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# Stabilisation et existence globale de quelques problèmes d'évolution avec un terme de dissipation

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# Dedication

I dedicate this modest work to my parents, my husband, my brothers, my sisters and my daughters.

*Aknowledgements*

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## ***Abstract***

The purpose of this graduation thesis is to establish the stabilization and the global existence of some evolution problems with a dissipation term. Being consisted of four chapters, this work is devoted to the study of the existence, stability and blow-up of some evolution equations with nonlinear dissipative terms, viscoelastic term, delay term and logarithmic nonlinear source terms. In the first chapter is introductory where we recall some notions related to the theory of stochastic partial differential equations such as Brownian motion and Ito formula. In the second chapter, we study a non-degenerate Kirchhoff equation with general nonlinear dissipation term and time varying delay term. This chapter deals with uniform stability by using Lyapunov functional. The third chapter we consider the initial value problem for a nonlinear equation in a bounded domain with dispersion, nonlinear damping and logarithmic source terms. Under some suitable conditions on the given parameters and by using Faedo-Galarkin method, we study the existence of solutions and we show the blow up of solutions when the energy is initially negative. In chapter four states some theorems on the existence of the solution for a stochastic hyperbolic equation and eventually the explosion of the solution. We consider an initial boundary value problem of stochastic viscoelastic wave equation with nonlinear damping and logarithmic nonlinear source terms. We proved a blow-up result for the solution with decreasing kernel.

**Key words and phrases:** Blow-up, Delay, Logarithmic source, Stability, Stochastic wave equation, Viscoelastic term.

## ***Publications***

1. A. Benramdane, N. Mezouar, M. S. Alqawba, S. M. Boulaaras and B. B. Cherif. Blow-Up for a Stochastic Viscoelastic Lamé Equation with Logarithmic Nonlinearity. *Journal of Function Spaces*, vol. **2021**, 2021, Article ID 9943969, 10 pages. https://doi.org/10.1155/2021/9943969.

## ***Presentations at conferences***

1. Estimateur de la distance minimale pour un processus AR(1), Tlemcen, Algeria, 9$^{ème}$ édition du colloque TAMTAM, (23-27 02 2019).

2. Global Existence and Energy Decay of Solutions to a Viscoelastic non-Degenerate Kirchhoff Equation with a time Varying Delay, Advances in Applied Mathematics , Association Tunisienne de Mathématiques Appliquées Industrielles, 15-19 12 2019.

# Contents

# Introduction

Since their appearance in the mid-17th century, differential equations have played a fundamental role in pure and applied mathematics. Even if their history has been well studied, they stand as a basic field of continuous researches. This is due to the emergence of new connections with other parts of mathematics and their rich interrelation with applied disciplines, as well as the interesting redevelopments of basic problems and theories in several periods and the emergence of new perspectives in the 20th century.

Our understanding of real-world phenomena and our technology today are largely based on partial differential equations (PDE). They were probably formulated for the first time during the birth of rational mechanics during 17th century (Leibniz, Newton...). Then the "catalog" of EDP grew as the sciences and in particular physics developed. In 1747, D'Alembert proposed the following wave equation:

$$u_{tt} - c^2 \Delta u = 0$$

To model the transverse vibration of a string fixed in its ends. Notice that in this model several restrictions was imposed on the physical problem, that is why Kirchhoff model **[40]** was proposed for the same problem with more considerations

$$u_{tt}-\left(\frac{P_0}{\rho h}+\frac{F}{2L\rho}\int_0^L\left|\frac{\partial u}{\partial x}(x,t)\right|^2\,dx\right)\Delta u=0,$$

Where the function $u(x,t)$ is the vertical displacement at the space coordinate $x$, varying in the segment $[0,L]$ and over time $t>0$, $\rho$ is the mass density, $h$ is the area of the cross section of the string, $P_0$ is the initial tension on the string, $L$ is the length of the string and $F$ is the Young modulus of the material.

Gabriel Lamé (1795-1870 ), the French mathematician made essential contributions to the theory of partial differential equations by the use of curvilinear coordinates, and to the mathematical theory of elasticity. The function that takes his name has a great interest which arises from the fact that we meet it in quite a number of applications, in particular in problems of potential, a point which brings them closer to the equations of hyper geometric type.

The waves take less visible forms, such as musical sounds, light and earthquakes (these are mechanical waves propagating in the ground). Whether electromagnetic or other kinds, waves take an intrinsic place in modern technology, such as television, radars, mobile telephones and microwave ovens, etc... So it is very important to model real wave phenomena. A better modeling consists in designing the delay systems in which differential equations intervene, which depend not only on the current value of their state variables at the present time $t$, but also part of their previous values. In addition to its interest in the application, the delay has a great influence in the stability of the system. Arbitrarily small delay in dissipation can destabilize a system and on the contrary it can also improve system performance, for example **[55]** and **[26]**). The first description of Brownian motion is due to the botanist Robert Brown In 1827 when observing the path of a pollen grain suspended in a liquid and subjected to successive molecular shocks. In 1905, the German Albert Einstein (in his Special Theory of Relativity) constructed a probabilistic model to describe the motion of a scattering particle. He shows in particular that the law of the position at time t of the particle, knowing that the initial state is x, admits a density which verifies the heat equation and therefore is Gaussian. The same year as Einstein, the Polish physicist Marian von Smoluchowki uses random walks to describe the Brownian motion of which it is the limits. In 1923, the American Norbert Wiener gave a first rigorous mathematical construction of Brownian motion as a stochastic process. In particular, he established the continuity of its trajectories. A mathematical problem that is important throughout science is to understand the influence of noise on differential equations, and on the long time behavior of the solutions. This problem was solved for ordinary differential equations by Itô in the 1940s. For partial differential equations, a comprehensive theory has proved to be more elusive, and only particular linear equations cases, had been treated satisfactorily. However, nonlinear PDEs are among the most difficult mathematical objects to understand. Hairer's work has caused a great deal of excitement because it develops a general theory that can be applied to a large class of nonlinear stochastic PDEs for exemple, Hairer [33] solved the KPZ equation, which is named for Mehran Kardar, Giorgio Parisi, and Yi-Cheng Zhang, the physicists who proposed the equation in 1986 for the motion of growing interfaces. Also in [32], Hairer and Mattingly , along with new methods , are establish the ergodicity of the two-dimensional stochastic Navier-Stokes equation. Building on the rough-path approach of Lyons for stochastic ordinary differential equations [34], The new theory allowed Hairer and Pillai to construct systematically solutions to singular non-linear SPDEs as fixed points of a renormalization procedure. Finally, Hairer was thus able to give, for the first time, a rigorous intrinsic meaning to many SPDEs arising in physics.

In recent years, the study of stochastic partial differential equations has become an area of research activity in fundamental and applied mathematics. The stochastic wave equation is one of the crucial stochastic partial differential equations (SPDEs) of hyperbolic type. The behavior of its solutions is significantly different from those of solutions to other SPDEs, such as the stochastic heat equation. Recently stochastic partial differential equations in a separable Hilbert space have been studied by many authors, and various results on the existence, uniqueness, invariant measures, stability, and other quantitative properties of solutions have been established.

The four major chapters of this PhD Thesis are:

- The **chapter 1**

  We recall some preliminaries used in the proofs of our results, we will recall eventually the notations used, as well as essential notions and fundamental results that concern the spaces $L^p$, the spaces of Sobolev, some inequalities, convex functions etc... . We shall also collect basic definitions and preliminary results on stochastic process such as Wiener process, stochastic integrals and Itô's formula in Hilbert space.

- The **chapter 2**
This chapter deals with the study of the global existence and decay properties of solutions for the initial boundary value problem of viscoelastic non-degenerate Kirchhoff equation of the form

$$\begin{cases} u_{tt} - M(\|\nabla u\|^2)\Delta u - \Delta u_{tt} + \int_0^t h(t-s)\Delta u(s)\,ds & \\ +\mu g(u_t(x,t-\tau(t))) = 0 & \text{in } \mathcal{D}\times]0,+\infty[, \\ u(x,t) = 0 & \text{on } \partial\mathcal{D}\times[0,+\infty[, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \mathcal{D}, \\ u_t(x,t-\tau(0)) = f_0(x,t-\tau(0)) & \text{in } \mathcal{D}\times]0,\tau(0)[, \end{cases} \tag{1}$$

where $\mathcal{D}$ is a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}^*$, with a smooth boundary $\partial\mathcal{D}$, $\mu$ is non-negative real number, $h$ is a non-negative function which decays exponentially, $\tau(t) > 0$ is a time varying delay, $g$ is function, and the initial data $(u_0, u_1, f_0)$ are in a suitable function space. $M(r) = a + br^\gamma$ is a $C^1$-function for $r \geq 0$, with $a, b > 0$, and $\gamma \geq 1$.
We considered the existence of global solutions in suitable Sobolev spaces under condition on the weight of the delay term in the feedback and the weight of the term without delay and the speed of delay. We study a general stability estimates by using some properties of convex functions.

- The **chapter 3**
In this chapter, We study the local existence for the following initial

boundary value problem:

$$
\begin{cases}
|u_t|^l u_{tt} - div(\rho(|\nabla u|^2)\nabla u) - \Delta u_{tt} + \mu_1 |u_t|^{m-1} u_t & \\
+\mu_2 |u_t(t-\tau)|^{m-1} u_t(t-\tau) = u|u|^{p-2}\ln|u|^k, & \text{in } \mathcal{D}\times(0,T),\\
u(x,t) = 0 & \text{on } \partial\mathcal{D}\times(0,T),\\
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \mathcal{D},\\
u_t(x,t-\tau) = f_0(x,t-\tau) & \text{in}\mathcal{D}\times]0,\tau[
\end{cases}
\tag{2}
$$

where $\mathcal{D}$ is a bounded domain with a sufficiently smooth boundary in $\mathbb{R}^n$, $(n \geq 1)$, $l$, $m$, $\mu_1$, $\mu_2$, $p$, $k$ are positive constants, $\tau > 0$ is a time delay and $(u_0, u_1, f_0)$ are the initial data in a suitable function space. We study blow up of solutions for a negative initial energy.

- The **chapter 4**
  This late chapter, is devoted to consider the existence and an initial boundary value problem of stochastic viscoelastic wave equation with nonlinear damping and logarithmic nonlinear source terms of the form

$$
\begin{cases}
u_{tt} - \mu\Delta u - (\lambda+\mu)\nabla(divu) + \int_0^t h(t-s)\Delta u(s)\,ds & \\
+|u_t|^{q-2}u_t = u|u|^{p-2}ln|u|^k + \epsilon\sigma(x,t)W_t(x,t) & \text{in } \mathcal{D}\times]0,+\infty[,\\
u(x,t) = 0 & \text{on } \partial\mathcal{D}\times[0,+\infty[,\\
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \overline{\mathcal{D}},
\end{cases}
\tag{3}
$$

where $\mathcal{D}$ is a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}^*$, with a smooth boundary $\partial\mathcal{D}$, $\mu$, $\lambda$ are the Lamé constants which satisfy $\mu > 0$, $\lambda+\mu \geq 0$, $h$ is a positive function, $p > q \geq 2$, and $H = L^2(\mathcal{D})$, the set of square integrable function on $\mathcal{D}$ equipped with the inner product $\langle .,. \rangle$, and its norm $\|.\|$.
$W(x,t)$ is an infinite dimensional Wiener process, $\sigma(x,t)$ is $L^2(\mathcal{D})$ valued progressively measurable and $\epsilon$ is positive constant which measures the strength of noise. We proved a blow up result for the solution with decreasing kernel.

# Chapter 1

# Preliminaries

In this chapter, we will introduce notations used in the whole work and recall some essential notions and fundamental results on Lebesgue spaces, Sobolev spaces, some important inequalities and also results on convex functions (see **[14, 28, 35, 39, 42, 43]**). Some necessary stochastic results are exposed such as Brownien notion and Ito formula (see **[68, 56]**).

## 1.1 Notations

- $\mathcal{D}$ : Open of $\mathbb{R}^n$, generic point $x = (x_1, x_2, ..., x_n)$.
- $\partial\mathcal{D}$ : Bord of $\mathcal{D}$.
- $\nabla u = (\frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n})$ : gradiant of $u$ for $n \geq 1$.
- $\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$ : Laplacian of $u$ for $n \geq 1$.
- $div\, u = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}$ : Divergence of $u$ for $n \geq 1$.
- $L^p(\mathcal{D}) = \left\{ f : \mathcal{D} \to \mathbb{R} \text{ is measurable and } \int_{\mathcal{D}} |f(x)|^p dx < +\infty \right\}$.
- $L^p(0, T, X) = \left\{ f : ]0, T[ \to X \text{ such that } \int_0^T \|f(x)\|_X^p dx < +\infty \right\}$.
- $L^\infty(\mathcal{D}) = \{ f : \mathcal{D} \to \mathbb{R} \text{ is measurable and } \exists\, C > 0, \text{ such that } |f(x)| \leq C \text{ a.s. sur } \mathcal{D} \}$.
- $L^\infty(0, T, X) = \left\{ f : (0, T) \to X \text{ is measurable and } \|f(x)\|_{L^p(0,T,X)} < \infty \right\}$.

- $W^{1,p}(\mathcal{D}) = \{f \in L^p(\mathcal{D}),\ \text{such that}\ \ \partial_i f \in L^p(\mathcal{D}), 1 \leq i \leq n\}.$
- $W^{m,p}(\mathcal{D}) = \{f \in L^p(\mathcal{D}),\ \text{such that}\ \ D^\alpha f \in L^p(\mathcal{D}), \forall \alpha, |\alpha| \leq m\}.$
- $D^\alpha f = \frac{\partial^{\alpha_1+\alpha_2+\ldots+\alpha_n}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}} f$
- $\|f\|_p = (\int_{\mathcal{D}} |f(x)|^p dx)^{\frac{1}{p}}$ : La norm of $f$ in $L^p(\mathcal{D})$.
- $\|f\|_\infty = \sup ess_{x \in \overline{\mathcal{D}}} |f(x)| = \inf\{C > 0; |f| \leq C\ a.s.\ \text{on}\ \mathcal{D}\}$ : the norm of $f$ in $L^\infty(\mathcal{D})$.
- $\langle f, g \rangle = \int_{\mathcal{D}} fg dx$ : the inner product in $L^2(\mathcal{D})$.

- $C_c^1(\mathcal{D})$ : the set of $C^1$ functions of compact support on $\mathcal{D}$.
- $C_c^\infty(\mathcal{D})$ : the set of $C^\infty$ functions of compact support on $\mathcal{D}$.
- $(\Omega, \mathcal{F}, P)$ Probability space: $\Omega$ is the set of possible outcomes of the experience ,

$\mathcal{F}$ $\sigma$-algebra of subsets of $\Omega$,

$P$ probability measure on $\mathcal{F}$.

- $\mathbb{E}(.)$ stands for expectation with respect to probability measure $P$.

- $\mathcal{B}_E$ is $\sigma-$algebra of Borel sets on the metric space $E$.
- $\mathbf{1}_A$ indicator of A :

$$\mathbf{1}_A(\omega) := \begin{cases} 1 & si\ \omega \in A \\ 0 & si\ non \end{cases}$$

## 1.2 Generalities on Functional Spaces

We begin by reviewing some basic facts from calculus in the most important class of normed spaces and linear spaces " Banach spaces". Throughout the rest of this chapter, $\mathbb{K}$ will denote $\mathbb{R}$ or $\mathbb{C}$.

**Definition 1.2.1.** *A Banach space is a complete normed linear vector space $X$.*

**Definition 1.2.2.** ***(Dual Space)***
*The set of all continuous linear functionals with domain the Banach space $X$ is called the dual space of $X$ and denoted by $X^*$ . In other words,*

$$X^* = \{T : X \to \mathbb{C} : T \textit{ is continous and linear}\}.$$

*The dual space is a Banach space even if $X$ is only a normed space.*

**Proposition 1.2.1.** *$X^*$ equipped with the norm $\|.\|_{X^*}$ defined by*

$$\|f\|_{X^*} = \sup_{\|u\|_X \le 1} |f(u)|, \tag{1.1}$$

*is also a Banach space. We shall denote the value of $f \in X^*$ at $u \in X$ by either $f(u)$ or $\langle f, u\rangle_{X^*,X}$.*

**Remark 1.2.1.** *From $X^*$, we construct the bidual or second dual $X^{**} = (X^*)^*$. Furthermore, with each $u \in X$ we can define $\varphi(u) \in X^{**}$ by $\varphi(u)(f) = f(u), f \in X^*$. This satisfies clearly $\|\varphi(u)\| \le \|u\|_X$. Moreover, for each $u \in X$ there is an $f \in X^*$ with $f(u) = \|u\|$ and $\|f\| = 1$. So it follows that $\|\varphi(u)\| = \|u\|$.*

### 1.2.1 The weak star topologies

From remark 1.2.1 we obtain a family $(\varphi_f)_{f\in X^*}$ of applications from $X$ in $\mathbb{R}$ defined as follow:

$$\begin{aligned} \varphi_f : X &\longrightarrow \mathbb{R} \\ x &\longmapsto \varphi_f(x), \end{aligned}$$

where $X$ be a Banach space.

**Definition 1.2.3.** *The weak topology $\sigma(X, X^*)$ on $X$, is the weakest topology on $X$ for which every $(\varphi_f)_{f \in X^*}$ is continuous.*

We will define the third topology on $X^*$, the weak star topology, denoted by $\sigma(X^*, X)$. For all $x \in X$. Denote by

$$\varphi_f : X^* \longrightarrow \mathbb{R}$$
$$f \longmapsto \varphi_x(f) = \langle f, x \rangle_{X^*, X},$$

when $x$ cover $X$, we obtain a family $(\varphi_x)_{x \in X^*}$ of applications to $X^*$ in $\mathbb{R}$.

**Definition 1.2.4.** *The weak star topology on $X^*$ is the weakest topology on $X^*$ for which every $(\varphi_x)_{x \in X^*}$ is continuous.*

**Remark 1.2.2.** *Since $X \subset X^{**}$, it is clear that, the weak star topology $\sigma(X^*, X)$ is weakest then the topology $\sigma(X^*, X^{**})$, and this later is weakest then the strong topology.*

### 1.2.2 Modes of convergence

**Definition 1.2.5.** *A sequence $(u_n)$ in $X$ is weakly convergent to $x$ if and only if*

$$\lim_{n \to \infty} \langle f, u_n \rangle = \langle f, u \rangle,$$

*for every $f \in X^*$, we write $u_n \rightharpoonup u$*

**Definition 1.2.6.** *Let $X$ be a Banach space, and let $(u_n)_{n \in N}$ be a sequence in $X$. Then $u_n$ converges strongly to $u$ in $X$ if and only if*

$$\lim \|u_n - u\|_X = 0,$$

*and this is denoted by $u_n \to u$, or $\lim_{n \to \infty} u_n = u$.*

**Remark 1.2.3.** *1. If the weak limit exist, it is unique.*

*2. If $u_n \to u \in X$(strongly) then $u_n \rightharpoonup u$(weakly).*

*3. If $dim X < +\infty$, then the weak convergent implies the strong convergent.*

### 1.2.3 Reflexive Spaces

**Definition 1.2.7.** *Since $\varphi$ is linear we see that*

$$\varphi : X \to X^{**},$$

*is a linear isometry of $X$ into a closed subspace of $X^{**}$, we denote this by*

$$X \hookrightarrow X^{**}.$$

**Definition 1.2.8.** *If $\varphi$ is into $X^{**}$ we say $X$ is reflexive, $X \cong X^{**}$.*

**Theorem 1.2.1.** ***(Kakutani)****.*
*Let $X$ be a Banach space. Then $X$ is reflexive if and only if*

$$B \equiv \{x \in X : \|x\| \leq 1\}, \tag{1.2}$$

*is compact in the weak topology $\sigma(X, X^{**})$.*

**Proposition 1.2.2.** *If $X$ is reflexive and if $Y$ is a closed vector subspace of $X$ then $Y$ is reflexive.*

**Corollaire 1.2.1.** *A Banach space $X$ is reflexive if and only if its dual space $X^*$ is reflexive.*

**Theorem 1.2.2.** ***(Eberlein-Smulian)***
*Suppose $X$ is a Banach space such that every bounded sequence in $X$ admits a weakly convergent subsequence (in $\sigma(X, X^*)$). Then $X$ is reflexive.*

### 1.2.4 Hilbert spaces

**Definition 1.2.9.** *Let $H$ be a vector space, an inner product is a function*

$$\langle .,. \rangle : H \times H \longrightarrow \mathbb{K}$$

*such that for $u, v, w \in H$ and $\lambda, \mu \in \mathbb{K}$*
*i/ $\langle u, v\rangle = \overline{\langle v, u\rangle}$*
*ii/ $\langle \lambda u + \mu v, w\rangle = \lambda\langle u, w\rangle + \mu\langle v, w\rangle$*
*iii/ $\langle u, u\rangle \geq 0$ and $\langle u, u\rangle = 0$ if and only if $u = 0$*

**Definition 1.2.10.** *A Hilbert space $H$ is a vector space supplied with inner product $\langle u, v\rangle$ such that $\|u\| = \langle u, u\rangle^{\frac{1}{2}}$ is the norm which let $H$ complete.*

**Theorem 1.2.3.** *(Riesz). If $(H; \langle .,. \rangle)$ is a Hilbert space, $\langle .,. \rangle$ being a scalar product on $H$, then $H^* = H$ in the following sense: to each $f \in H^*$ there corresponds a unique $x \in H$ such that $f(.) = \langle x, . \rangle$ and $\|f\|_{H^*} = \|x\|_H$.*

**Remark 1.2.4.** *From this theorem we deduce that $H^{**} = H$. This means that a Hilbert space is reflexive.*

**Theorem 1.2.4.** *Let $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence in the Hilbert space $H$, it posses a subsequence which converges in the weak topology of $H$.*

**Theorem 1.2.5.** *In the Hilbert space, all sequence which converges in the weak topology is bounded.*

**Theorem 1.2.6.** *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence which converges to $u$, in the weak topology and $(\upsilon_n)_{n\in\mathbb{N}}$ is an other sequence which converge weakly to $\upsilon$, then*

$$\lim_{n\to\infty}\langle \upsilon_n, u_n\rangle = \langle \upsilon, u\rangle. \tag{1.3}$$

### 1.2.5 The spaces $L^p(\mathcal{D})$

**Definition 1.2.11.** *Let $1 \le p < \infty$ and let $\mathcal{D}$ be an open domain in $\mathbb{R}^n$, $n \in \mathbb{N}$. Define the standard Lebesgue space $L^p(\mathcal{D})$ by*

$$L^p(\mathcal{D}) = \left\{ f : \mathcal{D} \to \mathbb{R} \text{ is measurable and } \int_{\mathcal{D}} |f(x)|^p dx < \infty \right\}. \tag{1.4}$$

*with the norm*

$$\|f\|_p = \left(\int_{\mathcal{D}} |f(x)|^p dx\right)^{1/p}.$$

*If $p = \infty$,*

$$L^\infty(\mathcal{D}) = \left\{ f : \mathcal{D} \to \mathbb{R} \text{ is measurable and } \|f\|_\infty < \infty \right\}, \tag{1.5}$$

*with the norm*

$$\|f\|_\infty = \inf\left\{ C > 0;\ |f| \le C \ a.e.\ on\ \mathcal{D} \right\}. \tag{1.6}$$

- The space $L^p(\mathcal{D})$ with the norm $\|.\|_p$ is a Banach space.
- The space $L^2(\mathcal{D})$ equipped with the scalar product

$$\langle f, g\rangle_{L^2(\mathcal{D})} = \int_{\mathcal{D}} f(x)g(x)dx,$$

is an Hilbert space.

**Theorem 1.2.7.** *For $1 < p < \infty$, $L^p(\mathcal{D})$ is a reflexive space.*

### 1.2.6 The spaces $L^p(0,T;X)$

**Definition 1.2.12.** *Let $X$ be a Banach space and $p$ a real such that $1 \leq p < \infty$ and $[0,T]$ is an interval of $\mathbb{R}$ . We call a $X$ valued Lebegue space and we denote by $L^p(0,T;X)$ the space of its measurable functions*

$$L^p(0,T;X) = \left\{ f : ]0,T[ \rightarrow X \text{ such that } \int_0^T \|f(x)\|_X^p dx < +\infty \right\}$$

*equipped with the norm*

$$\|f\|_{L^p(0,T;X)} = \left( \int_0^T \|f(t)\|_X^p dt \right)^{\frac{1}{p}} < \infty$$

*If $p = \infty$,*

$$\|f\|_{L^\infty(0,T;X)} = \sup_{t \in ]0,T[} ess \|f(t)\|_X.$$

**Theorem 1.2.8.** *The space $L^p(0,T;X)$ is complete.*

**Lemma 1.2.1.** *1) The space $L^2(0,T;X)$ is a Hilbert space for the inner product*

$$(f,g)_{L^2(0,T;X)} = \int_0^T (f(t),g(t))_X \, dt.$$

*2) $L^\infty(0,T;X) = (L^1(0,T;X))^*$.*

Now, we will introduce some basic results on the $L^p(0,T;X)$ space. These results, will be very useful in the other chapters of this thesis.

**Lemma 1.2.2.** *Let $f \in L^p(0,T;X)$ and $\frac{\partial f}{\partial t} \in L^p(0,T;X)$, $(1 \leq p \leq \infty)$, then the function $f$ is continuous from $[0,T]$ to $X$. i.e. $f \in C^1(0,T,X)$.*

**Theorem 1.2.9.** *$L^p(0,T;X)$ equipped with the norm $\|.\|_{L^p(0,T,X)}$, $1 \leq p \leq \infty$ is a Banach space.*

**Proposition 1.2.3.** *Let $X$ be a reflexive Banach space, $X^*$ it's dual, and $1 \leq p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then the dual of $L^p(0,T;X)$ is identify algebraically and topologically with $L^q(0,T;X^*)$.*

**Proposition 1.2.4.** *Let $X, Y$ be Banach spaces, $X \subset Y$ with continuous embedding, then we have*

$$L^p(0,T;X) \subset L^p(0,T;Y),$$

*with continuous embedding.*

The current compactness criterion will be useful for nonlinear evolution problem, especially in the limit of the nonlinear terms.

**Lemma 1.2.3.** *(Aubin -Lions lemma) Let $B_0, B, B_1$ be Banach spaces with $B_0 \subset B \subset B_1$. Assume that the embedding $B_0 \hookrightarrow B$ is compact and $B \hookrightarrow B_1$ are continuous. Let $1 < p, q < \infty$. Assume further that $B_0$ and $B_1$ are reflexive. Define*

$$W \equiv \{u \in L^p(0,T;B_0) : u' \in L^q(0,T;B_1)\}. \tag{1.7}$$

*Then, the embedding $W \hookrightarrow L^p(0,T;B)$ is compact.*

### 1.2.7 Sobolev Spaces

Let $\mathcal{D}$ be an arbitrary open of $\mathbb{R}^n$ , and let $p$ a real such that $1 \leq p \leq +\infty$.

**The $W^{1,p}(\mathcal{D})$ Spaces**

**Definition 1.2.13.** *The Sobolev space $W^{1,p}(\mathcal{D})$ is defined by*

$$W^{1,p}(\mathcal{D}) = \left\{ u \in L^p(\mathcal{D}) | \exists g_i \in L^p(\mathcal{D}),\ \textit{tels que}\ \int_{\mathcal{D}} u \frac{\partial \varphi}{\partial x_i} = -\int_{\mathcal{D}} g_i \varphi\ \forall \varphi \in C_c^\infty(\mathcal{D})\ \forall i = 1, 2, ..., n \right\}$$

*We say that $\varphi$ is a test function.*

*We put*

$$H^1(\mathcal{D}) = W^{1,2}(\mathcal{D}).$$

*For $u \in W^{1,p}(\mathcal{D})$, we denote $g_i = \frac{\partial u}{\partial x_i}$.*

The space $W^{1,p}(\mathcal{D})$ is equipped with the norm

$$\|u\|_{W^{1,p}} = \|u\|_p + \|\nabla u\|_p.$$

Sometimes, if $1 < p < \infty$, the norm is equivalent to

$$[\|u\|_p^p + \|\nabla u\|_p^p]^{\frac{1}{p}}.$$

The space $H^1(\mathcal{D})$ is equipped with the inner product.

$$\langle u, v \rangle_{H^1(\mathcal{D})} = \langle u, v \rangle_{L^2(\mathcal{D})} + \langle \nabla u, \nabla v \rangle_{L^2(\mathcal{D})}.$$

Its associated norm is given by

$$\|u\|_{H^1(\mathcal{D})} = [\|u\|_2^2 + \|\nabla u\|_2^2]^{\frac{1}{2}}$$

and it is equivalent to the norm of $W^{1,2}(\mathcal{D})$.

