الجمهوريـــة الجزائريـــة الديمقراطيـــة الشعبيـــة

République Algérienne Démocratique et Populaire

و البـحـث الـعلـمـي وزارة الـتعـلـيـم الـعـالي

Ministère de l'Enseignement Supérieur et de la Recherche Scientifique

Université MUSTAPHA Stambouli

Mascara

جامعة مصطفى اسطمبولي

معسكر

Faculté des Sciences Exactes

Département de Mathématiques

THESE de DOCTORAT

Spécialité : Analyse Mathématique

Intitulée

Existence globale, stabilité et explosion en temps fini des solutions de certaines équations d'évolution non linéaires

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Le : 12/07/2022

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Année Universitaire : 2021-2022

ACKNOWLEDGEMENTS

I would like to express my sincere gratitude towards my supervisor Pr. MOKHTARI Abdelkader, a professor at the University Amar Telidji of Laghouat. I thank him for his thoughtfulness, his patience and his kindness as well as for the trust he put in me.

I would also like to thank my co-director Pr. AMMARI Kais a professor at the University of Monastir (Tunisia) for his kindness and warm reception during my internship in Tunisia, as well as the guidance he provided me with during the preparation of this thesis. Thank you sir, for your orientations, your precious pieces of advice and for your support which I will never forget.

Here I express my gratitude to Pr. BENMERIEM Khaled a Professor at the University Mustapha Stambouli of Mascara for accepting to chair the jury members of this thesis, as it is a great honor for me, I would also want to express my gratitude for his efforts in making this doctoral training program a success.

I thank Mr. RAHMOUNE Abdelaziz Lecturer "A" at university of Amar Telidji Laghouat. It is a great honor for me to receive you as member of the jury.

I thank Mr. YAGOUB Ameur Lecturer "A" at the Amar Telidji University of Laghouat. for the great honor he is presenting me by accepting to be a member of the jury.

This thesis was prepared at the University of Mascara I would like to express my gratitude to the administration, led by the Rector of the University, Professor BENTATA Samir, for their efforts to advance science and aid in the development of its community. I am grateful for their financial and moral support throughout my training program.

Last but not least, my gratitude extends to my former professors at The University of Tiaret.

CONTENTS

RÉSUMÉ

Dans cette thèse, nous nous intéresserons à l'étude de l'existence globale, de la stabilité et de l'explosion en temps fini des solutions pour certaines équations d'évolution non linéaires. Cette thèse est divisée en six chapitres: Dans le premier chapitre, nous rassemblons quelques notions et résultats d'analyse fonctionnelle. Ces résultats sont nécessaires pour développer de nouveaux arguments. Nous présentons dans les cinq autres chapitres nos principaux résultats. Dans le deuxième chapitre, nous considérons une équation de Petrovsky quasi-linéaire avec une dissipation non linéaire localisée, nous prouvons l'existence et l'unicité de la solution globale, et nous étudions également le comportement asymptotique des solutions, sous des hypothèses de croissance appropriées. Dans le troisième chapitre, nous nous intéressons à l'étude dune équation d'onde localement amortie de type Kirchhoff dans un domaine borné. L'amortissement est non linéaire et est localisé dans un sous-ensemble ouvert approprié du domaine considéré, nous prouvons l'existence globale de solutions faibles du problème et nous estimons la décroissance de l'énergie.

Le chapitre quatre est consacré à une équation d'onde non linéaire d'ordre supérieur avec un terme source non linéaire et un terme de retard nous prouverons l'existence globale et donnons un résultat de décroissance générale de l'énergie.

Dans le cinquième chapitre nous considérons une équation de Petrovsky non linéaire avec amortissement interne non linéaire et un terme de retard, nous prouverons l'existence de solutions globales, et nous estimons la décroissance de l'énergie.

Dans le chapitre six, nous étudions l' équation de plaque viscoélastique non linéaire avec un terme de retard, nous prouverons qu'il existe des solutions à énergie initiale négative qui explosent en temps fini, et nous donnerons les estimations pour le temps dexplosion T*. L'approche la plus courante pour analyser nos problèmes est d'utiliser la méthode Faedo-Galerkin pour établir l'existence globale. Ensuite, en utilisant une énergie perturbée appropriée couplée à une technique de multiplicateur pour étudier le comportement asymptotique des solutions.

Mots clés: Existence globale, amortissement non linéaire, viscoélastique, retard, explosion, Faedo-Galerkin, méthode du multiplicateur, amortissement localisé, stabilisation exponentielle, stabilisation polynomiale.

الملخص

في هذه الرسالة، سنهتم بدراسة الوجود الشامل، الاستقرار، والانفجار في وقت محدد لنتائج بعض معادلات التطور غير الخطية.

هذه الأطروحة مقسمة إلى ستة فصول: في الفصل الأول ، نقوم بجمع بعض المفاهيم ونتائج التحليل الدالي. هذه النتائج ضرورية لتطوير مزيد من الحجج.

ثم نقدم نتائجنا الرئيسية في خمس فصول أخرى. في الفصل الثاني ، نأخذ في الاعتبار معادلة بتروفسكي شبه الخطية مع تبديد موضعي غير خطي ، ونثبت وجود حل شامل و وحيد ، ونبحث أيضًا في السلوك المقارب للحلول .

في الفصل الثالث، نعتبر معادلة موجة مثبطة محليا من نوع كيرشوف في مجال محدود. التخميد غير خطي ومتموقع في مجموعة فرعية مفتوحة مناسبة للنطاق قيد الدراسة ، ونثبت الوجود الشامل لحلول ضعيفة للمشكلة ونقدر انحلال الطاقة للحلول.

في الفصل الرابع ، نحن مهتمون بدراسة بعض معادلات الموجات غير الخطية ذات الترتيب الأعلى مع معامل مصدر غير خطي و معامل تأخير سنثبت الوجود الشامل ونتائج الاضمحلال العام. ندرس في الفصل الخامس معادلة بتروفسكي اللاخطية مع التخميد الداخلي غير الخطي وفترة التأخير سنثبت وجود حلول شاملة و كذا السلوك المقارب لها. الفصل السادس مخصص لمعادلة مرونة لزوجية الغير خطية مع فترة تأخير في ردود فعل المجال غير الخطي، سنثبت أن هناك حلول ذات طاقة ابتدائية سالبة تنفجر في وقت محدود وسنقدم تقديرات لوقت الانفجار.

النهج الأكثر شيوعًا لتحليل مشاكلنا هو استخدام طريقة فايدو-غلركين لتأسيس الوجود الشامل. ثم ، باستخدام الطاقة المضطربة المناسبة المقترنة بتقنية المضاعف للتحقيق في السلوك المقارب للحلول.

الكلمات المفتاحية: الوجود الشامل ، التخميد غير الخطي ، مرونة لزوجية ، التأخير ، فايدو-غلركين، طريقة المضاعف ، التخميد الموضعي ، الاستقرار الأسي ، الاستقرار متعدد الحدود.

ABSTRACT

In this thesis, we shall be concerned by the study of global existence, stability, and blow-up in finite time of the solutions for some nonlinear evolution equations.

This thesis is divided into six chapters:

In the first chapter, we collect some notions and results of functional analysis. These results are needed to develop further arguments.

We present in the five other chapters our main results.

In the second chapter, we consider the plate equation with a local dissipation involving the p-Laplacian. The dissipation is effective in a suitable nonvoid subset of the domain under consideration. We establish the existence and uniqueness of global solution, and we prove the exponential and polynomial decay estimates of the energy, under suitable growth assumptions.

In the third chapter, first we prove the existence of a global solution in Sobolev spaces for the initial boundary value problem of the nondegenerate Kirchhoff equation with a nonlinear term localized in a neighborhood of a suitable subset of the domain under consideration. Also we establish the exponential and polynomial stability of the solution.

In the chapter four, we are interested in the study of the initial-boundary value problem for some nonlinear higher-order wave equation with a nonlinear source term and a delay term, we will prove the global existence and general decay results. Chapter five is devoted to a

nonlinear Petrovsky equation with nonlinear internal damping and a delay term, we will prove the existence of global solutions in suitable Sobolev and the general stability estimates. In chapter six, we consider a nonlinear viscoelastic plate equation with a delay term in the nonlinear internal feedback, we will prove that there are solutions with negative initial energy that blow-up in finite time, and we will give the estimates for the blow-up time T^*. The most common approach to analyze our problems is to use the Faedo-Galerkin method to establish the global existence. Then, by using an appropriate perturbed energy coupled with multiplier technique to investigate the asymptotic behavior of the solutions.

Key words: Global existence, nonlinear damping, viscoelastic, delay, blow-up, Faedo-Galerkin, multiplier method, localized damping, exponential stabilization, polynomial stabilization.

AMS classification: 35A01, 35B35, 35B44, 35D30, 93D15.

INTRODUCTION

The emergence of partial differential equations has roots in the study of surfaces in geometry and it is also meant to solve a wide range of problems in mechanics, this dates all the way back to the second half of the nineteenth century. When a large number of mathematicians contributed to solving problems formulated in the form of partial differential equations.

Partial differential equations express and explain different fundamental laws of nature, and that contributed to the advancement of the mathematical research and its methodology. Consequently, this lead to the development of other fields of science, which includes engineering.

The following stage has been marked by the development of linear partial differential equations, by setting a general theory and also by developing methods in finding solutions for these equations. This became a departure point in solving problems of physics and in developing the theory of surfaces.

With the discovery of the basic concepts and properties of the theory of distributions, the modern theory of partial differential equations has become so entrenched, that it plays a central role in modern mathematics, physics, engineering and analysis.

Despite the relative ancientness of partial nonlinear differential equations, it still witnesses a noteworthy development. One of the main impulses for developing nonlinear partial differential equations has been the study of nonlinear wave propagation problems. These problems arise in different areas of applied mathematics, physics, and engineering, including fluid dynamics, nonlinear optics, solid mechanics, plasma physics, quantum field theory, and condensed-matter physics.

Nonlinear wave equations in particular have provided several examples of new solutions that are remarkably different from those obtained for linear wave problems. The most famous examples of these are the corresponding shock waves, water waves, solitons and solitary waves.

One of the most important questions that we had to answer in this study is whether for a nonlinear evolution equation with given initial data, there is a solution at least locally in time, and whether it is unique in the considered class.

Then study the possibility of extending the solution to become global in time. Furthermore, if there is a global solution for a given nonlinear evolution equation, one also wants to know about the asymptotic behavior of the solution as time goes to infinity. But in many cases, a certain norm of the solution tends to infinity as time t approaches T. Such a phenomenon is called blow-up, and T is called the blow-up time.

This thesis is divided into 6 Chapters.

Chapter 1 : General Notations and Preliminaries

In the first chapter, we collect some notions and results of functional analysis as well as some technical methods used to establish either existence or stability of some nonlinear evolution problems. These results are needed to develop further arguments.

Chapter 2 : Well-posedness and energy decay of solutions for a quasilinear Petrovsky with a localized nonlinear dissipation involving the p- Laplacian

In this chapter, we investigate the well-posedness as well as the polynomial and exponential decay rate estimates of the energy associated with the following beam equation with a nonlinear damping term

$$\begin{cases} u_{tt} + \Delta^2 u - \Phi(\|\nabla u\|_2^2)\Delta u - \operatorname{div}(a(x)|\nabla u_t|^{p-2}\nabla u_t) = 0, & \text{in} \quad \Omega \times [0, +\infty[, \\ u = \dfrac{\partial u}{\partial \nu} = 0, & \text{on} \quad \Gamma \times [0, +\infty[, \\ u(x,0) = u_0(x), \;\; u_t(x,0) = u_1(x), & \text{in} \quad \Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^n (n \geq 1)$ with smooth boundary $\partial\Omega = \Gamma$ and $p \geq 2$. The damping term $(\operatorname{div}(a(x)|\nabla u_t|^{p-2}\nabla u_t))$ is localised in a suitable nonvoid subset of the domain under consideration.
The function $\Phi \in \mathcal{C}^1$ satisfying

$$\Phi(s) = \alpha + \beta s^{2\gamma} \qquad (\alpha, \; \beta > 0 \quad \text{and} \quad \gamma \geq 0), \tag{2}$$

and
$a \in L^\infty(\Omega)$ be a nonnegative function satisfying

$$\exists a_0 > 0, \qquad a(x) \geq a_0 \qquad \forall x \in \omega, \tag{3}$$

where ω is an appropriate open set contained in Ω.

When $\Phi = 0$, the problem (1) was treated by L. Tebou [66], he proved existence, uniqueness and smoothness results. Owing to the perturbed energy coupled with multiplier technique, he obtained both exponential and polynomial decay estimates.
When $\Phi = 1$, Ammari et al. [9] studied the wave equation with a linear damping of Kelvin-Voingt type. They obtained a logarithmic decay of energy.

We use the Faedo-Galerkin method to prove the well-posedness of our system, and we prove the stability result by the perturbed energy with multiplier method.

Theorem 0.0.1 (Well-posedness)**.** *Assume that (2), (3) and $p \geq 2$ hold. If $u^0 \in H^4(\Omega) \cap H_0^2(\Omega)$ and $u^1 \in H^2(\Omega)$ then (1) admits a unique solution u such that*

$$u \in L^\infty(0, T, H^2(\Omega))$$

$$u_t \in L^\infty(0, T, H^2(\Omega)) \quad \textit{and} \quad u_{tt} \in L^\infty(0, T, L^2(\Omega)).$$

Further, one has the estimate:

$$\|u_{tt}^k(t)\|^2 + \|\Delta u_t^k(t)\|^2 \leq F_p,$$

where

$$F_p = \int_\Omega \left(|\Delta^2 u^0 - \Phi(\|\nabla u^0\|_2^2)\Delta u^0 - \operatorname{div}(a(x)|\nabla u^1|^{p-2}\nabla u^1)|^2 + |\Delta u^1|^2 \right) dx.$$

Introduce the energy

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\Delta u\|_2^2 + \frac{\alpha}{2}\|\nabla u\|_2^2 + \frac{\beta}{2(\gamma+1)}\|\nabla u\|_2^{2\gamma+2}, \quad \forall t \geq 0.$$

The energy E is a nonincreasing function of the variable t and we have for almost every $t \geq 0$

$$E'(t) = -\int_\Omega a(x)|\nabla u_t|^p dx \leq 0.$$

Theorem 0.0.2 (Stabilization). *(i) If $p = 2$, and $a \in L^\infty(\Omega)$ with (3), then there exist positive constants K_n, λ_1 and λ_2 such that for every weak solution of (1), one has the energy decay estimate:*

$$E(t) \leq \lambda_2 E(0) e^{-\frac{K_n}{2\lambda_1}t}, \quad \forall t \geq 0.$$

(ii) For $p > 2$, satisfying

$$p(n-2) < 2(n-1),$$

and $a \in L^\infty(\Omega)$ with (3). Then the energy E of solution of (1) satisfies:

$$E(t) \leq K(E(0), F_p)(1+t)^{-\frac{1}{\mu_{n,p}}},$$

where

$$\mu_{n,p} = \frac{p-2}{p} \max\left(1, \frac{n-2}{2(n-1)-(n-2)p}\right),$$

and K is a positive constant depending on the initial data as indicated.

Chapter 3 : Well-posedness and energy decay estimates of a non-degenerate Kirchhoff equation with localized nonlinear damping

In this chapter, we study the well-posedness and use a perturbed energy method combining with multiplier method to precise the decay estimates for the energy of non-degenerate Kirchhoff equation with nonlinear damping term

$$\begin{cases} u_{tt} - \Phi(\|\nabla u\|_2^2)\Delta u + a(x)g(u_t) = 0, & x \in \Omega, t \geq 0, \\ u = 0, & x \in \Gamma, t \geq 0, \\ u(x,0) = u_0(x), \ \ u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{4}$$

where Ω is bounded domain of $\mathbb{R}^n$ having a sufficently smooth boundary $\Gamma = \partial\Omega$.
The damping $a(x)g(u_t)$ is nonlinear and is effective only in a neighbourhood of a stuiable subset of the boundary.
The function $\Phi \in \mathcal{C}^1$ and for $s \geq 0$ satisfying

$$\Phi(s) = \alpha + \beta s^\gamma, \qquad \text{with } \alpha, \ \beta > 0 \quad \text{and} \quad \gamma > 0. \tag{5}$$

The nonnegative function $a : \Omega \to [0, \infty)$ is assumed bounded such that

$$\begin{aligned}
&\exists a_0 > 0, \quad a(x) \geq a_0 > 0, \quad \text{a.e} \quad \text{in} \quad \omega. \\
&a(x) \in W^{2,\infty}(\Omega). \\
&\exists a_1 > 0, \quad |\nabla a(x)| \leq a_1 a(x) \quad \text{a.e} \quad \text{in} \quad \Omega \\
&\exists a_2 > 0, \quad |\Delta a(x)| \leq a_2 a(x) \quad \text{a.e} \quad \text{in} \quad \Omega
\end{aligned} \tag{6}$$

where ω is an appropriate open set contained in Ω and $g \in C^1(\mathbb{R}, \mathbb{R})$ is nondecreasing function where $g(0) = 0$ and globally Lipschitz. Suppose that there exist $c_i > 0,\ i = 1, 2, 3, 4$ and $p \geq 1$ such that

$$c_1|s|^p \leq g(s) \leq c_2|s|^{\frac{1}{p}}, \quad \text{if} \quad |s| \leq 1 \tag{7}$$

$$c_3|s| \leq g(s) \leq c_4|s|, \quad \text{if} \quad |s| > 1 \tag{8}$$

$$\exists \tau_0,\ \tau_1 > 0,\ \ \tau_0 \leq g'(s) \leq \tau_1, \ \ \forall s \in \mathbb{R}. \tag{9}$$

When $\Phi = 1$ the problem (4) was trated by L.Tebou [64], he proved some decay estimates of the energy. The method of proof is direct and is based on the multiplier thechnique, on some integral inequalites due to Haraux and Komornik.
The estimations of the modified second energy plays an important role in the proof. Then they proved general stability estimates using multiplier method.
The main results of this chapter are the following

Theorem 0.0.3. *(Well-posedness). Let $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $u_1 \in H_0^1(\Omega)$ and assume that (6)-(9) hold and $\{u_0, u_1\}$ is small. Then the problem (4) has a unique weak solution u such that for any $T > 0$, we have*

$$\begin{aligned}
u &\in L^\infty(0, T; H^2(\Omega) \cap H_0^1(\Omega)), \\
u_t &\in L^\infty(0, T; H_0^1(\Omega)), \\
u_{tt} &\in L^\infty(0, T; L^2(\Omega)).
\end{aligned}$$

For the proof of this redult we use the Faedo-Galerkin method.
Introduce the energy

$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{\alpha}{2}\|\nabla u\|^2 + \frac{\beta}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)}, \qquad \forall t \geq 0.$$

The energy E is a nonincreasing function of the variable t and we have for almost every $t \geq 0$

$$E'(t) = -\int_\Omega a(x) u_t g(u_t) dx \leq 0 \qquad \forall t \geq 0.$$

Theorem 0.0.4. *(Stability). Let $(u_0, u_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$ and suppose that (6)-(9) hold. Then the solution u of the problem (4) satisfies the following energy decay estimates*

$$E(t) \leq CE(0)e^{-wt}, \quad \forall t > 0 \quad \textit{if } p = 1 \tag{10}$$

and

$$E(t) \leq C' t^{-2/(p-1)}, \quad \forall t > 0 \quad \textit{if } p > 1. \tag{11}$$

Here C and w are positive constants independent of the initial data, while C' is a positive constant depending only on the initial energy $E(0)$.

Chapter 4 : Existence and asymptotic behavior of global solutions for a nonlinear higher-order wave equation with a nonlinear source term and a delay term

In this chapter, we consider the higher-order hyperbolic equation with nonlinear source, damping term and delay term in the internal feedback.

$$\begin{cases} u_{tt}(x,t) + \mathcal{A}u(x,t) + \mu_1 g_1(u_t(x,t)) + \mu_2 g_2(u_t(x,t-\tau)) = a|u|^{p-2}u & \text{in } \Omega\times]0,+\infty[, \\ D^\alpha u(x,t) = 0, \quad |\alpha| \leq m-1 & \text{on } \partial\Omega \times [0,+\infty[, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau) & \text{in } \Omega\times]0,\tau[. \end{cases} \tag{12}$$

Where $\mathcal{A} = (-\Delta)^m$, $m \geq 1$, $p > 1$ are real numbers, Ω is a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}^*$, with a smooth boundary $\partial\Omega$, Δ is the Laplace operator in $\mathbb{R}^n$, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, $|\alpha| = \Sigma_{i=1}^n \alpha_i$, $D = \frac{\partial_i^\alpha}{\partial x_i^{\alpha_i}} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} x_2^{\alpha_2} \dots\dots x_n^{\alpha_n}}$, $x = (x_1, x_2 ..., x_n)$, μ_1 and μ_2 are positive real numbers, $\tau > 0$ is a time delay, the initial data (u_0, u_1, f_0) are in a suitable function space, g_1 and g_2 are two functions satisfying:

$g_1 : \mathbb{R} \to \mathbb{R}$ is nondecreasing function of class C^1 and $H : \mathbb{R}_+ \to \mathbb{R}_+$ is convex, increasing and of class $C^1(\mathbb{R}_+) \cap C^2(]0,+\infty[)$ satisfying

$$\begin{gathered} H(0) = 0 \text{ and } H \text{ is linear on } [0,\varepsilon] \text{ or} \\ H'(0) = 0 \text{ and } H'' > 0 \text{ on }]0,\varepsilon] \text{ such that} \\ c_1'|s| \leq |g_1(s)| \leq c_1|s| \quad \text{if } |s| \geq \varepsilon \\ s^2 + g_1^2(s) \leq H^{-1}(sg_1(s)) \quad \text{if } |s| \leq \varepsilon, \end{gathered} \tag{13}$$

where H^{-1} denotes the inverse function of H and ε, c_1, c_1' are positive constants.
$g_2 : \mathbb{R} \to \mathbb{R}$ is an odd nondecreasing function of class $C^1(\mathbb{R})$ such that there exist $c_2, \alpha_1, \alpha_2 > 0$,

$$|g_2'(s)| \leq c_2, \tag{14}$$

$$\alpha_1 s g_2(s) \leq G(s) \leq \alpha_2 s g_1(s), \tag{15}$$

where $G(s) = \int_0^s g_2(r)dr$.

When $a = 0$ and $A = (-\Delta)$, the problem (12) was treated by Benaissa and Louhibi [18]. They use the Faedo-Galerkin method to show the existence and uniqueness of a global solution and obtain a general decay of solution by introducing the multiplier method.
When $g_1(u_t) = |u_t|^{r-2}u_t$ and $\mu_2 = 0$, Ye [73] investigates the existence and asymptotic behaviour of global solution.
In this chapter, we prove the global existence of solutions for small initial data by appliying the potential well theory due to Payne and Sattinger [59] and Sattinger [61]. Meanwhile, we etablished the asymptotic behaviour of global solution as $t \to \infty$ by applying the Lyapunov functional for some perturbed energy.

We define the following functionals:

$$I(t) = I(u(t)) = \|\mathcal{A}^{\frac{1}{2}}u\|_2^2 - a\|u\|_p^p$$
$$J(t) = J(u(t)) = \frac{1}{2}\|\mathcal{A}^{\frac{1}{2}}u\|_2^2 - \frac{a}{p}\|u\|_p^p.$$

We define the stable set

$$\mathcal{W} = \{u \backslash u \in H_0^m(\Omega), I(u) > 0\} \cup \{0\}.$$

We denote the total energy by

$$\begin{aligned} E(u(t)) &= \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\mathcal{A}^{\frac{1}{2}}u(t)\|_2^2 + \xi\int_\Omega\int_0^1 G(z(x,\rho,t))\,d\rho\,dx - \frac{a}{p}\|u\|_p^p \\ &= \frac{1}{2}\|u_t\|_2^2 + \xi\int_\Omega\int_0^1 G(z(x,\rho,t))\,d\rho\,dx + J(u(t)) \qquad \forall t > 0. \end{aligned}$$

The energy E is a nonincreasing function of the variable t and its derivative satisfies

$$\begin{aligned} E'(t) \leq &-\Big(\mu_1 - \frac{\xi\alpha_2}{\tau} - \mu_2\alpha_2\Big)\int_\Omega u_t(x,t)g_1(u_t(x,t))\,dx \\ &-\Big(\frac{\xi\alpha_1}{\tau} - \mu_2(1-\alpha_1)\Big)\int_\Omega z(x,1,t)g_2(z(x,1,t))\,dx \leq 0, \end{aligned}$$

and

$$E(0) = \frac{1}{2}\|u_1\|_2^2 + \frac{1}{2}\|\mathcal{A}^{\frac{1}{2}}u_0\|_2^2 + \xi\int_\Omega\int_0^1 G(f_0(x,-\rho\tau))\,d\rho\,dx - \frac{a}{p}\|u_0\|_p^p$$

is the initial total energy.

Lemma 0.0.5. *Suppose that*

$$2 \leq p \leq \frac{2n}{n-2m}, \quad n > 2m, \tag{16}$$

holds. If $u_0 \in \mathcal{W}$, $u_1 \in L^2(\Omega)$ *and* $f_0 \in H_0^m(\Omega, H^m(0,1))$ *such that*

$$aC_s^p \left(\frac{2p}{p-2}E(0)\right)^{\frac{p-2}{2}} < 1. \tag{17}$$

Then, $u(t) \in \mathcal{W}$ *for all* $t \in [0,+\infty)$.

Theorem 0.0.6 (Global existence). *Let* $u_0 \in H^{2m}(\Omega) \cap \mathcal{W}$, $u_1 \in H_0^m(\Omega) \cap L^2(\Omega)$ *and* $f_0 \in H_0^m(\Omega, H^m(0,1))$ *satisfy the compatibility condition* $f(\cdot,0) = u_1$. *Assume that (13 - 15) hold. Then (12) admits a global weak solution* $u(x,t)$ *such that*

$$u \in L^\infty([0,\infty); H^{2m}(\Omega) \cap H_0^m(\Omega)), \quad u_t \in L^\infty([0,\infty); H_0^m(\Omega) \cap L^2(\Omega)),$$
$$u_{tt} \in L^2([0,\infty);\ L^2(\Omega)).$$

Theorem 0.0.7 (Stabilization). *Assume that (13 - 15) hold. Then, there exist positive constants* w_1, w_2, w_3 *and* ε_0 *such that the solution of (12) satisfies*

$$E(t) \le w_3 H_1^{-1}(w_1 t + w_2) \quad \forall t \ge 0,$$

where

$$H_1(t) = \int_t^1 \frac{1}{H_2(s)}\, ds \tag{18}$$

and

$$H_2(t) = \begin{cases} t & \text{if } H \text{ is linear on } [0,\varepsilon'], \\ tH'(\varepsilon_0 t), & \text{if } H'(0) = 0 \text{ and } H'' > 0 \text{ on }]0,\varepsilon']. \end{cases}$$

Chapter 5 : Well-posedness and energy decay of solutions for a Petrovsky equation with a delay term

In this chapter, we consider a nonlinear Petrovsky equation in a bounded domain with a delay term and a strong dissipation

$$\begin{aligned} u_{tt} + \Delta^2 u - \mu_1 g_1(\Delta_x(u_t(x,t))) - \mu_2 g_2(\Delta_x(u_t(x,t-\tau))) = 0 \quad \text{in } \Omega\times]0,+\infty[, \\ \Delta u(x,t) = u(x,t) = 0 \quad \text{on } \partial\Omega \times [0,+\infty[, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{in } \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau) \quad \text{in } \Omega\times]0,\tau[. \end{aligned} \tag{19}$$

We prove the existence of global solutions in suitable Sobolev spaces by using the energy method combined with Faedo-Galarkin method under condition on the weight of the delay term in the feedback and the weight of the term without delay. Furthermore, we study general stability estimates by using some properties of convex functions.

Theorem 0.0.8. *(Well-posedness). Let $u_0 \in W$, $u_1 \in V$ and $f_0 \in \mathcal{V}$ satisfies the compatibility condition $f(\cdot,0) = u_1$. Assume that (5.3), (5.4) and (5.5) hold. Then (5.1) admits a weak solution*

$$u \in L^\infty([0,\infty); H^4(\Omega)\cap V), \quad u_t \in L^\infty([0,\infty); V),$$
$$u_{tt} \in L^2([0,\infty);\ \mathcal{H}).$$

Theorem 0.0.9. *(Stabilization). Let $(u_0,u_1) \in W \times V$ and $f_0 \in \mathcal{V}$ satisfy the compatibility condition $f(.,0) = u_1$. Assume that (5.3), (5.4) and (5.5) hold, then the global solutions of the problem (5.1) have the following asymptotic property*

$$\psi(t) \leq \psi^{-1}\Big(h(t) + \psi(E(0))\Big), \ \forall t \geq 0 \tag{20}$$

where $\psi(t) = \displaystyle\int_t^1 \frac{1}{\omega\Psi(s)}\, ds$ *for* $t > 0$, $\quad h(t) = 0$ *for* $0 \leq t \leq \frac{E(0)}{\omega\Psi(E(0))}$ *and*

$$h^{-1}(t) = t + \frac{\psi(t+\psi(E(0)))}{\Psi(\psi^{-1}(t+\psi(E(0))))}, \quad \forall t \geq \frac{E(0)}{\Psi(E(0))}$$

$$\psi(t) = \begin{cases} t & \text{if } H \text{ is linear on } [0,\varepsilon] \\ tH'(\varepsilon_0 t) & \text{if } H'(0) = 0 \text{ and } H'' > 0 \text{ on }]0,\varepsilon], \end{cases}$$

where ω and ε are positive constants.

Chapter 6 : Blow-up of solutions to a viscoelastic plate equation with delay

In this chapter, we consider the viscoelastic plate equation in a bounded domain with a delay term in the nonlinear internal feedback, damping and source term

$$\begin{cases} u_{tt}(x,t) + \Delta^2 u(x,t) - \int_0^t h(t-s)\Delta^2 u(x,s)ds + \mu_1 |u_t|^{m-2}u_t & \\ +\mu_2|u_t(t-\tau)|^{m-2}u_t(t-\tau) = b|u|^{p-2}u & \text{in } \Omega\times]0,+\infty[, \\ u(x,t) = \frac{\partial u}{\partial \nu}(x,t) = 0, & \text{on } \partial\Omega \times [0,+\infty[, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau) & \text{in } \Omega\times]0,\tau[, \end{cases} \tag{21}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 1$, is a bounded domain with a smooth boundary $\partial\Omega$, $p > 2$, $m \geq 1$ and h is a positive nonincreasing function defined on $\mathbb{R}^+$, $\tau > 0$ represents the time delay, μ_1 and μ_2 are positive constants, and (u_0, u_1, f_0) are given functions belonging to suitable spaces.

We make the following hypotheses on the relaxation function

$h : \mathbb{R}_+ \to \mathbb{R}_+$ is a $\mathcal{C}^1$ function satisfying

$$1 - \int_0^\infty h(s)ds = l > 0,$$

$$h(s) \geq 0, \qquad h'(s) \leq 0, \qquad \int_0^\infty h(s)ds < \frac{(p/2)-1}{(p/2)-1+(1/2p)}.$$

Kafini et al. [33] consider the problem 21 without the viscoelastic term, i.e., $(h = 0)$, they prove that weak solution to 21 blow-up in finite time wherever the initial energy is negative and the exposent of the source term is more dominant than the exposent of the damping term $(p > m)$.
In this chapter, we obtain the blow-up results in finite time under suitible conditions on the relaxation function $h(.)$ and the negative initial energy.

Theorem 0.0.10 (Local existence)**.** *Suppose that* $m \geq 1$*,* $p > 2$ *and let* $u_0 \in H^4(\Omega) \cap H_0^2(\Omega)$*,* $u_1 \in H_0^2(\Omega) \cap L^2(\Omega)$ *and* $f_0 \in H_0^2(\Omega, H^2(0,1))$ *be given. Assume further that*

$$m \leq p \leq \frac{2(n-2)}{n-4}, \qquad n \geq 5. \tag{22}$$

The problem (21) has a unique local solution

$$u \in \mathcal{C}([0,T_m), H^4(\Omega) \cap H_0^2(\Omega)),$$

$$u_t \in \mathcal{C}([0,T_m), H_0^2(\Omega) \cap L^2(\Omega) \times (0,T_m)),$$

for some $T_m > 0$.

Introduce the energy

$$E(t) = \frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\left(1-\int_0^t h(s)ds\right)\|\Delta u\|_2^2+\frac{1}{2}(h\circ\Delta u)(t)-\frac{b}{p}\|u\|_p^p+\xi\int_\Omega\int_0^1 |z(x,\rho,t)|^m\,d\rho\,dx, \qquad \forall t \geq 0, \tag{23}$$

and

$$E'(t) \leq -C\Big\{\|z(x,1,t)\|_m^m + \|u_t(x,t)\|_m^m\Big\} \leq 0. \qquad \forall t \geq 0. \tag{24}$$

Theorem 0.0.11 (Blow-up)**.** *Suppose that* $m > 1$*,* $p > \max\{2,m\}$ *satisfying (22), let* $u_0 \in H^4(\Omega) \cap H_0^2(\Omega)$*,* $u_1 \in H_0^2(\Omega) \cap L^2(\Omega)$ *and* $f_0 \in \mathcal{C}^1([-\tau,0], L^2(\Omega))$*. Assume further that*

$$E(0) = \frac{1}{2}\|u_1\|_2^2 + \frac{1}{2}\|\Delta u_0\|_2^2 - \frac{b}{p}\|u_0\|_p^p + \xi\int_\Omega\int_0^1 |f_0(x,-\rho\tau|^m\,d\rho\,dx < 0.$$

Then the solution of (21) blow up in finite time

$$T^* \leq \frac{1-\alpha}{\Gamma\alpha[L(0)]^{\alpha/(1-\alpha)}},$$

where Γ *and* α *are positive constant with* $\alpha < 1$ *and* L *is given below.*

CHAPTER 1

GENERAL NOTATIONS AND PRELIMINARIES

The objective of this chapter is to give some general notations used throughout this work and recalling essential notions with classical results. More precisely, the first section is about some general notations. The second concerns some basic notions of functional analysis. The third one is about the existence method that we had used. In the fourth, we give some techniques which are useful for the stability.

1.1 General Notations

$\mathbb{R}$	The real line $(-\infty, +\infty)$
$\mathbb{R}^n$	the $n-$ dimensional Euclidean space
Ω	open set of $\mathbb{R}^n$
$\partial\Omega$	the boundary of Ω
$\frac{du}{dt},\ u'$	the first derivative of $u : [a, b] \to X$
$D^\alpha u = \frac{\partial^{\alpha_1+...+\alpha_n} u}{\partial x_1^{\alpha_1}...\partial x_n^{\alpha_n}}$	$\alpha = \{\alpha_1, ..., \alpha_n\} \in \mathbb{N}^n, \quad \lvert\alpha\rvert = \alpha_1 + ... + \alpha_n$
$L^p(\Omega)$	the space of p-summable functions $u : \Omega \to \mathbb{R},\ 1 \leq p < \infty.$
$W^{m,p}(\Omega)$	the Sobolev space $\{u \in L^p(\Omega),\ D^\alpha u \in L^p(\Omega),\ \lvert\alpha\rvert \leq m,\ 1 \leq p \leq \infty\}$
$W_0^{m,p}(\Omega)$	the closure of $C_c^\infty(\Omega)$ in the norm of $W^{m,p}(\Omega)$
$H^k(\Omega),\ H_0^k(\Omega)$	the spaces $W^{k,2}(\Omega)$ and $W_0^{k,2}(\Omega)$ respectively
$L^p(a, b, X)$	the space of p-summable functions from (a, b) to X

1.2 Functional Spaces

1.2.1 Sobolev Spaces $W^{m,p}(\Omega)$

Let Ω be a bounded or an unbounded domain of $\mathbb{R}^n$ with smooth boundary Γ.
For $m \in \mathbb{N}, 1 \leq p \leq \infty, W^{m,p}(\Omega)$ is defined to be the space of functions u in $L^p(\Omega)$ whose distribution derivatives of order up to m are also in $L^p(\Omega)$. Then, it is known (see, e.g., R.A. Adams [5], J.L. Lions and E. Magenes [43]) that $W^{m,p}(\Omega)$ is a Banach space for the norm

$$\|u\|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha \leq m} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}. \tag{1.1}$$

When $p = 2$ we usually denote $W^{m,p}(\Omega)$ by $H^m(\Omega)$ and this is a Hilbert space for the induced inner product.
We denote by $C_0^k(\Omega)$ the space of $C^k(\Omega)$ functions on Ω with compact support in Ω. The closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$ is denoted by $W_0^{m,p}(\Omega)$, which is a subspace of $W^{m,p}(\Omega)$. We now recall some important properties of the Sobolev spaces $W^{m,p}(\Omega)$ (see, e.g., R.A. Adams [5]).

Theorem 1.2.1 (Density Theorem). *If Ω is a C^m domain, $m \geq 1, 1 \leq p \leq \infty$, then $C^m(\Omega)$ is dense in $W^{m,p}(\Omega)$.*

Theorem 1.2.2 (Imbedding and Compactness Theorem). *Assume that Ω is a bounded domain of class C^m. Then we have* *(i) If $mp < n$, then $W^{m,p}(\Omega)$ is continuously imbedded in $L^{q^*}(\Omega)$ whith $\frac{1}{q^*} = \frac{1}{p} - \frac{m}{n}$:*

$$W^{m,p}(\Omega) \hookrightarrow L^{q^*}(\Omega). \tag{1.2}$$

Moreover, the imbedding operator is compact for any $q, 1 \le q < q^*$.
(ii) If $mp = n$, then $W^{m,p}(\Omega)$ is continuously imbedded in $L^q(\Omega)$, $\forall q, 1 \le q < \infty$:

$$W^{m,p}(\Omega) \hookrightarrow L^q(\Omega). \tag{1.3}$$

Moreover, the imbedding operator is compact, $\forall q, \quad 1 \le q < \infty$. If $p = 1, m = n$, then the above still holds for $q = \infty$.

Theorem 1.2.3 (Poincaré inequality). *Let Ω be a bounded domain in $\mathbb{R}^n$ and $u \in H_0^1(\Omega)$. Then there is a positive constant C depending only on Ω and n such that*

$$\|u\|_{L^2(\Omega)} \le C\|\nabla u\|_{L^2(\Omega)}, \qquad \forall u \in H_0^1(\Omega). \tag{1.4}$$

1.2.2 Abstract Functions Valued in Banach Spaces

For the study of evolution equations, it is convenient to introduce abstract functions valued in Banach spaces.
Let X be a Banach space, $1 \le p < \infty$, $-\infty \le a < b \le \infty$. Then $L^p((a,b);X)$ denotes the space of L^p functions from (a,b) into X. It is a Banach space for the norm

$$\|f\|_{L^p((a,b);X)} = \left(\int_a^b \|f(t)\|_X^p \, dt\right)^{\frac{1}{p}} \tag{1.5}$$

where the integral is understood in the Bochner sense.
For $p = \infty$, $L^\infty((a,b);X)$ is the space of measurable functions from (a,b) into X being essentially bounded. It is a Banach space for the norm

$$\|f\|_{L^\infty((a,b);X)} = \sup_{t\in(a,b)} ess\|f(t)\|_X. \tag{1.6}$$

Similarly, when $-\infty \le a < b \le \infty$ we can define Banach spaces $C^k([a,b];X)$ for the norm

$$\|f\|_{C^k([a,b];X)} = \sum_{i=0}^{k} \max_{t\in[a,b]} \left\| \frac{d^i f}{dt^i}(t) \right\|_X. \tag{1.7}$$

The following result (see, e.g., R. Temam [68]) is needed in the study of linear or nonlinear evolution equations.

Lemma 1.2.4. *Let V, H, V' be three Hilbert spaces with V' being the dual space of V and each space included and dense in the following one:*

$$V \hookrightarrow H \cong H' \hookrightarrow V'.$$

If an abstract function u belongs to $L^2([0,T], V)$ and its derivative u_t in the distribution sense belongs to $L^2([0,T], V')$, then u is almost everywhere equal to a function continuous from $[0,T]$ into H and we have the following equality, which holds in the scalar distribution sense on $(0,T)$:

$$\frac{d}{dt}\|u\|_H^2 = 2\langle u', u\rangle,$$

where $\langle ., . \rangle$ denotes the dual product between V and V'.

1.2.3 Some useful inequalities

The following elementary inequalities are very useful, will be frequently used.

1. The Young inequality

Let a, b and ϵ be positive constants and $p, q \geq 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$ab \leq \frac{\epsilon^p a^p}{p} + \frac{b^q}{q\epsilon^q}. \tag{1.8}$$

2. The Cauchy-Schwarz inequality

Every inner product satisfies the Cauchy-Schwarz inequality

$$\langle u, v\rangle \leq \|u\|\|v\|. \tag{1.9}$$

The equality sign holds if and only if u and v are proportional.

3. The Holder inequality

Let $1 \leq p \leq \infty$: Assume that $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, then $uv \in L^1(\Omega)$ and

$$\int_\Omega |uv|\, dx \leq \|u\|_p \|v\|_q. \tag{1.10}$$

4. The Gronwall inequality

Suppose that a, b are nonnegative constants and $u(t)$ is a nonnegative integrable function. Suppose that the following inequality holds for $0 \leq t \leq T$:

$$u(t) \leq a + b\int_0^t u(s)ds. \tag{1.11}$$

Then for $0 \leq t \leq T$:

$$u(t) \leq ae^{bt}. \tag{1.12}$$

1.3 Existence Methods

In this section we introduce one of important methods, namely the compactness method. This method was mainly developed in the 1960's, but it is still powerful tool today to deal with nonlinear evolution equations.

1.3.1 The Faedo-Galerkin method

The method is based on the three steps :
(i) choose certain basis of functions in an appropriate Sobolev space, and solve the approximate problems in any finite dimensional space spanned by finite basis functions. This often turns out to be an initial value problem for nonlinear ordinary differential equations. By the well-known local existence theorem for ordinary differential equations, local existence of solution to the approximate problem follows.
(ii) Obtain the compactness estimates for the solution of the approximate problem. It also turns out that the solution to the approximate problem globally exists.
(iii) Further use of the obtained compactness estimates allows one to choose a subsequence of solutions of the approximate problem obtained in the second step, converging to a solution of the original problem; uniqueness of solution for the original problem has to be proved separately, but the compactness estimates obtained in the second step are still very useful for this purpose.

1.4 Stabilty Methods

The purpose of stabilization is to attenuate the vibrations by feedback, therefore it is to ensure the decay of the energy solutions to 0 more or less quickly by a dissipation mechanism. More precisely, the stabilization problem in which we are interested amounts to determining the asymptotic behavior of the energy that we denote by $E(t)$ (this is the norm of solutions in the state space), to study its limit in order to determine if this limit is zero or not, and, if this limit is zero, to give an estimate of the decay rate of energy to zero. They are several type of stabilization :
1)Strong stabilization:

$$\lim_{t\to+\infty} E(t) = 0.$$

2)Exponential stabilization:

$$E(t) \le Ce^{-\delta t} \qquad \forall t > 0.$$

3)Polynomial stabilization:

$$E(t) \le \frac{C}{t^\alpha} \qquad \forall t > 0.$$

Where C, δ, and α are positive constans and C depends on the initial data.

1.4.1 Lyapunov's method

In 1982, Lyapunov introduced an energy function that he used to study the stability of some nonlinear systems without calculating explicitly their solutions. This method is known today by **Lyapunov's method** and it played an important role in the stability theory of partial differential equations.
The Lyapunov method consists in constructing (or finding) a functional L equivalent to the energy of the solution and tending to zero with some decay rate.

1.4.2 The multiplier method

We use this method to get a better estimate of the decay rate, A. Haraux and V. Komornik have improved and generalized this method. They introduced integral inequalities which make it possible to obtain very efficiently and very good decay estimates for many linear or nonlinear problems.
We will use these integral inequalities to study the decay rate of the energy of nonlinear dissipative problems.

Theorem 1.4.1 (Haraux [30])**.** *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a nonincreasing function and assume that there exists a constant $T > 0$ such that*

$$\int_t^{\infty} E(s)ds \leq TE(t), \qquad \forall t \in \mathbb{R}_+. \tag{1.13}$$

Then

$$E(t) \leq E(0)e^{1-t/T} \qquad \forall t \geq T. \tag{1.14}$$

Theorem 1.4.2 (Komornik [35], Theorem 9.1)**.** *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a nonincreasing function and assume that there are two constants $\alpha > 0$ and $T > 0$ such that*

$$\int_t^{\infty} E^{\alpha+1}(s)ds \leq TE(0)^{\alpha}E(t), \qquad \forall t \in \mathbb{R}_+. \tag{1.15}$$

Then we have

$$E(t) \leq E(0)\left(\frac{T+\alpha t}{T+\alpha T}\right)^{-1/\alpha} \qquad \forall t \geq T. \tag{1.16}$$

CHAPTER 2

WELL-POSEDNESS AND ENERGY DECAY OF SOLUTIONS FOR A QUASILINEAR PETROVSKY WITH A LOCALIZED NONLINEAR DISSIPATION INVOLVING THE P- LAPLACIAN

2.1 Introduction

We prove the existence and uniqueness of global solutions by Faedo-Galerkin method for the Cauchy problem concerning the evolution equation

$$u_{tt} + \Delta^2 u - \Phi(\|\nabla u\|_2^2)\Delta u - \operatorname{div}(a(x)|\nabla u_t|^{p-2}\nabla u_t) = 0, \tag{2.1}$$

suggested by the study of plates and beams, where $p \geq 2$ and Φ is a real function. We also investigate the asymptotic behavior of the solutions, under suitable growth assumptions. We will use for this task an appropriate perturbed energy coupled with multiplier technique.

