frac{h_0 C_s^2}{4\eta}(h' \circ \nabla\Delta u)(t),
\end{aligned}
$$

$$
\begin{aligned}
-\int_\Omega \nabla\Delta u(t) \int_0^t h(t-s)(\nabla\Delta u(t) - \nabla\Delta u(s))\ ds\ dx &\leq \eta \|\nabla\Delta u\| + \frac{l}{4\eta}(h \circ \nabla\Delta u)(t), \\
-\int_\Omega g(\Delta u_t) \int_0^t h(t-s)(\Delta u(t) - \Delta u(s))\ ds\ dx &\leq \eta \|g(\Delta u_t(t))\|^2 + \frac{l C_s^2}{4\eta}(h \circ \nabla\Delta u)(t),
\end{aligned}
$$

and

$$
\begin{aligned}
&-\int_\Omega \int_0^t h(t-s)\Big[(\nabla\Delta u(s)-\nabla\Delta u(t))\,ds\Big]\int_0^t h(t-s)(\Delta u(t)-\Delta u(s))\,ds\,dx\\
&-\int_\Omega \int_0^t h(t-s)\nabla\Delta u(t)\,ds\int_0^t h(t-s)(\Delta u(t)-\Delta u(s))\,ds\,dx\\
&\le \int_\Omega \Big(\int_0^t h(t-s)(\Delta u(t)-\Delta u(s))\,ds\,dx\Big)^2 dx\\
&+\int_\Omega |\nabla\Delta u(t)|\Big(\int_0^t h(s)\,ds\Big)\Big(\int_0^t h(t-s)|\Delta u(t)-\Delta u(s)|\,ds\,dx\Big)\\
&\le \Big(\int_0^t h(s)\,ds\Big)(h\circ\nabla\Delta u)(t)+\int_\Omega |\nabla\Delta u(t)|\Big(\int_0^t h(s)\,ds\Big)^{\frac12}\Big(\int_0^t h(t-s)|\Delta u(t)-\Delta u(s)|^2\,ds\,dx\Big)^{\frac12}\\
&\le \Big(\int_0^t h(s)\,ds\Big)(h\circ\nabla\Delta u)(t)+\|\nabla\Delta u(t)\|\Big(\int_0^t h(s)\,ds\Big)\Big[\Big(\int_0^t h(s)\,ds\Big)(h\circ\nabla\Delta u)(t)\Big]^{\frac12}\\
&\le \Big(\int_0^t h(s)\,ds\Big)(h\circ\nabla\Delta u)(t)+\Big(\int_0^t h(s)\,ds\Big)\Big[\eta\|\nabla\Delta u(t)\|^2+\frac{1}{4\eta}\Big(\int_0^t h(s)\,ds\Big)(h\circ\nabla\Delta u)(t)\Big]\\
&\le \eta l\|\nabla\Delta u(t)\|^2+\Big(l+\frac{l^2}{4\eta}\Big)(h\circ\nabla\Delta u)(t).
\end{aligned}
$$

Combining all the above estimates allows us to conclude

$$
\begin{aligned}
\chi'(t)\le{}& \eta(1+l)\|\nabla\Delta u\|^2+\Big(l+\frac{l}{4\eta}+\frac{lC_s^2}{4\eta}+\frac{l^2}{4\eta}\Big)(h\circ\nabla\Delta u)(t)\\
&-\Big(\Big(\int_0^t h(s)\,ds\Big)-C_s^2\eta\Big)\|\nabla u_t\|^2+\eta\|g(\Delta u_t)\|^2-\frac{h_0C_s^2}{4\eta}(h'\circ\nabla\Delta u)(t).
\end{aligned}
$$

□

End of Proof of Theorem 2.4.1: Since h is positive, for any $t_0>0$, we have $\int_0^t h(s)\,ds \ge \int_0^{t_0} h(s)\,ds=\tilde{h}_0$, for all $t>t_0$. Taking this into account and combining (2.40), (2.46) and (2.50), we deduce that

$$
\begin{aligned}
L'(t)\le{}& -a_1\|\nabla u_t\|^2-a_2\|\nabla\Delta u\|^2+a_3\ h(\circ\nabla\Delta u)(t)\\
&+a_4\ (h'\circ\nabla\Delta u)(t)+\varepsilon_2\eta\|g(\Delta u_t)\|^2
\end{aligned}
\tag{2.52}
$$

Now, we choose, $\varepsilon_1>0$ and $\eta>0$ so small that

$$a_1=\tilde{h}_0-C_s^2\eta-\varepsilon_1>0$$

$$a_2=\varepsilon(1-l-\eta-C_s^2\eta)-\eta(1+l)>0$$

and

$$a_3=\varepsilon\frac{l}{4\eta}+l+\frac{l}{4\eta}+\frac{l^2}{4\eta}+\frac{lC_s^2}{4\eta}$$

Then, we pick the constant $M>0$ sufficiently large such that

$$a_4=\frac{M}{2}-\frac{h_0C_s^2}{4\eta}.$$

Therefore, (2.52) takes the form

$$L'(t) \le -C_1E(t) + C_2(h \circ \nabla\Delta u)(t) + C_3\|g(\Delta u_t)\|^2, \tag{2.53}$$

where C_1, C_2 and C_3 are three positive constants.
Now, we estimate the last term in the right-hand side of (2.45). We define tow sets such that

$$\Omega_1 = \{x \in \Omega : |\Delta u_t| > \varepsilon\} \text{ and } \Omega_2 = \{x \in \Omega : |\Delta u_t| \le \varepsilon\}.$$

From (2.4), (2.5) and (2.40), we have

$$\int_{\Omega_1} |g(\Delta u_t)|^2\,dx \le C_s\int_{\Omega} \Delta u_t g(\Delta u_t)|\,dx \le -CE'(t) \tag{2.54}$$

and

$$\int_{\Omega_2} |g(\Delta u_t)|^2\,dx \le \int_{\Omega_2} G^{-1}\Big(\Delta u_t g(\Delta u_t)\Big)\,dx.$$

Case 1. G is linear on $[0,\varepsilon]$, we obtain

$$\int_{\Omega_2} |g(\Delta u_t)|^2\,dx \le -CE'(t). \tag{2.55}$$

Substitution of (2.54) and (2.55) into (2.53) gives

$$(L(t) + CE(t))' \le -C_1G_2(E(t)) + C_2(h \circ \nabla\Delta u)(t). \tag{2.56}$$

Case 2. G is nonlinear on $[0,\varepsilon]$, we exploit Jensen's inequality, it follows that

$$\begin{aligned}\int_{\Omega_2} |g(\Delta u_t)|^2\,dx &\le \int_{\Omega_2} G^{-1}\Big(\Delta u_t g(\Delta u_t)\Big)\,dx\\ &\le |\Omega|G^{-1}\Big(\frac{1}{|\Omega|}\int_{\Omega_2} \Delta u_t g(\Delta u_t)\,dx\Big)\\ &\le C_4G^{-1}(-C'E'(t)).\end{aligned} \tag{2.57}$$

A combination of (2.45), (2.54) and (2.57) yields

$$(L(t) + CE(t))' \le -C_1E(t) + C_2(h \circ \nabla\Delta u)(t) + C_4G^{-1}(-C'E'(t)), \quad t \ge t_0. \tag{2.58}$$

Making use of $E'(t) \le 0$, $G''(t) > 0$ (2.7), (2.8) and (2.58), we conclude for $\varepsilon_0 > 0$ small enough

$$\begin{aligned}&\Big[G'(\varepsilon_0E(t))\{L(t) + CE(t)\} + C_4C'E(t)\Big]'\\ &= \varepsilon_0E'(t)G''(\varepsilon_0E(t))\{L(t) + CE(t)\} + G'(\varepsilon_0E(t))\{L(t) + CE(t)\}' + C_4C'E'(t)\\ &\le -C_1G'(\varepsilon_0E(t))E(t) + C_2G'(\varepsilon_0E(t))(h \circ \nabla\Delta u)(t) + C_4G'(\varepsilon_0E(t))G^{-1}(-C'E'(t)) + C_4C'E'(t)\\ &\le -C_1G'(\varepsilon_0E(t))E(t) + C_4G^*(G'(\varepsilon_0E(t))) + C_2G'(\varepsilon_0E(t))(h \circ \nabla\Delta u)(t)\\ &\le -C_1G'(\varepsilon_0E(t))E(t) + C_4(G'(\varepsilon_0E(t)))\varepsilon_0E(t) + C_2G'(\varepsilon_0E(t))(h \circ \nabla\Delta u)(t)\\ &\le -C_1G'(\varepsilon_0E(t))E(t) + C_2G'(\varepsilon_0E(t))(h \circ \nabla\Delta u)(t)\end{aligned}$$

We have $0 \leq G'(\varepsilon_0 E(t)) \leq G'(\varepsilon_0 E(0))$ then we obtain

$$\Big[G'(\varepsilon_0 E(t))\{L(t)+CE(t)\}+C_4C'E(t)\Big]' \leq -C_1G_2(E(t))+C_2G'(\varepsilon_0E(0))(h\circ\nabla\Delta u)(t) \tag{2.59}$$

If G is linear

$$\widetilde{E}(t)=L(t)+CE(t) \tag{2.60}$$

and if G is non linear

$$\widetilde{E}(t)=G'(\varepsilon_0 E(t))\{L(t)+CE(t)\}+C_4C'E(t) \tag{2.61}$$

From (2.56) (2.58), (2.59), (2.60) and (2.61), we have

$$\widetilde{E}'(t)\leq -C_1G_2(E(t))+C_2(h\circ\nabla\Delta u)(t)$$

On the other hand, we can observe from Lemma 2.4.2 that $L(t)$ is equivalent to $E(t)$. So, $\widetilde{E}(t)$ is also equivalent to $E(t)$. Moreover, because $\nu(t)\leq\nu(0)$, there exists $\widetilde{\varepsilon}>0$ such that

$$\nu(t)\widetilde{E}(t)+2C_2E(t)\leq\widetilde{\varepsilon}E(t),\quad \forall t\geq t_0$$

Let

$$F(t)=\varepsilon(\nu(t)\widetilde{E}(t)+2C_2E(t)),\quad \text{for}\quad 0<\varepsilon<\frac{1}{\widetilde{\varepsilon}}$$

$$\begin{aligned}
F'(t)&=\varepsilon\Big(\nu(t)\widetilde{E}'(t)+\nu'(t)\widetilde{E}(t)+2C_2E'(t)\Big)\\
&\leq -C_1\varepsilon\nu(t)G_2(E(t))+C_2\varepsilon\nu(t)(h\circ\nabla\Delta u)(t)+2C_2\varepsilon E'(t)\\
&\leq -C_1\varepsilon\nu(t)G_2(E(t))-C_2\varepsilon(h'\circ\nabla\Delta u)(t)+2C_2\varepsilon E'(t)\\
&\leq -C_1\varepsilon\nu(t)G_2(E(t))\\
&\leq -C_1\varepsilon\nu(t)G_2\Big(\frac{1}{\widetilde{\varepsilon}}\Big(\nu(t)\widetilde{E}(t)+2C_2E(t)\Big)\Big)\\
&\leq -C_1\varepsilon\nu(t)G_2\Big(\nu(t)\widetilde{E}(t)+2C_2E(t)\Big)=-C_1\varepsilon\nu(t)G_2(F(t))
\end{aligned} \tag{2.62}$$

In the last two inequalities, we have used the fact that G_2 is increasing. Recalling that $G_1'=-\dfrac{1}{G_2}$, we infer from (2.62)

$$F'(t)G_1'(F(t))\geq C_1\varepsilon\nu(t),\ \ \forall t\geq t_0$$

A simple integration over (t_0,t) yields

$$G_1(F(t))\geq G_1(F(t_0))+C_1\varepsilon\int_0^t\nu(s)\,ds-C_1\varepsilon\int_0^{t_0}\nu(s)\,ds.$$

Choosing $\varepsilon > 0$ sufficiently small such that $G_1(F(t_0)) - C_1\varepsilon \int_0^{t_0} \nu(s)\,ds > 0$, and exploiting the fact that G_1^{-1} is decreasing, we infer

$$\begin{aligned} F(t) &\leq G_1^{-1}\Big(G_1(F(t_0)) + C_1\varepsilon \int_0^t \nu(s)\,ds - C_1\varepsilon \int_0^{t_0} \nu(s)\,ds\Big) \\ &\leq G_1^{-1}\Big(C_1\varepsilon \int_0^t \nu(s)\,ds\Big). \end{aligned}$$

Consequently, the equivalence of L, $\widetilde{E}$, F and E yields the estimate

$$E(t) \leq k_0 G_1^{-1}\Big(C_1\varepsilon \int_0^t \nu(s)\,ds\Big).$$

This concludes the proof of Theorem 2.4.1.

2.5 Examples

Example 2.1. *Let g given by $g(s) = s^p(-\ln s)^q$ where $p > 1$ and $q \in \mathbb{R}$ on $[0, \varepsilon]$ and the function G is defined in the neighborhood of 0 by*

$$G(s) = cs^{\frac{p+1}{2}}(-\ln\sqrt{s})^q,$$

we have

$$G'(s) = cs^{\frac{p-1}{2}}(-\ln\sqrt{s})^{q-1}[\frac{p+1}{2}(-\ln\sqrt{s}) - \frac{q}{2}]$$

$$G_1(t) = \frac{1}{c}\int_t^1 \frac{1}{s^{\frac{p+1}{2}}(-\ln\sqrt{s})^{q-1}[\frac{p+1}{2}(-\ln\sqrt{s}) - \frac{q}{2}]}.$$

Making the following changement of variable $z = \dfrac{1}{\sqrt{s}}$ we obtain

$$G_1(t) = \frac{1}{c}\int_t^{\frac{1}{\sqrt{t}}} \frac{z^{p-2}}{(\ln z)^{q-1}(\frac{p+1}{2}\ln z - \frac{q}{2})}dz.$$

We have three cases :

***The case 1** : if $p = q = 1$,we have*

$$G_1(t) = \frac{2}{c}\ln(-\ln\sqrt{et}),$$

we deduce that

$$G_1^{-1}(t) = \frac{1}{e}e^{-2e^{\frac{c}{2}t}},$$

then

$$E(t) \leq k_0\frac{1}{e}e^{-2e^{\frac{c}{2}C_1\varepsilon\int_0^t \nu(s)\,ds}}$$

The case 2 : *if* $p = 1, q < 1$ *,we have*

$$G_1(t) = \frac{2}{c}\int_t^{\frac{1}{\sqrt{t}}} \frac{1}{z(\ln z)^{q-1}(\ln z - \frac{q}{2})}dz \sim \frac{2}{c}\int_t^{\frac{1}{\sqrt{t}}} \frac{1}{z(\ln z)^q}dz$$

$$\sim \frac{2^q}{c(1-q)}(-\ln t)^{1-q} \quad as \quad t \to 0,$$

we deduce that

$$G_1^{-1}(t) \sim e^{-kt^{\frac{1}{1-q}}} \quad as \quad t \to \infty,$$

then

$$E(t) \leq k_0 e^{-k(C_1\varepsilon\int_0^t \nu(s))^{\frac{1}{1-q}}}$$

where $k = (\dfrac{c(1-q)}{2q})^{\frac{1}{1-q}}$

Chapter 3

Well-posedness and stability for a Petrovsky equation with properties of nonlinear localized for strong damping

3.1 Introduction

There has been significant advancement in the study of the stabilization of the hyperbolic equations with localized damping, which arise from many branches of applied sciences such as physics, mechanics, chemistry, material sciences and biological sciences. We have a number of detailed articles and reviews on this topic that relate to [61], where a localized frictional damping has been considered and exponential decay was obtained under an appropriate geometric control condition to impress a large class of damping regions. In particular, a semilinear wave equation with nonlinear localized damping and source terms was developed in [45]. The authors considered, in open bounded connected domain, the problem

$$u_{tt} + \Delta u - \chi g(u_t) = f(u). \tag{3.1}$$

The question was discussed in terms of topological and geometric aspects to extend previous work and find optimal decay rate (See [50]). As another type of such problem, we mention the Petrovsky equation with locally damping, considered in [29]

$$\begin{cases} u_{tt} + \Delta^2 u - \rho(x, u_t) = 0 \\ \dfrac{\partial u}{\partial \nu} = u = 0 \\ u(x,0) = u^0(x), \;\; u_t(x,0) = u^1(x). \end{cases} \tag{3.2}$$

The asymptotic behavior for solution was investigated by the authors and an explicit energy decay was established. We refer the reader to the following papers [20, 34, 46, 66, 67, 68].

Motivated by all above papers, we investigate the well-posedness and stability of the following damped beam equation

$$\begin{cases} u_{tt} + \Delta^2 u - a(x)g(\Delta u_t) = 0, & (x,t) \in \Omega \times [0, +\infty[\\ u = \Delta u = 0, & (x,t) \in \Gamma \times [0, \infty[\\ u(x,0) = u^0(x), \;\; u_t(x,0) = u^1(x), & x \in \Omega, \end{cases} \tag{3.3}$$

where Ω is a bounded domain of $\mathbb{R}^n$ with boundary Γ which assumed to be regular. The function $g : \mathbb{R} \to \mathbb{R}$ is a continuous nondecreasing, $a : \Omega \to \mathbb{R}$ is a nonnegative and bounded function. Let $x^0 \in \mathbb{R}^n$ be an arbitrary point of $\mathbb{R}^n$. We set

$$\Gamma(x^0) = \Big\{ x \in \Gamma; \quad m(x).\nu(x) > 0 \Big\}, \tag{3.4}$$

where ν is the unit normal vector pointing towards the exterior of Ω and

$$m(x) = x - x^0. \tag{3.5}$$

Let ω be a neighborhood of $\Gamma(x^0)$ in Ω and consider δ sufficiently small such that

$$\mathcal{M}_0 = \Big\{ x \in \Omega; d(x, \Gamma(x^0)) < \delta \Big\} \subset \omega \tag{3.6}$$

and

$$\mathcal{M}_1 = \Big\{ x \in \Omega; d(x, \Gamma(x^0)) < 2\delta \Big\} \subset \omega. \tag{3.7}$$

If $A \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, we have

$$d(x; A) = \inf_{y \in A} (|x - y|),$$

and $\mathcal{M}_0 \subset \mathcal{M}_1 \subset \omega$.