- The space $H^1(\mathcal{D})$ is an Hilbert space.
- The space $W^{1,p}(\mathcal{D})$ is a Banach space.

**Theorem 1.2.10.** *(Sobolev, Gagliardo, Nirenberg). Let* $1 \leq p < n$*, then*

$$W^{1,p}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n)$$

*where* $p^*$ *is given by* $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$*, and there exists a constant* $C = C(p,n)$ *such that*

$$\|u\|_{p^*} \leq C\|\nabla u\|_p, \quad \forall u \in W^{1,p}(\mathbb{R}^n).$$

**Corollaire 1.2.2.** *Let* $1 \leq p < n$*, then*

$$W^{1,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n), \quad \forall q \in [p, p^*]$$

*with continuous injection.*

For the limiting case $p = n$, we have

$$W^{1,n}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n), \quad \forall q \in [n, +\infty[$$

**Theorem 1.2.11.** *Let* $p > n$*. Then*

$$W^{1,p}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$$

*with continuous injection.*

**Corollaire 1.2.3.** *Let* $\mathcal{D}$ *a bounded domain in* $\mathbb{R}^n$ *of class* $C^1$ *with* $\Gamma = \partial\mathcal{D}$ *and* $1 \leq p \leq \infty$*. We have*

$$\begin{array}{ll} W^{1,p}(\mathcal{D}) \subset L^{p^*}(\mathcal{D}), \text{ where } \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n} & \text{if } p < n, \\ W^{1,p}(\mathcal{D}) \subset L^q(\mathcal{D}), \forall q \in [p, +\infty[, & \text{if } p = n, \\ W^{1,p}(\mathcal{D}) \subset L^\infty(\mathcal{D}), & \text{if } p > n, \end{array}$$

*and all these injections are continuous. Moreover, if* $p > n$ *we have:*

$$\forall u \in W^{1,p}(\mathcal{D}), \quad |u(x) - u(y)| \leq C|x-y|^\alpha \|u\|_{W^{1,p}(\mathcal{D})} \ a.e\ x, y \in \mathcal{D}$$

*with* $\alpha = 1 - \frac{n}{p} > 0$ *and* $C$ *is a constant which depend on* $p, n$ *and* $\mathcal{D}$*. In particular* $W^{1,p}(\mathcal{D}) \subset C(\overline{\mathcal{D}})$*.*

**Theorem 1.2.12.** *(Rellich-Kondrachov). Suppose that $\mathcal{D}$ is bounded and of class $C^1$. Then we have the following compact injections:*

$$\begin{array}{ll} W^{1,p}(\mathcal{D}) \subset L^q(\mathcal{D}) \forall q \in [1, p^*[ \text{ where } \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}, & \text{if } p < n, \\ W^{1,p}(\mathcal{D}) \subset L^q(\mathcal{D}), \forall q \in [p, +\infty[, & \text{if } p = n, \\ W^{1,p}(\mathcal{D}) \subset C(\overline{\mathcal{D}}), & \text{if } p > n, \end{array}$$

*In particular,*

$$W^{1,p}(\mathcal{D}) \subset L^q(\mathcal{D})$$

*with compact injection for all p (and all n).*

**Proposition 1.2.5.** *( Green's formula). For all $u \in H^2(\mathcal{D})$, $\upsilon \in H^1(\mathcal{D})$ we have*

$$-\int_{\mathcal{D}} \Delta u \upsilon dx = \int_{\mathcal{D}} \nabla u \nabla \upsilon dx - \int_{\partial\mathcal{D}} \frac{\partial u}{\partial \eta} \upsilon d\sigma,$$

*where $\frac{\partial u}{\partial \eta}$ is a normal derivation of u at $\Gamma = \partial\mathcal{D}$.*

**The Spaces** $W^{m,p}(\mathcal{D})$

**Definition 1.2.14.** *Let $m \geq 2$ be an integer and be p be a real number, such that $1 \leq p \leq +\infty$. We define*

$$W^{m,p}(\mathcal{D}) = \{u \in W^{m-1,p}(\mathcal{D}); \frac{\partial u}{\partial x_i} \in W^{m-1,p}(\mathcal{D}), \forall i = 1, ..., n\}.$$

*In the same*

$$W^{m,p}(\mathcal{D}) = \{f \in L^p(\mathcal{D}), \text{ such that } D^\alpha u \in L^p(\mathcal{D}), \forall \alpha, |\alpha| \leq m\}.$$

*where*

$$\alpha \in \mathbb{N}^n, \ |\alpha| = \sum_{i=1}^{n} \alpha_j \leq m$$

*and $D^\alpha u$ is obtained by $\int_{\mathcal{D}} u D^\alpha \varphi = (-1)^{|\alpha|} \int_{\mathcal{D}} (D^\alpha u) \varphi, \ \forall \varphi \in C_c^\infty(\mathcal{D})$, with $D^\alpha \varphi = \frac{\partial^{\alpha_1+\alpha_2+...+\alpha_n}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_n^{\alpha_n}} \varphi$.*

**Properties 1.2.1.** *The space $W^{m,p}(\mathcal{D})$ is a Banach space equipped with the norm*

$$\| \ u \ \|_{W^{m,p}(\mathcal{D})} = (\sum_{|\alpha|\leq m} \|D^{\alpha}u\|_p^p)^{\frac{1}{p}} \ , \ for \ p < +\infty$$

*and*

$$\|u\|_{W^{k,\infty}(\mathcal{D})} = \sum_{|\alpha|\leq m} \|D^{\alpha}u\|_{\infty} \ , \ for \ p = +\infty$$

*Moreover is a reflexive space for $1 < p < \infty$ and a separable space for $1 \leq p < \infty$.*
*The space $W^{m,2}(\mathcal{D}) = H^m(\mathcal{D})$ equipped with the scalar product*

$$\langle f, g\rangle_{H^m(\mathcal{D})} = \sum_{|\alpha|\leq m} \langle D^{\alpha} f, D^{\alpha} g\rangle_{L^2(\mathcal{D})}$$

*is an Hilbert space.*

**Spaces** $W^{m,p}(0,T;X)$

We define the Sobolev spaces with values in a Hilbert space $X$ by :

$$W^{m,p}(0,T;X) = \left\{ v \in L^p(0,T;X); \frac{\partial v}{\partial x_i} \in L^p(0,T;X). \ \ \forall i \leq m \right\} \ ,$$

For $m \in \mathbb{N}$, $p \in [1, \infty]$.
The space $W^{m,p}(0,T;X)$ is a Banach space with the norm

$$\|f\|_{W^{m,p}(0,T;X)} = \left( \sum_{i=0}^{m} \|\frac{\partial f}{\partial x_i}\|_{L^p(0,T;X)}^p \right)^{1/p} , \ \ \text{for} \ \ p < +\infty$$

and

$$\|f\|_{W^{m,\infty}(0,T;X)} = \sum_{i=0}^{m} \|\frac{\partial f}{\partial x_i}\|_{L^\infty(0,T;X)} \ , \quad \text{for } p = +\infty$$

$W^{m,2}(0,T;X)$ is an Hilbert space denoted $H^m(0,T;X)$ equipped with inner product

$$\langle u, v\rangle_{H^m(0,T;X)} = \sum_{i=0}^{m} \int_0^T \left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i}\right)_X dt \ .$$

### 1.2.8 Convex Function

**Definition 1.2.15.** *Let $f : I \to \mathbb{R}$. $f$ is said to be **convex** on $I$, for each $a, b \in I$, such that $a < b$, and for all $\alpha, \beta \in \mathbb{R}$ with $\alpha + \beta = 1$*

$$f(\alpha a + \beta b) \leq \alpha f(a) + \beta f(b)$$

*we also have the following similar definition:*
*$f$ is **convex** on $I$, if for alls $a, b \in I$, with $a < b$ and for all $t \in [0, 1]$:*

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b)$$

### 1.2.9 Differentiable Convex Fonctions

**Theorem 1.2.13.** *A differentiable function $f$ is convex if and only if its derivative is an increasing .*

**Corollaire 1.2.4.** *Let $f : I \to \mathbb{R}$ to be twice differentiable, then $f$ is convex if and only if its second derivative $f''$ is positive.*

### 1.2.10 Jensen inequality

**Theorem 1.2.14.** *( Jensen inequality (Discrete form)):*
*Let $f : I \to \mathbb{R}$ to be convex, then, for all $a_1, a_2, ..., a_n \in I$ and for any non negative reals $\lambda_1, \lambda_2, ..., \lambda_n$ such that $\sum_{i=1}^{n} \lambda_i = 1$, we have:*

$$f\Big(\sum_{i=1}^{n} \lambda_i a_i\Big) \leq \sum_{i=1}^{n} \lambda_i f(a_i), \quad \forall n \in \mathbb{N}^*.$$

**Theorem 1.2.15.** *( Jensen inequality (Continuous form)):*
*Let $f : I \to \mathbb{R}$ to be convex, then, for all $g$ integrable on $\mathcal{D}$ we have:*

$$f\left(\int_{\mathcal{D}} g \; dx\right) \leq \int_{\mathcal{D}} (f \circ g) \; dx.$$

### 1.2.11 Some inequalities

This paragraph will deal with some important integral inequalities in applied mathematics which are very useful in our next chapters.

**Theorem 1.2.16.** *( Hölder's inequality) Assume that $f \in L^p(\mathcal{D})$ and $g \in L^q(\mathcal{D})$ with $1 \leq p \leq \infty$ and $q$ the conjugate exponent, $\frac{1}{p} + \frac{1}{q} = 1$. Then $fg \in L^1(\mathcal{D})$ and*

$$\int_{\mathcal{D}} |fg| dx \leq \|f\|_p \|g\|_q.$$

### 1.2.12 Young Inequality

**Theorem 1.2.17.** *For all non negative reals $a$ and $b$ and for all positive reals $p$ et $q$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and $1 \leq p$, we have:*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

**Remark 1.2.5.** *1)A simple case of Young is for* $p = q = 2$

$$ab \leq \frac{a^2}{2} + \frac{b^2}{2}$$

*which identically gives Young inegality for all* $\delta > 0$*:*

$$ab \leq \delta a^2 + \frac{1}{4\delta} b^2.$$

*2) Young inegality can be written also under the form:*

$$ab \leq \delta a^p + c_\delta b^q, \quad \textit{où } c_\delta = \delta^{-\frac{1}{p}}.$$

### 1.2.13 Generalised Young inequality

**Theorem 1.2.18.** ***(First form of the generalised Young inegality )****:*
*Let* $c > 0$ *and* $f : [0, c] \to \mathbb{R}$ *be a strictly increasing continuous function such that* $f(0) = 0$*, and let* $a \in [0, c]$ *and* $b \in [0, f(c)]$*, then*

$$ab \leq \int_0^a f(x)dx + \int_0^b f^{-1}(x)dx. \tag{1.8}$$

<u>**Graphic Interpretation**</u>

**Theorem 1.2.19.** ***(Generalised Young inequality :second form)****:*
*Let* $c > 0$ *and* $\Phi : [0, c] \to \mathbb{R}$ *a strictly convex function of class* $C^1$ *such that* $\Phi(0) = \Phi'(0) = 0$*, for all* $a \in [0, c]$ *and* $b \in [0, \Phi'(c)]$ *we have*

$$ab \leq \Phi(a) + \Phi^*(b)$$

*where*

$$\Phi^*(x) := \int_0^x (\Phi')^{-1}(y)dy \quad \textit{pour } x \in \Phi([0, c])$$

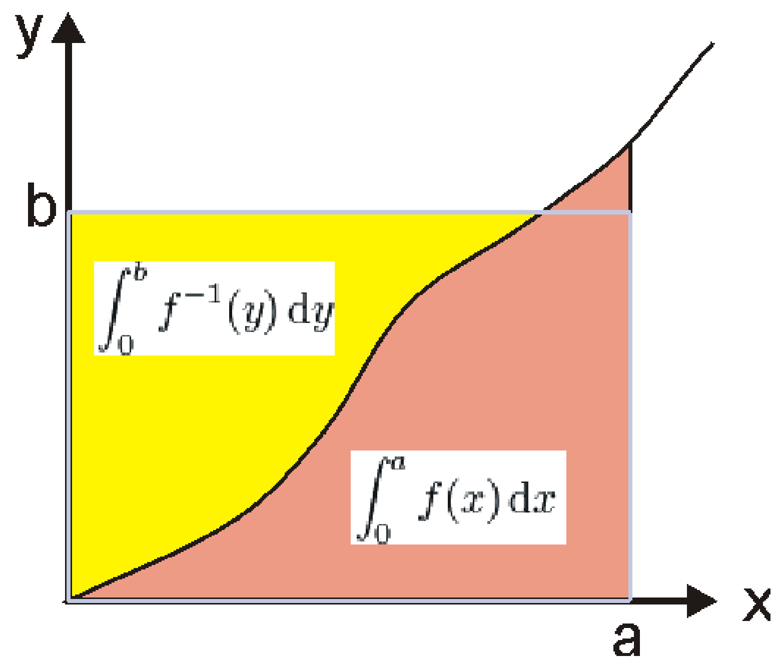


Figure 1.1: **Inégalité de Young**

## 1.3 Existence Methods

### 1.3.1 Faedo-Galerkin's approximations

The Galerkin method is a very general and robust to proof the existence of solution to some evolution problem. The idea of this method is as follows:
Starting from a posed problem in an infinite dimension space, we proceed first by an approximation on an increasing sequence of finite dimensional subspaces. Then we solve the approximated problem, which is in generally easier than to solve directly in infinite dimension. Finally and in order to construct a solution of the starting problem, we pass to the limit when making the approximation spaces dimension tend to infinity in a way or another. It should be noticed that, in addition to its theoretical interest, the Galerkin method provides also a constructive approximation process. We intend to apply this method to following Cauchy problem of a second order evolution equation in the separable Hilbert space with the inner

product $\langle .,. \rangle$ and the associated norm $\|.\|$ .

$$\begin{cases} u''(t) + A(t)u(t) = f(t) & \text{t} \in [0,T], \\ u(x,0) = u_0(x),\ u'(x,0) = u_1(x), \end{cases} \tag{1.9}$$

where $u$ and $f$ are unknown and given function, respectively, mapping the closed interval $[0,T] \subset \mathbb{R}$ into a real separable Hilbert space $H$. $A(t)$ $(0 \leq t \leq T\ )$ are linear bounded operators in $H$ acting in the energy space $V \subset H$.

Assume that $\langle A(t)u(t), v(t) \rangle = a(t; u(t), v(t))$, for all $u, v \in V$; where $a(t; ., .)$ is a bilinear continuous in $V$. The problem (1.9) can be formulated as: Found the solution $u(t)$ such that

$$\begin{cases} u \in C([0,T]; V), u' \in C([0,T]; H) \\ \langle u''(t), v \rangle + a(t; u(t), v) = \langle f, v \rangle & \text{t} \in D'(]0,T[), \\ u_0 \in V\ ,\ u_1 \in H, \end{cases} \tag{1.10}$$

The problem (1.9) can be resolved with the approximation process of Fadeo-Galerkin.

Now, we consider $V_m$ a sub-space of $V$ with the finite dimension $d_m$, and let $\{w_{jm}\}$ one basis of $V_m$ such that .

1. $V_m \subset V, \forall m \in \mathbb{N}$
2. $V_m \to V$ such that, there exist a dense subspace $\vartheta$ in $V$ and for all $v \in \vartheta$ we can get sequence $\{u_m\}_{m \in \mathbb{N}} \in V_m$ and $u_m \to u$ in V.
3. $V_m \subset V_{m+1}$ and $\overline{\cup_{m \in \mathbb{N}} V_m} = V$.

We define the solution $u_m$ of the approximate problem

$$\begin{cases} u_m(t) = \sum_{j=1}^{d_m} g_j(t) w_{jm}, \\ u_m \in C([0,T]; V_m), u'_m \in C([0,T]; V_m),\ u_m \in L^2(0,T; V_m) \\ \langle u''_m(t), w_{jm} \rangle + a(t; u_m(t), w_{jm}) = \langle f, w_{jm} \rangle,\ 1 \leq j \leq d_m \\ u_m(0) = \sum_{j=1}^{d_m} \xi_j(t) w_{jm}, u'_m(0) = \sum_{j=1}^{d_m} \eta_j(t) w_{jm}, \end{cases} \tag{1.11}$$

where

$$\sum_{j=1}^{d_m} \xi_j(t) w_{jm} \longrightarrow u_0 \text{ in V as } m \longrightarrow \infty$$

$$\sum_{j=1}^{d_m} \eta_j(t) w_{jm} \longrightarrow u_1 \text{ in V as } m \longrightarrow \infty$$

In pursuance of the theory of ordinary differential equations, the system (1.11) has unique local solution which is extend to a maximal interval $[0, t_m[$ by Zorn lemma since the non-linear terms have the suitable regularity. In the next step, we obtain a priori estimates for the solution, so that can be extended outside $[0, t_m[$ to obtain one solution defined for all $t > 0$.

### 1.3.2 A priori estimation and convergence

Using the following estimation

$$\|u_m\|^2 + \|u'_m\|^2 \leq C \left\{ \|u_m(0)\|^2 + \|u'_m(0)\|^2 + \int_0^T \|f(s)\|^2 ds \right\};\ 0 \leq t \leq T$$

and the Gronwall lemma we deduce that the solution $u_m$ of the approximate problem (1.11) converges to the solution $u$ of the initial problem (1.9). The uniqueness proves that $u$ is the solution.

### 1.3.3 Gronwall's lemma

**Lemma 1.3.1.** *Let $c_1$, $c_2$ be two positives constants and let $g, \varphi \in L^1(0,T)$ be non negative function such that $g\varphi \in L^1(0,T)$ and*

$$\varphi(t) \leq c_1 + c_2 \int_0^t g(s)\varphi(s) ds \quad a.e \text{ in } (0,T).$$

*Then, we have*

$$\varphi(t) \leq c_1 exp \left( c_2 \int_0^t g(s)ds \right) \quad a.e \ in \ (0,T).$$

## 1.4 Stabilization

The energie $E(t)$ of The following problem:

$$|u_t|^l u_{tt} + L(\nabla u) + g(u_t) = F(u) \ in \ \mathcal{D}\times]0,+\infty[.$$

are determined as:
1. $< |u_t|^l u_{tt}, u_t >= \int_D |u_t|^l u_{tt}, u_t dx = \frac{d}{dt}\left(\frac{1}{l+2}\int_D |u_t|^{l+2} dx\right) = \frac{d}{dt}\left(\frac{1}{l+2}\|u_t\|^{l+2}\right)$
2. $< L(\nabla u), u_t >= \frac{d}{dt}\left(\int_D B(\nabla u)dx\right)$
3. $< F(u), u_t >= \frac{d}{dt}\left(\int_D f(u)dx\right)$ .
Then $E(t) = \frac{1}{l+2}\|u_t\|^{l+2} + \int_D B(\nabla u)dx - \int_D f(u)dx.$
The problem of stabilization consists to determinate the asymptotic behaviour of the energy E(t), to study its limits in order to determine if this limit is null or not and if this limit is null, to give an estimate of the decay rate of the energy to zero.

**Definition 1.4.1.** ***( stability degree)****: We distinguish three degrees of stability We say that*
*1. The Stability is Strong if*

$$\lim_{t\to+\infty} E(t) = 0.$$

*2. The Stability is Uniform if*

$$\forall\, t \geq 0,\ E(t) \leq cf(t),$$

*where c is constant depending on the norm of the initial data and f is a decreasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy*

$$\lim_{t \to +\infty} f(t) = 0.$$

*3. The Stability is weak if*

$$u(x,t) \rightharpoonup 0 \textit{ (weakly)}$$

*and*

$$u_t(x,t) \rightharpoonup 0 \textit{ (weakly) when} t \to +\infty$$

*in some Hilbert space.*

## 1.5 Generalities on Stochastic Calculus

In this section, we will collect definitions and basic results of probability theory in infinite dimensions.

### 1.5.1 Measure and probability

**Definition 1.5.1.**

*A $\sigma$-algebra (or $\sigma$-field)(in French language it is said "une tribu" or "$\sigma$-algèbre") over a non-empty set $\Omega$ is a family $\mathcal{F}$ of subsets of $\Omega$ such that*
*i) $\Omega \in \mathcal{F}$*
*ii) $A \in \mathcal{F}$ implies $A^c \in \mathcal{F}$*
*iii) if $(A_n,\, n \geq 1)$ is a countable family of elements in $\mathcal{F}$ then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.*

**Definition 1.5.2.**

*Let $\Omega$ be a metric space, the Borel $\sigma$-algebra (or Borel field) $\mathcal{B}_\Omega$ of $\Omega$ is the* *smallest $\sigma$-algebra which contains all open sets. $\mathcal{B}_{\mathbb{R}}$ is also the smallest $\sigma$-algebra which contains all open intervals.*

**Definition 1.5.3.** ***(Measure Spaces)***
*A pair $(\Omega, \mathcal{F})$, where $\Omega$ is a set and $\mathcal{F}$ is a $\sigma-$algebra of subsets of $\Omega$, is called a measurable space. A function $\mu : \mathcal{F} \to [0, \infty]$ is a measure on $(\Omega, \mathcal{F})$ provided that*
*i) $\mu(\emptyset) = 0$*
*ii) If $\{A\}_{n=1}^{\infty} \subseteq \mathcal{F}$ is disjoint, then*

$$\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$$

*in which case we say that $\mu$ is countably additive.*
*The triple $(\Omega, \mathcal{F}, \mu)$ where $(\Omega, \mathcal{F})$ is a measurable space and $\mu$ a measure on $(\Omega, \mathcal{F})$ is called a measure space.*

**Definition 1.5.4.**
*The measure $\mu$ characterized by*

$$\mu(]a, b]) = b - a; a, b \in \mathbb{R}, a < b$$

*is the Lebesgue measure over $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ .*

**Definition 1.5.5.**
*A measure $\mu$ $\sigma-$finite if there exists $(A_n) \subset \mathcal{F}$ such that $A_n \subset A_{n+l}$, $\bigcup_{n=1}^{\infty} A_n = \Omega$ and $\mu(A_n) < \infty$, $n \geq 1$.*

**Definition 1.5.6.**
*Let $(\Omega_i, \mathcal{F}_i, \mu_i), i = 1, 2$ two measure spaces with $\mu_i, i = 1, 2$ are $\sigma-$finite. the Product space measure $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \otimes \mu_2)$ is defined as: $\mathcal{F}_1 \otimes \mathcal{F}_2$*

*is the smallest $\sigma-$algebra which contains all $A_1 \times A_2$, with $A_1 \in \mathcal{F}_1$, $A_2 \in \mathcal{F}_2$ and $\mu_1 \otimes \mu_2$ is the unique measure over $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$ such that*

$$(\mu_1 \otimes \mu_2)(A_1 \times A_2) = \mu_1(A_l)\mu_2(A_2)$$

**Definition 1.5.7.**
*A probability $P$ is a measure such that $P(\Omega) = 1$. If $P$ is a Probability over $(\Omega, \mathcal{F})$, then $(\Omega, \mathcal{F}, P)$ is a Probability space,*
*where $\Omega$ is the set of possible outcomes of the experience, $\mathcal{F}$ is called the set of events and $(\Omega, \mathcal{F}, P)$ is called sample space.*

**Definition 1.5.8.** ***(Measurable Functions)***
*Let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be measurable spaces. A function*

$$f : (\Omega_1, \mathcal{F}_1) \to (\Omega_2, \mathcal{F}_2)$$

*is $(\mathcal{F}_1, \mathcal{F}_2)-$measurable if $f^{-1}(\theta) \in \mathcal{F}_1$ for every $\theta \in \mathcal{F}_2$. If $\Omega_2 = \mathbb{R}$, we said $f$ is $\mathcal{F}_1-$measurable.*

### 1.5.2 Random variables

**Definition 1.5.9.**
*A real random variable defined over the probability space $(\Omega, \mathcal{F}, P)$ is a mapping*
*$X : \Omega \to \mathbb{R}$ such that $X^{-1}(B) \in \mathcal{F}$ for all $B \in \mathcal{B}_{\mathbb{R}}$ (i.e. a $\mathcal{F}-$ measurable mapping) where $\mathcal{B}_{\mathbb{R}}$ is $\sigma-$algebra of Borel sets on the metric space $\mathbb{R}$.*

With this definition, we will given law of random variable $X$ .

**Definition 1.5.10.**
*The map* $P_X : \mathcal{B}_{\mathbb{R}} \to [0,1]$ *defined by*

$$P_X(B) = P(X^{-1}(B))$$

*is a probability on* $\mathcal{B}_{\mathbb{R}}$. *It is called the law of* $X$ *and* $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, P_X)$ *the numerical replica of* $(\Omega, \mathcal{F}, P)$ *given by* $X$.

A notion related with the law of a random variable is that of the distribution function

**Definition 1.5.11.**
*the distribution function of random variable* $X$ *is a map* $F_X : \mathbb{R} \to [0,1]$ *defined by*

$$F_X(x) = P(X^{-1}((-\infty, x])).$$

In the sequel, we shall write $F$ instead of $F_X$.

**Properties 1.5.1.**
*The distribution function has the following properties:*
*(1)* $F$ *is non-decreasing,*
*(2)* $F$ *is right-continuous,*
*(3)* $\lim_{x\to-\infty} F(x) = 0$ *and* $\lim_{x\to+\infty} F(x) = 1$.

**Definition 1.5.12.**
*1. A random variable* $X$ *is discrete if it takes a countable number of values, it has the following form*

$$X = \sum_{n\in\mathbb{N}} x_n \mathbf{1}_{A_n}$$

*where $x_n \in \mathbb{R}$ and $A_n = \{X = x_n\}$ are disjoint events.*
*2. A random variable $X$ is continuous if its distribution function can be written as*

$$F(x) = \int_{-\infty}^{x} f(y)dy,$$

*where $f$ is a positive, Riemann integrable function, such that $\int_{-\infty}^{+\infty} f(y)dy = 1$.*
*The function $f$ is called the density of $F$ and, by extension, the density of $X$.*

**Definition 1.5.13.** *(expectation of a random variable)*
*The mathematical expectation of a random variable $X$ is given as*

$$\mathbb{E}(X) = \int_{\Omega} X dP = \int_{\Omega} X(\omega)P(d\omega)$$

*Hence*
*1. If $X$ a discrete random variable then the mathematical expectation of $X$ exists if and only if*

$$\sum_{n\in\mathbb{N}} |x_n|P(A_n) < \infty$$

*and it is defined as*

$$\mathbb{E}(X) = \sum_{n\in\mathbb{N}} x_n P(A_n).$$

*2. If $X$ a continuous random variable then the expectation of $X$ exists if and only if*

$$\int_{-\infty}^{+\infty} |x|f(x)dx < \infty$$

*and it is defined as*

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} xf(x)dx.$$

**Definition 1.5.14.**

*The random variables $Y$ and $X$ , with distribution functions $F_Y$ and $F_X$ respectively, are said to be independent if and only if*

$$F_{(X,Y)}(x,y) = F_X(x)F_Y(x).$$

*where $F_{(X,Y)} : \mathbb{R} \times \mathbb{R} \to [0,1]$ is distribution function $(X,Y)$ defined as*

$$F_{(X,Y)}(x,y) = P(X^{-1}((-\infty,x]), Y^{-1}((-\infty,y])).$$

**Definition 1.5.15.**

*Let $X$ is a real random variable,*

*1. If $\mathbb{E}(X^2) < \infty$ one defines the variance of $X$ as*

$$V(X) = \mathbb{E}(X - \mathbb{E}X)^2.$$

*2. The covariance of two real random variables $Y$ and $X$ is given as*

$$Cov(X,Y) := \mathbb{E}\left((X - \mathbb{E}X)(Y - \mathbb{E}Y)\right).$$

Similarly definitions in $\mathbb{R}^d$ valued random variables.

**Definition 1.5.16.** *Let $X = (X_1, ..., X_d)$ a random vecteur The expectation is defined coordinatewise (i.e.$\mathbb{E}(X) = (\mathbb{E}(X_1), ..., \mathbb{E}(X_d))$). Variance is replaced by the* ***covariance matrix*** *: If $\mathbb{E}(X_i^2) < \infty$; for $i = 1, ..., d$, then the covariance matrix of $X$ is defined as the $d \times d-$matrix*

$$C_X = \mathbb{E}\left((X_i - \mathbb{E}X_i)(X_j - \mathbb{E}X_j)\right) := Cov(X_i, X_j).$$

$$:= \begin{pmatrix} V(X_1) & Cov(X_1, X_2) & ... & Cov(X_1, X_d) \\ Cov(X_2, X_1) & V(X_2) & ... & Cov(X_2, X_d) \\ . & . & ... & . \\ . & . & ... & . \\ . & . & ... & . \\ Cov(X_d, X_1) & Cov(X_d, X_2) & ... & V(X_d) \end{pmatrix}$$

### 1.5.3 Necessary inequalities

**Theorem 1.5.1.** *(Hölder's inequality)*
*For any real-valued random variables $X$ and $Y$ defined on the probability space $(\Omega, \mathcal{F}, P)$, we have*

$$\mathbb{E}|XY| \leq \left(\mathbb{E}|X|^p\right)^{\frac{1}{p}} \left(\mathbb{E}|Y|^q\right)^{\frac{1}{q}}$$

*where $\frac{1}{p} + \frac{1}{q} = 1$ if $p \in (1, \infty)$.*

**Theorem 1.5.2.** *Jensen's inequality*
*Let $X$ is an integrable random variable(i.e. $\mathbb{E}(|X|) < \infty$) defined on the probability space $(\Omega, \mathcal{F}, P)$ and $\varphi$ is a convex function, then*

$$\varphi(\mathbb{E}(X)) \leq \mathbb{E}(\varphi(X)).$$

### 1.5.4 Random Processes

In what follow we will consider $(\Omega, \mathcal{F}, P)$ a complete probability space and $T$ the unit interval .