The equation 2.1 is a generalization of the problem associated with

$$\frac{\partial^2 u}{\partial t^2} + \frac{EI}{\rho A}\frac{\partial^4 u}{\partial x^4} - \left(\frac{H}{\rho} + \frac{E}{2\rho L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{2.2}$$

which was proposed by Woinowsky-Krieger [70] as a model for the deflection of an extensible beam of length L with hinged ends. The letters H, E, ρ, I and A denote, respectively, the tension in the rest position, the Young elasticity modulus, the density, the cross-sectional moment of inertia and the cross-sectional area. The nonlinear term in the brackets is a correction to the classical Euler-Bernoulli equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{EI}{\rho A}\frac{\partial^4 u}{\partial x^4} = 0,$$

which does not consider the changes of the tension induced by the variation of the length during the deflection. This kind of correction was proposed by Kirchhoff [34] to generalize D'Alembert's equation with clamped ends. For this reason, 2.1 is often called a Kirchhoff type beam equation. An interesting discussion on the various models for beam equations is presented in Arosio [13]. We refer the reader to Dickey [27] and Ball [14] for the earlier mathematical analysis of Eq. 2.2.

In our case, let $p \geq 2$, and we consider the following damped beam equation

$$\begin{cases} u_{tt} + \Delta^2 u - \Phi(\|\nabla u\|_2^2)\Delta u - \operatorname{div}(a(x)|\nabla u_t|^{p-2}\nabla u_t) = 0, & \text{in} \quad \Omega \times [0, +\infty[, \\ u = \dfrac{\partial u}{\partial \nu} = 0, & \text{on} \quad \Gamma \times [0, +\infty[, \\ u(x,0) = u_0(x), \;\; u_t(x,0) = u_1(x), & \text{in} \quad \Omega, \end{cases} \tag{2.3}$$

where Ω is a bounded domain of $\mathbb{R}^n$ whose boundary Γ is assumed regular. Φ is a real function and the function $a : \mathbb{R} \to \mathbb{R}$ is a continuous nondecreasing function. Let $x^0 \in \mathbb{R}^n$ be an arbitrary point of $\mathbb{R}^n$ and we set:

$$\Gamma(x^0) = \Big\{ x \in \Gamma; \quad m(x).\nu(x) > 0 \Big\}, \tag{2.4}$$

where ν represents the unit normal vector pointing towards the exterior of Ω and

$$m(x) = x - x^0. \tag{2.5}$$

Let ω be a neighborhood of $\Gamma(x^0)$ in Ω and consider δ sufficiently small such that

$$Q_0 = \Big\{ x \in \Omega; d(x, \Gamma(x^0)) < \delta \Big\} \subset \omega, \tag{2.6}$$

$$Q_1 = \Big\{ x \in \Omega; d(x, \Gamma(x^0)) < 2\delta \Big\} \subset \omega. \tag{2.7}$$

Here, if $A \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, $d(x; A) = \inf_{y \in A}(|x - y|)$. Then $Q_0 \subset Q_1 \subset \omega$.

Next, we make some remarks about early works in connection with our problem. When $\Phi = 0$ the problem (2.3) was treated by L. Tebou [**66**]. He proved the existence and uniqueness of a global solution by semigroup theory and he used the perturbed energy with multiplier method to show the polynomial and exponential decay of the solution. A locally distributed linear viscoelastic dissipation of Kelvin-Voigt type has been considered in the literature in connection with the one dimensional beam and wave equations ([24], [46]). Other works closely related to the present work are ([23], [21], [28], [40]).

Ammari et al. [9] studied the system

$$\begin{cases} u_{tt} + \Delta u - \operatorname{div}(a(x)\nabla u_t) = 0, & \text{in} \quad \Omega \times [0, +\infty[, \\ u = 0, & \text{on} \quad \Gamma \times [0, \infty[, \\ u(x,0) = u_0(x), \;\; u_t(x,0) = u_1(x), & \text{in} \quad \Omega, \end{cases} \tag{2.8}$$

where $a(x) = d\mathbf{1}_\omega(x)$, $d > 0$ is a constant. The authors obtained a logarithmic decay of energy. Their idea is to transform the resolvent problem of (2.8) to a transmission system to be easy to use the so-called Carleman estimate. In [32], the same problem has been considered and the polynomial energy estimate was showed. Liu and Rao in [44] and Tebou [65] proved the exponential stability.

The main purpose of this chapter is to study the well-posedness and to construct a perturbed energy method leading to precise decay estimates for the energy of (2.3) with a nonlinear damping term. The damping term is localized in a suitable nonvoid subset of the domain under consideration, and it involves the p-Laplacian.

2.2 Notation and Preliminaries

We begin by introducing some notations that will be used throughout this work. For the standard L^p space we write

$$\|u\|_p^p = \int_\Omega |u|^p \, dx.$$

Next, we give the precise assumptions on the functions $a(x)$.

(A.1) Let $a \in L^\infty(\Omega)$ be a nonnegative function satisfying

$$a(x) \geq a_0 > 0, \quad \text{a.e} \quad \text{in} \quad \omega. \tag{2.9}$$

(A.2) Let $\Phi(s)$ be a $\mathcal{C}^1$ real function. Assume that there exist α, $\beta > 0$ and $\gamma \geq 0$ such that

$$\Phi(s) = \alpha + \beta s^{2\gamma}.$$

Lemma 2.2.1. *Let $p > 2$. Suppose that $a \in L^\infty_+(\Omega)$ and $u \in W_0^{1,p}(\Omega)$. Then $\operatorname{div}(a|\nabla u|^{p-2}\nabla u) \in W^{-1,p}(\Omega)$ and*

$$\|\operatorname{div}(a|\nabla u|^{p-2}\nabla u)\|_{W^{-1,p}(\Omega)} \leq \|a\|^{1/p}_{L^\infty(\Omega)} \left(\int_\Omega a|\nabla u|^p dx \right)^{\frac{p-1}{p}}. \tag{2.10}$$

Furthermore, if $u \in H_0^2(\Omega)$ *and the parameter* p *satisfies* $(n-2)p < 2(n-1)$, *then* $\operatorname{div}(a|\nabla u|^{p-2}\nabla u) \in H^{-1}(\Omega)$ *and*

$$\|\operatorname{div}(a|\nabla u|^{p-2}\nabla u)\|_{H^{-1}(\Omega)} \leq \begin{cases} C\left(\int_\Omega a|\nabla u|^p dx\right)^{\frac{1}{p}} \|\Delta u\|_2^{p-2} & \text{if } n \in \{1,2\}, \\ C\left(\int_\Omega a|\nabla u|^p dx\right)^{\frac{2(n-1)-(n-2)p}{2n-(n-2)p}} \|\Delta u\|_2^{\frac{n(p-2)}{(2n-(n-2)p)}} & \text{if } n \geq 3. \end{cases} \tag{2.11}$$

Proof. **Case 1:** $n = \{1,2\}$. By using the generalized Hölder inequality and the Sobolev embedding theorem, we have

$$\begin{aligned} |\langle \operatorname{div}(a|\nabla u|^{p-2}\nabla u), \varphi\rangle| &= \left|\int_\Omega a|\nabla u|^{p-2}\nabla u.\nabla\varphi\, dx\right| \\ &\leq \Big(\int_\Omega a^p|\nabla u|^p\, dx\Big)^{\frac{1}{p}} \Big(\int_\Omega |\nabla u|^{2p}\, dx\Big)^{\frac{p-2}{2p}} \Big(\int_\Omega |\nabla \varphi|^2\, dx\Big)^{\frac{1}{2}} \\ &\leq C\Big(\int_\Omega a|\nabla u|^p\, dx\Big)^{\frac{1}{p}} \Big(\int_\Omega |\nabla u|^{2p}\, dx\Big)^{\frac{p-2}{2p}} \|\nabla\varphi\|_{H_0^1(\Omega)} \\ &\leq C\Big(\int_\Omega a|\nabla u|^p\, dx\Big)^{\frac{1}{p}} \Big(\int_\Omega |\Delta u|^{2}\, dx\Big)^{\frac{p-2}{2}} \|\nabla\varphi\|_{H_0^1(\Omega)}. \end{aligned} \tag{2.12}$$

Case 2: $n \geq 3$. We use the generalized Hölder inequality to get

$$\begin{aligned} |\langle \operatorname{div}(a|\nabla u|^{p-2}\nabla u), \varphi\rangle| &= \left|\int_\Omega a|\nabla u|^{p-2}\nabla u.\nabla\varphi\, dx\right| \leq \int_\Omega a|\nabla u|^{p-1-\xi}|\nabla u|^{\xi}.|\nabla\varphi|\, dx\Big| \\ &\leq \Big(\int_\Omega a^r|\nabla u|^{(p-1-\xi)r}\, dx\Big)^{\frac{1}{r}} \Big(\int_\Omega |\nabla u|^{\frac{2\xi r}{r-2}}\, dx\Big)^{\frac{r-2}{2r}} \Big(\int_\Omega |\nabla \varphi|^2\, dx\Big)^{\frac{1}{2}} \\ &\leq C\Big(\int_\Omega a|\nabla u|^p\, dx\Big)^{\frac{1}{r}} \Big(\int_\Omega |\nabla u|^{\frac{2((p-1)r-p)}{r-2}}\, dx\Big)^{\frac{r-2}{2r}} \|\nabla\varphi\|^2_{H_0^1(\Omega)} \\ &\leq C\Big(\int_\Omega a|\nabla u|^p\, dx\Big)^{\frac{2(n-1)-(n-2)p}{2n-(n-2)p}} \Big(\int_\Omega |\Delta u|^{2}\, dx\Big)^{\frac{2(n-1)-(n-2)p}{2(2n-(n-2)p)}} \|\nabla\varphi\|^2_{H_0^1(\Omega)}, \end{aligned} \tag{2.13}$$

with $\xi = \frac{(p-1)r-p}{r-2}$ and $r = \frac{2n-(n-2)p}{2(n-1)-(n-2)p}$. We derive the value of r in (2.13) using the Sobolev embedding theorem. $H^1(\Omega)$ is continuously embedded in $L^{\frac{2((p-1)r-p)}{r-2}}(\Omega)$ for $\frac{(p-1)r-p}{r-2} \leq \frac{n}{n-2}$. $\square$

2.3 Well-posedness of the problem

In this section, we prove the existence and uniqueness of solutions for (2.3) using Faedo-Galerkin method [41]. The main result of this section is presented by the following theorem

Theorem 2.3.1. *Assume that* (**A**.**1**), (**A**.**2**) *and* $p \geq 2$ *hold. If* $u^0 \in H^4(\Omega) \cap H_0^2(\Omega)$ *and* $u^1 \in H^2(\Omega)$ *then (2.3) admits a unique solution* u *such that*

$$u \in L^\infty(0,T,H^2(\Omega))$$

$$u_t \in L^\infty(0,T,H^2(\Omega)) \quad and \quad u_{tt} \in L^\infty(0,T,L^2(\Omega)).$$

Further, one has the estimate:

$$\|u_{tt}^k(t)\|^2 + \|\Delta u_t^k(t)\|^2 \le F_p,$$

where

$$F_p = \int_\Omega \left(|\Delta^2 u^0 - \Phi(\|\nabla u^0\|_2^2)\Delta u^0 - \operatorname{div}(a(x)|\nabla u^1|^{p-2}\nabla u^1)|^2 + |\Delta u^1|^2\right) dx. \tag{2.14}$$

Proof. We will use the Faedo-Galerkin method to prove the existence of global solutions. Let $(w^k)_{k\in\mathbb{N}}$ be normalized eigenfunctions of the operator Δ^2 corresponding to positive real eigenvalues λ_k tending to $+\infty$. Hence

$$\begin{cases} \Delta^2 w^k = \lambda_k w^k, & \text{in } \Omega \\ w^k = \dfrac{\partial w^k}{\partial \nu} = 0, & \text{in } \partial\Omega \end{cases}$$

Then $\{w^k | k \in \mathbb{N}\}$ is an orthogonal basis of $L^2(\Omega)$, $H^2(\Omega)$ and $H^4(\Omega)$, i.e, the set $V^n = span\{w^m | m = 1, 2, ..., n\}$ is dense in $L^2(\Omega)$, $H^2(\Omega)$ and $H^4(\Omega)$.

Step 1: Approximate solutions.

We construct approximate solutions u^k, $k = 1, 2, 3, \ldots$, in the form

$$u^k(t,x) = \sum_{j=1}^{k} c^{jk}(t) w^j(x),$$

where c^{jk} are determined by the ordinary differential equations

$$(u_{tt}^k + \Delta^2 u^k - \Phi(\|\nabla u^k\|_2^2)\Delta u^k - \operatorname{div}(a(x)|\nabla u_t^k|^{p-2}\nabla u_t^k), w^j) = 0, \tag{2.15}$$

with the initial conditions

$$u^k(0) = u_0^k = \sum_{j=1}^{k}(u_0, w^j)w^j \to u^0, \quad \text{in } H^4(\Omega)\cap H_0^2(\Omega) \text{ as } k \to +\infty, \tag{2.16}$$

$$u_t^k(0) = u_1^k = \sum_{j=1}^{k}(u_1, w^j)w^j \to u^1, \quad \text{in } H^2(\Omega) \text{ as } k \to +\infty. \tag{2.17}$$

$$\Delta^2 u_0^k - \Phi(\|\nabla u_0^k\|_2^2)\Delta u_0^k - \operatorname{div}(a(x)|\nabla u_1^k|^{p-2}\nabla u_1^k) \to \Delta^2 u^0 - \Phi(\|\nabla u^0\|_2^2)\Delta u^0 - \operatorname{div}(a(x)|\nabla u^1|^{p-2}\nabla u^1) \quad \text{in } L^2(\Omega). \tag{2.18}$$

The standard theory of ODE guarantees that the system (2.15)-(2.18) has a unique solution in $[0, t_k)$, with $0 < t_k < T$, by Zorn lemma since the nonlinear terms in (2.15) are locally Lipschitz continuous. Note that $u^k(t)$ is of class $\mathcal{C}^2$. In the next step, we obtain a priori estimates for the solution of the system (2.15)-(2.18), so that it can be extended outside $[0, t_k)$ to obtain one solution defined for all $T > 0$, using a standard compactness argument for the limiting procedure.

Step 2: The first estimate

Multiplying (2.15) by c_t^{jk} and summing over j from 1 to k, we find

$$\frac{1}{2}\frac{d}{dt}\left\{\|u_t^k\|_2^2 + \|\Delta u^k\|_2^2 + \alpha\|\nabla u^k\|_2^2 + \frac{\beta}{\gamma+1}\|\nabla u^k\|_2^{2\gamma+2}\right\} + \int_\Omega a(x)|\nabla u_t^k|^p\,dx = 0.$$

Integrating in $[0, t]$, $t < t_k$, since the sequences u_0^k, u_1^k converge, and from (2.16) and (2.17), we have

$$\begin{aligned} &\|u_t^k\|_2^2 + \|\Delta u^k\|_2^2 + \alpha\|\nabla u^k\|_2^2 + \frac{\beta}{\gamma+1}\|\nabla u^k\|_2^{2\gamma+2} + 2\int_0^t\int_\Omega a(x)|\nabla u_t^k|^p\,dx \\ &\le \|u_1^k\|_2^2 + \|\Delta u_0^k\|_2^2 + \alpha\|\nabla u_0^k\|_2^2 + \frac{\beta}{\gamma+1}\|\nabla u_0^k\|_2^{2\gamma+2} \\ &\le C_1. \end{aligned} \tag{2.19}$$

For some C_1 independent of k. These estimates imply that the solution u^k exists globally in $[0, +\infty]$. Invoking the estimate (2.19) we get

$$u^k \text{ is bounded in } L^\infty(0, T, H^2(\Omega)), \tag{2.20}$$

$$u_t^k \text{ is bounded in } L^\infty(0, T, L^2(\Omega)), \tag{2.21}$$

$$a(x)|\nabla u_t^k|^p \text{ is bounded in } L^1(\Omega \times (0, T)). \tag{2.22}$$

Step 3: The second estimate

First, we estimate $u_{tt}^k(0)$ by taking $t = 0$ in (2.15), we obtain

$$(u_{tt}^k(0) + \Delta^2 u^k(0) - \Phi(\|\nabla u^k(0)\|_2^2)\Delta u^k(0) - \operatorname{div}(a(x)|\nabla u_t^k(0)|^{p-2}\nabla u_t^k(0)), w^j) = 0, \tag{2.23}$$

multiplying by c_{tt}^{jk} and summing over j from 1 to k, we arrive at

$$\begin{aligned} &(u_{tt}^k(0), u_{tt}^k(0)) + (\Delta^2 u^k(0), u_{tt}^k(0)) - \Phi(\|\nabla u^k(0)\|_2^2)(\Delta u^k(0), u_{tt}^k(0)) \\ &= \operatorname{div}(a(x)|\nabla u_t^k(0)|^{p-2}\nabla u_t^k(0)), u_{tt}^k(0)). \end{aligned} \tag{2.24}$$

Using Hölder's inequality, (2.16) -(2.18), we have

$$\begin{aligned} \|u_{tt}^k(0)\| &\le \Big(\int_\Omega |\Delta^2 u_0^k - \Phi(\|\nabla u_0^k\|_2^2)\Delta u_0^k - \operatorname{div}(a(x)|\nabla u_1^k|^{p-2}\nabla u_1^k)|^2\,dx\Big)^{\frac{1}{2}} \\ &\le C_2, \end{aligned} \tag{2.25}$$

where C_2 is a positive constant independent of k.

Step 4: The third estimate

We assume first $t < T$ and apply (2.15) at points t and $t+\xi$ with ξ such that $0 < \xi < T - t$. By replacing w^j by $u_t^k(t+\xi) - u_t^k(t)$, we find, since $a(x)|\nabla u_t^k|^{p-2}\nabla u^k$ is nondecreasing function

$$\begin{aligned}&\Big(u_{tt}^k(t+\xi) - u_{tt}^k(t) + \Delta^2 u^k(t+\xi) - \Delta^2 u^k(t), u_t^k(t+\xi) - u_t^k(t)\Big)\\&+\Big(-\Phi(\|\nabla u^k(t+\xi)\|_2^2)\Delta u^k(t+\xi) + \Phi(\|\nabla u^k(t)\|_2^2)\Delta u^k(t), u_t^k(t+\xi) - u_t^k(t)\Big)\\&\leq 0.\end{aligned}$$

Hence we deduce that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big[\|u_t^k(t+\xi) - u_t^k(t)\|_2^2 + \|\Delta u^k(t+\xi) - \Delta u^k(t)\|_2^2\Big]\\&-\Phi(\|\nabla u^k(t+\xi)\|_2^2)\Big(\Delta u^k(t+\xi) - \Delta u^k(t), u_t^k(t+\xi) - u_t^k(t)\Big)\\&-\Big[\Phi(\|\nabla u^k(t+\xi)\|_2^2) - \Phi(\|\nabla u^k(t)\|_2^2)\Big]\times\Big(\Delta u^k(t), u_t^k(t+\xi) - u_t^k(t)\Big)\\&\leq 0.\end{aligned} \tag{2.26}$$

Set

$$\varphi^{\xi k}(t) = \|u_t^k(t+\xi) - u_t^k(t)\|_2^2 + \|\Delta u^k(t+\xi) - \Delta u^k(t)\|_2^2. \tag{2.27}$$

Using (2.26), Cauchy-Schwarz inequality and the fact that, Φ is $\mathcal{C}^1$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\varphi^{\xi k}(t) \leq\ &\Phi(\|\nabla u^k(t+\xi)\|_2^2)\Big(\Delta u^k(t+\xi) - \Delta u^k(t), u_t^k(t+\xi) - u_t^k(t)\Big)\\&+c\Big|\|\nabla u^k(t+\xi)\|_2^2 - \|\nabla u^k(t)\|_2^2\Big|\times\|\Delta u^k(t)\|_2\|u_t^k(t+\xi) - u_t^k(t)\|_2.\end{aligned}$$

The last term can be rewritten as

$$c\Big|\Big(\Delta(u^k(t+\xi)) - \Delta(u^k(t)), u^k(t+\xi) - u^k(t)\Big)\Big|\times\|\Delta u^k(t)\|_2\|u_t^k(t+\xi) - u_t^k(t)\|_2.$$

Using **(A.2)**, (2.20) and Young's inequality, it follows that

$$\varphi_t^{\xi k}(t) \leq c\varphi^{\xi k}(t).$$

Therefore, we deduce that

$$\varphi^{\xi k}(t) \leq \varphi^{\xi k}(0)e^{cT}; \qquad \forall t \in [0, T[.$$

Dividing the two sides by ξ^2, letting $\xi \to 0$, and using (2.27), we deduce that

$$\|u_{tt}^k(t)\|_2^2 + \|\Delta u_t^k(t)\|_2^2 \leq \|u_{tt}^k(0)\|^2 + \|\Delta u_1^k(0)\|^2,$$

by (2.17) and (2.25), we get

$$\|u_{tt}^k(t)\|_2^2 + \|\Delta u_t^k(t)\|_2^2 \leq F_p.$$

Therefore, we conclude that

$$u_{tt}^k \text{ is bounded in } L^\infty(0,T,L^2(\Omega)), \tag{2.28}$$

$$u_t^k \text{ is bounded in } L^\infty(0,T,H^2(\Omega)). \tag{2.29}$$

Using Lemma 2.2.1, (2.22) and (2.29), we deduce that

$$\operatorname{div}(a(x)|\nabla u_t^k|^{p-2}\nabla u_t^k) \text{ is bounded in } H^{-1}([0,T]\times\Omega). \tag{2.30}$$

Step 5: Passage to the limit

Applying Dunford-Petti's theorem, we conclude from (2.20), (2.21), (2.28), (2.29) and (2.30), after replacing the sequences u^k by subsequence if necessary, that

$$u^k \rightharpoonup u, \text{ weak-star in } L^\infty(0,T;H^2(\Omega)), \tag{2.31}$$

$$u_t^k \rightharpoonup u_t, \text{ weak-star in } L^\infty(0,T;H^2(\Omega)), \tag{2.32}$$

$$u_{tt}^k \rightharpoonup u_{tt}, \text{ weak-star in } L^\infty(0,T;L^2(\Omega)), \tag{2.33}$$

$$\operatorname{div}(a(x)|\nabla u_t^k|^{p-2}\nabla u_t^k) \rightharpoonup \operatorname{div}(a(x)|\nabla u_t|^{p-2}\nabla u_t), \text{ weak-star in } H^{-1}([0,T]\times\Omega), \tag{2.34}$$

$$\Phi(\|\nabla u^k\|_2^2)\Delta u^k \rightharpoonup \chi, \text{ weak-star in } L^\infty(0,T;L^2(\Omega)), \tag{2.35}$$

for suitable function $\chi \in L^\infty(0,T;L^2(\Omega))$.
We shall prove that, in fact, $\chi = \Phi(\|\nabla u\|_2^2)\Delta u$, i.e.

$$\Phi(\|\nabla u^k\|_2^2)\Delta u^k \rightharpoonup \Phi(\|\nabla u\|_2^2)\Delta u, \text{ weak-star in } L^\infty(0,T;L^2(\Omega)), \tag{2.36}$$

For $v \in L^2(0,T;L^2(\Omega))$, we have

$$\begin{aligned}\int_0^T (\chi - \Phi(\|\nabla u\|_2^2)\Delta u, v)\,dt = &\int_0^T (\chi - \Phi(\|\nabla u^k\|_2^2)\Delta u^k, v)\,dt + \int_0^T (\Phi(\|\nabla u\|_2^2)(\Delta u^k - \Delta u, v)\,dt \\ &+ \int_0^T (\Phi(\|\nabla u^k\|_2^2) - \Phi(\|\nabla u\|_2^2))(\Delta u^k, v)\,dt\end{aligned} \tag{2.37}$$

We deduce from (2.31) and (2.35) that the first and second terms in (2.37) tend to zero as $k \to +\infty$. For the third term, using the fact that Φ is $\mathcal{C}^1$ and (2.19), we can write (with c

positive constant)

$$\begin{aligned}\int_0^T (\Phi(\|\nabla u^k\|_2^2) - \Phi(\|\nabla u\|_2^2))(\Delta u^k, v)\,dt &\le c\int_0^T (\Phi(\|\nabla u^k\|_2^2) - \Phi(\|\nabla u\|_2^2))\|\Delta u^k\|_2\|v\|_2\,dt\\ &\le c\int_0^T |(\Delta(u^k+u), u^k-u)|\,dt\\ &\le c\Big(\int_0^T |u^k-u|^2\,dt\Big)^{\frac{1}{2}}.\end{aligned} \tag{2.38}$$

As (u^k) is bounded in $L^\infty(0,T,H^2(\Omega))$ (by (2.19)) and the injection of $H^2(\Omega)$ in $L^2(\Omega)$ is compact, we have

$$u^k \to u, \text{ strongly in } L^2(0,T;L^2(\Omega)). \tag{2.39}$$

From (2.37), (2.38) and (2.39), we deduce (2.36). It follows at once from (2.31)-(2.34), and (2.36), that for each fixed $v \in L^2(0,T,L^2(\Omega))$

$$\begin{aligned}&\int_0^T\int_\Omega (u_{tt}^k + \Delta^2 u^k - \Phi(\|\nabla u^k\|_2^2)\Delta u^k - \operatorname{div}(a(x)|\nabla u_t^k|^{p-2}\nabla u_t^k))v\,dx\,dt\\ &\qquad\to \int_0^T\int_\Omega (u_{tt} + \Delta^2 u - \Phi(\|\nabla u\|_2^2)\Delta u - \operatorname{div}(a(x)|\nabla u_t|^{p-2}\nabla u_t))v\,dx\,dt\end{aligned} \tag{2.40}$$

as $k \to +\infty$.

Step 6: Proof of uniqueness

Let u and v be two solutions of (2.3) with the same initial data and set $w = u - v$. Then we have

$$\begin{aligned}&\Big(w_{tt} + \Delta^2 w - \Phi(\|\nabla u\|_2^2)\Delta u + \Phi(\|\nabla v\|_2^2)\Delta v - \operatorname{div}(a(x)|\nabla u_t|^{p-2}\nabla u_t) + \operatorname{div}(a(x)|\nabla v_t|^{p-2}\nabla v_t), w_t\Big) = 0,\\ &\frac{1}{2}\frac{d}{dt}\{\|w_t\|_2^2 + \|\Delta w\|_2^2\} - \Phi(\|\nabla u\|_2^2)(\Delta w, w_t) - [\Phi(\|\nabla u\|_2^2) - \Phi(\|\nabla v\|_2^2)](\Delta v, w_t)\\ &+ (a(x)|\nabla u_t|^{p-2}\nabla u_t - a(x)|\nabla v_t|^{p-2}\nabla v_t, \nabla w_t) = 0.\end{aligned} \tag{2.41}$$

Using Cauchy-Schwarz and Hölder's inequalities, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\{\|w_t\|_2^2 + \|\Delta w\|_2^2\}\\ &\le \Phi(\|\nabla u\|_2^2)\|\Delta w\|_2\|w_t\|_2 + c\Big|\|\nabla u\|_2^2 - \|\nabla v\|_2^2\Big|\|\Delta v\|_2\|w_t\|_2\\ &\le c[\Phi(\|\nabla u\|_2^2)]^2\|\Delta w\|_2^2 + \frac{1}{2}\|w_t\|_2^2 + c|(\Delta w, u+v)|\|\Delta v\|_2\|w_t\|_2\\ &\le c[\Phi(\|\nabla u\|_2^2)]^2\|\Delta w\|^2 + \frac{1}{2}\|w_t\|^2 + c\|\Delta w\|_2^2\|u+v\|_2^2\|\Delta v\|_2^2 + \frac{1}{2}\|w_t\|^2\\ &\le c[\Phi(\|\nabla u\|_2^2)]^2 + \|\Delta v\|_2^2\|u+v\|_2^2]\|\Delta w\|_2^2 + \|w_t\|_2^2.\end{aligned} \tag{2.42}$$

Therefore, we deduce that

$$\|w_t(t)\|^2 + \|\Delta w(t)\|^2 \leq \|w_t(0)\|^2 + \|\Delta w(0)\|^2 \qquad \forall t \geq 0.$$

It follows that $u \equiv v$. This completes the proof.

□

2.4 Asymptotic behavior

In this section, we state and prove our stability result for the energy of the solution of system (2.3), using the perturbed energy coupled with multiplier technique.
Introduce the energy

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\Delta u\|_2^2 + \frac{\alpha}{2}\|\nabla u\|_2^2 + \frac{\beta}{2(\gamma+1)}\|\nabla u\|_2^{2\gamma+2}, \quad \forall t \geq 0. \tag{2.43}$$

Now, we are in a position to state our main result.

Lemma 2.4.1. *The energy functional defined by (2.43), satisfies*

$$E'(t) = -\int_\Omega a(x)|\nabla u_t|^p dx \leq 0, \quad \forall t \geq 0. \tag{2.44}$$

Proof. By multiplying equation (2.3) by u_t, integrating over Ω and using Green formula and the boundary conditions, we get the desired result. □

The stability result reads as follows

Theorem 2.4.2. *(i) If $p = 2$, and $a \in L^\infty(\Omega)$ with (2.9), then there exist positive constants K_n, λ_1 and λ_2 such that for every weak solution of (2.3), one has the energy decay estimate:*

$$E(t) \leq \lambda_2 E(0) e^{-\frac{K_n}{2\lambda_1}t}, \quad \forall t \geq 0. \tag{2.45}$$

(ii) For $p > 2$, satisfying

$$p(n-2) < 2(n-1),$$

and $a \in L^\infty(\Omega)$ with (2.9). Then the energy E of solution of (2.3) satisfies:

$$E(t) \leq K(E(0), F_p)(1+t)^{-\frac{1}{\mu_{n,p}}},$$

where

$$\mu_{n,p} = \frac{p-2}{p}\max\left(1, \frac{n-2}{2(n-1)-(n-2)p}\right),$$

and K is a positive constant depending on the initial data as indicated.

The proof of this new result relies on the following Lemmas.

Proof. We consider $\psi \in C_0^\infty(\mathbb{R}^n)$ such that

$$\begin{cases} 0 \leq \psi \leq 1, & \\ \psi = 1, & \text{in } \bar{\Omega} \backslash Q_1, \\ \psi = 0, & \text{in } Q_0. \end{cases} \tag{2.46}$$

For $M > 0$ and $\mu > 0$, define the perturbed energy

$$\widehat{E}(t) = ME(t) + E^\mu(t)\rho(t), \tag{2.47}$$

where

$$\rho(t) = 2\int_\Omega u_t(h.\nabla u)\,dx + \theta\int_\Omega u_t u\,dx,$$

$$h = m\psi, \tag{2.48}$$

and

$$\theta \in]n-4, n[.$$

Lemma 2.4.3. *There exist two positive constants λ_1 and λ_2 such that*

$$\lambda_1 E(t) \leq \widehat{E}(t) \leq \lambda_2 E(t), \quad \forall t \geq 0. \tag{2.49}$$

Proof. From the Cauchy-Schwarz inequality

$$|\rho(t)| \leq \theta C_s \|\nabla u\|\|u_t\| + 2R(x^0)\|\nabla u\|\|u_t\|, \tag{2.50}$$

where $C_s > 0$ and satisfies $\|u\|_2 \leq C_s\|\nabla u\|_2$ for all $u \in H_0^1(\Omega)$ and

$$R(x^0) = \max_{x\in\overline{\Omega}} |x - x^0|. \tag{2.51}$$

From (2.50) we obtain

$$|\rho(t)| \leq (\theta C_s + 2R(x^0))\left\{\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u\|^2\right\} \leq \frac{1+\alpha}{\alpha}(\theta C_s + 2R(x^0))E(t). \tag{2.52}$$

Then, for M large enough, we obtain (2.49), where $\lambda_1 = M - \frac{E^\mu(0)(\alpha+1)}{\alpha}(\theta C_s + 2R(x^0))$ and $\lambda_2 = M + \frac{E^\mu(0)(\alpha+1)}{\alpha}(\theta C_s + 2R(x^0))$. □

Lemma 2.4.4. *The functional $\rho(t)$ satisfies*

$$\begin{aligned}
\rho'(t) &= -(n-\theta)\int_\Omega |u_t|^2\,dx - (\theta-n+4)\int_\Omega |\Delta u|^2\,dx - (\theta-n+2)\Phi(\|\nabla u(t)\|_2^2)\int_\Omega |\nabla u|^2\,dx \\
&\quad - \int_{Q_1\setminus Q_0} m\nabla\psi|u_t|^2\,dx + n\int_{Q_1}(1-\psi)|u_t|^2\,dx + \int_\Gamma (h.\nu)\left(\frac{\partial^2 u}{\partial\nu^2}\right)^2 d\Gamma \\
&\quad + (n-4)\int_{Q_1}(\psi-1)|\Delta u|^2 dx - 4\sum_{i,k=0}^{n}\int_{Q_1} m_i\frac{\partial\psi_i}{\partial x_k}\frac{\partial^2 u}{\partial x_k^2}\frac{\partial^2 u}{\partial x_i^2}\,dx \\
&\quad - 4\int_{Q_1}\nabla\psi\Delta u\nabla u\,dx - 2\sum_{i,k=0}^{n}\int_{Q_1} m_i\frac{\partial^2\psi_i}{\partial x_k^2}\frac{\partial^2 u}{\partial x_i^2}\frac{\partial u}{\partial x_k}\,dx \\
&\quad + \int_{Q_1\setminus Q_0} m\nabla\psi|\Delta u|^2\,dx - 2\Phi(\|\nabla u\|_2^2)\sum_{i,k=1}^{n}\int_{Q_1} m_i\frac{\partial\psi_i}{\partial x_k}\frac{\partial u}{\partial x_k}\frac{\partial u}{\partial x_i}\,dx \\
&\quad + (n-2)\Phi(\|\nabla u\|_2^2)\int_{Q_1}(\psi-1)|\nabla u|^2\,dx + \Phi(\|\nabla u\|_2^2)\int_{Q_1\setminus Q_0} m\nabla\psi|\nabla u|^2\,dx \\
&\quad + \int_\Omega 2(h.\nabla u)\mathrm{div}(a(x)|\nabla u_t|^{p-2}\nabla u_t)\,dx \\
&\quad + \theta\int_\Omega u.\mathrm{div}(a(x)|\nabla u_t|^{p-2}\nabla u_t)\,dx.
\end{aligned} \tag{2.53}$$

Proof. Taking the derivative of ρ, using (2.3), we obtain

$$\begin{aligned}
\rho'(t) &= 2\int_\Omega u_t(h.\nabla u_t)\,dx + 2\int_\Omega u_{tt}(h.\nabla u)\,dx + \theta\int_\Omega u_{tt}u\,dx + \theta\int_\Omega u_t^2\,dx \\
&= 2\int_\Omega u_t(h.\nabla u_t)\,dx + 2\int_\Omega (h.\nabla u)(-\Delta^2 u + \Phi(\|\nabla u(t)\|^2)\Delta u + \mathrm{div}(a(x)|\nabla u_t|^{p-2}\nabla u_t))\,dx \\
&\quad + \theta\int_\Omega u_{tt}u\,dx + \theta\int_\Omega u_t^2\,dx.
\end{aligned} \tag{2.54}$$

Next, we will write some terms on the right-hand side of the identity (2.54).

Lemma 2.4.5.

$$2\int_\Omega u_t(h.\nabla u_t)\,dx = -n\int_\Omega |u_t|^2\,dx - \int_{Q_1\setminus Q_0} m\nabla\psi|u_t|^2\,dx + n\int_{Q_1}(1-\psi)|u_t|^2\,dx. \tag{2.55}$$

Proof. By integrating by parts and using (2.5), (2.46), (2.48) and noting that $u_t = 0$ on Γ, we have

$$\begin{aligned}
2\int_\Omega u_t(h.\nabla u_t)\,dx &= -\int_\Omega \mathrm{div}(h).|u_t|^2\,dx \\
&= -\int_{\Omega\setminus Q_1}\mathrm{div}(\psi.m).|u_t|^2\,dx - \int_{Q_1}\mathrm{div}(\psi.m).|u_t|^2\,dx \\
&= -n\int_{\Omega\setminus Q_1}|u_t|^2\,dx - \int_{Q_1} m\nabla\psi|u_t|^2\,dx - n\int_{Q_1}\psi|u_t|^2\,dx \\
&= -n\int_{\Omega\setminus Q_1}|u_t|^2\,dx - \int_{Q_1\setminus Q_0} m\nabla\psi|u_t|^2\,dx - n\int_{Q_1}\psi|u_t|^2\,dx.
\end{aligned}$$

□

Lemma 2.4.6. *We also have the following equality*

$$
\begin{aligned}
-2\int_{\Omega}(h.\nabla u)\Delta^2 u\,dx &= \int_{\Gamma}(h.\nu)\left(\frac{\partial^2 u}{\partial \nu^2}\right)^2 d\Gamma + (n-4)\int_{\Omega}|\Delta u|^2\,dx \\
&+ (n-4)\int_{Q_1}(\psi-1)|\Delta u|^2 dx - 4\sum_{i,k=0}^{n}\int_{Q_1} m_i \frac{\partial \psi_i}{\partial x_k}\frac{\partial^2 u}{\partial x_k^2}\frac{\partial^2 u}{\partial x_i^2}\,dx \\
&+ \int_{Q_1\setminus Q_0} m\nabla\psi|\Delta u|^2\,dx - 4\int_{Q_1}\nabla\psi\Delta u\nabla u\,dx \\
&- 2\sum_{i,k=0}^{n}\int_{Q_1} m_i\frac{\partial^2\psi_i}{\partial x_k^2}\frac{\partial^2 u}{\partial x_i^2}\frac{\partial u}{\partial x_k}\,dx.
\end{aligned}
\tag{2.56}
$$

Proof. We have $\frac{\partial u}{\partial x_k} = \frac{\partial u}{\partial \nu}\nu_k$ which implies

$$
h.\nabla u = (h.\nu)\frac{\partial u}{\partial \nu} = 0, \quad \text{and} \quad |\Delta u|^2 = \Big(\frac{\partial^2 u}{\partial \nu^2}\Big)^2 \quad \text{on} \quad \Gamma,
$$

then

$$
\begin{aligned}
I_1 &= -2\int_{\Omega}\Delta^2 u(h.\nabla u)\,dx \\
&= -2\int_{\Gamma}\frac{\partial u}{\partial \nu}(h.\nabla\Delta u)\,d\Gamma\ + 2\sum_{i,k=1}^{n}\int_{\Omega}\frac{\partial h_i}{\partial x_k}\frac{\partial u}{\partial x_k}\frac{\partial}{\partial x_i}\Big(\frac{\partial^2 u}{\partial x_i^2}\Big)\,dx + 2\int_{\Omega}h\Delta u.\nabla\Delta u\,dx \\
&= 2\sum_{i,k=1}^{n}\int_{\Omega}\frac{\partial h_i}{\partial x_k}\frac{\partial u}{\partial x_k}\frac{\partial}{\partial x_i}\Big(\frac{\partial^2 u}{\partial x_i^2}\Big)\,dx + 2\int_{\Omega}(h.\Delta u)\nabla\Delta u\,dx.
\end{aligned}
\tag{2.57}
$$

The first term on the right-hand side of (2.57) can be rewritten as follows

$$
\begin{aligned}
&2\sum_{i,k=1}^{n}\int_{\Omega}\frac{\partial h_i}{\partial x_k}\frac{\partial u}{\partial x_k}\frac{\partial}{\partial x_i}\Big(\frac{\partial^2 u}{\partial x_i^2}\Big)\,dx \\
&= 2\int_{\Gamma}\frac{\partial u}{\partial \nu}(\mathrm{div}h)\Delta u\,d\Gamma - 2\sum_{i,k=1}^{n}\int_{\Omega}\frac{\partial^2 h_i}{\partial x_k^2}\frac{\partial u}{\partial x_k}\frac{\partial^2 u}{\partial x_i^2}\,dx - 2\sum_{i,k=1}^{n}\int_{\Omega}\frac{\partial h_i}{\partial x_k}\frac{\partial^2 u}{\partial x_k^2}\frac{\partial^2 u}{\partial x_i^2}\,dx \\
&= -2\sum_{i,k=1}^{n}\int_{\Omega}\frac{\partial^2 h_i}{\partial x_k^2}\frac{\partial u}{\partial x_k}\frac{\partial^2 u}{\partial x_i^2}\,dx - 2\sum_{i,k=1}^{n}\int_{\Omega}\frac{\partial h_i}{\partial x_k}\frac{\partial^2 u}{\partial x_k^2}\frac{\partial^2 u}{\partial x_i^2}\,dx,
\end{aligned}
\tag{2.58}
$$

and the second term

$$
\begin{aligned}
2\int_{\Omega}(h.\Delta u).\nabla\Delta u\,dx &= 2\int_{\Gamma}(h.\nu)|\Delta u|^2\,d\Gamma - 2\sum_{i,k=1}^{n}\int_{\Omega}\frac{\partial h_i}{\partial x_k}\frac{\partial^2 u}{\partial x_k^2}\frac{\partial^2 u}{\partial x_i^2}\,dx - 2\int_{\Omega}h\Delta u.\nabla\Delta u\,dx \\
&= 2\int_{\Gamma}(h.\nu)\Big(\frac{\partial^2 u}{\partial \nu^2}\Big)^2 d\Gamma - 2\sum_{i,k=1}^{n}\int_{\Omega}\frac{\partial h_i}{\partial x_k}\frac{\partial^2 u}{\partial x_k^2}\frac{\partial^2 u}{\partial x_i^2}\,dx - \int_{\Omega}h\nabla(|\Delta u|^2)\,dx \\
&= \int_{\Gamma}(h.\nu)\Big(\frac{\partial^2 u}{\partial \nu^2}\Big)^2 d\Gamma - 2\sum_{i,k=1}^{n}\int_{\Omega}\frac{\partial h_i}{\partial x_k}\frac{\partial^2 u}{\partial x_k^2}\frac{\partial^2 u}{\partial x_i^2}\,dx + \int_{\Omega}\mathrm{div}(h)|\Delta u|^2\,dx.
\end{aligned}
\tag{2.59}
$$

Inserting (2.58) and (2.59) into (2.57), we obtain

$$I_1 = \int_\Gamma (h.\nu)\Big(\frac{\partial^2 u}{\partial \nu^2}\Big)^2 \, d\Gamma - 4\sum_{i,k=1}^n \int_\Omega \frac{\partial h_i}{\partial x_k}\frac{\partial^2 u}{\partial x_k^2}\frac{\partial^2 u}{\partial x_i^2}\, dx - 2\sum_{i,k=1}^n \int_\Omega \frac{\partial^2 h_i}{\partial x_k^2}\frac{\partial u}{\partial x_k}\frac{\partial^2 u}{\partial x_i^2}\, dx + \int_\Omega \operatorname{div}(h)|\Delta u|^2\, dx. \tag{2.60}$$

So, by using (2.5) and (2.46), the second term of (2.60) gives

$$\begin{aligned}
&-4\sum_{i,k=1}^n \int_\Omega \frac{\partial h_i}{\partial x_k}\frac{\partial^2 u}{\partial x_i^2}\frac{\partial^2 u}{\partial x_k^2}\, dx\\
&= -4\sum_{i,k=1}^n \int_{Q_1} \frac{\partial^2 u}{\partial x_i^2}\frac{\partial^2 u}{\partial x_k^2}\frac{\partial (m_i\psi_i)}{\partial x_k}\, dx - 4\sum_{i,k=1}^n \int_{\Omega\setminus Q_1} \frac{\partial^2 u}{\partial x_i^2}\frac{\partial^2 u}{\partial x_k^2}\frac{\partial (m_i\psi_i)}{\partial x_k}\, dx\\
&= -4\sum_{i,k=0}^n \int_{Q_1} \frac{\partial^2 u}{\partial x_i^2}\frac{\partial^2 u}{\partial x_k^2}\psi_i\frac{\partial m_i}{\partial x_k}\, dx - 4\sum_{i,k=0}^n \int_{Q_1} m_i\frac{\partial \psi_i}{\partial x_k}\frac{\partial^2 u}{\partial x_i^2}\frac{\partial^2 u}{\partial x_k^2}\, dx\\
&-4\sum_{i,k=0}^n \int_{\Omega\setminus Q_1} \frac{\partial^2 u}{\partial x_i^2}\frac{\partial^2 u}{\partial x_k^2}\, dx\\
&= -4\int_{Q_1}\psi|\Delta u|^2\, dx - 4\sum_{i,k=0}^n \int_{Q_1} m_i\frac{\partial \psi_i}{\partial x_k}\frac{\partial^2 u}{\partial x_i^2}\frac{\partial^2 u}{\partial x_k^2}\, dx - 4\int_{\Omega\setminus Q_1}|\Delta u|^2\, dx.
\end{aligned} \tag{2.61}$$