When $g(\Delta u_t) = |\Delta u_t|^{p-2} \Delta u_t$ the problem (3.3) was treated by Tebou [67]. The author proved the existence and uniqueness of global solution u for (3.3). Then, using an appropriate perturbed energy combined with multiplier technique, he directly proved exponential and polynomial decay estimates for the associated energy.

Very recently, Tebou [66] proved the existence of global solution, as well as, the exponential stability result for similar strong damping wave equation with a localized nonlinear source term.

Ammari et *al.* [2] studied the system

$$\begin{cases} u_{tt} - \Delta u - \operatorname{div}(a(x)\nabla u_t) = 0, & (x,t) \in \Omega \times [0, +\infty[\\ u = 0, & (x,t) \in \Gamma \times [0, \infty[\\ u(x,0) = u_0(x), \;\; u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{3.8}$$

where $a(x) = d\mathbf{1}_\omega(x)$, $d > 0$ and ω be an open, non-empty subset of Ω with smooth boundary. The authors obtained a logarithmic decay of energy. Their idea is to transform the resolvent problem of (3.8) to a transmission system to be easy to use the so-called Carleman estimate. In [28], the same problem has been considered and the polynomial energy estimate was showed. Liu and Rao in [46] and Tebou [68] proved the exponential stability. Moreover, when $a \equiv 1$ Komornik [32] treated the problem (3.3) for g having a polynomial growth near the origin, by using semigroup theory to prove the existence and uniqueness of solution and established energy decay results depending on g.
In the present paper, we prove the global existence of weak solution of (3.3) by using the Galerkin method (see Lions [41]) combined with a semigroup theory. Meanwhile, under suitable conditions on the function g with some ideas inspired from [18], we estimate the energy decay of the solution under some conditions on the nonlinear function g and nonnegative coefficient a.
The plan of this article is as follows. We present some notations and assumptions needed for our results and then establish the well-posedness of our problem by the semigroup theory in Section 2. Section 3 is devoted to use the Faedo-Galerkin method and prove the regularity of solution. In Section 4, we obtain the stability by introducing a suitable Lyapunov function.

3.2 Preliminaries

We begin by introducing some notations that will be used throughout this work. For the standard $L^q(\Omega)$ space, we write

$$(u, v) = \int_\Omega u(x)v(x)\,dx, \quad \|u\|_q^q = \int_\Omega |u(x)|^q\,dx.$$

Set

$$V = H_0^1(\Omega), \qquad \|u\|_V = \int_\Omega |\nabla u|^2\,dx,$$

$$W = \{u \in H^3(\Omega) \cap H_0^1(\Omega),\ \Delta u = 0 \text{ on } \Gamma\}, \quad \|u\|_W = \int_\Omega |\nabla \Delta u|^2\,dx,$$

and

$$\widetilde{W} = \{u \in H^5(\Omega) \cap H_0^1(\Omega),\ \Delta u = \Delta^2 u = 0 \text{ on } \Gamma\}, \quad \|u\|_{\widetilde{W}} = \int_\Omega |\nabla \Delta^2 u|^2\,dx.$$

First assume that a and g satisfies the following hypotheses:

(A1) The function $a : \Omega \to \mathbb{R}$ is a nonnegative and bounded such that

$$\begin{cases} \exists a_0 > 0, \quad a(x) \geq a_0 > 0 \quad \text{a.e} \quad \text{in} \quad \omega. \\ a(x) \in W^{1,\infty}(\Omega). \end{cases} \tag{3.9}$$

(A2) $g \in C^1(\mathbb{R}, \mathbb{R})$ is non-decreasing function with $g(0) = 0$ and assume that it is globally Lipschitz. Suppose that, for $c_1, c_2, c_3, c_4 > 0$ and $p \geq 1$, we have

$$c_1|s|^p \leq g(s) \leq c_2|s|^{\frac{1}{p}}, \quad \text{if} \quad |s| \leq 1, \tag{3.10}$$

$$c_3|s| \leq g(s) \leq c_4|s|, \quad \text{if} \quad |s| > 1, \tag{3.11}$$

$$\exists \tau > 0, \ |g'(s)| \leq \tau, \ \forall s \in \mathbb{R}. \tag{3.12}$$

We introduce the functional energy

$$\mathcal{E}(t) = \frac{1}{2}\|\nabla u_t(t)\|^2 + \frac{1}{2}\|\nabla \Delta u(t)\|^2. \tag{3.13}$$

Note that $\mathcal{E}$ is the natural energy for system (3.3), given the structure of the damping term.

Lemma 3.2.1. *Let u be a solution to the problem (3.3). Then $\mathcal{E}$ is a non-increasing function for all t on $\mathbb{R}_+$.*

Proof. Multiplying the first equation in (3.3) by $-\Delta u_t$, integrating over Ω, using Green formula and the boundary conditions, we get

$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla u_t(t)\|^2 + \|\nabla \Delta u(t)\|^2\right) = -\int_\Omega a(x)\Delta u_t(x,t) g(\Delta u_t(x,t))\,dx.$$

Then by **(A1)** and **(A2)**, we have

$$\mathcal{E}'(t) = -\int_\Omega a(x)\Delta u_t(x,t) g(\Delta u_t(x,t))\,dx \leq 0. \tag{3.14}$$

This completes the proof. □

3.3 Well-posedness

Let us introduce the vector function $U = (u, v)^T$, where $v = u_t$ and rewrite (3.3) as

$$\begin{cases} U_t + \mathcal{A}U = 0, & \text{in} \quad \Omega \\ U(0) = \begin{pmatrix} u^0 \\ u^1 \end{pmatrix}. \end{cases} \tag{3.15}$$

Here the nonlinear operator $\mathcal{A}$ is defined by

$$\mathcal{A} = \begin{pmatrix} 0 & -\mathcal{I} \\ \Delta^2 & -a(.)g(\Delta .) \end{pmatrix}. \tag{3.16}$$

The domain of $\mathcal{A}$ is given by

$$D(\mathcal{A}) = \left\{(u, v) \in W \times W; \Delta^2 u - ag(\Delta v) \in V\right\}.$$

Introduce the Hilbert space $\mathcal{H} = W \times V$, equipped with the norm

$$\|U\|_{\mathcal{H}}^2 = \int_\Omega |\nabla v|^2 \, dx + \int_\Omega |\nabla \Delta u|^2 dx \quad \forall \ (u, v) \in \mathcal{H}. \tag{3.17}$$

It is not hard to see that $D(\mathcal{A})$ is dense in $\mathcal{H}$.

We now state the existence and uniqueness result as follows.

Theorem 3.3.1. *Let $(u_0, u_1) \in W \times V$ and suppose that (3.9)-(3.12) hold. Then, there exists a solution for system (3.3) that satisfies*

$$u \in \mathcal{C}([0, \infty), W) \cap \mathcal{C}^1([0, \infty), V). \tag{3.18}$$

Proof. We show, in the first step, that $\mathcal{A}$ is a maximal monotone operator.

Let $U = \begin{pmatrix} u \\ v \end{pmatrix}$ and $\widetilde{U} = \begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix}$ be in $D(\mathcal{A})$. We have

$$\begin{aligned}
a \qquad & (\mathcal{A}U - \mathcal{A}\widetilde{U}, U - \widetilde{U}) \\
&= \left(\begin{pmatrix} -v + \widetilde{v} \\ \Delta^2 u - a(x) g(\Delta v) - \Delta^2 \widetilde{u} + a(x) g(\Delta \widetilde{v}) \end{pmatrix}, \begin{pmatrix} u - \widetilde{u} \\ v - \widetilde{v} \end{pmatrix} \right) \\
&= \left(\begin{pmatrix} -v + \widetilde{v} \\ \Delta^2 (u - \widetilde{u}) - a(x)(g(\Delta v) - g(\Delta \widetilde{v})) \end{pmatrix}, \begin{pmatrix} u - \widetilde{u} \\ v - \widetilde{v} \end{pmatrix} \right) \\
&= - \int_\Omega \nabla \Delta (v - \widetilde{v}) \nabla \Delta (u - \widetilde{u}) \, dx - \int_\Omega \Delta^2 (u - \widetilde{u}) \Delta (v - \widetilde{v}) \, d \\
&+ \int_\Omega a(x)(g(\Delta v) - g(\Delta \widetilde{v}))(\Delta v - \Delta \widetilde{v}) \, dx,
\end{aligned}$$

and thus, integrating by parts, to get

$$\begin{aligned}
& (\mathcal{A}U - \mathcal{A}\widetilde{U}, U - \widetilde{U}) \\
&= - \int_\Omega \nabla \Delta (v - \widetilde{v}) \nabla \Delta (u - \widetilde{u}) \, dx + \int_\Omega \nabla \Delta (u - \widetilde{u}) \nabla \Delta (v - \widetilde{v}) \, dx \\
&+ \int_\Omega a(x)(g(\Delta v) - g(\Delta \widetilde{v}))(\Delta v - \Delta \widetilde{v}) \, dx \\
&= \int_\Omega a(x)(g(\Delta v) - g(\Delta \widetilde{v}))(\Delta v - \Delta \widetilde{v}) \, dx \\
&\geq 0.
\end{aligned}$$

Then, the accretivity of nonbounded operator $\mathcal{A}$ is done.

Now, we prove that the operator $\mathcal{I} + \mathcal{A}$ is surjective.

Define $F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \ \mathcal{H}$, and show that there exists $U = \begin{pmatrix} u \\ v \end{pmatrix} \in D(\mathcal{A})$ satisfying

$$U_t + \mathcal{A}U = F,$$

that is

$$\begin{cases} u - v = f_1 \in W \\ v + \Delta^2 u - a(x)g(\Delta v) = f_2 \in V. \end{cases} \tag{3.19}$$

Equation $(3.19)_1$ gives u as a function of v and f_1. Substituting this in $(3.19)_2$, we get

$$v + \Delta^2 v - a(x)g(\Delta v) = f_2 - \Delta^2 f_1. \tag{3.20}$$

Let $\mathcal{B}$ the operator given by

$$\mathcal{B}v = f_2 - \Delta^2 f_1, \tag{3.21}$$

where we set

$$\mathcal{B}v = v + \Delta^2 v - a(x)g(\Delta v).$$

Since we are looking for v in W, the nonlinear operator $\mathcal{B}$ defined from W into its topological dual W', with $W \hookrightarrow V \hookrightarrow W'$ and V is the pivot space. This is doable because g is globally Lipschitz. Now, we prove that the operator $\mathcal{B}$ is monotone

$$\begin{aligned}
&\langle \mathcal{B}u - \mathcal{B}v, u - v\rangle \\
&= \langle u - v + \Delta^2(u - v) - a(x)(g(\Delta u) - g(\Delta v)), u - v\rangle \\
&= \int_\Omega \{|\nabla u - \nabla v|^2 + |\nabla \Delta u - \nabla \Delta v|^2 + a(x)(g(\Delta u) - g(\Delta v))(\Delta u - \Delta v)\}\, dx \\
&\geq 0.
\end{aligned}$$

So $\mathcal{B}$ is monotone.

Next, we prove that $\mathcal{B}$ is coercive

$$\begin{aligned}
\frac{\langle \mathcal{B}u, u\rangle}{\|u\|_W} &= \frac{\int_\Omega \{|\nabla u|^2 + |\nabla \Delta u|^2 + a(x)g(\Delta u)\Delta u\}\, dx}{\|u\|_W} \\
&\leq C\|u\|_W + C'\Big(\int_\Omega (a(x)g(\Delta u))^2\, dx\Big)^{\frac{1}{2}}.
\end{aligned}$$

Noting that $\lim \dfrac{\langle \mathcal{B}u, u\rangle}{\|u\|_W} \to \infty$ as $\|u\|_W \to \infty$, so $\mathcal{B}$ is coercive

Owing to the Minty-Browder theorem, (see, e.g., Theorem V.15 in [15]), the equation (3.21) has a unique solution v, which imply that (3.19) has a unique solution (u, v).

Since g is globally Lipschitz, the operator $\mathcal{I} + \mathcal{A}$ is surjective.

By using the nonlinear semigroup theory, the existence of a unique solution to the system (3.3) is ensured. The proof of Theorem 3.3.1 is completed. □

3.4 Regular solution

Here, we establish the regularity of the solutions of (3.3). We can use the Faedo-Galerkin method [41], we obtain the following result.

Theorem 3.4.1. *Let* $(u_0, u_1) \in \widetilde{W} \times W$ *and suppose that (3.9)-(3.12) hold. Then, there exists a solution of system (3.3) that satisfies*

$$u \in L^\infty([0,\infty), \widetilde{W}) \cap L^\infty([0,\infty), W).$$

Proof. We will use the Faedo-Galerkin method along with three a priori estimates to prove the existence of regular solutions.

i) *Approximate solutions:*

Let $T > 0$ be fixed and let $\{w^k\}, k \in \mathbb{N}$ be a basis of W, B^k the space generated by $w^1, w^2, \dots, w^k$, and λ^j are the eigenvalues of the operator Δ^2.

Hence,

$$\Delta^2 w^j = \lambda^j w^j$$
$$u^k(x,t) = \sum_{j=1}^{k} c^{jk}(t) w^j(x),$$

where c^{jk} is determined by the ordinary differential equations.

For any v in B^k, u^k satisfies the approximate equation

$$\int_\Omega (u^k_{tt}(t) + \Delta^2 u^k - a(x) g(\Delta u^k_t)) v \, dx = 0, \tag{3.22}$$

with initial conditions

$$u^k(0) = u^k_0 = \sum_{j=1}^{k} \langle u^0, w_j \rangle w_j \longrightarrow u^0 \quad \text{in } \widetilde{W}, \text{ as } k \to +\infty, \tag{3.23}$$

and

$$u^k_t(0) = u^k_1 = \sum_{j=1}^{k} \langle u^1, w_j \rangle w_j \longrightarrow u^1 \quad \text{in } W, \text{ as } k \to +\infty. \tag{3.24}$$

$$-\Delta^2 u^k_0 + a(x) g(\Delta u^k_1) \longrightarrow -\Delta^2 u^0 + a(x) g(\Delta u^1) \quad \text{in } V, \text{ as } k \to +\infty. \tag{3.25}$$

The standard theory of ODE guarantees that the system (3.22)-(3.25) has a unique solution $u^k \in H^3[0, t_k)$, with $0 < t_k < T$, owing to Zorn lemma since the nonlinear terms in (3.22) are locally Lipschitz continuous, and by using the embedding $H^m[0, t_k] \to C^{m-1}[0, t_k]$, we deduce that the solution $u^k \in C^2[0, t_k]$.