**Definition 1.5.17.**
*Let $X := (X_t,\, t \in T) = \left(X_t\right)_{t \in T}$ be a family of random variables defined*

*on $(\Omega, \mathcal{F}, P)$ and with values in a measurable space $(E, \mathcal{B})$. $X$ is called a* ***stochastic process*** *(or* ***random process****) with sample space $(\Omega, \mathcal{F}, P)$, state space $(E, \mathcal{B})$, and time set $T$.*
*The random variable $\omega \to X_t(\omega)$ is the state of at time $t$ and the mapping $t \to X_t(\omega)$ is the sample path of $X$ associated with $\omega$.*

**Definition 1.5.18.** *(processes types)*
*1. If $T$ is countable (as $T = \mathbb{N}$ or $\mathbb{Z}$), $X$ is a discrete-time process.*
*2. If $T$ is an interval in $\mathbb{R}$ (as $T = [a, b]$, $\mathbb{R}+$ or $\mathbb{R}$), $X$ is a continuous-time process.*
*3. If $(E, \mathcal{B}) = (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, $X$ is said to be a real stochastic process.*

**Definition 1.5.19.**
*A process $\left(X_t\right)_{t \in T}$ is said to be measurable if the map $X : T \times \Omega \to \mathbb{R}$, given by $X(t, \omega) = X_t(\omega)$ is $\mathcal{B}(t) \otimes \mathcal{F}-$measurable where $\mathcal{B}(t)$ is the Borel field of subsets of $T$.*

**Definition 1.5.20.**
*We say that the stochastic process $\left(X_t\right)_{t \in [0,+\infty)}$ ($T$ could be $[0, T_0]$ or $\mathbb{N}_0$ instead of $[0, \infty)$) is progressively measurable (or, simply, progressive) if, for every time $t$, the map $[0, t] \times \Omega \to E$ defined by $(s, \omega) \mapsto X_s(\omega) \to X_s(\omega)$ is $\mathcal{B}([0, t]) \otimes \mathcal{F}_t-$mesurable.*

**Definition 1.5.21.**
*A filtration $(\mathcal{F}_t,\, t \in T)$ on a measurable space $(\Omega, \mathcal{F})$ is a family of increasing sub $\sigma-$algebra of $\mathcal{F}$ ( i.e. $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ whenever $s \leq t$ ).*

**Definition 1.5.22.**
*Let $(\mathcal{F}_t,\, t \in T)$ be a filtration. A process $(X_t,\, t \in T)$ is said to be adapted*

*to the filtration $(\mathcal{F}_t,\, t \in T)$ if $X_t$ is $\mathcal{F}_t-$ measurable for all $t \in T$.*

**Definition 1.5.23.**

*For a fixed $\omega \in \Omega$, the function $t \to X_t(w)$ is called a trajectory of $X$ .*

**Proposition 1.5.1.**

*Suppose that the adapted stochastic process $X = (X_t,\, t \in T)$ has the property that all of its trajectories are continuous . Then, $X$ is progressively measurable.*

**Definition 1.5.24.**

*Let $\mu$ be a measure over $(\Omega, \mathcal{F})$. For $1 \leq p \leq \infty$, $L^P(\Omega, \mathcal{F}, \mu)$ is space of functions $f : \Omega \to \mathbb{R}$ such that*

$$\|f\|_p = \left(\int_\Omega |f|^p d\mu\right)^{\frac{1}{p}}, \; if\, p < \infty$$

*and*

$$\|f\|_\infty = \inf\{c : \mu\{|f| > c\}\}, \; if\, p = \infty.$$

All properties of lebesgue spaces cited in section 1.2.5 extended to $L^P(\Omega, \mathcal{F}, \mu)$.

**Definition 1.5.25.**

*Let $(\Omega, \mathcal{F}, P)$ be a Probability space and let $\mathcal{G}$ be a sub $\sigma$-algebra of $\mathcal{F}$. Conditional expectation with respect to $\mathcal{G}$ is the mapping $\mathbb{E}(. \diagup \mathcal{G})$ : $L^1(\Omega, \mathcal{F}, P) \to L^1(\Omega, \mathcal{G}, P)$ characterized by $\int_B \mathbb{E}(X \diagup \mathcal{G}) dP = \int_B X dP, B \in \mathcal{G}$, $X \in L^1(\Omega, \mathcal{F}, P)$.*

*Considered as a mapping from $L^2(\Omega, A, P)$ to $L^2(\Omega, \mathcal{G}, P)$ it becomes the orthogonal projector of $L^2(\Omega, \mathcal{G}, P)$ (i.e. for each $X$ in $L^2(\Omega, \mathcal{G}, P)$ , $\mathbb{E}(X \diagup \mathcal{G})$ is the best approximation of $X$ by a square-integrable $\mathcal{G}$-measurable random variable).*

*In particular if* $\mathcal{G} = \sigma(X_t, t \in T)$ *(i.e.* $\mathcal{G}$ *is the smallest* $\sigma$*-algebra such that each* $X_t$ *is* $\mathcal{G}$ *- measurable and if* $\mathcal{F}$ *a filtration we said it natural filtration ) then* $\mathbb{E}(X \diagup \mathcal{G})$ *is the conditional expectation of* $X$ *given* $(X_t, t \in T)$*.*

**Definition 1.5.26.**

*A process* $\big(X_t\big)_{t \in T}$ *is said to be a martingale with respect to a filtration* $(\mathcal{F}_t,\, t \in T)$ *(or simply* $\mathcal{F}_t$*-martingale) if*

- *It is* $(\mathcal{F}_t)_{t \in T}$ *adapted.*
- $\mathbb{E}(X_t) < \infty$ *for all* $t \in T$*.*
- $\mathbb{E}(X_t \diagup \mathcal{F}_s) = X_s$ *for all* $s < t,\ s, t \in T$

*where* $\mathbb{E}(. \diagup \mathcal{F}_s)$ *denotes the conditional expectation with respect to* $\mathcal{F}_s$*.*

In what follow we consider $H$ be a real separable Hilbert space with norm $\|.\|$ and scalar product $< ., . >$.

**Proposition 1.5.2.**

*Let* $\mathcal{R}$ *a linear selfadjoint and nonnegative operator on* $H$ *(i.e.*$\mathcal{R} = \mathcal{R}^* > 0$ *) then it admits a bounded sequence of nonnegative real numbers* $(\lambda_j)_{j \geq 1}$ *of eigenvalues, the corresponding eigenvectors* $(e_j)_{j \geq 1}$ *form an orthonormal basis in H, satisfies that* $\mathcal{R}e_j = \lambda_j e_j,\ j = 1, 2, ....$

**Definition 1.5.27.**

*An bounded linear operator* $\mathcal{R}$ *is nuclear (or trace-class) operator on* $H$ *if*

$$\sum_{j=1}^{\infty} | < \mathcal{R}e_j, e_j > | < \infty$$

*where* $(e_j)$ *is any orthonormal bases of* $H$ *.The class* $\mathcal{N}(H)$ *of nuclear operators is a Banach space with respect to the norm* $\|.\|_{\mathcal{N}} = \sum_{j=1}^{\infty} |\lambda_j|$*.*

*The trace operator*

$$Tr\mathcal{R} = \sum_{j=1}^{\infty} < \mathcal{R}e_j, e_j >$$

*is a well defined continuous linear functional on $\mathcal{N}(H)$.*

**Proposition 1.5.3.**

*An operator $\mathcal{R}$ selfadjoint and nonnegative is nuclear if*

$$Tr\mathcal{R} = \sum_{j=1}^{\infty} < \mathcal{R}e_j, e_j >= \sum_{j=1}^{\infty} < \lambda_j e_j, e_j >= \sum_{j=1}^{\infty} \lambda_j < \infty$$

*The space of this operators is denoted by $\mathcal{N}^+(H)$.*

In what follow we consider $\mathcal{R} \in \mathcal{N}^+(H)$.

**Definition 1.5.28.** *A probability measure $\mu$ on $(H, \mathcal{N}^+(H))$ is called Gaussian if its characteristic function equal to*

$$\varphi_\mu(u) := \exp\left(i < m, u > -\frac{1}{2} < \mathcal{R}u, u >\right) = \int_H e^{i<u,x>} \mu(dx)$$

*where $m(\in H)$ is called mean and $\mathcal{R}(\in \mathcal{N}^+(H)))$ is called covariance (operator), $\mu$ will be denoted $\mathcal{N}(m, \mathcal{R})$ and if the random variable $X$ follow $\mu$, we said that $X$ is a gaussian random and we note by $u \sim \mathcal{N}(m, \mathcal{R})$. Furthermore, for all $h, g \in H$*

$$\int < x, h > \mu(dx) =< m, h >$$

*and*

$$\int (< x, h > - < m, h >)(< x, g > - < m, g >)\mu(dx) =< \mathcal{R}h, g > .$$

*Moreover, for every $m \in H$ and $\mathcal{R} \in \mathcal{N}^+(H)$ there exists a Gaussian measure with mean $m$ and covariance $\mathcal{R}$*

**Definition 1.5.29.**
*A second order process $X = (X_t, t \in T)$ (i.e. $\mathbb{E}\|X\|^2 < \infty$) is called Gaussian if all the random variables $\sum_{i=1}^{k} \alpha_i X_{t_i}$, $\alpha_i \in \mathbb{R}, \tau_i \in T, i = 1, .., k$ are real Gaussian random variables.*

**Definition 1.5.30.**
*For a process $X = (X_t, t \in T)$ , the function $t \to X_t(w)$ is belongs to some function space $F$ for all $\omega \in \Omega$. We said $X$ a $F$-valued process.*

**Definition 1.5.31.** ***A Wiener process*** *(or Brownian motion process)*
*We call $W = (W_t)_{t \in T}$ a $\mathcal{R}$-Wiener process (or Brownian motion process) for the filtration $\mathcal{F}_t$ if a $H$-valued process and the following properties hold:*
*i/ $W_0 = 0$ and $W$ has continuous trajectories.*
*ii/For every $s < t$, $W_t - W_s$ is independent of $\mathcal{F}_s$ and $W_t - W_s \sim \mathcal{N}(0, (t-s)\mathcal{R})$,*
*where $\mathcal{N}(m, \Gamma)$ design the gaussian law with mean $m$ and covariance operator $\Gamma$.*

We have equivalent definitions for a $\mathcal{R}$-Wiener process.

**Theorem 1.5.3.** *Let $W = (W_t)_{t \in T}$ a $H$-valued process such that $W_0 = 0$ and $W$ has continuous trajectories. The following expressions are equivalent:*
*i/ $W_t$ is a $\mathcal{R}$-Wiener process*

*ii/ $W_t$ is a centered Gaussian process (or zero-mean Gaussian process) with covariance*

$$Cov(W_t(x), W_s(y)) = min(t, s)\mathcal{R}$$

*iii/ the increments of $W$ are independent, i.e. the random variables $W_{t_1}, W_{t_2} - W_{t_1}, ..., W_{t_n} - W_{t_{n-1}}$ are independent for all $0 \leq t_1 < ... < t_n \leq T, n \in \mathbb{N}$.*
*iv/ There exist real and independent Brownian motion $\{(\beta_k(t))_{t\in T}\}_{k\geq 1}$such that*

$$W_t = \sum_{j\geq 1} \sqrt{\lambda_j}\beta_j(t)e_j$$

.

**Proposition 1.5.4.** *1. Let $0 < T < \infty$ and denote by $\mathcal{M}_T^2(H)$ the space of all $H$-valued continuous, square integrable $\mathcal{F}_t$-martingales $W = (W_t)_{t\in[0,T]}$, this space equipped with the following norm*

$$\|M\|_{\mathcal{M}_T^2(H)} = \sup_{t\in[0,T]} \left(\mathbb{E}\left(\|M\|^2\right)\right)\frac{1}{2}.$$

*is a Banach space.*
*2. If $W = (W_t)_{t\in[0,T]}$ a $H$-valued $\mathcal{R}$-Wiener process for the filtration $\mathcal{F}_t$ then $w \in \mathcal{M}_T^2(H)$.*

## 1.6 Stochastic Integral

For the whole section we fix a positive real number T and a probability space $(\Omega; \mathcal{A}; P)$ and we define $\Omega_T := [0, T] \times \Omega$ and $P_T := dt \otimes P$ where

$dt$ is the Lebesgue measure. Moreover, let $W = (W_t)_{t\in[0,T]}$ a $H$-valued $\mathcal{R}$-Wiener process for the filtration $\mathcal{F}_t$ . We set

$$\mathcal{F}_t = \sigma(W_s : \Omega_T \to \mathbb{R}, 0 \leq s \leq t)$$

and we define the class $\mathcal{I}$ of stochastic processes $X = (X_t)_{t\in[0,T]}$ such that
(1) $(t,\omega) \to X_t(\omega)$ is $L^2(\Omega_T, \mathcal{I} \otimes \mathcal{A}, P_T)$.
(2) $X$ is adapted to $(\mathcal{F}_t)_{t\in[0,T]}$ .
$\mathcal{I}$ is called the class of stochastically integrable processes on $[0,T]$. Now let $\mathcal{E}$ be the subclass of $\mathcal{I}$ constituted by the elementary (step) process: $X \in \mathcal{E}$ if $X \in \mathcal{I}$ and

$$X_t(\omega) = \sum_{i=0}^{k-1} f_i(\omega)\mathbf{1}_{[t_i,t_{i+1})}(t), \omega \in \Omega, 0 \leq t \leq T.$$

Where $0 = t_0 < t_1 < ... < t_k = T$, $[t_i, t_{i+1})$ open on the right except $[t_{k-1}, t_k)$ which is closed, $f_i$ is member of $L^2(\Omega, \mathcal{F}_{t_i}, P)$, $0 \leq i \leq k-1$. For such a process one defines the stochastic integral as

$$Int(X)(t) := \int_0^t X_s dW_s = \int_0^t X(s)dW(s) = \sum_{i=0}^{k-1} f_i \left(W(\min\{t_{i+1}, t\}) - W(\min\{t_i, t\})\right)$$

This Stochastic integral is a continuous square integrable martingale with respect to $\mathcal{F}_t$, $t \in [0,T]$, i.e.

$$Int :\in \mathcal{E} \to \mathcal{M}^2_T(H).$$

and it is linear isomorphism. Since $\mathcal{M}^2_T(H)$ is a Banach space, this implies that Int can be extended to the abstract completion $\bar{\mathcal{E}} = \mathcal{I}$ (i.e. $\mathcal{E}$ dense in $\mathcal{I}$), by continuity, $Int(X)(t)$ has a unique linear extension to $\mathcal{I}$ which is called stochastic integral on $[0, T]$ with respect to $W$ denoted by

$$Int(X)(t) := \int_0^t X_s dW_s = \int_0^t X(s)dW(s), X \in \mathcal{I}$$

Stochastic integral has the following proprieties

### 1.6.1 Properties of the stochastic integral

The stochastic integral has the following properties:

- $\int_0^T (aH_1 + bH_2)(s)dW(s) = a\int_0^T H_1(s)dW(s) + b\int_0^T H_2(s)dW(s)$
- $\mathbb{E}(\int_0^T H(s)dW(s)) = 0.$
- $\mathbb{E}\big((\int_0^T H(s)dW(s))^2\big) = \mathbb{E}\big(\int_0^T H(s)^2 ds\big).$
- $\mathbb{E}\Big(\int_0^T H_1(s)dW(s))(\int_0^T H_2(s)dW(s))\Big) = \mathbb{E}(\int_0^T H_1H_2(s)ds).$
- $\mathbb{E}(\int_0^t H(s)dW(s)/\mathcal{F}u) = \int_0^u H(s)dW(s).$

Now we introduce Itô's formula, which will play a key role for our stability analysis.

**Definition 1.6.1.** *(Itô processus):*
*Let $\mathcal{F}_t$ be an increasing family of sub $\sigma$-fields of $\mathcal{F}$. A stochastic process $Y_t$ is said to have a stochastic differential on $[0, T]$, if for $t \leq T$*

$$Y_t = Y_0 + \int_0^t K_s ds + \int_0^t G(s)dW(s), \tag{1.12}$$

*where*
*$Y_0$ is $\mathcal{F}_0$ measurable, $K_s$ is adapted to $\mathcal{F}_t$ with $\int_0^T |K_t|dt < \infty$ with the probability 1 and $\int_0^T |G_t|^2 dt < \infty$ with the probability 1.*

**Theorem 1.6.1.** *(Itô's formula)*
*Suppose that* $f \in C^2(H;\mathbb{R}^+)$ *and* $(Y_t)_{0\leq t\leq T}$ *Itô processus :*

$$Y_t = Y_0 + \int_0^t K_s ds + \int_0^t G(s)dW(s),$$

*then*

$$f(Y_t) = f(Y_0) + \int_0^t f'(Y_s)K_s ds + \int_0^t f'(Y_s)G(s)dW(s) + \frac{1}{2}tr(f''(Y_s)G(s)\mathcal{R}G^*(s)).$$

**Example 1.6.1.** *(Brownian stochastic integral example)*
*We apply Itô's theorem to ordinary standard Brownian motion with* $f(x) = x^2$*. We start in ( 1.12 ) being* $dW$ *(we have* $K_s = 0$ *and* $G(s) = 1$*) we obtaine*

$$W_t^2 = 2\int_0^t W(s)dW(s) + \frac{1}{2}\int_0^t 2ds.$$

*We get*

$$W_t^2 - t = 2\int_0^t W(s)dW(s).$$

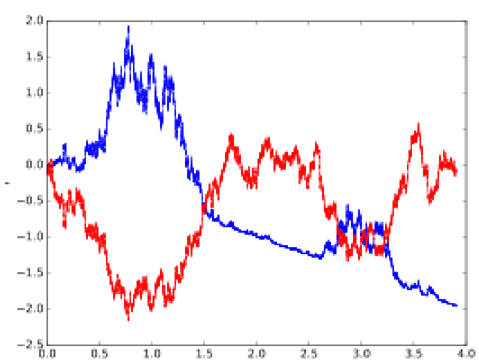


Figure 1.2: Itô integral $\int_0^t W(s)dW(s)$ (blue) of Brownian motion $W_t$ (red) with respect to itself, i.e. Both the integrand and the integrator are Brownian. It turns out $\int_0^t W(s)dW(s) = \frac{1}{2}(W_t^2 - t)$.

# Chapter 2

# Energy Decay of Solutions of Viscoelastic non-Degenerate Kirchhoff Equation with a Time Varying Delay Term

## 2.1 Introduction

In this chapter, we study the decay properties of solutions for the initial boundary value the problem of viscoelastic non-degenerate Kirchhoff equation of the form

$$\begin{cases} u_{tt} - M(\|\nabla u\|^2)\Delta u - \Delta u_{tt} + \int_0^t h(t-s)\Delta u(s)\,ds & \\ +\mu g(u_t(x,t-\tau(t))) = 0 & \text{in } \mathcal{D}\times]0,+\infty[, \\ u(x,t) = 0 & \text{on } \partial\mathcal{D}\times[0,+\infty[, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \mathcal{D}, \\ u_t(x,t-\tau(0)) = f_0(x,t-\tau(0)) & \text{in } \mathcal{D}\times]0,\tau(0)[ \end{cases} \tag{2.1}$$

where $\mathcal{D}$ is a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}^*$, with a smooth boundary $\partial\mathcal{D}$, $\mu$ is positive real number, $h$ is a positive function which decays exponentially, $\tau(t) > 0$ is a time varying delay, $g$ is function and the data

$(u_0, u_1, f_0)$ are initial conditions supposed to be in a suitable function space. $M(s) = a + b\,s^\gamma$ is a $C^1$-function for $s \geq 0$, with $a, b > 0$ and $\gamma \geq 1$.

Due to their special properties in keeping the memory of their past trace, viscoelastic materials present a natural damping. Effects of this later are mathematically modeled by integrodifferential operators. Several important physical processes like quasilinear bidirectional waves of shallow water **[66]**, the velocity evolution of ion-acoustic waves in a collisionless plasma when an ion viscosity is invoked **[59]**, the heat conduction in viscoelastic materials analysis, electric signals in nonlinear telegraph line with nonlinear damping, vibration of nonlinear elastic rod with viscosity **[41]** and viscous flow in viscoelastic materials **[5]** are described by such equations with the viscoelastic term.

For quite a long time, many authors have given attention to the nonlinear viscoelastic wave equations with homogeneous Dirichlet boundary condition and then several works were achieved about the existence or the nonexistence of global solutions, blow up results in finite time, and the asymptotic behavior of the solutions for the viscoelastic equations. For instance, Berrimi and Messaoudi **[13]** studied the problem when $M(s) = 1$ without dispersion term or delay one but with the presence of a source term:

$$u_{tt} - \Delta u_{tt} + \int_0^t h(t-s)\Delta u(s)\,ds = |u|^\gamma u.$$

They gave a local existence result and under suitable conditions on $h$ with specific initial data proved that the solution became global with exponentially or polynomially decaying energy at the same time as the rate of the decay of the relaxation function $h$. While, Messaoudi **[48]** established a particular decay result which is not necessarily of exponential or polynomial type for the following similar problem in a bounded domain

$$u_{tt} - \Delta u_{tt} + \int_0^t h(t-s)\Delta u(s)\,ds = 0$$

Generally speaking the problems of form

$$u_{tt} - \sigma\Delta u_{tt} + \Delta u + \int_0^t h(t-s)\Delta u(s)\,ds = \mathfrak{F}(u)$$

are really overworked, we refer the reader for example, but not limited to Munoz Rivera et al. **[54]**, Santos **[65]**, Cavalcanti and Oquendo **[17]**, and so on.

In collaboration with Benaissa **[11]**, Massoudi established the global existence of solution and the decay rate of the energy for the following problem :

$$u_{tt} - \Delta u + \mu_1\sigma(t)g_1(u_t(x,t)) + \mu_2\sigma(t)g_2(u_t(x,t-\tau(t))) = 0 \text{ in } \mathcal{D}\times]0,+\infty[$$

To be more precise in modeling real systems, there was a great interest to partial differential equations with time delay which appears naturally in many practical situations where it was shown that even an arbitrarily delay can cause instability to the system. There were a lot of works that deals with problem of constant delay perturbed systems, we can mention the work of Kafini and Messaoudi **[37]** and the one of Yüksekkaya and Pişkin **[72]**. Yüksekkaya et al. **[73]** treated the same issu in the case of higher-order Kirchoff equation. Others authors like Mezouar et al. **[52]**, **[51]** took into consideration that the delay can depend on time .Recently, Pişkin et al. **[62]** have gave a more general case where they delt with a

coupled nonlinear viscoelastic Kirchhoff system with distributed delay and source term. In our work we are interested to the Kirchhoff-type $M(\|\nabla u\|^2)$ viscoelastic equation (2.1) with distributed delay $(\mu g(u_t(x, t-\tau(t)))$ without source terms. We intent to we obtain the decay properties of solutions for the equation (2.1) and using Lyapunov functional, we are meant for proving the stability of solution.
This chapter is organized as follows: In Section 2, we set necessary hypotheses and results. In Section 3, we state our main result. Finally, in Section 4, we prove the uniform energy decay.

## 2.2 Preliminaries and Assumptions

Using the Sobolev spaces $H^2(\mathcal{D})$, $H_0^1(\mathcal{D})$ and the Hilbert space $L^2(\mathcal{D})$ with their usual scalar products and norms.
The prime $'$ and the subscript $t$ will denote time differentiation.
We introduce, as in the work of in Nicaise and Pignotti **[55]**, the new variable

$$z(x, \rho, t) = u_t(x, t - \rho\tau(t)), \; x \in \mathcal{D}, \quad \rho \in (0, 1), \; t > 0.$$

Then, we get

$$\tau(t)z_t(x, \rho, t) + (1 - \rho\tau'(t))z_\rho(x, \rho, t) = 0, \; \text{ in } \mathcal{D} \times (0, 1) \times (0, +\infty). \quad (2.2)$$

We rewrite the problem (2.1) as follows :

$$\begin{cases} u_{tt} - M(\|\nabla u\|^2)\Delta u - \Delta u_{tt} + \int_0^t h(t-s)\Delta u(s)\,ds & \\ +\mu g(z(x,1,t)) = 0 & \text{in } \mathcal{D}\times]0,+\infty[, \\ \tau(t)z_t(x,\rho,t) + (1-\rho\tau'(t))z_\rho(x,\rho,t) = 0, & \text{in } \mathcal{D}\times]0,1[\times]0,+\infty[ \\ u(x,t) = 0, & \text{on } \partial\mathcal{D}\times[0,\infty[ \\ z(x,0,t) = u_t(x,t), & \text{on } \mathcal{D}\times[0,\infty[ \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & \text{in } \mathcal{D} \\ z(x,\rho,0) = f_0(x,-\rho\tau(0)), & \text{in } \mathcal{D}\times]0,1[. \end{cases} \tag{2.3}$$

We present some assumptions needed in the proof of our results.

**(A1)** For the relaxation function $h : \mathbb{R}_+ \to \mathbb{R}_+$ it is a bounded $C^1$ function satisfying

$$a - \int_0^\infty h(s)\,ds = k > 0,$$

and suppose that there exist a constant $\zeta > 0$ such that

$$h'(t) + \zeta h(t) \leq 0.$$

**(A2)** $g : \mathbb{R} \to \mathbb{R}$ is non decreasing function of class $C^1(\mathbb{R})$ such that there exist $\alpha_1$, $\alpha_2$, $c_1$, $c_2$ are non-negative constants, such that

$$\begin{cases} \alpha_1 sg(s) \leq G(s) \leq \alpha_2 sg(s), \\ G(s) = \int_0^s g(r)dr \end{cases} \tag{2.4}$$

and

$$\begin{cases} c_1|s| \leq |g(s)| \leq c_2|s| & \text{if } |s| \geq \varepsilon \\ s^2 + g(s)^2 \leq H^{-1}(sg(s)) & \text{if } |s| \leq \varepsilon \end{cases} \tag{2.5}$$

where $H : \mathbb{R}_+ \to \mathbb{R}_+$ is convex, increasing function of class $C^1(\mathbb{R}_+) \cap C^2(\mathbb{R}_+^*)$ satisfying $H(0) = 0$ if $H$ is linear on $[0,\varepsilon]$ or $H'(0) = 0$ and

$H'' > 0$ on $]0, \varepsilon]$ and $H^{-1}$ denotes the inverse function of $H$.

**(A3)** $\tau$ is a function in $W^{2,+\infty}([0,T])$, $T > 0$, such that

$$\begin{cases} 0 < \tau_0 \leq \tau(t) \leq \tau_1, & \forall t > 0 \\ \tau'(t) \leq d < 1, & \forall t > 0 \end{cases}$$

where $\tau_0$ and $\tau_1$ are positive numbers.

**(A4)** We also assume that

$$\frac{\alpha_2(1-\alpha_2)}{\alpha_1^2(d-1)} < 1.$$

We define the energy associated to the solution of system (2.3) by

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{b}{2(\gamma+1)}\|\nabla u\|_2^{2(\gamma+1)} + \frac{1}{2}\Big(a - \int_0^t h(s)ds\Big)\|\nabla u\|_2^2 + \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{1}{2}(h\, o \,\nabla u)(t) + \xi\tau(t)\int_{\mathcal{D}}\int_0^1 G(z(x,\rho,t))\, d\rho\, dx \tag{2.6}$$

where $\xi$ is positive constant such that

$$\frac{\mu(1-\alpha_2)}{\alpha_1(d-1)} < \xi < \frac{\mu\alpha_1}{\alpha_2}, \tag{2.7}$$

and

$$(h\, o\, \varphi)(t) = \int_0^t h(t-s)\|\varphi(.,t) - \varphi(.,s)\|^2\, ds.$$

A crucial property of the convolution operator is stated in the following lemma.