Similarly, the third term of (2.60) can be rewritten as follows

$$-2\sum_{i,k=1}^n \int_\Omega \frac{\partial^2 h_i}{\partial x_k^2}\frac{\partial u}{\partial x_k}\frac{\partial^2 u}{\partial x_i^2}\, dx = -4\int_{Q_1}\nabla\psi\Delta u\nabla u\, dx - 2\sum_{i,k=0}^n \int_{Q_1} m_i\frac{\partial^2 \psi_i}{\partial x_k^2}\frac{\partial^2 u}{\partial x_i^2}\frac{\partial u}{\partial x_k}\, dx \tag{2.62}$$

and the fourth term of (2.60) can be rewritten as follows

$$\begin{aligned}
\int_\Omega \operatorname{div}(h)|\Delta u|^2\, dx &= \int_{\Omega\setminus Q_1}\operatorname{div}(\psi m)|\Delta u|^2\, dx + \int_{Q_1}\operatorname{div}(\psi m)|\Delta u|^2\, dx\\
&= n\int_{\Omega\setminus Q_1}|\Delta u|^2\, dx + \int_{Q_1\setminus Q_0} m\nabla\psi|\Delta u|^2\, dx + n\int_{Q_1}\psi|\Delta u|^2\, dx.
\end{aligned} \tag{2.63}$$

Inserting (2.61), (2.62) and (2.63) in (2.60), we arrive at

$$\begin{aligned}
I_1 &= \int_\Gamma (h.\nu)\Big(\frac{\partial^2 u}{\partial \nu^2}\Big)^2 \, d\Gamma + (n-4)\int_{\Omega\setminus Q_1}|\Delta u|^2\, dx + (n-4)\int_{Q_1}\psi|\Delta u|^2\, dx\\
&-4\sum_{i,k=0}^n \int_{Q_1} m_i\frac{\partial \psi_i}{\partial x_k}\frac{\partial^2 u}{\partial x_i^2}\frac{\partial^2 u}{\partial x_k^2}\, dx + \int_{Q_1\setminus Q_0} m\nabla\psi|\Delta u|^2\, dx\\
&-4\int_{Q_1}\nabla\psi\Delta u\nabla u\, dx - 2\sum_{i,k=0}^n \int_{Q_1} m_i\frac{\partial^2 \psi_i}{\partial x_k^2}\frac{\partial^2 u}{\partial x_i^2}\frac{\partial u}{\partial x_k}\, dx.
\end{aligned}$$

□

Lemma 2.4.7.

$$
\begin{aligned}
2\Phi(\|\nabla u(t)\|_2^2)\int_\Omega (h.\nabla u)\Delta u\,dx = {}& (n-2)\Phi(\|\nabla u(t)\|_2^2)\int_\Omega |\nabla u|^2\,dx \\
& - 2\Phi(\|\nabla u(t)\|_2^2)\sum_{i,k=1}^{n}\int_{Q_1} m_i\frac{\partial\psi_i}{\partial x_k}\frac{\partial u}{\partial x_k}\frac{\partial u}{\partial x_i}\,dx \\
& + (n-2)\Phi(\|\nabla u(t)\|_2^2)\int_{Q_1}(\psi-1)|\nabla u|^2\,dx \\
& + \Phi(\|\nabla u(t)\|_2^2)\int_{Q_1\backslash Q_0} m\nabla\psi|\nabla u|^2\,dx.
\end{aligned}
\tag{2.64}
$$

Proof. Integrating by parts, we obtain

$$
\begin{aligned}
I_2 &= 2\Phi(\|\nabla u(t)\|_2^2)\int_\Omega (h.\nabla u)\Delta u\,dx \\
&= -2\Phi(\|\nabla u(t)\|_2^2)\sum_{i,k=1}^{n}\int_\Omega \frac{\partial h_i}{\partial x_k}\frac{\partial u}{\partial x_k}\frac{\partial u}{\partial x_i}\,dx \\
&\quad - 2\Phi(\|\nabla u(t)\|_2^2)\int_\Omega (h.\nabla u)\Delta u\,dx.
\end{aligned}
\tag{2.65}
$$

Similarly to (2.61). So, by using (2.5) and (2.46), the first term of (2.65) gives

$$
\begin{aligned}
-2\Phi(\|\nabla u(t)\|_2^2)\sum_{i,k=1}^{n}\int_\Omega \frac{\partial h_i}{\partial x_k}\frac{\partial u}{\partial x_k}\frac{\partial u}{\partial x_i}\,dx = {}& -2\Phi(\|\nabla u(t)\|_2^2)\sum_{i,k=1}^{n}\int_{\Omega\backslash Q_1}|\nabla u|^2\,dx \\
& - 2\Phi(\|\nabla u(t)\|_2^2)\sum_{i,k=1}^{n}\int_{Q_1} m_i\frac{\partial\psi_i}{\partial x_k}\frac{\partial u}{\partial x_k}\frac{\partial u}{\partial x_i}\,dx \\
& - 2\Phi(\|\nabla u(t)\|_2^2)\int_{Q_1}\psi|\nabla u|^2\,dx.
\end{aligned}
\tag{2.66}
$$

The second term can be rewritten as follows:

$$
\begin{aligned}
&-2\Phi(\|\nabla u(t)\|_2^2)\int_\Omega h\nabla u.\Delta u\,dx \\
&= 2\Phi(\|\nabla u(t)\|_2^2)\int_\Omega \operatorname{div}(h)|\nabla u|^2\,dx + \Phi(\|\nabla u(t)\|_2^2)\int_\Omega h\nabla(|\nabla u|^2)\,dx \\
&= 2\Phi(\|\nabla u(t)\|_2^2)\int_\Omega \operatorname{div}(h)|\nabla u|^2\,dx - \Phi(\|\nabla u(t)\|_2^2)\int_\Omega \operatorname{div}(h)|\nabla u|^2\,dx \\
&= \Phi(\|\nabla u(t)\|_2^2)\int_\Omega \operatorname{div}(h)|\nabla u|^2\,dx \\
&= n\Phi(\|\nabla u(t)\|_2^2)\int_{\Omega\backslash Q_1}|\nabla u|^2\,dx + \Phi(\|\nabla u(t)\|_2^2)\int_{Q_1\backslash Q_0} m\nabla\psi|\nabla u|^2\,dx \\
&+ n\Phi(\|\nabla u(t)\|_2^2)\int_{Q_1}\psi|\nabla u|^2\,dx.
\end{aligned}
\tag{2.67}
$$

Taking into account (2.66) and (2.67) into (2.65) yields

$$I_2 = (n-2)\Phi(\|\nabla u(t)\|_2^2)\sum_{i,k=1}^{n}\int_{\Omega\setminus Q_1}|\nabla u|^2\,dx - 2\Phi(\|\nabla u\|_2^2)\sum_{i,k=1}^{n}\int_{Q_1} m_i\frac{\partial\psi_i}{\partial x_k}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_k}\,dx$$
$$+ (n-2)\Phi(\|\nabla u(t)\|_2^2)\int_{Q_1}\psi|\nabla u|^2\,dx + \Phi(\|\nabla u(t)\|_2^2)\int_{Q_1\setminus Q_0} m\nabla\psi|\nabla u|^2\,dx.$$

□

Lemma 2.4.8.

$$\theta\int_\Omega uu_{tt}\,dx = -\theta\int_\Omega|\Delta u|^2\,dx - \theta\Phi(\|\nabla u(t)\|^2)\int_\Omega|\nabla u|^2\,dx + \theta\int_\Omega u.\mathrm{div}(a(x)|\nabla u_t|^{p-2}\nabla u_t)\,dx. \quad (2.68)$$

Proof. Using the first equation of (2.3) and applying the Green formula, we obtain

$$\theta\int_\Omega uu_{tt}\,dx = \theta\int_\Omega u(-\Delta^2 u + \Phi(\|\nabla u(t)\|_2^2)\Delta u + \mathrm{div}(a(x)|\nabla u_t|^{p-2}\nabla u_t))\,dx$$
$$= -\theta\int_\Omega u\Delta^2 u\,dx + \theta\Phi(\|\nabla u(t)\|_2^2)\int_\Omega u\Delta u\,dx + \theta\int_\Omega u.\mathrm{div}(a(x)|\nabla u_t|^{p-2}\nabla u_t))\,dx$$
$$= -\theta\int_\Omega|\Delta u|^2\,dx - \theta\Phi(\|\nabla u(t)\|_2^2)\int_\Omega|\nabla u|^2\,dx + \theta\int_\Omega u.\mathrm{div}(a(x)|\nabla u_t|^{p-2}\nabla u_t)\,dx.$$

□

Taking into account (2.55), (2.56), (2.64) and (2.68) into (2.54) we obtain (2.53). The proof of Lemma 2.4.4 is complete. □

Lemma 2.4.9.

$$\rho'(t) \le -K_nE(t) + (A+n)\int_\omega|u_t|^2\,dx + 5A\int_\Omega|\Delta u|^2\,dx + 3A\Phi(\|\nabla u(t)\|^2)\int_\Omega|\nabla u|^2\,dx$$
$$+ 2\int_\Omega(h.\nabla u)div(a(x)|\nabla u_t|^{p-2}\nabla u_t)\,dx + \theta\int_\Omega u.div(a(x)|\nabla u_t|^{p-2}\nabla u_t)\,dx, \quad (2.69)$$

where

$$K_n = \min\Big\{2(n-\theta), 2(\theta-n+4), 2(\theta-n+2), 2(\theta-n+2)(\gamma+1)\Big\},$$
$$\theta\in]n-4,n[,$$

and

$$A = R(x^0)\max_{x\in\overline{\Omega}}|\nabla\psi(x)|.$$

Proof. Next, we will estimate some terms on the right-hand side of identity (2.53).
Taking (2.4), (2.6), (2.7) and (2.46) into account

$$\int_\Gamma(h.\nu)\left(\frac{\partial^2 u}{\partial\nu^2}\right)^2 d\Gamma = \int_{\Gamma(x^0)}(m.\nu)\psi\left(\frac{\partial^2 u}{\partial\nu^2}\right)^2 d\Gamma + \int_{\Gamma\setminus\Gamma(x^0)}(m.\nu)\psi\left(\frac{\partial^2 u}{\partial\nu^2}\right)^2 d\Gamma$$
$$\le 0 \quad (2.70)$$

$$\int_{Q_1\setminus Q_0} m\nabla\psi|u_t|^2\,dx \le R(x^0)\max_{x\in\overline{\Omega}}|\nabla\psi(x)|\int_\omega |u_t|^2\,dx, \tag{2.71}$$

$$n\int_{Q_1}(1-\psi)|u_t|^2\,dx \le n\int_\omega |u_t|^2\,dx, \tag{2.72}$$

$$4\Big|\sum_{i,k=0}^{n}\int_{Q_1}\frac{\partial^2 u}{\partial x_k^2}\frac{\partial^2 u}{\partial x_i^2}m_i\frac{\partial\psi_i}{\partial x_i}\,dx\Big| \le 4R(x^0)\max_{x\in\overline{\Omega}}|\nabla\psi(x)|\int_\Omega|\Delta u|^2\,dx, \tag{2.73}$$

$$\int_{Q_1\setminus Q_0} m\nabla\psi|\Delta u|^2\,dx \le R(x^0)\max_{x\in\overline{\Omega}}|\nabla\psi(x)|\int_\Omega|\Delta u|^2\,dx, \tag{2.74}$$

$$2\Phi(\|\nabla u(t)\|^2)\Big|\sum_{i,k=0}^{n}\int_{Q_1}\frac{\partial u}{\partial x_k}\frac{\partial u}{\partial x_k}m_i\frac{\partial\psi_i}{\partial x_i}\,dx\Big| \le 2R(x^0)\max_{x\in\overline{\Omega}}|\nabla\psi(x)|\Phi(\|\nabla u(t)\|^2)\int_\Omega|\nabla u|^2\,dx, \tag{2.75}$$

$$\Phi(\|\nabla u(t)\|^2)\int_{Q_1\setminus Q_0} m\nabla\psi|\nabla u|^2\,dx \le R(x^0)\max_{x\in\overline{\Omega}}|\nabla\psi(x)|\Phi(\|\nabla u(t)\|^2)\int_\Omega|\nabla u|^2\,dx. \tag{2.76}$$

$$-4\int_{Q_1}\nabla\psi\Delta u\nabla u\,dx \le 2\max_{x\in\overline{\Omega}}|\nabla\psi(x)|\int_\Omega|\nabla u|^2\,dx + 2\max_{x\in\overline{\Omega}}|\nabla\psi(x)|\int_\Omega|\Delta u|^2\,dx \tag{2.77}$$

$$-2\sum_{i,k=0}^{n}\int_{Q_1} m_i\frac{\partial^2\psi_i}{\partial x_k^2}\frac{\partial^2 u}{\partial x_i^2}\frac{\partial u}{\partial x_k}\,dx \le R(x^0)\max_{x\in\overline{\Omega}}|\Delta\psi(x)|\int_\Omega|\nabla u|^2\,dx + R(x^0)\max_{x\in\overline{\Omega}}|\Delta\psi(x)|\int_\Omega|\Delta u|^2\,dx \tag{2.78}$$

Taking into account (2.70)-(2.78) into (2.53) we obtain (2.69). The proof of Lemma 2.4.9 is complete.

□

End of proof of Theorem 2.4.2 Taking the derivative of (2.47) with respect to t, we have

$$\widehat{E'}(t) = ME'(t) + \mu E'(t)E^{\mu-1}(t)\rho(t) + E^\mu(t)\rho'(t), \tag{2.79}$$

using the above Lemma, (2.43) and (2.52), we obtain

$$\begin{aligned}
&\widehat{E'}(t) \le ME'(t) + C_\mu|E'(t)|E^\mu(0) - K_nE^{\mu+1}(t) + (A+n)E^\mu(t)\int_\omega|u_t|^2\,dx\\
&+ (5AE^\mu(t) + 2\max_{x\in\overline{\Omega}}|\nabla\psi(x)| + R(x^0)\max_{x\in\overline{\Omega}}|\Delta\psi(x)|)\int_\Omega|\Delta u|^2\,dx\\
&+ E^\mu(t)\Phi(\|\nabla u(t)\|^2)(3A + \frac{2}{\alpha}\max_{x\in\overline{\Omega}}|\nabla\psi(x)| + \frac{R(x^0)}{\alpha}\max_{x\in\overline{\Omega}}|\Delta\psi(x)|))\int_\Omega|\nabla u|^2\,dx\\
&+ E^\mu(t)\int_\Omega 2(h.\nabla u)\mathrm{div}(a(x)|\nabla u_t|^{p-2}\nabla u_t)\,dx - \theta E^\mu(t)\int_\Omega \nabla u.a(x)|\nabla u_t|^{p-2}\nabla u_t\,dx\\
&\le ME'(t) + C_\mu|E'(t)|E^\mu(0) - \frac{K_n}{2}E^{\mu+1}(t)\\
&+ E^\mu(t)\int_\Omega 2(h.\nabla u)\mathrm{div}(a(x)|\nabla u_t|^{p-2}\nabla u_t)\,dx - \theta E^\mu(t)\int_\Omega \nabla u.a(x)|\nabla u_t|^{p-2}\nabla u_t\,dx,
\end{aligned}$$

where $C_\mu = \mu(\theta C_s + 2R(x^0))\frac{\alpha+1}{\alpha}$. From now on, we separate the case $p = 2$ from the case $p > 2$.

Case $p = 2$. Choosing $\mu = 0$, and M large enough, we derive from (2.79)

$$\begin{aligned}\widehat{E'}(t) \le ME'(t) + C_\mu|E'(t)| - \frac{K_n}{2}E(t) + (A+n)\int_\omega |u_t|^2\,dx \\ + \int_\Omega 2(h.\nabla u)\mathrm{div}(a(x)\nabla u_t)\,dx + \theta\int_\Omega u.\mathrm{div}(a(x)\nabla u_t)\,dx.\end{aligned} \tag{2.80}$$

Integrating by parts, using Young's inequality and (2.43), we have

$$\begin{aligned}\theta\int_\Omega u.\mathrm{div}(a(x)\nabla u_t)\,dx &= -\theta\int_\Omega \nabla u.a(x)\nabla u_t\,dx \\ &\le \frac{\theta}{2}\int_\Omega a^2(x)|\nabla u_t|^2\,dx + \frac{\theta}{2}\int_\Omega |\nabla u|^2\,dx \\ &\le \frac{\theta}{2}\|a\|_{L^\infty(\Omega)}(-E'(t)) + \frac{\theta}{\alpha}E(t).\end{aligned} \tag{2.81}$$

Similarly,

$$\begin{aligned}&\int_\Omega 2(h.\nabla u)\mathrm{div}(a(x)\nabla u_t)\,dx = -2\int_\Omega a(x)\mathrm{div}(h)\nabla u.\nabla u_t\,dx - 2\int_\Omega h\Delta u a(x)\nabla u_t\,dx \\ &\le \|a\|_{L^\infty(\Omega)}\Big(n + \frac{2A}{\alpha} + 2R(x^0)\Big)(-E'(t)) + \|a\|_{L^\infty(\Omega)}\Big(\frac{2n}{\alpha} + \frac{2A}{\alpha} + 2R(x^0)\Big)E(t).\end{aligned} \tag{2.82}$$

Which imply the following Poincaré inequality

$$(A+n)\int_\omega |u_t|^2\,dx \le \frac{A+n}{a_0}(-E'(t)). \tag{2.83}$$

Taking into account (2.81)-(2.83) into (2.80), we obtain

$$\widehat{E'}(t) \le ME'(t) + C|E'(t)| - \frac{K_n}{2}E(t) \le -\frac{K_n}{2}E(t),$$

from which one derives (2.45) thanks to (2.49), and a density argument.

Case $p > 2$

$$\begin{aligned}&\widehat{E'}(t) \le ME'(t) + C_\mu|E'(t)|E^\mu(0) - (K_n - 6A(\gamma+1))E^{\mu+1}(t) + (A+n)E^\mu(t)\int_\omega |u'|^2\,dx \\ &+ E^\mu(t)\int_\Omega 2(h.\nabla u)\mathrm{div}(a(x)|\nabla u_t|^{p-2}\nabla u_t)\,dx + \theta E^\mu(t)\int_\Omega u.\mathrm{div}(a(x)|\nabla u_t|^{p-2}\nabla u_t)\,dx.\end{aligned} \tag{2.84}$$

Using Young's inequality (2.43) and (2.44), we obtain

$$\begin{aligned}\int_\omega |u_t|^2\,dx &\le \Big(\int_\Omega |u_t|^p\,dx\Big)^{\frac{1}{p}}\Big(\int_\Omega 1^{2p}\,dx\Big)^{\frac{p-2}{2p}}\Big(\int_\Omega |u_t|^2\,dx\Big)^{\frac{1}{2}} \\ &\le C_s\Big(\int_\Omega |\nabla u_t|^2\,dx\Big)^{\frac{1}{2}}\Big(\int_\Omega |u_t|^2\,dx\Big)^{\frac{1}{2}} \\ &\le C\Big(\int_\Omega 1^{\frac{p}{p-2}}\,dx\Big)^{\frac{p-2}{p}}\Big(\int_\Omega a(x)|\nabla u_t|^p\,dx\Big)^{\frac{1}{p}}\Big(\int_\Omega |u_t|^2\,dx\Big)^{\frac{1}{2}} \\ &\le C|E'(t)|^{\frac{1}{p}}E^{\frac{1}{2}}(t).\end{aligned} \tag{2.85}$$

It follows from Lemma 2.2.1, (2.84) and (2.85)

$$\begin{aligned}\widehat{E'}(t) &\leq ME'(t) + C_\mu |E'(t)| E^\mu(0) - \frac{K_n}{2} E^{\mu+1}(t) + CE^{\mu+\frac{1}{2}}(t) |E'(t)|^{\frac{1}{p}} \\ &+ CE^{\mu+\frac{1}{2}}(t) |E'(t)|^{\frac{1}{p}} F_p^{\frac{p-2}{2}}, \quad \text{if} \quad n = \{1, 2\},\end{aligned} \tag{2.86}$$

and

$$\begin{aligned}\widehat{E'}(t) &\leq ME'(t) + C_\mu |E'(t)| E^\mu(0) - \frac{K_n}{2} E(t)^{\mu+1} + CE(t)^{\mu+\frac{1}{2}} |E'(t)|^{\frac{1}{p}} \\ &+ CE(t)^{\mu+\frac{1}{2}}(t) |E'(t)|^{\frac{2(n-1)-(n-2)p}{2n-(n-2)p}} F_p^{\frac{n(p-2)}{2(2n-(n-2)p)}}, \quad \text{if} \quad n \geq 3.\end{aligned} \tag{2.87}$$

Now,

$$\begin{aligned}CE(t)^{\mu+\frac{1}{2}}(t) |E'(t)|^{\frac{1}{p}} &= E(t)^{(\mu+1)\frac{p-1}{p}} E(t)^{\frac{2\mu-(p-2)}{2p}} |E'(t)|^{\frac{1}{p}} \\ &\leq \frac{K_n}{8} E(t)^{\mu+1} + CE(0)^{\frac{2\mu-(p-2)}{2}} |E'(t)|\end{aligned} \tag{2.88}$$

$$CE^{\mu+\frac{1}{2}}(t) |E'(t)|^{\frac{1}{p}} F_p^{\frac{p-2}{2}} \leq \frac{K_n}{8} E(t)^{\mu+1} + CE(0)^{\frac{2\mu-(p-2)}{2}} F_p^{\frac{p(p-2)}{2}} |E'(t)| \tag{2.89}$$

$$\begin{aligned}CE(t)^{\mu+\frac{1}{2}}(t) |E'(t)|^{\frac{2(n-1)-(n-2)p}{2n-(n-2)p}} F_p^{\frac{n(p-2)}{2(2n-(n-2)p)}} &\leq \frac{K_n}{4} E(t)^{\mu+1} \\ &+ CE(0)^{\frac{2\mu(2(n-1)-(n-2)p)-(n-2)(p-2)}{2(n-1)-(n-2)p}} F_p^{\frac{n(p-2)}{2(2(n-1)-(n-2)p)}} |E'(t)|.\end{aligned} \tag{2.90}$$

Reporting (2.88)-(2.90) in (2.87), we find for M large enough, and $\mu = \mu_{n,p}$

$$\widehat{E'}(t) \leq -\frac{K_n}{4} E'(t) \leq -\frac{K_n}{4(2M)^{\mu+1}} \widehat{E'}(t)^{\mu+1}. \tag{2.91}$$

This concludes the proof of Theorem 2.4.2. □

CHAPTER 3

WELL-POSEDNESS AND ENERGY DECAY ESTIMATES OF A NON-DEGENERATE KIRCHHOFF EQUATION WITH LOCALIZED NONLINEAR DAMPING

The aim of this work is to analyze a damped wave equation of Kirchhoff type in bounded domain taking into account nonlinear lacal damping effects. We prove the well-posedness and regularity of solution, explained using the theory of the Faedo-Galerkin scheme. Owing to the energy method combined with the piecewise multipliers method, we obtain both exponential and polynomial decay estimates.

3.1 Introduction and statement of main results

Nonlinear wave phenomena were the subject of research by such outstanding scientists. However, as a unified science, the theory of nonlinear waves developed in the late 1960s early 1970s, which became the years of its rapid development. The theory of nonlinear damped waves is still a young science, although research in this direction was carried out even very recently. To begin with, we consider the nonlinear damped wave system

$$\begin{cases} u_{tt} - \Phi(\|\nabla u\|_2^2)\Delta u + a(x)g(u_t) = 0, & x \in \Omega, t \geq 0 \\ u = 0, & x \in \Gamma, t \geq 0 \\ u(x,0) = u_0(x), \;\; u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{3.1}$$

here Ω is a bounded domain of $\mathbb{R}^n$ with regular boundary Γ. For an arbitrary point $x^0 \in \mathbb{R}^n$, we set

$$\Gamma(x^0) = \left\{ x \in \Gamma; \quad \kappa(x).\nu(x) > 0 \right\}, \tag{3.2}$$

where ν represents the unit normal vector pointing towards the exterior of Ω and

$$\kappa(x) = x - x^0. \tag{3.3}$$

With δ sufficiently small, let ω be a neighborhood of $\Gamma(x^0)$ in Ω, such that

$$Q_0 = \left\{ x \in \Omega; d(x, \Gamma(x^0)) < \delta \right\} \subset \omega, \tag{3.4}$$

$$Q_1 = \left\{ x \in \Omega; d(x, \Gamma(x^0)) < 2\delta \right\} \subset \omega. \tag{3.5}$$

If $A \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, we have

$$d(x; A) = \inf_{y \in A} (|x - y|).$$

Then $Q_0 \subset Q_1 \subset \omega$, $\Phi(s)$ is a nonnegative $\mathcal{C}^1$-function for $s \geq 0$ satisfying

$$\Phi(s) = \alpha + \beta s^\gamma,$$

with $\alpha,\ \beta > 0$ and $\gamma > 0$.

Here we have a number of detailed articles and reviews, among which we note the work by Kouémou-Patcheu [38], where the Kirchhoff equation with a nonlinear dissipative term was studied

$$u_{tt} + A^2 u + \Phi(\|A^{\frac{1}{2}} u(t)\|_H^2) A u - g(u_t) = 0,$$

where A is a linear operator in a Hilbert space H. Φ and g are real functions. The author proved the global existence of solutions by the Faedo-Galerkin method by using a new method introduced by Martinez [47] to study the decay rate of solution.

When $g(v) = v$ and $\Phi(s) = 1$, E. Zuazua [75] proved that the associate energy decays exponentially in the case where $a(x) \geq a_0 > 0$ by using the multiplier method combined with the compactness argument. On the other hand, C. Bardos et *al.* [15] obtained a necessary and sufficient condition by microlocal analysis, to guarantee that the energy decays exponentially if and only if the damping region satisfies the geometric optics condition. In addition, under some geometrical conditions and without assuming that the feedback has a polynomial growth in zero, the multiplier method gives a more explicit decay rate estimate to improve the results in [39], where more general case of a semilinear wave equation damped with a nonlinear velocity feedback acting on a part of the boundary was treated. Without any geometrical condition and without assuming that the feedback has a polynomial growth in zero, the authors showed that the energy decays as fast as the solution of some associated differential equation.

When $\Phi(s) = 1$ the problem (3.1) has been studied in [76]. The author proved that the energy decays to zero by using the multiplier method.

Very recently, L. Tebou [67] considered the problem

$$u_{tt} - \Delta u - a(x)g(\Delta u_t) = 0,$$

and used semigroup combined with Faedo-Galerkin approach to obtain the well possedness and also showed the uniform exponential decay of solution by introducing a multiplier method combined with a nonlinear integral inequalities given by Martinez [47].

when $\Phi(s) = 1$ the problem (3.1) was treated by L. Tebou [64], he proved the exponential and polynomial decay estimates, he based on the multipliers technique.

When $\alpha = 0$, $\beta = 1$, $a(x) = 1$ and $g(v) = v$, the problem (3.1) was treated by Nishihara and Yamada [57]. The existence, uniqueness of global solution $u(t)$ for small data $(u_0, u_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^2(\Omega)$ with $u_0 \neq 0$ and the polynomial decay of the solution is proved. When $\alpha = 0$, $\beta = 1$, $a(x) = 1$ and $g(v) = |v|^{p-1}v, p > 1$, Ono in [58] studied the decay property of the energy of solution without proving the global existence which is not known at present.

The novelty in this work is to prove the global existence of weak solution of the problem (3.1) by using the Galerkin method (see Lions [41]).

Meanwhile, under suitable conditions on $g(.)$ with some ideas inspired from [66], we estimate the energy decay of the solution under some conditions on g and a.

First assume that $a(x)$ and $g(s)$ satisfy the following hypotheses

(A1) The nonnegative function $a : \Omega \to [0, \infty)$ is assumed bounded such that

$$\begin{aligned} &\exists a_0 > 0, \quad a(x) \geq a_0 > 0, \quad \text{a.e} \quad \text{in} \quad \omega. \\ &a(x) \in W^{2,\infty}(\Omega). \\ &\exists a_1 > 0, \quad |\nabla a(x)| \leq a_1 a(x) \quad \text{a.e} \quad \text{in} \quad \Omega \\ &\exists a_2 > 0, \quad |\Delta a(x)| \leq a_2 a(x) \quad \text{a.e} \quad \text{in} \quad \Omega \end{aligned} \tag{3.6}$$

(A2) Assume that $g \in C^1(\mathbb{R}, \mathbb{R})$ is nondecreasing function where $g(0) = 0$ and globally Lipschitz.

Suppose that there exist $c_i > 0,\ i = 1, 2, 3, 4$ and $p \geq 1$ such that

$$c_1|s|^p \leq g(s) \leq c_2|s|^{\frac{1}{p}}, \ \text{ if } \ |s| \leq 1 \tag{3.7}$$

$$c_3|s| \leq g(s) \leq c_4|s|, \ \text{ if } \ |s| > 1 \tag{3.8}$$

$$\exists \tau_0,\ \tau_1 > 0,\ \tau_0 \leq g'(s) \leq \tau_1,\ \ \forall s \in \mathbb{R}. \tag{3.9}$$

We introduce the functional energy

$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{\alpha}{2}\|\nabla u\|^2 + \frac{\beta}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)}, \qquad \forall t \geq 0. \tag{3.10}$$

The function E is a nonincreasing of the time variable t and its derivative satisfies

$$E'(t) = -\int_\Omega a(x) u_t g(u_t) dx \leq 0 \qquad \forall t \geq 0. \tag{3.11}$$

Theorem 3.1.1. *(Well-posedness). Let $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $u_1 \in H_0^1(\Omega)$ and assume that (3.6)-(3.9) hold and $\{u_0, u_1\}$ is small. Then the problem (3.1) has a unique weak solution u such that for any $T > 0$, we have*

$$u \in L^\infty(0, T; H^2(\Omega) \cap H_0^1(\Omega)),$$

$$u_t \in L^\infty(0, T; H_0^1(\Omega)),$$

$$u_{tt} \in L^\infty(0, T; L^2(\Omega)).$$

Theorem 3.1.2. *(Stability). Let $(u_0, u_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$ and suppose that (3.6)-(3.9) hold. Then the solution u of the problem (3.1) satisfies the following energy decay estimates*

$$E(t) \leq CE(0)e^{-wt}, \ \ \forall t > 0 \ \ \textit{if } p = 1 \tag{3.12}$$

and

$$E(t) \leq C' t^{-2/(p-1)}, \ \ \forall t > 0 \ \ \textit{if } p > 1. \tag{3.13}$$

Here C and w are positive constants independent of the initial data, while C' is a positive constant depending only on the initial energy $E(0)$.

3.2 Some technical tools and proof of main results

Lemma 3.2.1. *(Gronwall inequality) Let $T > 0$ and $C \geq 0$. Let $G, \alpha,\ \beta : [0, T] \to \mathbb{R}$ be continuous and nonnegative functions*

$$0 \leq G(s) \leq C + 2\int_0^s \alpha(r)\sqrt{G(r)}\,dr + \int_0^s \beta(r)G(r)\,dr.$$

Then

$$G(t) \leq \Big(\sqrt{C} + \int_0^t \alpha(s)\,ds\Big)^2 e^{\int_0^t \beta(s)\,ds}.$$

Lemma 3.2.2. *(Modified Gronwall inequality) Let G and f be nonnegative functions on $[0,+\infty)$ satisfying*

$$0 \leq G(t) \leq K + \int_0^t f(s)G(s)^{r+1}\,ds,$$

with $K > 0$ and $r > 0$. Then

$$G(t) \leq \Big\{K^{-r} - r\int_0^t f(s)\,ds\Big\}^{-1/r},$$

as long as the RHS exists.

3.2.1 Proof of Theorem 3.1.1

The Faedo-Galerkin method will be the key to prove the existence of a global solution.
For this end, for $T > 0$ be fixed and let $\{w^k\}, k \in \mathbb{N}$ be a basis of $H^2(\Omega) \cap H_0^1(\Omega)$, V_k the space generated by $w^1, w^2, \ldots, w^k$, and λ^j are the eigenvalues of the operator $-\Delta$.
Hence

$$-\Delta w^j = \lambda^j w^j.$$

We construct the approximate solutions $u^k, k = 1, 2, 3, \ldots$, in the form

$$u^k(t,x) = \sum_{j=1}^k c^{jk}(t)w^j(x),$$

where c^{jk} are determined by the ODEs

$$(u_{tt}^k - \Phi(\|\nabla u^k\|^2)\Delta u^k + a(x)g(u_t^k), v) = 0, \tag{3.14}$$

with the initial conditions

$$u^k(0) = u_0^k = \sum_{j=1}^k (u_0, w^j)w^j \longrightarrow u_0, \quad \text{in } H^2(\Omega) \cap H_0^1(\Omega) \tag{3.15}$$

$$u_t^k(0) = u_1^k = \sum_{j=1}^k (u_1, w^j)w^j \longrightarrow u_1, \quad \text{in } H_0^1(\Omega) \tag{3.16}$$

$$\Phi(\|\nabla u_0^k\|_2^2)\Delta u_0^k - a(x)g(u_1^k) \longrightarrow \Phi(\|\nabla u_0\|_2^2)\Delta u_0 - a(x)g(u_1), \quad \text{in } L^2(\Omega). \tag{3.17}$$

Choosing $v = 2u_t^k$ in (3.14), we get

$$\frac{d}{dt}\left\{\|u_t^k\|^2 + \alpha\|\nabla u^k\|^2 + \frac{\beta}{\gamma+1}\|\nabla u^k\|^{2(\gamma+1)}\right\} + 2\int_\Omega a(x)u_t^k g(u_t^k)\,dx = 0.$$

Integrating in $[0,t], t < t_k$, since the sequences u_0^k, u_1^k converge, by (3.15) and (3.16), we have

$$E^k(t) + \int_0^t\int_\Omega a(x)u_t^k g(u_t^k)\,dx \le E^k(0) \le I_0, \tag{3.18}$$

where

$$E^k(t) = \frac{1}{2}\|u_t^k\|^2 + \frac{\alpha}{2}\|\nabla u^k\|^2 + \frac{\beta}{2(\gamma+1)}\|\nabla u^k\|^{2(\gamma+1)},$$

and

$$I_0 = \frac{1}{2}\|u_1\|^2 + \frac{\alpha}{2}\|\nabla u_0\|^2 + \frac{\beta}{2(\gamma+1)}\|\nabla u_0\|^{2(\gamma+1)}.$$

Which implies that the solution u^k exists globally in $[0,+\infty]$. Invoking the estimate (3.18) we get

$$u^k \text{ is bounded in } L^\infty(0,T,H_0^1(\Omega)), \tag{3.19}$$

$$u_t^k \text{ is bounded in } L^\infty(0,T,L^2(\Omega)), \tag{3.20}$$

$$a(x)u_t^k g(u_t^k) \text{ is bounded in } L^1(\mathcal{A}), \tag{3.21}$$

where $\mathcal{A} = \Omega \times (0,T)$. We prove that $\sqrt{a(x)}g(u_t^k)$ is bounded, using (3.8) and (3.21), we have

$$\int_0^T\int_{|u_t^k|>1} a(x)g^2(u_t^k)\,dx\,dt \le \int_{\mathcal{A}} a(x)|u_t^k g(u_t^k)|\,dx\,dt \le K.$$

Using (3.7) and Hölder's inequality, we have

$$\begin{aligned}
\int_{|u_t|\le 1} a(x)g^2(u_t^k)\,dx &\le \int_{|u_t^k|\le 1} a(x)|g(u_t^k)u_t^k|^{\frac{2}{p+1}}\,dx \\
&= \int_{|u_t^k|\le 1} a^{1-\frac{2}{p+1}}(x)|a(x)g(u_t^k)u_t^k|^{\frac{2}{p+1}}\,dx \\
&= \int_{|u_t|\le 1} a^{\frac{p-1}{p+1}}(x)|a(x)g(u_t^k)u_t^k|^{\frac{2}{p+1}}\,dx \\
&\le \left(\int_\Omega a(x)\,dx\right)^{\frac{p-1}{p+1}}\left(\int_\Omega |a(x)g(u_t^k)u_t^k|\,dx\right)^{\frac{2}{p+1}} \\
&\le \|a\|_\infty^{\frac{p-1}{p+1}}\left(\int_\Omega a(x)|g(u_t^k)u_t^k|\,dx\right)^{\frac{2}{p+1}}.
\end{aligned}$$

By (3.21), we get

$$\begin{aligned}
\int_0^T\int_{|u_t^k|\le 1} a(x)g^2(u_t^k)\,dx\,dt &\le \|a\|_\infty^{\frac{p-1}{p+1}} T^{\frac{p-1}{p+1}}\left(\int_{\mathcal{A}} a(x)|g(u_t^k)u_t^k|\,dx\,dt\right)^{\frac{2}{p+1}}. \\
&\le \widetilde{K}.
\end{aligned}$$

Then

$$\sqrt{a(x)}g(u_t^k) \text{ is bounded in } L^2(\mathcal{A}). \tag{3.22}$$

We now proceed with further a priori estimates.

Choosing $v = -2\Delta u_t^k$ in (3.14), we find

$$\frac{d}{dt}\{\|\nabla u_t^k\|^2 + (\alpha + \beta\|\nabla u^k\|^{2\gamma})\|\Delta u^k\|^2\} - 2\int_\Omega a(x)g(u_t^k).\Delta u_t^k dx = \|\Delta u^k\|^2 \frac{d}{dt}(\alpha + \beta\|\nabla u^k\|^{2\gamma}). \tag{3.23}$$

Set

$$E_*^k(t) = \|\nabla u_t^k\|^2 + (\alpha + \beta\|\nabla u^k\|^{2\gamma})\|\Delta u^k\|^2 \tag{3.24}$$

and

$$I_1 = \|\nabla u_1\|^2 + (\alpha + \beta\|\nabla u_0\|^{2\gamma})\|\Delta u_0\|^2.$$

Since $g(0) = 0$ and $u_t^k = 0$ on Γ, applying the Green formula, we obtain

$$\begin{aligned}\int_\Omega a(x)g(u_t^k).\Delta u_t^k dx &= -\int_\Omega a(x)|\nabla u_t^k|^2 g'(u_t^k)\,dx - \int_\Omega \nabla a(x)g(u_t^k).\nabla u_t^k dx\\ &= -\int_\Omega a(x)|\nabla u_t^k|^2 g'(u_t^k)\,dx + \int_\Omega \Delta a(x)u_t^k g(u_t^k)\,dx\\ &+ \int_\Omega \nabla a(x)u_t^k \nabla u_t^k g'(u_t^k)\,dx.\end{aligned}$$

Using (3.6) and (3.9), we have

$$\int_\Omega a(x)g'(u_t^k)|\nabla u_t^k|^2\,dx \geq \tau_0\int_\Omega a(x)|\nabla u_t^k|^2\,dx, \tag{3.25}$$

using (3.6) and (3.18), we get

$$\begin{aligned}\int_\Omega \Delta a(x)u_t^k g(u_t^k)\,dx &\leq \Big\|\frac{|\Delta a|}{a}\Big\|_\infty \int_\Omega a(x)u_t^k g(u_t^k)\,dx\\ &\leq \Big\|\frac{|\Delta a|}{a}\Big\|_\infty (-E^k)'\\ &\leq a_2(-E^k)'.\end{aligned} \tag{3.26}$$

Now, using the Cauchy-Schwarz inequality (3.6) and (3.9), we get

$$\begin{aligned}\int_\Omega \nabla a(x)u_t^k\nabla u_t^k g'(u_t^k)\,dx &\leq \tau_1 C_s\Big\|\frac{|\nabla a|}{a}\Big\|_\infty \int_\Omega a(x)|\nabla u_t^k|^2\,dx\\ &\leq C_s\tau_1 a_1\int_\Omega a(x)|\nabla u_t^k|^2\,dx\end{aligned} \tag{3.27}$$

where $C_s > 0$ (depending only on the geometry of (Ω) is the constant such that $\|u_t\| \leq C_s\|\nabla u_t\|,\ \forall u_t \in H_0^1(\Omega)$.

Owing to (3.18) and (3.24), we obtain

$$\begin{aligned}\|\Delta u^k\|^2\frac{d}{dt}(\alpha + \beta\|\nabla u^k\|^{2\gamma}) &\leq 2\beta\gamma\|\Delta u^k\|^2\|\nabla u_t^k\|\|\nabla u^k\|^{2\gamma-1}\\ &\leq C\|\Delta u^k\|^2\|\nabla u_t^k\|\\ &\leq C(E_*^k(t))^{\frac{3}{2}}.\end{aligned} \tag{3.28}$$

Replacing (3.25)-(3.28) in (3.23), we obtain

$$\frac{d}{dt}\Big\{E_*^k(t)+a_2E^k(t)\Big\}+\Big(\tau_0-C_s\tau_1a_1\Big)\int_\Omega a(x)|\nabla u_t^k|^2\,dx\le C(E_*^k(t))^{\frac{3}{2}},$$

integrating the previous inequality over $(0,t)$, we get

$$\begin{aligned}E_*^k(t)+a_2E^k(t)+\Big(\tau_0-C_s\tau_1ai_1\Big)\int_0^t\int_\Omega a(x)|\nabla u_t^k(s)|^2\,ds\,dx\\ \le I_1+a_2I_0+C\int_0^t(E_*^k(s))^{1+\frac{1}{2}}\,ds.\end{aligned}$$

We suppose that $\tau_0-C_s\tau_1a_1>0$, we have

$$E_*^k(t)\le I_1+a_2I_0+C\int_0^t(E_*^k(s))^{1+\frac{1}{2}}\,ds.$$

By Lemma 3.2.2, we have

$$E_*^k(t)\le\Big\{\Big(I_1+a_2I_0\Big)^{-\frac{1}{2}}-C\frac{1}{2}\int_0^t\,ds\Big\}^{-2}.$$

Therefore, if initial data $\{u_0,u_1\}$ is small, we deduce that

$$E_*^k(t)\le C.$$

We conclude that

$$u_t^k \text{ is bounded in } L^\infty(0,T,H_0^1(\Omega)), \tag{3.29}$$

$$(\alpha+\beta\|\nabla u^k\|^{2\gamma})\Delta u^k \text{ is bounded in } L^\infty(0,T,L^2(\Omega)), \tag{3.30}$$

and

$$\Delta u^k \text{ is bounded in } L^\infty(0,T,L^2(\Omega)). \tag{3.31}$$

We estimate $u_{tt}^k(0)$, for this end we choose $v=u_{tt}^k$ in (3.14) and set $t=0$. It gives

$$(u_{tt}^k(0)-\Phi(\|\nabla u^k(0)\|_2^2)\Delta u^k(0)+a(x)g(u_t^k(0)),u_{tt}^k(0))=0.$$

Thanks to Hölder's inequality, (3.15) and (3.17), we have

$$\begin{aligned}\|u_{tt}^k(0)\|&\le\Big(\int_\Omega|\Phi(\|\nabla u_0^k\|_2^2)\Delta u_0^k-a(x)g(u_1^k)|^2\,dx\Big)^{\frac{1}{2}}\\&\le C_2,\end{aligned} \tag{3.32}$$

where C_2 is a positive constant independent of k.