In the next step, we obtain a priori estimates for the solution of the system (3.22)-(3.25), so that it can be extended outside $[0, t_k)$ to obtain one solution defined for all $T > 0$, using a standard compactness argument for the limiting procedure.

ii) *A priori estimates:*

First estimate. First, we estimate $u_{tt}^k(0)$. Taking $v = -\Delta u_{tt}^k$ in (3.22) and choosing $t = 0$, we obtain that

$$\|\nabla u_{tt}^k(0)\|^2 = \int_\Omega \nabla u_{tt}^k(x,0)\nabla(-\Delta^2 u_0^k + a(x)g(\Delta u_1^k))\,dx.$$

Using Cauchy-Schwarz's inequality, we have

$$\|\nabla u_{tt}^k(0)\| \leq \Big(\int_\Omega |\nabla(-\Delta^2 u_0^k + a(x)g(\Delta u_1^k))|^2\,dx\Big)^{\frac{1}{2}}.$$

By (3.23)-(3.25), we get

$$u_{tt}^k(0) \quad \text{is bounded in} \quad V. \tag{3.26}$$

Second estimate. We assume first $t < T$ and let $0 < a < T - t$. Set

$$u^{ka}(x,t) = u^k(x,t+a),$$

and

$$U^{ka} = u^k(x,t+a) - u^k(x,t),$$

which solves the next differential equation

$$(U_{tt}^{ka} + \Delta^2 U^{ka} - a(x)(g(\Delta u_t^{ka}) - g(\Delta u_t^k)), v) = 0, \quad \forall v \in B^k.$$

By taking $v = -\Delta U^{ka}$, we find, since g is non-decreasing, that

$$\frac{d}{dt}\int_\Omega \{|\nabla U_t^{ka}(x,t)|^2 + |\nabla\Delta U^{ka}(x,t)|^2\}\,dx \leq 0, \text{ for all } t \geq 0.$$

Integrating in $[0,t]$, we get

$$\int_\Omega \{|\nabla U_t^{ka}(x,t)|^2 + |\nabla\Delta U^{ka}(x,t)|^2\}\,dx$$
$$\leq \int_\Omega \{|\nabla U_t^{ka}(x,0)|^2 + |\nabla\Delta U^{ka}(x,0)|^2\}\,dx, \text{ for all } t \geq 0.$$

Dividing by a^2 and letting $a \to 0$, we find

$$\int_\Omega \{|\nabla u_{tt}^k(x,t)|^2 + |\nabla\Delta u_t^k(x,t)|^2\}\,dx$$
$$\leq \int_\Omega \{|\nabla u_{tt}^k(x,0)|^2 + |\nabla\Delta u_1^k(x)|^2\}\,dx, \text{ for all } t \geq 0.$$

By (3.24) and (3.26), we deduce that

$$\int_\Omega \{|\nabla u_{tt}^k(x,t)|^2 + |\nabla\Delta u_t^k(x,t)|^2\}\,dx \leq C_0, \quad \forall t \geq 0, \tag{3.27}$$

where C_0 is a positive constant independent of $k \in \mathbb{N}$. Therefore, we conclude that

$$u_t^k \quad \text{is bounded in} \qquad L^\infty(0,T;W) \tag{3.28}$$

and

$$u_{tt}^k \quad \text{is bounded in} \qquad L^\infty(0,T;V). \tag{3.29}$$

Third estimate. Differentiating (3.22) with respect to x and taking $v = \nabla\Delta^2 u^k$, we have

$$\begin{aligned}\|\nabla\Delta^2 u^k\|^2 &= \int_\Omega \nabla\Delta^2 u^k(-\nabla u_{tt}^k + \nabla(a(x)g(\Delta u_t^k))\,dx \\ &= \int_\Omega \nabla\Delta^2 u^k(-\nabla u_{tt}^k + \nabla a(x)g(\Delta u_t^k) + a(x)\nabla\Delta u_t^k g'(\Delta u_t^k))\,dx.\end{aligned}$$

Using Cauchy-Schwarz's inequality, we get

$$\|\nabla\Delta^2 u^k\| \le 2\Big(\int_\Omega \{|\nabla u_{tt}^k|^2 + |\nabla a(x)g(\Delta u_t^k|^2 + |a(x)\nabla\Delta u_t^k g'(\Delta u_t^k)|^2\}\,dx\Big)^{\frac{1}{2}}. \tag{3.30}$$

By Hölder's inequality and Sobolev embedding, we obtain

$$\begin{aligned}\int_{|\Delta u_t^k|\le 1} |\nabla a(x)g(\Delta u_t^k|^2\,dx &\le c_1\|\nabla a\|_\infty^2 \int_{|\Delta u_t^k|\le 1} |\Delta u_t^k|^{\frac{2}{p}}\,dx \\ &\le c_1\|\nabla a\|_\infty^2 \Big(\int_\Omega 1^{\frac{p}{p-1}}\,dx\Big)^{\frac{p-1}{p}} \Big(\int_\Omega |\Delta u_t^k|^2\,dx\Big)^{\frac{1}{p}} \\ &\le c_1 {C'}_s^2 \|\nabla a\|_\infty^2 \|\nabla\Delta u_t^k\|^{\frac{2}{p}}, \end{aligned} \tag{3.31}$$

and

$$\begin{aligned}\int_{|\Delta u_t^k|> 1} |\nabla a(x)g(\Delta u_t^k|^2\,dx &\le c_1\|\nabla a\|_\infty^2 \int_{|\Delta u_t^k|> 1} |\Delta u_t^k|^2\,dx \\ &\le c_1 {C'}_s^2 \|\nabla a\|_\infty^2 \|\nabla\Delta u_t^k\|^2,\end{aligned}$$

where $C'_s > 0$ and satisfies $\|\Delta u_t^k\| \le C'_s\|\nabla\Delta u_t^k\|$. Then

$$\int_\Omega |a(x)\nabla\Delta u_t^k g'(\Delta u_t^k)|^2\,dx \le \tau^2\|a\|_\infty^2\|\nabla\Delta u_t^k\|^2. \tag{3.32}$$

Taking into account (3.31)-(3.32) in (3.30) and using (3.27), we obtain

$$\|\nabla\Delta^2 u^k\| \le C_1, \quad \forall t \ge 0,$$

where C_1 is a positive constant independent of $k \in \mathbb{N}$. Therefore, we conclude that

$$u^k \quad \text{is bounded in} \qquad L^\infty(0,T;\widetilde{W}). \tag{3.33}$$

iii) *Passing to the limit:*

Applying Dunford-Pettis and Banach-Alaoglu-Bourbaki theorems, we conclude from (3.28), (3.29) and (3.33) that there exists a subsequence $\{u^m\}$ of $\{u^k\}$ and a function u such that

$$u^m \rightharpoonup^* u, \text{ in } L^\infty(0,T;\widetilde{W}), \tag{3.34}$$

$$u_t^m \rightharpoonup^* u_t, \text{ in } L^\infty(0,T;W), \tag{3.35}$$

$$u_{tt}^m \rightharpoonup^* u_{tt}, \text{ in } L^\infty(0,T;V). \tag{3.36}$$

It follows at once from (3.34) and (3.36), for each fixed $v \in L^2(0,T,L^2(\Omega))$, that

$$\int_0^T \int_\Omega (u_{tt}^m + \Delta^2 u^m) v \, dx \, dt \longrightarrow \int_0^T \int_\Omega (u_{tt} + \Delta^2 u) v \, dx \, dt. \tag{3.37}$$

It remains to show the convergence

$$\int_0^T \int_\Omega a(x) g(\Delta u_t^m) v \, dx \, dt \longrightarrow \int_0^T \int_\Omega a(x) g(\Delta u_t) v \, dx \, dt. \tag{3.38}$$

Then, we have

$$\sqrt{a} g(\Delta u_t^m) \longrightarrow \sqrt{a} g(\Delta u_t). \tag{3.39}$$

For two positive integers m, n with $m > n$, we set $U^{mn} = u^m - u^n$.
The function U^{mn} satisfies

$$\begin{cases} U_{tt}^{mn} + \Delta^2 U^{mn} - a(x)\Big(g(\Delta u_t^m) - g(\Delta u_t^n), v\Big) = 0 \\ U^{mn}(0) = u_0^m - u_0^n, \; U_t^{mn}(0) = u_1^m - u_1^n. \end{cases}$$

Taking $v = -2\Delta U_t^{mn}$ and using integration by parts, we get

$$\begin{aligned} \|\nabla U_t^{mn}(t)\|^2 + \|\nabla \Delta U^{mn}(t)\|^2 \quad &+ \quad 2\int_0^T \int_\Omega a(x)\Big(g(\Delta u_t^m) - g(\Delta u_t^n)\Big)\Delta U_t^{mn} \, dx \, dt \\ &= \quad \|\nabla U_t^{mn}(0)\|^2 + \|\nabla \Delta U^{mn}(0)\|^2. \end{aligned} \tag{3.40}$$

Using (3.10), we have

$$\begin{aligned} &\frac{1}{c_4} \int_0^T \int_{|\Delta u_t^m|>1} a(x) |g(\Delta u_t^m) - g(\Delta u_t^n)|^2 \, dx \, dt \\ \leq &\int_0^T \int_{|\Delta u_t^m|>1} a(x) (g(\Delta u_t^m) - g(\Delta u_t^n)) \Delta U_t^{mn} \, dx \, dt, \end{aligned}$$

and

$$\begin{aligned} &\int_0^T \int_{|\Delta u_t^m|>1} a(x) |g(\Delta u_t^m) - g(\Delta z_t)|^2 \, dx \, dt \\ &\leq c_4^2 \int_0^T \int_{|\Delta u_t^m|>1} a(x) |\Delta u_t^m - \Delta z_t|^2 \, dx \, dt \\ &\leq c_4^2 \int_0^T \int_\Omega a(x) |\Delta u_t^m - \Delta z_t|^2 \, dx \, dt, \quad \forall z \in W^{1,\infty}(0,\infty,L^2(\Omega)), \end{aligned} \tag{3.41}$$

and using (3.11), we have

$$\frac{1}{c_2}\int_0^T\int_{|\Delta u_t^m|\leq 1} a(x)|g(\Delta u_t^m)-g(\Delta u_t^n)|^{p+1}\,dx\,dt$$
$$\leq \int_0^T\int_{|\Delta u_t^m|\leq 1} a(x)(g(\Delta u_t^m)-g(\Delta u_t^n))\Delta U_t^{mn}\,dx\,dt,$$

and

$$\begin{aligned}&\int_0^T\int_{|\Delta u_t^m|\leq 1} a(x)|g(\Delta u_t^m)-g(\Delta z_t)|^{p+1}\,dx\,dt\\ &\leq c_2^{p+1}\int_0^T\int_{|\Delta u_t^m|\leq 1} a(x)|\Delta u_t^m)-\Delta z_t|^{\frac{p+1}{p}}\,dx\,dt\\ &\leq C(T)\|a(x)\|_\infty^{\frac{p-1}{p}}\Big(\int_0^T\int_\Omega a(x)|\Delta u_t^m-\Delta z_t|^{p+1}\,dx\,dt\Big)^{\frac{1}{p}}.\end{aligned}\tag{3.42}$$

Therefore, the convergence (3.23)-(3.25), combined with (3.40)-(3.42), shows that the sequences $(\sqrt{a}\Delta u_t^m)_m, (\sqrt{a}g(\Delta u_t^m))_m$ are Cauchy sequences in $L^2(0,T;L^2(\Omega))$. By the middle convergence in (3.35), we derive

$$\sqrt{a}\Delta u_t^m \longrightarrow \sqrt{a}\Delta u_t \text{ in } L^2(0,T;L^2(\Omega)).$$

Then, choosing $z=u$ in (3.41) and (3.42) and for $m\to\infty$, we get (3.39), which completes the proof.

□

3.5 Stability result

We state and prove our stability result as follows.

Theorem 3.5.1. *Let $(u_0,u_1)\in\widetilde{W}\times W$ and suppose that (3.9)-(3.12) hold. Then, any weak solution of (3.3) satisfies the estimate*

$$\mathcal{E}(t)\leq C\mathcal{E}(0)e^{-kt}\ \ \forall t>0,\quad \textit{and}\quad p=1,\tag{3.43}$$

and

$$\mathcal{E}(t)\leq C't^{-2/(p-1)}\ \ \forall t>0,\quad \textit{and}\quad p>1,\tag{3.44}$$

where C and k are positive constants independent of the initial data, while C' is a positive constant only depending on the initial energy $\mathcal{E}(0)$.

In order to prove Theorem 3.5.1, we first consider $\psi\in C_0^\infty(\mathbb{R}^n)$ such that

$$\begin{cases}0\leq\psi\leq 1,\\ \psi=1,\ \text{in } \bar{\Omega}\backslash\mathcal{M}_1,\\ \psi=0,\ \text{in } \mathcal{M}_0.\end{cases}\tag{3.45}$$

For $M > 0$ and $\mu > 0$, define the perturbed energy

$$\widehat{\mathcal{E}}(t) = M\mathcal{E}(t) + \mathcal{E}^{\mu}(t)\rho(t), \tag{3.46}$$

where

$$\rho(t) = -2\int_{\Omega} u_t(h.\nabla\Delta u)\,dx - \theta\int_{\Omega} u_t\Delta u\,dx, \tag{3.47}$$

$$h = m\psi, \tag{3.48}$$

and

$$\theta \in]n-2, 3n[.$$

Lemma 3.5.2. *There exists two positive constants λ_1 and λ_2 such that*

$$\lambda_1\mathcal{E}(t) \le \widehat{\mathcal{E}}(t) \le \lambda_2\mathcal{E}(t), \quad \forall t \ge 0. \tag{3.49}$$

Proof. We have the obvious estimates

$$\|u_t\| \le C_s\|\nabla u_t\|,$$

and

$$\|\Delta u\| \le C_s'\|\nabla\Delta u\|,$$

where C_s, C_s' are a positive constants (depending only on the geometry of Ω).
Thanks to Cauchy-Schwarz's inequality, we get

$$|\rho(t)| \le 2C_s\mathcal{R}(x^0)\|\nabla\Delta u\|\|\nabla u_t\| + \theta C_sC_s'\|\nabla\Delta u\|\|\nabla u_t\|, \tag{3.50}$$

where

$$\mathcal{R}(x^0) = \max_{x\in\overline{\Omega}}|x - x^0|. \tag{3.51}$$

From (3.50) we obtain

$$\begin{aligned} |\rho(t)| \quad &\le C_s(\theta C_s' + 2\mathcal{R}(x^0))\left\{\frac{1}{2}\|\nabla u_t\|^2 + \frac{1}{2}\|\nabla\Delta u\|^2\right\} \\ &\le C_s(\theta C_s' + 2\mathcal{R}(x^0))\mathcal{E}(t). \end{aligned} \tag{3.52}$$

Then, for M large enough, we obtain (3.49), where $\lambda_1 = M - C_s\mathcal{E}^{\mu}(0)(\theta C_s' + 2\mathcal{R}(x^0))$ and $\lambda_2 = M + C_s\mathcal{E}^{\mu}(0)(\theta C_s' + 2\mathcal{R}(x^0))$. □

Lemma 3.5.3. *The functional ρ, defined in (3.48) satisfies*

$$\begin{aligned}
\rho'(t) &= \int_\Gamma (h.\nu)\left(\frac{\partial^3 u}{\partial \nu^3}\right)^2 d\Gamma + \int_\Gamma (h.\nu)\left(\frac{\partial u_t}{\partial \nu}\right)^2 d\Gamma \\
&- (3n-\theta)\int_\Omega |\nabla u_t|^2\,dx - (\theta - n + 2)\int_\Omega |\nabla \Delta u|^2\,dx \\
&- \int_{\mathcal{M}_1\backslash\mathcal{M}_0} m\nabla\psi|\nabla u_t|^2\,dx + 3n\int_{\mathcal{M}_1}(1-\psi)|\nabla u_t|^2\,dx \\
&+ 2\int_{\mathcal{M}_1\backslash\mathcal{M}_0} m\nabla\psi u_t\Delta u_t dx - 2n\int_{\mathcal{M}_1\backslash\mathcal{M}_0}\nabla\psi u_t\nabla u_t dx \\
&+ (n-2)\int_{\mathcal{M}_1}(\psi-1)|\nabla\Delta u|^2 dx + \int_{\mathcal{M}_1\backslash\mathcal{M}_0} m\nabla\psi|\nabla\Delta u|^2\,dx \\
&- 2\sum_{i,k=0}^{n}\int_{\mathcal{M}_1} m_i\frac{\partial\psi_i}{\partial x_k}\frac{\partial^3 u}{\partial x_k^3}\frac{\partial^3 u}{\partial x_i^3}\,dx - \theta\int_\Omega \Delta u.a(x)g(\Delta u_t)\,dx \\
&+ \int_\Omega 2(h.\nabla\Delta u)a(x)g(\Delta u_t)\,dx.
\end{aligned} \tag{3.53}$$

Proof. Taking the derivative of ρ, with get

$$\begin{aligned}
\rho'(t) &= -2\int_\Omega u_{tt}(h\nabla\Delta u)\,dx - 2\int_\Omega u_t(h\nabla\Delta u_t)\,dx - \theta\int_\Omega u_{tt}(\Delta u)\,dx - \theta\int_\Omega u_t(\Delta u_t)\,dx \\
&= +\ 2\int_\Omega h.\nabla\Delta u.\Delta^2 u\,dx - 2\int_\Omega h.\nabla\Delta u.a(x)g(\Delta u_t)\,dx - 2\int_\Omega u_t(h\nabla\Delta u_t)\,dx \\
&-\ \theta\int_\Omega u_{tt}\Delta u\,dx + \theta\int_\Omega |\nabla u_t|^2\,dx.
\end{aligned}$$

To complete the proof of Lemma 3.5.3, we will need following three Lemmas.