**Lemma 2.2.1.** *[2]* **(Sobolev-Poincarés inequality )** *Let $q$ be a number with*

$$2 \leq q < +\infty \ (if\, n = 1, 2) \ \ or \ \ 2 \leq q \leq 2n/(n-2) \ (if\, n \geq 3),$$

*then there exists a constant* $C_s = C_s(\mathcal{D}, q)$ *such that*

$$\|u\|_q \leq C_s \|\nabla u\|_2 \quad for \;\; u \in H_0^1(\mathcal{D}).$$

**Lemma 2.2.2.** *[58] For* $h, \varphi \in C^1([0, +\infty[, \mathbb{R})$ *we have*

$$\int_{\mathcal{D}} h{*}\varphi\varphi_t \, dx = -\frac{1}{2}h(t)\|\varphi(t)\|^2 + \frac{1}{2}(h' o\, \varphi)(t) - \frac{1}{2}\frac{d}{dt}[(h\, o\, \varphi)(t) - (\int_0^t h(s)\, ds)\|\varphi\|^2].$$

**Remark 2.2.1.** *[6] Let us denote by* $\Phi^*$ *the conjugate function of the differentiable convex function* $\Phi$*, i.e.,*

$$\Phi^*(s) = \sup_{t \in \mathbb{R}} (st - \Phi(t)).$$

*Then* $\Phi^*$ *is the Legendre-Fenchel (LF) transform of* $\Phi$*, which is given by*

$$\Phi^*(s) = s(\Phi')^{-1}(s) - \Phi[(\Phi')^{-1}(s)], \quad if \;\; s \in (0, \Phi'(r)], \tag{2.8}$$

*and* $\Phi^*$ *satisfies the generalized Young inequality*

$$AB \leq \Phi^*(A) + \Phi(B), \quad if \;\; A \in (0, \Phi'(r)] \;\;\; B \in (0, r]. \tag{2.9}$$

**Remark 2.2.2.** *[69] The Legendre-Fenchel (LF) transforms can also be defined using an infimum (min) rather than a supremum (max)*

$$\Phi^*(s) = \inf_{t \in \mathbb{R}} (st - \Phi(t)).$$

Now, the following theorem indicates the existence of global solution of our problem. Its proof derives from the combination of the proofs given in [51].

**Theorem 2.2.1. (Global existence)**
*Let* $(u_0, u_1, f_0) \in H^2(\mathcal{D}) \cap H_0^1(\mathcal{D}) \times H_0^1(\mathcal{D}) \times H_0^1(\mathcal{D}, H^1(0,1))$ *satisfy the compatibility condition*

$$f_0(., 0) = u_1.$$

*Assume That* **(A1)-(A4)** *hold. Then the problem (2.1) admits a weak solution*

$$u \in L^{\infty}([0,\infty); H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})), u_t \in L^{\infty}([0,\infty); H_0^1(\mathcal{D})), u_{tt} \in L^{\infty}([0,\infty); L^2(\mathcal{D})).$$

## 2.3 Main results

We have the following theorem concerning Uniform decay rates of energy

.

**Theorem 2.3.1.** *Assume That* **(A1)-(A4)** *hold. Then, there exist positive constants $w_1,\ w_2,\ w_3$ and $\varepsilon_0$ such that the solution energy of (2.1) satisfies*

$$E(t) \leq w_3 H_1^{-1}(w_1 t + w_2) \quad \forall t \geq 0 \tag{2.10}$$

*where*

$$H_1(t) = \int_t^1 \frac{1}{sH'(\varepsilon_0 s)}\, ds. \tag{2.11}$$

*Here, $H_1$ is strictly decreasing and convex on $(0,1]$ with $\lim_{t\to 0} H_1(t) = +\infty$,*

**Lemma 2.3.1.** *Let $(u,z)$ be a solution of the problem (2.3). Then, the energy functional defined by (2.6) satisfies*

$$E'(t) \leq -\lambda \int_{\mathcal{D}} u_t g(u_t)\, dx - \beta \int_{\mathcal{D}} z(x,1,t) g(z(x,1,t))\, dx - \frac{1}{2} h(t)\|\nabla u(t)\|^2 + \frac{1}{2}(h' o \nabla u)(t) \leq$$

*where $\lambda = \mu\alpha_1 - \xi\alpha_2$ and $\beta = \xi(1-d)\alpha_1 + \mu(1-\alpha_2)$.*

*Proof.* By multiplying the first equation in (2.3) by $u_t$ and integrating by parts over $\mathcal{D}$, we obtain

$$\frac{d}{dt}\Big[\frac{1}{2}\|u_t\|_2^2 + \frac{b}{2(\gamma+1)}\|\nabla u\|_2^{2(\gamma+1)} + \frac{1}{2}a\|\nabla u\|_2^2 + \frac{1}{2}\|\nabla u_t\|_2^2\Big]$$
$$-\int_{\mathcal{D}}\int_0^t h(t-s)\nabla u(s)\nabla u_t(t)\,ds\,dx + \mu\int_{\mathcal{D}} u_t(x,t)g(z(x,1,t))\,dx = 0. \tag{2.12}$$

Then, according to Lemma 2.2.2, the equation (2.12) becomes

$$\frac{d}{dt}\Big[\frac{1}{2}\|u_t\|_2^2 + \frac{b}{2(\gamma+1)}\|\nabla u\|_2^{2(\gamma+1)} + \frac{1}{2}\Big(a - \int_0^t h(s)ds\Big)\|\nabla u\|_2^2 + \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{1}{2}(ho\nabla u)(t$$
$$+\frac{1}{2}h(t)\|\nabla u(t)\|^2 - \frac{1}{2}(h'o\nabla u)(t) + \mu\int_{\mathcal{D}} u_t(x,t)g(z(x,1,t))\,dx = 0. \tag{2.13}$$

Now we multiply the second equation in (2.3) by $\xi g(z)$ and integrate the result over
$\mathcal{D}\times(0,1)$

$$\xi\tau(t)\int_{\mathcal{D}}\int_0^1 z_t(x,\rho,t)g(z(x,\rho,t))\,d\rho\,dx = -\xi\int_{\mathcal{D}}\int_0^1(1-\rho\tau'(t))\frac{\partial}{\partial\rho}G(z(x,\rho,t))d\rho\,dx.$$

So, we obtain

$$\begin{aligned}\frac{d}{dt}\Big(\xi\tau(t)\int_{\mathcal{D}}\int_0^1 G(z(x,\rho,t))\,d\rho\,dx\Big) &= \xi\tau'(t)\int_{\mathcal{D}}\int_0^1 G(z(x,\rho,t))\,d\rho\,dx\\ &\quad-\xi\int_{\mathcal{D}}\int_0^1(1-\rho\tau'(t))\frac{\partial}{\partial\rho}G(z(x,\rho,t))\,d\rho\,dx\\ &= -\xi\int_{\mathcal{D}}\int_0^1\frac{\partial}{\partial\rho}\Big((1-\rho\tau'(t))G(z(x,\rho,t))\Big)\,d\rho\,dx\\ &= -\xi(1-\tau'(t))\int_{\mathcal{D}} G(z(x,1,t))\,dx + \xi\int_{\mathcal{D}} G(u_t(x,t))\,dx.\end{aligned} \tag{2.14}$$

Combining (2.13) and (2.14), we get

$$E'(t) = -\xi(1-\tau'(t))\int_{\mathcal{D}} G(z(x,1,t))\,dx + \xi\int_{\mathcal{D}} G(u_t(x,t))\,dx - \frac{1}{2}h(t)\|\nabla u(t)\|^2$$
$$+\frac{1}{2}(h'o\nabla u)(t) - \mu\int_{\mathcal{D}} u_t(x,t)g(z(x,1,t))\,dx.$$

From (2.4) and ($A3$), it follows that

$$E'(t) \le -\xi(1-d)\alpha_1\int_{\mathcal{D}} z(x,1,t)g(z(x,1,t))\,dx + \xi\alpha_2\int_{\mathcal{D}} u_t(x,t)g(u_t(x,t))$$
$$-\mu\int_{\mathcal{D}} u_t(x,t)g(z(x,1,t))\,dx - \frac{1}{2}h(t)\|\nabla u(t)\|^2 + \frac{1}{2}(h'o\nabla u)(t). \tag{2.15}$$

From the definition of $G$ and (2.8) we have

$$G^*(g(z(x,1,t))) = z(x,1,t)g(z(x,1,t)) - G(z(x,1,t)).$$

From (2.4), we obtain

$$G^*(g(z(x,1,t))) \ge (1-\alpha_2)z(x,1,t)g(z(x,1,t)). \tag{2.16}$$

and

$$G(u_t(x,t)) \ge \alpha_1 u_t(x,t)g(u_t(x,t)). \tag{2.17}$$

By using the Remark 2.2.2, we get

$$u_t g(z(x,1,t)) \ge G^*(g(z(x,1,t))) + G(u_t(x,t)).$$

By combining (2.16) and (2.17), we deduce that

$$-\mu u_t g(z(x,1,t)) \le -\mu(1-\alpha_2)z(x,1,t))(g(z(x,1,t))) - \mu\alpha_1 u_t(x,t)g(u_t(x,t)). \tag{2.18}$$

By replacing (2.18) in (2.15), we obtain

$$\begin{aligned}
E'(t) \leq & -\xi(1-d)\alpha_1 \int_{\mathcal{D}} z(x,1,t)g(z(x,1,t))\,dx + \xi\alpha_2 \int_{\mathcal{D}} u_t(x,t)g(u_t(x,t))\,dx \\
& - \mu(1-\alpha_2)\int_{\mathcal{D}} z(x,1,t))(g(z(x,1,t)))\,dx - \mu\alpha_1 \int_{\mathcal{D}} u_t(x,t)g(u_t(x,t))\,dx \\
& - \frac{1}{2}h(t)\|\nabla u(t)\|^2 + \frac{1}{2}(h' o \nabla u)(t) \\
\leq & -(\xi(1-d)\alpha_1 + \mu(1-\alpha_2))\int_{\mathcal{D}} z(x,1,t)g(z(x,1,t))\,dx \\
& - (\mu\alpha_1 - \xi\alpha_2)\int_{\mathcal{D}} u_t(x,t)g(u_t(x,t)) - \frac{1}{2}h(t)\|\nabla u(t)\|^2 + \frac{1}{2}(h' o \nabla u)(t).
\end{aligned}$$

According to (2.7) we get non-increasing energy function. This completes the proof. □

## 2.4 Uniform Decay: Proof of Theorem2.3.1

In the current section we study the solution's asymptotic behavior of system (2.1).

In order to prove our main result, we construct a Lyapunov functional $F$ equivalent to $E$ by defining some functionals which allow us to obtain the desired estimate.

**Lemma 2.4.1.** *Let (u,z) be a solution of the problem (2.3). Then, the functional*

$$I(t) = \tau(t)\int_{\mathcal{D}}\int_0^1 e^{-2\tau(t)\rho} G(z(x,\rho,t))\,d\rho\,dx, \tag{2.19}$$

*satisfies the estimate*

*i)*

$$|I(t)| \leq \frac{1}{\xi}E(t).$$

*ii)*

$$I'(t) \leq -2\tau(t)e^{-2\tau_1}\int_{\mathcal{D}}\int_0^1 G(z(x,\rho,t))\,d\rho\,dx - \alpha_1(1-d)e^{-2\tau_1}\int_{\mathcal{D}} z(x,1,t)g(z(x,1,t))\,dx$$
$$+\,\alpha_2\int_{\mathcal{D}} u_t(x,t)g(u_t(x,t))\,dx.$$

*Proof.* i) Since $e^{-2\tau(t)\rho}$ is a decreasing function for $\tau(t) \in [\tau_0, \tau_1]$ and by using the definition of energy $E(t)$, we have

$$|I(t)| \leq \tau(t)\int_{\mathcal{D}}\int_0^1 e^{-2\tau_0\rho}G(z(x,\rho,t))\,d\rho\,dx$$
$$\leq \frac{1}{\xi}E(t).$$

ii) Taking the derivative of $I$ with respect to $t$ we obtain

$$\frac{d}{dt}\,I(t) = \int_{\mathcal{D}}\int_0^1 e^{-2\tau(t)\rho}\Big[\tau'(t)G(z(x,\rho,t)) + \tau(t)z_t(x,\rho,t)g(z(x,\rho,t))\Big]\,d\rho\,dx$$
$$-\,2\int_{\mathcal{D}}\int_0^1 \tau(t)\tau'(t)\rho e^{-2\tau(t)\rho}G(z(x,\rho,t))\,d\rho\,dx$$
$$= -\int_{\mathcal{D}}\int_0^1 e^{-2\tau(t)\rho}\frac{\partial}{\partial\rho}\Big((1-\rho\tau'(t))G(z(x,\rho,t))\Big)\,d\rho\,dx$$
$$-\,2\int_{\mathcal{D}}\int_0^1 \tau(t)\tau'(t)\rho e^{-2\tau(t)\rho}G(z(x,\rho,t))\,d\rho\,dx$$

where the second equality has been gotten from (2.2), so $I'(t)$ becomes

$$\frac{d}{dt}\,I(t) = -\int_{\mathcal{D}}\int_0^1 \frac{\partial}{\partial\rho}\Big(e^{-2\tau(t)\rho}(1-\tau'(t)\rho)G(z(x,\rho,t))\Big)\,d\rho\,dx$$
$$-\,2\int_{\mathcal{D}}\int_0^1 \tau(t)e^{-2\tau(t)\rho}(1-\tau'(t)\rho)G(z(x,\rho,t))\,d\rho\,dx$$
$$-\,2\tau(t)\tau'(t)\int_{\mathcal{D}}\int_0^1 \rho e^{-2\tau(t)\rho}G(z(x,\rho,t))\,d\rho\,dx$$
$$= \int_{\mathcal{D}} G(u_t(x,t))\,dx - e^{-2\tau(t)}(1-\tau'(t))\int_{\mathcal{D}} G(z(x,1,t))\,dx$$
$$-\,2\tau(t)\int_{\mathcal{D}}\int_0^1 [(1-\tau'(t)\rho) + \tau'(t)\rho]e^{-2\tau(t)\rho}G(z(x,\rho,t))\,d\rho\,dx.$$

Finally by using (2.4) and $(A3)$ we find

$$I'(t) \leq -2I(t) + \alpha_2\int_{\mathcal{D}} u_t(x,t)g(u_t(x,t))\,dx - e^{-2\tau(t)}(1-d)\alpha_1\int_{\mathcal{D}} z(x,1,t)g(z(x,1,t))\,dx.$$

As $e^{-2\tau(t)\rho}$ is a decreasing function over $[0,1]\times[\tau_0,\tau_1]$, we have

$$I(t)\geq \tau(t)\int_{\mathcal{D}}\int_0^1 e^{-2\tau_1}G(z(x,\rho,t))\,d\rho\,dx.$$

Which achieves the proof. □

**Lemma 2.4.2.** *Let (u,z) be a solution of the problem (2.3). Then, the functional*

$$\phi(t)=\int_{\mathcal{D}}u_t u\,dx+\int_{\mathcal{D}}\nabla u_t\nabla u\,dx,$$

*satisfies the estimate*

*i)*

$$|\phi(t)|\leq \frac{1}{2}\|u_t\|^2+\frac{1}{2}\Big(c_s^2+1\Big)\|\nabla u\|^2+\frac{1}{2}\|\nabla u_t\|^2.$$

*ii)*

$$\begin{aligned}\phi'(t)\leq\ &\|u_t\|^2-M(\|\nabla u\|^2)\|\nabla u\|^2+(1+\eta)(a-k)\|\nabla u\|^2+\frac{1}{4\eta}(ho\nabla u)(t)+\|\nabla u_t\|^2\\&-\mu\int_{\mathcal{D}}u(x,t)g(z(x,1,t))\,dx\end{aligned}$$

*where $\eta>0$ and $c_s$ is the Sobolev embedding constant.*

*Proof.* i) Using Young's inequality, the Sobolov embedding, we deduce

$$\begin{aligned}|\phi(t)|&\leq \frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\|u\|^2+\frac{1}{2}\|\nabla u_t\|^2+\frac{1}{2}\|\nabla u\|^2\\&\leq \frac{1}{2}\|u_t\|_2^2+\frac{1}{2}c_s^2\|\nabla u\|^2+\frac{1}{2}\|\nabla u_t\|^2+\frac{1}{2}\|\nabla u\|^2\\&\leq \frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\Big(c_s^2+1\Big)\|\nabla u\|^2+\frac{1}{2}\|\nabla u_t\|^2.\end{aligned}$$

ii) We derivative $\phi(t)$ with respect to $t$ and we use the first equation of

(2.3), we obtain

$$\begin{aligned}\phi'(t) &= \int_{\mathcal{D}} u_{tt}u\,dx + \int_{\mathcal{D}} |u_t|^2\,dx + \int_{\mathcal{D}} \nabla u_{tt}\nabla u\,dx + \int_{\mathcal{D}} \nabla u_t\nabla u_t\,dx \\ &= \|u_t\|^2 + \int_{\mathcal{D}} \Big[u_{tt} - \Delta u_{tt}\Big]u\,dx + \|\nabla u_t\|^2 \\ &= \|u_t\|^2 - M(\|\nabla u\|^2)\|\nabla u\|^2 + \int_{\mathcal{D}} \nabla u(t)\int_0^t h(t-s)\nabla u(s)\,ds\,dx \\ &- \mu\int_{\mathcal{D}} ug(z(x,1,t))\,dx + \|\nabla u_t\|^2.\end{aligned} \tag{2.20}$$

Young's inequality, Sobolev embedding and **(A1)**, allows to estimate the third term in the right side as follow:

$$\begin{aligned}\int_{\mathcal{D}} \nabla u(t)\int_0^t h(t-s)\nabla u(s)\,ds\,dx &\le \int_0^t h(t-s)\int_{\mathcal{D}} \Big|\nabla u(t)(\nabla u(s) - \nabla u(t))\Big|\,dx\,ds \\ &+ \|\nabla u(t)\|^2\int_0^t h(t-s)\,ds \\ &\le \|\nabla u(t)\|^2\int_0^t h(s)\,ds + \eta\|\nabla u(t)\|^2\int_0^t h(s)\,ds \\ &+ \frac{1}{4\eta}\int_0^t h(t-s)\|\nabla u(s) - \nabla u(t))\|^2\,ds \\ &\le (1+\eta)(a-k)\|\nabla u(t)\|^2 + \frac{1}{4\eta}(ho\nabla u)(t).\end{aligned}$$

which achieves the proof. □

**Lemma 2.4.3.** *Let (u,z) be a solution of the problem (2.3). Then, the functional*

$$\psi(t) = \int_{\mathcal{D}} \Big(\Delta u_t - u_t\Big)\int_0^t h(t-s)(u(t) - u(s))\,ds\,dx,$$

*satisfies the estimates*

*i)*

$$|\psi(t)| \leq \frac{1}{2}\|\nabla u_t\|^2 + \frac{1}{2}(a-k)\Big(1+c_s^2\Big)(ho\nabla u)(t) + \frac{1}{2}\|u_t\|^2.$$

*ii)*

$$\begin{aligned}\psi'(t) \leq\ & \delta(a-k)M(\|\nabla u\|^2)\|\nabla u\|^2 + 2\delta(a-k)^2\|\nabla u\|^2 \\ & + \left(\frac{M_0}{4\delta} + \left(2\delta + \frac{1}{4\delta} + \mu\frac{c_s^2}{4\delta}\right)(a-k)\right)(ho\nabla u)(t) \\ & - \frac{h(0)}{4\delta}\Big(1+c_s^2\Big)(h'o\nabla u)(t) + \left(\delta + \delta c_s^2 - \int_0^t h(s)\,ds\right)\|\nabla u_t\|^2 \\ & + \mu\delta\|g(z(x,1,t))\|^2 - \int_0^t h(s)\,ds\|u_t\|_2^2\end{aligned}$$

*where* $M_0 = a + b\left(\frac{2E(0)}{a}\right)^{\gamma}$*,* $c_s$ *is the Sobolev embedding constant and* $\delta$ *is non-negative real number.*

*Proof.* i) From definition of $\psi$ we have

$$\begin{aligned}\psi(t) = & -\int_{\mathcal{D}} \nabla u_t \int_0^t h(t-s)(\nabla u(t) - \nabla u(s))\,ds\,dx \\ & - \int_{\mathcal{D}} u_t \int_0^t h(t-s)(u(t)-u(s))\,ds\,dx.\end{aligned}$$

By using Young's and Hölder's inequality, we find

$$\begin{aligned}& \Big| - \int_{\mathcal{D}} u_t \int_0^t h(t-s)(u(t)-u(s))\,ds\,dx\Big| \\ & \leq \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\int_{\mathcal{D}}\Big[\int_0^t \Big(h(t-s)\Big)^{\frac{1}{2}}\Big((h(t-s))^{\frac{1}{2}}|u(t)-u(s)|\Big)\,ds\Big]^2\,dx \\ & \leq \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\Big(\int_0^t h(t-s)\,ds\Big)\int_{\mathcal{D}}\int_0^t h(t-s)|u(t)-u(s)|^2\,ds\,dx \\ & \leq \frac{1}{2}\|u_t\|^2 + \frac{1}{2}(a-k)c_s^2(ho\nabla u)(t).\end{aligned} \tag{2.21}$$

Similarly

$$\Big| - \int_{\mathcal{D}} \nabla u_t \int_0^t h(t-s)(\nabla u(t) - \nabla u(s))\, ds\, dx \Big| \le \frac{1}{2} \int_{\mathcal{D}} \Big( \int_0^t h(t-s)|\nabla u(t) - \nabla u(s)|\, ds \Big)^2 dx + \frac{1}{2}\|\nabla u_t\|^2 \le \frac{1}{2}(a-k)(ho\nabla u)(t) + \frac{1}{2}\|\nabla u_t\|^2. \tag{2.22}$$

Combining (2.21) and (2.22), we obtain

$$|\psi(t)| \le \frac{1}{2}\|\nabla u_t\|^2 + \frac{1}{2}(a-k)\Big(1 + c_s^2\Big)(ho\nabla u)(t) + \frac{1}{2}\|u_t\|^2.$$

ii) We derivative $\psi$ by using the Liebnitz formula and by recalling the first equation of (2.3), we get

$$\begin{aligned}
\psi'(t) &= \int_{\mathcal{D}} M(\|\nabla u\|^2)\nabla u(t) \int_0^t h(t-s)(\nabla u(t) - \nabla u(s))\, ds\, dx \\
&- \int_{\mathcal{D}} \int_0^t h(t-s)\nabla u(s)\, ds \int_0^t h(t-s)(\nabla u(t) - \nabla u(s))\, ds\, dx \\
&+ \mu \int_{\mathcal{D}} g(z(x,1,t)) \int_0^t h(t-s)(u(t) - u(s))\, ds\, dx \\
&- \int_{\mathcal{D}} \nabla u_t \int_0^t h'(t-s)(\nabla u(t) - \nabla u(s))\, ds\, dx \\
&- \int_{\mathcal{D}} u_t \int_0^t h'(t-s)(u(t) - u(s))\, ds\, dx - \|\nabla u_t\|^2 \int_0^t h(s)\, ds \\
&- \|u_t\|^2 \int_0^t h(s)\, ds \\
&= I_1 + I_2 + I_3 + I_4 + I_5 - \|\nabla u_t\|^2 \int_0^t h(s)\, ds - \|u_t\|^2 \int_0^t h(s)\, ds.
\end{aligned} \tag{2.23}$$

Now we will estimate $I_1, ..., I_5$ by applying Hölder's and Young's inequalities. We get

$$\begin{aligned}
|I_1| &\le M(\|\nabla u\|^2) \int_{\mathcal{D}} |\nabla u(t)| \Big( \int_0^t h(s)\, ds \Big)^{\frac{1}{2}} \Big( \int_0^t h(t-s)|\nabla u(t) - \nabla u(s)|^2\, ds \Big)^{\frac{1}{2}} dx \\
&\le M(\|\nabla u\|^2) \Big( \delta \|\nabla u(t)\|^2 \int_0^t h(s)\, ds + \frac{1}{4\delta}(ho\nabla u)(t) \Big).
\end{aligned}$$

From (2.6) and lemma 2.3.1 we obtain

$$|I_1| \le \delta M(\|\nabla u\|^2)\|\nabla u(t)\|^2(a-k) + \frac{M_0}{4\delta}(ho\nabla u)(t) \tag{2.24}$$

where $M_0 = a + b(\frac{2E(0)}{a})^\gamma$.

$$\begin{aligned}|I_2| &\leq \delta \int_{\mathcal{D}} \Big( \int_0^t h(t-s)(|\nabla u(s) - \nabla u(t)| + |\nabla u(t)|)\, ds \Big)^2 dx \\ &+ \frac{1}{4\delta} \int_{\mathcal{D}} \Big( \int_0^t h(t-s)|\nabla u(t) - \nabla u(s)|\, ds \Big)^2 dx \\ &\leq 2\delta \|\nabla u(t)\|^2 \left( \int_0^t h(s)\, ds \right)^2 + \left( 2\delta + \frac{1}{4\delta} \right) \int_{\mathcal{D}} \Big( \int_0^t h(t-s)|\nabla u(t) - \nabla u(s)|\, ds \Big)^2 dx \\ &\leq 2\delta \|\nabla u(t)\|^2 (a-k)^2 + (2\delta + \frac{1}{4\delta})(a-k)(ho\nabla u)(t).\end{aligned} \tag{2.25}$$

$$|I_3| \leq \mu \left( \delta \|g(z(x,1,t))\|^2 + \frac{c_s^2}{4\delta}(a-k)(ho\nabla u)(t) \right). \tag{2.26}$$

From **(A1)** we get

$$\begin{aligned}|I_4| &\leq \delta \int_{\mathcal{D}} |\nabla u_t|^2\, dx + \frac{1}{4\delta} \int_{\mathcal{D}} \Big( \int_0^t |h'(t-s)||\nabla u(t) - \nabla u(s)|\, ds \Big)^2 dx \\ &\leq \delta \|\nabla u_t\|^2 + \frac{1}{4\delta} \int_0^t (-h'(t-s))\, ds \int_{\mathcal{D}} \int_0^t (-h'(t-s))|\nabla u(t) - \nabla u(s)|^2\, ds\, dx \\ &\leq \delta \|\nabla u_t\|^2 - \frac{h(0)}{4\delta}(h'o\nabla u)(t).\end{aligned} \tag{2.27}$$

And recalling lemma 2.2.1, we get

$$\begin{aligned}|I_5| &\leq \delta \| u_t\|_2^2 + \frac{1}{4\delta} \int_{\mathcal{D}} \Big( \int_0^t |h'(t-s)||u(t) - u(s)|\, ds \Big)^2 dx \\ &\leq \delta c_s^2 \| \nabla u_t\|_2^2 - \frac{h(0)c_s^2}{4\delta}(h'o\nabla u)(t).\end{aligned} \tag{2.28}$$

Then, we completes the proof by substituting (2.24)-(2.28) in (2.23) . □

Now, for $N$, $\varepsilon_1 > 0$, we introduce the following functional

$$F(t) = NE(t) + \varepsilon_1 \phi(t) + \psi(t) + I(t). \tag{2.29}$$

**Lemma 2.4.4.** *Let $(u, z)$ be a solution of the problem (2.3). Assume that* **(A1)**- **(A4)** *hold then $F(t)$ satisfies,*
*i)*

$$F'(t) \leq -mE(t) + c\Big[\|g(z(x,1,t))\|^2 + \int_{\mathcal{D}} |u(x,t)g(z(x,1,t))|\, dx\Big], \tag{2.30}$$

*with $m$, $c$ are non-negative constants.*
*ii) $F(t)$ is equivalent to $E(t)$.*

*Proof.* By using ii) of lemmas 2.3.1-2.4.3 and by **(A1)**, we obtain

$$\begin{aligned} F'(t) \leq & -b_1 \int_{\mathcal{D}} z(x,1,t)g(z(x,1,t))\, dx - b_2 \int_{\mathcal{D}} u_t(x,t)g(u_t(x,t))\, dx \\ & - 2\tau(t)e^{-2\tau_1} \int_{\mathcal{D}} \int_0^1 G(z(x,\rho,t))\, d\rho\, dx - b_3 M(\|\nabla u\|^2)\|\nabla u\|^2 \\ & - b_4\|u_t\|_2^2 - b_5\|\nabla u_t\|^2 - b_6\|\nabla u(t)\|^2 - b_7(ho\nabla u)(t) \\ & + \mu\delta\|g(z(x,1,t))\|^2 - \varepsilon_1\mu \int_{\mathcal{D}} u(x,t)g(z(x,1,t))\, dx \end{aligned}$$

for $t \geq t_0 > 0$ where

$$\begin{aligned} b_1 &= N\beta + \alpha_1(1-d)e^{-2\tau_1} > 0, \\ b_2 &= N\lambda - \alpha_2, \\ b_3 &= \varepsilon_1 - \delta(a-k), \\ b_4 &= h_0 - \varepsilon_1, \\ b_5 &= h_0 - \delta(1+c_s^2) - \varepsilon_1, \\ b_6 &= \frac{Nh_1}{2} - \varepsilon_1(1+\eta)(a-k) - 2\delta(a-k)^2, \\ b_7 &= \zeta\left(\frac{N}{2} - \frac{h(0)}{4\delta}\left(1+c_s^2\right)\right) - \left(\frac{\varepsilon_1}{4\eta} + \frac{M_0}{4\delta} + \left(2\delta + \frac{1}{4\delta} + \mu\frac{c_s^2}{4\delta}\right)(a-k)\right), \\ h_0 &= \int_0^{t_0} h(s)ds \text{ and } h_1 = \min\{h(t), t \geq t_0 > 0\}. \end{aligned}$$

We take $h_0 > \varepsilon_1$ and $\delta$ is non-negative reel sufficiently small such that $b_3$, $b_4$ and $b_5$ are non-negative. After that we choose $N$ large enough such that $b_2$, $b_6$ and $b_7$ are non-negative.
Hence

$$\begin{aligned} F'(t) &\leq -b_3 M(\|\nabla u\|^2)\|\nabla u\|^2 - b_4\|u_t\|_2^2 - b_5\|\nabla u_t\|^2 - b_7(ho\nabla u)(t) \\ &- 2\tau(t)e^{-2\tau_1}\int_{\mathcal{D}}\int_0^1 G(z(x,\rho,t))\,d\rho\,dx \\ &+ \mu\delta\|g(z(x,1,t))\|^2 + \mu\varepsilon_1\int_{\mathcal{D}}|u(x,t)g(z(x,1,t))|\,dx \\ &\leq -mE(t) + c\Big[\|g(z(x,1,t))\|^2 + \int_{\mathcal{D}}|u(x,t)g(z(x,1,t))|\,dx\Big] \end{aligned}$$

where $m = 2\min\{b_3, \frac{e^{-2\tau_1}}{\xi}, b_5, b_4, b_7\}$ and $c = \mu\max\{\delta, \varepsilon_1\}$.
To show that $F(t)$ is equivalent to $E(t)$, we prove that there exist two positive constants $\lambda_1$ and $\lambda_2$ such that

$$\lambda_1 E(t) \leq F(t) \leq \lambda_2 E(t). \tag{2.31}$$

By recalling i) of lemmas 2.4.1-2.4.3, we obtain $\lambda_3$ non-negative constant depending of $\varepsilon_1$, $a$, $c_s$, $E(0)$, $k$, $\xi$ such that

$$\Big|F(t) - NE(t)\Big| \leq \lambda_3 E(t).$$

By choosing $N$ large enough such that $\lambda_1 = N - \lambda_3$ non-negative, This finishes the proof. □

### 2.4.1 Proof of Theorem 2.3.1

In the same way as in Mezouar and Boulaaras **[51]**, we consider the following partition of $\mathcal{D}$

$$\mathcal{D}_1 = \{x \in \mathcal{D} :\ |z(x,1,t)| \leq \varepsilon\}, \quad \mathcal{D}_2 = \{x \in \mathcal{D} :\ |z(x,1,t)| > \varepsilon\}.$$

By using Young's inequality and from (2.5), (2.6) and lemma 2.2.1, we find

$$\begin{aligned}\int_{\mathcal{D}} |ug(z(x,1,t))|\,dx + \|g(z(x,1,t))\|_2^2 &\leq \delta\|u\|_2^2 + \left(\frac{1}{4\delta}+1\right)\|g(z(x,1,t))\|_2^2 \\ &\leq \left(\frac{1}{4\delta}+1\right)\int_{\mathcal{D}_1} H^{-1}\left(z(x,1,t)g\left(z(x,1,t)\right)\right)\,dx \\ &+ \left(\frac{1}{4\delta}+1\right)c_2\int_{\mathcal{D}_2} z(x,1,t)g(z(x,1,t))\,dx \\ &+ \delta C_s^2\|\nabla u\|_2^2 \\ &\leq c_\delta \int_{\mathcal{D}_1} H^{-1}\left(z(x,1,t)g\left(z(x,1,t)\right)\right)\,dx \\ &+ \frac{2\delta C_s^2}{a}E(t) - C_\delta E'(t)\end{aligned} \tag{2.32}$$

where $H^{-1}$ is inverse function of $H$.