Now, differentiating (3.14) with respect to t

$$(u_{ttt}^k-(\alpha+\beta\|\nabla u^k\|^{2\gamma})\Delta u_t^k-2\gamma\beta(\nabla u_t^k,\nabla u^k)\|\nabla u^k\|^{2\gamma-2}\Delta u^k+a(x)u_{tt}^kg'(u_t^k),v)=0. \tag{3.33}$$

Choosing $v = -2\Delta u_t^k$ in (3.33), using (3.29) and (3.31), we find

$$\begin{aligned}
&\frac{d}{dt}\left\{\|u_{tt}^k\|^2 + (\alpha + \beta\|\nabla u^k\|^{2\gamma})\|\nabla u_t^k\|^2\right\} + 2\int_\Omega a(x)g'(u_t^k)(u_{tt}^k)^2 dx \\
&= 2\gamma\beta(\nabla u_t^k, \nabla u^k)\|\nabla u^k\|^{2\gamma-2}\|\nabla u_t^k\|^2 + 2\gamma\beta(\nabla u_t^k, \nabla u^k)\|\nabla u^k\|^{2\gamma-2}\int_\Omega u_{tt}^k \Delta u^k dx \\
&\leq 2\gamma\beta\|\nabla u_t^k\|^3\|\nabla u^k\|^{2\gamma-1} + 2\gamma\beta\|\nabla u_t^k\|\|\nabla u^k\|^{2\gamma-1}\|u_{tt}^k\|\|\Delta u^k\| \\
&\leq C_3 + C_3'\|u_{tt}^k\|^2.
\end{aligned} \tag{3.34}$$

Integrating from 0 to t, using the Gronwall's inequality and (3.32) to obtain

$$u_{tt}^k \text{ is bounded in } L^\infty(0, T, L^2(\Omega)). \tag{3.35}$$

We deduce for each $T > 0$:

$$u^k \rightharpoonup^* u, \text{ in } L^\infty(0, T; H^2(\Omega) \cap H_0^1(\Omega)), \tag{3.36}$$

$$u_t^k \rightharpoonup^* u_t, \text{ in } L^\infty(0, T; H_0^1(\Omega)), \tag{3.37}$$

$$u_{tt}^k \rightharpoonup^* u_{tt}, \text{ in } L^\infty(0, T; L^2(\Omega)), \tag{3.38}$$

$$u_t^k \to u_t, \text{ almost everywhere in } \Omega \times [0, +\infty), \tag{3.39}$$

$$\Phi(\|\nabla u^k\|_2^2)\Delta u^k \rightharpoonup^* \chi, \text{ in } L^\infty(0, T; L^2(\Omega)). \tag{3.40}$$

$$\sqrt{a(x)}g(u_t^k) \rightharpoonup \Psi, \text{ in } L^2(\mathcal{A}), \tag{3.41}$$

for a suitable function $u \in L^\infty(0, T; H^2(\Omega) \cap H_0^1(\Omega)), \chi \in L^\infty(0, T; L^2(\Omega))$ and $\Psi \in L^2(\mathcal{A})$. We shall prove that, in fact, $\chi = \Phi(\|\nabla u\|_2^2)\Delta u$, i.e.

$$\Phi(\|\nabla u^k\|_2^2)\Delta u^k \rightharpoonup^* \Phi(\|\nabla u\|_2^2)\Delta u, \text{ in } L^\infty(0, T; L^2(\Omega)). \tag{3.42}$$

For $v \in L^2(0, T; L^2(\Omega))$, we have

$$\begin{aligned}
\int_0^T (\chi - \Phi(\|\nabla u\|_2^2)\Delta u, v)\, dt &= \int_0^T (\chi - \Phi(\|\nabla u^k\|_2^2)\Delta u^k, v)\, dt + \int_0^T (\Phi(\|\nabla u\|_2^2)(\Delta u^k - \Delta u, v)\, dt \\
&\quad + \int_0^T (\Phi(\|\nabla u^k\|_2^2 - \Phi(\|\nabla u\|_2^2)(\Delta u^k, v)\, dt.
\end{aligned} \tag{3.43}$$

We deduce from (3.36) and (3.40) that (3.43) and (3.43) tend to zero as $k \to +\infty$. For $(3.43)_3$, using the fact that Φ is $\mathcal{C}^1$ and (3.31), we can write (with c positive constant)

$$\begin{aligned}\int_0^T (\Phi(\|\nabla u^k\|_2^2 - \Phi(\|\nabla u\|_2^2)(\Delta u^k, v)\,dt &\le c\int_0^T (\Phi(\|\nabla u^k\|_2^2 - \Phi(\|\nabla u\|_2^2)\|\Delta u^k\|_2\|v\|_2\,dt \\ &\le c\int_0^T (\|\nabla u^k\|_2^2 - \|\nabla u\|_2^2)\,dt \\ &= c\int_0^T |(\Delta(u^k+u), u^k-u)|\,dt \\ &\le c\Big(\int_0^T |u^k-u|^2\,dt\Big)^{\frac{1}{2}}.\end{aligned} \tag{3.44}$$

As (u^k) is bounded in $L^\infty(0,T,H^2(\Omega))$ (by (3.31)) and the embedding of $L^2(\Omega)$ in $H^2(\Omega)$ is compact, we have

$$u^k \longrightarrow u, \text{ strongly in } L^2(0,T;L^2(\Omega)). \tag{3.45}$$

From (3.43), (3.44) and (3.45), we deduce (3.42). It follows from (3.36)-(3.38), and (3.42). that for each fixed $v \in L^2(0,T,L^2(\Omega))$

$$\int_0^T\int_\Omega (u_{tt}^k + \Phi(\|\nabla u^k\|_2^2)\Delta u^k)v\,dx\,dt \longrightarrow \int_0^T\int_\Omega (u_{tt} + \Phi(\|\nabla u\|_2^2)\Delta u)v\,dx\,dt. \tag{3.46}$$

It remains now to prove

$$\int_0^T\int_\Omega a(x)g(u_t^k)v\,dx\,dt \longrightarrow \int_0^T\int_\Omega a(x)g(u_t)v\,dx\,dt, \quad \forall v \in L^2(0,T,L^2(\Omega)). \tag{3.47}$$

We prove that

$$\sqrt{a(x)}g(u_t) \in L^1([0,T]\times\Omega).$$

Indeed, since g is continuous, we deduce from (3.39), that

$$\sqrt{a(x)}g(u_t^k) \longrightarrow \sqrt{a(x)}g(u_t) \quad \text{almost everywhere in} \quad [0,T]\times\Omega. \tag{3.48}$$

$$a(x)u_t^k g(u_t^k) \longrightarrow a(x)u_t g(u_t) \quad \text{almost everywhere in} \quad [0,T]\times\Omega.$$

Using (3.21) and Fatou's Lemma, we deduce that

$$\int_0^T\int_\Omega a(x)u_t g(u_t)\,dx\,dt \le K. \tag{3.49}$$

Thanks to the Cauchy-Schwarz's inequality and (3.49), we have

$$\int_0^T\int_\Omega |\sqrt{a(x)}g(u_t)|\,dx\,dt \le c|\mathcal{A}|^{\frac{1}{2}}\Big(\int_0^T\int_\Omega |\sqrt{a(x)}g(u_t)|^2\,dx\,dt\Big)^{\frac{1}{2}} \tag{3.50}$$

if $|u_t| \ge 1$

$$\Big(\int_0^T\int_\Omega |\sqrt{a(x)}g(u_t)|^2\,dx\,dt\Big)^{\frac{1}{2}} \le K^{\frac{1}{2}},$$

and if $|u_t| < 1$.

$$\Big(\int_0^T \int_\Omega |\sqrt{a(x)}g(u_t)|^2 \, dx \, dt\Big)^{\frac{1}{2}} \leq \|a\|_\infty^{\frac{p-1}{2(p+1)}} T^{\frac{p-1}{2(p+1)}} K^{\frac{1}{p+1}}.$$

Then

$$\int_0^T \int_\Omega |\sqrt{a(x)}g(u_t)| \, dx \, dt \leq \widetilde{K}. \tag{3.51}$$

Let $E \subset [0,T] \times \Omega$

$$E_1 = \Big\{(t,x) \in [0,T] \times \Omega : |g(u_t^k)| \leq |E|^{-1/2}\Big\}; \text{ and } E_2 = E \setminus E_1.$$

Let $J(r) = \inf \Big\{|s| : \ s \in \mathbb{R}, \ |g(s)| \geq r\Big\}$, then we have

$$\begin{aligned}\int_E \sqrt{a(x)}g(u_t^k) \ dx \ dt &= \int_{E_1} \sqrt{a(x)}g(u_t^k) \, dx \, dt + \int_{E_2} \sqrt{a(x)}g(u_t^k) \, dx \, dt \\ &\leq \|\sqrt{a}\|_\infty |E|^{1/2} + J(|E|^{\frac{-1}{2}})^{-1} \int_{E_2} \sqrt{a(x)}|u_t^k g(u_t^k)| \, dx \, dt.\end{aligned}$$

Applying (3.21), we find

$$\sup_k \int_E \sqrt{a(x)}g(u_t^k) \ dx \ dt \longrightarrow 0, \text{ when } |E| \longrightarrow 0,$$

and from (3.48), we deduce thanks to Vitali's Theorem that

$$\sqrt{a(x)}g(u_t^k) \longrightarrow \sqrt{a(x)}g(u_t) \text{ in } L^1([0,T] \times \Omega),$$

hence (3.41) yields $\sqrt{a(x)}g(u_t) = \Psi \in L^2(\mathcal{A})$ and

$$\sqrt{a(x)}g(u_t^k) \rightharpoonup^* \sqrt{a(x)}g(u_t), \text{ in } L^2(\mathcal{A}).$$

We deduce, for all $v \in L^2([0,T] \times L^2(\Omega))$, that

$$\int_0^T \int_\Omega \sqrt{a(x)}g(u_t^k)v \, dx \, dt \longrightarrow \int_0^T \int_\Omega \sqrt{a(x)}g(u_t)v \, dx \, dt.$$

Finally we have shown that, for all $v \in L^2([0,T] \times L^2(\Omega))$

$$\int_0^T \int_\Omega \Big(u_{tt} - \Phi(\|\nabla u(t)\|_2^2)\Delta u + a(x)g(u_t)\Big)v \, dx \, dt = 0.$$

Therefore, u is a solution for the problem (3.1).

Proof of uniqueness. Let u and v be two solutions of (3.1) with the same initial data and $w = u - v$. Then we have

$$w_{tt} - \Phi(\|\nabla u\|^2)\Delta w + a(x)(g(u_t) - g(v_t)) = \beta(\|\nabla u\|^{2\gamma} - \|\nabla v\|^{2\gamma})\Delta v \tag{3.52}$$

with $w = 0$ on $[0, +\infty) \times \Gamma$ and $w(0) = w_t(0) = 0$ in Ω. Taking the $L^2(\Omega)$ inner product of (3.2.1) with w_t, we get

$$\begin{aligned} &\frac{d}{dt}[\|w_t\|^2 + \Phi(\|\nabla u\|^2)\|\nabla w\|^2] + 2\int_\Omega a(x)(g(u_t) - g(v_t))(u_t - v_t)\,dx \\ &\quad = 4\beta\gamma\|\nabla u\|^{2\gamma-2}\|\nabla w\|^2\int_\Omega \nabla u \nabla v\,dx + 2\beta(\|\nabla u\|^{2\gamma} - \|\nabla v\|^{2\gamma})\int_\Omega \Delta v w_t\,dx. \end{aligned} \tag{3.53}$$

The first and second terms in the right-hand side of (3.53) are bounded by

$$C\|\nabla w\|^2 \text{ and } C\|\nabla w\|\|w_t\|,$$

using the monotonicity of g and integrating it over $(0, t)$ hence we conclude that

$$\|w_t\|^2 + \alpha\|\nabla w\|^2 \le C\int_0^t \left\{\|w_t(s)\|^2 + \alpha\|\nabla w(s)\|^2\right\} ds,$$

which, by Lemma 3.2.1, implies $w \equiv 0$. The proof of Theorem 3.1.1 is now completed.

3.2.2 Proof of Theorem 3.1.2

We will use now the perturbed energy method, (Lyapunov function method). To this end, let $M,\ \mu \ge 0$ and set for every $t \ge 0$

$$\widehat{E}(t) = ME(t) + E^\mu(t)\rho(t), \tag{3.54}$$

where

$$\rho(t) = 2\int_\Omega u_t(h.\nabla u)\,dx + \theta\int_\Omega u_t u\,dx, \tag{3.55}$$

$$h = \kappa\psi, \tag{3.56}$$

and

$$\theta \in]n-2, n[.$$

We consider $\psi \in C_0^\infty(\mathbb{R}^n)$ such that

$$\begin{cases} 0 \le \psi \le 1, & \\ \psi = 1, & \text{in } \bar{\Omega}\backslash Q_1, \\ \psi = 0, & \text{in } Q_0. \end{cases} \tag{3.57}$$

Proposition 3.2.3. *There exist λ_1 and λ_2 such that*

$$\lambda_1 E(t) \le \widehat{E}(t) \le \lambda_2 E(t), \quad \forall t \ge 0. \tag{3.58}$$

Proof. From the Young's inequality

$$\begin{aligned} 2\int_\Omega u'(h.\nabla u)\,dx &\le R(x^0)\left(\|u_t\|^2+\|\nabla u\|^2\right) \\ &\le 2R(x^0)\left(1+\frac{1}{\alpha}\right)E(t), \end{aligned}$$

where

$$R(x^0)=\max_{x\in\overline{\Omega}}|x-x^0|, \tag{3.59}$$

and

$$\begin{aligned} \theta\int_\Omega u_t u\,dx &\le \frac{\theta}{2}\left(\|u_t\|^2+\|u\|^2\right) \\ &\le \frac{\theta}{2}\left(\|u_t\|^2+C_s^2\|\nabla u\|^2\right) \\ &\le \theta\left(1+\frac{C_s^2}{\alpha}\right)E(t), \end{aligned}$$

where $C_s>0$ (depending only on the geometry of (Ω) is the constant such that
$\|u\|\le C_s\|\nabla u\|,\ \forall u\in H_0^1(\Omega)$

$$|\rho(t)|\le CE(t). \tag{3.60}$$

Then, for M large enough, we obtain (3.58), where

$$\lambda_1=M-E^\mu(0)\Big\{2R(x^0)\left(1+\frac{1}{\alpha}\right)+\theta\left(1+\frac{C_s^2}{\alpha}\right)\Big\}$$

and

$$\lambda_2=M+E^\mu(0)\Big\{2R(x^0)\left(1+\frac{1}{\alpha}\right)+\theta\left(1+\frac{C_s^2}{\alpha}\right)\Big\}.$$

□

Lemma 3.2.4. *The functional $\rho(t)$ satisfies*

$$\begin{aligned} \rho'(t) &= \Phi(\|\nabla u(t)\|^2)\int_\Gamma (h.\nu)\Big(\frac{\partial u}{\partial\nu}\Big)^2 d\Gamma-(n-\theta)\int_\Omega |u_t|^2\,dx-(\theta-n+2)\Phi(\|\nabla u\|_2^2)\int_\Omega|\nabla u|^2\,dx \\ &-\int_{Q_1\setminus Q_0}\kappa\nabla\psi|u_t|^2\,dx+n\int_{Q_1}(1-\psi)|u_t|^2\,dx \\ &-2\Phi(\|\nabla u(t)\|^2)\sum_{i,k=1}^n\int_{Q_1}\kappa_i\frac{\partial\psi_i}{\partial x_k}\frac{\partial u}{\partial x_k}\frac{\partial u}{\partial x_i}\,dx \\ &+(n-2)\Phi(\|\nabla u(t)\|^2)\int_{Q_1}(\psi-1)|\nabla u|^2\,dx+\Phi(\|\nabla u(t)\|^2)\int_{Q_1\setminus Q_0}\kappa\nabla\psi|\nabla u|^2\,dx \\ &-2\int_\Omega(h.\nabla u)a(x)g(u_t)\,dx-\theta\int_\Omega u.a(x)g(u_t)\,dx. \end{aligned} \tag{3.61}$$

Proof. Taking the derivative of ρ, using (3.1), we obtain

$$\begin{aligned}\rho'(t) &= 2\int_\Omega u_t(h.\nabla u_t)\,dx + 2\int_\Omega u_{tt}(h.\nabla u)\,dx + \theta\int_\Omega u_{tt}u\,dx + \theta\int_\Omega u_t^2\,dx\\ &= 2\int_\Omega u_t(h.\nabla u_t)\,dx + 2\int_\Omega (h.\nabla u)(\Phi(\|\nabla u(t)\|^2)\Delta u - a(x)g(u_t))\,dx \\ &+ \theta\int_\Omega u_{tt}u\,dx + \theta\int_\Omega u_t^2\,dx.\end{aligned} \tag{3.62}$$

Next, we will calculate the terms on the RHS of (3.62)

Lemma 3.2.5. *We have*

$$2\int_\Omega u_t(h.\nabla u_t)\,dx = -n\int_\Omega |u_t|^2\,dx - \int_{Q_1\backslash Q_0}\kappa\nabla\psi|u_t|^2\,dx + n\int_{Q_1}(1-\psi)|u_t|^2\,dx. \tag{3.63}$$

Proof. By integrating by parts, using (3.3), (3.57), (3.56) and noting that $u_t = 0$ on Γ, we have

$$\begin{aligned}2\int_\Omega u_t(h.\nabla u_t)\,dx &= -\int_\Omega div(h).|u_t|^2\,dx\\ &= -\int_{\Omega\backslash Q_1} div(\psi.\kappa).|u_t|^2\,dx - \int_{Q_1} div(\psi.\kappa).|u_t|^2\,dx\\ &= -n\int_{\Omega\backslash Q_1}|u_t|^2\,dx - \int_{Q_1}\kappa\nabla\psi|u_t|^2\,dx - n\int_{Q_1}\psi|u_t|^2\,dx\\ &= -n\int_{\Omega\backslash Q_1}|u_t|^2\,dx - \int_{Q_1\backslash Q_0}\kappa\nabla\psi|u_t|^2\,dx - n\int_{Q_1}\psi|u_t|^2\,dx.\end{aligned}$$

□

Lemma 3.2.6. *We have*

$$\begin{aligned}2\Phi(\|\nabla u(t)\|^2)\int_\Omega (h.\nabla u)\Delta u\,dx &= \Phi(\|\nabla u(t)\|^2)\int_\Gamma (h.\nu)\Big(\frac{\partial u}{\partial \nu}\Big)^2\,d\Gamma + (n-2)\Phi(\|\nabla u(t)\|^2)\int_\Omega |\nabla u|^2\,dx\\ &- 2\Phi(\|\nabla u(t)\|^2)\sum_{i,k=1}^n\int_{Q_1}\kappa_i\frac{\partial\psi_i}{\partial x_k}\frac{\partial u}{\partial x_k}\frac{\partial u}{\partial x_i}\,dx\\ &+ (n-2)\Phi(\|\nabla u(t)\|^2)\int_{Q_1}(\psi-1)|\nabla u|^2\,dx\\ &+ \Phi(\|\nabla u(t)\|^2)\int_{Q_1\backslash Q_0}\kappa\nabla\psi|\nabla u|^2\,dx.\end{aligned} \tag{3.64}$$

Proof. We have $\frac{\partial u}{\partial x_k} = \frac{\partial u}{\partial \nu}\nu_k$ which implies

$$h.\nabla u = (h.\nu)\frac{\partial u}{\partial \nu}, \quad \text{on} \quad \Gamma.$$

Integrating by parts, we obtain

$$\begin{aligned}2\Phi(\|\nabla u(t)\|^2)\int_\Omega (h.\nabla u)\Delta u\,dx &= 2\Phi(\|\nabla u(t)\|^2)\int_\Gamma (h.\nu)|\nabla u|^2\,d\Gamma - 2\Phi(\|\nabla u(t)\|^2)\int_\Omega (h.\nabla u)\nabla(\nabla u)\,dx\\ &- 2\Phi(\|\nabla u(t)\|^2)\sum_{i,k=1}^{n}\int_\Omega \frac{\partial h_i}{\partial x_k}\frac{\partial u}{\partial x_k}\frac{\partial u}{\partial x_i}\,dx\\ &= 2\Phi(\|\nabla u(t)\|^2)\int_\Gamma (h.\nu)\Big(\frac{\partial u}{\partial \nu}\Big)^2\,d\Gamma - \Phi(\|\nabla u(t)\|^2)\int_\Omega h\nabla(|\nabla u|^2)\,dx\\ &- 2\Phi(\|\nabla u(t)\|^2)\sum_{i,k=1}^{n}\int_\Omega \frac{\partial h_i}{\partial x_k}\frac{\partial u}{\partial x_k}\frac{\partial u}{\partial x_i}\,dx\\ &= \Phi(\|\nabla u(t)\|^2)\int_\Gamma (h.\nu)\Big(\frac{\partial u}{\partial \nu}\Big)^2\,d\Gamma + \Phi(\|\nabla u(t)\|^2)\int_\Omega (divh)|\nabla u|^2\,dx\\ &- 2\Phi(\|\nabla u(t)\|^2)\sum_{i,k=1}^{n}\int_\Omega \frac{\partial h_i}{\partial x_k}\frac{\partial u}{\partial x_k}\frac{\partial u}{\partial x_i}\,dx.\end{aligned} \tag{3.65}$$

By (3.56) and (3.57), Equation (3.65) gives

$$\begin{aligned}\Phi(\|\nabla u(t)\|^2)\int_\Omega (divh)|\nabla u|^2\,dx &= n\Phi(\|\nabla u(t)\|^2)\int_{\Omega\setminus Q_1} |\nabla u|^2\,dx + \Phi(\|\nabla u(t)\|^2)\int_{Q_1\setminus Q_0} \kappa\nabla\psi|\nabla u|^2\,dx\\ &+ n\Phi(\|\nabla u(t)\|^2)\int_{Q_1}\psi|\nabla u|^2\,dx.\end{aligned} \tag{3.66}$$

By using (3.3), (3.57) and (3.56), the third term can be rewritten as follows

$$\begin{aligned}-2\Phi(\|\nabla u(t)\|^2)\sum_{i,k=1}^{n}\int_\Omega \frac{\partial h_i}{\partial x_k}\frac{\partial u}{\partial x_k}\frac{\partial u}{\partial x_i}\,dx &= -2\Phi(\|\nabla u(t)\|^2)\int_{\Omega\setminus Q_1}|\nabla u|^2\,dx\\ &- 2\Phi(\|\nabla u(t)\|^2)\sum_{i,k=1}^{n}\int_{Q_1}\kappa_i\frac{\partial \psi_i}{\partial x_k}\frac{\partial u}{\partial x_k}\frac{\partial u}{\partial x_i}\,dx\\ &- 2\Phi(\|\nabla u\|^2)\int_{Q_1}\psi|\nabla u|^2\,dx.\end{aligned} \tag{3.67}$$

Taking into account (3.66) and (3.67) into (3.65) yields

$$\begin{aligned}2\Phi(\|\nabla u(t)\|^2)\int_\Omega (h.\nabla u)\Delta u\,dx &= \Phi(\|\nabla u(t)\|^2)\int_\Gamma (h.\nu)\Big(\frac{\partial u}{\partial \nu}\Big)^2\,d\Gamma + (n-2)\Phi(\|\nabla u(t)\|^2)\int_{\Omega\setminus Q_1}|\nabla u|^2\,dx\\ &- 2\Phi(\|\nabla u\|_2^2)\sum_{i,k=1}^{n}\int_{Q_1}\kappa_i\frac{\partial \psi_i}{\partial x_k}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_k}\,dx\\ &+ (n-2)\Phi(\|\nabla u(t)\|^2)\int_{Q_1}\psi|\nabla u|^2\,dx\\ &+ \Phi(\|\nabla u(t)\|^2)\int_{Q_1\setminus Q_0}\kappa\nabla\psi|\nabla u|^2\,dx.\end{aligned}$$

□

Lemma 3.2.7. *We have*

$$\theta \int_\Omega u u_{tt}\,dx = -\theta\Phi(\|\nabla u\|_2^2)\int_\Omega |\nabla u|^2\,dx - \theta\int_\Omega u.a(x)g(u_t))\,dx. \tag{3.68}$$

Proof. Applying the Green formula, we obtain

$$\begin{aligned}\theta \int_\Omega u u_{tt}\,dx &= \theta\Phi(\|\nabla u(t)\|^2)\int_\Omega \Delta u.u\,dx - \int_\Omega u.a(x)g(u_t))\,dx\\ &= -\theta\Phi(\|\nabla u\|^2)\int_\Omega |\nabla u|^2\,dx - \theta\int_\Omega u.a(x)g(u_t))\,dx.\end{aligned}$$

□

Taking into account (3.63), (3.64) and (3.68) into (3.62) we obtain (3.61). The proof of Lemma 3.2.4 is completed. □

Lemma 3.2.8. *We have*

$$\begin{aligned}\rho'(t) \le &-K_nE(t) + (A+n)\int_\omega |u'|^2\,dx + (3A+n-2)\Phi(\|\nabla u(t)\|^2)\int_\Omega |\nabla u|^2\,dx\\ &-2\int_\Omega (h.\nabla u)a(x)g(u_t)\,dx - \theta\int_\Omega u.a(x)g(u_t)\,dx\end{aligned} \tag{3.69}$$

where

$$K_n = \min\Big\{2(n-\theta), 2(\theta-n+2)(\gamma+2)\Big\},$$
$$\theta \in]n-2, n[,$$

and

$$A = R(x^0)\max_{x\in\overline{\Omega}} |\nabla\psi(x)|.$$

Proof. Next, we will estimate some terms on the RHS of (3.61).
Taking (3.2), (3.4), (3.5), (3.57) and (3.56) into account

$$\begin{aligned}\int_\Gamma (h.\nu)\left(\frac{\partial u}{\partial\nu}\right)^2 d\Gamma &= \int_{\Gamma(x^0)} (\kappa.\nu)\psi\left(\frac{\partial u}{\partial\nu}\right)^2 d\Gamma + \int_{\Gamma\setminus\Gamma(x^0)} (\kappa.\nu)\psi\left(\frac{\partial u}{\partial\nu}\right)^2 d\Gamma\\ &\le 0\end{aligned} \tag{3.70}$$

$$\int_{Q_1\setminus Q_0} \kappa\nabla\psi|u_t|^2\,dx \le R(x^0)\max_{x\in\overline{\Omega}}|\nabla\psi(x)|\int_\omega |u_t|^2\,dx, \tag{3.71}$$

$$n\int_{Q_1}(1-\psi)|u_t|^2\,dx \le n\int_\omega |u_t|^2\,dx, \tag{3.72}$$

$$2\Phi(\|\nabla u(t)\|^2)\Big|\sum_{i,k=0}^n \int_{Q_1}\frac{\partial u}{\partial x_k}\frac{\partial u}{\partial x_k}\kappa_i\frac{\partial\psi_i}{\partial x_i}\,dx\Big| \le 2R(x^0)\max_{x\in\overline{\Omega}}|\nabla\psi(x)|\Phi(|\nabla u\|^2)\int_\Omega |\nabla u|^2\,dx, \tag{3.73}$$

$$\Phi(\|\nabla u\|^2)\int_{Q_1\setminus Q_0}\kappa\nabla\psi|\nabla u|^2\,dx \leq R(x^0)\max_{x\in\overline{\Omega}}|\nabla\psi(x)|\Phi(\|\nabla u\|^2)\int_{\Omega}|\nabla u|^2\,dx. \tag{3.74}$$

$$(n-2)\Phi(\|\nabla u(t)\|^2)\int_{Q_1}(\psi-1)|\nabla u|^2\,dx \leq (n-2)\Phi(\|\nabla u(t)\|^2)\int_{\Omega}|\nabla u|^2\,dx. \tag{3.75}$$

Substitution of (3.70)-(3.75) into (3.61) and using (3.10) give (3.69), which concludes the proof. □

Taking the derivative of $\widehat{E}(t)$ with respect to t, we obtain

$$\widehat{E'}(t) = ME'(t) + \mu E'(t)E^{\mu-1}(t)\rho(t) + E^{\mu}(t)\rho'(t), \tag{3.76}$$

using the above Lemma 3.2.8, (3.10) and (3.60), we obtain

$$\begin{aligned}\widehat{E'}(t) \leq\ & ME'(t) + C_\mu|E'(t)|E^{\mu}(0) - K_nE^{\mu+1}(t) + (3A+n-2)\Phi(\|\nabla u\|^2)E^{\mu}(t)\int_{\Omega}|\nabla u|^2\,dx\\ & + (A+n)E^{\mu}(t)\int_{\omega}|u_t|^2\,dx - 2E^{\mu}(t)\int_{\Omega}(h.\nabla u)a(x)g(u_t)\,dx\\ & - \theta E^{\mu}(t)\int_{\Omega}a(x)u.g(u_t)\,dx\end{aligned} \tag{3.77}$$

where $C_\mu = \mu C$. Using (3.10), we get

$$(3A+n-2)E^{\mu}(t)\Phi(\|\nabla u\|^2)\int_{\Omega}|\nabla u|^2\,dx \leq (3A+n-2)(\gamma+2)E^{\mu+1}(t). \tag{3.78}$$

We suppose that $(3A+n-2)(\gamma+2) \leq \frac{K_n}{12} \leq \frac{K_n}{6}$.

From now on, we separate the case $p=1$ from the case $p>1$.

Case $p=1$: **Proof of (3.12)**. Using (3.11), we get

$$\begin{aligned}(A+n)E^{\mu}(t)\int_{\omega}|u_t|^2\,dx &\\ &\leq C\frac{A+n}{a_0}E^{\mu}(t)\int_{\Omega}a(x)u_tg(u_t)\,dx\\ &\leq CE^{\mu}(t)(-E'(t))\\ &\leq CE^{\mu}(0)|E'(t)|.\end{aligned} \tag{3.79}$$

Thanks to Hölder's inequality, we get

$$\begin{aligned}2E^{\mu}(t)\int_{\Omega}h.\nabla u.a(x)g(u_t)\,dx &\leq 2R(x^0)E^{\mu}(t)\|\nabla u\|\Big(\int_{\Omega}a^2(x)g^2(u_t)\,dx\Big)^{\frac{1}{2}}\\ &\leq 2\sqrt{\frac{2}{\alpha}}R(x^0)\|a\|_{\infty}E^{\mu+\frac{1}{2}}(t)\Big(\int_{\Omega}a(x)u_t(t)g(u_t)\,dx\Big)^{\frac{1}{2}}\\ &\leq 2\sqrt{\frac{2}{\alpha}}R(x^0)\|a\|_{\infty}E^{\mu+\frac{1}{2}}(t)(-E'(t))^{\frac{1}{2}}.\end{aligned}$$

Applying Young's inequality, we obtain

$$
\begin{aligned}
2E^{\mu}(t)\int_{\Omega} h\nabla u.a(x)g(u_t)\,dx &\le \frac{2R(x^0)\|a\|_\infty}{\alpha}E^{\mu+1}(t)+R(x^0)\|a\|_\infty E^{\mu}(t)|E'(t)| \\
&\le \frac{2R(x^0)\|a\|_\infty}{\alpha}E^{\mu+1}(t)+R(x^0)\|a\|_\infty E^{\mu}(0)|E'(t)| \\
&\le \frac{K_n}{6}E^{\mu+1}(t)+R(x^0)\|a\|_\infty E^{\mu}(0)|E'(t)|
\end{aligned} \tag{3.80}
$$

and

$$
\begin{aligned}
\theta E^{\mu}(t)\int_{\Omega} a(x)ug(u_t)\,dx &\le C_s\theta E^{\mu}(t)\|\nabla u\|\Big(\int_{\Omega} a^2(x)g^2(u_t)\,dx\Big)^{\frac{1}{2}} \\
&\le C_s\frac{2\theta\|a\|_\infty}{\alpha}E^{\mu+1}(t)+C_s\theta\|a\|_\infty E^{\mu}(0)|E'(t)| \\
&\le \frac{K_n}{6}E^{\mu+1}(t)+C_s\theta\|a\|_\infty E^{\mu}(0)|E'(t)|.
\end{aligned} \tag{3.81}
$$

Reporting (3.78)-(3.81), into (3.2.2), we find

$$
\widehat{E'}(t)\le ME'(t)-K_nE^{\mu+1}(t)+CE^{\mu}(0)|E'(t)|+\frac{K_n}{2}E^{\mu+1}(t).
$$

For M large enough and $E(0)$ small enough, we obtain

$$
\widehat{E'}(t)\le -\frac{K_n}{2}E^{\mu+1}(t).
$$

Choosing $\mu=0$, we have

$$
\begin{aligned}
\widehat{E'}(t) &\le -\frac{K_n}{2}E(t) \\
&\le -\frac{K_n}{2\lambda_1}\widehat{E}(t).
\end{aligned} \tag{3.82}
$$

Finally, by combining (3.58) and (3.82) we obtain (3.12).

Case $p>1$: **Proof of (3.13)** Thanks to Hölder's inequality and (3.11), we get

$$
\begin{aligned}
(A+n)E^{\mu}(t)\int_{\omega}|u_t|^2\,dx &\le \frac{A+n}{a_0}\int_{\Omega} a(x)|u_t|^2\,dx \\
&\le CE^{\mu}(t)\int_{\Omega_1} a(x)u_tg(u_t)\,dx+C'E^{\mu}(t)\int_{\Omega_2} a(x)\Big(u_tg(u_t)\Big)^{\frac{2}{p+1}}\,dx \\
&\le C(\Omega)E^{\mu}(t)\int_{\Omega} a(x)u_tg(u_t)\,dx+C'(\Omega)\|a\|_\infty^{\frac{p-1}{p+1}}E^{\mu}(t)\int_{\Omega}\Big(a(x)u_tg(u_t)\Big)^{\frac{2}{p+1}}\,dx \\
&\le C(\Omega)E^{\mu}(t)(-E'(t))+C'(\Omega)\|a\|_\infty^{\frac{p-1}{p+1}}E^{\mu}(t)\Big(-E'(t)\Big)^{\frac{2}{p+1}}.
\end{aligned}
$$

Now, for fixed arbitrarily small $\varepsilon>0$ (to be chosen later), applying Young's inequality to obtain

$$
\begin{aligned}
(A+n)E^{\mu}(t)\int_{\omega}|u_t|^2\,dx &\le C(\Omega)E^{\mu}(0)|E'(t)|+C'(\Omega)\frac{2}{p+1}\varepsilon^{\frac{p+1}{2}}|E'(t)| \\
&\quad +C'(\Omega)\|a\|_\infty\frac{p-1}{p+1}\frac{1}{\varepsilon^{\frac{p+1}{p-1}}}E^{\mu\frac{p+1}{p-1}}(t) \\
&\le CE^{\mu}(0)|E'(t)|+C\varepsilon^{\frac{p+1}{2}}|E'(t)|+\frac{K_n}{12}E^{\mu\frac{p+1}{p-1}}(t).
\end{aligned} \tag{3.83}
$$

Using Hölder's and Young's inequalities, we obtain

$$2E^{\mu}(t)\int_{\Omega_1} h.a(x)\nabla u.g(u_t)\,dx \le \frac{K_n}{12}E^{\mu+1}(t) + R(x^0)\|a\|_{\infty}E^{\mu}(0)|E'(t)|. \tag{3.84}$$

By Hölder's inequality, we have

$$\begin{aligned}
2E^{\mu}(t)\int_{\Omega_2} h.a(x)\nabla u.g(u_t)\,dx &\le 2R(x^0)E^{\mu}(t)\|\nabla u\|\Big(\int_{\Omega_2} a^2(x)g^2(u_t)\,dx\Big)^{\frac{1}{2}} \\
&\le 2R(x^0)\|a\|_{\infty}^{\frac{p}{p+1}}E^{\mu+\frac{1}{2}}(t)\Big(\int_{\Omega_2}(a(x)u_t(g(u_t))^{\frac{2}{p+1}}\,dx\Big)^{\frac{1}{2}} \\
&\le 2R(x^0)\|a\|_{\infty}^{\frac{p}{p+1}}E^{\mu+\frac{1}{2}}(t)(-E'(t))^{\frac{1}{p+1}}.
\end{aligned}$$

Set $\varepsilon_1 > 0$, thanks to Young's inequality, we obtain

$$\begin{aligned}
&2E^{\mu}(t)\int_{\Omega_2} h.a(x)\nabla u.g(u_t)\,dx \\
&\quad\le C(\Omega)R(x^0)\frac{1}{p+1}\varepsilon_1^{p+1}|E'(t)| + C'(\Omega)R(x^0)\|a\|_{\infty}\frac{p}{p+1}\frac{1}{\varepsilon_1^{\frac{p+1}{p}}}E^{(\mu+\frac{1}{2})\frac{p+1}{p}}(t) \\
&\quad\le C\varepsilon_1^{p+1}|E'(t)| + \frac{K_n}{12}E^{(\mu+\frac{1}{2})\frac{p+1}{p}}(t).
\end{aligned} \tag{3.85}$$

We deduce from (3.84) and (3.85)

$$\begin{aligned}
2E^{\mu}(t)\int_{\Omega} h.a(x)\nabla u.g(u_t)\,dx \le \frac{K_n}{12}E^{\mu+1}(t) + \frac{K_n}{12}E^{(\mu+\frac{1}{2})\frac{p+1}{p}}(t) + C\varepsilon_1^{p+1}|E'(t)| \\
+ R(x^0)\|a\|_{\infty}E^{\mu}(0)|E'(t)|.
\end{aligned} \tag{3.86}$$

Thanks to Hölder's and Young's inequalities, we get

$$\begin{aligned}
\theta E^{\mu}(t)\int_{\Omega_1} a(x)ug(u_t)\,dx &\le C_s\theta E^{\mu}(t)\|\nabla u\|\Big(\int_{\Omega} a^2(x)g^2(u_t)\,dx\Big)^{\frac{1}{2}} \\
&\le C_s\frac{2\theta\|a\|_{\infty}}{\alpha}E^{\mu+1}(t) + C_s\theta\|a\|_{\infty}E^{\mu}(0)|E'(t)| \\
&\le \frac{K_n}{12}E^{\mu+1}(t) + C_s\theta\|a\|_{\infty}E^{\mu}(0)|E'(t)|
\end{aligned} \tag{3.87}$$

and

$$\begin{aligned}
&\theta E^{\mu}(t)\int_{\Omega_2} a(x).ug(u_t)\,dx \\
&\quad\le C(\Omega)R(x^0)\frac{1}{p+1}\varepsilon_2^{p+1}|E'(t)| + C(\Omega)R(x^0)\|a\|_{\infty}\frac{p}{p+1}\frac{1}{\varepsilon_2^{\frac{p+1}{p}}}E^{(\mu+\frac{1}{2})\frac{p+1}{p}}(t) \\
&\quad\le C\varepsilon_2^{p+1}|E'(t)| + \frac{K_n}{12}E^{(\mu+\frac{1}{2})\frac{p+1}{p}}(t).
\end{aligned} \tag{3.88}$$

We deduce from (3.87) and (3.88)

$$E^{\mu}(t)\int_{\Omega} a(x)ug(u_t)\,dx \le \frac{K_n}{12}E^{\mu+1}(t) + \frac{K_n}{12}E^{(\mu+\frac{1}{2})\frac{p+1}{p}}(t) + C\varepsilon_2^{p+1}|E'(t)| + C_s\theta\|a\|_{\infty}E^{\mu}(0)|E'(t)|. \tag{3.89}$$

Reporting (3.83), (3.86) and (3.89), into (3.2.2) and choosing $\varepsilon_1 < \varepsilon_2$ we find

$$\begin{aligned} \widehat{E'}(t) &\leq ME'(t) - K_n E^{\mu+1}(t) + \frac{3K_n}{12} E^{\mu+1}(t) + \frac{K_n}{12} E^{\mu \frac{p+1}{p-1}}(t) \\ &+ \frac{2K_n}{12} E^{(\mu+\frac{1}{2})\frac{p+1}{p}}(t) + CE^{\mu}(0)|E'(t)| + C\varepsilon_2^{p+1}|E'(t)| + C\varepsilon^{\frac{p+1}{2}}|E'(t)|. \end{aligned} \tag{3.90}$$

We choose now μ such that

$$\left(\mu + \frac{1}{2}\right)\frac{p+1}{p} = \mu + 1.$$

Thus we find $\mu = \frac{p-1}{2}$ and

$$\mu \frac{p+1}{p-1} = \mu + 1 + \lambda$$

with $\lambda = 0$.

Choosing ε_2, ε and $E(0)$ small enough and M large enough, we derive from (3.90)

$$\begin{aligned} \widehat{E'}(t) &\leq -\frac{K_n}{2} E^{\mu+1} \\ &\leq -\frac{K_n}{2\lambda_1} \widehat{E}^{\mu+1}(t). \end{aligned} \tag{3.91}$$

Finally, by combining (3.58) and (3.91) we obtain (3.13).

CHAPTER 4

EXISTENCE AND ASYMPTOTIC BEHAVIOR OF GLOBAL SOLUTIONS FOR A NONLINEAR HIGHER-ORDER WAVE EQUATION WITH A NONLINEAR SOURCE TERM AND A DELAY TERM

We consider in this chapter the initial-boundary value problem for some nonlinear higher-order wave equation in a bounded domain. The existence of global weak solutions for this problem is established by using the potential well theory combined with Faedo-Galerkin method. We also established the asymptotic behavior of global solutions as $t \to +\infty$ by applying the Lyapunov method.

4.1 Introduction

$$\begin{cases} u_{tt}(x,t) + \mathcal{A}u(x,t) + \mu_1 g_1(u_t(x,t)) + \mu_2 g_2(u_t(x,t-\tau)) = a|u|^{p-2}u & \text{in } \Omega\times]0,+\infty[, \\ D^\gamma u(x,t) = 0, \ \ |\gamma| \leq m-1 & \text{on } \partial\Omega \times [0,+\infty[, \\ u(x,0) = u_0(x), \ \ u_t(x,0) = u_1(x) & \text{in } \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau) & \text{in } \Omega\times]0,\tau[, \end{cases} \tag{4.1}$$

where $\mathcal{A} = (-\Delta)^m$, $m \geq 1$, $p > 1$ are real numbers, Ω is a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}^*$, with a smooth boundary $\partial\Omega$, Δ is the Laplace operator in $\mathbb{R}^n$, $\gamma = (\gamma_1, \gamma_2, ...\gamma_n)$, $|\gamma| = \Sigma_{i=1}^n \gamma_i$, $D = \frac{\partial_i^\gamma}{\partial x_i^{\gamma_i}} = \frac{\partial^{|\gamma|}}{\partial x_1^{\gamma_1} x_2^{\gamma_2} ....x_n^{\gamma_n}}$, $x = (x_1, x_2.......x_n)$, μ_1 and μ_2 are positive real numbers, g_1 and g_2 are two functions, $\tau > 0$ is a time delay, and the initial data (u_0, u_1, f_0) are in a suitable function space.

For the system

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + a_0 u_t(x,t) + a u_t(x, t-\tau) = 0 & \text{in } \Omega\times]0, +\infty[, \\ u(x,t) = 0 & \text{on } \partial\Omega \times [0, +\infty[, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \Omega, \end{cases} \tag{4.2}$$

it is well-known, in the absence of delay $(a = 0, a_0 > 0)$, that this system is exponentially stable, see [45] and [75]. In the presence of delay $(a > 0)$, Nicaise and Pignotti [55] examined system (4.2) and proved, under the assumption that the weight of the feedback with delay is smaller than the one without delay (i.e., $0 < a < a_0$), that the energy is exponentially stable. However, in the opposite case, they could produce a sequence of delays for which the corresponding solution is instable.

In the case for $m = 1$, Benaissa and Louhibi [18] studied the following problem

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + \mu_1 g_1(u_t(x,t)) + \mu_2 g_2(u_t(x, t-\tau)) = 0 & \text{in } \Omega\times]0, +\infty[, \\ u(x,t) = 0 & \text{on } \partial\Omega \times [0, +\infty[, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \Omega, \\ u_t(x, t-\tau) = f_0(x, t-\tau) & \text{in } \Omega\times]0, \tau[, \end{cases}$$

they showed the global existence of weak solutions using the Faedo-Galerkin method, and obtained general stability estimates by introducing multiplier method and general weighted integral inequalities.

For the initial-boundary value problem of a single higher order nonlinear hyperbolic equation

$$\begin{cases} u_{tt}(x,t) + \mathcal{A}u(x,t) + a|u_t|^{r-2}u_t = b|u|^{p-2}u & \text{in } \Omega\times]0, +\infty[, \\ D^\gamma u(x,t) = 0, \quad |\gamma| \leq m-1 & \text{on } \partial\Omega \times [0, +\infty[, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \Omega, \end{cases} \tag{4.3}$$

Nakao [53] has used Galerkin's method to present the existence and uniqueness of the bounded solutions, periodic and almost periodic solutions to the problem (4.3) as the dissipative term is a linear function νu_t. Nakao and Kuwahara [54] study decay estimates of global solutions to the problem (4.3) with the degenerate dissipative term $a(x)u_t$ by using a difference inequality.

In the case of $m \geq 1$ and $\mu_1 = \mu_2 = 0$, the problem (4.1) becomes the following initial-boundary value problem

$$\begin{cases} u_{tt}(x,t) + (-\Delta)^m u(x,t) = a|u|^{p-2}u & \text{in } \Omega\times]0, +\infty[, \\ D^\gamma u(x,t) = 0, \quad |\gamma| \leq m-1 & \text{on } \partial\Omega \times [0, +\infty[, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \Omega, \end{cases} \tag{4.4}$$

P. Brenner and W. Von Wahl [19] proved the existence and uniqueness of classical solutions to (4.4) in Hilbert space. H. Pecher [60] investigated the existence and uniqueness of Cauchy problem for the equation in (4.4) by use of the potential well method due to L. Payne and D.H. Sattinger [59] and D.H. Sattinger [61]. B.X. Wang [69] showed that the scattering operators map a band in H^s into H^s if the nonlinearities have critical or subcritical powers in H^s.
Y. Yanbing et al [72] studied solutions of

$$\begin{cases} u_{tt}(x,t) + \Delta^2 u(x,t) - \Delta u(x,t) - \alpha \Delta u_t(x,t) = f(u) & \text{in } \Omega \times]0,+\infty[, \\ \Delta u(x,t) = u(x,t) = 0, & \text{on } \partial\Omega \times [0,+\infty[, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \Omega, \end{cases} \tag{4.5}$$

and proved a global well-posedness result, asymptotic behaviour and finite time blow up for some strongly damped nonlinear wave equation.
In this chapter, we prove the global existence of solutions for the problem (4.1) by applying the potential well theory and Faedo-Galerkin method. Meanwhile, we study the asymptotic behavior of global solutions by the Lyapunov method.