Lemma 3.5.4. *We have*

$$\begin{aligned}
-2\int_\Omega u_t(\nabla\Delta u_t) &= \int_\Gamma\left(\frac{\partial u_t}{\partial\nu}\right)^2 d\Gamma - 3n\int_\Omega|\nabla u_t|^2\,dx + 3n\int_{\mathcal{M}_1}(1-\psi)|\nabla u_t|^2\,dx \\
&- \int_{\mathcal{M}_1\backslash\mathcal{M}_0} m\nabla\psi|\nabla u_t|^2\,dx + 2\int_{\mathcal{M}_1\backslash\mathcal{M}_0} m\nabla\psi u_t\Delta u_t dx \\
&- 2n\int_{\mathcal{M}_1\backslash\mathcal{M}_0}\nabla\psi u_t\nabla u_t dx.
\end{aligned} \tag{3.54}$$

Proof. Integrating by parts and noting that $u_t = 0$ and $|\nabla u_t|^2 = \left(\frac{\partial u_t}{\partial\nu}\right)^2$ on Γ, we have

$$\begin{aligned}
&-2\int_\Omega u_t(h\nabla\Delta u_t)dx \\
&= 2\int_\Omega div(h)u_t\Delta u_t dx + 2\int_\Omega h\nabla u_t\Delta u_t\,dx \\
&- 2\int_\Omega div[div(h)]u_t\nabla u_t dx - 2\int_\Omega div(h)|\nabla u_t|^2 dx + 2\int_\Omega h\nabla u_t\Delta u_t\,dx \\
&= 2\int_\Omega div(h)u_t\Delta u_t dx + \int_\Gamma h\nu\left(\frac{\partial u_t}{\partial\nu}\right)^2 d\Gamma - \int_\Omega div(h)|\nabla u_t|^2 dx.
\end{aligned} \tag{3.55}$$

Using (3.5), (3.45) and (3.48), we obtain

$$\begin{aligned}
& 2\int_{\Omega}(\mathrm{div} h)u_t\Delta u_t dx \\
= & \ 2\int_{\Omega\setminus\mathcal{M}_1}\mathrm{div}(\psi.m)u_t\Delta u_t dx + 2\int_{\mathcal{M}_1}\mathrm{div}(\psi.m)u_t\Delta u_t dx \\
= & \ 2n\int_{\Omega\setminus\mathcal{M}_1}u_t\Delta u_t dx + 2\int_{\mathcal{M}_1}m\nabla\psi u_t\Delta u_t dx + 2n\int_{\mathcal{M}_1}\psi u_t\Delta u_t dx \\
= & \ -2n\int_{\Omega\setminus\mathcal{M}_1}|\nabla u_t|^2 dx + 2\int_{\mathcal{M}_1\setminus\mathcal{M}_0}m\nabla\psi u_t\Delta u_t dx + 2n\int_{\mathcal{M}_1}\psi u_t\Delta u_t dx \\
= & \ -2n\int_{\Omega\setminus\mathcal{M}_1}|\nabla u_t|^2 dx + 2\int_{\mathcal{M}_1\setminus\mathcal{M}_0}m\nabla\psi u_t\Delta u_t dx - 2n\int_{\mathcal{M}_1\setminus\mathcal{M}_0}\nabla\psi u_t\nabla u_t dx \\
- & \ 2n\int_{\mathcal{M}_1}\psi|\nabla u_t|^2 dx, \qquad (3.56)
\end{aligned}$$

and

$$\begin{aligned}
& -\int_{\Omega}\mathrm{div}(h)|\nabla u_t|^2 dx \\
= & \ -\int_{\Omega\setminus\mathcal{M}_1}\mathrm{div}(\psi.m)|\nabla u_t|^2 dx - \int_{\mathcal{M}_1}\mathrm{div}(\psi.m)|\nabla u_t|^2 dx \qquad (3.57) \\
= & \ -n\int_{\Omega\setminus\mathcal{M}_1}|\nabla u_t|^2 dx - \int_{\mathcal{M}_1\setminus\mathcal{M}_0}m\nabla\psi|\nabla u_t|^2 dx - n\int_{\mathcal{M}_1}\psi|\nabla u_t|^2 dx.
\end{aligned}$$

Taking into account (3.56) and (3.57) into (3.55), we get

$$\begin{aligned}
& -2\int_{\Omega}u_t(h\nabla\Delta u_t)dx \\
& = \int_{\Gamma}\left(\frac{\partial u_t}{\partial\nu}\right)^2 d\Gamma - 3n\int_{\Omega\setminus\mathcal{M}_1}|\nabla u_t|^2 dx + 2\int_{\mathcal{M}_1\setminus\mathcal{M}_0}m\nabla\psi u_t\Delta u_t dx \\
& -2n\int_{\mathcal{M}_1\setminus\mathcal{M}_0}\nabla\psi u_t\nabla u_t dx - \int_{\mathcal{M}_1\setminus\mathcal{M}_0}m\nabla\psi|\nabla u_t|^2 dx \\
& -3n\int_{\mathcal{M}_1}\psi|\nabla u_t|^2 dx.
\end{aligned}$$

□

Lemma 3.5.5. *We have the following equality*

$$\begin{aligned}
-2\int_{\Omega}(h.\nabla u)\Delta^2 u\,dx \ = & \ \int_{\Gamma}(h.\nu)\left(\frac{\partial^3 u}{\partial\nu^3}\right)^2 d\Gamma + (n-2)\int_{\Omega}|\nabla\Delta u|^2\,dx \\
+ & \ (n-2)\int_{\mathcal{M}_1}(\psi-1)|\nabla\Delta u|^2 dx - 2\sum_{i,k=0}^{n}\int_{\mathcal{M}_1}m_i\frac{\partial\psi_i}{\partial x_k}\frac{\partial^3 u}{\partial x_k^3}\frac{\partial^3 u}{\partial x_i^3}\,dx \\
+ & \ \int_{\mathcal{M}_1\setminus\mathcal{M}_0}m\nabla\psi|\nabla\Delta u|^2\,dx. \qquad (3.58)
\end{aligned}$$

Proof. We have $\dfrac{\partial u}{\partial x_k} = \dfrac{\partial u}{\partial \nu}\nu_k$, which implies

$$h.\Delta u = (h.\nu)\frac{\partial^2 u}{\partial \nu^2} = 0, \text{ and } |\nabla \Delta u|^2 = \Big(\frac{\partial^3 u}{\partial \nu^3}\Big)^2 \quad \text{on} \quad \Gamma.$$

Then

$$\begin{aligned}
& 2\int_\Omega (h.\nabla\Delta u).\Delta^2 u\,dx \\
= \;& 2\int_\Gamma (h.\nu)|\nabla\Delta u|^2\,d\Gamma - 2\sum_{i,k=1}^{n}\int_\Omega \frac{\partial h_i}{\partial x_k}\frac{\partial^3 u}{\partial x_k^3}\frac{\partial^3 u}{\partial x_i^3}\,dx - 2\int_\Omega h(\nabla\Delta u).\nabla(\nabla\Delta u)\,dx \\
= \;& 2\int_\Gamma (h.\nu)\Big(\frac{\partial^3 u}{\partial \nu^3}\Big)^2 d\Gamma - 2\sum_{i,k=1}^{n}\int_\Omega \frac{\partial h_i}{\partial x_k}\frac{\partial^3 u}{\partial x_k^3}\frac{\partial^3 u}{\partial x_i^3}\,dx - \int_\Omega h\nabla(|\nabla\Delta u|^2)\,dx \\
= \;& \int_\Gamma (h.\nu)\Big(\frac{\partial^3 u}{\partial \nu^3}\Big)^2 d\Gamma - 2\sum_{i,k=1}^{n}\int_\Omega \frac{\partial h_i}{\partial x_k}\frac{\partial^3 u}{\partial x_k^3}\frac{\partial^3 u}{\partial x_i^3}\,dx + \int_\Omega \operatorname{div}(h)|\nabla\Delta u|^2\,dx. \qquad (3.59)
\end{aligned}$$

So, by using (3.5), (3.45) and (3.48), the second term of (3.59) gives

$$\begin{aligned}
& -2\sum_{i,k=1}^{n}\int_\Omega \frac{\partial h_i}{\partial x_k}\frac{\partial^3 u}{\partial x_i^3}\frac{\partial^3 u}{\partial x_k^3}\,dx \\
= \;& -2\sum_{i,k=1}^{n}\int_{\mathcal{M}_1} \frac{\partial^3 u}{\partial x_i^3}\frac{\partial^3 u}{\partial x_k^3}\frac{\partial(m_i\psi_i)}{\partial x_k}\,dx - 2\sum_{i,k=1}^{n}\int_{\Omega\setminus\mathcal{M}_1} \frac{\partial^3 u}{\partial x_i^3}\frac{\partial^3 u}{\partial x_k^3}\frac{\partial(m_i\psi_i)}{\partial x_k}\,dx \\
= \;& -2\sum_{i,k=0}^{n}\int_{\mathcal{M}_1} \frac{\partial^3 u}{\partial x_i^3}\frac{\partial^3 u}{\partial x_k^3}\psi_i\frac{\partial m_i}{\partial x_k}\,dx - 2\sum_{i,k=0}^{n}\int_{\mathcal{M}_1} m_i\frac{\partial \psi_i}{\partial x_k}\frac{\partial^3 u}{\partial x_i^3}\frac{\partial^2 u}{\partial x_k^3}\,dx \\
- \;& 2\sum_{i,k=0}^{n}\int_{\Omega\setminus\mathcal{M}_1} \frac{\partial^3 u}{\partial x_i^3}\frac{\partial^3 u}{\partial x_k^3}\,dx \\
= \;& -2\int_{\mathcal{M}_1}\psi|\nabla\Delta u|^2\,dx - 2\sum_{i,k=0}^{n}\int_{\mathcal{M}_1} m_i\frac{\partial \psi_i}{\partial x_k}\frac{\partial^3 u}{\partial x_i^3}\frac{\partial^3 u}{\partial x_k^3}\,dx - 2\int_{\Omega\setminus\mathcal{M}_1}|\nabla\Delta u|^2\,dx. \qquad (3.60)
\end{aligned}$$

Similarly, the third term of (3.59) can be rewritten as follows

$$\begin{aligned}
& \int_\Omega \operatorname{div}(h)|\nabla\Delta u|^2\,dx \\
= \;& \int_{\Omega\setminus\mathcal{M}_1}\operatorname{div}(\psi m)|\nabla\Delta u|^2\,dx + \int_{\mathcal{M}_1}\operatorname{div}(\psi m)|\nabla\Delta u|^2\,dx \qquad (3.61) \\
= \;& n\int_{\Omega\setminus\mathcal{M}_1}|\nabla\Delta u|^2\,dx + \int_{\mathcal{M}_1\setminus\mathcal{M}_0} m\nabla\psi|\nabla\Delta u|^2\,dx + n\int_{\mathcal{M}_1}\psi|\nabla\Delta u|^2\,dx.
\end{aligned}$$

Inserting (3.60) and (3.61) in (3.59), we arrive at

$$\begin{aligned}
& 2\int_\Omega (h.\nabla\Delta u).\Delta^2 u\,dx \\
= \;& \int_\Gamma (h.\nu)\Big(\frac{\partial^3 u}{\partial \nu^3}\Big)^2 d\Gamma + (n-2)\int_{\Omega\setminus\mathcal{M}_1}|\nabla\Delta u|^2 + (n-2)\int_{\mathcal{M}_1}\psi|\nabla\Delta u|^2 \\
- \;& 2\sum_{i,k=1}^{n}\int_{\mathcal{M}_1} m_i\frac{\partial \psi_i}{\partial x_k}\frac{\partial^3 u}{\partial x_k^3}\frac{\partial^3 u}{\partial x_i^3} + \int_{\mathcal{M}_1\setminus\mathcal{M}_0} m\nabla\psi|\nabla\Delta u|^2 dx.
\end{aligned}$$

□

Lemma 3.5.6. *We have*

$$-\theta\int_{\Omega} u_{tt}\Delta u\,dx = -\theta\int_{\Omega}|\nabla\Delta u|^2\,dx - \theta\int_{\Omega} a(x)\Delta u g(\Delta u_t)\,dx. \tag{3.62}$$

Proof. Using the first equation of (4.1) and applying the Green formula, we obtain

$$\begin{aligned}-\theta\int_{\Omega} u_{tt}\Delta u\,dx &= -\theta\int_{\Omega}\Delta u(-\Delta^2 u + a(x)g(\Delta u_t))\,dx\\ &= -\theta\int_{\Omega}|\nabla\Delta u|^2\,dx - \theta\int_{\Omega} a(x)\Delta u g(\Delta u_t)\,dx.\end{aligned}$$

□

By (3.54), (3.58) and (3.62) give (3.53). This completes the proof. □

Lemma 3.5.7. *We have*

$$\begin{aligned}\rho'(t) &\le -K_n\mathcal{E}(t) + \{3A+n-2\}\int_{\Omega}|\nabla\Delta u|^2\,dx + A\int_{\omega}|\Delta u_t|^2\,dx\\ &+ \{A(1+C_s)+3n+n(1+C_s)\max_{x\in\overline{\Omega}}|\nabla\psi(x)|\}\int_{\Omega}|\nabla u_t|^2\,dx\\ &- \theta\int_{\Omega}\Delta u.a(x)g(\Delta u_t)\,dx + 2\int_{\Omega}(h.\nabla\Delta u)a(x)g(\Delta u_t)\,dx,\end{aligned} \tag{3.63}$$

where

$$K_n = \min\Big\{2(3n-\theta), 2(\theta-n+2)\Big\},$$
$$\theta\in]n-2,3n[,$$

and

$$A = \mathcal{R}(x^0)\max_{x\in\overline{\Omega}}|\nabla\psi(x)|.$$

Proof. Next, we estimate some terms on the RHS of identity (3.53).
Taking (3.4), (3.6), (3.7), (3.45) and (3.48), we have

$$\begin{aligned}\int_{\Gamma}(h.\nu)\left(\frac{\partial^3 u}{\partial\nu^3}\right)^2 d\Gamma &= \int_{\Gamma(x^0)}(m.\nu)\psi\left(\frac{\partial^3 u}{\partial\nu^3}\right)^2 d\Gamma + \int_{\Gamma\backslash\Gamma(x^0)}(m.\nu)\psi\left(\frac{\partial^3 u}{\partial\nu^3}\right)^2 d\Gamma\\ &\le 0,\end{aligned} \tag{3.64}$$

and

$$\begin{aligned}\int_{\Gamma}(h.\nu)\left(\frac{\partial u_t}{\partial\nu}\right)^2 d\Gamma &= \int_{\Gamma(x^0)}(m.\nu)\psi\left(\frac{\partial u_t}{\partial\nu}\right)^2 d\Gamma + \int_{\Gamma\backslash\Gamma(x^0)}(m.\nu)\psi\left(\frac{\partial u_t}{\partial\nu}\right)^2 d\Gamma\\ &\le 0,\end{aligned} \tag{3.65}$$

and

$$\int_{\mathcal{M}_1\setminus\mathcal{M}_0} m\nabla\psi|\nabla u_t|^2\,dx \le \mathcal{R}(x^0)\max_{x\in\overline{\Omega}}|\nabla\psi(x)|\int_\Omega|\nabla u_t|^2\,dx,$$

and

$$3n\int_{\mathcal{M}_1}(1-\psi)|\nabla u_t|^2\,dx \le 3n\int_\Omega|\nabla u_t|^2\,dx,$$

and

$$\begin{aligned}
2\int_{\mathcal{M}_1\setminus\mathcal{M}_0} m\nabla\psi u_t\Delta u_t dx &\le \mathcal{R}(x^0)\max_{x\in\overline{\Omega}}|\nabla\psi(x)|\Big(\int_\Omega|u_t|^2\,dx+\int_\omega|\Delta u_t|^2\,dx\Big)\\
&\le C_s\mathcal{R}(x^0)\max_{x\in\overline{\Omega}}|\nabla\psi(x)|\int_\Omega|\nabla u_t|^2\,dx\\
&+ \mathcal{R}(x^0)\max_{x\in\overline{\Omega}}|\nabla\psi(x)|\int_\omega|\Delta u_t|^2\,dx,
\end{aligned}$$

and

$$2n\int_{\mathcal{M}_1\setminus\mathcal{M}_0}\nabla\psi u_t\nabla u_t dx \le n\max_{x\in\overline{\Omega}}|\nabla\psi(x)|(C_s+1)\int_\Omega|\nabla u_t|^2\,dx,$$

and

$$2\Big|\sum_{i,k=0}^{n}\int_{\mathcal{M}_1}\frac{\partial^3 u}{\partial x_k^3}\frac{\partial^3 u}{\partial x_i^3}m_i\frac{\partial\psi_i}{\partial x_i}\,dx\Big| \le 2\mathcal{R}(x^0)\max_{x\in\overline{\Omega}}|\nabla\psi(x)|\int_\Omega|\nabla\Delta u|^2\,dx,$$

and

$$\int_{\mathcal{M}_1\setminus\mathcal{M}_0} m\nabla\psi|\nabla\Delta u|^2\,dx \le \mathcal{R}(x^0)\max_{x\in\overline{\Omega}}|\nabla\psi(x)|\int_\Omega|\nabla\Delta u|^2\,dx,$$

and

$$(n-2)\int_{\mathcal{M}_1}(\psi-1)|\nabla\Delta u|^2\,dx \le (n-2)\int_\Omega|\nabla\Delta u|^2\,dx. \tag{3.66}$$

Taking into account (3.64)-(3.66) into (3.53) we obtain (3.63). The proof of Lemma 3.5.7 is completed. □

Proof. (of Theorem 3.5.1) Taking the derivative of (3.46) with respective to t, we get

$$\widehat{\mathcal{E}}'(t) = M\mathcal{E}'(t) + \mu\mathcal{E}'(t)\mathcal{E}^{\mu-1}(t)\rho(t) + \mathcal{E}^{\mu}(t)\rho'(t). \tag{3.67}$$

Using (3.13) and (3.63), we have

$$\begin{aligned}
\widehat{\mathcal{E}}'(t) &\leq M\mathcal{E}'(t) + C_\mu \mathcal{E}^\mu(0)|\mathcal{E}'(t)| - K_n \mathcal{E}^{\mu+1}(t) \\
&+ A\mathcal{E}^\mu(t)\int_\omega |\Delta u_t|^2\,dx + (3A+n-2)\mathcal{E}^\mu(t)\int_\Omega |\nabla\Delta u|^2\,dx \\
&+ \Big\{A(1+C_s) + 3n + n(C_s+1)\max_{x\in\overline{\Omega}}|\nabla\psi(x)|\Big\}\mathcal{E}^\mu(t)\int_\Omega |\nabla u_t|^2\,dx \\
&+ 2\mathcal{E}^\mu(t)\int_\Omega (h.\nabla\Delta u)a(x)g(\Delta u_t)\,dx - \theta\mathcal{E}^\mu(t)\int_\Omega \Delta u.a(x)g(\Delta u_t)\,dx \\
&\leq M\mathcal{E}'(t) + C_\mu\mathcal{E}^\mu(0)|\mathcal{E}'(t)| - K_n\mathcal{E}^{\mu+1}(t) + \frac{K_n}{2}\mathcal{E}^{\mu+1}(t) \\
&+ A\mathcal{E}^\mu(t)\int_\omega |\Delta u_t|^2\,dx + 2\mathcal{E}^\mu(t)\int_\Omega (h.\nabla\Delta u)a(x)g(\Delta u_t)\,dx \\
&- \theta\mathcal{E}^\mu(t)\int_\Omega \Delta u.a(x)g(\Delta u_t)\,dx. \qquad (3.68)
\end{aligned}$$

Using (3.9), we have

$$\begin{aligned}
A\mathcal{E}^\mu(t)\int_w |\Delta u_t|^2\,dx &\leq \frac{A}{a_0}\mathcal{E}^\mu(t)\int_w a(x)|\Delta u_t|^2\,dx \\
&\leq \frac{A}{a_0}\mathcal{E}^\mu(t)\int_\Omega a(x)|\Delta u_t|^2\,dx.
\end{aligned}$$

As in Komornik [34], we consider the following partition of Ω,

$$\Omega_1 = \{x\in\Omega : |\Delta u_t| > 1\}, \quad \Omega_2 = \{x\in\Omega : |\Delta u_t| \leq 1\}.$$

From now on, we distinguish two cases: $p = 1$ and $p > 1$.