By substituting in (2.30) we obtain

$$F'(t) \leq -\left(m - \frac{2\delta C_s^2}{a}\right)E(t) - C_\delta E'(t) + c_\delta \int_{\mathcal{D}_1} H^{-1}(z(x,1,t)g(z(x,1,t)))\,dx.$$

We put

$$\beta_1 = m - \frac{2\delta C_s^2}{a} > 0, \text{ for } \delta \text{ small enough}$$

and

$$J(t) = F(t) + C_\delta E(t).$$

So

$$J'(t) \leq -\beta_1 E(t) + c_\delta \int_{\mathcal{D}_1} H^{-1}(z(x,1,t)g(z(x,1,t)))\,dx \tag{2.33}$$

and

$$J(t) \text{ still equivalent to } E(t).$$

Now, under (2.5) we distinguish two cases corresponding to linearity of $H$

- First Case: $H$ is linear on $[0, \varepsilon]$.
  From Lemma 2.3.1, we deduce that

$$J'(t) \leq -\beta_1 E(t) - cE'(t).$$

Thus

$$(J(t) + cE(t))' \leq -\beta_1 E(t).$$

We put $L = J + cE$ which still equivalent to $E$ and from the above estimation $L(t)$ satisfies

$$L(t) \leq L(0)e^{-c't}$$

Hence,

$$E(t) \leq C(E(0))e^{-c't}.$$

Since $H$ is linear and from (2.11) we get $H_1(t) = -c' ln\, t$ then $H_1^{-1}(t) = e^{-c't}$, this implies that

$$E(t) \leq w_3 H_1^{-1}(w_1 t)$$

- Second Case: $H$ is nonlinear on $[0, \varepsilon]$, in this case we benefit the concavity of $H^{-1}$ and we apply Jensen's inequality to obtain

$$H^{-1}\left(\frac{1}{|\mathcal{D}_1|}\int_{\mathcal{D}_1} z(x,1,t)g(z(x,1,t))\,dx\right) \geq c\int_{\mathcal{D}_1} H^{-1}\left(z(x,1,t)g(z(x,1,t))\right)\,dx.$$

Hence (2.33) leads

$$J'(t) \leq -\beta_1 E(t) + cH^{-1}\left(\frac{1}{|\mathcal{D}_1|}\int_{\mathcal{D}_1} z(x,1,t)g(z(x,1,t))\,dx\right). \tag{2.34}$$

Now, we define the function $J_0$ by:

$$J_0(t) := H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) J(t) + \beta_2 E(t)$$

with $\varepsilon_0 < \varepsilon$ and $\beta_2 > 0$. Then, we can find two non-negative constants $\gamma_1$, $\gamma_2$ such that

$$\gamma_1 J_0(t) \leq E(t) \leq \gamma_2 J_0(t) \Leftrightarrow E(t) \sim J_0(t). \tag{2.35}$$

Taking a derivative of $J_0$, we obtain

$$J_0'(t) = \varepsilon_0 \frac{E'(t)}{E(0)} H''\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) J(t) + H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) J'(t) + \beta_2 E'(t).$$

Since $E' \leq 0$, $H' > 0$, $H'' > 0$ on $(0, \varepsilon]$ and (2.34) we get

$$\begin{aligned} J_0'(t) \leq & \, cH'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) H^{-1}\left(\frac{1}{|\mathcal{D}_1|}\int_{\mathcal{D}_1} z(x,1,t) g(z(x,1,t))\, dx\right) \\ & + \beta_2 E'(t) - \beta_1 E(t) H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right), \end{aligned}$$

According to the remark 2.2.1, we will get

$$\begin{aligned} H^{-1}\left(\frac{1}{|\mathcal{D}_1|}\int_{\mathcal{D}_1} z(x,1,t) g(z(x,1,t))\, dx\right) H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) & \leq H^*\left(H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\right) \\ & \quad + \frac{1}{|\mathcal{D}_1|}\int_{\mathcal{D}_1} z(x,1,t) g(z(x,1,t))\, dx \\ & \leq \varepsilon_0 \frac{E(t)}{E(0)} H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) \\ & \quad + \frac{1}{|\mathcal{D}_1|}\int_{\mathcal{D}_1} z(x,1,t) g(z(x,1,t))\, dx. \end{aligned}$$

Hence

$$\begin{aligned} J_0'(t) \leq & -\left(\beta_1 E(0) - c\varepsilon_0\right) \frac{E(t)}{E(0)} H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) \\ & + \frac{c}{|\mathcal{D}_1|}\int_{\mathcal{D}_1} z(x,1,t) g(z(x,1,t))\, dx + \beta_2 E'(t). \end{aligned}$$

From lemma 2.3.1, we get

$$J_0'(t) \leq -\left(\beta_1 E(0) - c\varepsilon_0\right) \frac{E(t)}{E(0)} H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + \left(\beta_2 - \frac{c}{|\mathcal{D}_1|\beta}\right) E'(t).$$

We put $\beta_2 = \frac{c}{|\mathcal{D}_1|\beta}$. By choosing $\varepsilon_0$ small enough, we have

$$J_0'(t) \leq -w \frac{E(t)}{E(0)} H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) \tag{2.36}$$

where $w = \beta_1 E(0) - c\varepsilon_0 > 0$.

Now, let $J_1(t) = \frac{\gamma_1 J_0(t)}{E(0)}$ still equivalent to $E(t)$.

As $tH'(\varepsilon_0 t)$ a positive increasing function on $(0, 1]$ then from (2.35), (2.36) gives

$$J_1'(t) \leq -w_1 J_1(t) H'\left(\varepsilon_0 J_1(t)\right).$$

Therefore, from (2.11), we deduce that

$$J_1'(t) \leq w_1 \frac{1}{H_1'\left(J_1(t)\right)},$$

which leads to

$$\Big[H_1(J_1(t))\Big]' \leq w_1.$$

Then by a simple integration we obtain

$$H_1\left(J_1(t)\right) \leq w_1 t + H_1\left(J_1(0)\right),$$

thus,

$$J_1(t) \leq H_1^{-1}(w_1 t + w_2). \tag{2.37}$$

Finally, from the equivalence between $J_1(t)$ and $E(t)$, (2.10) is deduce from (2.37).

# Chapter 3

# Local well-posedness and blow up of solution to a Logarithmic quasilinear equation with delay term

## 3.1 Introduction

### 3.1.1 The model

In this chapter we investigate the following initial boundary value problem:

$$\begin{cases} |u_t|^l u_{tt} - div(\rho(|\nabla u|^2)\nabla u) - \Delta u_{tt} + \mu_1 |u_t|^{m-1}u_t & \\ +\mu_2 |u_t(t-\tau)|^{m-1}u_t(t-\tau) = u|u|^{p-2}\ln|u|^k, & \text{in } \mathcal{D}\times(0,T), \\ u(x,t) = 0 & \text{on } \partial\mathcal{D}\times(0,T), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \mathcal{D}, \\ u_t(x,t-\tau) = f_0(x,t-\tau) & \text{in}\mathcal{D}\times]0,\tau[ \end{cases} \tag{3.1}$$

where $\mathcal{D}$ is a bounded domain with a sufficiently smooth boundary in $\mathbb{R}^n$, $(n \geq 1)$, $l$, $m$, $\mu_1$, $\mu_2$, $p$, $k$ are positive constants, $\tau > 0$ is a time delay and $(u_0, u_1, f_0)$ are the initial data in a suitable function space. Here, $u(x,t)$ represent displacement. $-\Delta u_{tt}$ and $|u_t|^{m-1}u_t$ are the dispersion term and the nonlinear damping term respectively.

In the presence of viscoelastic term on wave equation many authors studied the following problem:

$$\begin{cases} |u_t|^l u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t h(t-s)\Delta u(s)\, ds + f(u_t) = g(u) & \text{in } \mathcal{D}\times]0,+\infty[, \\ u(x,t) = 0 & \text{on } \partial\mathcal{D}\times[0,+\infty[, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \mathcal{D}. \end{cases} \tag{3.2}$$

In the absence of source term $\big(i.e.\, g(u) = 0\big)$, Hang and Wang [35] obtained the global existence and established uniform decay results for $f(u_t) = u_t(x,t)$ and for $f(u_t) = -\gamma\Delta u_t$ Cavalcanti et al [15] proved a global existence of weak solutions and showed the relaxation function decays exponentially. In the absence of damping term $\big(i.e.\, f(u_t) = 0\big)$ and for $g(u) = b|u|^{p-2}u$ with $b > 0$ and $p > 2$, Liu [45] discussed the general decay result for the global solution with positive energy that blow up in finite time. However, Messaoudi and Tatar [50] used the potential well method, they studied that the viscoelastic term is strong enough to ensure the global existence and uniform decay of solutions provided that the initial data are in same stable set. Song [67] considered the problem (3.2) without dispersion term ($i.e. -\Delta u_{tt} = 0$), for $f(u_t) = |u|_t^{m-2}u_t$ and $g(u) = |u|^{p-2}u$. He studied the global nonexistence with positive initial energy solution. The same issue was treated by Messaoudi [46] for $l = 0$, he showed a blow up result for the solution with negative initial energy if $p > m$, and a global result for $p \le m$.
Pişkin [60] studied the following non linear hyperbolic equation

$$\begin{cases} u_{tt} - div(|\nabla u|^m \nabla u) - \Delta u_t + |u_t|^{q-1}u_t = u|u|^{p-1}, & \text{in } \mathcal{D},\, t > 0, \\ u(x,t) = 0 & \text{on } \partial\mathcal{D},\, t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \mathcal{D}. \end{cases} \tag{3.3}$$

He investigated the global existence, decay and blow up of solution.

In the absence of dispersion term, Park [57] treated problem (3.1) for $l = 0$, $\rho \equiv 1$, $m = 1$ and $p = 2$

$$\begin{cases} u_{tt} - \triangle u + \mu_1 u_t + \mu_2 u_t(t-\tau) = u \ln |u|^k, & \text{in } \mathcal{D} \times (0,T), \\ u(x,t) = 0 & \text{on } \partial\mathcal{D} \times (0,T), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \mathcal{D}, \\ u_t(x,t-\tau) = f_0(x,t-\tau) & \text{in} \mathcal{D} \times ]0,\tau[. \end{cases} \tag{3.4}$$

She showed the local and global existence of solutions using Faedo-Galerkin 's method and the logarithmic Sobolev Inequality and they establish the de cay rates and infinite time blow-up for the solution using the potential well and perturbed energy methods.

In **[37]**, Kafini and Messaoudi have treated a similar problem with the absence of viscoelastic term for $l = 0$, $m = 1$ and $\rho \equiv 1$

$$\begin{cases} u_{tt} - \triangle u + \mu_1 u_t + \mu_2 u_t(t-\tau) = u|u|^{p-2} \ln |u|^k, & \text{in } \mathcal{D} \times (0,T), \\ u(x,t) = 0 & \text{on } \partial\mathcal{D} \times (0,T), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \mathcal{D}, \\ u_t(x,t-\tau) = f_0(x,t-\tau) & \text{in} \mathcal{D} \times ]0,\tau[. \end{cases} \tag{3.5}$$

They proved the local existence result by using the semigroup theory and they showed a blow up result for the solution with negative initial energy.

In **[51]**, Mezouar and Boulaaras have studied the global existence and stability of solutions of a nonlinear viscoelastic kirchhoff equation in a bounded domain with a time varying delay in the weakly nonlinear internal feedback in suitable Sobolev spaces by means of the energy method combined with Faedo-Galarkin procedure with respect to the condition of the weight of the delay term:

$$\begin{cases} |u_t|^l u_{tt} - M(\|\nabla u\|^2)\Delta u - \Delta u_{tt} + \int_0^t h(t-s)\Delta u(s)\,ds & \\ +\mu_1 g_1(u_t(x,t)) + \mu_2 g_2(u_t(x,t-\tau(t))) = 0 & \text{in } \mathcal{D} \times ]0,+\infty[, \\ u(x,t) = 0 & \text{on } \partial\mathcal{D} \times [0,+\infty[, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \mathcal{D}, \\ u_t(x,t-\tau(0)) = f_0(x,t-\tau(0)) & \text{in } \mathcal{D} \times ]0,\tau(0)[. \end{cases} \tag{3.6}$$

After that in **[52]** they studied with the collaboration of Allahem the following problem:

$$\begin{cases} |u_t|^l u_{tt} - M(\|\nabla u\|^2)\Delta u - \Delta u_{tt} + \int_0^t h(t-s)\Delta u(s)\,ds \\ +\mu_1 g_1(u_t(x,t)) + \mu_2 g_2(u_t(x,t-\tau(t))) = ku\ln|u| & \text{in } \mathcal{D}\times]0,+\infty[, \\ u(x,t) = 0 & \text{on } \partial\mathcal{D}\times[0,+\infty[, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \mathcal{D}, \\ u_t(x,t-\tau(0)) = f_0(x,t-\tau(0)) & \text{in } \mathcal{D}\times]0,\tau(0)[. \end{cases} \tag{3.7}$$

Note that, the naturally appearance of logarithmic nonlinearity in inflation cosmology and supersymmetric filed theories, quantum mechanics and nuclear physics (see **[10]**, **[31]**) makes of it to be of much use in physics. Such kind of problem can be applied in many different areas of physics such as nuclear physics, optics and geophysics (see **[9]**).
Recently, Piskin and Irikil **[61]** show the finite-time blow up of solutions negative initial energy to the following problem with $p-$laplacian and logarithmic nonlinearity:

$$\begin{cases} u_{tt} - \Delta u - div(|\nabla u|^{p-2}\nabla u) - \Delta u_t + |u_t|^{m-2}u_t = u|u|^{p-2}\ln|u|, & \text{in } \mathcal{D}\times(0,T), \\ u(x,t) = 0 & \text{on } \partial\mathcal{D}\times(0,T), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \mathcal{D}. \end{cases} \tag{3.8}$$

The organization of this chapter as follows: In Section 2, we give some hypotheses and state local existence. We finish by Section 3 when we prove the blow up of solutions.

## 3.2 Preliminaries

Throughout this paper, we denote by $\|.\|$ and $\|.\|_p$ norms of $L^2(\mathcal{D})$ and $L^p(\mathcal{D})$ respectively. The prime $'$ and the subscript $t$ will denote time differentiation.

Now we introduce, as in the work of in Nicaise and Pignotti **[55]**, the new variable

$$z(x,\rho,t) = u_t(x,t-\tau\rho),\ x\in\mathcal{D},\quad \rho\in(0,1), t>0.$$

Then, we have

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0,\ \text{in } \mathcal{D}\times(0,1)\times(0,+\infty). \tag{3.9}$$

Therefore, the problem (3.1) is equivalent to

$$\begin{cases} |u_t|^l u_{tt} - div(\rho(|\nabla u|^2)\nabla u) - \Delta u_{tt} + \mu_1|u_t|^{m-1}u_t & \\ +\mu_2|z(x,1,t)|^{m-1}z(x,1,t) = u|u|^{p-2}\ln|u|^k, & \text{in } \mathcal{D}\times]0,+\infty[, \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, & \text{in } \mathcal{D}\times]0,1[\times]0,+\infty[ \\ u(x,t) = 0, & \text{on } \partial\mathcal{D}\times[0,\infty[ \\ z(x,0,t) = u_t(x,t), & \text{on } \mathcal{D}\times[0,\infty[ \\ u(x,0) = u_0(x),\quad u_t(x,0) = u_1(x), & \text{in } \mathcal{D} \\ z(x,\rho,0) = f_0(x,-\rho\tau), & \text{in } \mathcal{D}\times]0,1[. \end{cases} \tag{3.10}$$

To state and prove our result, we need the following assumptions.

**(A1)** Assume that $l$ satisfies

$$\begin{cases} 0 < l \le \frac{2}{n-2} & \text{if } n\ge 3 \\ 0 < l < \infty & \text{if } n = 1,2. \end{cases}$$

**(A2)** Let $\rho(s) = b_1 + b_2 s^q$, $q\ge 0$ where $b_1, b_2$ are non-negative constants and $b_1 + b_2 > 0$, for $s>0$.

**(A3)**

$$\begin{cases} 2\le p < \frac{n}{n-2} & \text{if } n\ge 3 \\ 2\le p<\infty & \text{if } n=1,2. \end{cases} \tag{3.11}$$

**(A4)** $\mu_2 \leq \mu_1$.

**(A5)** $k \leq \frac{\pi p b_1}{c} e^{\frac{2(np+2)}{np}}$,

where $c = \frac{\|\nabla u\|_p^p}{\|\nabla u\|^2}$ is a positive constant.
We define the energy associated to the solution of system (3.10) by

$$\begin{aligned} E(t) = & \frac{1}{l+2}\|u_t\|_{l+2}^{l+2} + \frac{1}{2}\int_{\mathcal{D}} P(|\nabla u|^2)\, dx + \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{k}{p^2}\|u\|_p^p \\ & + \xi \int_{\mathcal{D}}\int_0^1 z^{m+1}(x,\rho,t)\, d\rho\, dx - \frac{1}{p}\int_{\mathcal{D}} |u|^p \ln |u|^k\, dx \end{aligned} \tag{3.12}$$

where $\xi$ is a positive constant such that

$$\tau \frac{\mu_2 m}{m+1} < \xi < \tau(\mu_1 - \frac{\mu_2}{m+1})$$

and $P(s) = \displaystyle\int_0^s \rho(\xi) d\xi$, $s \geq 0$.
Now, we state the following lemmas needed later.

**Lemma 3.2.1.** *(Sobolev-Poincaré's inequality ) Let $q$ be a number with*

$$2 \leq q < +\infty (n = 1, 2) \;\; or \;\; 2 \leq q \leq 2n/(n-2)(n \geq 3),$$

*then there exists a constant $C_s = C_s(\mathcal{D}, q)$ such that*

$$\|u\|_q \leq C_s \|\nabla u\| \quad for \;\; u \in H_0^1(\mathcal{D}).$$

**Lemma 3.2.2.** *(The general logarithmic Sobolev inequality )[18, 1]*

$$2\int_{\mathcal{D}} |u(x)|^p \log\Big(\frac{|u(x)|}{\|u\|_{L^p(\mathcal{D})}}\Big) dx + n(1+\log a)\|u\|_{L^p(\mathcal{D})}^p \leq \frac{a^2}{\pi}\|\nabla u\|_{L^p(\mathcal{D})}^p, \; \forall u \in H_0^1(\mathcal{D})$$

*where $a$ is any positive number and $\mathcal{D} \subset \mathbb{R}^n$.*

**Lemma 3.2.3.** *[37] There exists a positive constant $C > 0$ depending on $\mathcal{D}$ only such that*

$$\|u\|_p^p \leq C\Big[\int_{\mathcal{D}} |u|^p ln|u|^k dx + \|\nabla u\|^2\Big],$$

*for any $u \in L^p(\mathcal{D})$, provided that $\int_{\mathcal{D}} |u|^p ln|u|^k dx \geq 0$.*

**Lemma 3.2.4.** *[37] There exists a positive constant $C > 0$ depending on $\mathcal{D}$ only such that*

$$\|u\|_p^s \leq C\Big[\|u\|_p^p + \|\nabla u\|^2\Big],$$

*for any $u \in L^p(\mathcal{D})$ and $2 \leq s \leq p$.*

**Lemma 3.2.5.** *[19] ( logarithmic Gronwall inequality)*
*Let $c > 0$, $y \in L^1(0,\, T,\, \mathbb{R}^+)$ and*

$$w:\ [0,T] \to [1,\infty)$$

*satisfies*

$$w(t) \leq c\big(1 + \int_0^t y(s)w(s)ln(w(s))ds\big),\ \forall t \in (0,T].$$

*Then*

$$w(t) \leq c\exp\Big(c\int_0^t y(s)ds\Big),\ \forall t \in (0,T].$$

## 3.3 Local existence of solution

**Theorem 3.3.1. (Local existence)** *Let $u_0 \in W_0^{1,2(q+1)}(\mathcal{D}) \cap L^{p+1}(\mathcal{D})$ and $u_1 \in L^2(\mathcal{D})$. Assume that* **(A1)**-**(A5)** *hold. Then, for some $T_m > 0$, the problem (3.1) admits a weak solution*

$$u \in C([0,T_m); W^{1,2(q+1)}(\mathcal{D}) \cap L^{p+1}(\mathcal{D})),$$

*and*

$$u_t \in C([0,T_m);L^2(\mathcal{D})) \cap L^{m+1}(\mathcal{D}\times(0,T_m)).$$

we proof the local existence after showing that the energy functional (3.12) is uniformly bounded by $E(0)$ and is decreasing in the following lemma.

**Lemma 3.3.1.** *Let $u$ is a solution of the problem (3.1). Then, there exists a positive constance $C_0$ such that the energy functional (3.12) satisfies*

$$E'(t) \le -C_0\left[\|u_t(t)\|_{m+1}^{m+1} + \|z(x,1,t)\|_{m+1}^{m+1}\right] \le 0. \tag{3.13}$$

*Proof.* By multiplying the first equation in (3.10) by $u_t$, integrating by parts over $\mathcal{D}$, we obtain

$$\frac{d}{dt}\Big(\frac{1}{l+2}\|u_t\|_{l+2}^{l+2} + \frac{1}{2}\Big(b_1\|\nabla u\|^2 + \frac{b_2}{q+1}\|\nabla u\|_{2(q+1)}^{2(q+1)}\Big) + \frac{1}{2}\|\nabla u_t\|_2^2\Big) + \mu_1\|u_t\|_{m+1}^{m+1} + \mu_2\int_{\mathcal{D}} u_t|z(x,1,t)|^{m-1}z(x,1,t)\,dx = \int_{\mathcal{D}} u_t u|u|^{p-2}\ln|u|^k\,dx. \tag{3.14}$$

We have

$$\begin{aligned}\int_{\mathcal{D}} u_t u|u|^{p-2}\ln|u|^k\,dx &= \frac{d}{dt}\Big[\frac{1}{p}\int_{\mathcal{D}}|u|^p\ln|u|^k\,dx\Big] - \frac{k}{p}\int_{\mathcal{D}} u_t u|u|^{p-2}\,dx\\ &= \frac{d}{dt}\Big[\frac{1}{p}\int_{\mathcal{D}}|u|^p\ln|u|^k\,dx - \frac{k}{p^2}\|u\|_p^p\Big].\end{aligned}$$

Consequently, equation (3.14) becomes

$$\frac{d}{dt}\Big(\frac{1}{l+2}\|u_t\|_{l+2}^{l+2} + \frac{1}{2}\int_{\mathcal{D}} P(|\nabla u|^2)\,dx + \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{k}{p^2}\|u\|_p^p - \frac{1}{p}\int_{\mathcal{D}}|u|^p\ln|u|^k\,dx\Big) = -\mu_1\|u_t\|_{m+1}^{m+1} - \mu_2\int_{\mathcal{D}} u_t|z(x,1,t)|^{m-1}z(x,1,t)\,dx. \tag{3.15}$$

We multiply the second equation in (3.10) by $\xi|z(x,\rho,t)|^{m-1}z(x,\rho,t)$, we integrate the result over $\mathcal{D}\times(0,1)$, to obtain

$$\begin{aligned}\xi\int_{\mathcal{D}}\int_0^1 z_t(x,\rho,t)|z(x,\rho,t)|^{m-1}z(x,\rho,t)\,d\rho\,dx &= -\frac{\xi}{\tau}\int_{\mathcal{D}}\int_0^1 z_\rho(x,\rho,t)|z(x,\rho,t)|^{m-1}z(x,\rho,t)\,d\rho\,dx\\ &= -\frac{\xi}{(m+1)\tau}\int_{\mathcal{D}}\int_0^1\frac{\partial}{\partial\rho}\Big(|z(x,\rho,t)|^{m+1}\Big)\,d\rho\,dx\\ &= -\frac{\xi}{(m+1)\tau}\Big(\|z(x,1,t)\|_{m+1}^{m+1}-\|z(x,0,t)\|_{m+1}^{m+1}\Big).\end{aligned}$$

Hence

$$\xi\frac{d}{dt}\int_{\mathcal{D}}\int_0^1|z(x,\rho,t)|^{m+1}\,d\rho\,dx = -\frac{\xi}{\tau}\Big(\|z(x,1,t)\|_{m+1}^{m+1}-\|u_t(x,t)\|_{m+1}^{m+1}\Big). \tag{3.16}$$

Summing up (3.15)-(3.16) and using Young's inequality to estimate the last term in (3.15) the assertion (3.13) is established where $C_0=\min\{\mu_1-\frac{\xi}{\tau}-\frac{\mu_2}{m+1},\frac{\xi}{\tau}-\frac{\mu_2 m}{m+1}\}$ positive. □

*Proof.* **Local Existence**
Let $u_0\in W_0^{1,2(q+1)}(\mathcal{D})$, $u_1\in H_0^1(\mathcal{D})$ and $f_0\in H_0^1(\mathcal{D},H^1(0,1))$ we will use Faedo-Galerkin method to prove the existence of solution of (3.1).
Let $T>0$ and $\{w^j\}_{j\in\mathbb{N}}$ be a basis of $W_0^{1,2(q+1)}(\mathcal{D})$ and $V_r=Vect\,\{w^j,\ j=\overline{1,r}\}$.
We construct an approximate solution leads to this basis as follow

$$u^r(t)=\sum_{j=1}^{r}c^{rj}(t)w^j(x).$$

Now we define, for $1\le j\le r$, the sequence $\phi^j(x,\rho)$ as follows:

$$\phi^j(x,0)=w^j.$$

Then, we may extend $\phi^j(x,0)$ by $\phi^j(x,\rho)$ over $L^2(\mathcal{D}\times(0,1))$ such that $(\phi^j)_j$ forms a basis of $L^2(\mathcal{D},H^1(0,1))$ and $Z_r=Vect\{\phi^j;\ j=\overline{1,r}\}$; so the approximate solutions $z_r$ take the following form

$$z^r(t) = \sum_{j=1}^{r} d^{rj}(t)\phi^j(x).$$

Hence $(u^k, z^k)$ satisfying, for all $1 \leq j \leq r$, the following ordinary differential equations:

$$\begin{cases} (|u_t^r|^l u_{tt}^r, w^j) - (div(\rho(|\nabla u^r|^2)\nabla u^r), w^j) + (\nabla u_{tt}^r, \nabla w^j) + \mu_1(|u_t^r|^{m-1}u_t^r, w^j) \\ +(\mu_2|z^r(x,1,t)|^{m-1}z^r(x,1,t), w^j) = (u^r|u^r|^{p-2}\ln|u^r|^k, w^j), \\ (\tau z_t^r + z_\rho^r, \phi^j) = 0, \end{cases} \tag{3.17}$$

corresponding with the following initials conditions:

$$\begin{cases} z^r(x,0,t) = u_t^r(x,t), \\ u^r(0) = u_0^r = \sum_{j=1}^{r}(u_0, w^j)w^j \to u_0, \text{ in } W^{1,2(q+1)}(\mathcal{D}), \\ u_t^r(0) = u_1^r = \sum_{j=1}^{r}(u_1, w^j)w^j \to u_1, \text{ in } H_0^1(\mathcal{D}), \\ z^r(\rho,0) = z_0^r = \sum_{j=1}^{r}(f_0, \phi^j)\phi^j \to f_0, \text{ in } H_0^1(\mathcal{D}, H^1(0,1)). \end{cases} \tag{3.18}$$

From the generalized Hölder inequality and according to Sobolev embedding this term $(|u_t^r|^l u_{tt}^r, w^j)$ make sense.