This chapter is organized as follows: in the next section, we give some preliminaries and main results. Then Section 3 contains the proofs of the global existence and general decay results.

4.2 Preliminaries and main results

To state and prove our result, we use the following assumptions:

(A1) $g_1 : \mathbb{R} \to \mathbb{R}$ is nondecreasing function of class C^1 and $H : \mathbb{R}_+ \to \mathbb{R}_+$ is convex, increasing and of class $C^1(\mathbb{R}_+) \cap C^2(]0,+\infty[)$ satisfying

$$\begin{gathered} H(0) = 0 \text{ and } H \text{ is linear on } [0,\varepsilon] \text{ or} \\ H'(0) = 0 \text{ and } H'' > 0 \text{ on }]0,\varepsilon] \text{ such that} \\ c_1'|s| \le |g_1(s)| \le c_1|s| \quad \text{if } |s| \ge \varepsilon \\ s^2 + g_1^2(s) \le H^{-1}(sg_1(s)) \quad \text{if } |s| \le \varepsilon, \end{gathered} \tag{4.6}$$

where H^{-1} denotes the inverse function of H and ε, c_1, c_1' are positive constants.

(A2) $g_2 : \mathbb{R} \to \mathbb{R}$ is an odd nondecreasing function of class $C^1(\mathbb{R})$ such that there exist $c_2, \alpha_1, \alpha_2 > 0$,

$$|g_2'(s)| \le c_2, \tag{4.7}$$

$$\alpha_1 s g_2(s) \le G(s) \le \alpha_2 s g_1(s), \tag{4.8}$$

where $G(s) = \int_0^s g_2(r)dr$;

(A3)

$$\alpha_2\mu_2 < \alpha_1\mu_1.$$

(A4) $m \geq 1$ is a natural number, p satisfies $2 \leq p < +\infty$ if $n \leq 2m$ and $2 \leq p \leq \frac{2(n-m)}{n-2m}$ if $n > 2m$.

Lemma 4.2.1. *Let q be a real number with $2 \leq q < +\infty$ if $n \leq 2m$ and $2 \leq q \leq \frac{2n}{n-2m}$ if $n > 2m$. Then there is a constant C_s depending on Ω and q such that*

$$\|u\|_q \leq C_s \|\mathcal{A}^{\frac{1}{2}} u\|_2, \quad \forall u \in H_0^m(\Omega).$$

Remark 4.2.1. Let us denote by Φ^* the conjugate function of the differentiable convex function Φ, i.e.,

$$\Phi^*(s) = \sup_{t\in\mathbb{R}^+} (st - \Phi(t)).$$

Then Φ^* is the Legendre transform of Φ, which is given by (see Arnold [12, p. 61-62])

$$\Phi^*(s) = s(\Phi')^{-1}(s) - \Phi[(\Phi')^{-1}(s)], \quad \text{if } s \in (0, \Phi'(r)],$$

and Φ^* satisfies the generalized Young inequality

$$AB \leq \Phi^*(A) + \Phi(B), \quad \text{if } A \in (0, \Phi'(r)],\ B \in (0, r]. \tag{4.9}$$

We introduce, as in Nicaise and Pignotti [55], the new variable

$$z(x, \rho, t) = u_t(x, t - \rho\tau),\ x \in \Omega, \quad \rho \in (0, 1), t > 0.$$

Then, we have

$$\tau z_t(x, \rho, t) + z_\rho(x, \rho, t) = 0,\ \text{ in } \Omega \times (0, 1) \times (0, +\infty). \tag{4.10}$$

Therefore, problem (4.1) is equivalent to

$$\begin{cases} u_{tt}(x,t) + \mathcal{A}u(x,t) + \mu_1 g_1(u_t(x,t)) + \mu_2 g_2(z(x,1,t)) = a|u|^{p-2}u, & \text{in } \Omega\times]0,+\infty[, \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, & \text{in } \Omega\times]0,1[\times]0,+\infty[, \\ D^\gamma u(x,t) = 0, |\gamma| \leq m-1, & \text{on } \partial\Omega \times [0,+\infty[, \\ z(x,0,t) = u_t(x,t), & \text{on } \Omega \times [0,\infty[, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & \text{in } \Omega, \\ z(x,\rho,0) = f_0(x,-\rho\tau), & \text{in } \Omega\times]0,1[. \end{cases} \tag{4.11}$$

We first define the following functionals:

$$\begin{aligned} I(t) &= I(u(t)) = \|\mathcal{A}^{\frac{1}{2}}u\|_2^2 - a\|u\|_p^p \\ J(t) &= J(u(t)) = \frac{1}{2}\|\mathcal{A}^{\frac{1}{2}}u\|_2^2 - \frac{a}{p}\|u\|_p^p. \end{aligned} \tag{4.12}$$

We denote the total energy by

$$\begin{aligned} E(u(t)) &= \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\mathcal{A}^{\frac{1}{2}}u(t)\|_2^2 + \xi \int_\Omega \int_0^1 G(z(x,\rho,t))\, d\rho\, dx - \frac{a}{p}\|u\|_p^p \\ &= \frac{1}{2}\|u_t\|_2^2 + \xi \int_\Omega \int_0^1 G(z(x,\rho,t))\, d\rho\, dx + J(u(t)), \end{aligned} \tag{4.13}$$

where ξ is a positive constant such that

$$\tau\frac{\mu_2(1-\alpha_1)}{\alpha_1} < \xi < \tau\frac{\mu_1-\alpha_2\mu_2}{\alpha_2}$$

and

$$E(0) = \frac{1}{2}\|u_1\|_2^2 + \frac{1}{2}\|\mathcal{A}^{\frac{1}{2}}u_0\|_2^2 + \xi \int_\Omega \int_0^1 G(f_0(x,-\rho\tau))\, d\rho\, dx - \frac{a}{p}\|u_0\|_p^p.$$

Then, for the problem (4.11), we can define the stable set as

$$\mathcal{W} = \{u \backslash u \in H_0^m(\Omega), I(u) > 0\} \cup \{0\}.$$

Theorem 4.2.2. *(Local existence) Assume that* (A1)-(A4) *hold, if* $u_0 \in H^{2m}(\Omega) \cap H_0^m(\Omega)$, $u_1 \in H_0^m(\Omega)$ *and* $f_0 \in H_0^m(\Omega, H^m(0,1))$ *satisfy the compatibility condition* $f(\cdot,0) = u_1$*. Then there exists* $T > 0$ *such that the problem* (4.1) *has a unique local solution* $u(t)$ *which satisfies*

$$u \in \mathcal{C}([0,\infty); H_0^m(\Omega)), \quad u_t \in \mathcal{C}([0,\infty); L^2(\Omega)).$$

Now we have the existence of a global solution.

Theorem 4.2.3. *Let* $u_0 \in H^{2m}(\Omega) \cap \mathcal{W}$, $u_1 \in H_0^m(\Omega) \cap L^2(\Omega)$ *and* $f_0 \in H_0^m(\Omega, H^m(0,1))$ *satisfy the compatibility condition* $f(\cdot,0) = u_1$*. Assume that* (A1)-(A4) *hold. Then* (4.1) *admits a global weak solution* $u(x,t)$ *such that*

$$\begin{gathered} u \in L^\infty([0,\infty); H^{2m}(\Omega) \cap H_0^m(\Omega)), \quad u_t \in L^\infty([0,\infty); H_0^m(\Omega) \cap L^2(\Omega)), \\ u_{tt} \in L^2([0,\infty);\ L^2(\Omega)). \end{gathered}$$

Also we have a uniform decay rates for the energy.

Theorem 4.2.4. *Assume that* (A1)-(A4) *hold. Then, there exist positive constants* w_1, w_2, w_3 *and* ε_0 *such that the solution of* (4.1) *satisfies*

$$E(t) \le w_3 H_1^{-1}(w_1 t + w_2) \quad \forall t \ge 0,$$

where

$$H_1(t) = \int_t^1 \frac{1}{H_2(s)}\, ds \tag{4.14}$$

and

$$H_2(t) = \begin{cases} t & \text{if}\, H\, \textit{is linear on}\, [0,\varepsilon'], \\ tH'(\varepsilon_0 t), & \textit{if} \;\; H'(0) = 0\; \textit{and}\, H'' > 0\; \textit{on}\,]0,\varepsilon']. \end{cases}$$

Lemma 4.2.5. *Let (u, z) be a solution to the problem (4.11). Then, the energy functional defined by (4.13) satisfies*

$$\begin{aligned} E'(t) &\leq -\Big(\mu_1 - \frac{\xi\alpha_2}{\tau} - \mu_2\alpha_2\Big)\int_\Omega u_t(x,t)g_1(u_t(x,t))\,dx \\ &\quad - \Big(\frac{\xi\alpha_1}{\tau} - \mu_2(1-\alpha_1)\Big)\int_\Omega z(x,1,t)g_2(z(x,1,t))\,dx \\ &\leq 0. \end{aligned} \tag{4.15}$$

Proof. By multiplying the first equation in (4.11) by u_t, integrating over Ω and using integration by parts, we obtain

$$\begin{aligned} \frac{d}{dt}\left\{\frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\|\mathcal{A}^{\frac{1}{2}}u(t)\|_2^2 - \frac{a}{p}\|u(t)\|_p^p\right\} + \mu_1\int_\Omega u_t(x,t)g_1(u_t(x,t))\,dx \\ + \mu_2\int_\Omega u_t(x,t)g_2(z(x,1,t))\,dx = 0. \end{aligned} \tag{4.16}$$

We multiply the second equation in (4.11) by $\xi g_2(z)$, we integrate the result over $\Omega\times(0,1)$, to obtain

$$\begin{aligned} \xi\int_\Omega\int_0^1 z_t(x,\rho,t)g_2(z(x,\rho,t))\,d\rho\,dx &= -\frac{\xi}{\tau}\int_\Omega\int_0^1 z_\rho(x,\rho,t)g_2(z(x,\rho,t))\,d\rho\,dx \\ &= -\frac{\xi}{\tau}\int_\Omega\int_0^1 \frac{\partial}{\partial\rho}\Big(G(z(x,\rho,t))\Big)\,d\rho\,dx \\ &= -\frac{\xi}{\tau}\int_\Omega (G(z(x,1,t)) - G(z(x,0,t)))\,dx. \end{aligned}$$

Hence

$$\xi\frac{d}{dt}\int_\Omega\int_0^1 G(z(x,\rho,t))\,d\rho\,dx = -\frac{\xi}{\tau}\int_\Omega G(z(x,1,t))\,dx + \frac{\xi}{\tau}\int_\Omega G(u_t(x,t))\,dx. \tag{4.17}$$

By combining (4.16) and (4.17), we obtain

$$\begin{aligned} E'(t) = &-\mu_1\int_\Omega u_t(x,t)g_1(u_t(x,t))\,dx - \mu_2\int_\Omega u_t(x,t)g_2(z(x,1,t))\,dx \\ &- \frac{\xi}{\tau}\int_\Omega G(z(x,1,t))\,dx + \frac{\xi}{\tau}\int_\Omega G(u_t(x,t))\,dx, \end{aligned}$$

and by recalling (4.8), we obtain

$$\begin{aligned} E'(t) \leq &-(\mu_1 - \frac{\xi\alpha_2}{\tau})\int_\Omega u_t(x,t)g_1(u_t(x,t))\,dx - \mu_2\int_\Omega u_t(x,t)g_2(z(x,1,t))\,dx \\ &- \frac{\xi}{\tau}\int_\Omega G(z(x,1,t))\,dx. \end{aligned} \tag{4.18}$$

From the definition of G and by using remark 4.2.1, we obtain

$$G^*(s) = sg_2^{-1}(s) - G(g_2^{-1}(s)), \quad \forall s \geq 0.$$

Hence

$$\begin{aligned} G^*(g_2(z(x,1,t))) &= z(x,1,t)g_2(z(x,1,t)) - G(z(x,1,t)) \\ &\leq (1-\alpha_1)z(x,1,t)g_2(z(x,1,t)). \end{aligned}$$

By using (4.8) and (4.9) with $A = g_2(z(x,1,t))$ and $B = u_t(x,t)$, from (4.18) we obtain

$$\begin{aligned} E'(t) \leq & -\Big(\mu_1 - \frac{\xi\alpha_2}{\tau}\Big) \int_\Omega u_t(x,t) g_1(u_t(x,t))\,dx \quad + \mu_2 \int_\Omega (G(u_t(x,t)) + G^*(g_2(z(x,1,t)))\,dx \\ & - \frac{\xi}{\tau}\int_\Omega G(z(x,1,t))\,dx \\ \leq & -\Big(\mu_1 - \frac{\xi\alpha_2}{\tau} - \mu_2\alpha_2\Big)\int_\Omega u_t(x,t)g_1(u_t(x,t))\,dx \\ & - \Big(\frac{\xi\alpha_1}{\tau} - \mu_2(1-\alpha_1)\Big)\int_\Omega z(x,1,t)g_2(z(x,1,t))\,dx \leq 0. \end{aligned}$$

□

Lemma 4.2.6. *Suppose that*

$$2 \leq p \leq \frac{2n}{n-2m}, \quad n > 2m, \tag{4.19}$$

holds. If $u_0 \in \mathcal{W}$, $u_1 \in L^2(\Omega)$ *and* $f_0 \in H_0^m(\Omega, H^m(0,1))$ *such that*

$$aC_s^p\left(\frac{2p}{p-2}E(0)\right)^{\frac{p-2}{2}} < 1. \tag{4.20}$$

Then, $u(t) \in \mathcal{W}$ *for all* $t \in [0,+\infty)$.

Proof. Since $I(u(0)) > 0$, then it follows from the continuity of $u(t)$

$$I(t) \geq 0, \tag{4.21}$$

for some interval near $t = 0$. Let t_{max} be the maximum time (possible $t_{max} = T$) when (4.21) holds on $[0, t_{max})$. From (4.12), we have

$$J(t) = \frac{p-2}{2p}\|\mathcal{A}^{\frac{1}{2}}u(t)\|_2^2 + \frac{1}{p}I(u). \tag{4.22}$$

From (4.12), (4.13) and (4.22), we deduce

$$\begin{aligned} \|\mathcal{A}^{\frac{1}{2}}u(t)\|_2^2 &\leq \frac{2p}{p-2}J(t) \leq \frac{2p}{p-2}E(t) \\ &\leq \frac{2p}{p-2}E(0) \quad \text{on} \quad [0, t_{max}). \end{aligned} \tag{4.23}$$

Then by (4.20), (4.23) and the embedding, we obtain

$$\begin{aligned} a\|u\|_p^p &\leq aC_s^p\|\mathcal{A}^{\frac{1}{2}}u(t)\|_2^p = aC_s^p\|\mathcal{A}^{\frac{1}{2}}u(t)\|_2^{p-2}\|\mathcal{A}^{\frac{1}{2}}u(t)\|_2^2 \\ &\leq aC_s^p\left(\frac{2p}{p-2}E(0)\right)^{\frac{p-2}{2}}\|\mathcal{A}^{\frac{1}{2}}u(t)\|_2^2 \\ &< \|\mathcal{A}^{\frac{1}{2}}u(t)\|_2^2 \quad \text{on} \quad [0, t_{max}). \end{aligned}$$

Thus $I(t) = \|\mathcal{A}^{\frac{1}{2}}u(t)\|_2^2 - a\|u\|_p^p > 0$ on $[0, t_{max})$. This implies that we can take $t_{max} = T$. □

Theorem 4.2.7. *Suppose that (4.19) holds. If $u_0 \in \mathcal{W}$ and $u_1 \in L^2(\Omega)$ satisfying (4.20). Then the solution of (4.11) is uniformly bounded and global in time.*

Proof. It suffices to show that $\|\mathcal{A}^{\frac{1}{2}}\|_2^2 + \|u_t\|_2^2$ is bounded independently of t. Clearly,

$$\begin{aligned} E(0) \geq E(t) &= \frac{1}{2}\|u_t\|_2^2 + \xi \int_\Omega \int_0^1 G(z(x,\rho,t))\, d\rho\, dx + J(u(t)) \\ &= \frac{1}{2}\|u_t\|_2^2 + \xi \int_\Omega \int_0^1 G(z(x,\rho,t))\, d\rho\, dx + \frac{p-2}{2p}\|\mathcal{A}^{\frac{1}{2}}u(t)\|_2^2 + \frac{1}{p}I(u(t)) \\ &\geq \frac{p-2}{2p}\|\mathcal{A}^{\frac{1}{2}}u(t)\|_2^2 + \frac{1}{2}\|u_t\|_2^2, \end{aligned}$$

since $I(u(t))$ and $\xi \int_\Omega \int_0^1 G(z(x,\rho,t))\, d\rho\, dx$ are nonnegative. Therefore,

$$\|\mathcal{A}^{\frac{1}{2}}u(t)\|_2^2 + \|u_t\|_2^2 \leq CE(0),$$

where C is a positive constant. □

4.3 Proofs of main results

4.3.1 Proof of Theorem 4.2.3

Proof. Throughout this section we assume $u_0 \in H^{2m}(\Omega) \cap \mathcal{W}$, $u_1 \in H_0^m(\Omega) \cap L^2(\Omega)$ and $f_0 \in H_0^m(\Omega, H^m(0,1))$. We employ the Galerkin method to construct a global solution. Let $T > 0$ be fixed and denote by V_k the space generated by $\{w^1, w^2, \ldots\ldots, w^k\}$, where the set $\{w^k, k \in \mathbb{N}\}$ is a basis of $H^{2m}(\Omega) \cap H_0^m(\Omega)$. Now, we define, for $1 \leq j \leq k$, the sequence $\phi^j(x,\rho)$ as follows:

$$\phi^j(x,0) = w^j.$$

Then, we may extend $\phi^j(x,0)$ by $\phi^j(x,\rho)$ over $L^2(\Omega \times (0,1))$ such that $(\phi^j)_j$ forms a basis of $L^2(\Omega, H^m(0,1))$ and denote Z_k the space generated by $\{\phi^k\}$.

Step 1: Approximate solutions.

We construct approximate solutions (u^k, z^k), $k = 1, 2, 3, \ldots$, in the form

$$u^k(t) = \sum_{j=1}^{k} c^{jk}(t) w^j(x), \quad z^k(t) = \sum_{j=1}^{k} d^{jk}(t)\phi^j,$$

where c^{jk} and $d^{jk}(j = 1, 2, \ldots, k)$ are determined by the ordinary differential equations

$$(u_{tt}^k(t), w^j) + (\mathcal{A}^{\frac{1}{2}}u^k(t), \mathcal{A}^{\frac{1}{2}}w^j) + \mu_1(g_1(u_t^k), w^j) + \mu_2(g_2(z^k(.,1)), w^j) = a(|u^k|^{p-2}u^k, w^j), \quad (4.24)$$

$$z^k(x,0,t) = u_t^k(x,t), \quad (4.25)$$

$$u^k(0) = u_0^k = \sum_{j=1}^{k}(u_0, w^j)w^j \to u_0, \quad \text{in } H^{2m}(\Omega) \cap \mathcal{W} \text{ as } k \to +\infty, \tag{4.26}$$

$$u_t^k(0) = u_1^k = \sum_{j=1}^{k}(u_1, w^j)w^j \to u_1, \quad \text{in } H_0^m(\Omega) \cap L^2(\Omega) \text{ as } k \to +\infty, \tag{4.27}$$

and

$$(\tau z_t^k + z_\rho^k, \phi^j) = 0, \quad 1 \le j \le k, \tag{4.28}$$

$$z^k(\rho, 0) = z_0^k = \sum_{j=1}^{k}(f_0, \phi^j)\phi^j \to f_0 \quad \text{in } H_0^m(\Omega, H^m(0,1)) \text{ as } k \to +\infty. \tag{4.29}$$

By virtue of the theory of ordinary differential equations, the systems (4.24)-(4.29) have a unique local solution which is extended to a maximal interval $[0, T_k]$ (with $0 < T_k < +\infty$) by Zorn lemma since the nonlinear terms in (4.24) are locally Lipschitz continuous. Note that $u_k(t)$ is from the class C^2. In the next step, we obtain a priori estimates for the solution, such that it can be extended outside $[0, T_k]$ to obtain one solution defined for all $t > 0$. In order to use a standard compactness argument for the limiting procedure, it suffices to derive some a priori estimates for (u_k, z_k).

Step 2: The first estimate

Since the sequences u_0^k, u_1^k and z_0^k converge, the standard calculations, using (4.24)-(4.29), similar to those used to derive (4.15), yield

$$E^k(t) - E^k(0) \le -\beta_1 \int_0^t \int_\Omega u_t^k g_1(u_t^k)\,dx\,ds - \beta_2 \int_0^t \int_\Omega z^k(x,1,s) g_2(z^k(x,1,s))\,dx\,ds,$$

where $\beta_1 = \mu_1 - \frac{\xi\alpha_2}{\tau} - \mu_2\alpha_2$ and $\beta_2 = \frac{\xi\alpha_1}{\tau} - \mu_2(1-\alpha_1)$. So we obtain

$$\begin{aligned} &E^k(t) + \beta_1 \int_0^t \int_\Omega u_t^k g_1(u_t^k)\,dx\,ds + \beta_2 \int_0^t \int_\Omega z^k(x,1,s) g_2(z^k(x,1,s))\,dx\,ds \\ &\le E^k(0), \end{aligned}$$

where

$$E^k(t) = \frac{1}{2}\|u_t^k\|_2^2 + \xi \int_\Omega \int_0^1 G(z^k(x,\rho,t))\,d\rho\,dx + J(u^k(t)),$$

$$J(u^k(t)) = \frac{1}{2}\|\mathcal{A}^{\frac{1}{2}} u^k\|_2^2 - \frac{a}{p}\|u^k\|_p^p,$$

and

$$\begin{aligned} E^k(0) &= \frac{1}{2}\|u_1^k\|_2^2 + \xi \int_\Omega \int_0^1 G(z^k(x,\rho,0))\,d\rho\,dx + \frac{1}{2}\|\mathcal{A}^{\frac{1}{2}} u_0^k\|_2^2 - \frac{a}{p}\|u_0^k\|_p^p \\ &\le C_1. \end{aligned}$$

For some C_1 independent of k. We obtain the first estimate:

$$\|u_t^k\|_2^2 + \int_\Omega \int_0^1 G(z^k(x,\rho,t))\, d\rho\, dx + J(u^k(t)) + \int_0^t \int_\Omega u_t^k g_1(u_t^k)\, dx\, ds + \int_0^t \int_\Omega z^k(x,1,s) g_2(z^k(x,1,s))\, dx\, ds \leq C_1.$$

These estimates imply that the solution (u^k, z^k) exists globally in $[0,+\infty[$.

$$u^k \text{ is bounded in } L^\infty_{\text{loc}}(0,\infty, H_0^m(\Omega)), \tag{4.30}$$

$$u_t^k \text{ is bounded in } L^\infty_{\text{loc}}(0,\infty, L^2(\Omega)), \tag{4.31}$$

$$G(z^k(x,\rho,t)) \text{ is bounded in } L^\infty_{\text{loc}}(0,\infty, L^1(\Omega\times(0,1))), \tag{4.32}$$

$$u_t^k(t) g_1(u_t^k(t)) \text{ is bounded in } L^1(\Omega\times(0,T)), \tag{4.33}$$

$$z^k(x,1,t) g_2(z^k(x,1,t)) \text{ is bounded in } L^1(\Omega\times(0,T))). \tag{4.34}$$

Step 3: The second estimate

First, we estimate $u_{tt}^k(0)$ taking $t=0$ in (4.24) , we obtain

$$(u_{tt}^k(0), w^j) + (\mathcal{A}^{\frac{1}{2}} u^k(0), \mathcal{A}^{\frac{1}{2}} w^j) + \mu_1 (g_1(u_t^k)(0), w^j) + \mu_2 (g_2(z^k(.,1)(0), w^j) = a(|u^k(0)|^{p-2} u^k(0), w^j),$$

multiplying by c_{tt}^{jk} and summing over j from 1 to k,

$$(u_{tt}^k(0), u_{tt}^k(0)) + (\mathcal{A}u^k(0), u_{tt}^k(0)) + \mu_1(g_1(u_t^k)(0), u_{tt}^k(0)) + \mu_2(g_2(z^k(.,1)(0), u_{tt}^k(0)) = a(|u^k(0)|^{p-2}u^k(0), u_{tt}^k(0)).$$

Using Hölder's inequality, we have

$$\|u_{tt}^k(0)\| \leq \|\mathcal{A}u^k(0)\| + \mu_1 \|g_1(u_1^k)\| + \mu_2 \|g_2(z_1^k)\| + a\| |u_0^k|^{p-2} u_0^k\|.$$

Since $g_1(u_1^k)$, $g_2(z_1^k)$ are bounded in $L^2(\Omega)$, (4.26), (4.27) and (4.29) yield

$$\|u_{tt}^k(0)\| \leq C,$$

where C is a positive constant independent of k.

Now, differentiating (4.24) with respect to t

$$(u_{ttt}^k(t), w^j) + (\mathcal{A}u_t^k(t), w^j) + \mu_1(u_{tt}^k g_1'(u_t^k), w^j) + \mu_2(z_t^k g_2'(z^k(.,1)), w^j) = a(p-1)(|u^k|^{p-2} u_t^k, w^j),$$

multiplying by c_{tt}^{jk} and summing over j from 1 to k,

$$\frac{1}{2}\frac{d}{dt}\left\{\|u_{tt}^k(t)\|_2^2 + \|\mathcal{A}^{\frac{1}{2}} u_t^k(t)\|_2^2\right\} + \mu_1 \int_\Omega (u_{tt}^k(t))^2 g_1'(u_t^k(t))dx + \mu_2 \int_\Omega u_{tt}^k(t) z_t^k(x,1,t) g_2'(z^k(x,1,t))dx = a(p-1)\int_\Omega |u^k(t)|^{p-2} u_t^k(t) u_{tt}^k(t) dx. \tag{4.35}$$

We have from Hölder's inequality,

$$a(p-1)\int_\Omega |u^k(t)|^{p-2}u_t^k(t)u_{tt}^k(t)dx \le a(p-1)\|u^k(t)\|_{2(p-1)}^{p-2}\|u_t^k(t)\|_{2(p-1)}\|u_{tt}^k(t)\|_2,$$

where

$$\frac{p-2}{2(p-1)}+\frac{1}{2(p-1)}+\frac{1}{2}=1.$$

Using Lemma 4.2.1, Young's inequality and (4.30), we have

$$\begin{aligned} a(p-1)\int_\Omega |u^k(t)|^{p-2}u_t^k(t)u_{tt}^k(t)dx &\le a(p-1)C_s^{p-1}\|\mathcal{A}^{\frac{1}{2}}u^k(t)\|_2^{p-2}\|\mathcal{A}^{\frac{1}{2}}u_t^k(t)\|_2\|u_{tt}^k(t)\|_2 \\ &\le C(\epsilon)\|\mathcal{A}^{\frac{1}{2}}u_t^k(t)\|_2^2+\epsilon\|u_{tt}^k(t)\|_2^2. \end{aligned} \tag{4.36}$$

Differentiating (4.28) with respect to t, we obtain

$$(\tau z_{tt}^k + z_{t\rho}^k, \phi^j) = 0.$$

Multiplying by d_t^{jk} and summing over j from 1 to k, it follows that

$$\frac{\tau}{2}\frac{d}{dt}\|z_t^k\|_2^2+\frac{1}{2}\frac{d}{d\rho}\|z_t^k\|_2^2=0.$$

Integrating over $(0,1)$ with respect to ρ, we obtain

$$\frac{\tau}{2}\frac{d}{dt}\int_0^1 \|z_t^k\|_2^2\,d\rho+\frac{1}{2}\|z_t^k(x,1,t)\|_2^2-\frac{1}{2}\|u_{tt}^k(x,t)\|_2^2=0. \tag{4.37}$$

Taking the sum of (4.35) and (4.37), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\{\|u_{tt}^k(t)\|_2^2+\|\mathcal{A}^{\frac{1}{2}}u_t^k(t)\|_2^2+\tau\int_0^1\|z_t^k\|_2^2\,d\rho\right\}+\mu_1\int_\Omega (u_{tt}^k(t))^2g_1'(u_t^k(t))dx+\frac{1}{2}\|z_t^k(x,1,t)\|_2^2 \\ &=a(p-1)\int_\Omega |u^k(t)|^{p-2}u_t^k(t)u_{tt}^k(t)dx-\mu_2\int_\Omega u_{tt}^k(t)z_t^k(x,1,t)g_2'(z^k(x,1,t))dx+\frac{1}{2}\|u_{tt}^k(x,t)\|_2^2 \end{aligned} \tag{4.38}$$

Using (4.7) and Young's inequality, we conclude

$$\int_\Omega |u_{tt}^k(t)||z_t^k(x,1,t)||g_2'(z^k(x,1,t))|dx \le \varepsilon\|z_t^k(x,1,t)\|_2^2+\frac{c_2^2}{4\varepsilon}\|u_{tt}^k\|_2^2. \tag{4.39}$$

A combination of (4.36), (4.38) and (4.39), then yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\{\|u_{tt}^k(t)\|_2^2+\|\mathcal{A}^{\frac{1}{2}}u_t^k(t)\|_2^2+\tau\int_0^1\|z_t^k\|_2^2\,d\rho\right\}+\mu_1\int_\Omega (u_{tt}^k(t))^2g_1'(u_t^k(t))\,dx+(\frac{1}{2}-\varepsilon)\|z_t^k(x,1,t)\|_2^2 \\ &\le(\varepsilon+\frac{c_3^2}{4\varepsilon}+\frac{1}{2})\|u_{tt}^k\|_2^2+C(\epsilon)\|A^{\frac{1}{2}}u_t^k(t)\|^2, \end{aligned} \tag{4.40}$$

integrating over $[0,t]$ for all $t\in[0,T]$ with arbitrary fixed T,

$$\frac{1}{2}\left\{\|u^k_{tt}(t)\|_2^2+\|\mathcal{A}^{\frac{1}{2}}u^k_t(t)\|_2^2+\tau\|z^k_t(x,\rho,t)\|^2_{L^2(\Omega\times(0,1))}\right\}+\mu_1\int_0^t\int_\Omega (u^k_{tt}(t))^2 g_1'(u^k_t(t))\,dx\,dt$$
$$+(\frac{1}{2}-\varepsilon)\int_0^t\|z^k_t(x,1,t)\|_2^2\,dt$$
$$\leq\frac{1}{2}\left\{\|u^k_{tt}(0)\|_2^2+\|\mathcal{A}^{\frac{1}{2}}u^k_t(0)\|_2^2+\tau\|z^k_t(x,\rho,0)\|^2_{L^2(\Omega\times(0,1))}\right\}+(\varepsilon+\frac{c_3^2}{4\varepsilon}+\frac{1}{2})\int_0^t\|u^k_{tt}\|_2^2\,dt$$
$$+C(\epsilon)\int_0^t\|A^{\frac{1}{2}}u^k_t(t)\|^2\,dt. \tag{4.41}$$

Then from (4.41), after choosing ε small enough and using Gronwall's Lemma, we obtain

$$\|u^k_{tt}(t)\|_2^2+\|\mathcal{A}^{\frac{1}{2}}u^k_t(t)\|_2^2+\tau\|z^k_t(x,\rho,t)\|^2_{L^2(\Omega\times(0,1))}+\mu_1\int_0^t\int_\Omega (u^k_{tt}(t))^2 g_1'(u^k_t(t))\,dx\,dt$$
$$+(\frac{1}{2}-\varepsilon)\int_0^t\|z^k_t(x,1,t)\|_2^2\,dt\leq M$$

for all $t\in[0,T]$, where M is a positive constant independent of $k\in\mathbb{N}$. Therefore, we conclude that

$$u^k_{tt}\text{ is bounded in }L^\infty_{\text{loc}}(0,\infty,L^2(\Omega)), \tag{4.42}$$
$$u^k_t\text{ is bounded in }L^\infty_{\text{loc}}(0,\infty,H^m_0(\Omega)), \tag{4.43}$$
$$z^k_t\text{ is bounded in }L^\infty_{\text{loc}}(0,\infty,L^2(\Omega\times(0,1))). \tag{4.44}$$

Step 4: The third estimate

Replacing w^j by $\mathcal{A}w^j$ in (4.24), multiplying by c^{jk}_t and summing over j from 1 to k, it follows that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\mathcal{A}^{\frac{1}{2}}u^k_t(t)\|_2^2+\|\mathcal{A}u^k(t)\|_2^2\right)+\mu_1\int_\Omega|\mathcal{A}^{\frac{1}{2}}u^k_t(t)|^2g_1'(u^k_t)dx\\&+\mu_2\int_\Omega\mathcal{A}^{\frac{1}{2}}z^k(x,1,t)\mathcal{A}^{\frac{1}{2}}u^k_t g_2'(z^k(x,1,t))dx\\&=a\int_\Omega\mathcal{A}^{\frac{1}{2}}(|u^k|^{p-2}.u^k).\mathcal{A}^{\frac{1}{2}}u^k_t dx.\end{aligned} \tag{4.45}$$

Replacing ϕ^j by $\mathcal{A}\phi^j$ in (4.28), multiplying by d^{jk} and summing over j from 1 to k, it follows that

$$\frac{\tau}{2}\frac{d}{dt}\|\mathcal{A}^{\frac{1}{2}}z^k\|_2^2+\frac{1}{2}\frac{d}{d\rho}\|\mathcal{A}^{\frac{1}{2}}z^k\|_2^2=0,$$

we integrate over $(0,1)$ to find

$$\frac{\tau}{2}\frac{d}{dt}\int_0^1\|\mathcal{A}^{\frac{1}{2}}z^k(t)\|_2^2\,d\rho+\frac{1}{2}\|\mathcal{A}^{\frac{1}{2}}z^k(x,1,t)\|_2^2-\frac{1}{2}\|\mathcal{A}^{\frac{1}{2}}u^k_t(t)\|_2^2=0. \tag{4.46}$$

Combining (4.45), (4.46), using (4.7), Cauchy-Schwarz and Young's inequalities produce the estimate

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\mathcal{A}^{\frac{1}{2}}u_t^k(t)\|_2^2+\|\mathcal{A}u^k(t)\|_2^2+\int_0^1\|\mathcal{A}^{\frac{1}{2}}z^k(x,\rho,t)\|_2^2\,d\rho\right)\\
&+\mu_1\int_\Omega|\mathcal{A}^{\frac{1}{2}}u_t^k(t)|^2g_1^{'}(u_t^k)dx+(\frac{1}{2}-\varepsilon)\|\mathcal{A}^{\frac{1}{2}}z_t^k(x,1,t)\|_2^2\\
&\leq C(\varepsilon)\|\mathcal{A}^{\frac{1}{2}}u_t^k(t)\|_2^2+a\int_\Omega\mathcal{A}^{\frac{1}{2}}(|u^k|^{p-2}.u^k).\mathcal{A}^{\frac{1}{2}}u_t^kdx.
\end{aligned}$$

Integrating the last inequality over $(0,t)$, we have

$$\begin{aligned}
&\|\mathcal{A}^{\frac{1}{2}}u_t^k(t)\|_2^2+\|\mathcal{A}u^k(t)\|_2^2+\int_0^1\|\mathcal{A}^{\frac{1}{2}}z^k(x,\rho,t)\|_2^2\,d\rho\\
&+2\mu_1\int_0^t\int_\Omega|\mathcal{A}^{\frac{1}{2}}u_t^k(s)|^2g_1^{'}(u_t^k)\,dx\,dt+2(\frac{1}{2}-\varepsilon)\int_0^t\|\mathcal{A}^{\frac{1}{2}}z_t^k(x,1,s)\|_2^2\,dt\\
&\leq A^k(0)+C(\varepsilon)\int_0^t\|\mathcal{A}^{\frac{1}{2}}u_t^k(s)\|_2^2\,dt+c\int_0^t\|\mathcal{A}^{\frac{1}{2}}u^k(s)\|^{p-1}.\|\mathcal{A}^{\frac{1}{2}}u_t^k(s)\|\,dt\\
&\leq A^k(0)+C(\varepsilon)\int_0^t\|\mathcal{A}^{\frac{1}{2}}u_t^k(s)\|_2^2\,dt+\frac{c}{2}\int_0^t\|\mathcal{A}^{\frac{1}{2}}u^k(s)\|^{2(p-1)}\,dt+\frac{c}{2}\int_0^t\|\mathcal{A}^{\frac{1}{2}}u_t^k(s)\|\,dt\\
&\leq A^k(0)+\max\{(C(\varepsilon)+\frac{c}{2}),C(E^k(0))^p\}\int_0^t(\|\mathcal{A}^{\frac{1}{2}}u_t^k(s)\|_2^2+\|\mathcal{A}^{\frac{1}{2}}u^k(s)\|^2)\,dt,
\end{aligned}$$

where

$$A^k(0)=\|\mathcal{A}^{\frac{1}{2}}u_t^k(t)\|_2^2+\|\mathcal{A}u^k(t)\|_2^2+\|\mathcal{A}^{\frac{1}{2}}z^k(x,\rho,t)\|_{L^2(\Omega\times(0,1))}^2,$$

and using Gronwall's Lemma, we have

$$\|\mathcal{A}^{\frac{1}{2}}u_t^k(t)\|_2^2+\|\mathcal{A}u^k(t)\|_2^2+\int_0^1\|\mathcal{A}^{\frac{1}{2}}z^k(x,\rho,t)\|_2^2\,d\rho\leq A^k(0)e^{cT}, \tag{4.47}$$

for all $t\in\mathbb{R}_+$, therefore, we conclude that

$$u^k \text{ is bounded in } L^\infty_{\text{loc}}(0,\infty,H^{2m}(\Omega)), \tag{4.48}$$

$$z^k \text{ is bounded in } L^\infty_{\text{loc}}(0,\infty,H_0^m(\Omega,L^2(0,1))). \tag{4.49}$$

Step 5: Passage to the limit

Applying Dunford-Petti's theorem, we conclude from (4.30)-(4.34), (4.42)-(4.44) and (4.48)-(4.49), after replacing the sequences u^k and z^k by subsequence if necessary, that

$$u^k \rightharpoonup u \text{ weak-star in } L^\infty\left(0,\infty;H^{2m}(\Omega)\right) \tag{4.50}$$

$$u_t^k \rightharpoonup u_t \text{ weak-star in } L^\infty(0,\infty;H_0^m(\Omega)) \tag{4.51}$$

$$u_{tt}^k \rightharpoonup u_{tt} \text{ weak-star in } L^\infty\left(0,\infty;L^2(\Omega)\right) \tag{4.52}$$

$$g_1(u_t^k) \rightharpoonup \chi \text{ weak-star in } L^2(\Omega \times (0,T)) \tag{4.53}$$

$$z^k \rightharpoonup z \text{ weak-star in } L^\infty(0,\infty, H_0^m(\Omega, L^2(0,1))). \tag{4.54}$$

$$z_t^k \rightharpoonup z_t \text{ weak-star in } L^\infty(0,\infty, L^2(\Omega \times (0,T)). \tag{4.55}$$

$$g_2(u_t^k) \rightharpoonup \psi \text{ weak-star in } L^2(\Omega \times (0,T)). \tag{4.56}$$

On the other hand, from Aubin-Lions theorem, (see Lions [41]) we deduce that there exists a subsequence $\{u^m\}$ of $\{u^k\}$ such that

$$u^m \to u \text{ strongly in } L^2(0,T,L^2(\Omega)) \tag{4.57}$$

$$u_t^m \to u_t \text{ strongly in } L^2(0,T,L^2(\Omega)) \tag{4.58}$$

which implies

$$u^m \to u \text{ almost everywhere in } \mathcal{B},$$

and

$$u_t^m \to u_t \text{ almost everywhere in } \mathcal{B}. \tag{4.59}$$

Hence

$$|u^m|^{p-2}u^m \to |u|^{p-2}u \text{ almost everywhere in } \mathcal{B}, \tag{4.60}$$

where $\mathcal{B} = \Omega \times (0,T)$.

$$\int_{\mathcal{B}} (|u^m|^{p-2}u^m)^{\frac{p}{p-1}}\, dx\, dt \le \int_{\mathcal{B}} |u^m|^p\, dx\, dt \le C\|\mathcal{A}^{\frac{1}{2}}u^m\|^p_{L^2(\mathcal{B})},$$

using (4.30) we obtain

$$\|u^m\|_{L^{p/(p-1)}(\mathcal{B})} \le C. \tag{4.61}$$

Thus, using (4.60), (4.61) and Lions Lemma, we derive

$$|u^m|^{p-2}u^m \rightharpoonup |u|^{p-2}u \text{ weakly in } L^{p/(p-1)}(\mathcal{B}), \tag{4.62}$$

and

$$z^m \to z \text{ strongly in } L^2(0,T,L^2(\Omega))$$

which implies

$$z^m \to z \text{ almost everywhere in } \mathcal{B}.$$

Lemma 4.3.1. *For each $T > 0$, $g_1(u_t), g_2(z(x,1,t)) \in L^1(\mathcal{B})$ and $\|g_1(u_t)\|_{L^1(\mathcal{B})} \leq K$, $\|g_2(z(x,1,t))\|_{L^1(\mathcal{B})} \leq K$, where K is a constant independent of t.*

Proof. By (A1) and (4.59), we have

$$g_1(u_t^m(x,t)) \to g_1(u_t(x,t)) \quad \text{almost everywhere in } \mathcal{B},$$
$$0 \leq u_t^k(x,t) g_1(u_t^m(x,t)) \to u_t(x,t) g_1(u_t(x,t)) \quad \text{almost everywhere in } \mathcal{B}.$$

Hence, by (4.33) and Fatou's Lemma, we have

$$\int_0^T \int_\Omega u_t(x,t) g_1(u_t(x,t))\, dx\, dt \leq K_1 \quad \text{for } T > 0. \tag{4.63}$$

Now, we can estimate $\int_0^T \int_\Omega |g_1(u_t(x,t))|\, dx\, dt$. By Cauchy-Schwarz inequality and using (4.63), we have

$$\begin{aligned}\int_0^T \int_\Omega |g_1(u_t(x,t))|\, dx\, dt &\leq c|\mathcal{B}|^{1/2} \Big(\int_0^T \int_\Omega u_t(x,t) g_1(u_t(x,t))\, dx\, dt \Big)^{1/2} \\ &\leq c|\mathcal{B}|^{1/2} K_1^{1/2} \equiv K.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\int_0^T \int_\Omega |g_2(z(x,1,t)))|\, dx\, dt &\leq c|\mathcal{B}|^{1/2} \Big(\int_0^T \int_\Omega z(x,1,t) g_2(z(x,1,t))\, dx\, dt \Big)^{1/2} \\ &\leq c|\mathcal{B}|^{1/2} K_1^{1/2} \equiv K.\end{aligned}$$

□

Lemma 4.3.2. *$g_1(u_t^k) \to g_1(u_t)$ in $L^1(\Omega \times (0,T))$ and $g_2(z^k) \to g_2(z)$ in $L^1(\Omega \times (0,T))$.*

Proof. Let $E \subset \Omega \times [0,T]$ and set

$$E_1 = \Big\{ (x,t) \in E : |g_1(u_t^k(x,t))| \leq \frac{1}{\sqrt{|E|}} \Big\}, \quad E_2 = E \backslash E_1,$$

where $|E|$ is the measure of E. If $M(r) = \inf\{|s| : s \in \mathbb{R} \text{ and } |g(s)| \geq r\}$

$$\int_E |g_1(u_t^k)|\, dx\, dt \leq c\sqrt{|E|} + \Big(M\Big(\frac{1}{\sqrt{|E|}}\Big)\Big)^{-1} \int_{E_2} |u_t^k g_1(u_t^k)|\, dx\, dt.$$

By applying (4.33) we deduce that $\sup_k \int_E |g_1(u_t^k)|\, dx\, dt \to 0$ as $|E| \to 0$. From Vitali's convergence theorem we deduce that

$$g_1(u_t^k) \to g_1(u_t) \quad \text{in } L^1(\Omega \times (0,T)).$$

Similarly, we have

$$g_2(z^k) \to g_2(z) \quad \text{in } L^1(\Omega \times (0,T)).$$

This completes the proof. □

Hence

$$g_1(u_t^k) \rightharpoonup g_1(u_t) \quad \text{weak in } L^2(\Omega \times (0,T)), \tag{4.64}$$

$$g_2(z^k) \rightharpoonup g_2(z) \quad \text{weak in } L^2(\Omega \times (0,T)). \tag{4.65}$$

By multiplying (4.24) by $\theta(t) \in \mathcal{D}(0,T)$ and by integrating over $(0,T)$, it follows that

$$\begin{aligned} &\int_0^T (u_t^k(t), w^j)\theta'(t)\,dt + \int_0^T (\mathcal{A}^{\frac{1}{2}} u^k(t), \mathcal{A}^{\frac{1}{2}} w^j)\theta(t)\,dt \\ &+ \mu_1 \int_0^T (g_1(u_t^k), w^j)\theta(t)\,dt + \mu_2 \int_0^T (g_2(z^k(.,1)), w^j)\theta(t)\,dt \\ &= \int_0^T (|u^k|^{p-2}u^k, w^j)\theta(t)\,dt, \end{aligned} \tag{4.66}$$

and multiplying (4.26) by $\theta(t) \in \mathcal{D}(0,T)$ and integrating over $(0,T) \times (0,1)$, it follows that

$$\int_0^T \int_0^1 (\tau z_t^k + z_\rho^k, \phi^j)\theta(t)\,dt\,d\rho = 0. \tag{4.67}$$

The convergence of (4.50)–(4.56), (4.62), (4.64) and (4.65) are sufficient to pass to the limit in (4.66) and (4.67) to obtain

$$\begin{aligned} &\int_0^T (u_t^k(t), w)\theta'(t)\,dt + \int_0^T (\mathcal{A}^{\frac{1}{2}} u^k(t), \mathcal{A}^{\frac{1}{2}} w)\theta(t)\,dt \\ &+ \mu_1 \int_0^T (g_1(u_t^k), w)\theta(t)\,dt + \mu_2 \int_0^T (g_2(z^k(.,1)), w)\theta(t)\,dt \\ &= \int_0^T (|u^k|^{p-2}u^k, w)\theta(t)\,dt \end{aligned}$$

and

$$\int_0^T \int_0^1 (\tau z_t + z_\rho, \phi)\theta(t)\,dt\,d\rho = 0.$$

By integrating, we have

$$\int_0^T \Big(u_{tt} + \mathcal{A}u^k(t) + \mu_1 g_1(u_t) + \mu_2 g_2(z(.,1)), w\Big)\theta(t)\,dt = \int_0^T |u|^{p-2}u\theta(t)\,dt.$$

This completes the proof of Theorem 4.2.3. □

4.4 Asymptotic behavior

4.4.1 Proof of Theorem 4.2.4

Proof. In this section, we prove the energy decay result by constructing a suitable Lyapunov functional. Now we define the following functional

$$L(t) = NE(t) + N_1F_1(t) + F_2(t), \tag{4.68}$$

where

$$F_1(t) = \int_\Omega u u_t dx, \tag{4.69}$$

$$F_2(t) = \int_\Omega \int_0^1 e^{-2\tau\rho} G(z(x,\rho,t))\, d\rho\, dx \tag{4.70}$$

and we also need the following lemma

Lemma 4.4.1. *Let (u,z) be a solution of problem (4.11). Then there exist two positive constants λ_1, λ_2 such that*

$$\lambda_1 E(t) \le L(t) \le \lambda_2 E(t), \qquad t \ge 0, \tag{4.71}$$

for N sufficiently large and N_1 is a positive real number to be chosen appropriately later.