Case $p = 1$: (Proof of (3.43)). Using (3.14), we get

$$\begin{aligned}
&A\mathcal{E}^\mu(t)\int_\Omega a(x)|\Delta u_t|^2\,dx \\
&\leq C\frac{A}{a_0}\mathcal{E}^\mu(t)\int_\Omega a(x)\Delta u_t g(\Delta u_t)\,dx \\
&\leq C\mathcal{E}^\mu(t)(-\mathcal{E}'(t)) \\
&\leq C\mathcal{E}^\mu(0)|\mathcal{E}'(t)|. \qquad (3.69)
\end{aligned}$$

Using Cauchy-Schwarz's inequality, we get

$$\begin{aligned}
&2\mathcal{E}^\mu(t)\int_\Omega ha(x)\nabla\Delta u g(\Delta u_t)\,dx \\
&\leq 2\mathcal{R}(x^0)\mathcal{E}^\mu(t)\|\nabla\Delta u\|\Big(\int_\Omega a^2(x)g^2(\Delta u_t)\,dx\Big)^{\frac{1}{2}} \\
&\leq 2c\mathcal{R}(x^0)\|a\|_\infty \mathcal{E}^{\mu+\frac{1}{2}}(t)\Big(\int_{\Omega_1} a(x)\Delta u_t(t)g(\Delta u_t)\,dx\Big)^{\frac{1}{2}} \\
&\leq 2c\mathcal{R}(x^0)\|a\|_\infty \mathcal{E}^{\mu+\frac{1}{2}}(t)(-\mathcal{E}'(t))^{\frac{1}{2}}.
\end{aligned}$$

Applying Young's inequality, we obtain

$$\begin{aligned}&2\mathcal{E}^{\mu}(t)\int_{\Omega} h.a(x)\nabla\Delta u g(\Delta u_t)\,dx\\&\leq c\mathcal{R}(x^0)\|a\|_{\infty}\mathcal{E}^{2\mu+1}(t)+c\mathcal{R}(x^0)\|a\|_{\infty}|\mathcal{E}'(t)|\\&\leq c\mathcal{R}(x^0)\|a\|_{\infty}\mathcal{E}^{\mu}(0)\mathcal{E}^{\mu+1}(t)+c\mathcal{R}(x^0)\|a\|_{\infty}|\mathcal{E}'(t)|\\&\leq \frac{K_n}{4}\mathcal{E}^{\mu+1}(t)+c\mathcal{R}(x^0)\|a\|_{\infty}|\mathcal{E}'(t)|.\end{aligned}\tag{3.70}$$

Using Cauchy-Schwarz and Young's inequalities, we get

$$\begin{aligned}&\theta\mathcal{E}^{\mu}(t)\int_{\Omega} a(x)\Delta u g(\Delta u_t)\,dx\\&\leq \theta C_s'\mathcal{E}^{\mu}(t)\|\nabla\Delta u\|\Big(\int_{\Omega_1} a^2(x)g^2(\Delta u_t)\,dx\Big)^{\frac{1}{2}}\\&\leq C\frac{\|a\|_{\infty}}{2}\mathcal{E}^{\mu}(0)\mathcal{E}^{\mu+1}(t)+C'\frac{\|a\|_{\infty}}{2}|\mathcal{E}'(t)|\\&\leq \frac{K_n}{8}\mathcal{E}^{\mu+1}(t)+C'\frac{\|a\|_{\infty}}{2}|\mathcal{E}'(t)|.\end{aligned}\tag{3.71}$$

By (3.69)-(3.71) and (3.68), we find

$$\widehat{\mathcal{E}}'(t)\leq M\mathcal{E}'(t)+C\mathcal{E}^{\mu}(0)|\mathcal{E}'(t)|+C|\mathcal{E}'(t)|-\frac{K_n}{8}\mathcal{E}^{\mu+1}(t).$$

Choosing $\mu=0$ and M large enough, we obtain

$$\begin{aligned}\widehat{\mathcal{E}}'(t) &\leq -\frac{K_n}{8}\mathcal{E}(t)\\&\leq -\frac{K_n}{8\lambda_1}\widehat{\mathcal{E}}(t).\end{aligned}\tag{3.72}$$

Finally, by combining (3.49) and (3.72), we obtain (3.43).

Case $p>1$: (Proof of (3.44)). By using Hölder's inequality and (3.14), we get

$$\begin{aligned}&\mathcal{E}^{\mu}(t)\int_{\Omega} a(x)|\Delta u_t|^2\,dx\\ \leq\ & C\mathcal{E}^{\mu}(t)\int_{\Omega_1} a(x)\Delta u_t g(\Delta u_t)\,dx+C'\mathcal{E}^{\mu}(t)\int_{\Omega_2} a(x)\Big(\Delta u_t g(\Delta u_t)\Big)^{\frac{2}{p+1}}\,dx\\ \leq\ & C\mathcal{E}^{\mu}(t)\int_{\Omega} a(x)\Delta u_t g(\Delta u_t)\,dx+C(\Omega)\|a\|_{\infty}^{\frac{p-1}{p+1}}\mathcal{E}^{\mu}(t)\int_{\Omega}\Big(a(x)\Delta u_t g(\Delta u_t)\Big)^{\frac{2}{p+1}}\,dx\\ \leq\ & C\mathcal{E}^{\mu}(t)(-\mathcal{E}'(t))+C(\Omega,p)\|a\|_{\infty}^{\frac{p-1}{p+1}}\mathcal{E}^{\mu}(t)\Big(-\mathcal{E}'(t)\Big)^{\frac{2}{p+1}}.\end{aligned}$$

Now, fixing an arbitrarily small $\varepsilon>0$, by applying Young's inequality, we obtain

$$\begin{aligned}\frac{1}{a_0}\mathcal{E}^{\mu}(t)\int_{\Omega} a(x)|\Delta u_t|^2\,dx \leq\ & C\frac{1}{a_0}\mathcal{E}^{\mu}(0)|\mathcal{E}'(t)|+\tfrac{\|a\|_{\infty}}{a_0}\tfrac{p-1}{p+1}\tfrac{1}{\varepsilon^{\frac{p+1}{p-1}}}\mathcal{E}^{\mu\frac{p+1}{p-1}}(t)\\&+\tfrac{C(\Omega,p)}{a_0}\tfrac{2}{p+1}\varepsilon^{\frac{p+1}{2}}|\mathcal{E}'(t)|.\end{aligned}\tag{3.73}$$

Then, we have

$$\begin{aligned}\frac{A}{a_0}\mathcal{E}^{\mu}(t)\int_w |\Delta u_t|^2\,dx &\leq CA\frac{1}{a_0}\mathcal{E}^{\mu}(0)|\mathcal{E}'(t)| + A\frac{\|a\|_\infty}{a_0}\frac{p-1}{p+1}\frac{1}{\varepsilon^{\frac{p+1}{p-1}}}\mathcal{E}^{\mu\frac{p+1}{p-1}}(t)\\ &\quad +A\frac{C(\Omega,p)}{a_0}\frac{2}{p+1}\varepsilon^{\frac{p+1}{2}}|\mathcal{E}'(t)\\ &\leq C\mathcal{E}^{\mu}(0)|\mathcal{E}'(t)| + \tfrac{K_n}{8}\mathcal{E}^{\mu\frac{p+1}{p-1}}(t) + C\varepsilon^{\frac{p+1}{2}}|\mathcal{E}'(t)|. \end{aligned}\tag{3.74}$$

Using Cauchy-Schwarz's inequality, we get

$$2\mathcal{E}^{\mu}(t)\int_{\Omega_1} ha(x)\nabla\Delta u g(\Delta u_t)\,dx \leq 2c\mathcal{R}(x^0)\|a\|_\infty\mathcal{E}^{\mu+\frac{1}{2}}(t)(-\mathcal{E}'(t))^{\frac{1}{2}}.\tag{3.75}$$

Applying Young's inequality, we obtain

$$\begin{aligned}&2\mathcal{E}^{\mu}(t)\int_{\Omega_1} h.a(x)\nabla\Delta u g(\Delta u_t)\,dx\\ &\leq c\mathcal{R}(x^0)\|a\|_\infty\mathcal{E}^{\mu}(0)\mathcal{E}^{\mu+1}(t) + c\mathcal{R}(x^0)\|a\|_\infty|\mathcal{E}'(t)|.\end{aligned}\tag{3.76}$$

By Cauchy-Schwarz and Hölder's inequalities, we have

$$\begin{aligned}&2\mathcal{E}^{\mu}(t)\int_{\Omega_2} h.a(x)\nabla\Delta u g(\Delta u_t)\,dx\\ &\leq \mathcal{R}(x^0)\mathcal{E}^{\mu}(t)\|\nabla\Delta u\|\Big(\int_{\Omega_2} a^2(x)g^2(\Delta u_t)\,dx\Big)^{\frac{1}{2}}\\ &\leq c\mathcal{R}(x^0)\|a\|_\infty^{\frac{p}{p+1}}\mathcal{E}^{\mu+\frac{1}{2}}(t)\Big(\int_{\Omega}(a(x)\Delta u_t(t)g(\Delta u_t))^{\frac{2}{p+1}}\,dx\Big)^{\frac{1}{2}}\\ &\leq C(\Omega,p)\mathcal{R}(x^0)\|a\|_\infty^{\frac{p}{p+1}}\mathcal{E}^{\mu+\frac{1}{2}}(t)(-\mathcal{E}'(t))^{\frac{1}{p+1}}.\end{aligned}$$

Set $\varepsilon_1 > 0$, thanks to Young's inequality, we obtain

$$\begin{aligned}&2\mathcal{E}^{\mu}(t)\int_{\Omega_2} h.a(x)\nabla\Delta u g(\Delta u_t)\,dx\\ &\leq C(\Omega,p)\mathcal{R}(x^0)\|a\|_\infty\frac{p}{p+1}\frac{1}{\varepsilon_1^{\frac{p+1}{p}}}\mathcal{E}^{(\mu+\frac{1}{2})\frac{p+1}{p}}(t)\\ &+\frac{C(\Omega,p)}{p+1}\mathcal{R}(x^0)\varepsilon_1^{p+1}|\mathcal{E}'(t)|.\end{aligned}\tag{3.77}$$

Then, we deduce from (3.76) and (3.77), that

$$\begin{aligned}&2\mathcal{E}^{\mu}(t)\int_{\Omega} h.a(x)\nabla\Delta u g(\Delta u_t)\,dx \qquad (3.78)\\ &\leq \frac{K_n}{4}\mathcal{E}^{\mu+1}(t) + \frac{K_n}{8}\mathcal{E}^{(\mu+\frac{1}{2})\frac{p+1}{p}}(t) + C|\mathcal{E}'(t)| + C\mathcal{E}^{\mu}(0)|\mathcal{E}'(t)|.\end{aligned}$$

Using Hölder's and Young's inequalities, we get

$$\theta\mathcal{E}^{\mu}(t)\int_{\Omega_1} a(x)\Delta u g(\Delta u_t)\,dx \leq C\frac{\|a\|_\infty}{2}\mathcal{E}^{\mu}(0)\mathcal{E}^{\mu+1}(t) + C'\frac{\|a\|_\infty}{2}|\mathcal{E}'(t)|,\tag{3.79}$$

and

$$\begin{aligned}\theta\mathcal{E}^{\mu}(t)\int_{\Omega_2} a(x)\Delta u g(\Delta u_t)\,dx &\leq C\|a\|_\infty \frac{p}{p+1}\frac{1}{\varepsilon_2^{\frac{p+1}{p}}}\mathcal{E}^{(\mu+\frac{1}{2})\frac{p+1}{p}}(t)\\ &+ \frac{C(\Omega,p)}{p+1}\varepsilon_2^{p+1}|\mathcal{E}'(t)|. \end{aligned} \tag{3.80}$$

We deduce from (3.79) and (3.80)

$$\theta\mathcal{E}^{\mu}(t)\int_{\Omega} a(x)\Delta u g(\Delta u_t)\,dx \leq \frac{K_n}{4}\mathcal{E}^{\mu+1}(t) + \frac{K_n}{8}\mathcal{E}^{(\mu+\frac{1}{2})\frac{p+1}{p}}(t) + C|\mathcal{E}'(t)|. \tag{3.81}$$

Reporting (4.30), (3.78) and (3.81) into (3.68), we find

$$\begin{aligned}\widehat{\mathcal{E}}'(t) &\leq M\mathcal{E}'(t) + C\mathcal{E}^{\mu}(0)|\mathcal{E}'(t)| + C|\mathcal{E}'(t)| - \frac{K_n}{2}\mathcal{E}^{\mu+1}(t)\\ &+ \frac{K_n}{4}\mathcal{E}^{(\mu+\frac{1}{2})\frac{p+1}{p}}(t) + \frac{K_n}{8}\mathcal{E}^{\mu\frac{p+1}{p-1}}(t).\end{aligned}$$

We choose μ such that

$$\Big(\mu+\frac{1}{2}\Big)\frac{p+1}{p} = \mu+1.$$

Thus, we find $\mu = \dfrac{p-1}{2}$ and

$$\mu\frac{p+1}{p-1} = \mu+1+\alpha.$$

with $\alpha = 0$.

We find, for M large enough, the following inequality

$$\begin{aligned}\widehat{\mathcal{E}}'(t) &\leq -\frac{K_n}{8}\mathcal{E}^{\mu+1}\\ &\leq -\frac{K_n}{8\lambda_1^{\mu+1}}\widehat{\mathcal{E}}^{\mu+1}(t).\end{aligned} \tag{3.82}$$

Finally, by combining (3.49) and (3.82), we obtain (3.44). This completes the proof. □

Chapter 4

Well-posedness and exponential stability of coupled non-degenrate Kirchhoff equation and the heat equation

4.1 Introduction

Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth enough boundary. Let α and β be two nonzero real numbers with the same sign. Consider the coupled wave/heat system

$$\begin{cases} y_{tt} - \phi(\|\nabla y(t)\|^2)\Delta y - \gamma\Delta y_{tt} + \alpha\Delta\theta = 0, & \text{in } \Omega \times (0, +\infty) \\ \theta_t - \sigma\Delta\theta - \beta\Delta y_t = 0, & \text{in } \Omega \times (0, +\infty) \\ y = \theta = 0, & \text{on } \partial\Omega \times (0, +\infty) \\ y(\cdot, 0) = y_0,\ y_t(\cdot, 0) = y_1,\ \theta(\cdot, 0) = \theta_0, & \text{in } \Omega \end{cases} \tag{4.1}$$

where γ and σ are positive physical constants representing respectively, the rotational force constant, thermal conductivity, and ϕ is given function. The functions (y_0, y_1, θ_0) are the given initial data. When $\gamma = 0$ and $\phi(s) = m_0 + m_1 s$, with $m_0 > 0$ and $m_1 > 0$, Ben Aissa [8] has studied the global existence for small data and the uniform exponential decay rate of the energy.

Tebeau et al [49] considered the two and three-dimensional system of linear thermoelasticity in a bounded smooth domain with Dirichlet boundary conditions

$$\begin{cases} y_{tt} - \mu\Delta y - (\lambda+\mu)\nabla \text{div} y + \alpha\nabla\theta = 0, & \text{in } \Omega\times(0,+\infty) \\ \theta_t - \Delta\theta + \beta \text{div} y_t = 0, & \text{in } \Omega\times(0,+\infty) \\ y = \theta = 0, & \text{on } \partial\Omega\times(0,+\infty) \\ y(\cdot,0) = y_0,\ y_t(\cdot,0) = y_1,\ \theta(\cdot,0) = \theta_0, & \text{in } \Omega \end{cases}$$

where λ and μ are the Lame coefficients, which are assumed to satisfy $\mu > 0,\ \lambda + 2\mu > 0$. The constants $\alpha,\ \beta > 0$ are the coupling parameters. The authors analyzed whether the energy of solutions decays exponentially or uniformly to zero as $t \to \infty$. They showed that when the domain is convex, the decay rate is never uniform. In fact, the lack of uniform decay may also be due to a critical polarization of the energy on the transversal component of the displacement.

Mansouri et al. [57] considered a coupled system consisting of a Kirchhoff thermoelastic plate and an undamped wave equation

$$\begin{cases} y_{tt} - \gamma\Delta y_{tt} + a\Delta^2 y + \alpha\Delta\theta + \mu z = 0, & \text{in } \Omega\times(0,+\infty) \\ \theta_t - \sigma\Delta\theta - \beta\Delta y_t = 0, & \text{in } \Omega\times(0,+\infty) \\ z_{tt} - \mu\Delta z + \mu y = 0, & \text{in } \Omega\times(0,+\infty) \\ y = \partial_\nu y = 0, \quad z = \theta = 0, & \text{on } \partial\Omega\times(0,+\infty) \\ y(\cdot,0) = y_0,\ y_t(\cdot,0) = y_1,\ \theta(\cdot,0) = \theta_0, & \text{in } \Omega, \\ z(\cdot,0) = z_0,\ z_t(\cdot,0) = z_1 & \text{in } \Omega \end{cases}$$

They showed that the coupled system is not exponentially stable. Afterwards, they proved that the coupled system is polynomially stable, and provided an explicit polynomial decay rate of the associated semigroup.