The ODE theory ensure the existence and uniquess of solution of system (3.17), the solution is $(c^{rj},\, d^{rj})$ is twice differentiating in $[0,\, t_r)$, for some $0 < t_r < T$. We can extend $t_r$ to $T$ by this first a priori estimate.

From lemma 3.3.1, we have

$$\Big(E^r(t)\Big)' \leq -C_0\Big[\|u_t^r(t)\|_{m+1}^{m+1} + \|z^r(x,1,t)\|_{m+1}^{m+1}\Big] \leq 0.$$

By integration, we get

$$E^r(t) - E^r(0) \leq -C_0 \int_0^t \Big[\|u_t^r(s)\|_{m+1}^{m+1} + \|z^r(x,1,s)\|_{m+1}^{m+1}\Big] ds.$$

Then, we obtain

$$E^r(t) + C_0 \int_0^t \Big[\|u_t^r(s)\|_{m+1}^{m+1} + \|z^r(x,1,s)\|_{m+1}^{m+1}\Big] ds \leq E^r(0).$$

the (3.18) implies that

$$E^r(t) + C_0 \int_0^t \Big[\|u_t^r(s)\|_{m+1}^{m+1} + \|z^r(x,1,s)\|_{m+1}^{m+1}\Big] ds \leq c_1 \tag{3.19}$$

where

$$\begin{aligned} E^r(t) = {} & \frac{1}{l+2}\|u_t^r\|_{l+2}^{l+2} + \frac{1}{2}\Big(b_1\|\nabla u^r\|^2 + \frac{b_2}{q+1}\|\nabla u^r\|_{2(q+1)}^{2(q+1)}\Big) + \frac{1}{2}\|\nabla u_t^r\|^2 + \frac{k}{p^2}\|u^r\|_p^p \\ & + \xi \int_{\mathcal{D}}\int_0^1 |z^r(x,\rho,t)|^{m+1}\, d\rho\, dx - \frac{1}{p}\int_{\mathcal{D}} |u^r|^p \ln |u^r|^k\, dx. \end{aligned}$$

Consequently

$$\begin{aligned} & \frac{1}{l+2}\|u_t^r\|_{l+2}^{l+2} + \frac{1}{2}\Big(b_1\|\nabla u^r\|^2 + \frac{b_2}{q+1}\|\nabla u^r\|_{2(q+1)}^{2(q+1)}\Big) + \frac{1}{2}\|\nabla u_t^r\|^2 + \frac{k}{p^2}\|u^r\|_p^p \\ & + \xi \int_{\mathcal{D}}\int_0^1 |z^r(x,\rho,t)|^{m+1}\, d\rho\, dx + C_0\int_0^t \Big[\|u_t^r(s)\|_{m+1}^{m+1} + \|z(x,1,s)\|_{m+1}^{m+1}\Big] ds \\ & \leq c_1 + \frac{1}{p}\int_{\mathcal{D}} |u^r|^p \ln |u^r|^k\, dx. \end{aligned}$$

Now by applying Lemma 3.2.2 we get

$$\begin{aligned} \frac{1}{p}\int_{\mathcal{D}} |u^r|^p \ln |u^r|^k\, dx &= \frac{k}{p}\int_{\mathcal{D}} |u^r|^p \ln |u^r|\, dx \\ &\leq \frac{k}{p}\left[\frac{1}{p}\|u^r\|_p^p \ln \|u^r\|_p + \frac{a^2}{2\pi}\|\nabla u^r\|_p^p - \frac{n}{2}(1+\ln\, a)\|u^r\|_p^p\right]. \end{aligned}$$

We introduce this estimation on the above calculus, we obtain

$$\begin{aligned}&\frac{1}{l+2}\|u_t^r\|_{l+2}^{l+2}+\frac{1}{2}\Big(b_1\|\nabla u^r\|^2+\frac{b_2}{q+1}\|\nabla u^r\|_{2(q+1)}^{2(q+1)}\Big)+\frac{1}{2}\|\nabla u_t^r\|^2\\&+\frac{k}{p}\Big(\frac{1}{p}+\frac{n}{2}(1+\ln\,a)\Big)\|u^r\|_p^p-\frac{a^2k}{2\pi p}\|\nabla u^r\|_p^p\le c_1+\frac{k}{p}\|u^r\|_p^p\ln\|u^r\|_p.\end{aligned} \tag{3.20}$$

As $2\le p$, hence $\|\nabla u^r\|^2\le C_{\mathcal{D}}\|\nabla u^r\|_p^p$ where $C_{\mathcal{D}}=|\mathcal{D}|^{\frac{p-2}{2p}}$.
So we choose $\theta$ a positive number such that

$$b_1<\theta\le-\frac{\left(b_1C_{\mathcal{D}}-\frac{a^2k}{\pi p}\right)\|\nabla u^r\|_p^p}{\|\nabla u^r\|^2-C_{\mathcal{D}}\|\nabla u^r\|_p^p}, \tag{3.21}$$

Such that

$$0<(b_1-\theta)\left(\|\nabla u^r\|^2-C_{\mathcal{D}}\|\nabla u^r\|_p^p\right)\le b_1\|\nabla u^r\|^2-\frac{a^2k}{\pi p}\|\nabla u^r\|_p^p.$$

Then (3.20) leads

$$\begin{aligned}&\frac{1}{l+2}\|u_t^r\|_{l+2}^{l+2}+\frac{b_2}{2(q+1)}\|\nabla u^r\|_{2(q+1)}^{2(q+1)}+\frac{1}{2}\|\nabla u_t^r\|^2\\&+\frac{k}{p}\Big(\frac{1}{p}+\frac{n}{2}(1+\ln\,a)\Big)\|u^r\|_p^p\le c_1+\frac{k}{p}\|u^r\|_p^p\ln\|u^r\|_p.\end{aligned} \tag{3.22}$$

Note that (3.21) holds true if

$$\frac{a^2k}{\pi p}<b_1\frac{\|\nabla u^r\|^2}{\|\nabla u^r\|_p^p}\Rightarrow a<\sqrt{\frac{\pi p b_1\|\nabla u^r\|^2}{k\|\nabla u^r\|_p^p}}. \tag{3.23}$$

Now, we choose a small enough such that

$$\frac{1}{p}+\frac{n}{2}(1+\ln a)>0\Rightarrow a<e^{-\frac{np+2}{np}}. \tag{3.24}$$

In conclusion, we will choose

$$e^{-\frac{np+2}{np}}<a<\sqrt{\frac{\pi p b_1\|\nabla u^r\|^2}{k\|\nabla u^r\|_p^p}}$$

and this selection is possible from **(A5)**.

Let us note that:

$$p\int_0^t |u^r(s)|^{p-1}u_s(s)ds = |u^r(t)|^p - |u^r(0)|^p$$

By integration over $\mathcal{D}$ we get

$$\|u^r(t)\|_p^p = \|u^r(0)\|_p^p + p\int_{\mathcal{D}}\int_0^t |u^r(s)|^{p-1}u_s(s)ds\,dx.$$

We apply the Young's inequality with exponents $\frac{p}{p-1}$ and $p$, we find

$$\|u^r(t)\|_p^p \leq \|u^r(0)\|_p^p + p\left[\int_0^t \delta_1\|u^r(s)\|_p^p ds + \frac{1}{4\delta_1}\int_0^t \|u_s(s)\|_p^p ds\right].$$

Under **(A3)** and lemma 4.2.1 this estimation becomes

$$\|u^r(t)\|_p^p \leq \|u^r(0)\|_p^p + p\left[\int_0^t \delta_1\|\nabla u^r(s)\|_p^p ds + \frac{C_s}{4\delta_1}\int_0^t \|\nabla u_s^r(s)\|^2 ds\right].$$

From (3.22), we obtain

$$\|u^r(t)\|_p^p \leq \|u^r(0)\|_p^p + pc\int_0^t \|\nabla u^r(s)\|_p^p \ln \|u^r(s)\|_p^p ds.$$

Applying lemma 3.2.5 , we get

$$\|u^r\|^p \leq Ce^{CT}. \tag{3.25}$$

So, from (3.19), we get the first estimate:

$$\begin{aligned}
&\|u_t^r\|_{l+2}^{l+2} + \|\nabla u^r\|^2 + \|\nabla u_t^r\|^2 + \|u^r\|_p^p + \|\nabla u^r\|_{2(q+1)}^{2(q+1)} \\
&\quad + C_0\int_0^t \left[\|u_t^r(s)\|_{m+1}^{m+1} + \|z^r(x,1,s)\|_{m+1}^{m+1}\right]ds \\
&\quad \leq c\left(1 + Ce^{CT}ln(Ce^{CT})\right) = A_1.
\end{aligned} \tag{3.26}$$

The estimate implies that the solution $(u^r, z^r)$ exists in $[0, T)$ and it yields

$$u^r \text{ is bounded in } L^\infty_{loc}\left(0, \infty, H^1(\mathcal{D})\right), \tag{3.27}$$

$$u^r_t \text{ is bounded in } L^\infty_{loc}\left(0, \infty, H^1_0(\mathcal{D})\right), \tag{3.28}$$

$$z^r(x, 1, t), u^r_t \text{ are bounded in } L^\infty_{loc}\left(0, \infty, L^{m+1}(\mathcal{D})\right), \tag{3.29}$$

$$u^r \text{ is bounded in } L^\infty_{loc}\left(0, \infty, W_0^{1,2(q+1)}(\mathcal{D})\right), \tag{3.30}$$

and once $A$ is a bounded operator from $W_0^{1,2(q+1)}(\mathcal{D}) \to (W_0^{1,2(q+1)}(\mathcal{D}))'$ defined by $A\omega = div\left(\rho(|\nabla\omega|^2)\nabla\omega\right)$. It follows from (3.27) that

$$A(u^r) \text{ is bounded in } L^\infty_{loc}\left([0, T]; (W_0^{1,2(q+1)}(\mathcal{D}))'\right). \tag{3.31}$$

Applying a similar priori estimate II to [64], we get
$u \in L^\infty\left([0, T]; W_0^{1,2(q+1)}(\mathcal{D}) \cap L^{p+1}(\mathcal{D})\right)$, $u_t \in L^\infty\left([0, T]; L^2(\mathcal{D})\right) \cap L^{m+1}\left(\mathcal{D} \times (0, T)\right)$.
Using a well-known result (Lemmas 8.1-8.2, Lions and Magenes [44]) it follows that $u \in C_w\left([0, T]; W_0^{1,2(q+1)}(\mathcal{D}) \cap L^{p+1}(\mathcal{D})\right)$ and $u_t \in C_w\left([0, T]; L^2(\mathcal{D})\right) \cap L^{m+1}\left(\mathcal{D} \times (0, T)\right)$. By using the lemma 2.11 in [63] we get regularity. The proof of Theorem 3.3.1 is finished. □

## 3.4 Blow up of solutions

In this section, we study the blow up of solutions of (3.1).
Before starting the result of blow up of solutions we need to stating the following lemmas which will be used in the proof.

**Lemma 3.4.1.** *There exists a positive constant $C > 0$ depending on $\mathcal{D}$ only such that*

$$\left(\int_{\mathcal{D}} |u|^p \ln |u|^k dx\right)^{\mathcal{N}} \leq C\left[\int_{\mathcal{D}} |u|^p \ln |u|^k dx + \|\nabla u\|^2\right]$$

*for any* $u \in L^{p+1}(\mathcal{D})$ *and* $\frac{2}{p+1} < \mathcal{N} \leq 1$, *provided that* $\int_{\mathcal{D}} |u|^p \ln |u|^k dx \geq 0$.

*Proof.* We distingue two cases :

• Case 1. If $\int_{\mathcal{D}} |u|^p \ln |u|^k dx > 1$,
then as $\mathcal{N} \leq 1$, we have $\Big(\int_{\mathcal{D}} |u|^p \ln |u|^k dx\Big)^{\mathcal{N}} \leq \int_{\mathcal{D}} |u|^p \ln |u|^k dx$.

• Case 2. If $\int_{\mathcal{D}} |u|^p \ln |u|^k dx \leq 1$, then as $\frac{2}{p+1} < \mathcal{N} \leq 1$ we have

$$\begin{aligned}\Big(\int_{\mathcal{D}} |u|^p \ln |u|^k dx\Big)^{\mathcal{N}} &\leq \Big(\int_{\mathcal{D}} |u|^p \ln |u|^k dx\Big)^{\frac{2}{p+1}} \\ &\leq \Big(\int_{\mathcal{D}_1} |u|^p \ln |u|^k dx\Big)^{\frac{2}{p+1}},\end{aligned}$$

where $\mathcal{D}_1 = \{x \in \mathcal{D} : |u| > 1\}$.
Hence

$$\Big(\int_{\mathcal{D}} |u|^p \ln |u|^k dx\Big)^{\mathcal{N}} \leq \|u\|_{p+1}^2 \leq C_s^2 \|\nabla u\|^2.$$

So the result of lemma 3.4.1. □

**Lemma 3.4.2.** *There exist a positive constant* $C > 0$ *depending on* $\mathcal{D}$ *only such that:*

$$\|u\|^2 \leq C\Big[\Big(\int_{\mathcal{D}} |u|^p \ln |u|^k dx\Big)^{\frac{2}{p}} + \|\nabla u\|^{\frac{4}{p}}\Big]$$

*provided that* $\int_{\mathcal{D}} |u|^p \ln |u|^k dx \geq 0$.

*Proof.* First, we set a partition the $\mathcal{D}$ as follow:
$\mathcal{D}_+ = \{x \in \mathcal{D} : |u| > e\}$ and $\mathcal{D}_- = \{x \in \mathcal{D} : |u| \leq e\}$.

From **(A3)** and the increasing of logarithmic function on $\mathcal{D}_+$ we have

$$\begin{aligned}
\|u\|_p^p &\leq \frac{1}{k}\int_{\mathcal{D}_+} |u|^p \ln|u|^k dx + e^p \int_{\mathcal{D}_-} |\frac{u}{e}|^p dx \\
&\leq \frac{1}{k}\int_{\mathcal{D}_+} |u|^p \ln|u|^k dx + e^p \int_{\mathcal{D}_-} |\frac{u}{e}|^2 dx \\
&\leq C\Big[\int_{\mathcal{D}_+} |u|^p \ln\,|u|^k dx + \|\nabla u\|^2\Big]
\end{aligned}$$

where $C = \max\{\frac{1}{k},\, C_s^2 e^{p-2}\}$.
Using the fact that

$$\begin{aligned}
\|u\|^2 &\leq C_{\mathcal{D}}\|u\|_p^2 \\
&\leq C'\Big[\int_{\mathcal{D}} |u|^p \ln|u|^k dx + \|\nabla u\|^2\Big]^{\frac{2}{p}} \\
&\leq C'\Big[\Big(\int_{\mathcal{D}} |u|^p \ln|u|^k dx\Big)^{\frac{2}{p}} + \|\nabla u\|^{\frac{4}{p}}\Big].
\end{aligned}$$

The proof is finished.

□

**Theorem 3.4.1. (Blow up of solution)** *Suppose that* **(A1)**- **(A4)** *hold, furthermore assume that* $l \geq 2$, $p > \max\{m+1,\, 2(q+1)\frac{b_1+b_2}{b_2},\, l+2\}$ *and the initial energy* $E(0)$ *define by*

$$\begin{aligned}
E(0) = &\frac{1}{l+2}\|u_1\|_{l+2}^{l+2} + \frac{1}{2}\|\nabla u_1\|_2^2 + \frac{1}{2}\int_{\mathcal{D}} P(|\nabla u_0|^2)\, dx \\
&+ \frac{k}{p^2}\|u_0\|_p^p + \xi\int_{\mathcal{D}}\int_0^1 z^{m+1}(x,\rho,0)\, d\rho\, dx - \frac{1}{p}\int_{\mathcal{D}} |u_0|^p \ln|u_0|^k\, dx,
\end{aligned}$$

*is negative. Then, the solution of (3.1) blows up in finite time.*

*Proof.* Let $u$ be a solution to (3.1).
We assume that $l \geq 2$, $p > \max\{m+1,\, 2(q+1)\frac{b_1+b_2}{b_2},\, l+2\}$ and $E(0) < 0$.

We define

$$\phi(t) = \frac{1}{l+1}\int_{\mathcal{D}} |u_t|^l u_t u\,dx + \int_{\mathcal{D}} \nabla u_t \nabla u\,dx, \tag{3.32}$$

$$G(t) = \int_{\mathcal{D}} |u|^p \ln|u|^k\,dx \tag{3.33}$$

and

$$H(t) = -E(t). \tag{3.34}$$

The assumption $E(0) < 0$, (3.13) and (3.12) implies that $H$ is a positive increasing function and

$$\begin{cases} H'(t) \geq C_0(\|u_t(t)\|_{m+1}^{m+1} + \|z(x,1,t)\|_{m+1}^{m+1}), \\ H(0) < H(t) < G(t). \end{cases} \tag{3.35}$$

We set

$$\gamma = \min\{\frac{1}{m+1} - \frac{1}{p}, \frac{2}{l+2}, \frac{q}{2(q+1)}\} < \frac{1}{2} \tag{3.36}$$

Differentiating (3.32) with respect to $t$ and using the first of (3.10), we obtain

$$\begin{aligned}\phi'(t) &= \int_{\mathcal{D}} \Big[div(\rho(|\nabla u|^2)\nabla u) - \mu_1|u_t|^{m-1}u_t - \mu_2|z(x,1,t)|^{m-1}z(x,1,t) \\ &\quad + u|u|^{p-2}\ln|u|^k\Big]u\,dx + \frac{1}{l+1}\|u_t\|_{l+2}^{l+2} + \|\nabla u_t\|^2 \\ &= \frac{1}{l+1}\|u_t\|_{l+2}^{l+2} + \|\nabla u_t\|^2 - \int_{\mathcal{D}} \rho(|\nabla u|^2)|\nabla u|^2\,dx \\ &\quad - \mu_1\int_{\mathcal{D}} |u_t|^{m-1}u_t u\,dx - \mu_2\int_{\mathcal{D}} |z(x,1,t)|^{m-1}z(x,1,t)u\,dx + \int_{\mathcal{D}} |u|^p\ln|u|^k\,dx.\end{aligned} \tag{3.37}$$

From the definition of $E(t)$ and (3.34), we note that

$$\begin{aligned}\int_{\mathcal{D}} \rho(|\nabla u|^2)|\nabla u|^2\,dx &\leq (b_1+b_2)\|\nabla u\|_{2q+2}^{2q+2} \\ &\leq 2(q+1)\frac{b_1+b_2}{b_2}\left(-H(t) + \frac{1}{p}G(t)\right).\end{aligned} \tag{3.38}$$

Also by using Hölder's inequality, we have

$$\Big|\int_{\mathcal{D}} |u_t|^{m-1}u_t u\,dx\Big| \le \int_{\mathcal{D}} |u_t|^m |u|\,dx \le \|u\|_{m+1}\|u_t\|_{m+1}^m.$$

Since $p \ge m+1$, by using the embedding $\|u\|_{m+1} \le C_{\mathcal{D}}\|u\|_p$ and Lemma 3.2.3, we obtain

$$\begin{aligned}\Big|\int_{\mathcal{D}} |u_t|^{m-1}u_t u\,dx\Big| &\le C_{\mathcal{D}}\Big(\int_{\mathcal{D}} |u|^p \ln|u|^k\,dx + \|\nabla u\|^2\Big)^{1/p}\|u_t\|_{m+1}^{m+1} \\ &\le C_{\mathcal{D}}G(t)^{\frac{1}{p}-\frac{1}{m+1}}\Big(\int_{\mathcal{D}} |u|^p \ln|u|^k\,dx + \|\nabla u\|^2\Big)^{\frac{1}{m+1}}\|u_t\|_{m+1}^{m+1}.\end{aligned} \tag{3.39}$$

By recalling (3.36), (3.35) and using Young's inequality, (3.39) leads

$$\begin{aligned}\Big|\int_{\mathcal{D}} |u_t|^{m-1}u_t u\,dx\Big| &\le C_{\mathcal{D}}H(t)^{\frac{1}{p}-\frac{1}{m+1}}\Big[\delta\Big(\int_{\mathcal{D}} |u|^p \ln|u|^k\,dx + \|\nabla u\|^2\Big) + \delta^{-\frac{1}{m}}\|u_t\|_{m+1}^{m+1}\Big] \\ &\le C_{\mathcal{D}}H(0)^{\frac{1}{p}-\frac{1}{m+1}}\delta\Big(\int_{\mathcal{D}} |u|^p \ln|u|^k\,dx + \|\nabla u\|^2\Big) \\ &\quad + C\delta^{-\frac{1}{m}}H(0)^{\frac{1}{p}-\frac{1}{m+1}+\gamma}H(t)^{-\gamma}H'(t).\end{aligned} \tag{3.40}$$

Similarly, we obtain

$$\begin{aligned}\Big|\int_{\mathcal{D}} |z(x,1,t)|^{m-1}z(x,1,t)u\,dx\Big| &\le C_{\mathcal{D}}H(0)^{\frac{1}{p}-\frac{1}{m+1}}\delta\Big(\int_{\mathcal{D}} |u|^p \ln|u|^k\,dx + \|\nabla u\|^2\Big) \\ &\quad + C\delta^{-\frac{1}{m}}H(0)^{\frac{1}{p}-\frac{1}{m+1}+\gamma}H(t)^{-\gamma}H'(t).\end{aligned} \tag{3.41}$$

Combining (3.38), (3.40) and (3.41) in (3.37), and recalling the definition of energy we obtain

$$\begin{aligned}\phi'(t) \geq \frac{1}{l+1}\|u_t\|_{l+2}^{l+2} + \|\nabla u_t\|^2 + (2(q+1)\frac{b_1+b_2}{b_2} + C_{\mathcal{D}}(\mu_1+\mu_2)H(0)^{\frac{1}{p}-\frac{1}{m+1}}\frac{2}{b_1})H(t) \\ + \Big[1 - 2(q+1)\frac{b_1+b_2}{pb_2} - \delta C_{\mathcal{D}}(\mu_1+\mu_2)H(0)^{\frac{1}{p}-\frac{1}{m+1}}(\frac{2}{pb_1}+1)\Big]G(t) \\ - \delta^{-\frac{1}{m}}C(\mu_1+\mu_2)H(0)^{\frac{1}{p}-\frac{1}{m+1}+\gamma}H'(t)H^{-\gamma}(t).\end{aligned} \tag{3.42}$$

Since $p > 2(q+1)\frac{b_1+b_2}{b_2}$, by taking $\delta$ small enough such that

$$\alpha := 1 - 2(q+1)\frac{b_1+b_2}{pb_2} - \delta C_{\mathcal{D}}(\mu_1+\mu_2)H(0)^{\frac{1}{p}-\frac{1}{m+1}}(\frac{2}{pb_1}+1) > 0.$$

Consequently

$$\begin{aligned}\phi'(t) \geq C\Big[\|u_t\|_{l+2}^{l+2} + \|\nabla u_t\|^2 + H(t) + G(t)\Big] \\ - \delta^{-\frac{1}{m}}C(\mu_1+\mu_2)H(0)^{\frac{1}{p}-\frac{1}{m+1}+\gamma}H'(t)H^{-\gamma}(t).\end{aligned} \tag{3.43}$$

Now, we set as in [6, 8, 29]

$$\psi(t) = H(t)^{1-\gamma} + \epsilon\phi(t) \tag{3.44}$$

where $\epsilon$ is a positive constant to determine later.

We have

$$\psi'(t) \geq LH'(t)H^{-\gamma}(t) + \epsilon C\Big[\|u_t\|_{l+2}^{l+2} + \|\nabla u_t\|^2 + H(t) + G(t)\Big]. \tag{3.45}$$

For a choice of $\epsilon < 1$ small enough such that

$$L = 1 - \gamma - \epsilon\delta^{-\frac{1}{m}}C(\mu_1+\mu_2)H(0)^{\frac{1}{p}-\frac{1}{m+1}+\gamma} > 0$$

and

$$H(0) + \epsilon\int_{\mathcal{D}} |u_1|^l u_1 u_0\,dx > 0,$$

and as $H$ is a positive increasing function (3.45) becomes

$$\psi'(t) \geq \epsilon C\Big[\|u_t\|_{l+2}^{l+2} + \|\nabla u_t\|^2 + H(t) + G(t)\Big] \tag{3.46}$$

which implies $\psi(t) > \psi(0) > 0$.

Now we will show that there exists a positive constant $c$ such that

$$\psi'(t) \geq c\epsilon\psi(t)^{\nu} \ \text{ for } \ t \in [0,T), \tag{3.47}$$

where $\nu = \frac{1}{1-\gamma} > 1$ .