Proof. Let $L(t) = N_1 F_1(t) + F_2(t)$

$$|L(t)| \le N_1 \int_\Omega |u u_t| dx + \int_\Omega \int_0^1 e^{-2\tau\rho} G(z(x,\rho,t))\, d\rho\, dx.$$

By Young's inequality, Lemma 4.2.1, (4.13) and the fact that $e^{-2\tau\rho} \le 1$ for all $\rho \in [0,1]$, we obtain

$$\begin{aligned} |L(t)| &\le \frac{N_1}{2}\|u_t\|_2^2 + \frac{N_1 C_s}{2}\|A^{\frac{1}{2}}u\|_2^2 + \int_\Omega \int_0^1 G(z(x,\rho,t))\, d\rho\, dx \\ &\le cE(t). \end{aligned}$$

Consequently, $|L(t) - NE(t)| \le cE(t)$, which yields

$$(N-c)E(t) \le L(t) \le (N+c)E(t).$$

Choosing N large enough, we obtain estimate (4.71). □

Lemma 4.4.2. *Let (u,z) be a solution of (4.11). Then the functional $F_1(t)$ defined by (4.69) satisfies, for any $\eta > 0$, the estimate*

$$\begin{aligned} F_1'(t) \le{}& \|u_t\|_2^2 + a\|u\|_p^p - \Big(1 - \eta C_s^2(\mu_1+\mu_2)\Big)\|\mathcal{A}^{\frac{1}{2}}u\|_2^2 \\ &+ \frac{\mu_1}{4\eta}\int_\Omega |g_1(u_t)|^2\, dx + \frac{\mu_2 c_2}{4\eta}\int_\Omega z(x,1,t) g_2(z(x,1,t))\, dx. \end{aligned} \tag{4.72}$$

Proof. Taking the derivative of $F_1(t)$ with respect to t and using the first equation of (4.11), we obtain

$$\begin{aligned} F_1'(t) &= \|u_t\|_2^2 + \int_\Omega u_{tt} u\, dx \\ &= \|u_t\|_2^2 + a\|u\|_p^p - \|\mathcal{A}^{\frac{1}{2}}u\|_2^2 - \mu_1\int_\Omega u g_1(u_t(x,t))\, dx - \mu_2 \int_\Omega u g_2(z(x,1,t))\, dx. \end{aligned} \tag{4.73}$$

Now, we estimate the terms in the right hand side of (4.73) using Young's inequality and Lemma 4.2.1, we obtain

$$\int_{\Omega} u g_1(u_t)\,dx \le \eta C_s^2 \|\mathcal{A}^{\frac{1}{2}}u\|_2^2 + \frac{1}{4\eta}\int_{\Omega} |g_1(u_t)|^2\,dx, \tag{4.74}$$

$$\int_{\Omega} u g_2(z(x,1,t))\,dx \le \eta C_s^2 \|\mathcal{A}^{\frac{1}{2}}u\|_2^2 + \frac{1}{4\eta}\int_{\Omega} |g_2(z(x,1,t))|^2\,dx\,. \tag{4.75}$$

From (4.7), (4.75) becomes

$$\int_{\Omega} u g_2(z(x,1,t))\,dx \le \eta C^2 \|\mathcal{A}^{\frac{1}{2}}u\|_2^2 + \frac{c_2}{4\eta}\int_{\Omega} z(x,1,t) g_2(z(x,1,t))\,dx. \tag{4.76}$$

Estimate (4.72) follows by substituting (4.74) and (4.76) into (4.73). □

Lemma 4.4.3. *Let (u,z) be the solution to (4.11). Then the functional $F_2(t)$ defined by (4.70) satisfies, the estimate*

$$\begin{aligned} F_2'(t) \le {}& -2e^{-2\tau}\int_{\Omega}\int_0^1 G(z(x,\rho,t))\,d\rho\,dx - \frac{\alpha_1 e^{-2\tau}}{\tau}\int_{\Omega} z(x,1,t) g_2(z(x,1,t))\,dx \\ & + \frac{\alpha_2}{\tau}\int_{\Omega} u_t(x,t) g_1(u_t(x,t))\,dx. \end{aligned} \tag{4.77}$$

Proof. By differentiating (4.70) with respect to t and using (4.8) and (4.10), we obtain

$$\begin{aligned} F_2'(t) &= -\frac{1}{\tau}\int_{\Omega}\int_0^1 e^{-2\tau\rho}\frac{\partial}{\partial\rho}G(z(x,\rho,t))\,d\rho\,dx \\ &= -\frac{1}{\tau}\int_{\Omega}\int_0^1 \frac{\partial}{\partial\rho}\Big(e^{-2\tau\rho}G(z(x,\rho,t))\Big)\,dx + 2\tau e^{-2\tau\rho}G(z(x,\rho,t))\Big]\,d\rho\,dx \\ &= -\frac{1}{\tau}\int_{\Omega}\Big[e^{-2\tau}G(z(x,1,t) - G(u_t(x,t))\Big]\,dx \\ &\quad - 2\int_{\Omega}\int_0^1 e^{-2\tau\rho}G(z(x,\rho,t))\,d\rho\,dx \\ &= -\frac{1}{\tau}\int_{\Omega} e^{-2\tau}G(z(x,1,t)\,dx + \frac{1}{\tau}\int_{\Omega} G(u_t(x,t))\,dx \\ &\quad - 2\int_{\Omega}\int_0^1 e^{-2\tau\rho}G(z(x,\rho,t))\,d\rho\,dx \\ &= -2F_2(t) + \frac{1}{\tau}\int_{\Omega} G(u_t(x,t))\,dx - \frac{e^{-2\tau}}{\tau}\int_{\Omega} G(z(x,1,t))\,dx \\ &\le -2F_2(t) + \frac{\alpha_2}{\tau}\int_{\Omega} u_t(x,t) g_1(u_t(x,t))\,dx \\ &\quad - \frac{\alpha_1 e^{-2\tau}}{\tau}\int_{\Omega} z(x,1,t) g_2(z(x,1,t))\,dx. \end{aligned}$$

Since $-e^{-2\tau\rho}$ is an increasing function, we have $-e^{-2\tau\rho} \le -e^{-2\tau}$ for all $\rho \in [0,1]$, we deduce

$$-2F(t) \le -2e^{-2\tau}\int_{\Omega}\int_0^1 G(z(x,\rho,t))\,d\rho\,dx.$$

The proof is complete. □

Lemma 4.4.4. *Let* (u, z) *be a solution of* (4.11)*and assume that (A1)-(A4) hold. Then the functional defined by (4.68) satisfies*

$$L'(t) \leq -mE(t) + C_1(\|u_t\|_2^2 + \|g_1(u_t)\|_2^2) \tag{4.78}$$

for some positive constants m *and* C_1.

Proof. By differentiating (4.68) and recalling (4.15), (4.72) and (4.77), we obtain

$$\begin{aligned} L'(t) &= NE'(t) + N_1F_1'(t) + F_2'(t) \\ &\leq -(N\beta_1 - \frac{\alpha_2}{\tau}) \int_\Omega u_t(x,t)g_1(u_t(x,t))\,dx \\ &\quad - \Big(N\beta_2 - N_1\frac{\mu_2 c_2}{4\eta} + \frac{\alpha_1 e^{-2\tau}}{\tau}\Big) \int_\Omega z(x,1,t)g_2(z(x,1,t))\,dx \\ &\quad - 2e^{-2\tau}\int_\Omega\int_0^1 G(z(x,\rho,t))\,d\rho\,dx + N_1\frac{\mu_1}{4\eta}\int_\Omega |g_1(u_t)|^2\,dx \\ &\quad + N_1\|u_t\|_2^2 + N_1 a\|u\|_p^p - N_1\Big(1 - \eta C_s^2(\mu_1+\mu_2)\Big)\|\mathcal{A}^{\frac{1}{2}}u\|_2^2. \end{aligned} \tag{4.79}$$

We choose N large enough such that

$$N\beta_1 - \frac{\alpha_2}{\tau} > 0 \text{ and } N\beta_2 - N_1\frac{\mu_2 c_2}{4\eta} > 0.$$

Thus (4.79) becomes

$$\begin{aligned} L'(t) &\leq -2e^{-2\tau}\int_\Omega\int_0^1 G(z(x,\rho,t))\,d\rho\,dx - N_1\Big(1 - \eta C^2(\mu_1+\mu_2)\Big)\|\mathcal{A}^{\frac{1}{2}}u\|_2^2 + N_1 a\|u\|_p^p \\ &\quad + N_1\|u_t\|_2^2 + N_1\frac{\mu_1}{4\eta}\int_\Omega |g_1(u_t)|^2\,dx. \end{aligned}$$

We choose η small enough so that $(1 - \eta C^2(\mu_1+\mu_2) > 0$. Noting by

$$m = \min\left\{2N_1\Big(1 - \eta C^2(\mu_1+\mu_2)\Big), \frac{2e^{-2\tau}}{\xi}\right\},$$

and choosing N_1 small enough so that $pN_1 \leq m$ we obtain

$$\begin{aligned} L'(t) &\leq -mE(t) + \frac{m}{2}\|u_t\|_2^2 + N_1\|u_t\|_2^2 + N_1\frac{\mu_1}{4\eta}\int_\Omega |g_1(u_t)|^2\,dx \\ &\leq -mE(t) + c(\|u_t\|_2^2 + \int_\Omega |g_1(u_t)|^2\,dx). \end{aligned}$$

This completes the proof. □

As in Komornik [35], we consider the following partition of Ω

$$\Omega_1 = \{x \in \Omega: \ |u_t| \geq \varepsilon\}, \quad \Omega_2 = \{x \in \Omega: \ |u_t| \leq \varepsilon\}.$$

By using (4.6), we have

$$\int_{\Omega_1} |u_t|^2 \, dx + \int_{\Omega_1} |g_1(u_t)|^2 \, dx \le (c_1' + c_1) \int_{\Omega_1} u_t g_1(u_t) \, dx \le -\mu_1 E'(t), \tag{4.80}$$

and

$$\int_{\Omega_2} |u_t|^2 \, dx + \int_{\Omega_2} |g_1(u_t)|^2 \, dx \le \int_{\Omega_2} H^{-1}(u_t g_1(u_t)) \, dx.$$

- Case 1. H is linear on $[0, \varepsilon']$, in this case, one can easily check that there exists $\mu_2 > 0$, such that and thus,

  $$\int_{\Omega_2} |u_t|^2 \, dx + \int_{\Omega_2} |g_1(u_t)|^2 \, dx \le -\mu_2 E'(t). \tag{4.81}$$

  Substitution of (4.80) and (4.81) into (4.78) gives

  $$(L(t) + \mu E(t))' \le m H_2(E(t)), \tag{4.82}$$

  where $\mu = C_1(\mu_1 + \mu_2)$

- Case 2. $H'(0) > 0$ and $H'' > 0$ on $]0, \varepsilon']$ we define

  $$I_1(t) = \frac{1}{|\Omega_2|} \int_{\Omega_2} u_t g(u_t) \, dx,$$

  and use Jensen's inequality and the concavity of H^{-1} to obtain

  $$H^{-1}(I_1(t)) \ge \widetilde{C} \int_{\Omega_2} H^{-1}(u_t g(u_t)) \, dx.$$

  By using (4.6), we obtain

  $$\begin{aligned} \int_{\Omega_2} (|u_t|^2 + |g_1(u_t)|^2) \, dx &\le \int_{\Omega_2} H^{-1}(u_t g_1(u_t)) \, dx \\ &\le \widetilde{C} H^{-1}(I_1(t)) \le \widetilde{C} H^{-1}(-C_2 E'(t)). \end{aligned} \tag{4.83}$$

  Combining (4.78), (4.80) and (4.83), we get

  $$(L(t) + C_1\mu_1 E(t))' \le -mE(t) + \widetilde{C} H^{-1}(-C_2 E'(t)). \tag{4.84}$$

  By recalling that $E' \le 0, \; H' > 0, \; H'' > 0$ on $(0, \varepsilon]$ and making use of (4.84), we obtain

  $$\begin{aligned} &\Big(H'(\varepsilon_0 E(t))\{L(t) + C_1\mu_1 E(t)\} + \widetilde{C} C_2 E(t)\Big)' \\ &= \varepsilon_0 E'(t) H''(\varepsilon_0 E(t))(L(t) + C_1\mu_1 E(t)) + H'(\varepsilon_0 E(t))(L(t) + C_1\mu_1 E(t))' + \widetilde{C} C_2 E'(t) \\ &\le -mH'(\varepsilon_0 E(t))E(t) + \widetilde{C} H'(\varepsilon_0 E(t)) H^{-1}(-C_2 E'(t)) + \widetilde{C} C_2 E'(t), \end{aligned}$$

  by using Remark 4.2.1 with H^*, the convex conjugate of H in the sense of Young, we obtain

  $$\begin{aligned} &\Big(H'(\varepsilon_0 E(t))\{L(t) + C_1\mu_1 E(t)\} + \widetilde{C} C_2 E(t)\Big)' \\ &\qquad \le -mH'(\varepsilon_0 E(t))E(t) + \widetilde{C} H^*(H'(\varepsilon_0 E(t))) \\ &\qquad \le -mH'(\varepsilon_0 E(t))E(t) + \widetilde{C} \varepsilon_0 H'(\varepsilon_0 E(t))E(t) \\ &\qquad \le -C_3 H'(\varepsilon_0 E(t))E(t) = -C_3 H_2(E(t)). \end{aligned} \tag{4.85}$$

Let

$$\widetilde{L}(t) = \begin{cases} L(t) + \mu E(t) & \text{If } H \text{ is linear on } [0,\varepsilon], \\ \\ H'(\varepsilon_0 E(t))\{L(t) + C_1\mu_1 E(t))\} + \widetilde{C}C_2 E(t) & \text{If } H'(0) > 0 \text{ and } H'' > 0 \text{ on }]0,\varepsilon], \end{cases}$$

From (4.82) and (4.85), it follows

$$\frac{d}{dt}\widetilde{L}(t) \leq -C_3 H_2(E(t)), \ \ \forall t \geq t_0. \tag{4.86}$$

From Lemma 4.4.1, we have $L(t)$ is equivalent to $E(t)$. So, $\widetilde{L}(t)$ is also equivalent to $E(t)$, for some positive constants $\widetilde{\epsilon_1}$ and $\widetilde{\epsilon_2}$

$$\widetilde{\epsilon_1} E(t) \leq \widetilde{L}(t) \leq \widetilde{\epsilon_2} E(t). \tag{4.87}$$

Let

$$L(t) = \frac{1}{\widetilde{\epsilon_2}}\widetilde{L}(t), \tag{4.88}$$

then we observe, from (4.86) and (4.88), that

$$\begin{aligned} L'(t) &\leq -\frac{C_3}{\widetilde{\epsilon_2}} H_2(E(t)) \\ &\leq -\frac{C_3}{\widetilde{\epsilon_2}} H_2\Big(\frac{1}{\widetilde{\epsilon_2}}\widetilde{L}(t)\Big) = -\frac{C_3}{\widetilde{\epsilon_2}} H_2(L(t)), \end{aligned}$$

then

$$\frac{L'(t)}{H_2(L(t))} \leq -\frac{C_3}{\widetilde{\epsilon_2}}.$$

By recalling (4.14), we deduce $H_2(t) = -1/H_1'(t)$, hence

$$L'(t)H_1'(L(t)) \geq \frac{C_3}{\widetilde{\epsilon_2}}.$$

A simple integration over $(0,t)$ yields

$$H_1(L(t)) \geq H_1(L(0)) + \frac{C_3}{\widetilde{\epsilon_2}}t.$$

Exploiting the fact that H_1^{-1} is decreasing, we infer

$$L(t) \leq H_1^{-1}\Big(\frac{C_3}{\widetilde{\epsilon_2}}t + H_1(L(0))\Big).$$

Consequently, the equivalence of L, $\widetilde{L}$ and E yields the estimate

$$E(t) \leq w_3 H_1^{-1}(w_1 t + w_2),$$

This completes the proof of Theorem 4.2.4. □

CHAPTER 5

WELL-POSEDNESS AND ENERGY DECAY OF SOLUTIONS FOR A PETROVSKY EQUATION WITH A DELAY TERM

The aim of this chapter is to established existence of global solutions to a nonlinear Petrovsky equation in a bounded domain with a delay term in suitable Sobolev spaces by using the energy method combined with Faedo-Galerkin method under condition on the weight of the delay term in the feedback and the weight of the term without delay. Furthermore, we study general stability estimates by using some properties of convex functions.

5.1 Introduction

In this work, we consider the existence and decay properties of global solutions for the initial boundary value problem of viscoelastic Petrovsky equation

$$\begin{aligned} &u_{tt} + \Delta^2 u - \mu_1 g_1(\Delta_x(u_t(x,t))) - \mu_2 g_2(\Delta_x(u_t(x,t-\tau))) = 0 \quad \text{in } \Omega\times]0,+\infty[, \\ &\Delta u(x,t) = u(x,t) = 0 \quad \text{on } \partial\Omega \times [0,+\infty[, \\ &u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{in } \Omega, \\ &u_t(x,t-\tau) = f_0(x,t-\tau) \quad \text{in } \Omega\times]0,\tau[, \end{aligned} \tag{5.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}^*$, $\partial\Omega$ is a smooth boundary.
When $\mu_2 = 0$, the Problem (5.1) was treated by Komornik [36] using semigroup approach for sitting

the well-posedness and he studied the strong stability of this system by introducing a multiplier method combined with a nonlinear integral inequalities given by Martinez [47].

Benaissa et al. [17] proved the existence of global solution, as well as, a general stability result for the equation

$$u_{tt} - \Delta u + \int_0^t h(t-s)\Delta u(s)\, ds + \mu_1 g_1(u_t(x,t)) + \mu_2 g_2(u_t(x,t-\tau)) = 0, \tag{5.2}$$

for $x \in \Omega$ and $t > 0$, when h is decays at a certain rate.

In the absence of the viscoelastic term (i.e. if $h \equiv 0$), problem (5.2) has been studied by Benaissa et al. [18]. It is well known that in the further absence of a damping mechanism, the delay term causes instability of the system (see, for instance, Datko et al. [26]). On the contrary, in the absence of the delay term, the damping term ensures global existence for arbitrary initial data and energy decay is estimated depending on the rate of growth of g_1 (see Alabau-Boussouira, [7], Haraux [31], Lasiecka and Tataru [39]).

Time delay is the property of a physical system by which the response to an applied force is delayed in its effect (see Shinskey [62]). Whenever material, information or energy is physically transmitted from one place to another, there is a delay associated with the transmission. Time delays so often arise in many physical, chemical, biological, and economical phenomena. In recent years, the control of PDEs with time delay effects has become an active area of research (see Abdallah et al [6] and Zhong [74]). To stabilize a hyperbolic system involving delay terms, additional control terms are necessary (see Ait Benhassi and al [4], Ammari and Nicaise [10], Nicaise and Pignotti [55], Nicaise and Pignotti [56]). In Nicaise and Pignotti [55], the authors examined the problem (5.2) in the linear situation (i.e. if $g_1(s) = g_2(s) = s$ for all $s \in \mathbb{R}$) and determined suitable relations between μ_1 and μ_2, for which the stability or alternatively instability takes place. More precisely, they showed that the energy is exponentially stable if $\mu_2 < \mu_1$ and they found a sequence of delays for which the corresponding solution of (5.2) will be instable if $\mu_2 \geq \mu_1$. The main approach used in Nicaise and Pignotti [55] is an observability inequality obtained with a Carleman estimate. The same results were obtained if both the damping and the delay were acting in the boundary domain. We also recall the result by Xu et al. [71], where the authors proved the same result as in Nicaise and Pignotti [55] for the one space dimension by adopting the spectral analysis approach. When $\mu_2 = 0$ and $g_1(u_t) = div(a(x)\nabla u_t)$ where $a(x) = d\,\mathbb{1}_\omega(x)$ and $d > 0$ is a constant Ammari et al. [8] proved the logarithm stabilization of problem (5.1). Recently, Benaissa and Louhibi [18] extended the result of Nicaise and Pignotti [55] to the nonlinear case.

Our purpose in this chapter is to give the global solvability in Sobolev spaces and energy decay estimates of solutions to the problem (5.1) for a nonlinear damping and a delay term. To obtain global solutions to the problem (5.1), we use the argument combining the Galerkin approximation scheme (see [41]). To prove decay estimates, we use the multiplier method and some properties of convex functions. These arguments of convexity were introduced and developed by Lasiecka et al. [39] and Alabau-Boussouira [7].

5.2 Notation and Preliminaries

We begin by introducing some notation that will be used throughout this work.
Let us introduce the following real Hilbert spaces $\mathcal{H}$, V, $\mathcal{V}$ and W by setting

$$\mathcal{H} = H_0^1(\Omega), \quad \|v\|_{\mathcal{H}}^2 = \int_\Omega |\nabla v|^2 \, dx,$$

$$V = \{v \in H^3(\Omega) | v = \Delta v = 0 \text{ on } \Gamma\}, \quad \|v\|_V^2 = \int_\Omega |\nabla \Delta v|^2 \, dx$$

$$\mathcal{V} = \{v \in H^3(\Omega \times (0,1)) | v = \Delta v = 0 \text{ on } \Gamma\}, \quad \|v\|_{\mathcal{V}}^2 = \int_\Omega \int_0^1 |\nabla \Delta v|^2 \, d\rho \, dx$$

and

$$W = \{v \in H^5(\Omega) | v = \Delta v = \Delta^2 v = 0 \text{ on } \Gamma\}, \quad \|v\|_W^2 = \int_\Omega |\nabla \Delta^2 v|^2 \, dx.$$

Identifying $\mathcal{H}$ with its dual $\mathcal{H}'$ we have

$$W \subset V \subset \mathcal{H} = \mathcal{H}' \subset V' \subset W',$$

with dense and compact imbedings. If $v \in L^2(\Omega)$, we denote by $\|v\|_{L^2(\Omega)}^2 = \|v\|^2$.

We impose the following assumptions on g_1 and g_2
$g_1 : \mathbb{R} \to \mathbb{R}$ is non decreasing function of class $\mathcal{C}^1$ and $H : \mathbb{R}_+ \to \mathbb{R}_+$ is convex, increasing and of class $\mathcal{C}^1(\mathbb{R}_+) \cap \mathcal{C}^2(]0, +\infty[)$ satisfying

$$\begin{gathered} H(0) = 0 \text{ and } H \text{ is linear on } [0, \epsilon] \text{ or} \\ H'(0) = 0 \text{ and } H'' > 0 \text{ on }]0, \epsilon] \text{ such that} \\ |g_1(s)| \le c_1 |s| \quad \text{if } |s| \ge \epsilon \\ g_1^2(s) \le H^{-1}(s g_1(s)) \quad \text{if } |s| \le \epsilon, \end{gathered} \tag{5.3}$$

where H^{-1} denotes the inverse function of H and ε, c_2 are positive constants. $g_2 : \mathbb{R} \to \mathbb{R}$ is an odd no decreasing function of class $\mathcal{C}^1(\mathbb{R})$ such that there exist $c_2, \alpha_1, \alpha_2 > 0$,

$$|g_2'(s)| \le c_2, \tag{5.4}$$

$$\alpha_1 s g_2(s) \le G(s) \le \alpha_2 s g_1(s), \tag{5.5}$$

where $G(s) = \int_0^s g_2(r)dr$.

Now we introduce, as in the work of Dafermos [25] and in Nicaise and Pignotti [55], the new variable

$$z(x,\rho,t) = u_t(x, t-\tau\rho), \quad x \in \Omega,\ \rho \in (0,1),\ t > 0.$$

Then, we have

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \quad \text{in } \Omega \times (0,1) \times (0,+\infty).$$

Therefore, problem (5.1) takes the form

$$\begin{aligned} u_{tt} + \Delta^2 u - \mu_1 g_1(\Delta_x(u_t(x,t))) - \mu_2 g_2(\Delta_x(u_t(x,t-\tau))) = 0 \quad \text{in } \Omega\times]0,+\infty[,\\ \tau\Delta z_t(x,\rho,t) + \Delta z_\rho(x,\rho,t) = 0, \quad \text{in } \Omega\times]0,1[\times]0,+\infty[,\\ \Delta u(x,t) = u(x,t) = 0, \quad \text{on } \partial\Omega \times [0,\infty[,\\ z(x,0,t) = u_t(x,t), \quad \text{on } \Omega \times [0,\infty[,\\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \text{in } \Omega,\\ z(x,\rho,0) = f_0(x,-\rho\tau), \quad \text{in } \Omega\times]0,1[. \end{aligned} \tag{5.6}$$

We define the energy associated with the solution of problem (5.6) by the following formula:

$$E(t) = \frac{1}{2}\|\nabla u_t\|^2 + \frac{1}{2}\|\nabla\Delta u\|^2 + \xi \int_\Omega \int_0^1 G(\Delta_x(z(x,\rho,t)))\, d\rho\, dx \tag{5.7}$$

where ξ is a positive constant such that

$$\tau\frac{\mu_2(1-\alpha_1)}{\alpha_1} \le \xi \le \tau\frac{\mu_1 - \alpha_2\mu_2}{\alpha_2}. \tag{5.8}$$

Lemma 5.2.1. *Let (u,z) be a solution of the problem (5.6). Then, the energy functional defined by (5.7) satisfies*

$$E'(t) \le -\beta_1 \int_\Omega \Delta u_t g_1(\Delta u_t)\, dx - \beta_2 \int_\Omega \Delta z(x,1,t) g_2(\Delta z(x,1,t))\, dx \le 0,$$

where $\beta_1 = \mu_1 - \frac{\xi\alpha_2}{\tau} - \mu_2\alpha_2$ and $\beta_2 = \frac{\xi\alpha_1}{\tau} - \mu_2(1-\alpha_1)$.

Before stating our mains results, we balance the following lemma due to Guesmia [29] which will be so useful.

Lemma 5.2.2. *Let $E : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-increasing differentiable function and $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ a convex and increasing function such that $\Psi(0) = 0$. Assume that*

$$\int_S^T \Psi(E(t))\, dt \le E(s), \quad \forall 0 \le S \le T$$

Then E satisfies the following estimate:

$$\psi(t) \leq \psi^{-1}\Big(h(t) + \psi(E(0))\Big), \ \forall t \geq 0 \tag{5.9}$$

where $\psi(t) = \int_t^1 \frac{1}{\Psi(s)}\, ds$ *for* $t > 0$, $h(t) = 0$ *for* $0 \leq t \leq \frac{E(0)}{\Psi(E(0))}$ *and*

$$h^{-1}(t) = t + \frac{\psi(t + \psi(E(0)))}{\Psi(\psi^{-1}(t + \psi(E(0))))}, \quad \forall t \geq \frac{E(0)}{\Psi(E(0))}.$$

5.3 Statement of main results

Theorem 5.3.1. *(Well-posedness). Let $u_0 \in W$, $u_1 \in V$ and $f_0 \in \mathcal{V}$ satisfies the compatibility condition $f(\cdot, 0) = u_1$. Assume that (5.3), (5.4) and (5.5) hold. Then (5.1) admits a weak solution*

$$u \in L^\infty([0,\infty); H^4(\Omega) \cap V), \quad u_t \in L^\infty([0,\infty); V),$$
$$u_{tt} \in L^2([0,\infty);\ \mathcal{H}).$$

Theorem 5.3.2. *(Stabilization). Let $(u_0, u_1) \in W \times V$ and $f_0 \in \mathcal{V}$ satisfy the compatibility condition $f(.,0) = u_1$. Assume that (5.3), (5.4) and (5.5) hold, then the global solutions of the problem (5.1) have the following asymptotic property*

$$\psi(t) \leq \psi^{-1}\Big(h(t) + \psi(E(0))\Big), \ \forall t \geq 0 \tag{5.10}$$

where $\psi(t) = \int_t^1 \frac{1}{\omega\Psi(s)}\, ds$ *for* $t > 0$, $h(t) = 0$ *for* $0 \leq t \leq \frac{E(0)}{\omega\Psi(E(0))}$ *and*

$$h^{-1}(t) = t + \frac{\psi(t + \psi(E(0)))}{\Psi(\psi^{-1}(t + \psi(E(0))))}, \quad \forall t \geq \frac{E(0)}{\Psi(E(0))}$$

$$\psi(t) = \begin{cases} t & \text{if } H \text{ is linear on } [0,\varepsilon] \\ tH'(\varepsilon_0 t) & \text{if } H'(0) = 0 \text{ and } H'' > 0 \text{ on }]0,\varepsilon], \end{cases}$$

where ω and ε are positive constants.

5.4 Proof of Lemma 5.2.1

By multiplying the first equation in (5.6) by $-\Delta u_t$, integrating over Ω and using integration by parts, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big[\|\nabla u_t\|^2 + \|\nabla \Delta u\|^2\Big] + \mu_1 \int_\Omega \Delta u_t(x,t) g_1(\Delta u_t(x,t))\, dx \\ &+ \mu_2 \int_\Omega \Delta u_t(x,t) g_2(\Delta z(x,1,t))\, dx \\ &= 0. \end{aligned} \tag{5.11}$$

We multiply the second equation in (5.6) by $\xi g_2(\Delta z)$, we integrate the result over $\Omega \times (0,1)$, to obtain

$$\begin{aligned}
\xi \int_\Omega \int_0^1 \Delta z_t(x,\rho,t) g_2(\Delta z(x,\rho,t))\, d\rho\, dx &= -\frac{\xi}{\tau} \int_\Omega \int_0^1 \Delta z_\rho(x,\rho,t) g_2(\Delta z(x,\rho,t))\, d\rho\, dx \\
&= -\frac{\xi}{\tau} \int_\Omega \int_0^1 \frac{\partial}{\partial \rho}\Big(G(\Delta z(x,\rho,t))\Big)\, d\rho\, dx \\
&= -\frac{\xi}{\tau} \int_\Omega (G(\Delta z(x,1,t)) - G(\Delta z(x,0,t)))\, dx.
\end{aligned}$$

Hence

$$\xi \frac{d}{dt} \int_\Omega \int_0^1 G(\Delta z(x,\rho,t))\, d\rho\, dx = -\frac{\xi}{\tau} \int_\Omega G(\Delta z(x,1,t))\, dx + \frac{\xi}{\tau} \int_\Omega G(\Delta u_t(x,t))\, dx. \tag{5.12}$$

By combining (5.11) and (5.12), we obtain

$$\begin{aligned}
E'(t) = &-\mu_1 \int_\Omega \Delta u_t(x,t) g_1(\Delta u_t(x,t))\, dx + \frac{\xi}{\tau} \int_\Omega G(\Delta u_t(x,t))\, dx \\
&- \mu_2 \int_\Omega \Delta u_t(x,t) g_2(\Delta z(x,1,t))\, dx - \frac{\xi}{\tau} \int_\Omega G(\Delta z(x,1,t))\, dx.
\end{aligned} \tag{5.13}$$

From the definition of G and by using remark 4.2.1, we obtain

$$G^*(s) = s g_2^{-1}(s) - G(g_2^{-1}(s)), \quad \forall s \geq 0.$$

Hence

$$\begin{aligned}
G^*(g_2(\Delta z(x,1,t))) &= z(x,1,t) g_2(\Delta z(x,1,t)) - G(\Delta z(x,1,t)) \\
&\leq (1-\alpha_1) \Delta z(x,1,t) g_2(\Delta z(x,1,t)).
\end{aligned}$$

By using (5.5) and (4.2.1) with $A = g_2(\Delta z(x,1,t))$ and $B = \Delta u_t(x,t)$, from (5.13) we obtain

$$\begin{aligned}
E'(t) \leq &-\Big(\mu_1 - \frac{\xi \alpha_2}{\tau}\Big) \int_\Omega \Delta u_t(x,t) g_1(\Delta u_t(x,t))\, dx \\
&+ \mu_2 \int_\Omega (G(\Delta u_t(x,t)) + G^*(\Delta g_2(\Delta z(x,1,t)))\, dx - \frac{\xi}{\tau} \int_\Omega G(z(x,1,t))\, dx \\
\leq &-\Big(\mu_1 - \frac{\xi \alpha_2}{\tau} - \mu_2 \alpha_2\Big) \int_\Omega \Delta u_t(x,t) g_1(\Delta u_t(x,t))\, dx \\
&- \Big(\frac{\xi \alpha_1}{\tau} - \mu_2 (1-\alpha_1)\Big) \int_\Omega \Delta z(x,1,t) g_2(\Delta z(x,1,t))\, dx \\
\leq &\ 0.
\end{aligned}$$

5.5 Proof of Theorem 5.3.1

In order to establish the well-posedness of problem (5.1), we will use the Faedo-Galerkin method as it is well explained in Lions[41]. So, we assume $u_0 \in W$, $u_1 \in V$ and $f_0 \in \mathcal{V}$. Let $T > 0$ be fixed and

$\{w_j\}, j \in \mathbb{N}$ be a basis of $\mathcal{H}$, V and W, that's, the space spanned by $\mathcal{B}_k = \{w_1, w_2, \dots, w_k\}$ is dense in $\mathcal{H}$, V and W.
Now, we define, for $1 \le j \le k$, the sequence $\phi^j(x,\rho)$ as follows:

$$\phi^j(x,0) = w^j.$$

Then, we may extend $\phi^j(x,0)$ by $\phi^j(x,\rho)$ over $L^2(\Omega \times (0,1))$ such that $(\phi^j)_j$ forms a basis of $L^2(\Omega, H^2(0,1))$ and denote Z_k the space spanned by $\{\phi^k\}$. We construct approximate solutions (u^k, z^k), $k = 1, 2, 3, \dots$, in the form

$$u^k(x,t) = \sum_{j=1}^{k} c^{jk}(t) w^j(x), \quad z^k(x,t) = \sum_{j=1}^{k} d^{jk}(t)\phi^j,$$

where c^{jk} and $d^{jk}(j = 1, 2, \dots, k)$ are determined by the ordinary differential equations

$$\begin{gathered}(u^k_{tt}(t), w^j) + (\Delta^2 u^k(t), w^j) - \mu_1(g_1(\Delta u^k_t), w^j) - \mu_2(g_2(\Delta z^k(.,1)), w^j) = 0, \\ z^k(x,0,t) = u^k_t(x,t),\end{gathered} \tag{5.14}$$

$$u^k(0) = u^k_0 = \sum_{j=1}^{k}(u_0, w^j)w^j \to u_0, \quad \text{in } W \text{ as } k \to +\infty, \tag{5.15}$$

$$u^k_t(0) = u^k_1 = \sum_{j=1}^{k}(u_1, w^j)w^j \to u_1, \quad \text{in } V \text{ as } k \to +\infty, \tag{5.16}$$

and

$$(\tau \Delta z^k_t + \Delta z^k_\rho, \phi^j) = 0, \quad 1 \le j \le k, \tag{5.17}$$

$$z^k(\rho, 0) = z^k_0 = \sum_{j=1}^{k}(f_0, \phi^j)\phi^j \to f_0 \quad \text{in } \mathcal{V} \text{ as } k \to +\infty. \tag{5.18}$$

The standard theory of ODE guarantees that the system (5.14)-(5.18) has a unique solution in $[0, t_k)$, with $0 < t_k < T$, by Zorn Lemma since the nonlinear terms in (5.14) are locally Lipschitz continuous. Note that $u^k(t)$ is of class $\mathcal{C}^2$.

In the next step, we obtain a priori estimates for the solution of the system (5.14)-(5.18), so that it can be extended outside $[0, t_k)$ to obtain one solution defined for all $t > 0$, using a standard compactness argument for the limiting procedure.
In order to use a standard compactness argument for the limiting procedure, it suffices to derive some a priori estimates for (u^k, z^k).
The first estimate. Since the sequences u^k_0, u^k_1 and z^k_0 converge and from Lemma 5.2.1, we can find

a positive constant C_1 independent of k such that

$$\begin{aligned} &E^k(t)+\beta_1\int_0^t\int_\Omega \Delta u_t^k g_1(\Delta u_t^k)\,dx\,ds+\beta_2\int_0^t\int_\Omega \Delta z^k(x,1,s)g_2(\Delta z^k(x,1,s))\,dx\,ds\\ &\le E^k(0)\le C_1, \end{aligned}\tag{5.19}$$

where

$$E^k(t)=\frac{1}{2}\|\nabla u_t^k\|^2+\frac{1}{2}\|\nabla\Delta u^k\|^2+\xi\int_\Omega\int_0^1 G(\Delta z^k(x,\rho,t))\,d\rho\,dx,$$

$$\beta_1=\mu_1-\frac{\xi\alpha_2}{\tau}-\mu_2\alpha_2\quad \beta_2=\frac{\xi\alpha_1}{\tau}-\mu_2(1-\alpha_1)$$

and C_1 is a positive constant depending only on $\|u_1\|_V$ and $\|u_0\|_W$.