Tebou [64] studied a coupled system of the wave and heat equations given by

$$\begin{cases} y_{tt} - c^2\Delta y + \alpha(-\Delta)^\mu\theta = 0, & \text{in } \Omega\times(0,+\infty) \\ \theta_t - \nu\Delta\theta - \beta y_t = 0, & \text{in } \Omega\times(0,+\infty) \\ y = \theta = 0, & \text{on } \partial\Omega\times(0,+\infty) \\ y(\cdot,0) = y_0,\ y_t(\cdot,0) = y_1,\ \theta(\cdot,0) = \theta_0, & \text{in } \Omega \end{cases}$$

where c and ν are positive physical constants. For $0 \le \mu < 1$, he showed that the semigroup associated to the system is not uniformly stable, and he proposed an explicit non-uniform decay rate. For $\mu = 1$ the above coupled system is reduced to the thermoelasticity equations, the author showed that in this case, the semigroup is exponentially stable. In addition, he examined a partially clamped

Kirchhoff thermoelastic plate without mechanical feedback controls, and he proved that the semigroup is also exponentially stable in this case, using a constructive frequency domain method to prove the stabilization result, along with an explicit decay rate.
Tebou et al. [36] considered thermoelastic plate with rotational forces, in a bounded domain Ω. This rotational forces involve the spectral fractional Laplacian, with power parameter $0 \leq \theta \leq 1$

$$\begin{cases} y_{tt} + (-\Delta)^{\theta} y_{tt} + \Delta^2 y + \alpha \Delta z = 0, & \text{in } \Omega \times (0, +\infty) \\ z_t - \kappa \Delta z - \beta \Delta y_t = 0, & \text{in } \Omega \times (0, +\infty) \\ y(\cdot, 0) = y_0, \; y_t(\cdot, 0) = y_1, \; z(\cdot, 0) = z_0, & \text{in } \Omega \end{cases}$$

The authors distinguished two particular cases of this problem that models for thermoelastic plate, either the Euler-Bernoulli when $\theta = 0$ or Kirchhoff if $\theta = 1$. They showed that the semigroup studied in this case is of Gevrey class δ for every $\delta > (2-\theta)/(2-4\theta)$ and proved that it is exponentially stable.

The main purpose of this chapter is to prove global solvability and energy decay estimates of the solutions of problem (4.1). We extend the results obtained by Ben Aissa by giving more precise decay rates. We use a new method recently introduced by Benaissa and Guesmia [9] to study the decay rate of solutions

The plan of the chapter is as follows. In Section 2, we give some hypotheses, and we announce the main results of this chapter. In section 3 we use the Faedo-Galerkin method to study the existence of the solutions of system (4.1). In section 4, we prove exponential stability estimates using multiplier method.

4.2 Hypothesis and main results

In this section we prepare some hypotheses which will be needed in the proof of our result.
Let ϕ is a $\mathcal{C}^1$-class function on $\mathbb{R}_+$ and bijective. Assume that there exist $m_0, \; m_1 > 0$ such that and satisfies

$$\phi(s) \geq m_0, \quad \text{and} \quad s\phi(s) \geq m_1 \widetilde{\phi}(s), \;\; \forall s \geq 0, \quad \text{where} \;\; \widetilde{\phi}(s) = \int_0^s \phi(r)\, dr. \tag{4.2}$$

Remark 4.2.1. *1) We have $\int_0^{+\infty} \phi(r)\, dr = +\infty$ and then $s \to \int_0^s \phi(r)\, dr$ is a bijection from $\mathbb{R}_+$ to $\mathbb{R}_+$*

2) The function $\widetilde{\phi}(s) = \frac{1}{2} \int_0^s \phi(r)\, dr$ is a convex function. Indeed, let $x_1 \neq 0$ and $x_2 \neq 0$ such that $x_1 < x_2$, as ϕ is of class $\mathcal{C}^1[x_1, x_2]$ and a non decreasing function, then $\widetilde{\phi}$ is a convex function.

Now if $x_1 = 0$, we have for all $0 \leq \lambda \leq 1$

$$\widetilde{\phi}(\lambda x_2) = \frac{1}{2}\int_0^{\lambda x_2} \phi(s)\,ds = \frac{1}{2}\lambda\int_0^{x_2} \phi(\lambda z)\,dz,$$

where we have the change of variable $s = z$. As ϕ is a non decreasing function and $\lambda x_2 \leq x_2$ for all $\lambda \in [0,1]$, then

$$\widetilde{\phi}(\lambda x_2) \leq \lambda\widetilde{\phi}(x_2).$$

Introduce the energy

$$E(t) = \frac{1}{2}\int_\Omega |y_t(t)|^2\,dx + \widetilde{\phi}(\|\nabla y(t)\|^2) + \frac{\gamma}{2}\int_\Omega |\nabla y_t(t)|^2\,dx + \frac{\alpha}{2\beta}\int_\Omega |\theta(t)|^2\,dx, \quad \forall t \geq 0. \tag{4.3}$$

Lemma 4.2.1. *Let (y, θ) be a solution to the problem (4.1). Then, the energy functional defined by (4.3) satisfies*

$$E'(t) = -\sigma\frac{\alpha}{\beta}\int_\Omega |\nabla\theta(t)|^2\,dx \leq 0, \quad \forall t \geq 0. \tag{4.4}$$

Proof. Multiplying the first equation (4.1) by y_t, integrating over Ω and using integration by parts, we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |y_t|^2\,dx + \frac{1}{2}\phi(\|\nabla y(t)\|^2)\frac{d}{dt}\int_\Omega |\nabla y(t)|^2\,dx + \frac{\gamma}{2}\frac{d}{dt}\int_\Omega |\nabla y_t|^2\,dx = \alpha\int_\Omega \nabla y_t \nabla\theta\,dx \tag{4.5}$$

and using (4.2), we have

$$\frac{1}{2}\phi(\|\nabla y(t)\|^2)\frac{d}{dt}\int_\Omega |\nabla y(t)|^2\,dx = \frac{d}{dt}\widetilde{\phi}(\|\nabla y(t)\|^2). \tag{4.6}$$

Multiplying the second equation (4.1) by $\alpha\theta$, integrating over Ω and using Green's formula, we find

$$\frac{\alpha}{2\beta}\frac{d}{dt}\int_\Omega |\theta(t)|^2\,dx + \sigma\frac{\alpha}{\beta}\int_\Omega |\nabla\theta(t)|^2\,dx = -\alpha\int_\Omega \nabla y_t \nabla\theta\,dx \tag{4.7}$$

Reporting (4.6) and (4.7) in (4.5), we get

$$\frac{d}{dt}\Big[\frac{1}{2}\int_\Omega |y_t(t)|^2\,dx + \widetilde{\phi}(\|\nabla y(t)\|^2) + \frac{\alpha}{2\beta}\int_\Omega |\theta(t)|^2\,dx + \frac{\gamma}{2}\int_\Omega |\nabla y_t(t)|^2\,dx\Big] = -\sigma\frac{\alpha}{\beta}\int_\Omega |\nabla\theta(t)|^2\,dx.$$

□

We are now in the position to state our results

Theorem 4.2.2. *(Well-posedness) Let $\phi : [0,+\infty[\to [0,+\infty[$ a $\mathcal{C}^1$-class function and*

$$(y_0, y_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H^2(\Omega) \cap H_0^1(\Omega)$$

,

$$\theta_0 \in H^2(\Omega) \cap H_0^1(\Omega)$$

. *Then the problem (4.1) has a unique weak solution (y, θ) such that for any $T > 0$, we have*

$$y \in L^\infty(0,T; H^2(\Omega) \cap H_0^1(\Omega))$$

$$y_t \in L^\infty(0,T; H^2(\Omega) \cap H_0^1(\Omega)), \quad y_{tt} \in L^\infty(0,T; H_0^1(\Omega))$$

$$\theta \in L^\infty(0,T; H_0^1(\Omega)) \cap L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega)).$$

$$\theta_t \in L^\infty(0,T; L^2(\Omega)).$$

Theorem 4.2.3. *(Stabilization) Let $(y_0, y_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H^2(\Omega) \cap H_0^1(\Omega)$, $\theta_0 \in H^2(\Omega) \cap H_0^1(\Omega)$. Assume that ϕ satisfies (4.2) and $\frac{1}{\beta}$ small enough.*
The energy of the unique solution of system (4.1), given by (4.3), decays exponentially to zero, there exist positive constants M and λ, independent of the initial data, with

$$E(t) \leq M \exp(-\lambda t) E(0). \tag{4.8}$$

4.3 Proof of Theorem 4.2.2

We will use the Faedo-Galerkin method to prove the existence of a global solutions. Let e^k, $k \in \mathbb{N}$ be normalized eigenfunctions of the operators Δ

$$\begin{cases} -\Delta e^k = \lambda^k e^k \\ e^k = 0 \quad in \quad \partial\Omega \end{cases}$$

Let us denote by W^m the linear hull of $e^1, ..., e^m$. Note that $(e^k)_k$ is a basis of $H^2(\Omega)$, $H_0^1(\Omega)$ and $L^2(\Omega)$; i.e., the set $e^1, .., e^m, ...$ is dense in $H^2(\Omega)$, $H_0^1(\Omega)$ and $L^2(\Omega)$.

Step 1: Approximate solutions. We construct approximate solutions y^m and θ^m, $m = 1, 2, 3, \ldots$, in the form

$$\begin{cases} y^m(x,t) = \sum_{k=1}^{m} h^{m,k}(t) e^k(x) \\ \theta^m(x,t) = \sum_{k=1}^{n} c^{m,k}(t) e^k(x) \end{cases} \tag{4.9}$$

where $h^{m,k}$ and $c^{m,k}$ $(k = 1, 2, ...m)$ are determined by the following ordinary differential equations

$$\begin{cases} (y_{tt}^m - \phi(\|\nabla y^m(t)\|^2)\Delta y^m - \gamma \Delta y_{tt}^m + \alpha \Delta \theta^m, w) = 0 \quad \forall w \in W^m \\ (\theta_t^m - \sigma \Delta \theta^m - \beta \Delta y_t^m, v) = 0, \quad \forall v \in W^m \end{cases} \tag{4.10}$$

with initial conditions

$$y^m(x,0) = y_0^m = \sum_{k=1}^{m} \langle f, e^m \rangle e^m \to y_0, \quad \text{in } H^2(\Omega) \cap H_0^1(\Omega) \ \text{ as } m \to +\infty, \tag{4.11}$$

$$y_t^m(x,0) = y_1^m = \sum_{k=1}^{n} \langle f_t, e^m \rangle e^m \to y_1, \quad \text{in } H^2(\Omega) \cap H_0^1(\Omega) \text{ as } m \to +\infty. \tag{4.12}$$

$$\theta^m(x,0) = \theta_0^m = \sum_{k=1}^{n} \langle g, e^m \rangle e^m \to \theta_0, \quad \text{in } H^2(\Omega) \cap H_0^1(\Omega) \text{ as } m \to +\infty, \tag{4.13}$$

$$\phi(\|\nabla y_0^m\|^2)\Delta y_0^m - \alpha \Delta \theta_0^m \to \ \phi(\|\nabla y_0\|^2)\Delta y_0 - \alpha \Delta \theta_0, \quad \text{in } L^2(\Omega) \text{ as } m \to +\infty, \tag{4.14}$$

Step 2: A priori estimates.

Choosing $w = y_t^m$ and $v = \theta^m$ in (4.10) and using Green's formula, we find

$$\begin{cases} \dfrac{d}{dt}\displaystyle\int_\Omega |y_t^m|^2\,dx + \gamma \dfrac{d}{dt}\displaystyle\int_\Omega |\nabla y_t^m|^2\,dx + 2\dfrac{d}{dt}\widetilde{\phi}(\|\nabla y^m(t)\|^2) = 2\alpha \displaystyle\int_\Omega \nabla y_t^m \nabla \theta^m\,dx \\ \dfrac{\alpha}{\beta}\dfrac{d}{dt}\displaystyle\int_\Omega |\theta^m(t)|^2 dx + 2\sigma\dfrac{\alpha}{\beta}\displaystyle\int_\Omega |\nabla\theta^m(t)|^2 dx = -2\alpha \displaystyle\int_\Omega \nabla y_t^m \nabla\theta^m\,dx, \end{cases} \tag{4.15}$$

integrating (4.15) over $(0,t)$ and using (4.11)-(4.13), we obtain

$$\begin{aligned} &\int_\Omega |y_t^m|^2 dx + 2\widetilde{\phi}(\|\nabla y^m(t)\|^2) + \frac{\alpha}{\beta}\int_\Omega |\theta^m|^2\,dx + \gamma\int_\Omega |\nabla y_t^m|^2 dx + 2\sigma\frac{\alpha}{\beta}\int_0^t\int_\Omega |\nabla\theta^m(s)|^2\,dx\,ds \\ &= \int_\Omega |y_1^m|^2 dx + 2\widetilde{\phi}(\|\nabla y_0^m\|^2) + \gamma\int_\Omega |\nabla y_1^m|^2 dx + \frac{\alpha}{\beta}\int_\Omega |\theta_0^m|^2\,dx \\ &= 2E^m(0) \le C_0 \end{aligned} \tag{4.16}$$

where

$$E^m(0) = \frac{1}{2}\int_\Omega |y_1^m|^2 dx + \widetilde{\phi}(\|\nabla y_0^m\|^2) + \frac{\gamma}{2}\int_\Omega |\nabla y_1^m|^2 dx + \frac{\alpha}{2\beta}\int_\Omega |\theta_0^m|^2\,dx,$$

and C_0 is a positive constant independent of m.

In the other hand, $\widetilde{\phi}^{-1}$ is non decreasing

$$\int_\Omega |\nabla y^m|^2\,dx \le \widetilde{\phi}^{-1}(E^m(0)) \le \widetilde{\phi}^{-1}(C_0).$$

These estimates imply that the solution (y^m, θ^m) exists globally in $[0, +\infty[$.

Estimates (4.16) yields

$$y^m \text{ is bounded in } L^\infty(0,T,H_0^1(\Omega)), \tag{4.17}$$

$$y_t^m \text{ is bounded in } L^\infty(0,T,H_0^1(\Omega)), \tag{4.18}$$

$$\theta^m \text{ is bounded in } L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)). \tag{4.19}$$

Choosing $w = -2\Delta y_t^m$ and $v = -2\alpha\Delta\theta^m$ in (4.10) and using Green's formula, we obtain

$$\begin{cases} \dfrac{d}{dt}\displaystyle\int_\Omega |\nabla y_t^m(t)|^2\,dx + \phi(\|\nabla y^m(t)\|^2)\frac{d}{dt}\int_\Omega |\Delta y^m(t)|^2\,dx + \gamma\frac{d}{dt}\int_\Omega |\Delta y_t^m(t)|^2\,dx = 2\alpha\int_\Omega \Delta\theta^m(t)\Delta y_t^m(t)\,dx \\ \dfrac{\alpha}{\beta}\dfrac{d}{dt}\displaystyle\int_\Omega |\nabla\theta^m(t)|^2\,dx + 2\sigma\frac{\alpha}{\beta}\int_\Omega |\Delta\theta^m(t)|^2\,dx = -2\alpha\int_\Omega \Delta y_t^m(t)\Delta\theta^m(t)\,dx. \end{cases} \tag{4.20}$$

The second term of (4.20) can be rewritten as follows

$$\begin{aligned} &\phi(\|\nabla y^m(t)\|^2)\frac{d}{dt}\int_\Omega |\Delta y^m(t)|^2\,dx \\ &= \frac{d}{dt}\left[\phi(\|\nabla y^m(t)\|^2)\int_\Omega |\Delta y^m(t)|^2 dx\right] - \int_\Omega |\Delta y^m(t)|^2\,dx\frac{d}{dt}\left[\phi(\|\nabla y^m(t)\|^2)\right] \\ &= \frac{d}{dt}\left[\phi(\|\nabla y^m(t)\|^2)\int_\Omega |\Delta y^m(t)|^2\,dx\right] - 2\phi'(\|\nabla y^m(t)\|^2)\int_\Omega \nabla y^m(t)\nabla y_t^m(t)\,dx\int_\Omega |\Delta y^m(t)|^2\,dx. \end{aligned} \tag{4.21}$$

Reporting (4.21) in (4.20) and integrating over $(0.t)$, we get

$$\begin{aligned} &\int_\Omega |\nabla y_t^m(t)|^2\,dx + \phi(\|\nabla y^m(t)\|^2)\int_\Omega |\Delta y^m(t)|^2\,dx + \gamma\int_\Omega |\Delta y_t^m(t)|^2\,dx \\ &+ \frac{\alpha}{\beta}\int_\Omega |\nabla\theta^m(t)|^2\,dx + 2\sigma\frac{\alpha}{\beta}\int_0^t\int_\Omega |\Delta\theta^m(s)|^2\,dx\,ds \\ &= \int_\Omega |\nabla y_1^m|^2\,dx + \phi(\|\nabla y_0^m\|^2)\int_\Omega |\Delta y_0^m|^2\,dx + \gamma\int_\Omega |\Delta y_1^m|^2\,dx \\ &+ \frac{\alpha}{\beta}\int_\Omega |\nabla\theta_0^m|^2\,dx + 2\int_0^t \phi'(\|\nabla y^m(s)\|^2)\int_\Omega |\Delta y^m(s)|^2\,dx\int_\Omega \nabla y^m(s)\nabla y_t^m(s)\,dx\,ds. \end{aligned} \tag{4.22}$$