To do so, we distingue two cases:

- Case 1. If $\phi(t) \leq 0$ for some $t \in (0,T]$, then we have the following result

$$\psi(t)^{\nu} = \big(H(t)^{1-\gamma} + \epsilon\phi(t)\big)^{\nu} \leq H(t) \ \text{ for } \ t \in [0,T). \tag{3.48}$$

  Hence

$$\psi'(t) \geq C\epsilon H(t) \geq C\epsilon\psi(t)^{\nu} \ \text{ for } \ t \in [0,T). \tag{3.49}$$

- Case 2. If $\phi(t) > 0$ for some $t \in (0,T]$, we note that

$$\psi(t)^{\nu} \leq 2^{\nu-1}(H(t) + \epsilon\phi(t)^{\nu}) \ \text{ for } \ t \in [0,T). \tag{3.50}$$

  By using Hölder's, Young's inequalities, $p > l+2$ and noting that $1 < \nu < 2$ for some $t \in (0,T]$, we have

$$\begin{aligned} \phi(t)^{\nu} &\leq \Big[\frac{1}{l+1}\|u_t\|_{l+2}^{l+1}\|u\|_{l+2} + \|\nabla u_t\|\|\nabla u\|\Big]^{\nu} \\ &\leq C\Big(\|u_t\|_{l+2}^{l+1}\|u\|_p + \|\nabla u_t\|\|\nabla u\|\Big)^{\nu} \\ &\leq C\Big(\|u_t\|_{l+2}^{(l+1)\nu}\|u\|_p^{\nu} + \|\nabla u_t\|^{\nu}\|\nabla u\|^{\nu}\Big) \\ &\leq C\Big(\|u_t\|_{l+2}^{l+2} + \|u\|_p^{\frac{\nu(l+2)}{l+2-(l+1)\nu}} + \|\nabla u_t\|^2 + \|\nabla u\|^{\frac{2\nu}{2-\nu}}\Big). \end{aligned} \tag{3.51}$$

Since $2 \le s = \frac{\nu(l+2)}{l+2-(l+1)\nu} < p$, we apply successively lemma 3.2.4 and Lemma 3.2.3, we obtain

$$\begin{aligned} \|u\|_p^s &\le C(\|u\|_p^p + \|\nabla u\|^2) \\ &\le C\left(\int_{\mathcal{D}} |u|^p \ln|u|^k \, dx + \|\nabla u\|^2\right). \end{aligned} \tag{3.52}$$

From Lemma 3.3.1 we have $E(t) \le E(0) < 0$, hence

$$\begin{aligned} \|\nabla u\|^{\frac{2\nu}{2-\nu}} &\le C\left(\|\nabla u\|_{2(q+1)}^{2(q+1)}\right)^{\frac{\nu}{(2-\nu)(q+1)}} \\ &\le C\frac{2(q+1)}{b_2}(E(t)+G(t))^{\frac{\nu}{(2-\nu)(q+1)}} \\ &\le C\frac{2(q+1)}{b_2}(G(t))^{\frac{\nu}{(2-\nu)(q+1)}}. \end{aligned} \tag{3.53}$$

Since $\gamma \le \frac{q}{2(q+1)}$, so we have $\frac{\nu}{(2-\nu)(q+1)} \le 1$, then by using Lemma 3.4.1 the assertion (3.53) becomes

$$\|\nabla u\|_2^{\frac{2\nu}{2-\nu}} \le C'\left(G(t) + \|\nabla u\|^2\right). \tag{3.54}$$

Combining (3.52) and (3.54) in (3.51) we obtain

$$\begin{aligned} \phi(t)^\nu &\le C\left(\|u_t\|_{l+2}^{l+2} + \|\nabla u_t\|^2 + \|\nabla u\|^2 + G(t)\right) \\ &\le C\left(\|u_t\|_{l+2}^{l+2} + \|\nabla u_t\|^2 + \frac{2}{b_1}\left(-H(t) + \frac{1}{p}G(t)\right) + G(t)\right) \\ &\le C\left(\|u_t\|_{l+2}^{l+2} + \|\nabla u_t\|^2 - \frac{2}{b_1}H(t) + \left(\frac{2}{pb_1}+1\right)G(t)\right). \end{aligned} \tag{3.55}$$

By substituting (3.55) into (3.50) we get

$$\Psi(t)^\nu \le 2^{\nu-1}\left[\left(1 - \frac{2\epsilon C}{b_1}\right)H(t) + C\epsilon\left(\|u_t\|_{l+2}^{l+2} + \|\nabla u_t\|^2 + G(t)\right)\right]. \tag{3.56}$$

We choose $\epsilon$ small enough such that $1 - \frac{2\epsilon C}{b_1} > 0$ we can find a positive constant $c$ such that

$$\Psi(t)^{\nu} \leq c\epsilon\Big[H(t) + \|u_t\|_{l+2}^{l+2} + \|\nabla u_t\|^2 + G(t)\Big]. \tag{3.57}$$

From (3.46) and (3.57) we have

$$\psi'(t) \geq c\epsilon\psi(t)^{\nu}, \ \forall t \in [0, T). \tag{3.58}$$

Hence, (3.47) is established and $\psi(t)$ blows up in finite time $T$, where

$$T < c\epsilon^{-1}\psi(0)^{-\gamma/(1-\gamma)}.$$

The proof is completed. □

# Chapter 4

# Blow-Up for a Stochastic Viscoelastic Lamé Equation with Logarithmic Nonlinearity

## 4.1 Introduction

In recent years, stochastic partial differential equations in a separable Hilbert space have been studied by many authors and various results on the existence, uniqueness, stability, blow-up, other quantitative and qualitative properties of solutions have been established.
In this chapter, we consider the following problem of stochastic wave equation :

$$\begin{cases} u_{tt} - \mu\Delta u - (\lambda+\mu)\nabla(divu) + \int_0^t h(t-s)\Delta u(s)\,ds & \\ +|u_t|^{q-2}u_t = u|u|^{p-2}ln|u|^k + \epsilon\sigma(x,t)W_t(x,t) & \text{in } \mathcal{D}\times]0,+\infty[, \\ u(x,t) = 0 & \text{on } \partial\mathcal{D}\times[0,+\infty[, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \overline{\mathcal{D}}, \end{cases} \tag{4.1}$$

where $\mathcal{D}$ is a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}^*$, with a smooth boundary $\partial\mathcal{D}$ , $\mu$, $\lambda$ are the Lamé constants which satisfy $\mu > 0$, $\lambda + \mu \geq 0$, $h$ is a positive function, $p > q \geq 2$, the constant $k$ is a small non-negative real

number; and $L^2(\mathcal{D})$ is the set of square integrable function on $\mathcal{D}$ equipped with the inner product $\langle ., . \rangle$ and its norm $\|.\|_2$.
$W(x,t)$ is an infinite dimensional Wiener process, $\sigma(x,t)$ is $L^2(\mathcal{D})$ valued progressively measurable and $\epsilon$ is positive constant which measures the strength of noise.

It is commonly to observe a wave motion as physical phenomenon which is mathematically modeled by a partial differential equation of hyperbolic type. Much has been written about such equations regarding their wide spread applications to engineering and sciences. However, for more realistic models, the random fluctuation had been taken into consideration which led to introduced stochastic wave equation in 1960's. Several examples of linear stochastic wave propagation and applications can be found in [24]. Mueller [53] was the first who investigate the existence of explosive solutions for some stochastic wave equation. Motivated by Mueller [53], Chow [23] was interested by knowing how does a random perturbation affect the solution behavior for a wave equation with a polynomial nonlinearity. He was concerned with the existence of a local and global solutions the stochastic equation:

$$\begin{cases} u_{tt} = \Delta u + f(u) + \sigma(u) W_t(x,t) & \text{in } x \in \mathbb{R}^d,\ t > 0, \\ u(x,0) = g(x), \quad u_t(x,0) = h(x), \end{cases} \tag{4.2}$$

where the initial data $g$ and $h$ are given functions, the nonlinear terms $f(u)$ and $\sigma(u)$ are assumed to be polynomials in $u$. Four years later, he [22] established an energy inequality and the exponential bound for a linear stochastic equation and gave the existence theorem for a unique global solution for the randomly perturbed wave equation:

$$\begin{cases} u_{tt} + 2\alpha u_t - A(x, \partial x) u(x,t) = f(x,t) + \sigma(x,t) W_t(x,t) & \text{in } x \in \mathbb{R}^d,\ t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x). \end{cases} \tag{4.3}$$

In 2009, Chow [21] studied the problem of explosive solutions for a class of nonlinear stochastic wave equation in a domain $\mathcal{D} \subset \mathbb{R}^d$ for $d \geq 3$,

$$\begin{cases} u_{tt} = (c^2\Delta - \alpha)u + f(u) + \sigma(u,x,t)W_t(x,t) & \text{in } x \in \mathcal{D},\ t > 0, \\ u(x,0) = g(x), \quad u_t(x,0) = h(x). \end{cases} \tag{4.4}$$

We can mention some other works such as Cheng et al. [20] who studied the existence of a global solution and blow-up solutions for the nonlinear stochastic viscoelastic wave equation with nonlinear damping and source terms:

$$\begin{cases} u_{tt} - \Delta u + \int_0^t h(t-s)\Delta u(s)\,ds & \\ +|u_t|^{q-2}u_t = u|u|^{p-2} + \epsilon\sigma(x,t)W_t(x,t) & \text{in } \mathcal{D}\times]0,+\infty[, \\ u(x,t) = 0 & \text{on } \partial\mathcal{D} \times [0,+\infty[, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \overline{\mathcal{D}}. \end{cases} \tag{4.5}$$

The authors proved that finite time blow-up with non-negative probability or it is explosive or it is explosive in energy sense for $p > q$. Moreover, Kim et al. [38] considered the stochastic quasi-linear viscoelastic wave equation with nonlinear damping and source terms:

$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t h(t-s)\Delta u(s)\,ds & \\ +|u_t|^{q-2}u_t = u|u|^{p-2} + \epsilon\sigma(x,t)W_t(x,t) & \text{in } \mathcal{D}\times]0,+\infty[, \\ u(x,t) = 0 & \text{on } \partial\mathcal{D} \times [0,+\infty[, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \overline{\mathcal{D}}. \end{cases} \tag{4.6}$$

They showed the existence of a global solution and blow-up in finite time.

Recently, Yang et al. [71] treated the following stochastic nonlinear viscoelastic wave equation:

$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t h(t-s)\Delta u(s)\,ds = \sigma(x,t)W_t(x,t) & \text{in } \mathcal{D}\times]0,+\infty[, \\ u(x,t) = 0 & \text{on } \partial\mathcal{D} \times [0,+\infty[, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \overline{\mathcal{D}}. \end{cases} \tag{4.7}$$

They established the existence of global solution and asymptotic stability of the solution by using some properties of the convex function.
However, it was noticed that the logarithmic nonlinearity appears naturally in many branches of physics such as nuclear physics, optics and geophysics (see [31] and [27]). These specific applications in physics and other fields attract a lot of mathematical scientists to work with such problems. In the deterministic case, Al-Gharabli [4] investigated the stability of the solution of a viscoelastic plate equation with a logarithmic nonlinearity source term for the following problem:

$$\begin{cases} u_{tt} + \Delta^2 u + u + \displaystyle\int_0^t h(t-s)\Delta u^2(s)\,ds = uln|u|^k & \text{in } \mathcal{D}\times]0,+\infty[, \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{in } \partial\mathcal{D}\times]0,+\infty[ \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \overline{\mathcal{D}}, \end{cases} \tag{4.8}$$

where $\mathcal{D} \subseteq \mathbb{R}^2$ is a bounded domain with a smooth boundary $\partial\mathcal{D}$. The vector $\nu$ is the unit outer normal to $\partial\mathcal{D}$, and $h$ is nondecreasing nonnegative function.
Mezouar et al. [52] treated a more general problem where they considered the following nonlinear viscoelastic Kirchhoff equation with a time-varying delay term:

$$\begin{cases} |u_t|^l u_{tt} - M(\|\nabla u\|^2)\Delta u - \Delta u_{tt} + \int_0^t h(t-s)\Delta u(s)\,ds & \\ +\mu_1 g_1(u_t(x,t)) + \mu_2 g_2(u_t(x,t-\tau(t))) = ku\ln|u| & \text{in } \mathcal{D}\times]0,+\infty[, \\ u(x,t) = 0 & \text{on } \partial\mathcal{D}\times[0,+\infty[, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \mathcal{D}, \\ u_t(x,t-\tau(0)) = f_0(x,t-\tau(0)) & \text{in } \mathcal{D}\times]0,\tau(0)[. \end{cases} \tag{4.9}$$

The chapter is organized as follows: In section 2, we introduce some basic definitions, necessary assumptions, and lemmas that are helpful in proving our main result. Section 3 is devoted to show the blow-up of the solution of our problem.

## 4.2 Preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space for which a filtration $\{\mathcal{F}_t, t \geq 0\}$ of increasing sub $\sigma-$fields $\mathcal{F}_t$ is given and $W(x,t)$ be a continuous Wiener random field in this space with mean zero and the covariance operator $Q$ satisfying

$$Tr(Q) = \sum_{i \geq 1} \lambda_i < \infty.$$

$W(x,t)$ is defined by

$$W(x,t) = \sum_{j=1}^{\infty} \sqrt{\lambda_j} \beta_j(t) e_j(t), \; j \in \mathbb{N}^*, \; \mathrm{t} \geq 0,$$

where $\beta_j(t)$ is a sequence of real-valued standard Brownian motions mutually independent on the probability space $(\Omega, \mathcal{F}, P)$, $\lambda_j$ are the eigenvalues of $Q$, and $e_j$ are the corresponding eigenvectors. That is,

$$Qe_j = \lambda_j e_j.$$

Note $\mathbb{E}(.)$ stands for expectation with respect to probability measure $P$. Let $\mathcal{H}$ be the set of $L_2^0 = L^2(Q^{\frac{1}{2}}V, V)$-valued processes with the norm

$$\|\phi(t)\|_{\mathcal{H}}^2 = \mathbb{E} \int_0^t \|\phi(s)\|_{L_2^0}^2 ds = \mathbb{E} \int_0^t Tr(\phi(s) Q \phi^*(s)) ds < \infty,$$

where $\phi^*(s)$ denotes the adjoint operator of $\phi(s)$ and $V = H_0^1(\mathcal{D})$ which is equivalent to $H^1(\mathcal{D})$. For any process $\phi(s) \in \mathcal{H}$, we can define the stochastic integral with respect to the $Q$-Wiener process as $\int_0^t \phi(s) dW(s)$ which is a martingale. For more details about the infinite dimension Wiener process and stochastic integral, we refer to Da Prato et al. [25] (p. 90-96).

To state and prove our result, we need some assumptions.

**(A1)** Assume that $h : \mathbb{R}^+ \to \mathbb{R}^+$ is a $C^1$ nonincreasing function satisfying

$$h(0) > 0,\ \mu - \int_0^\infty h(s)ds = l > 0$$

and there exist tow nonnegative constants $\varsigma_1$ and $\varsigma_2$ such that

$$-\varsigma_1 h(t) \leq h'(t) \leq -\varsigma_2 h(t), \quad t \geq 0.$$

**(A2)**

$$\int_0^\infty h(s)ds < \mu \frac{(p-2)p}{(p-1)^2}.$$

**(A3)** $p > q \geq 2$ and

$$\begin{cases} 2 < p \leq \dfrac{2(n-1)}{n-2} & \text{if } n \geq 3 \\ 2 < p \leq +\infty & \text{if } n = 1, 2. \end{cases} \tag{4.10}$$

The following theorem state the existence and uniqueness of a local solution of our problem; the proof can be established by combining the proof given in [20, 52].

**Theorem 4.2.1.** *Assume that* **(A1)** *and* **(A3)** *hold. if* $(u_0, u_1) \in H_0^1(\mathcal{D}) \times L^2(\mathcal{D})$ *and* $\mathbb{E} \int_0^t \|\sigma(t)\|_2^2 dt < \infty$*, then there exists a solution in which holds (4.1) on the interval* $[0, T]$ *in the sense of distributions over* $(0, T) \times \mathcal{D}$ *for almost all* $w$ *a test function such that*

$$(u, u_t) \in L^2(\Omega; L^\infty([0, T]; (H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})) \times H_0^1(\mathcal{D})))$$

$$\cap L^2(\Omega; C([0, T]; H_0^1(\mathcal{D}) \times L^2(\mathcal{D}))).$$

We define the energy associated to the solution of system (4.1) by

$$\begin{aligned} e(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\Big(\mu - \int_0^t h(s)ds\Big)\|\nabla u\|_2^2 + \frac{\lambda+\mu}{2}\|divu\|_2^2 \\ + \frac{1}{2}(h\, o\, \nabla u)(t) + \frac{k}{p^2}\|u\|_p^p - \frac{1}{p}\int_{\mathcal{D}} |u|^p ln|u|^k dx \end{aligned} \tag{4.11}$$

where

$$(h\, o\, v)(t) = \int_0^t h(t-s)\|v(.,t) - v(.,s)\|^2\, ds.$$

We rewrite (4.1) as an equivalent Itô's system

$$\begin{cases} du = vdt, \\ dv = \Big[\mu\Delta u + (\lambda+\mu)\nabla(divu) - \displaystyle\int_0^t h(t-s)\Delta u(s)\, ds \\ -|v|^{q-2}v + u|u|^{p-2}ln|u|^k\Big]dt + \epsilon\sigma(x,t)dW_t(x,t) & \text{in } \mathcal{D}\times]0,+\infty[, \\ u(x,t) = 0 & \text{on } \partial\mathcal{D}\times[0,+\infty[, \\ u(x,0) = u_0(x), \quad v(x,0) = u_1(x) & \text{in } \overline{\mathcal{D}}, \end{cases} \tag{4.12}$$

which can be written as the integral equation

$$\begin{cases} u(t) = u_0 + \int_0^t v(s)ds, \\ v(t) = v(0) + \int_0^t \Big[\mu\Delta u + (\lambda+\mu)\nabla(divu) - \displaystyle\int_0^t h(s-r)\Delta u(r)\, dr \\ -|v|^{q-2}v + u|u|^{p-2}ln|u|^k\Big]ds + \int_0^t \epsilon\sigma(x,s)dW_s(x,t) & \text{in } \mathcal{D}\times]0,+\infty[, \\ u(x,t) = 0 & \text{on } \partial\mathcal{D}\times[0,+\infty[, \\ u(x,0) = u_0(x), \quad v(x,0) = u_1(x) & \text{in } \overline{\mathcal{D}}. \end{cases} \tag{4.13}$$

**Lemma 4.2.1.** *[2](Sobolev-Poincaré' s inequality ). Let $m$ be a number with*

$$2 \leq m \leq +\infty (n = 1,2) \;\; or \;\; 2 \leq m \leq 2n/(n-2)(n \geq 3)$$

*then there exists a constant $C_s = C_s(\mathcal{D}, m)$ such that*

$$\|u\|_m \leq C_s\|\nabla u\|_2 \;\; for \;\; u \in H_0^1(\mathcal{D}).$$

**Lemma 4.2.2.** [58]. *For $h, \varphi \in C^1([0,+\infty[, \mathbb{R})$ we have*

$$\int_{\mathcal{D}} h{*}\varphi\varphi_t\, dx = -\frac{1}{2}h(t)\|\varphi(t)\|_2^2+\frac{1}{2}(h' o\,\varphi)(t)-\frac{1}{2}\frac{d}{dt}\Big[(h\, o\, \varphi)(t)-(\int_0^t h(s)\, ds)\|\varphi\|^2\Big]$$

**Lemma 4.2.3.** *Let $(u, v)$ be a solution of the problem (4.12) with the initial data $(u_0, v_0) \in H_0^1(\mathcal{D}) \times L^2(\mathcal{D})$, $\mathbb{E}\int_0^t \|\sigma(s)\|_2^2 ds < \infty$. Then, the energy functional defined by (4.11) satisfies*

$$\begin{aligned} e(t) =& e(0) - \int_0^t \|v\|_q^q ds - \frac{1}{2}\int_0^t h(s)\|\nabla u(s)\|_2^2 ds + \frac{1}{2}\int_0^t (h' o \nabla u)(s) ds \\ &+ \int_0^t \langle v(s), \epsilon\sigma(x,s)dW_s\rangle + \frac{\epsilon^2}{2}\sum_{j=1}^{\infty}\int_0^t\int_{\mathcal{D}} \lambda_j e_j^2(x)\sigma^2(x,s)dxds. \end{aligned} \tag{4.14}$$

*Proof.* We can apply the Itô formula to (4.12) for each $x \in \mathcal{D}$ after integrating the above equation over $\mathcal{D}$ to get

$$\begin{aligned} \|v(t)\|_2^2 =& \|v(0)\|_2^2 + 2\int_{\mathcal{D}}\int_0^t v(s)\Big[\mu\Delta u + (\lambda+\mu)\nabla(divu) - \int_0^s h(s-\tau)\Delta u(\tau)\, d\tau \\ &- |v|^{q-2}v + u|u|^{p-2}ln|u|^k\Big] dsdx + 2\int_0^t \langle v(s), \epsilon\sigma(x,s)dW_s\rangle \\ &+ \epsilon^2\sum_{j=1}^{\infty}\int_0^t\int_{\mathcal{D}} \lambda_j e_j^2(x)\sigma^2(x,s)dxds. \end{aligned} \tag{4.15}$$

By using integration by parts, we get

$$\begin{aligned} \mu\int_{\mathcal{D}}\int_0^t \Delta u v(s) dxds &= -\mu\int_{\mathcal{D}}\int_0^t \nabla u\nabla v dsdx \\ &= -\frac{\mu}{2}\Big(\|\nabla u(t)\|_2^2 - \|\nabla u(0)\|_2^2\Big), \end{aligned} \tag{4.16}$$

$$\int_{\mathcal{D}}\int_0^t(\lambda+\mu)\nabla(divu(s))v(s)dsdx = -(\lambda+\mu)\int_{\mathcal{D}}\int_0^t divu(s)divv(s)dsdx$$
$$= -\frac{\lambda+\mu}{2}\Big[\|divu(t)\|_2^2 - \|divu(0)\|_2^2\Big]. \tag{4.17}$$

By applying Lemma 4.2.2, we have

$$\begin{aligned}
&\int_0^t\int_{\mathcal{D}}\int_0^s h(s-\tau)\Delta u(\tau)v(s)d\tau dxds\\
&\quad= -\int_0^t\int_{\mathcal{D}}\int_0^s h(s-\tau)\nabla u(\tau)\nabla v(s)d\tau dxds\\
&\quad= \int_0^t\Big(\frac{1}{2}h(s)\|\nabla u(s)\|_2^2 - \frac{1}{2}(h'\,o\,\nabla u)(s)\\
&\quad+\frac{1}{2}\frac{d}{ds}\Big[(h\,o\,\nabla u)(s) - \int_0^s h(\tau)d\tau)\|\nabla u(s)\|_2^2\Big]\Big)ds.
\end{aligned} \tag{4.18}$$

We have

$$\begin{aligned}
\int_{\mathcal{D}}\int_0^t u|u|^{p-2}ln|u|^k u_s dsdx &= \int_0^t\int_{\mathcal{D}}\frac{1}{p}\frac{d}{ds}\Big(|u(s)|^p\Big)ln|u|^k dxds\\
&= \int_0^t\Big\{\int_{\mathcal{D}}\Big\{\frac{1}{p}\frac{d}{ds}(|u(s)|^p ln|u|^k)\\
&\quad-\frac{1}{p}|u(s)|^p\frac{d}{ds}(ln|u|^k)\Big\}dx\Big\}ds\\
&= \int_{\mathcal{D}}\Big(\frac{1}{p}(|u|^p ln|u|^k)\Big)dx - \frac{k}{p^2}\|u\|_p^p.
\end{aligned} \tag{4.19}$$

By replacing (4.16)-(4.19) in (4.15) and multiplying equation (4.15) by $\frac{1}{2}$, we arrive at (4.14). □

## 4.3 Blow Up

We prove our main result for $p > q$ we purpose

$$\mathbb{E}\int_0^\infty \int_{\mathcal{D}} \sigma^2(x,t)dxdt < \infty,$$

$$G(t) = \frac{\epsilon^2}{2}\sum_{j=1}^{\infty}\mathbb{E}\int_0^t \int_{\mathcal{D}} \lambda_j e_j^2(x)\sigma^2(x,s)dxds,$$

$$\begin{aligned} G(\infty) &= \frac{\epsilon^2}{2}\sum_{j=1}^{\infty}\mathbb{E}\int_0^\infty \int_{\mathcal{D}} \lambda_j e_j^2(x)\sigma^2(x,s)dxds \\ &\leq \frac{\epsilon^2}{2}Tr(Q)c_0^2\mathbb{E}\int_0^\infty \int_{\mathcal{D}} \sigma^2(x,s)dxds := E_1 < \infty, \end{aligned} \tag{4.20}$$

where

$$Tr(Q) = \sum_{j=1}^{\infty}\lambda_j < \infty \text{ and } c_0 = \sup_{j\geq 1}\|e_j\|_\infty < \infty.$$

**Lemma 4.3.1.** *Let $(u, v)$ be a solution of system (4.12) with initial data $(u_0, v_0) \in H_0^1(\mathcal{D}) \times L^2(\mathcal{D})$. Then, we have*

$$\begin{aligned} \frac{d}{dt}\mathbb{E}e(t) =& -\mathbb{E}\|v(t)\|_q^q - \frac{1}{2}h(t)\mathbb{E}\|\nabla u(t)\|_2^2 + \frac{1}{2}\mathbb{E}(h' o \nabla u)(t) \\ &+ \frac{\epsilon^2}{2}\sum_{j=1}^{\infty}\mathbb{E}\int_{\mathcal{D}} \lambda_j e_j^2(x)\sigma^2(x,t)dx, \end{aligned} \tag{4.21}$$

$$\begin{aligned}\mathbb{E}\langle u(t), v(t)\rangle = & \mathbb{E}\langle u_0, u_1\rangle - \mu \int_0^t \mathbb{E}\|\nabla u(s)\|_2^2 ds \\ & - (\lambda+\mu) \int_0^t \mathbb{E}\|divu(s)\|_2^2 ds \\ & + \mathbb{E}\int_0^t \int_0^s h(s-r)\langle \nabla u(r), \nabla u(s)\rangle dr ds \\ & - \mathbb{E}\int_0^t \langle u(s), |v(s)|^{q-2} v(s)\rangle ds \\ & + \mathbb{E}\int_0^t \langle u(s), u(s)|u(s)|^{p-2} ln|u(s)|^k\rangle ds \\ & + \mathbb{E}\int_0^t \|v(s)\|_2^2 ds.\end{aligned} \tag{4.22}$$

*Proof.* Using Itô formula and by following the same way as our discussions in lemma 4.2.3 with taking the expectations, we obtain (4.21) .
We multiply the second equation in (4.13) by $u$ and integrate the result over $\mathcal{D}$, and we take expectation; we obtain (4.22). □

We set $H(t) = G(t) - \mathbb{E}e(t)$. As $h$ is a positive decreasing function so

$$\begin{aligned}H'(t) = & G'(t) - \frac{d}{dt}\mathbb{E}e(t) = \mathbb{E}\|v\|_q^q + \frac{1}{2}h(t)\mathbb{E}\|\nabla u(t)\|_2^2 \\ & - \frac{1}{2}\mathbb{E}(h' o \nabla u)(t) \geq \mathbb{E}\|v\|_q^q.\end{aligned} \tag{4.23}$$

Consequently,

$$H'(t) \geq 0. \tag{4.24}$$

**Lemma 4.3.2.** *Let $(u, v)$ be a solution of system (4.12). Assume that* **(A1)** *holds. Then, there exists a positive constant $C$ such that*

$$\begin{aligned}\mathbb{E}\|u(t)\|_{p+1}^s \leq & C\Big(G(t) - H(t) - \frac{1}{2}\mathbb{E}\|v\|_2^2 + \frac{1}{p}\mathbb{E}\int_{\mathcal{D}} |u|^p ln|u|^k dx \\ & - \frac{1}{2}\mathbb{E}(h\, o\, \nabla u)(t) - \frac{\lambda+\mu}{2}\mathbb{E}\|divu\|_2^2 + \mathbb{E}\|u\|_{p+1}^{p+1}\Big),\end{aligned} \tag{4.25}$$

*where* $2 \leq s \leq p+1$.

*Proof.*

$$
\begin{aligned}
& G(t) - H(t) - \frac{1}{2}\mathbb{E}\|v\|_2^2 + \frac{1}{p}\mathbb{E}\int_{\mathcal{D}} |u|^p ln|u|^k dx \\
& - \frac{1}{2}\mathbb{E}(h\,o\,\nabla u)(t) - \frac{\lambda+\mu}{2}\mathbb{E}\|divu\|_2^2 + \mathbb{E}\|u\|_{p+1}^{p+1} \\
& = \mathbb{E}e(t) - \frac{1}{2}\mathbb{E}\|v\|_2^2 + \frac{1}{p}\mathbb{E}\int_{\mathcal{D}} |u|^p ln|u|^k dx \\
& - \frac{1}{2}\mathbb{E}(h\,o\,\nabla u)(t) + \mathbb{E}\|u\|_{p+1}^{p+1} - \frac{\lambda+\mu}{2}\mathbb{E}\|divu\|_2^2 \\
& = \frac{1}{2}\mathbb{E}\|u_t\|_2^2 + \frac{1}{2}\mathbb{E}\Big(\mu - \int_0^t h(s)ds\Big)\|\nabla u\|_2^2 \\
& + \frac{\lambda+\mu}{2}\mathbb{E}\|divu\|_2^2 + \frac{1}{2}\mathbb{E}(h\,o\,\nabla u)(t) + \frac{k}{p^2}\mathbb{E}\|u\|_p^p \\
& - \frac{1}{p}\mathbb{E}\int_{\mathcal{D}} |u|^p ln|u|^k dx - \frac{1}{2}\mathbb{E}\|v\|_2^2 \\
& + \frac{1}{p}\mathbb{E}\int_{\mathcal{D}} |u|^p ln|u|^k dx - \frac{1}{2}\mathbb{E}(h\,o\,\nabla u)(t) \\
& - \frac{\lambda+\mu}{2}\mathbb{E}\|divu\|_2^2 + \mathbb{E}\|u\|_{p+1}^{p+1} \\
& = \frac{1}{2}\Big(\mu - \int_0^t h(s)ds\Big)\mathbb{E}\|\nabla u\|_2^2 + \frac{k}{p^2}\mathbb{E}\|u\|_p^p + \mathbb{E}\|u\|_{p+1}^{p+1} \\
& \geq \frac{1}{2}l\mathbb{E}\|\nabla u\|_2^2 + \mathbb{E}\|u\|_{p+1}^{p+1}.
\end{aligned}
\tag{4.26}
$$

The last inequality is getting from **(A1)**.