This estimate implies that the solution u^k exists globally in $[0,+\infty)$. Estimate (5.19) yields

$$u^k \text{ is bounded in } L^\infty(0,\infty,V),\tag{5.20}$$

$$u_t^k \text{ is bounded in } L^\infty(0,\infty,\mathcal{H}),\tag{5.21}$$

$$G(\Delta z^k(x,\rho,t)) \text{ is bounded in } L^\infty(0,\infty,L^1(\Omega\times(0,1))),\tag{5.22}$$

$$\Delta u_t^k(t)g_1(\Delta u_t^k(t)) \text{ is bounded in } L^1(\Omega\times(0,T)),\tag{5.23}$$

$$\Delta z^k(x,1,t)g_2(\Delta z^k(x,1,t)) \text{ is bounded in } L^1(\Omega\times(0,T))).\tag{5.24}$$

The second estimate. Differentiating (5.14) with respect to x, replacing w^j by ∇w^j in (5.14), multiplying by c_t^{jk}, summing over j from 1 to k, and choosing $t=0$, we obtain

$$\|\nabla u_{tt}^k(0)\|^2+\Big(\nabla u_{tt}^k(0),\nabla\Delta^2u_0^k-\mu_1\nabla\Big(g_1(\Delta u_1^k)\Big)-\mu_2\nabla\Big(g_2(\Delta z_0^k)\Big)\Big)=0.$$

Using Cauchy-Schwarz inequality, (5.3) and (5.4), we have

$$\begin{aligned} \|\nabla u_{tt}^k(0)\| &\le \|\nabla\Delta^2u_0^k\|+\mu_1\|\nabla\Delta u_1^k g_1'(\Delta u_1^k)\|+\mu_2\|\nabla\Delta z_0^k g_2'(\Delta z_0^k)\|\\ &\le \|\nabla\Delta^2u_0^k\|+c_3\|\nabla\Delta u_1^k\|+c_4\|\nabla\Delta z_0^k\|. \end{aligned}\tag{5.25}$$

By (5.15), (5.16), and (5.18) yields

$$u_{tt}^k(0)\quad\text{is bounded in}\quad \mathcal{H}.\tag{5.26}$$

The third estimate. Differentiating (5.14) with respect to t gives

$$\begin{aligned} &(u_{ttt}^k(t),w^j)+(\Delta^2u_t^k(t),w^j)-\mu_1(\Delta(u_{tt}^k))g_1'(\Delta(u_t^k)),w^j)\\ &\qquad -\mu_2(\Delta(z_t^k(.,1))g_2'(\Delta(z^k(.,1))),w^j)=0. \end{aligned}$$

Replacing w^j by $-\Delta w^j$, multiplying by c_{tt}^{jk}, summing over j from 1 to k, it follows that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big[\|\nabla u_{tt}^k\|^2+\|\nabla\Delta u_t^k\|^2\Big]+\mu_1\int_\Omega(\Delta u_{tt}^k)^2g'(\Delta u_t^k)\,dx\\ &\qquad +\mu_2\int_\Omega\Delta(z_t^k(.,1))g_2'(\Delta(z^k(.,1)))\Delta u_{tt}^k\,dx=0. \end{aligned}\tag{5.27}$$

Differentiating (5.17) with respect to t gives

$$(\Delta\tau z_{tt}^k + \frac{\partial}{\partial\rho}\Delta z_t^k, \phi^j) = 0.$$

Replacing ϕ^j by $\Delta\phi^j$ in (5.17), multiplying by d_t^{jk} and summing over j from 1 to k, it follows that

$$\tau\int_\Omega \Delta z_{tt}^k \Delta z_t^k\,dx + \int_\Omega \frac{d}{d\rho}\Delta z_t^k \Delta z_t^k\,dx = 0.$$

Then, we obtain

$$\frac{\tau}{2}\frac{d}{dt}\|\Delta z_t^k\|^2 + \frac{1}{2}\frac{d}{d\rho}\|\Delta z_t^k\|^2 = 0.$$

We integrate over $(0,1)$ to find

$$\frac{\tau}{2}\frac{d}{dt}\int_0^1 \|\Delta z_t^k(x,\rho,t)\|^2\,d\rho + \frac{1}{2}\|\Delta z_t^k(x,1,t)\|^2 - \frac{1}{2}\|\Delta u_{tt}^k(t)\|^2 = 0. \tag{5.28}$$

Taking the sum of (5.27) and (5.28), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big[\|\nabla u_{tt}^k\|^2+\|\nabla\Delta u_t^k\|^2 + \tau\int_0^1 \|\Delta z_t^k(x,\rho,t)\|^2\,d\rho\Big] + \frac{1}{2}\|\Delta z_t^k(x,1,t)\|^2\\
&\qquad + \mu_1\int_\Omega (\Delta u_{tt}^k)^2 g_1'(\Delta u_t^k)\,dx\\
&\qquad = -\mu_2\int_\Omega \Delta(z_t^k(.,1))g_2'(\Delta(z^k(.,1)))\Delta u_{tt}^k\,dx + \frac{1}{2}\|\Delta u_{tt}^k(t)\|^2.
\end{aligned}$$

Using Young's inequality, we conclude that

$$\mu_2\int_\Omega \Delta(z_t^k(.,1))g_2'(\Delta(z^k(.,1)))\Delta u_{tt}^k\,dx \le \frac{1}{2}\|\Delta u_{tt}^k\|^2 + \frac{\mu_2^2 c_2^2}{2}\|\Delta z_t^k(x,1,t)\|^2.$$

Integrating the last inequality over $(0,t)$, choosing $g_1'(\Delta u_t^k) - \frac{1}{\mu_1} > 0$ and $1-\mu_2^2 c_2^2 > 0$, we obtain

$$\begin{aligned}
&\|\nabla u_{tt}^k\|^2+\|\nabla\Delta u_t^k\|^2 + \tau\int_0^1 \|\Delta z_t^k(x,\rho,t)\|^2\,d\rho\\
&\qquad + \frac{1}{2}(1-\mu_2^2 c_2^2)\int_0^t\int_\Omega |\Delta z_t^k(x,1,s)|^2\,ds + \mu_1\int_0^t\int_\Omega (g_1'(\Delta u_t^k) - \frac{1}{\mu_1})(\Delta u_{tt}^k)^2\,dx\,ds\\
&\qquad = \|\nabla u_{tt}^k(0)\|^2 + \|\nabla\Delta u_1^k\|^2 + \tau\int_0^1 \left\|\Delta z_t^k(x,\rho,0)\right\|^2 d\rho\\
&\qquad \le C_2,
\end{aligned}$$

for all $t\in[0.T]$, where C_2 is a positive constant independent of k. Therefore, we conclude that

$$u_t^k \text{ is bounded in } L^\infty(0,\infty,V), \tag{5.29}$$

$$u_{tt}^k \text{ is bounded in } L^\infty(0,\infty,\mathcal{H}), \tag{5.30}$$

$$z_t^k \text{ is bounded in } L^\infty(0,\infty,H_0^1(\Omega\times(0,1))). \tag{5.31}$$

The fourth estimate. Replacing w^j by $2\Delta^2 w^j$ in (5.14), multiplying by c_t^{jk} and summing over j from 1 to k, it follows that

$$\begin{aligned} 2\int_\Omega u_{tt}^k \Delta^2 u_t^k \, dx + \frac{d}{dt}\Big[\|\Delta^2 u^k\|^2\Big] - 2\mu_1 \int_\Omega g_1(\Delta u_t^k).\Delta^2 u_t^k \, dx \\ - 2\mu_2 \int_\Omega g_2(\Delta z^k(.,1)).\Delta^2 u_t^k \, dx = 0. \end{aligned}$$

Therefore by using the Green's formula, we have

$$\begin{aligned} \frac{d}{dt}\Big[\|\Delta^2 u^k\|^2\Big] &= -2\int_\Omega \Delta u_{tt}^k \Delta u_t^k \, dx + 2\mu_1 \int_\Omega g_1'(\Delta u_t^k).(\nabla \Delta u_t^k)^2 \, dx \\ &+ 2\mu_2 \int_\Omega \nabla \Delta z^k(x,1,t) g_2'(\Delta(z^k(x,1,t))).\nabla \Delta u_t^k \, dx. \\ &= -\frac{d}{dt}\Big[\|\Delta u_t^k\|^2\Big] + 2\mu_1 \int_\Omega g_1'(\Delta u_t^k).(\nabla \Delta u_t^k)^2 \, dx \\ &+ 2\mu_2 \int_\Omega \nabla \Delta z^k(x,1,t) g_2'(\Delta(z^k(x,1,t))).\nabla \Delta u_t^k \, dx. \end{aligned} \tag{5.32}$$

Replacing ϕ^j by $\Delta^2 \phi^j$ in (5.17), multiplying by d^{jk} and summing over j from 1 to k, it follows that

$$\frac{\tau}{2}\frac{d}{dt}\|\nabla \Delta z^k\|^2 + \frac{1}{2}\frac{d}{d\rho}\|\nabla \Delta z^k\|^2 = 0. \tag{5.33}$$

From (5.32) and (5.33), it follows that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\|\Delta^2 u^k(t)\|^2 + \|\Delta u_t^k(t)\|^2 + \|\nabla \Delta z^k(x,\rho,t)\|^2_{L^2(\Omega\times(0,1))}\Big) \\ &+ \frac{1}{2}\int_\Omega |\nabla \Delta z^k(x,1,t)|^2 \, dx + 2\mu_1 \int_\Omega g_1'(\Delta u_t^k).(\nabla \Delta u_t^k)^2 \, dx \, ds \\ &= 2\mu_2 \int_\Omega \nabla \Delta z^k(x,1,t) g_2'(\Delta(z^k(x,1,t))).\nabla \Delta u_t^k \, dx + \frac{1}{2}\|\nabla \Delta u_t^k(t)\|^2. \end{aligned}$$

Using (5.29), Cauchy-Schwarz and Young's inequalities produce the estimate

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\|\Delta^2 u^k(t)\|^2 + \|\Delta u_t^k(t)\|^2 + \|\nabla \Delta z^k(x,\rho,t)\|^2_{L^2(\Omega\times(0,1))}\Big) \\ &+ c\int_\Omega |\nabla \Delta z^k(x,1,t)|^2 \, dx + \int_\Omega g_1'(\Delta u_t^k).(\nabla \Delta u_t^k)^2 \, dx \, ds \\ &\leq c'\|\nabla \Delta u_t^k(t)\|^2 \leq C. \end{aligned}$$

Integrating it over $(0,t)$ we arrive at

$$\begin{aligned} \|\Delta^2 u^k(t)\|^2 + \|\Delta u_t^k(t)\|^2 &+ \|\nabla \Delta z^k(x,\rho,t)\|^2_{L^2(\Omega\times(0,1))} \\ &+ 2\int_0^t \int_\Omega g_1'(\Delta u_t^k).(\nabla \Delta u_t^k)^2 \, dx \, ds \\ &\leq C.T + \|\Delta^2 u_0^k\|^2 + \|\Delta u_1^k\|^2 + \|\nabla \Delta z^k(x,\rho,0)\|^2_{L^2(\Omega\times(0,1))} \end{aligned}$$

then

$$\Delta^2 u^k \quad \text{is bounded in} \quad L^\infty(0,T;L^2(\Omega)) \tag{5.34}$$

$$z^k \quad \text{is bounded in} \quad L^\infty(0,T;\mathcal{V}). \tag{5.35}$$

Passage to the limit. Applying Dunford-Petit theorem we conclude from (5.20)-(5.24), (5.29)-(5.31), (5.34), and (5.35), replacing the sequence u^k, and z^k with a subsequence if needed, that

$$u^k \rightharpoonup u, \text{ weak-star in } L^\infty(0,T;V\cap H^4(\Omega)) \tag{5.36}$$

$$u_t^k \rightharpoonup u_t, \text{ weak-star in } L^\infty(0,T;V) \tag{5.37}$$

$$u_{tt}^k \rightharpoonup u_{tt}, \text{ weak-star in } L^\infty(0,T;\mathcal{H}) \tag{5.38}$$

$$g(\Delta u_t^k) \rightharpoonup \phi, \text{ weak-star in } L^2(\mathcal{A}) \tag{5.39}$$

$$z^k \longrightarrow z, \text{ weak-star in } L^\infty(0,T;\mathcal{V}) \tag{5.40}$$

$$z_t^k \longrightarrow z_t, \text{ weak-star in } L^\infty(0,T;H_0^1(\Omega\times(0,1))) \tag{5.41}$$

$$g(\Delta z^k(x,1,t)) \rightharpoonup \psi, \text{ weak-star in } L^2(\mathcal{A}) \tag{5.42}$$

where $\mathcal{A} = \Omega\times[0,T]$. For suitable functions $u \in L^\infty(0,\infty,V\cap H^4(\Omega))$ and $z \in L^\infty(0,\infty,\mathcal{V})$, $\chi \in L^2(\Omega\times(0,T))$ $\psi \in L^2(\Omega\times(0,T))$ for all $T \geq 0$. We have to show that (u,z) is a solution of (5.6). From (5.29) and (5.30), we have that u_t^k is bounded in $L^\infty(0,\infty,V)$. Then u_t^k in bounded in $L^2(0,\infty,V)$.
Since u_{tt}^k is bounded in $L^\infty(0,\infty,\mathcal{H})$, then u_{tt}^k is bounded in $L^2(0,\infty,\mathcal{H})$. Consequently, u_t^k is bounded in $H^1(\mathcal{A})$. Moreover, as the embedding $H^1(\mathcal{A}) \hookrightarrow L^2(\mathcal{A})$ is compact, using Aubin-Lions' Theorem[41], we can extract a subsequence (u^ν) of (u^k) such that

$$u_t^\nu \longrightarrow u_t, \text{ strongly in } L^2(\mathcal{A}).$$

Therefore,

$$u_t^\nu \longrightarrow u_t, \text{ almost everywhere in } \mathcal{A}. \tag{5.43}$$

Similarly, we obtain

$$z^\nu \longrightarrow z, \text{ almost everywhere in } \mathcal{A}. \tag{5.44}$$

Lemma 5.5.1. *For each $T > 0$, $g_1(\Delta u_t), g_2(\Delta z(x,1,t)) \in L^1(\mathcal{A})$ and $\|g_1(\Delta u_t)\|_{L^1(\mathcal{A})} \leq K$, $\|g_2(\Delta z(x,1,t))\|_{L^1(\mathcal{A})} \leq K$, where K is a constant independent of t.*

Proof. By (5.3) and (5.43), we have

$$g_1(\Delta u_t^k(x,t)) \to g_1(\Delta u_t(x,t)) \quad \text{almost everywhere in } \mathcal{A},$$
$$0 \leq \Delta u_t^k(x,t) g_1(\Delta u_t^k(x,t)) \to \Delta u_t(x,t) g_1(\Delta u_t(x,t)) \quad \text{almost everywhere in } \mathcal{A}.$$

Hence, by (5.23) and Fatou's Lemma, we have

$$\int_0^T \int_\Omega \Delta u_t(x,t) g_1(\Delta u_t(x,t))\, dx\, dt \leq K_1 \quad \text{for } T > 0. \tag{5.45}$$

Now, we can estimate $\int_0^T \int_\Omega |g_1(\Delta u_t(x,t))|\, dx\, dt$. By Cauchy-Schwarz inequality and using (5.43), we have

$$\begin{aligned}\int_0^T \int_\Omega |g_1(\Delta u_t(x,t))|\, dx\, dt &\leq c|\mathcal{A}|^{1/2}\Big(\int_0^T \int_\Omega \Delta u_t(x,t) g_1(\Delta u_t(x,t))\, dx\, dt\Big)^{1/2} \\ &\leq c|\mathcal{A}|^{1/2} K_1^{1/2} \equiv K.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\int_0^T \int_\Omega |g_2(\Delta z(x,1,t)))|\, dx\, dt &\leq c|\mathcal{A}|^{1/2}\Big(\int_0^T \int_\Omega \Delta z(x,1,t) g_2(\Delta z(x,1,t))\, dx\, dt\Big)^{1/2} \\ &\leq c|\mathcal{A}|^{1/2} K_1^{1/2} \equiv K.\end{aligned}$$

□

Lemma 5.5.2. *$g_1(\Delta u_t^k) \to g_1(\Delta u_t)$ in $L^1(\Omega \times (0,T))$ and $g_2(\Delta z^k) \to g_2(\Delta z)$ in $L^1(\Omega \times (0,T))$.*

Proof. Let $E \subset \Omega \times [0,T]$ and set

$$E_1 = \Big\{(x,t) \in E : |g_1(\Delta u_t^k(x,t))| \leq \frac{1}{\sqrt{|E|}}\Big\}, \quad E_2 = E \backslash E_1,$$

where $|E|$ is the measure of E. If $M(r) = \inf\{|s| : s \in \mathbb{R} \text{ and } |g(s)| \geq r\}$

$$\int_E |g_1(u_t^k)|\, dx\, dt \leq c\sqrt{|E|} + \Big(M\Big(\frac{1}{\sqrt{|E|}}\Big)\Big)^{-1} \int_{E_2} |u_t^k g_1(\Delta u_t^k)|\, dx\, dt.$$

By applying (5.23) we deduce that $\sup_k \int_E |g_1(u_t^k)|\, dx\, dt \to 0$ as $|E| \to 0$. From Vitali's convergence theorem we deduce that

$$g_1(\Delta u_t^k) \to g_1(\Delta u_t) \quad \text{in } L^1(\Omega \times (0,T)).$$

Similarly, we have

$$g_2(\Delta z^k) \to g_2(\Delta z) \quad \text{in } L^1(\Omega \times (0,T)).$$

This completes the proof. □

Hence

$$g_1(\Delta u_t^k) \rightharpoonup g_1(\Delta u_t) \quad \text{weak in } L^2(\Omega \times (0,T)),$$
$$g_2(\Delta z^k) \rightharpoonup g_2(\Delta z) \quad \text{weak in } L^2(\Omega \times (0,T)).$$

By multiplying (5.14) by $\theta(t) \in \mathcal{D}(0,T)$ and by integrating over $(0,T)$, it follows that

$$\begin{aligned}&\int_0^T (u_{tt}^k(t), w^j)\theta'(t)\,dt + \int_0^T (\nabla\Delta u^k(t), \nabla\Delta w^j)\theta(t)\,dt + \\ &\mu_1 \int_0^T (g_1(\Delta u_t^k), w^j)\theta(t)\,dt + \mu_2 \int_0^T (g_2(\Delta z^k(.,1)), w^j)\theta(t)\,dt = 0\end{aligned} \tag{5.46}$$

and multiplying (5.17) by $\theta(t) \in \mathcal{D}(0,T)$ and integrating over $(0,T)\times(0,1)$, it follows that

$$\int_0^T \int_0^1 (\tau\Delta z_t^k + \Delta z_\rho^k, \phi^j)\theta(t)\,dt\,d\rho = 0. \tag{5.47}$$

The convergence of (5.36), (5.38), (5.39) and (5.42) are sufficient to pass to the limit in (5.46) and (5.47) to obtain

$$\begin{aligned}&\int_0^T (u_{tt}, w)\theta'(t)\,dt + \int_0^T (\nabla\Delta u, \nabla\Delta w)\theta(t)\,dt \\ &+ \mu_1 \int_0^T (g_1(\Delta u_t), w)\theta(t)\,dt + \mu_2 \int_0^T (g_2(\Delta z(.,1)), w)\theta(t)\,dt = 0,\end{aligned}$$

and

$$\int_0^T \int_0^1 (\tau\Delta z_t + \Delta z_\rho, \phi)\theta(t)\,dt\,d\rho = 0.$$

By integrating, we have

$$\int_0^T \Big(u_{tt} + \Delta^2 u + \mu_1 g_1(\Delta u_t) + \mu_2 g_2(\Delta z(.,1)), w\Big)\theta(t)\,dt = 0.$$

Thus the problem (5.1) admits a global weak solution u. This completes the proof of Theorem 5.3.1.

5.6 Proof of Theorem 5.3.2

Using the multiplier technique as in [36, 35, 47]. From now on, we denote by c various positive constants which may be different on different occurrences. Multiplying the first equation of (5.6) by $-\frac{\varphi(E)}{E}\Delta u$, we obtain

$$\begin{aligned}0 &= \int_S^T -\frac{\varphi(E)}{E} \int_\Omega \Delta u(u_{tt} + \Delta^2 u - \mu_1 g_1(\Delta(u_t(x,t))) - \mu_2 g_2(\Delta(z(x,1,t))))\,dx\,dt \\ &= -\Big[\frac{\varphi(E)}{E}\int_\Omega u_t \Delta u\,dx\Big]_S^T + \int_S^T \Big(\frac{\varphi(E)}{E}\Big)' \int_\Omega \Delta u u_t\,dx\,dt \\ &- 2\int_S^T \frac{\varphi(E)}{E}\int_\Omega |\nabla u_t|^2\,dx\,dt + \int_S^T \frac{\varphi(E)}{E}\int_\Omega (|\nabla u_t|^2 + |\nabla\Delta u|^2)\,dx\,dt \\ &+ \mu_1 \int_S^T \frac{\varphi(E)}{E}\int_\Omega \Delta u.g_1(\Delta u_t(x,t))\,dx\,dt + \mu_2 \int_S^T \frac{\varphi(E)}{E}\int_\Omega \Delta u.g_2(\Delta z(x,1,t))\,dx\,dt.\end{aligned}$$

Similarly, if we multiply the second equation of (5.6) by $\frac{\varphi(E)}{E} e^{-2\tau\rho} g_2(\Delta z(x,\rho,t))$, we have

$$
\begin{aligned}
0 &= \int_S^T \frac{\varphi(E)}{E} \int_\Omega \int_0^1 e^{-2\tau\rho} g_2(\Delta z(x,\rho,t))(\tau\Delta z_t + \Delta z_\rho)\,dx\,d\rho\,dt \\
&= \Big[\int_\Omega \int_0^1 \tau e^{-2\tau\rho} G(\Delta z)\,d\rho\,dx\Big]_S^T - \tau \int_S^T \Big(\frac{\varphi(E)}{E}\Big)' \int_\Omega \int_0^1 e^{-2\tau\rho} G(\Delta z)\,dx\,d\rho\,dt \\
&+ \int_S^T \frac{\varphi(E)}{E} \int_\Omega \int_0^1 (e^{-2\tau\rho} G(\Delta z(x,1,t)) - G(\Delta z(x,0,t)))\,dx\,dt \\
&+ 2\tau \int_S^T \frac{\varphi(E)}{E} \int_\Omega \int_0^1 e^{-2\tau\rho} G(\Delta z)\,dx\,d\rho\,dt.
\end{aligned}
$$

Taking their sum, we obtain

$$
\begin{aligned}
A \int_S^T \varphi(E)\,dt \le{}& \Big[\frac{\varphi(E)}{E} \int_\Omega u_t \Delta u\,dx\Big]_S^T - \int_S^T \Big(\frac{\varphi(E)}{E}\Big)' \int_\Omega \Delta u u_t\,dx\,dt \\
&+ 2\int_S^T \frac{\varphi(E)}{E} \int_\Omega |\nabla u_t|^2\,dx\,dt - \mu_1 \int_S^T \frac{\varphi(E)}{E} \int_\Omega \Delta u . g_1(\Delta u_t(x,t))\,dx\,dt \\
&- \mu_2 \int_S^T \frac{\varphi(E)}{E} \int_\Omega \Delta u . g_2(\Delta z(x,1,t))\,dx\,dt \\
&- \Big[\frac{\varphi(E)}{E} \int_\Omega \int_0^1 \tau e^{-2\tau\rho} G(\Delta z)\,d\rho\,dx\Big]_S^T \\
&+ \tau \int_S^T \Big(\frac{\varphi(E)}{E}\Big)' \int_\Omega \int_0^1 e^{-2\tau\rho} G(\Delta z)\,dx\,d\rho\,dt \\
&- \int_S^T \frac{\varphi(E)}{E} \int_\Omega (e^{-2\tau\rho} G(\Delta z(x,1,t) - G(\Delta z(x,0,t)))\,dx\,dt,
\end{aligned}
\tag{5.48}
$$

where $A = 2\min\{1, \tau e^{-2\tau}/\xi\}$. Since E is nonincreasing, we find that

$$
\Big[\frac{\varphi(E)}{E} \int_\Omega u_t \Delta u\,dx\Big]_S^T \le C_s \varphi(E(S))
$$

$$
\Big| \int_S^T \Big(\frac{\varphi(E)}{E}\Big)' \int_\Omega \Delta u u_t\,dx\,dt \Big| \le C_s \varphi(E(S))
$$

$$
\begin{aligned}
-\Big[\frac{\varphi(E)}{E} \int_\Omega \int_0^1 \tau e^{-2\tau\rho} G(\Delta z)\,d\rho\,dx\Big]_S^T &= \frac{\varphi(E(S))}{E(S)} \int_\Omega \int_0^1 \tau e^{-2\tau\rho} G(\Delta z(x,\rho,S))\,d\rho\,dx \\
&- \frac{\varphi(E(T))}{E(T)} \int_\Omega \int_0^1 \tau e^{-2\tau\rho} G(\Delta z(x,\rho,T))\,d\rho\,dx \\
&\le C\varphi(E(S))
\end{aligned}
$$

$$
\begin{aligned}
\tau \int_S^T \Big(\frac{\varphi(E)}{E}\Big)' \int_\Omega \int_0^1 e^{-2\tau\rho} G(\Delta z)\,dx\,d\rho\,dt& \\
&\le \tau \int_S^T \Big(- \Big(\frac{\varphi(E)}{E}\Big)'\Big) E\,dt \\
&\le C\varphi(E(S))
\end{aligned}
$$

$$
\int_S^T \frac{\varphi(E)}{E} \int_\Omega e^{-2\tau\rho} G(\Delta z(x,1,t)\,dx\,dt \le c \int_S^T \frac{\varphi(E)}{E} E'\,dt \le C\varphi(E(S))
$$

$$\int_S^T \frac{\varphi(E)}{E} \int_\Omega G(\Delta z(x,0,t)))\, dx\, dt = \int_S^T \frac{\varphi(E)}{E} \int_\Omega G(\Delta u_t(x,t))\, dx\, dt$$
$$\leq \int_S^T \frac{\varphi(E)}{E} E'\, dt \leq C\varphi(E(S)).$$

Using these estimates, we conclude from (5.48) that

$$\begin{aligned} A\int_S^T \varphi(E)\, dt \leq C\varphi(E(S)) &+ \mu_1 \int_S^T \frac{\varphi(E)}{E} \int_\Omega |\Delta u|.|g_1(\Delta u_t(x,t))|\, dx\, dt \\ &+ \mu_2 \int_S^T \frac{\varphi(E)}{E} \int_\Omega |\Delta u|.|g_2(\Delta z(x,1,t))|\, dx\, dt \\ &+ 2\int_S^T \frac{\varphi(E)}{E} \int_\Omega |\nabla u_t|^2\, dx\, dt. \end{aligned} \tag{5.49}$$

Now, we estimate the terms of the right-hand side of (5.49) in order to apply the results of Lemma 5.2.2.

As in Komornik [35], we consider the following partition of Ω,

$$\Omega_t^1 = \{x \in \Omega : |\Delta u_t| > \epsilon\}, \quad \Omega_t^2 = \{x \in \Omega : |\Delta u_t| \leq \epsilon\},$$
$$\widetilde{\Omega}_t^1 = \{x \in \Omega : |\Delta z(x,1,t)| > \epsilon\}, \quad \widetilde{\Omega}_t^2 = \{x \in \Omega : |\Delta z(x,1,t)| \leq \epsilon\}.$$

We distinguish two cases :

Case 1. H is linear on $[0,\epsilon]$. By using Sobolev embedding, we have

$$\begin{aligned} \int_S^T \frac{\varphi(E)}{E} \int_{\Omega_t^1} |\nabla u_t|^2\, dx\, dt &\leq C_s \int_S^T \frac{\varphi(E)}{E} \int_\Omega |\Delta u_t|^2\, dx\, dt \\ &\leq C_s \int_S^T \frac{\varphi(E)}{E} \int_\Omega \Delta u_t g_1(\Delta u_t)\, dx\, dt \\ &\leq -C_s \int_S^T \frac{\varphi(E)}{E} E'(t)\, dt \leq C\varphi(E(S)), \end{aligned}$$

and

$$\begin{aligned} \int_S^T \frac{\varphi(E)}{E} \int_{\Omega_t^2} |\nabla u_t|^2\, dx\, dt &\leq C_s \int_S^T \frac{\varphi(E)}{E} \int_\Omega (|\Delta u_t|^2 + |g_1(\Delta u_t)|^2\, dx\, dt \\ &\leq C_s \int_S^T \frac{\varphi(E)}{E} \int_\Omega \Delta u_t g_1(\Delta u_t)\, dx\, dt \\ &\leq C\varphi(E(S)). \end{aligned}$$

Exploiting Young's and Poincaré inequalities, we obtain

$$\begin{aligned} &\int_S^T \frac{\varphi(E)}{E} \int_{\Omega_t^1} |\Delta u|.|g_1(\Delta(u_t(x,t)))|\, dx\, dt \\ &\quad\leq \varepsilon \int_S^T \frac{\varphi(E)}{E} \int_{\Omega_t^1} |\Delta u|^2\, dx\, dt + C(\varepsilon)\int_S^T \frac{\varphi(E)}{E} \int_{\Omega_t^1} |g_1(\Delta u_t(x,t))|^2\, dx\, dt \\ &\quad\leq \varepsilon C_s \int_S^T \frac{\varphi(E)}{E} \int_\Omega |\nabla \Delta u|^2\, dx\, dt + C(\varepsilon)\int_S^T \frac{\varphi(E)}{E} \int_{\Omega_t^1} \Delta u_t g_1(\Delta u_t(x,t))\, dx\, dt \\ &\quad\leq \varepsilon C_s \int_S^T \varphi(E)\, dt + C(\varepsilon)\varphi(E(S)), \end{aligned}$$

and

$$\int_S^T \frac{\varphi(E)}{E} \int_{\Omega_t^2} |\Delta u|.|g_1(\Delta u_t(x,t))|\, dx\, dt \le \varepsilon C_s \int_S^T \varphi(E)\, dt + C(\varepsilon)\varphi(E(S)).$$

Similarly,

$$\begin{aligned}
&\int_S^T \frac{\varphi(E)}{E} \int_{\widetilde{\Omega}_t^1} |\Delta u|.|g_2(\Delta z(x,1,t))|\, dx\, dt \\
&\le \varepsilon \int_S^T \frac{\varphi(E)}{E} \int_{\widetilde{\Omega}_t^1} |\Delta u|^2\, dx\, dt + C(\varepsilon) \int_S^T \frac{\varphi(E)}{E} \int_{\widetilde{\Omega}_t^1} |g_2(\Delta z(x,1,t))|^2\, dx\, dt \\
&\le \varepsilon C_s \int_S^T \frac{\varphi(E)}{E} \int_{\Omega} |\nabla \Delta u|^2\, dx\, dt + C(\varepsilon) \int_S^T \frac{\varphi(E)}{E} \int_{\widetilde{\Omega}_t^1} \Delta z(x,1,t) g_2(\Delta z(x,1,t))\, dx\, dt \\
&\le \varepsilon C_s \int_S^T \varphi(E)\, dt + C(\varepsilon)\varphi(E(S)),
\end{aligned}$$

and

$$\int_S^T \frac{\varphi(E)}{E} \int_{\widetilde{\Omega}_t^2} |\Delta u|.|g_2(\Delta z(x,1,t))|\, dx\, dt \le \varepsilon C_s \int_S^T \varphi(E)\, dt + C(\varepsilon)\varphi(E(S)).$$

Case 2. $H'(0) = 0, \;\; H'' > 0$ **on** $]0, \epsilon]$

$$\int_S^T \frac{\varphi(E)}{E} \int_{\Omega_t^1} |\nabla u_t|^2\, dx\, dt \le C\varphi(E(S))$$

$$\begin{aligned}
\int_S^T \frac{\varphi(E)}{E} \int_{\Omega_t^2} |\nabla u_t|^2\, dx\, dt &\le C_s \int_S^T \frac{\varphi(E)}{E} \int_{\Omega_t^2} (|\Delta u_t|^2 + |g_1(\Delta u_t)|^2)\, dx\, dt \\
&\le C_s \int_S^T \frac{\varphi(E)}{E} \int_{\Omega} H^{-1}(\Delta u_t g_1(\Delta u_t))\, dx\, dt \\
&\le C_s \int_S^T \frac{\varphi(E)}{E} |\Omega| H^{-1}\Big(\frac{1}{|\Omega|} \int_{\Omega} \Delta u_t g_1(\Delta u_t)\Big)\, dx\, dt.
\end{aligned}$$

Using remark 4.2.1, we obtain

$$\begin{aligned}
&\int_S^T \frac{\varphi(E)}{E} \int_{\Omega_t^2} (|\Delta u_t|^2 + |g_1(\Delta u_t)|^2)\, dx\, dt \\
&\qquad \le c \int_S^T H^*(\frac{\varphi(s)}{s})\, dt + c \int_S^T \int_{\Omega} \Delta u_t g_1(\Delta u_t)\Big)\, dx\, dt.
\end{aligned} \tag{5.50}$$

Choosing $\varphi(s) = sH'(\epsilon s)$ and using remark 4.2.1, we obtain

$$H^*(\frac{\varphi(s)}{s}) = s\epsilon H'(\epsilon s) = \epsilon s H'(\epsilon s) - H(\epsilon s) \le \epsilon \varphi(s). \tag{5.51}$$

Making use of (5.50)-(5.51), we have

$$\begin{aligned}
&\int_S^T \frac{\varphi(E)}{E} \int_{\Omega_t^2} (|\Delta u_t|^2 + |g_1(\Delta u_t)|^2)\, dx\, dt \\
&\qquad \le c\epsilon \int_S^T \varphi(E)\, dt + cE(S)
\end{aligned}$$

$$\begin{aligned}
&\int_S^T \frac{\varphi(E)}{E} \int_{\Omega_t^1} |\Delta u|.|g_1(\Delta u_t(x,t))|\,dx\,dt \\
&\qquad \le \varepsilon \int_S^T \frac{\varphi(E)}{E} \int_{\Omega_t^1} |\Delta u|^2\,dx\,dt + C(\varepsilon)\int_S^T \frac{\varphi(E)}{E} \int_{\Omega_t^1} |g_1(\Delta u_t(x,t))|^2\,dx\,dt \\
&\qquad \le \varepsilon C_s \int_S^T \frac{\varphi(E)}{E} \int_{\Omega} |\nabla\Delta u|^2\,dx\,dt + C(\varepsilon)\int_S^T \frac{\varphi(E)}{E} \int_{\Omega_t^1} \Delta u_t g_1(\Delta u_t(x,t))\,dx\,dt \\
&\qquad \le \varepsilon C_s \int_S^T \varphi(E)\,dt + C(\varepsilon)\varphi(E(S)),
\end{aligned}$$

and

$$\begin{aligned}
&\int_S^T \frac{\varphi(E)}{E} \int_{\Omega_t^2} |\Delta u|.|g_1(\Delta u_t(x,t))|\,dx\,dt \\
&\le \varepsilon \int_S^T \frac{\varphi(E)}{E} \int_{\Omega_t^2} |\Delta u|^2\,dx\,dt + C(\varepsilon)\int_S^T \frac{\varphi(E)}{E} \int_{\Omega_t^2} |g_1(\Delta u_t(x,t))|^2\,dx\,dt \\
&\le \varepsilon C_s \int_S^T \frac{\varphi(E)}{E} \int_{\Omega} |\nabla\Delta u|^2\,dx\,dt + C(\varepsilon)\int_S^T \frac{\varphi(E)}{E} \int_{\Omega_t^2} (|\Delta u_t|^2 + |g_1(\Delta u_t(x,t))|^2)\,dx\,dt \\
&\le \varepsilon C_s \int_S^T \frac{\varphi(E)}{E} \int_{\Omega} |\nabla\Delta u|^2\,dx\,dt + C(\varepsilon)\int_S^T \frac{\varphi(E)}{E} |\Omega| H^{-1}\Big(\frac{1}{|\Omega|}\int_\Omega \Delta u_t g_1(\Delta u_t)\Big)\,dx\,dt \\
&\le \varepsilon C \int_S^T \varphi(E)\,dt + C(\varepsilon)\varphi(E(S)),
\end{aligned}$$

also

$$\begin{aligned}
&\int_S^T \frac{\varphi(E)}{E} \int_{\widetilde{\Omega}_t^1} |\Delta u|.|g_2(\Delta z(x,1,t))|\,dx\,dt \\
&\quad \le \varepsilon \int_S^T \frac{\varphi(E)}{E} \int_{\widetilde{\Omega}_1^t} |\Delta u|^2\,dx\,dt + C(\varepsilon)\int_S^T \frac{\varphi(E)}{E} \int_{\widetilde{\Omega}_t^1} |g_2(\Delta z(x,1,t))|^2\,dx\,dt \\
&\quad \le \varepsilon C_s \int_S^T \frac{\varphi(E)}{E} \int_{\Omega} |\nabla\Delta u|^2\,dx\,dt + C(\varepsilon)\int_S^T \frac{\varphi(E)}{E} \int_{\widetilde{\Omega}_t^1} \Delta z(x,1,t) g_2(\Delta z(x,1,t))\,dx\,dt \\
&\quad \le \varepsilon C_s \int_S^T \varphi(E)\,dt + C(\varepsilon)\varphi(E(S)),
\end{aligned}$$

and

$$\begin{aligned}
&\int_S^T \frac{\varphi(E)}{E} \int_{\widetilde{\Omega}_t^2} |g_2(\Delta z(x,1,t))|^2\,dx\,dt \\
&\quad \le \int_S^T \frac{\varphi(E)}{E} |\Omega| H^{-1}\Big(\frac{1}{|\Omega|}\int_{\widetilde{\Omega}_t^2} \Delta z(x,1,t)) g_2(\Delta((z(x,1,t)))\Big)\,dx\,dt \\
&\quad \le c\int_S^T H^*(\frac{\varphi(s)}{s})\,dt + c\int_S^T \int_\Omega \Delta((z(x,1,t))) g_2(\Delta((z(x,1,t)))\Big)\,dx\,dt \\
&\quad \le c\int_S^T \varphi(E)\,dt + cE(S).
\end{aligned}$$

Then, choosing $\varepsilon > 0$ small enough and using (5.49), we obtain in both cases

$$\begin{aligned}\int_S^T \varphi(E(t))\,dt &\le c(E(S)+\varphi(E(S)))\\ &\le c\Big(1+\frac{\varphi(E(S))}{E(S)}\Big)E(S) \le cE(S),\ \ \forall S \ge 0.\end{aligned}$$

Using Lemma 5.2.2 in the particular case where $\Psi(s) = \omega\varphi(s)$ we deduce from (5.9) our estimate (5.10). The proof of Theorem 5.3.2 is now complete.

5.7 Examples

Example 5.1. *Let g given by $g(s) = s^p(-\ln s)^q$, where $p \ge 1$ and $q \in \mathbb{R}$ on $[0,\epsilon]$. Then*

$$g'(s) = s^{p-1}(-\ln s)^{q-1}(p(-\ln s) - q)$$

which is an increasing function in the right neighborhood of 0. The function H is defined in the neighborhood of 0 by

$$H(s) = cs^{\frac{p+1}{2}}(-\ln\sqrt{s})^q,$$

we have

$$H'(s) = cs^{\frac{p-1}{2}}(-\ln\sqrt{s})^{q-1}\Big(\frac{p+1}{2}(-\ln\sqrt{s}) - \frac{q}{2}\Big).$$

Thus

$$\begin{aligned}\varphi(s) &= cs^{\frac{p+1}{2}}(-\ln\sqrt{s})^{q-1}\Big(\frac{p+1}{2}(-\ln\sqrt{s}) - \frac{q}{2}\Big)\\ \psi(s) &= c\int_t^1 \frac{1}{s^{\frac{p+1}{2}}(-\ln\sqrt{s})^{q-1}\Big(\frac{p+1}{2}(-\ln\sqrt{s}) - \frac{q}{2}\Big)}\,ds\\ &= c\int_1^{\frac{1}{\sqrt{t}}} \frac{z^{p-2}}{(\ln z)^{q-1}\Big(\frac{p+1}{2}\ln z - \frac{q}{2}\Big)}\,dz.\end{aligned}$$

We obtain in the neighborhood of 0

$$\psi(t) = \begin{cases} c\dfrac{1}{t^{\frac{p-1}{2}}(-\ln t)^q} & \text{if } \ p > 1\\ c(-\ln t)^{1-q}, & \text{if } p = 1,\ q < 1\\ c(\ln(-\ln t)), & \text{if } \ p = 1,\ \ q = 1\end{cases}$$

and then in the neighborhood of $+\infty$

$$\psi^{-1}(t) = \begin{cases} ct^{-\frac{2}{p-1}}(\ln t)^{-\frac{2q}{p-1}} & \text{if } \ p > 1\\ ce^{-t^{\frac{1}{1-q}}}, & \text{if } p = 1,\ q < 1\\ ce^{-e^t}, & \text{if } \ p = 1,\ \ q = 1\end{cases}$$

Since $h(t) = t$ as t tends to infinity, we obtain

$$E(t) \leq \begin{cases} ct^{-\frac{2}{p-1}}(\ln t)^{-\frac{2q}{p-1}} & \text{if} \quad p > 1 \\ ce^{-t^{\frac{1}{1-q}}}, & \text{if } p = 1, \ q < 1 \\ ce^{-e^t}, \quad \text{if } p = 1, \ q = 1 \end{cases}$$

Example 5.2. *Let g given by $g(s) = e^{-(-\ln s)^\gamma}$ where $1 < \gamma < 2$. Then*

$$H(s) = c\sqrt{s}e^{-(-\ln\sqrt{s})^\gamma}$$

we have

$$H'(s) = \frac{c}{2\sqrt{s}}e^{-(-\ln\sqrt{s})^\gamma}\Big(1 + \gamma(-\ln\sqrt{s})^{\gamma-1}\Big).$$

Thus

$$\varphi(s) = \frac{c\sqrt{s}}{2}e^{-(-\ln\sqrt{s})^\gamma}\Big(1 + \gamma(-\ln\sqrt{s})^{\gamma-1}\Big).$$

So

$$\psi(t) = c\int_t^1 \frac{e^{(-\ln\sqrt{s})^\gamma}}{\sqrt{s}\Big(1 + \gamma(-\ln\sqrt{s})^{\gamma-1}\Big)}\,ds.$$

Making the following change of variable: $z\frac{1}{\sqrt{s}}$, we obtain

$$\psi(t) = c\int_1^{\frac{1}{\sqrt{t}}} \frac{e^{(\ln z)^\gamma}}{z^2\Big(1 + \gamma(\ln z)^{\gamma-1}\Big)}\,dz$$

we find that

$$\psi(t) \equiv c\frac{\sqrt{t}e^{(-\ln\sqrt{t})^\gamma}}{(-\ln\sqrt{t})^{2(\gamma-1)}}, \quad \text{as } t \to 0.$$

Hence the energy E satisfies the decay estimate

$$E(t) \leq C(E(0))\psi^{-1}(t), \quad \text{for sufficiently large } t,$$

where $C(E(0))$ is a constant which depends continuously on $E(0)$.

CHAPTER 6

BLOW-UP OF SOLUTIONS TO A VISCOELASTIC PLATE EQUATION WITH DELAY

In this chapter, we consider the nonlinear viscoelastic plate equation in a bounded domain with a delay term in the nonlinear internal feedback, we prove that there are solutions with negative initial energy that blow-up in finite time.

6.1 Introduction

It is well known that viscoelastic materials indicate natural damping, which is due to the property of these substances to keep the memory of their past history. From the mathematical point of view, these damping effects are modeled by integro-differential operators. The Petrovsky type of equation

$$u_{tt} + \Delta^2 u = g(x, t, u, u_t)$$

originated from the study of beams and plates, and it can also be used in many branches of physics such as nuclear physics, optics, geophysics, and ocean acoustics. However in most instances, these phenomena do not depend only on current state but also on certain past events.

In this work we consider the following abstract system of nonlinear damped with delay:

$$\begin{cases} u_{tt}(x,t) + \Delta^2 u(x,t) - \int_0^t h(t-s)\Delta^2 u(x,s)ds + \mu_1 |u_t|^{m-2}u_t & \\ +\mu_2 |u_t(t-\tau)|^{m-2}u_t(t-\tau) = b|u|^{p-2}u & \text{in } \Omega\times]0,+\infty[, \\ u(x,t) = \frac{\partial u}{\partial \nu}(x,t) = 0, & \text{on } \partial\Omega \times [0,+\infty[, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau) & \text{in } \Omega\times]0,\tau[, \end{cases} \tag{6.1}$$

where $\Omega \subset R^n$, $n \geq 1$, is a bounded domain with a smooth boundary $\partial\Omega$, $p > 2$, $m \geq 1$ and h is a positive nonincreasing function defined on R^+, $\tau > 0$ represents the time delay, μ_1 and μ_2 are positive constants, and (u_0, u_1, f_0) are given functions belonging to suitable spaces. In the absence of the viscoelastic term, i.e., $(h = 0)$, the equation

$$u_{tt}(x,t) + \Delta^2 u(x,t) + \mu_1 |u_t|^{m-2}u_t + \mu_2 |u_t(t-\tau)|^{m-2}u_t(t-\tau) = b|u|^{p-2}u. \tag{6.2}$$

Kafini, Messaoudi and Nicaise [33] showed that, under suitable conditions on the initial data, the energy of solutions blows up in finite-time with negative initial energy if $p > m$.

When $\mu_2 = 0$, Messaoudi [48] studied problem (6.2). He proved the existence of a local weak solution and in [49] he showed that this solution blows up in finite time with negative initial energy if $p > m$. In the presence of the viscoelastic terms, Rivera et al. [52] considered the plate model:

$$u_{tt}(x,t) + \gamma\Delta_{tt}u(x,t) + \Delta^2 u(x,t) - \int_0^t h(t-s)\Delta^2 u(x,s)ds = 0, \quad \text{in} \quad \Omega \times (0,T), \tag{6.3}$$

together with initial and dynamical boundary conditions and proved that the sum of the first and second energies decays exponentially (polynomially) if the kernel h decays exponentially (polynomially). When $\gamma = 0$ and $\Omega = (0,\pi) \times (-l,l) \subset R^2$, problem (6.3) has been studied by Messaoudi and Mukiawa [51]. They proved a global existence result and the general decay result.

Tahamtani and Shahrouzi [63] have also studied the following system

$$u_{tt}(x,t) + \Delta^2 u(x,t) - \int_0^t h(t-s)\Delta^2 u(x,s)ds = b|u|^{p-2}u,$$

they proved the global existence of solutions by the Faedo-Galerkin method and showed that this solution blows up in finite time with in finite time with non-positive initial energy as well as positive initial energy. The work is organized as follows. In section 2, we present some preliminary results and we state a local existence theorem and in section 3, we establish a blow-up result for solutions with negative initial energy and $p > m$. Our technique of proof is similar to the one in [33]. Estimates for the blow-up time T^* are also given.

6.2 Preliminary results

We denote by $\|.\|_k$ the L^k-norm over Ω. In particular, the L^2-norm is denoted $\|.\|_2$

Lemma 6.2.1. *(Sobolev-Poincaré inequality) [5] Let q be a number with*

$$2 \leq q < +\infty \ \ (n = 1, 2, 3, 4) \ \ or \ \ 2 \leq q \leq 2n/(n-4) \ (n \geq 5),$$

then there exists a constant $C_s = C_s(\Omega, q)$ such that

$$\|u\|_q \leq C_s \|\Delta u\|_2 \ \ for \ \ u \in H_0^2(\Omega).$$

Lemma 6.2.2. *Suppose that*

$$2 \leq p < +\infty \ \ (n = 1, 2, 3, 4) \ \ or \ \ 2 \leq p \leq 2n/(n-4) \ \ (n \geq 5), \tag{6.4}$$

holds. Then there exists a positive constant C depending on Ω only such that

$$\|u\|_p^s \leq C(\|u\|_p^p + \|\Delta u\|_2^2),$$

for any $u \in H_0^2(\Omega)$ and $2 \leq s \leq p$.