Applying the Cauchy-Schwarz inequality, (4.17) and (4.18) in the last term of the right-hand side of (4.22), we find

$$\begin{aligned} &\int_0^t \phi'(\|\nabla y^m(s)\|^2)\int_\Omega |\Delta y^m(s)|^2\,dx\int_\Omega \nabla y^m(s)\nabla y_t^m(s)\,dx\,ds \\ &\qquad \le C\max_{0\le r\le E^m(0)}|\phi'(r)|\int_0^t \phi(\|\nabla y^m(s)\|^2)\int_\Omega |\Delta y^m(s)|^2\,dx\,ds. \end{aligned} \tag{4.23}$$

Reporting (4.23) in (4.22) and using the Gronwall's lemma, we have

$$y^m \text{ is bounded in } L^\infty(0,T,H^2(\Omega)), \tag{4.24}$$

$$y_t^m \text{ is bounded in } L^\infty(0,T,H^2(\Omega)), \tag{4.25}$$

$$\theta^m \text{ is bounded in } L^\infty(0,T;H_0^1(\Omega))\cap L^2(0,T;H^2(\Omega)). \tag{4.26}$$

Choosing $w = y_{tt}^m(t)$, $v = \theta^m(t)$ in (4.10) and choosing $t = 0$, we obtain that

$$\begin{cases} \left(y_{tt}^m(0) - \phi(\|\nabla y_0^m\|^2)\Delta y_0^m - \gamma\Delta y_{tt}^m(0) + \alpha\Delta\theta_0^m, y_{tt}^m(0)\right) = 0 \\ (\theta_t^m(0) - \sigma\Delta\theta_0^m + \beta\Delta y_1^m, \theta_t^m(0)) = 0 \end{cases}$$

Using Cauchy-Schwarz's inequality, we have

$$(\gamma - c_s^2)\|\nabla y_{tt}^m(0)\| \le c_s\Big(\int_\Omega |\phi(\|\nabla y_0^m\|^2)\Delta y_0^m - \alpha\Delta\theta_0^m|^2\,dx\Big)^{\frac{1}{2}},$$

where $c_s > 0$ and satisfies $\|z\| \le c_s\|\nabla z\|$ for all $z \in H_0^1(\Omega)$ and

$$\|\theta_t^m(0)\| \le c\Big(\|\Delta\theta_0^m\| + \|\Delta y_1^m\|\Big).$$

We choose $\gamma > c_s^2$ and using (4.11)-(4.14), we get

$$y_{tt}^m(0) \text{ is bounded in } H_0^1(\Omega), \tag{4.27}$$

$$\theta_t^m(0) \text{ is bounded in } L^2(\Omega). \tag{4.28}$$

We assume first $t < T$ and apply (4.10) at points t and $t+\xi$ with ξ such that $0 < \xi < T-t$. By taking the difference and $w = y_t^m(t+\xi) - y_t^m$ and $\theta^m(t+\xi) - \theta^m(t)$, we find

$$\begin{cases} (y_{tt}^m(t+\xi) - y_{tt}^m(t) - \phi(\|\nabla y^m(t+\xi)\|^2)\Delta y^m(t+\xi) + \phi(\|\nabla y^m(t)\|^2)\Delta y^m(t), y_t^m(t+\xi) - y_t^m(t)) \\ -\gamma(\Delta y_{tt}^m(t+\xi) - \Delta y_{tt}^m(t), y_t^m(t+\xi) - y_t^m(t)) + \alpha(\Delta\theta^m(t+\xi) - \Delta\theta^m(t), y_t^m(t+\xi) - y_t^m(t)) = 0 \\ (\theta_t^m(t+\xi) - \theta_t^m(t) - \sigma\Delta\theta^m(\xi+t) + \sigma\Delta\theta^m(t), \theta^m(t+\xi) - \theta^m(t)) \\ -\beta(\Delta y_t^m(t+\xi) - \Delta y_t^m, \theta^m(t+\xi) - \theta^m(t)) = 0. \end{cases}$$

Now, applying Green's formula, we find

$$\begin{aligned} &\frac{d}{dt}\int_\Omega |y_t^m(t+\xi) - y_t^m(t)|^2\,dx + \phi(\|\nabla y^m(t+\xi)\|^2)\frac{d}{dt}\int_\Omega |\nabla y^m(t+\xi) - \nabla y^m(t)|^2\,dx \\ &+ \gamma\frac{d}{dt}\int_\Omega |\nabla y_t^m(t+\xi) - \nabla y_t^m(t)|^2\,dx + \frac{\alpha}{\beta}\frac{d}{dt}\int_\Omega |\theta^m(t+\xi) - \theta^m(t)|^2\,dx + \sigma\frac{2\alpha}{\beta}\int_\Omega |\nabla\theta^m(t+\xi) - \nabla\theta^m(t)|^2\,dx \\ &= 2(\phi(\|\nabla y^m(t+\xi)\|^2) - \phi(\|\nabla y^m(t)\|^2))\int_\Omega \Delta y^m(t)(y_t^m(t+\xi) - y_t^m(t))\,dx. \end{aligned} \tag{4.29}$$

Set

$$\begin{aligned} \Psi_{\xi m}(t) = {}& \|y_t^m(t+\xi) - y_t^m(t)\|^2 + \phi(\|\nabla y^m(t+\xi)\|^2)\|\nabla y^m(t+\xi) - \nabla y^m(t)\|^2 \\ &+ \gamma\|\nabla y_t^m(t+\xi) - \nabla y_t^m(t)\|^2 + \|\theta^m(t+\xi) - \theta^m(t)\|^2. \end{aligned} \tag{4.30}$$

Using Cauchy-Schwarz's inequality, (4.29) and the fact that, ϕ is $\mathcal{C}^1$, we obtain

$$\begin{aligned} &\frac{d}{dt}\Psi_{\xi m}(t) + \sigma\frac{2\alpha}{\beta}\int_\Omega |\nabla\theta^m(t+\xi) - \nabla\theta^m(t)|^2\,dx \\ &= 2\|\nabla y^m(t+\xi) - \nabla y^m(t)\|^2\phi'(\|\nabla y^m(t+\xi)\|^2)\int_\Omega \nabla y^m(t+\xi)\nabla y_t^m(t+\xi)\,dx + \\ &2(\phi(\|\nabla y^m(t+\xi)\|^2) - \phi(\|\nabla y^m(t)\|^2))\int_\Omega \Delta y^m(t)(y_t^m(t+\xi) - y_t^m(t))\,dx \\ &\le \frac{2}{m_0}\phi'(\|\nabla y^m(t+\xi)\|^2))\phi(\|\nabla y^m(t+\xi)\|^2)(\|\nabla y^m(t+\xi) - \nabla y^m(t)\|^2)\|\nabla y^m(t+\xi)\|\|\nabla y_t^m(t+\xi)\| \\ &+ c(\|\nabla y^m(t+\xi)\|^2 - \|\nabla y^m(t)\|^2)\|\Delta y^m(t)\|\|y_t^m(t+\xi) - y_t^m(t)\|. \end{aligned} \tag{4.31}$$

The last term can be rewritten as

$$c\|\Delta y^m(t)\|\|y_t^m(t+\xi)-y_t^m(t)\|\int_\Omega(\nabla y^m(t+\xi)-\nabla y^m(t))(\nabla y^m(t+\xi)+\nabla y^m(t))\,dx, \quad (4.32)$$

Reporting (4.17), (4.18), (4.24), (4.32) in (4.31), and using the Cauchy-Schwarz's inequality we find

$$\frac{d}{dt}\Psi_{\xi m}(t)+\sigma\frac{2\alpha}{\beta}\int_\Omega|\nabla\theta^m(t+\xi)-\nabla\theta^m(t)|^2\,dx\le c\Psi_{\xi m}(t).$$

Therefore, we deduce that

$$\Psi_{\xi m}(t)\le\Psi_{\xi m}(0)\exp(cT),\quad \forall t\in[0,T].$$

Dividing the two sides by ξ^2, letting $\xi\to 0$, and using (4.30), we deduce that

$$\begin{aligned}&\|y_{tt}^m(t)\|^2+m_0\|\nabla y_t^m(t)\|^2+\gamma\|\nabla y_{tt}^m(t)\|^2+\|\theta_t^m(t)\|^2\\&\le c(\|y_{tt}^m(0)\|^2+\phi(\|\nabla y_0^m\|^2)\|\nabla y_1^m(t)\|^2+\gamma\|\nabla y_{tt}^m(0)\|^2+\|\theta_t^m(0)\|^2)\end{aligned}$$

By (4.12), (4.27) and (4.28), we deduce that

$$\|y_{tt}^m(t)\|^2+\|\nabla y_t^m(t)\|^2+\|\nabla y_{tt}^m(t)\|^2+\|\theta_t^m(t)\|^2\le C_2,$$

for all $t\in[0.T]$, where C_2 is a positive constant independent of m. Therefore, we conclude that

$$y_{tt}^m \text{ is bounded in } L^\infty(0,T,H_0^1(\Omega)), \quad (4.33)$$

$$\theta_t^m \text{ is bounded in } L^\infty(0,T;L^2(\Omega)). \quad (4.34)$$

Step 3: Passage to the limit

Applying Dunford-Petit theorem we conclude from (4.17)-(4.19), (4.24)-(4.26), (4.33) and (4.34) replacing the sequence (y^m,θ^m) with a subsequence (y^k,θ^k) we have

$$y^k\rightharpoonup y,\ \text{weak-star in } L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega)) \quad (4.35)$$

$$y_t^k\rightharpoonup y_t,\ \text{weak-star in } L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega)) \quad (4.36)$$

$$y_{tt}^k\rightharpoonup y_{tt},\ \text{weak-star in } L^\infty(0,T;H_0^1(\Omega)) \quad (4.37)$$

$$\phi(\|\nabla y^k(t)\|^2)\Delta y^k\rightharpoonup\chi \text{ weak-star in } L^\infty(0,T;L^2(\Omega)) \quad (4.38)$$

$$\theta^k\rightharpoonup\theta \text{ weak-star in } L^\infty(0,T;H_0^1(\Omega))\cap L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega)) \quad (4.39)$$

$$\theta_t^k \rightharpoonup \theta_t, \text{ weak-star in } L^\infty(0,T;L^2(\Omega)) \tag{4.40}$$

We shall prove that, $\chi = \phi(\|\nabla y(t)\|^2)\Delta y$ i.e.,

$$\phi(\|\nabla y^k(t)\|^2)\Delta y^k(t) \rightharpoonup \phi(\|\nabla y(t)\|^2)\Delta y(t) \text{ weak-star in } L^\infty(0,T;L^2(\Omega)) \tag{4.41}$$

For $v \in L^2(0,T;L^2(\Omega))$, we have

$$\begin{aligned}
&\int_0^T \int_\Omega (\chi - \phi(\|\nabla y(t)\|^2)\Delta y) v \, dx \, dt \\
&= \int_0^T \int_\Omega (\chi - \phi(\|\nabla y^k(t)\|^2)\Delta y^k) v \, dx \, dt + \int_0^T \phi(\|\nabla y(t)\|^2) \int_\Omega (\Delta y^k - \Delta y) v \, dx \, dt \\
&+ \int_0^T (\phi(\|\nabla y^k(t)\|^2) - \phi(\|\nabla y(t)\|^2) \int_\Omega \Delta y^k v \, dx \, dt
\end{aligned} \tag{4.42}$$

We deduce from (4.35) and (4.38) that the first and second terms in (4.42) tend to zero as $k \to +\infty$.

Using that ϕ is a $\mathcal{C}^1$-class function on $\mathbb{R}_+$, we have

$$\begin{aligned}
&\int_0^T (\phi(\|\nabla y^k(t)\|^2) - \phi(\|\nabla y(t)\|^2)) \int_\Omega \Delta y^k v \, dx \, dt \\
&\leq c \int_0^T |\|\nabla y^k(t)\|^2 - \|\nabla y(t)\|^2| \|\Delta y^k\| \|v\| \, dt \\
&\leq c \int_0^T \int_\Omega |\Delta(y^k + y)(y^k - y)| \, dx \, dt \\
&\leq c \int_0^T \|y^k(t) - y(t)\| \, dt
\end{aligned} \tag{4.43}$$

As y^k is bounded in $L^\infty(0,T;H^2(\Omega))$ and the injection of $H^2(\Omega)$ in $L^2(\Omega)$ is compact, we have we have

$$y^k \to y \text{ strongly in } L^2(0,T;L^2(\Omega)). \tag{4.44}$$

From(4.42), (4.66)and (4.44) , we deduce (4.41). It follows at once from (4.35), (4.37), (4.39) and (4.40) that for each fixed $v \in L^2(0,T;L^2(\Omega))$,

$$\begin{aligned}
&\int_0^T \int_\Omega (y_{tt}^k - \phi(\|\nabla y^k(t)\|^2)\Delta y^k - \gamma \Delta y_{tt}^k + \alpha \Delta \theta^k) v \, dx \, dt \\
&\to \int_0^T \int_\Omega (y_{tt} - \phi(\|\nabla y(t)\|^2)\Delta y - \gamma \Delta y_{tt} + \alpha \Delta \theta) v \, dx \, dt
\end{aligned}$$

and

$$\int_0^T \int_\Omega (\theta_t^k - \sigma \Delta \theta^k - \beta \Delta y_t^k) v \, dx \, dt \to \int_0^T \int_\Omega (\theta_t - \sigma \Delta \theta - \beta \Delta y_t) v \, dx \, dt$$

as $k \to +\infty$

4.4 Proof of Theorem 4.2.3

In this section, we prove our stability,using the multiplier technique
and the following Lemma due to [27]

Lemma 4.4.1. *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing differentiable function and $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ a convex and increasing function such that $\varphi(0) = 0$. Assume that*

$$\int_S^T \varphi(E(t))\,dt \le E(s), \ \ \forall 0 \le S \le T$$

Then E satisfies the following estimate:

$$E(t) \le \psi^{-1}\Big(h(t) + \psi(E(0))\Big), \ \forall t \ge 0 \tag{4.45}$$

where $\psi(t) = \int_t^1 \frac{1}{\varphi(s)}\,ds$ for $t > 0$, $\ \ h(t) = 0$ for $0 \le t \le \frac{E(0)}{\varphi(E(0))}$ and

$$h^{-1}(t) = t + \frac{\psi(t + \psi(E(0)))}{\varphi(\psi^{-1}(t + \psi(E(0))))}, \ \forall t \ge \frac{E(0)}{\varphi(E(0))}$$

Step 1: We multiplying the first equation of (4.1) by $\dfrac{\varphi(E)}{E}y$ where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is convex, increasing and of class $\mathcal{C}^1$ on $]0, +\infty[$ such that $\varphi(0) = 0$ and we integrate by parts, we have for all $0 \le S \le T$

$$\begin{aligned}
0 &= \int_S^T \frac{\varphi(E)}{E} \int_\Omega y \left[y_{tt} + \phi(\|\nabla y\|^2)\Delta y - \gamma \Delta y_{tt} + \alpha \Delta\theta\right]\,dx\,dt \\
&= \left[\frac{\varphi(E)}{E} \int_\Omega y y_t\,dx\right]_S^T - \int_S^T \left(\frac{\varphi(E)}{E}\right)' \int_\Omega y y_t\,dx\,dt - \int_S^T \frac{\varphi(E)}{E} \int_\Omega |y_t|^2\,dx\,dt \\
&+ \gamma \left[\frac{\varphi(E)}{E} \int_\Omega \nabla y \nabla y_t\,dx\right]_S^T - \gamma \int_S^T \left(\frac{\varphi(E)}{E}\right)' \int_\Omega \nabla y \nabla y_t\,dx\,dt - \gamma \int_S^T \frac{\varphi(E)}{E} \int_\Omega |\nabla y_t|^2\,dx\,dt \\
&+ \int_S^T \frac{\varphi(E)}{E} \phi(\|\nabla y\|^2) \int_\Omega |\nabla y(s)|^2\,dx\,dt \\
&+ \alpha \int_S^T \frac{\varphi(E)}{E} \int_\Omega y \Delta\theta\,dx\,dt.
\end{aligned} \tag{4.46}$$

Then, we have

$$\begin{aligned}
&2 \int_S^T \frac{\varphi(E)}{E} \phi(\|\nabla y(t)\|^2) \int_\Omega |\nabla y(t)|^2\,dx\,dt = \\
&- 2 \left[\frac{\varphi(E)}{E} \int_\Omega y y_t\,dx\right]_S^T + 2 \int_S^T \left(\frac{\varphi(E)}{E}\right)' \int_\Omega y y_t\,dx\,dt + 2 \int_S^T \frac{\varphi(E)}{E} \int_\Omega |y_t(t)|^2\,dx\,dt \\
&- 2\gamma \left[\frac{\varphi(E)}{E} \int_\Omega \nabla y \nabla y_t\,dx\right]_S^T + 2\gamma \int_S^T \left(\frac{\varphi(E)}{E}\right)' \int_\Omega \nabla y \nabla y_t\,dx\,dt + 2\gamma \int_S^T \frac{\varphi(E)}{E} \int_\Omega |\nabla y_t(t)|^2\,dx\,dt \\
&- 2\alpha \int_S^T \frac{\varphi(E)}{E} \int_\Omega y \Delta\theta\,dx\,dt.
\end{aligned}$$