- Case 1. If $\|u\|_{p+1} \leq 1$, then $\|u\|_{p+1}^s \leq \|u\|_{p+1}^2$.

By applying lemma 4.2.1, we obtain $\|u\|_{p+1}^s \leq c\|\nabla u\|_2^2$, then

$$\frac{1}{2}l\mathbb{E}\|\nabla u\|_2^2 + \frac{k}{p^2}\mathbb{E}\|u\|_{p+1}^{p+1} \geq \frac{1}{2}l\mathbb{E}\|u\|_{p+1}^s + \frac{k}{p^2}\mathbb{E}\|u\|_{p+1}^{p+1} \geq \mathbb{E}\|u\|_{p+1}^s.$$

- Case 2. If $\|u\|_{p+1} \geq 1$ then $\|u\|_{p+1}^{p+1} \geq \|u\|_{p+1}^s$.

Hence,

$$\frac{1}{2}l\mathbb{E}\|\nabla u\|_2^2 + \mathbb{E}\|u\|_{p+1}^{p+1} \geq \frac{1}{2}l\mathbb{E}\|\nabla u\|_2^2 + \mathbb{E}\|u\|_{p+1}^{s} \geq \mathbb{E}\|u\|_{p+1}^{s}.$$

Consequently we obtain (4.25). □

We are ready to state and prove our main result for $p > q$. For this purpose, we define

$$L(t) := H^{1-\alpha}(t) + \delta\mathbb{E}\langle u, v\rangle,$$

where

$$0 < \alpha < min\{\frac{p-1}{2(p+1)}, \frac{p+1-q}{(p+1)q}\} \tag{4.27}$$

and $\delta$ is a very small constant determined later.

**Theorem 4.3.1.** *Assume* **(A1)** *and* **(A2)** *hold. Let* $(u, v)$ *be a solution of system (4.12) with initial data* $(u_0, v_0) \in H_0^1(\mathcal{D}) \times L^2(\mathcal{D})$ *satisfying*

$$\mathbb{E}e(0) \leq -(1+\beta)E_1, \tag{4.28}$$

*where* $\beta$ *is nonnegative constant and* $E_1$ *is given (4.20). If* $p > q$*, then there exists a positive time* $T_0 \in [0, T]$ *such that*

$$\lim_{t\to T_0^-} \mathbb{E}(e(t)) = +\infty,$$

*where*

$$T_0 = \frac{1-\alpha}{\alpha K L^{\frac{\alpha}{1-\alpha}}(0)},$$

$$L(0) = H^{1-\alpha}(0) + \delta\mathbb{E}\langle u_0, u_1\rangle > 0,$$

*and* $K$ *is given later.*

*Proof.* Let

$$L(t) = H^{1-\alpha}(t) + \delta\mathbb{E}\langle u, v\rangle.$$

A direct differentiation of $L(t)$ gives

$$\begin{aligned}
L'(t) &= (1-\alpha)H^{-\alpha}(t)H'(t) + \delta\Big[-\mu\mathbb{E}\|\nabla u(t)\|_2^2 \\
&\quad - (\lambda+\mu)\mathbb{E}\|divu\|_2^2 + \mathbb{E}\int_0^t h(t-r)\langle\nabla u(r), \nabla u(t)\rangle dr \\
&\quad - \mathbb{E}\langle u(t), |v(t)|^{q-2}v(t)\rangle \\
&\quad + \mathbb{E}\langle u(t), u(t)|u(t)|^{p-2}ln|u(t)|^k\rangle + \mathbb{E}\|v(t)\|_2^2\Big] \\
&= (1-\alpha)H^{-\alpha}(t)H'(t) + \delta\Big[-\mu\mathbb{E}\|\nabla u(t)\|_2^2 \\
&\quad - (\lambda+\mu)\mathbb{E}\|divu\|_2^2 + \mathbb{E}\int_0^t h(t-r)\langle\nabla u(r), \nabla u(t)\rangle dr \\
&\quad - \mathbb{E}\langle u(t), |v(t)|^{q-2}v(t)\rangle + \mathbb{E}\langle u(t), u(t)|u(t)|^{p-2}ln|u(t)|^k\rangle \\
&\quad + \mathbb{E}\|v(t)\|_2^2\Big] + \delta p\Big[H(t) - G(t) + \mathbb{E}e(t)\Big].
\end{aligned} \tag{4.29}$$

Recalling (4.23) and (4.11), (4.29) leads to

$$\begin{aligned}
L'(t) &\geq (1-\alpha)H^{-\alpha}(t)\mathbb{E}\|v\|_q^q + \delta p\big(H(t) - G(t)\big) \\
&\quad + \delta(\frac{\mu p}{2} - \mu)\mathbb{E}\|\nabla u(t)\|_2^2 + \delta(\frac{p}{2}+1)\mathbb{E}\|v\|_2^2 \\
&\quad - \delta\mathbb{E}\langle u(t), |v(t)|^{q-2}v(t)\rangle - \frac{\delta p}{2}\mathbb{E}\int_0^t h(s)ds\|\nabla u\|_2^2 \\
&\quad + \delta\mathbb{E}\int_0^t h(t-r)\langle\nabla u(r), \nabla u(t)\rangle dr \\
&\quad + (\lambda+\mu)\delta(\frac{p}{2} - 1)\mathbb{E}\|divu\|_2^2 + \frac{\delta k}{p}\mathbb{E}\|u\|_p^p \\
&\quad + \frac{\delta p}{2}\mathbb{E}(h\, o\, \nabla u)(t).
\end{aligned} \tag{4.30}$$

By using Young's and Hölder's inequalities, we get

$$\begin{aligned}\mathbb{E}\int_0^t h(t-r)\langle\nabla u(r),\nabla u(t)\rangle dr &= \mathbb{E}\int_0^t h(t-r)\langle\nabla u(r)-\nabla u(t),\nabla u(t)\rangle dr\\ &+\mathbb{E}\int_0^t h(s)ds\|\nabla u(t)\|_2^2\\ &\geq -\frac{p}{2}\mathbb{E}(h\,o\,\nabla u)(t)-\frac{1}{2p}\mathbb{E}\int_0^t h(s)ds\|\nabla u(t)\|_2^2\\ &+\mathbb{E}\int_0^t h(s)ds\|\nabla u(t)\|_2^2.\end{aligned} \tag{4.31}$$

Hence,

$$\begin{aligned}L'(t) &\geq (1-\alpha)H^{-\alpha}(t)\mathbb{E}\|v\|_q^q+\delta p\big(H(t)-G(t)\big)\\ &+\delta(\frac{\mu p}{2}-\mu)\mathbb{E}\|\nabla u(t)\|_2^2+\delta(\frac{p}{2}+1)\mathbb{E}\|v\|_2^2\\ &-\delta\mathbb{E}\langle u(t),|v(t)|^{q-2}v(t)\rangle\\ &+\delta(1-\frac{1}{2p}-\frac{p}{2})\mathbb{E}\int_0^t h(s)ds\|\nabla u(t)\|_2^2\\ &-\frac{\delta p}{2}\mathbb{E}(h\,o\,\nabla u)(t)+\frac{\delta p}{2}\mathbb{E}(h\,o\,\nabla u)(t)\\ &+(\lambda+\mu)\delta(\frac{p}{2}-1)\mathbb{E}\|divu\|_2^2+\frac{\delta k}{p}\mathbb{E}\|u\|_p^p \qquad (4.32)\\ &\geq (1-\alpha)H^{-\alpha}(t)\mathbb{E}\|v\|_q^q+\delta p\big(H(t)-G(t)\big)\\ &+\delta\big(\frac{\mu p}{2}-\mu\big)\mathbb{E}\|\nabla u(t)\|_2^2\\ &+\delta(\frac{p}{2}+1)\mathbb{E}\|v\|_2^2-\delta\mathbb{E}\langle u(t),|v(t)|^{q-2}v(t)\rangle\\ &+\delta\big(1-\frac{1}{2p}-\frac{p}{2}\big)\mathbb{E}\int_0^t h(s)ds\|\nabla u(t)\|_2^2\\ &+(\lambda+\mu)\delta(\frac{p}{2}-1)\mathbb{E}\|divu\|_2^2+\frac{\delta k}{p}\mathbb{E}\|u\|_p^p.\end{aligned}$$

As $q < p+1$ then $\mathbb{E}\|u(t)\|_q^q \leq c\mathbb{E}\|u(t)\|_{p+1}^{p+1}$ so by using Young's and Hölder's inequality; we obtain

$$\begin{aligned}
\mathbb{E}\langle u(t), |v(t)|^{q-2}v(t)\rangle &\leq \left(\mathbb{E}\|v(t)\|_q^q\right)^{\frac{q-1}{q}} \left(\mathbb{E}\|u(t)\|_q^q\right)^{\frac{1}{q}} \\
&\leq c(\mathbb{E}\|v(t)\|_q^q)^{\frac{q-1}{q}} (\mathbb{E}\|u(t)\|_{p+1}^{p+1})^{\frac{1}{p+1}} \\
&\leq c(\mathbb{E}\|v(t)\|_q^q)^{\frac{q-1}{q}} (\mathbb{E}\|u(t)\|_{p+1}^{p+1}))^{\frac{1}{p+1}-\frac{1}{q}} \left(\mathbb{E}\|u(t)\|_{p+1}^{p+1}\right)^{\frac{1}{q}} \\
&\leq c\left(\frac{q-1}{q}\xi(\mathbb{E}\|v(t)\|_q^q) + \frac{\xi^{1-q}}{q}(\mathbb{E}\|u(t)\|_{p+1}^{p+1})\right) \left(\mathbb{E}\|u(t)\|_{p+1}^{p+1}\right)^{\frac{1}{p+1}-\frac{1}{q}},
\end{aligned} \tag{4.33}$$

where $\xi$ and $c$ are constants.

We consider the following partition of $\mathcal{D}$:

$$\mathcal{D}_1 = \{x \in \mathcal{D} : \ |u| > 1\}, \quad \mathcal{D}_2 = \{x \in \mathcal{D} : \ |u| \leq 1\}.$$

We have

$$\begin{aligned}
\mathbb{E}\int_{\mathcal{D}} |u|^p ln|u|^k dx &= \mathbb{E}\int_{\mathcal{D}_1} |u|^p ln|u|^k dx + \mathbb{E}\int_{\mathcal{D}_2} |u|^p ln|u|^k dx \\
&\leq \mathbb{E}\int_{\mathcal{D}_1} |u|^p ln|u|^k dx \\
&\leq \mathbb{E}\int_{\mathcal{D}_1} k|u|^{p+1} dx \\
&\leq k\mathbb{E}\|u\|_{p+1}^{p+1}.
\end{aligned} \tag{4.34}$$

By (4.24), (4.28) and $-\mathbb{E}e(0) = H(0)$ we have

$$\begin{aligned}
(1+\beta)G(t) &< (1+\beta)E_1 \leq H(0) \leq H(t) \\
&\leq G(t) + \frac{1}{p}\mathbb{E}\int_{\mathcal{D}} |u|^p ln|u|^k dx.
\end{aligned} \tag{4.35}$$

Therefore

$$G(t) \leq \frac{1}{1+\beta} H(t). \tag{4.36}$$

From (4.34), (4.35) and (4.36), we get

$$k\mathbb{E}\|u(t)\|_{p+1}^{p+1} \geq \mathbb{E}\int_{\mathcal{D}_1} k|u|^{p+1} dx \geq p(H(t) - G(t)) \geq p\frac{\beta}{1+\beta} H(t). \tag{4.37}$$

As $H$ is increasing positive nonnegative function and by recalling ( 4.27), we get

$$\begin{aligned}\left(\mathbb{E}\|u(t)\|_{p+1}^{p+1}\right)^{\frac{1}{p+1}-\frac{1}{q}} &\leq \left(p\frac{\beta}{k(1+\beta)}\right)^{(\frac{1}{p+1}-\frac{1}{q})} H^{(\frac{1}{p+1}-\frac{1}{q})}(t) \\ &\leq \left(p\frac{\beta}{k(1+\beta)}\right)^{(\frac{1}{p+1}-\frac{1}{q})} H^{-\alpha}(t) \\ &\leq \left(p\frac{\beta}{k(1+\beta)}\right)^{(\frac{1}{p+1}-\frac{1}{q})} H^{-\alpha}(0).\end{aligned} \tag{4.38}$$

Taking into account (4.38) in (4.33) we find

$$\begin{aligned}\mathbb{E}\langle u(t), |v(t)|^{q-2}v(t)\rangle \leq &\left(c(\frac{p\beta}{k(1+\beta)})^{\frac{1}{p+1}-\frac{1}{q}}\right)\frac{q-1}{q}\xi\left(\mathbb{E}\|v(t)\|_q^q\right)H^{-\alpha}(t) \\ &+\left(c(\frac{p\beta}{k(1+\beta)})^{\frac{1}{p+1}-\frac{1}{q}}\right)\frac{\xi^{1-q}}{q}\left(\mathbb{E}\|u(t)\|_{p+1}^{p+1}\right)H^{-\alpha}(0).\end{aligned} \tag{4.39}$$

Substituting (4.39) into (4.32), we get

$$\begin{aligned}L'(t) \geq & (1-\alpha)H^{-\alpha}(t)\mathbb{E}\|v\|_q^q + \delta p\big(H(t)-G(t)\big) + \delta\mu(\frac{p}{2}-1)E\|\nabla u(t)\|_2^2 \\ & + \delta(\frac{p}{2}+1)\mathbb{E}\|v\|_2^2 + \delta(1-\frac{1}{2p}-\frac{p}{2})\mathbb{E}\int_0^t h(s)ds\|\nabla u(t)\|_2^2 \\ & + (\lambda+\mu)\delta(\frac{p}{2}-1)\mathbb{E}\|divu\|_2^2 - \delta\frac{a_1(q-1)}{q}\xi(\mathbb{E}\|v\|_q^q)H^{-\alpha}(t) \\ & - \delta\frac{a_1}{q}\xi^{1-q}(\mathbb{E}\|u\|_{p+1}^{p+1})H^{-\alpha}(0) + \frac{\delta k}{p}\mathbb{E}\|u\|_p^p\end{aligned} \tag{4.40}$$

where $a_1 = c(\frac{p\beta}{k(1+\beta)})^{\frac{1}{p+1}-\frac{1}{q}}$.
Using lemma 4.3.2 we arrive at

$$\begin{aligned}L'(t) \geq & (1-\alpha-\delta\frac{a_1(q-1)}{q}\xi)H^{-\alpha}(t)\mathbb{E}\|v\|_q^q + \delta p(H(t)-G(t)) + \delta\mu(\frac{p}{2}-1)\mathbb{E}\|\nabla u(t)\|_2^2 \\ & + \delta(\frac{p}{2}+1)\mathbb{E}\|v\|_2^2 + \delta(1-\frac{1}{2p}-\frac{p}{2})\mathbb{E}\int_0^t h(s)ds\|\nabla u(t)\|_2^2 + (\lambda+\mu)\delta(\frac{p}{2}-1)\mathbb{E}\|divu \\ & - \delta\frac{a_1}{q}\xi^{1-q}H^{-\alpha}(0)C\Big(G(t)-H(t)-\frac{1}{2}\mathbb{E}\|v\|_2^2+\frac{1}{p}\mathbb{E}\int_{\mathcal{D}}|u|^p ln|u|^k dx + \mathbb{E}\|u\|_{p+1}^{p+1} \\ & - \frac{1}{2}\mathbb{E}(h\,o\,\nabla u)(t) - \frac{\lambda+\mu}{2}\mathbb{E}\|divu\|_2^2\Big) + \frac{\delta k}{p}\mathbb{E}\|u\|_p^p.\end{aligned} \tag{4.41}$$

Once $\xi$ is fixed, we pick $\delta$ small enough so that

$$1-\alpha-\delta\frac{a_1(q-1)}{q}\xi \geq 0,$$

it implies that

$$L'(t) \geq \delta(p + a_2\xi^{1-q})\big(H(t) - G(t)\big) + \delta(\frac{p}{2} + 1 + a_2\frac{1}{2}\xi^{1-q})\mathbb{E}\|v\|_2^2$$
$$- \delta a_2\xi^{1-q}\frac{1}{p}\mathbb{E}\int_{\mathcal{D}} |u|^p ln|u|^k dx + \delta(\lambda+\mu)\left(\xi^{1-q}\frac{a_2}{2} + (\frac{p}{2} - 1)\right)\mathbb{E}\|divu\|_2^2$$
$$+ \delta a_2\xi^{1-q}\frac{1}{2}\mathbb{E}(h\, o\, \nabla u)(t) + \delta a_3\mathbb{E}\|\nabla u(t)\|_2^2 + \frac{\delta k}{p}\mathbb{E}\|u\|_p^p - \delta a_2\xi^{1-q}\mathbb{E}\|u\|_{p+1}^{p+1}. \tag{4.42}$$

where $a_2 = C\frac{a_1}{q}H^{-\alpha}(0)$ and $a_3 = \mu(\frac{p}{2} - 1) + (1 - \frac{1}{2p} - \frac{p}{2})\int_0^\infty h(s)ds$ which is positive from **(A2)**.
From **(A1)**, (4.11) and Lemma 4.2.1, we have

$$H(t) - G(t) \geq -\frac{1}{2}\mathbb{E}\|v\|_2^2 - (\frac{\mu}{2} + 2C_s)\mathbb{E}\|\nabla u\|_2^2 - \frac{\lambda+\mu}{2}\mathbb{E}\|divu\|_2^2 - \frac{1}{2}\mathbb{E}(h\, o\, \nabla u)(t)$$
$$- \frac{k}{p^2}\mathbb{E}\|u\|_p^p + \frac{1}{p}\mathbb{E}\int_{\mathcal{D}} |u|^p ln|u|^k dx + \mathbb{E}\|u\|_{p+1}^{p+1}. \tag{4.43}$$

Now we add and subtract $\delta a_4\big(H(t) - G(t)\big)$ in (4.42) and using (4.43), we find

$$\begin{aligned} L'(t) \geq\ & \delta\big(p - a_4 + a_2\xi^{1-q}\big)\big(H(t) - G(t)\big) \\ & + \delta\big(\frac{p}{2} + 1 - \frac{a_4}{2} + a_2\frac{1}{2}\xi^{1-q}\big)\mathbb{E}\|v\|_2^2 \\ & + \delta(\lambda+\mu)\left(\xi^{1-q}\frac{a_2}{2} + (\frac{p}{2} - 1) - \frac{a_4}{2}\right)\mathbb{E}\|divu\|_2^2 \\ & + \delta\big(a_2\xi^{1-q}\frac{1}{2} - a_4\big)\mathbb{E}(h\, o\, \nabla u)(t) \\ & + \delta\Big(a_3 - a_4(\frac{\mu + 4C_s}{2})\Big)\mathbb{E}\|\nabla u(t)\|_2^2 \\ & + \frac{\delta k}{p}(1 - \frac{a_4}{p})\mathbb{E}\|u\|_p^p + \frac{\delta}{p}\big(a_4 - a_2\xi^{1-q}\big)\mathbb{E}\int_{\mathcal{D}} |u|^p ln|u|^k dx \\ & + \delta\big(a_4 - a_2\xi^{1-q}\big)\mathbb{E}\|u\|_{p+1}^{p+1}, \end{aligned} \tag{4.44}$$

where $a_4 = min\{a_2\xi^{1-q}, \frac{2a_3}{\mu+4C_s}\} > 0$.
Using (4.37), we obtain

$$\begin{aligned} L'(t) \geq \delta p \frac{\beta}{1+\beta} H(t) + \delta(\frac{p}{2} + 1 - \frac{a_4}{2} + a_2 \frac{1}{2}\xi^{1-q})\mathbb{E}\|v\|_2^2 \\ + \delta(\lambda+\mu)\left(\xi^{1-q}\frac{a_2}{2} + (\frac{p}{2}-1) - \frac{a_4}{2}\right)\mathbb{E}\|divu\|_2^2 \\ + \frac{\delta}{2}(a_2\xi^{1-q} - a_4)\mathbb{E}(h\,o\,\nabla u)(t) \\ + \delta\Big(a_3 - a_4(\frac{\mu+4C_s}{2})\Big)\mathbb{E}\|\nabla u(t)\|_2^2 \\ + \frac{\delta k}{p}(1-\frac{a_4}{p})\mathbb{E}\|u\|_p^p + \delta(a_4 - a_2\xi^{1-q})\mathbb{E}\|u\|_{p+1}^{p+1} \\ \geq \gamma(H(t) + \mathbb{E}\|v\|_2^2 + \mathbb{E}\|divu\|_2^2 + \mathbb{E}(h\,o\,\nabla u)(t) \\ + \mathbb{E}\|\nabla u(t)\|_2^2 + \mathbb{E}\|u\|_p^p + \mathbb{E}\|u\|_{p+1}^{p+1}) \\ \geq 0, \end{aligned} \tag{4.45}$$

where $\gamma > 0$ is the minimum of the coefficients of $H(t)$, $\mathbb{E}\|v\|_2^2$, $\mathbb{E}\|divu\|_2^2$, $\mathbb{E}(h\,o\,\nabla u)(t)$, $\mathbb{E}\|\nabla u(t)\|_2^2$ and $\mathbb{E}\|u\|_p^p$ in (4.45).
Consequently

$$L(t) \geq L(0) > 0, \forall t > 0.$$

Next, we have

$$\begin{aligned} (L(t))^{\frac{1}{1-\alpha}} &= (H^{1-\alpha}(t) + \delta\mathbb{E}\langle u, v\rangle)^{\frac{1}{1-\alpha}} \\ &\leq 2^{\frac{1}{1-\alpha}}(H(t) + \delta^{\frac{1}{1-\alpha}}|\mathbb{E}\int_{\mathcal{D}} uvdx|^{\frac{1}{1-\alpha}}). \end{aligned} \tag{4.46}$$

Therefore, by using Hölder's and Young's inequalities, we obtain

$$
\begin{aligned}
|\mathbb{E}\int_{\mathcal{D}} uvdx|^{\frac{1}{1-\alpha}} &\leq \Big(c(\mathbb{E}\|u\|_{p+1}^2)^{\frac{1}{2}}(\mathbb{E}\|v\|_2^2)^{\frac{1}{2}}\Big)^{\frac{1}{1-\alpha}} \\
&\leq c(\mathbb{E}\|u\|_{p+1}^2)^{\frac{1}{2(1-\alpha)}}(\mathbb{E}\|v\|_2^2)^{\frac{1}{2(1-\alpha)}} \\
&\leq c\left[\frac{(\mathbb{E}\|u\|_{p+1}^2)^{\frac{\eta}{2(1-\alpha)}}}{\eta} + \frac{(\mathbb{E}\|v\|_2^2)^{\frac{\zeta}{2(1-\alpha)}}}{\zeta}\right]
\end{aligned}
\tag{4.47}
$$

with $\frac{1}{\eta} + \frac{1}{\zeta} = 1$.

We choose $\zeta = 2(1-\alpha)$, $\eta = \frac{2(1-\alpha)}{1-2\alpha}$ and we use (4.27) so (4.47) becomes

$$
\begin{aligned}
|\mathbb{E}\int_{\mathcal{D}} uvdx|^{\frac{1}{1-\alpha}} &\leq c\left[(1-2\alpha)\mathbb{E}\|u\|_{p+1}^{\frac{2}{(1-2\alpha)}} + \mathbb{E}\|v\|_2^2\right] \\
&\leq c\left[\mathbb{E}\|u\|_{p+1}^{\frac{2}{(1-2\alpha)}} + \mathbb{E}\|v\|_2^2\right].
\end{aligned}
\tag{4.48}
$$

By applying Lemma 4.3.2 with $s = \frac{2}{1-2\alpha}$ and recalling (4.11), we obtain

$$
\begin{aligned}
|\mathbb{E}\int_{\mathcal{D}} uvdx|^{\frac{1}{1-\alpha}} &\leq c\Big[G(t) - H(t) - \frac{1}{2}\mathbb{E}\|v\|_2^2 \\
&+ \frac{1}{p}\mathbb{E}\int_{\mathcal{D}} |u|^p ln|u|^k dx - \frac{1}{2}\mathbb{E}(h\, o\, \nabla u)(t) \\
&- \frac{\lambda+\mu}{2}\mathbb{E}\|divu\|_2^2 + \mathbb{E}\|u\|_{p+1}^{p+1} + \mathbb{E}\|v\|_2^2\Big] \\
&\leq c\Big[\frac{1}{2}\mathbb{E}\|v\|_2^2 + \frac{1}{2}(\mu - \int_0^t h(s)ds)\mathbb{E}\|\nabla u\|_2^2 \\
&+ \frac{\lambda+\mu}{2}\mathbb{E}\|divu\|_2^2 + \frac{k}{p^2}\mathbb{E}\|u\|_p^p \\
&+ \frac{1}{2}\mathbb{E}(h\, o\, \nabla u)(t) \\
&- \frac{1}{p}\mathbb{E}\int_{\mathcal{D}} |u|^p ln|u|^k dx - \frac{1}{2}\mathbb{E}\|v\|_2^2 + \frac{1}{p}\mathbb{E}\int_{\mathcal{D}} |u|^p ln|u|^k dx \\
&- \frac{1}{2}\mathbb{E}(h\, o\, \nabla u)(t) - \frac{\lambda+\mu}{2}\mathbb{E}\|divu\|_2^2 + \mathbb{E}\|v\|_2^2 + \mathbb{E}\|u\|_{p+1}^{p+1}\Big] \\
&\leq c\Big[\mathbb{E}\|v\|_2^2 + \frac{1}{2}\mu\mathbb{E}\|\nabla u\|_2^2 + \frac{k}{p^2}\mathbb{E}\|u\|_p^p + \frac{\lambda+\mu}{2}\mathbb{E}\|divu\|_2^2 \\
&+ \frac{1}{2}\mathbb{E}(h\, o\, \nabla u)(t) + \mathbb{E}\|u\|_{p+1}^{p+1}\Big].
\end{aligned} \tag{4.49}
$$

Hence,

$$
\begin{aligned}
(L(t))^{\frac{1}{1-\alpha}} &\leq 2^{\frac{1}{1-\alpha}}\Big(H(t) + \delta^{\frac{1}{1-\alpha}} c\Big[\mathbb{E}\|v\|_2^2 + \frac{1}{2}\mu\mathbb{E}\|\nabla u\|_2^2 + \frac{\lambda+\mu}{2}\mathbb{E}\|divu\|_2^2 \\
&+ \frac{k}{p^2}\mathbb{E}\|u\|_p^p + \frac{1}{2}\mathbb{E}(h\, o\, \nabla u)(t) + \mathbb{E}\|u\|_{p+1}^{p+1}\Big]\Big) \\
&\leq \widetilde{C}\Big[H(t) + \mathbb{E}\|v\|_2^2 + \mathbb{E}\|\nabla u\|_2^2 + \mathbb{E}\|divu\|_2^2 + \mathbb{E}\|u\|_p^p \\
&+ \mathbb{E}(h\, o\, \nabla u)(t) + \mathbb{E}\|u\|_{p+1}^{p+1}\Big],
\end{aligned} \tag{4.50}
$$

where $\widetilde{C} = 2^{\frac{1}{1-\alpha}} max\{1, c\delta^{\frac{1}{1-\alpha}}, c\delta^{\frac{1}{1-\alpha}}\frac{\lambda+\mu}{2}, c\delta^{\frac{1}{1-\alpha}}\frac{k}{p^2}\}$.

According to (4.45) and (4.50), we have

$$(L(t))^{\frac{1}{1-\alpha}} \leq \frac{\widetilde{C}}{\gamma} L'(t) \leq \widetilde{K} L'(t). \tag{4.51}$$

In a direct integration of (4.51), we get

$$(L(t))^{\frac{\alpha}{1-\alpha}} \geq \frac{1}{(L(0))^{\frac{-\alpha}{1-\alpha}} - \frac{\widetilde{K}\alpha t}{1-\alpha}}.$$

Therefore, L(t) blows up in time $T \leq T_0 = \frac{1-\alpha}{\alpha K L^{\frac{\alpha}{1-\alpha}}(0)}$, and the proof is completed.

□

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