Now we introduce, as in Nicaise and Pignotti [55], the new variable

$$z(x, \rho, t) = u_t(x, t - \rho\tau), \ \ x \in \Omega, \quad \rho \in (0, 1), \ \ t > 0.$$

Then, we have

$$\tau z_t(x, \rho, t) + z_\rho(x, \rho, t) = 0, \ \text{ in } \Omega \times (0, 1) \times (0, +\infty).$$

Therefore, problem (6.1) is equivalent to

$$\begin{cases} u_{tt}(x,t) + \Delta^2 u(x,t) - \int_0^t h(t-s)\Delta^2 u(x,s)ds + \mu_1|u_t(x,t)|^{m-2}u_t(x,t) & \\ +\mu_2|z(x,1,t)|^{m-2}z(x,1,t) = b|u(x,t)|^{p-2}u(x,t) & \text{in } \Omega \times]0, +\infty[, \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, & \text{in } \Omega \times]0, 1[\times]0, +\infty[, \\ u(x,t) = \frac{\partial u}{\partial \nu}(x,t) = 0, & \text{on } \partial\Omega \times [0, \infty[, \\ z(x,0,t) = u_t(x,t), & \text{on } \Omega \times [0, \infty[, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & \text{in } \Omega, \\ z(x,\rho,0) = f_0(x, -\rho\tau), & \text{in } \Omega \times]0, 1[. \end{cases} \tag{6.5}$$

Now we define the energy associated to the solution of system (6.5) by

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\left(1 - \int_0^t h(s)ds\right)\|\Delta u\|_2^2 + \frac{1}{2}(h \circ \Delta u)(t) - \frac{b}{p}\|u\|_p^p + \xi \int_\Omega \int_0^1 |z(x,\rho,t)|^m \, d\rho \, dx, \tag{6.6}$$

where ξ is a positive constant such that

$$\tau\Big(\frac{\mu_2}{m}(m-1)\Big) < \xi < \tau\Big(\mu_1 - \frac{\mu_2}{m}\Big), \tag{6.7}$$

and

$$(h \circ v)(t) = \int_0^t h(t-\tau)\|v(t)-v(\tau)\|_2^2 d\tau, \tag{6.8}$$

and make the following extra assumptions on h

$$h(s) \geq 0, \qquad h'(s) \leq 0, \qquad \int_0^\infty h(s)ds < \frac{(p/2)-1}{(p/2)-1+(1/2p)}. \tag{6.9}$$

Lemma 6.2.3. *Let (u,z) be a solution of the problem (6.5). Then, there exists $C>0$ such that*

$$E'(t) \leq -C\Big\{\|z(x,1,t)\|_m^m + \|u_t(x,t)\|_m^m\Big\} \leq 0. \tag{6.10}$$

Multiplying the first equation in (6.5) by u_t and integrating over Ω, using integration by parts, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\|u_t\|_2^2 + \|\Delta_x u\|_2^2 - \frac{2b}{p}\|u\|_p^p\Big) + \mu_1\|u_t\|_m^m + \mu_2\int_\Omega |z(x,1,t)|^{m-2}z(x,1,t)u_t(x,t)\,dx \\ &= \int_\Omega\int_0^t h(t-s)\Delta u(x,s)\Delta u_t(x,t)dsdx. \end{aligned} \tag{6.11}$$

The term on the right-hand side of (6.11) can be rewritten as follows:

$$\begin{aligned} &\int_\Omega\int_0^t h(t-s)\Delta u(x,s)\Delta u_t(x,t)dsdx + \frac{1}{2}h(t)\|\Delta u(x,t)\|_2^2 \\ &= \frac{1}{2}\frac{d}{dt}\Big[\int_0^t h(s)ds\|\Delta u(x,t)\|_2^2 - (h\circ\Delta u)(t)\Big] + \frac{1}{2}(h'\circ\Delta u)(t). \end{aligned}$$

Consequently, equation (6.11) becomes

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\|u_t\|_2^2 + \Big(1-\int_0^t h(s)ds\Big)\|\Delta_x u\|_2^2 + (h\circ\Delta u)(t) - \frac{2b}{p}\|u\|_p^p\Big) = -\mu_1\|u_t\|_m^m \\ &-\mu_2\int_\Omega |z(x,1,t)|^{m-2}z(x,1,t)u_t(x,t)\,dx - \frac{1}{2}h(t)\|\Delta u\|_2^2 + \frac{1}{2}(h'\circ\Delta u)(t). \end{aligned} \tag{6.12}$$

We multiply the second equation in (6.5) by $\xi|z(x,\rho,t)|^{m-2}z(x,\rho,t)$ and integrate the result over $\Omega\times(0,1)$, to obtain

$$\begin{aligned} \xi\int_\Omega\int_0^1 z_t(x,\rho,t)|z(x,\rho,t)|^{m-2}z(x,\rho,t)\,d\rho\,dx &= -\frac{\xi}{\tau}\int_\Omega\int_0^1 z_\rho(x,\rho,t)|z(x,\rho,t)|^{m-2}z(x,\rho,t)\,d\rho\,dx \\ &= -\frac{\xi}{\tau}\int_\Omega\int_0^1 \frac{\partial z(x,\rho,t)}{\partial\rho}\,|z(x,\rho,t)|^{m-2}z(x,\rho,t)\,d\rho\,dx \\ &= -\frac{\xi}{\tau m}\int_\Omega\int_0^1 \frac{\partial}{\partial\rho}\,|z(x,\rho,t)|^m\,d\rho\,dx \\ &= -\frac{\xi}{\tau m}\int_\Omega(|z(x,1,t)|^m - |z(x,0,t)|^m)\,dx. \end{aligned}$$

Hence

$$\xi \frac{d}{dt} \int_\Omega \int_0^1 |z(x,\rho,t)|^m \, d\rho \, dx = -\frac{\xi}{\tau} \int_\Omega |z(x,1,t)|^m \, dx + \frac{\xi}{\tau} \int_\Omega |u_t(x,t)|^m \, dx. \tag{6.13}$$

Combining (6.12) and (6.13), we obtain

$$\begin{aligned} E'(t) = & -\frac{\xi}{\tau}\|z(x,1,t)\|_m^m - (\mu_1 - \frac{\xi}{\tau})\|u_t(x,t)\|_m^m - \mu_2 \int_\Omega |z(x,1,t)|^{m-2} z(x,1,t) u_t(x,t) \, dx \\ & -\frac{1}{2} h(t)\|\Delta u\|_2^2 + \frac{1}{2}(h' \circ \Delta u)(t). \end{aligned} \tag{6.14}$$

By using Young's inequality, we get

$$\mu_2 \left| \int_\Omega |z(x,1,t)|^{m-2} z(x,1,t) u_t dx \right| \leq \frac{\mu_2(m-1)}{m} \|z(x,1,t)\|_m^m + \frac{\mu_2}{m}\|u_t\|_m^m.$$

Hence, we get from (6.14)

$$\begin{aligned} E'(t) & \leq -\left(\frac{\xi}{\tau} - \frac{\mu_2(m-1)}{m}\right)\|z(x,1,t\|_m^m - \left(\mu_1 - \frac{\xi}{\tau} - \frac{\mu_2}{m}\right)\|u_t\|_m^m \\ & \leq -C\Big\{\|z(x,1,t)\|_m^m + \|u_t(x,t)\|_m^m\Big\}, \end{aligned}$$

where

$$C = \min\left\{\frac{\xi}{\tau} - \frac{\mu_2}{m}(m-1), \mu_1 - \frac{\xi}{\tau} - \frac{\mu_2}{m}\right\},$$

which is positive by (6.7). This completes the proof of the Lemma.

6.3 Local solutions

Theorem 6.3.1. *Suppose that $m \geq 1$, $p > 2$ and let $u_0 \in H^4(\Omega) \cap H_0^2(\Omega)$, $u_1 \in H_0^2(\Omega) \cap L^2(\Omega)$ and $f_0 \in H_0^2(\Omega, H^2(0,1))$ be given. Assume further that*

$$m \leq p \leq \frac{2(n-2)}{n-4}, \qquad n \geq 5 \tag{6.15}$$

and h is a $\mathcal{C}^1$ function satisfying

$$1 - \int_0^\infty h(s) ds = l > 0. \tag{6.16}$$

The problem (6.1) has a unique local solution

$$u \in \mathcal{C}([0,T), H^4(\Omega) \cap H_0^2(\Omega)),$$

$$u_t \in \mathcal{C}([0,T), H_0^2(\Omega) \cap L^2(\Omega) \times (0,T)),$$

for some $T > 0$.

Remark 6.3.1. *This theorem can be easily established by combining the arguments of [16] and [73].*

Now, we are able to state the result of local existence. To do this, we take a related simpler problem into account. Then, we prove the local existence of solutions to the problem (6.5) by contraction mapping principle. For v given, considering the following problem:

$$\begin{cases} u_{tt}(x,t) + \Delta^2 u(x,t) - \int_0^t h(t-s)\Delta^2 u(x,s)ds + \mu_1|u_t(x,t)|^{m-2}u_t(x,t) & \\ +\mu_2|z(x,1,t)|^{m-2}z(x,1,t) = b|v(x,t)|^{p-2}v(x,t) & \text{in } \Omega\times]0,T[, \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, & \text{in } \Omega\times]0,1[\times]0,+T[, \\ u(x,t) = \frac{\partial u}{\partial \nu}(x,t) = 0, & \text{on } \partial\Omega\times[0,T[, \\ z(x,0,t) = u_t(x,t), & \text{on } \Omega\times[0,T[, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & \text{in } \Omega, \\ z(x,\rho,0) = f_0(x,-\rho\tau), & \text{in } \Omega\times]0,1[. \end{cases} \tag{6.17}$$

in which $T > 0$.

Remark 6.3.2. *The following lemma plays an important role in studying the local existence of solutions for the problem (6.5). We can prove this lemma by using the Galerkin method. This lemma is a direct result of [41] J.L. Lions, Theorem 3.1 and the remark 3.4, Chap.1 .*

Lemma 6.3.2. *Assume that $m \geq 1$, $p > 2$ and let $u_0 \in H^4(\Omega) \cap H_0^2(\Omega)$, $u_1 \in H_0^2(\Omega)$ and $f_0 \in H_0^2(\Omega, H^2(0,1))$ be given. Assume further that*

$$m \leq p \leq \frac{2(n-2)}{n-4}, \qquad n \geq 5 \tag{6.18}$$

and h is a $\mathcal{C}^1$ function satisfying

$$1 - \int_0^\infty h(s)ds = l > 0. \tag{6.19}$$

The problem (6.17) admits a unique solution u such that

$$\begin{aligned} u &\in L^\infty([0,T), H^4(\Omega) \cap H_0^2(\Omega)), \\ u_t &\in L^\infty([0,T), H_0^2(\Omega)), \\ u_{tt} &\in L^\infty([0,T), L^2(\Omega)). \end{aligned} \tag{6.20}$$

Theorem 6.3.3. *Suppose that (6.18) and (6.19) hold. If $u_0 \in H_0^2(\Omega)$, $u_1 \in L^2(\Omega)$, $f_0 \in H_0^2(\Omega, H^2(0,1))$ and $v \in \mathcal{C}([0,T]; H_0^2(\Omega))$, then the problem (6.17) has a unique weak solution u fulfilling*

$$u \in \mathcal{C}([0,T), H_0^2(\Omega)), \qquad u_t \in \mathcal{C}([0,T), L^2(\Omega)). \tag{6.21}$$

Proof. We approximate u_0, u_1 and f_0 by sequences $\{u_0^\mu\}$, $\{u_1^\mu\}$ and $\{f_0^\mu\}$ in $\mathcal{C}_0^\infty(\Omega)$, respectively, and v by a sequence $\{v^\mu\}$ in $(\mathcal{C}[0,T];\mathcal{C}_0^\infty)$. We then consider the following problem

$$\begin{cases} u_{tt}^\mu(x,t) + \Delta^2 u^\mu(x,t) - \int_0^t h(t-s)\Delta^2 u^\mu(x,s)ds + \mu_1 |u_t^\mu(x,t)|^{m-2} u_t^\mu(x,t) & \\ +\mu_2 |z^\mu(x,1,t)|^{m-2} z^\mu(x,1,t) = b|v^\mu(x,t)|^{p-2} v^\mu(x,t) & \text{in } \Omega\times]0,T[, \\ \tau z_t^\mu(x,\rho,t) + z_\rho^\mu(x,\rho,t) = 0, & \text{in } \Omega\times]0,1[\times]0,+T[, \\ u^\mu(x,t) = \frac{\partial u^\mu}{\partial \nu}(x,t) = 0, & \text{on } \partial\Omega \times [0,T[, \\ z^\mu(x,0,t) = u_t^\mu(x,t), & \text{on } \Omega \times [0,T[, \\ u^\mu(x,0) = u_0^\mu(x), \quad u_t^\mu(x,0) = u_1^\mu(x), & \text{in } \Omega, \\ z^\mu(x,\rho,0) = f_0^\mu(x,-\rho\tau), & \text{in } \Omega\times]0,1[. \end{cases} \tag{6.22}$$

It follows from Lemma 6.3.2 that for every u, the problem (6.22) has a unique solution $\{u^\mu\}$ which fulfills (6.20). Next, we proceed to show that the sequences $\{u^\mu\}$ and $\{u_t^\mu\}$ are Cauchy sequences in W, where

$$W = \left\{ w : w \in \mathcal{C}([0,T); H_0^2(\Omega)), \quad w_t \in \mathcal{C}([0,T); L^2(\Omega)) \cap L^m(\Omega \times (0,T)) \right\}.$$

Let

$U = u^\mu - u^\nu, \quad V = v^\mu - v^\nu,$

then it is easy to see that U satisfies

$$\begin{aligned} & U_{tt} + \Delta^2 U - \int_0^t h(t-s)\Delta^2 U(s)ds + \mu_1 \left(u_t^\mu |u_t^\mu|^{m-2} - (u_t^\nu |u_t^\nu|^{m-2} \right) \\ & + \mu_2 \left(z^\mu(1)|z^\mu(1)|^{m-2} - z^\nu(1)|z^\nu(1)|^{m-2} \right) = b\left(|v^\mu|^{p-2} v^\mu - |v^\nu|^{p-2} v^\nu \right), \\ & U(x,t) = 0, \qquad x \in \partial\Omega, \quad t > 0, \\ & U(x,0) = U_0(x) = u_0^\mu(x) - u_0^\nu(x), \\ & U_t(x,0) = U_1(x) = u_1^\mu(x) - u_1^\nu(x). \end{aligned} \tag{6.23}$$

We multiply Eq.(6.23) by U_t and integrate over $\Omega \times (0,t)$ to get

$$\begin{aligned} & \frac{1}{2}\|U_t\|^2 + \frac{1}{2}\left(1 - \int_0^t h(s)ds\right) \|\Delta U(t)\|^2 + \frac{1}{2}(h \circ \Delta U)(t) \\ & - \int_0^t \left[\frac{1}{2}(h' \circ \Delta U)(s) - \frac{1}{2} h(s) \|\Delta U(s)\|^2 \right] ds \\ & + \mu_1 \int_0^t \int_\Omega \left(u_t^\mu |u_t^\mu|^{m-2} - (u_t^\nu |u_t^\nu|^{m-2} \right) U_t(x,s)dxds \\ & + \mu_2 \int_0^t \int_\Omega \left(z^\mu(1)|z^\mu(1)|^{m-2} - z^\nu(1)|z^\nu(1)|^{m-2} \right) U_t(x,s)dxds \\ & = \frac{1}{2}\|U_1\|^2 + \frac{1}{2}\|\Delta U_0\|^2 + b\int_0^t \int_\Omega \left[|v^\mu|^{p-2} v^\mu - |v^\nu|^{p-2} v^\nu \right] U_t(x,s)dxds. \end{aligned} \tag{6.24}$$

Using Holder's inequality to estimate the last term in (6.24)

$$\begin{aligned}\int_\Omega \left[|v^\mu|^{p-2}v^\mu - |v^\nu|^{p-2}v^\nu\right] U_t(x,s)dx &\le \int_\Omega \left||v^\mu|^{p-2}v^\mu - |v^\nu|^{p-2}v^\nu\right| |U_t(x,s)|dx \\ &\le C_1 \int_\Omega |V(x,s)| \left(|v^\mu|^{p-2} + |v^\nu|^{p-2}\right) |U_t(x,s)|dx \\ &\le C_2\|U_t\|\|V\|_{2n/n-4}\left[\|v^\mu\|^{p-2}_{n(p-2)/2} + \|v^\nu\|^{p-2}_{n(p-2)/2}\right].\end{aligned} \tag{6.25}$$

We have from the assumption (6.25) and Lemma 6.2.1 that

$$\begin{aligned}&\int_\Omega \left[|v^\mu|^{p-2}v^\mu - |v^\nu|^{p-2}v^\nu\right] U_t(x,s)dx \\ &\le C\|U_t\|_2\|\Delta V\|_2\left[\|\Delta v^\mu\|_2^{p-2} + \|\Delta v^\nu\|_2^{p-2}\right].\end{aligned} \tag{6.26}$$

Since

$$\left[\mu_1\left(u_t^\mu|u_t^\mu|^{m-2} - (u_t^\nu|u_t^\nu|^{m-2}\right) + \mu_2\left(z^\mu(1)|z^\mu(1)|^{m-2} - z^\nu(1)|z^\nu(1)|^{m-2}\right)\right](u_t^\mu - u_t^\nu) \ge 0$$

then (6.24) yields

$$\frac{1}{2}\|U_t\|^2 + \frac{l}{2}\|\Delta U\|^2 \le \frac{1}{2}(\|U_1\|^2 + \|\Delta U_0\|^2) + C_3\int_0^t \|U_t(s)\|\|\Delta V(s)\|ds \tag{6.27}$$

where C_3 is a generic positive constant depending on C and the radius of the ball in $\mathcal{C}([0,T];H_0^2(\Omega))$ containing v^μ and v^ν .Young's inequality then gives

$$\max_{0\le t\le T}\|U_t\|^2 + \|\Delta U\|^2 \le 2(\|U_1\|^2 + \|\Delta U_0\|^2) + 4C_3^2T^2\max_{0\le t\le T}\|\Delta V\|^2. \tag{6.28}$$

Since $\{u_0^\mu\}$ is Cauchy sequence in $H_0^2(\Omega)$, $\{u_1^\mu\}$ is Cauchy sequence in $L^2(\Omega)$, and $\{v^\mu\}$ is Cauchy sequence in $\mathcal{C}([0,T];H_0^2(\Omega))$, we conclude that (u^μ, u_t^μ) is Cauchy in $\mathcal{C}([0,T];H_0^2(\Omega))\times\mathcal{C}([0,T];L^2(\Omega))$.To show that u_t is Cauchy in $L^m(\Omega\times(0,T))$, we use

$$\|U_t\|^m_{L^m(\Omega\times(0,T))} \le C\int_0^t\int_\Omega \left(u_t^\mu|u_t^\mu|^{m-2} - (u_t^\nu|u_t^\nu|^{m-2}\right)U_t(x,s)dxds \tag{6.29}$$

which yields, by (6.24),

$$\|U_t\|^m_{L^m(\Omega\times(0,T))} \le C(\|U_1\|^2 + \|\Delta U_0\|^2) + C\int_0^T \|U_t(.,s)\|\|\Delta V(.,s)\|ds. \tag{6.30}$$

Therefore $\{u_t^\mu\}$is Cauchy in $L^m(\Omega\times(0,T))$ and hence $\{u^\mu\}$is Cauchy in W. We now show that the limit u is a weak solution of (6.17) in the sense of [41]. That is for each θ in H_0^2 we must show that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega u_t(x,t)\theta(x)dx + \int_\Omega \Delta u(x,t)\Delta\theta(x)dx - \int_\Omega\int_0^t h(t-s)\Delta u(x,s)\Delta\theta(x)dxds \\ &+\mu_1\int_\Omega |u_t(x,t)|^{m-2}u_t(x,t)\theta(x)dx + \mu_2\int_\Omega |z(x,1,t)|^{m-2}z(x,1,t)\theta(x)dx = b\int_\Omega |v(x,t)|^{p-2}v(x,t)\theta(x)dx\end{aligned} \tag{6.31}$$

Multiplying both sides of Eq. (6.22) by θ and integrating over Ω, we get

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega u_t^\mu(x,t)\theta(x)dx + \int_\Omega \Delta u^\mu(x,t)\Delta\theta(x)dx - \int_\Omega\int_0^t h(t-s)\Delta u^\mu(x,s)\Delta\theta(x)dxds \\
&+\mu_1\int_\Omega |u_t^\mu(x,t)|^{m-2}u_t^\mu(x,t)\theta(x)dx + \mu_2\int_\Omega |z^\mu(x,1,t)|^{m-2}z^\mu(x,1,t)\theta(x)dx = b\int_\Omega |v^\mu(x,t)|^{p-2}v^\mu(x,t)\theta(x)dx
\end{aligned} \tag{6.32}
$$

As $\mu \to +\infty$, we find that

$$
\begin{aligned}
&\int_\Omega \Delta u^\mu(x,t)\Delta\theta(x)dx \to \int_\Omega \Delta u(x,t)\Delta\theta(x)dx \\
&\int_\Omega\int_0^t h(t-s)\Delta u^\mu(x,s)\Delta\theta(x)dxds \to \int_\Omega\int_0^t h(t-s)\Delta u(x,s)\Delta\theta(x)dxds \\
&\int_\Omega |v^\mu(x,t)|^{p-2}v^\mu(x,t)\theta(x)dx \to \int_\Omega |v(x,t)|^{p-2}v(x,t)\theta(x)dx
\end{aligned}
$$

in $\mathcal{C}([0,T])$ and

$$
\begin{aligned}
&\int_\Omega |u_t^\mu(x,t)|^{m-2}u_t^\mu(x,t)\theta(x)dx \to \int_\Omega |u_t(x,t)|^{m-2}u_t(x,t)\theta(x)dx \\
&\int_\Omega |z^\mu(x,1,t)|^{m-2}z^\mu(x,1,t)\theta(x)dx \to \int_\Omega |z(x,1,t)|^{m-2}z(x,1,t)\theta(x)dx
\end{aligned}
$$

in $L^1([0,T])$. Hence, we have

$$
\lim_{\mu\to+\infty}\int_\Omega u_t^\mu(x,t)\theta(x)dx = \int_\Omega u_t(x,t)\theta(x)dx
$$

so (6.31) holds for almost all t in $[0,T]$.

□

Proof. For $T>0$, to prove the existence of local solution of (6.5), we define a class of functions $X_{R,T}$ which consists of all functions w in W satisfying the initial conditions in (6.5) and

$$
\max_{0\le t\le T}\frac{1}{2}\left(\|w_t(t)\|^2 + l\|\Delta w(t)\|^2\right) + \mu_1\int_0^T\int_\Omega |w_t(x,s)|^m dxds + \mu_2\int_0^T\int_\Omega |w_t(x,s-\tau)|^m dxds \le R^2 \tag{6.33}
$$

$X_{R,T}$ is nonempty if R is large enough. This follows from the trace theorem (see [42]). We also define the map f from $X_{R,T}$ into W by $u=f(v)$, where u is the unique solution of the linear problem (6.17). We would like to show, for R sufficiently large and T sufficiently small, that f is a contraction from $X_{R,T}$ into itself.

By using the energy equality we get

$$\begin{aligned}
&\|u_t(t)\|^2 + l\|\Delta u(t)\|^2 + 2\mu_1 \int_0^t \int_\Omega |u_t(x,s)|^m dxds + 2\mu_2 \int_0^t \int_\Omega |u_t(x,s-\tau)|^m dxds \\
&\leq \|u_1\|^2 + l\|\Delta u_0\|^2 + 2b \int_0^t \int_\Omega |v|^{p-1} |u_t(x,s)| dxds \\
&\leq \|u_1\|^2 + l\|\Delta u_0\|^2 + 2b \int_0^t \|u_t(s)\| \|\Delta v(s)\|^{p-1} ds \qquad \forall t \in [0,T],
\end{aligned} \tag{6.34}$$

consequently

$$\|u\|_W^2 \leq C\|u_1\|^2 + l\|\Delta u_0\|^2 + CR^{p-1}T\|u\|_W,$$

where C is independant of R. By choosing r large enough and T sufficiently small, (6.33) is satisfied; hence $u \in X_{R,T}$. This shows that f maps $X_{R,T}$ into itself.

Next we verify that f is a contraction. For this aim we set $U = u - \tilde{u}$ and $V = v - \tilde{v}$, where $u = f(v)$ and $\tilde{u} = f(\tilde{v})$. It is straightforward to see that U satisfies

$$\begin{aligned}
&U_{tt} + \Delta^2 U - \int_0^t h(t-s)\Delta^2 U(s) ds + \mu_1 u_t |u_t|^{m-2} - \mu_1 \tilde{u}_t |\tilde{u}_t|^{m-2} \\
&+ \mu_2 z(1)|z(1)|^{m-2} - \mu_2 \tilde{z}(1)|\tilde{z}(1)|^{m-2} = b|v|^{p-2}v - b|\tilde{v}|^{p-2}\tilde{v}, \\
&U(x,t) = 0, \qquad x \in \partial\Omega, \quad t > 0, \\
&U(x,0) = U_t(x,0) = 0 \qquad x \in \Omega.
\end{aligned} \tag{6.35}$$

We multiply the first equation of (6.35) by U_t and integrate it over $\Omega \times [0,t]$ to get

$$\begin{aligned}
&\frac{1}{2}\|U_t\|^2 + \frac{1}{2}\left(1 - \int_0^t h(s) ds\right) \|\Delta U(t)\|^2 + \frac{1}{2}(h \circ \Delta U)(t) \\
&- \int_0^t \left[\frac{1}{2}(h' \circ \Delta U)(s) - \frac{1}{2} h(s) \|\Delta U(s)\|^2\right] ds \\
&+ \mu_1 \int_0^t \int_\Omega \left(u_t|u_t|^{m-2} - (\tilde{u}_t|\tilde{u}_t|^{m-2}\right) U_t(x,s) dxds \\
&+ \mu_2 \int_0^t \int_\Omega \left(z(1)|z(1)|^{m-2} - \tilde{z}(1)|\tilde{z}(1)|^{m-2}\right) U_t(x,s) dxds \\
&\leq C \int_0^t \int_\Omega \left[|v|^{p-2}v - |\tilde{v}|^{p-2}\tilde{v}\right] U_t(x,s) dxds.
\end{aligned} \tag{6.36}$$

By using (6.15), (6.24), and (6.29), we obtain

$$\|U_t\| + \|\Delta U\| + \int_0^t \int_\Omega \left(|U_t(x,s)|^m + |U_t(x,s-\tau)|^m\right) dxds \leq C \int_0^t \|U_t\| \|\Delta V\| \left(\|\Delta v\|^{p-2} \|\Delta \tilde{v}\|^{p-2}\right) ds. \tag{6.37}$$

Thus we have

$$\|U\|_W^2 \le CTR^{p-2}\|V\|_W^2. \tag{6.38}$$

By choosing T so small that $CTR^{p-2} < 1$, (6.38) shows that f is a contraction. The contraction mapping theorem then guarantees the existence of a unique u satisfying $u = f(u)$. Obviously it is a solution of (6.5). The proof is completed.

□

6.4 Blow-up

Theorem 6.4.1. *Suppose that $m > 1$, $p > \max\{2, m\}$ satisfying (6.15), let $u_0 \in H^4(\Omega) \cap H_0^2(\Omega)$, $u_1 \in H_0^2(\Omega) \cap L^2(\Omega)$ and $f_0 \in \mathcal{C}^1([-\tau, 0], L^2(\Omega))$. Assume further that*

$$E(0) = \frac{1}{2}\|u_1\|_2^2 + \frac{1}{2}\|\Delta u_0\|_2^2 - \frac{b}{p}\|u_0\|_p^p + \xi \int_\Omega \int_0^1 |f_0(x, -\rho\tau|^m \, d\rho \, dx < 0.$$

Then the solution of (6.1) blows up in finite time

$$T^* \le \frac{1-\alpha}{\Gamma\alpha[L(0)]^{\alpha/(1-\alpha)}},$$

where Γ and α are positive constant with $\alpha < 1$ and L is given by (6.41) below.

We set

$$H(t) := -E(t). \tag{6.39}$$

So, by (6.6) and (6.10) we arrive at

$$0 < H(0) \le H(t) \le \frac{b}{p}\|u\|_p^p. \tag{6.40}$$

Now we define the function

$$L(t) := H^{1-\alpha}(t) + \epsilon \int_\Omega u u_t(x, t) dx, \tag{6.41}$$

for ϵ small to be chosen later and

$$0 < \alpha \le min\left\{\frac{(p-2)}{2p}, \frac{(p-m)}{p(m-1)}\right\}. \tag{6.42}$$

By differentiating $L(t)$ and using the first equation in (6.5), we get

$$\begin{aligned}
L'(t) &= (1-\alpha)H^{-\alpha}(t)H'(t) + \epsilon\|u_t\|_2^2 - \epsilon\|\Delta u\|_2^2 + \epsilon \int_0^t h(t-s) \int_\Omega \Delta u(t)\Delta u(s) \, dx \, ds \\
&+ \epsilon b\|u\|_p^p - \epsilon\mu_1 \int_\Omega |u_t|^{m-2} u_t u \, dx - \epsilon\mu_2 \int_\Omega |z(x,1,t)|^{m-2} z(x,1,t) u(x,t) \, dx \\
&= (1-\alpha)H^{-\alpha}(t)H'(t) + \epsilon\|u_t\|_2^2 - \epsilon\|\Delta u\|_2^2 + \epsilon b\|u\|_p^p \\
&- \epsilon\mu_1 \int_\Omega |u_t|^{m-2} u_t u \, dx - \epsilon\mu_2 \int_\Omega |z(x,1,t)|^{m-2} z(x,1,t) u(x,t) \, dx \\
&- \epsilon \int_0^t h(t-s) \int_\Omega \Delta u(t)[\Delta u(t) - \Delta u(s)] \, dx \, ds + \epsilon \int_0^t h(t-s)\|\Delta u(t)\|_2^2 \, ds.
\end{aligned} \tag{6.43}$$

By using Schwarz inequality, (6.43) takes the form

$$
\begin{aligned}
L'(t) \geq\ & (1-\alpha)H^{-\alpha}(t)H'(t) + \epsilon\|u_t\|_2^2 - \epsilon\|\Delta u\|_2^2 + \epsilon b\|u\|_p^p + \epsilon\int_0^t h(t-s)\|\Delta u(t)\|_2^2\,ds \\
& - \epsilon\mu_1\int_\Omega |u_t|^{m-2}u_t u\,dx - \epsilon\mu_2\int_\Omega |z(x,1,t)|^{m-2}z(x,1,t)u(x,t)\,dx \\
& - \epsilon\int_0^t h(t-s)\|\Delta u(t)\|_2\|\Delta u(t)-\Delta u(s)\|_2\,ds.
\end{aligned} \tag{6.44}
$$

We then use (6.8) and Young's inequality to estimate the last three terms:

$$
\mu_1\int_\Omega |u_t(x,t)|^{m-2}u_t(x,t)u(x,t)\,dx \leq \mu_1\Big[\frac{\delta^m}{m}\|u\|_m^m + \frac{(m-1)\delta^{\frac{-m}{m-1}}}{m}\|u_t\|_m^m\Big],
$$

$$
\mu_2\int_\Omega |z(x,1,t)|^{m-2}z(x,1,t)u(x,t)\,dx \leq \mu_2\Big[\frac{\delta^m}{m}\|u\|_m^m + \frac{(m-1)\delta^{\frac{-m}{m-1}}}{m}\|z(x,1,t)\|_m^m\Big],
$$

and

$$
\epsilon\int_0^t h(t-s)\|\Delta u(t)\|_2\|\Delta u(t)-\Delta u(s)\|_2\,ds \leq \epsilon\beta(h\circ\Delta u)(t) + \frac{\epsilon}{4\beta}\int_0^t h(s)\|\Delta u(t)\|_2^2\,ds.
$$

So, (6.44) becomes:

$$
\begin{aligned}
L'(t) \geq\ & (1-\alpha)H^{-\alpha}(t)H'(t) - \epsilon(\mu_1+\mu_2)\frac{(m-1)\delta^{\frac{-m}{m-1}}}{m}\{\|z(x,1,t)\|_m^m + \|u_t(x,t)\|_m^m\} \\
& + \epsilon\|u_t\|_2^2 - \epsilon\left(1-\int_0^t h(s)ds\right)\|\Delta u(t)\|_2^2 + \epsilon b\|u\|_p^p - \epsilon(\mu_1+\mu_2)\frac{\delta^m}{m}\|u\|_m^m \\
& - \epsilon\beta(h\circ\Delta u)(t) - \frac{\epsilon}{4\beta}\int_0^t h(s)\|\Delta u(t)\|_2^2\,ds,
\end{aligned}
$$

by using the Lemma 6.2.3 we obtain:

$$
\begin{aligned}
L'(t) \geq\ & \left[(1-\alpha)H^{-\alpha}(t) - \epsilon(\mu_1+\mu_2)\frac{(m-1)\delta^{\frac{-m}{m-1}}}{mC}\right]H'(t) + \epsilon\|u_t\|_2^2 \\
& - \epsilon\left(1-\int_0^t h(s)ds\right)\|\Delta u(t)\|_2^2 + \epsilon b\|u\|_p^p - \epsilon(\mu_1+\mu_2)\frac{\delta^m}{m}\|u\|_m^m \\
& - \epsilon\beta(h\circ\Delta u)(t) - \frac{\epsilon}{4\beta}\int_0^t h(s)ds\|\Delta u(t)\|_2^2.
\end{aligned} \tag{6.45}
$$

We use (6.6) and (6.39) to substitute for $b\|u\|_p^p$, hence (6.45) becomes

$$
\begin{aligned}
L'(t) \geq & \left[(1-\alpha)H^{-\alpha}(t) - \epsilon(\mu_1+\mu_2)\frac{(m-1)\delta^{\frac{-m}{m-1}}}{mC}\right] H'(t) + \epsilon\|u_t\|_2^2 - \epsilon\left(1 - \int_0^t h(s)ds\right)\|\Delta u(t)\|_2^2 \\
& - \epsilon(\mu_1+\mu_2)\frac{\delta^m}{m}\|u\|_m^m - \epsilon\beta(h\circ\Delta u)(t) - \frac{\epsilon}{4\beta}\int_0^t h(s)ds\|\Delta u(t)\|_2^2 \\
& + \epsilon\left(pH(t) + \frac{p}{2}\|u_t\|_2^2 + \frac{p}{2}\left(1 - \int_0^t h(s)ds\right)\|\Delta u\|_2^2 + \frac{p}{2}(h\circ\Delta u)(t) + \xi p\int_\Omega\int_0^1 |z(x,\rho,t)|^m\, d\rho\, dx\right) \\
\geq & \left[(1-\alpha)H^{-\alpha}(t) - \epsilon(\mu_1+\mu_2)\frac{(m-1)\delta^{\frac{-m}{m-1}}}{mC}\right] H'(t) + \epsilon\left(1+\frac{p}{2}\right)\|u_t\|_2^2 + \epsilon pH(t) \\
& + \epsilon\left(\frac{p}{2}-\beta\right)(h\circ\Delta u)(t) + \epsilon\left(\left(\frac{p}{2}-1\right) - \left(\frac{p}{2}-1+\frac{1}{4\beta}\right)\int_0^t h(s)\, ds\right)\|\Delta u\|_2^2 \\
& - \epsilon(\mu_1+\mu_2)\frac{\delta^m}{m}\|u\|_m^m + \epsilon\xi p\int_\Omega\int_0^t |z(x,\rho,t)|^m\, d\rho\, dx,
\end{aligned} \tag{6.46}
$$

for some β with $0 < \beta < p/2$. By recalling (6.9), the estimate (6.46) reduces to

$$
\begin{aligned}
L'(t) \geq & \left[(1-\alpha)H^{-\alpha}(t) - \epsilon(\mu_1+\mu_2)\frac{(m-1)\delta^{\frac{-m}{m-1}}}{mC}\right] H'(t) + \epsilon\left(1+\frac{p}{2}\right)\|u_t\|_2^2 + \epsilon pH(t) \\
& + \epsilon a_1(h\circ\Delta u)(t) + \epsilon a_2\|\Delta u\|_2^2 - \epsilon(\mu_1+\mu_2)\frac{\delta^m}{m}\|u\|_m^m + \epsilon\xi p\int_\Omega\int_0^t |z(x,\rho,t)|^m\, d\rho\, dx,
\end{aligned} \tag{6.47}
$$

where

$$
a_1 = \frac{p}{2} - \beta, \;\; a_2 = \left(\frac{p}{2}-1\right) - \left(\frac{p}{2}-1+\frac{1}{4\beta}\right)\int_0^t h(s)ds > 0.
$$

Of course (6.47) remains valid even if δ is time dependent. Therefore by taking δ so that

$$
\delta^{\frac{-m}{m-1}} = kH^{-\alpha}(t),
$$

for large k to be specified later, and substituting in (6.47) we arrive at

$$
\begin{aligned}
L'(t) \geq & \left[(1-\alpha)H^{-\alpha}(t) - \epsilon(\mu_1+\mu_2)\frac{(m-1)}{mC}k\right] H^{-\alpha}(t)H'(t) \\
& + \epsilon\left(1+\frac{p}{2}\right)\|u_t\|_2^2 + \epsilon a_1(h\circ\Delta u)(t) + \epsilon a_2\|\Delta u\|_2^2 + \epsilon\xi p\int_\Omega\int_0^t |z(x,\rho,t)|^m\, d\rho\, dx \\
& + \epsilon\left[pH(t) - \frac{k^{1-m}}{m}(\mu_1+\mu_2)H^{\alpha(m-1)}(t)\|u\|_m^m\right].
\end{aligned} \tag{6.48}
$$

Using (6.40) and the inequality $\|u\|_m^m \leq C_1\|u\|_p^m$, we obtain

$$
H^{\alpha(m-1)}(t)\|u\|_m^m \leq \left(\frac{b}{p}\right)^{\alpha(m-1)} C_1\|u\|_p^{m+\alpha p(m-1)}, \;\; \text{for some } C_1 > 0.
$$

Hence (6.48) yields

$$
\begin{aligned}
L'(t) \geq & \left[(1-\alpha)H^{-\alpha}(t) - \epsilon(\mu_1+\mu_2)\frac{(m-1)}{mC}k\right] H^{-\alpha}(t)H'(t) \\
& + \epsilon\left(1+\frac{p}{2}\right)\|u_t\|_2^2 + \epsilon a_1(h\circ\Delta u)(t) + \epsilon a_2\|\Delta u\|_2^2 + \epsilon\xi p\int_\Omega\int_0^t |z(x,\rho,t)|^m\,d\rho\,dx \\
& + \epsilon\left[pH(t) - \frac{k^{1-m}}{m}(\mu_1+\mu_2)\left(\frac{b}{p}\right)^{\alpha(m-1)} C_1\|u\|_p^{m+\alpha p(m-1)}\right].
\end{aligned} \tag{6.49}
$$

To finish the proof we shall use the following Corollary

Corollary 6.4.2. *Let the assumptions of the Lemma 6.2.2 hold. Then we have the following*

$$
\|u\|_p^s \leq C\Big(-H(t) - \|u_t\|_2^2 - (h\circ\Delta u)(t) - \xi\int_\Omega\int_0^1 |z(x,\rho,t)|^m\,d\rho\,dx + \|u\|_p^p\Big), \quad \textit{for all} \quad t\in[0,T),
$$

for any $u(.,t)\in H_0^2(\Omega)$ *and* $2\leq s\leq p$.

By the corollary 6.4.2 and (6.42) for $s = m+\alpha p(m-1)\leq p$, (6.49) becomes:

$$
\begin{aligned}
L'(t) \geq & \left[(1-\alpha) - \epsilon(\mu_1+\mu_2)\frac{(m-1)}{mC}k\right] H^{-\alpha}(t)H'(t) \\
& + \epsilon\left(1+\frac{p}{2}\right)\|u_t\|_2^2 + \epsilon a_1(h\circ\Delta u)(t) + \epsilon a_2\|\Delta u\|_2^2 + \epsilon\xi p\int_\Omega\int_0^1 |z(x,\rho,t)|^m\,d\rho\,dx \\
& + \epsilon\left[pH(t) - ck^{1-m}\left\{-H(t) - \|u_t\|_2^2 - (h\circ\Delta u)(t) + \|u\|_p^p - \xi\int_\Omega\int_0^1 |z(x,\rho,t)|^m\,d\rho\,dx\right\}\right] \\
\geq & \left[(1-\alpha) - \epsilon(\mu_1+\mu_2)\frac{(m-1)}{mC}k\right] H^{-\alpha}(t)H'(t) + \epsilon\left(1+\frac{p}{2}+ck^{1-m}\right)\|u_t\|_2^2 \\
& + \epsilon\left(a_1+ck^{1-m}\right)(h\circ\Delta u)(t) + \epsilon a_2\|\Delta u\|_2^2 + \epsilon(p+ck^{1-m})H(t) - \epsilon ck^{1-m}\|u\|_p^p \\
& + \epsilon\xi(ck^{1-m}+p)\int_\Omega\int_0^1 |z(x,\rho,t)|^m\,d\rho\,dx.
\end{aligned} \tag{6.50}
$$

By noting that

$$
H(t) \geq \frac{b}{p}\|u\|_p^p - \frac{1}{2}\|u_t\|_2^2 - \frac{1}{2}\|\Delta u\|_2^2 - \frac{1}{2}(h\circ\Delta u)(t) - \xi\int_\Omega\int_0^1 |z(x,\rho,t)|^m\,d\rho\,dx,
$$

and writing $p = 2a_3 + (p-2a_3)$, where $a_3 = min\{a_1, a_2\}$, the estimate (6.50) yields

$$
\begin{aligned}
L'(t) \geq & \left[(1-\alpha) - \epsilon(\mu_1+\mu_2)\frac{(m-1)}{mC}k\right] H^{-\alpha}(t)H'(t) \\
& + \epsilon\left(1+\frac{p}{2}+C_1k^{1-m}-a_3\right)\|u_t\|_2^2 + \epsilon\left(a_1+ck^{1-m}-a_3\right)(h\circ\Delta u)(t) \\
& + \epsilon(a_2-a_3)\|\Delta u\|_2^2 + \epsilon\left(\frac{2ba_3}{p} - ck^{1-m}\right)\|u\|_p^p \\
& + \epsilon\xi(ck^{1-m}+p-2a_3)\int_\Omega\int_0^1 |z(x,\rho,t)|^m\,d\rho\,dx + \epsilon(p-2a_3+ck^{1-m})H(t).
\end{aligned}
$$

At this point, we choose k large enough so that

$$\frac{2ba_3}{p} - ck^{1-m} > 0,$$

we pick ϵ small enough so that

$$(1-\alpha) - \epsilon(\mu_1 + \mu_2)\frac{(m-1)}{mC}k > 0,$$

and

$$L(0) = H^{1-\alpha}(0) + \epsilon \int_\Omega u_0 u_1(x)\, dx > 0.$$

$$\begin{aligned} L'(t) \geq & \left[(1-\alpha) - \epsilon(\mu_1 + \mu_2)\frac{(m-1)}{mC}k\right] H^{-\alpha}(t)H'(t) \\ & + \epsilon\gamma\left[H(t) + \|u_t\|_2^2 + \|u\|_p^p + (h \circ \Delta u)(t) + \xi \int_\Omega \int_0^1 |z(x,\rho,t)|^m \, d\rho \, dx\right], \end{aligned} \tag{6.51}$$

Thus, for some $\gamma > 0$, estimate (6.51) becomes

$$L'(t) \geq \gamma\left[H(t) + \|u_t\|_2^2 + \|u\|_p^p + (h \circ \Delta u)(t) + \xi \int_\Omega \int_0^1 |z(x,\rho,t)|^m \, d\rho \, dx\right]. \tag{6.52}$$

Consequently we have

$$L(t) \geq L(0) > 0, \quad \text{for all} \quad t \geq 0.$$

Now we estimate

$$\left|\int_\Omega uu_t(x,t)dx\right| \leq \|u\|_2\|u_t\|_2 \leq C\|u\|_p\|u_t\|_2$$

which implies

$$\left|\int_\Omega uu_t(x,t)dx\right|^{1/(1-\alpha)} \leq C\|u\|_p^{1/(1-\alpha)}\|u_t\|_2^{1/(1-\alpha)}$$

and by Young's inequality we obtain

$$\left|\int_\Omega uu_t(x,t)dx\right|^{1/(1-\alpha)} \leq C\left[\|u\|_p^{\mu/(1-\alpha)} + \|u_t\|_2^{\theta/(1-\alpha)}\right]$$

for $1/\mu + 1/\theta = 1$ we take $\theta = 2(1-\alpha)$ te get $\mu/(1-\alpha) = 2/(1-2\alpha) \leq p$ by (6.42)

$$\left|\int_\Omega uu_t(x,t)dx\right|^{1/(1-\alpha)} \leq C\left[\|u\|_p^s + \|u_t\|_2^2\right]$$

where $s = 2/(1-2\alpha) \leq p$. By using Corollary 6.4.2 we obtain:

$$\left|\int_\Omega uu_t(x,t)dx\right|^{1/(1-\alpha)} \leq C\left[H(t) + \|u_t\|_2^2 + \|u\|_p^p + (h \circ \Delta u)(t) + \xi \int_\Omega \int_0^1 |z(x,\rho,t)|^m \, d\rho \, dx\right].$$

Therefore,

$$\begin{aligned} L^{1/(1-\alpha)}(t) &= \left(H^{1-\alpha}(t) + \epsilon \int_\Omega u u_t(x,t)dx\right)^{1/(1-\alpha)} \\ &\leq 2^{1/(1-\alpha)} \left(H(t) + \left|\int_\Omega u u_t(x,t)dx\right|^{1/(1-\alpha)}\right) \\ &\leq C\left[H(t) + \|u_t\|_2^2 + \|u\|_p^p + (h \circ \Delta u)(t) + \xi \int_\Omega \int_0^1 |z(x,\rho,t)|^m \, d\rho \, dx\right], \quad t \geq 0. \end{aligned} \tag{6.53}$$

By combining (6.52) and (6.53) we arrive at

$$L'(t) \geq \Gamma L^{1/(1-\alpha)}(t), \quad t \geq 0, \tag{6.54}$$

where Γ is a positive constant depending only on ϵ, γ and C.

A simple integration of (6.54) yields

$$L^{\alpha/(1-\alpha)}(t) \geq \frac{1}{L^{-\alpha/(1-\alpha)}(0) - \Gamma t \alpha/(1-\alpha)}.$$

This concludes the proof of Theorem 6.4.1.

CONCLUSION AND PROSPECTS

We have studied in the last part of this thesis the case when the solution of ordinary differential equation (ODE) or partial differential equation (PDE) diverges at a finite time. Such a phenomenon is said to be blow-up phenomenon.

If blow-up does occur in a finite time, one may further ask: What is the set of the blow-up points? What is the blow-up rate of the solution when time approaches the blow-up time? These problems, which are also important from the point of view of applications.

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