Applying Green's formula and Sobolev embedding inequality (4.3) and (4.2), we obtain

$$\begin{aligned}
&2\int_S^T \varphi(E) + \int_S^T \frac{\varphi(E)}{E}\phi(\|\nabla y(t)\|^2)\int_\Omega |\nabla y(t)|^2\,dx\,dt \\
&\leq \frac{2}{m_1}\left[\frac{\varphi(E)}{E}\int_\Omega yy_t\,dx\right]_S^T + \frac{2}{m_1}\int_S^T \left(\frac{\varphi(E)}{E}\right)'\int_\Omega yy_t\,dx\,dt \\
&+ \frac{2+m_1}{m_1}\int_S^T \frac{\varphi(E)}{E}\int_\Omega |y_t|^2\,dx\,dt + \frac{2\gamma}{m_1}\left[\frac{\varphi(E)}{E}\int_\Omega \nabla y\nabla y_t\,dx\right]_S^T \\
&+ \frac{2\gamma}{m_1}\int_S^T \left(\frac{\varphi(E)}{E}\right)'\int_\Omega \nabla y\nabla y_t\,dx\,dt + \gamma\frac{2+m_1}{m_1}\int_S^T \frac{\varphi(E)}{E}\int_\Omega |\nabla y_t|^2\,dx\,dt \\
&+ \frac{2\alpha}{m_1}\int_S^T \frac{\varphi(E)}{E}\int_\Omega \nabla y\nabla\theta\,dx\,dt + \frac{\alpha}{\beta}\int_S^T \frac{\varphi(E)}{E}\int_\Omega |\theta|^2\,dx\,dt.
\end{aligned} \tag{4.47}$$

Since E is nonincreasing, $s \to \dfrac{\varphi(s)}{s}$ is non decreasing, using Young's, Sobolev embedding inequalities and (4.2), we have

$$\begin{aligned}
\left|\int_\Omega yy_t\,dx\right| &\leq \frac{\varepsilon}{2}\|y_t\|_2^2 + 2C(\epsilon)\|y\|_2^2 \\
&\leq \frac{\varepsilon}{2}\|y_t\|_2^2 + 2C(\varepsilon)\|\nabla y\|_2^2 \\
&\leq \varepsilon E(t) + 2C(\varepsilon)\widetilde{\phi}^{-1}(E(t)),
\end{aligned}$$

$$\frac{2}{m_1}\left|\left[\frac{\varphi(E)}{E}\int_\Omega yy_t dx\right]_S^T\right| \leq \varepsilon\varphi(E(S)) + C(\varepsilon)\frac{\varphi(E(S))}{E(S)}\widetilde{\phi}^{-1}(E(S)), \tag{4.48}$$

and

$$\begin{aligned}
\frac{2}{m_1}\int_S^T \left(\frac{\varphi(E)}{E}\right)'\int_\Omega yy_t\,dx\,dt &\leq \varepsilon\int_S^T \Big(-\left(\frac{\varphi(E)}{E}\right)'\Big)E\,dt + C(\varepsilon)\int_S^T \Big(-\left(\frac{\varphi(E)}{E}\right)'\Big)\widetilde{\phi}^{-1}(E)\,dt \\
&\leq \varepsilon\varphi(E(S)) + C(\varepsilon)\frac{\varphi(E(S))}{E(S)}\widetilde{\phi}^{-1}(E(S)).
\end{aligned} \tag{4.49}$$

Similarly,

$$\left|\int_\Omega \nabla y\nabla y_t\,dx\right| \leq \varepsilon E(t) + 2C(\varepsilon)\widetilde{\phi}^{-1}(E(t)),$$

$$\frac{2\gamma}{m_1}\left|\left[\frac{\varphi(E)}{E}\int_\Omega \nabla y\nabla y_t dx\right]_S^T\right| \leq \varepsilon\varphi(E(S)) + C(\varepsilon)\frac{\varphi(E(S))}{E(S)}\widetilde{\phi}^{-1}(E(S)), \tag{4.50}$$

and

$$\begin{aligned}
\frac{2\gamma}{m_1}\int_S^T \left(\frac{\varphi(E)}{E}\right)'\int_\Omega \nabla y\nabla y_t\,dx\,dt &\leq \varepsilon\int_S^T \Big(-\left(\frac{\varphi(E)}{E}\right)'\Big)E\,dt + C(\varepsilon)\int_S^T \Big(-\left(\frac{\varphi(E)}{E}\right)'\Big)\widetilde{\phi}^{-1}(E)\,dt \\
&\leq \varepsilon\varphi(E(S)) + C(\varepsilon)\frac{\varphi(E(S))}{E(S)}\widetilde{\phi}^{-1}(E(S)),
\end{aligned} \tag{4.51}$$

and

$$\frac{\alpha}{\beta}\int_S^T \frac{\varphi(E)}{E}\int_\Omega |\theta|^2\,dx\,dt \le C_0\int_S^T \frac{\varphi(E)}{E}(-E')\,dt \le C_0\varphi(E(S)), \tag{4.52}$$

and

$$\frac{2\alpha}{m_1}\left|\int_S^T \frac{\varphi(E)}{E}\int_\Omega \nabla y\nabla\theta\,dx\,dt\right| \le \varepsilon\int_S^T \frac{\varphi(E)}{E}\widetilde{\phi}^{-1}(E)\,dt + C_0(\varepsilon)\int_S^T \frac{\varphi(E)}{E}(-E')\,dt \le \varepsilon\int_S^T \frac{\varphi(E)}{E}\widetilde{\phi}^{-1}(E)\,dt + C_0(\varepsilon)\varphi(E(S)). \tag{4.53}$$

Reporting (4.48)-(4.53) in (4.47), we get

$$\begin{aligned}
&2\int_S^T \varphi(E) + \int_S^T \frac{\varphi(E)}{E}\phi(\|\nabla y(t)\|^2)\int_\Omega |\nabla y(t)|^2\,dx\,dt\\
&\le C_1\varphi(E(S)) + C(\varepsilon)\frac{\varphi(E(S))}{E(S)}\widetilde{\phi}^{-1}(E(S)) + \varepsilon\int_S^T \frac{\varphi(E)}{E}\widetilde{\phi}^{-1}(E)\,dt\\
&+ \frac{2+m_1}{m_1}(c_s+\gamma)\int_s^T \frac{\varphi(E)}{E}\int_\Omega |\nabla y_t|^2\,dx\,dt.
\end{aligned} \tag{4.54}$$

Step 2: In this step, we are going to estimate the term $\int_S^T \frac{\varphi(E)}{E}\int_\Omega |\nabla y_t|^2\,dx\,dt$.
We Multiplying the second Eq of (4.1) by $\dfrac{\varphi(E)}{E}y_t$ integrating by parts over $\Omega\times(S,T)$, we obtain

$$\begin{aligned}
0 &= \int_S^T \frac{\varphi(E)}{E}\int_\Omega y_t\left(\theta_t - \sigma\Delta\theta - \beta\Delta y_t\right)dx\,dt\\
&= \left[\frac{\varphi(E(t))}{E(t)}\int_\Omega y_t\theta\,dx\right]_S^T - \int_S^T \frac{\varphi(E)}{E}\int_\Omega y_{tt}\theta\,dx\,dt - \int_S^T \left(\frac{\varphi(E)}{E}\right)'\int_\Omega y_t\theta\,dx\,dt\\
&+ \sigma\int_S^T \frac{\varphi(E)}{E}\int_\Omega \nabla y_t\nabla\theta\,dx\,dt + \beta\int_S^T \frac{\varphi(E)}{E}\int_\Omega |\nabla y_t|^2\,dx\,dt.
\end{aligned}$$

Then,

$$\begin{aligned}
\beta\int_S^T \frac{\varphi(E)}{E}\int_\Omega |\nabla y_t|^2\,dx\,dt &= -\left[\frac{\varphi(E(t))}{E(t)}\int_\Omega y_t\theta\,dx\right]_S^T + \int_S^T \frac{\varphi(E)}{E}\int_\Omega y_{tt}\theta\,dx\,dt\\
&+ \int_S^T \left(\frac{\varphi(E)}{E}\right)'\int_\Omega y_t\theta\,dx\,dt - \sigma\int_S^T \frac{\varphi(E)}{E}\int_\Omega \nabla y_t\nabla\theta\,dx\,dt.
\end{aligned} \tag{4.55}$$

Since E is nonincreasing, using the Cauchy-Schwarz inequality and (4.3), we have

$$\left|\left[\frac{\varphi(E(t))}{E(t)}\int_\Omega y_t\theta dx\right]_S^T\right| \le C_0\varphi(E(S)), \tag{4.56}$$

$$\int_S^T \left(\frac{\varphi(E(t))}{E(t)}\right)'\int_\Omega y_t\theta\,dx\,dt \le \int_S^T \left(-\left(\frac{\varphi(E(t))}{E(t)}\right)'\right)\int_\Omega y_t\theta\,dx\,dt \le C_0\varphi(E(S)). \tag{4.57}$$

Thanks to Young inequality, we obtain

$$\begin{aligned}\sigma\int_S^T\frac{\varphi(E(t))}{E(t)}\int_\Omega\nabla y_t\nabla\theta\,dx\,dt &\le \frac{\beta}{2}\int_S^T\frac{\varphi(E(t))}{E(t)}\int_\Omega|\nabla y_t|^2\,dx\,dt+\frac{\sigma^2}{2\beta}\int_S^T\frac{\varphi(E(t))}{E(t)}\int_\Omega|\nabla\theta|^2dxdt\\ &\le \frac{\beta}{2}\int_S^T\frac{\varphi(E(t))}{E(t)}\int_\Omega|\nabla y_t|^2\,dx\,dt+\frac{\sigma^2}{2\alpha}\int_S^T\frac{\varphi(E(t))}{E(t)}(-E')\,dt\\ &\le \frac{\beta}{2}\int_S^T\frac{\varphi(E(t))}{E(t)}\int_\Omega|\nabla y_t|^2\,dx\,dt+C_0\varphi(E(S)).\end{aligned}\tag{4.58}$$

Substituting (4.56)-(4.58) in (4.55), we get

$$\frac{\beta}{2}\int_S^T\frac{\varphi(E(t))}{E(t)}\int_\Omega|\nabla y_t|^2\,dx\,dt\le C_0\varphi(E(S))+\int_S^T\frac{\varphi(E(t))}{E(t)}\int_\Omega y_{tt}\theta\,dx\,dt.$$

Choosing $\gamma > c_s$, we obtain

$$\frac{2+m_1}{m_1}(c_s+\gamma)\int_S^T\frac{\varphi(E(t))}{E(t)}\int_\Omega|\nabla y_t|^2\,dx\,dt\le C\varphi(E(S))+\frac{\gamma}{\beta}C'\int_S^T\frac{\varphi(E(t))}{E(t)}\int_\Omega y_{tt}\theta\,dx\,dt.\tag{4.59}$$

Step 3: In this step, we are going to estimate the term $C'\int_S^T\frac{\varphi(E(t))}{E(t)}\int_\Omega y_{tt}\theta\,dx\,dt$. Exploiting Young's, Poincaré inequalities and (3.17), we obtain

$$\begin{aligned}C'\Big|\int_S^T\frac{\varphi(E(t))}{E(t)}\int_\Omega y_{tt}\theta\,dx\,dt\Big| &\le \frac{1}{2}\int_S^T\frac{\varphi(E(t))}{E(t)}\int_\Omega|y_{tt}|^2\,dx\,dt+\frac{C'^2}{2}\int_S^T\frac{\varphi(E(t))}{E(t)}\int_\Omega|\theta|^2\,dx\,dt\\ &\le \frac{c_s}{2}\int_S^T\frac{\varphi(E(t))}{E(t)}\int_\Omega|\nabla y_{tt}|^2\,dx\,dt+C\int_S^T\frac{\varphi(E(t))}{E(t)}\int_\Omega|\nabla\theta|^2\,dx\,dt\\ &\le \frac{\gamma}{2}\int_S^T\frac{\varphi(E(t))}{E(t)}\int_\Omega|\nabla y_{tt}|^2\,dx\,dt+C\int_S^T\frac{\varphi(E(t))}{E(t)}(-E'(t))\,dx\,dt.\end{aligned}\tag{4.60}$$

Step 4: In this step, estimate for $\frac{\gamma}{2}\int_S^T\frac{\varphi(E(t))}{E(t)}\int_\Omega|\nabla y_{tt}|^2\,dx\,dt$. We multiplying the first Eq of (4.1) by $\frac{\varphi(E)}{E}y_{tt}$ integrating over $\Omega\times(S,T)$, we obtain

$$\int_S^T\frac{\varphi(E(t))}{E(t)}\int_\Omega y_{tt}(y_{tt}-\phi(\|\nabla y(t)\|^2)\Delta y-\gamma\Delta y_{tt}+\alpha\Delta\theta)\,dx\,dt=0,$$

applying Green's formula, we derive

$$\begin{aligned}\gamma\int_S^T\frac{\varphi(E(t))}{E(t)}\int_\Omega|\nabla y_{tt}|^2\,dx\,dt &= -\int_S^T\frac{\varphi(E(t))}{E(t)}\int_\Omega|y_{tt}|^2\,dx\,dt-\int_S^T\frac{\varphi(E(t))}{E(t)}\phi(\|\nabla y(t)\|^2\int_\Omega\nabla y\nabla y_{tt}\,dx\,dt\\ &\quad+\alpha\int_S^T\frac{\varphi(E(t))}{E(t)}\int_\Omega\nabla\theta\nabla y_{tt}\,dx\,dt.\end{aligned}\tag{4.61}$$

The application of Young inequality shows

$$\int_S^T \frac{\varphi(E(t))}{E(t)} \phi(\|\nabla y(t)\|^2 \int_\Omega \nabla y \nabla y_{tt}\, dx\, dt \le \frac{\gamma}{4} \int_S^T \frac{\varphi(E(t))}{E(t)} \int_\Omega |\nabla y_{tt}|^2\, dx\, dt + \frac{1}{\gamma} \int_S^T \frac{\varphi(E(t))}{E(t)} \Big(\phi(\|\nabla y(t)\|^2)\Big)^2 \int_\Omega |\nabla y|^2\, dx\, dt. \tag{4.62}$$

Similarly, we have

$$\begin{aligned} \alpha \int_S^T \frac{\varphi(E(t))}{E(t)} \int_\Omega \nabla\theta \nabla y_{tt}\, dx\, dt &\le \frac{\gamma}{4} \int_S^T \frac{\varphi(E(t))}{E(t)} \int_\Omega |\nabla y_{tt}|^2\, dx\, dt + \frac{\alpha^2}{\gamma} \int_S^T \frac{\varphi(E(t))}{E(t)} \int_\Omega |\nabla \theta|^2\, dx\, dt \\ &\le \frac{\gamma}{4} \int_S^T \frac{\varphi(E(t))}{E(t)} \int_\Omega |\nabla y_{tt}|^2\, dx\, dt + C \int_S^T \frac{\varphi(E(t))}{E(t)} (-E)'\, dt \end{aligned} \tag{4.63}$$

The use of (4.62) and (4.63) in (4.61), gives

$$\frac{\gamma}{2} \int_S^T \frac{\varphi(E(t))}{E(t)} \int_\Omega |\nabla y_{tt}|^2\, dx\, dt \le \frac{1}{\gamma} \int_S^T \frac{\varphi(E(t))}{E(t)} \Big(\phi(\|\nabla y(t)\|^2)\Big)^2 \int_\Omega |\nabla y|^2\, dx\, dt + C \int_S^T \frac{\varphi(E(t))}{E(t)} (-E)'\, dt. \tag{4.64}$$

By replacing (4.64) in (4.60), we have

$$C' \int_S^T \frac{\varphi(E(t))}{E(t)} \int_\Omega \theta y_{tt}\, dx\, dt \le \frac{1}{\gamma} \int_S^T \frac{\varphi(E(t))}{E(t)} \Big(\phi(\|\nabla y(t)\|^2)\Big)^2 \int_\Omega |\nabla y|^2\, dx\, dt + C\varphi(E(S)). \tag{4.65}$$

To complete the proof of Theorem 4.2.2. reporting (4.59), (4.65) in (4.54), we get

$$\begin{aligned} &2 \int_S^T \varphi(E) + \int_S^T \frac{\varphi(E)}{E} \phi(\|\nabla y(t)\|^2) \Big(1 - \frac{1}{\beta} \phi(\|\nabla y(t)\|^2)\Big) \int_\Omega |\nabla y(t)|^2\, dx\, dt \\ &\le C_1 \varphi(E(S)) + C(\varepsilon) \frac{\varphi(E(S))}{E(S)} \widetilde{\phi}^{-1}(E(S)) + \varepsilon \int_S^T \frac{\varphi(E)}{E} \widetilde{\phi}^{-1}(E)\, dt. \end{aligned} \tag{4.66}$$

Using the fact that $\widetilde{\phi}^{-1}(s) \le cs$ and choosing $\frac{1}{\beta}$ and ε small enough, we deduce from (4.66)

$$\int_S^T \varphi(E(t))\, dt \le C_2 \varphi(E(S)).$$

Hence, the claimed uniform exponential decay estimate, thanks to Lemma 4.4.1 with $\psi(s) = \varphi(s) = s$.

Conclusion and prospects

After we prove the existence and uniqueness of the solution it crosses our minds the most important question which is asymptotic behavior. That means: Does it exist for all time? And what is its behavior in big time (exponential decay, polynomial decay, logarithmic decay, ... etc)? We have studied in the last part of this thesis the coupled system consisting of the Kirchhoff equation and the heat equation ,we establish the well-posedness result by using the Faedo-Galerkin scheme. In the future we can show the solution's existence by the the nonlinear semigroup theory. .

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