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University Mustapha Stambouli
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كلية العلوم الدقيقة
قسم الرياضيات

Doctorate Thesis
Specialty: Mathematics
Option: Harmonic analysis

Title:

Harmonic geometry of structures on tangent bundle

Presented by: Medjadj Abdallah
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جامعة مصطفى إسطمبولي
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Titre:

Géométrie harmoniques des structures sur le fibré tangent

Presenter par: Medjadj Abdallah
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شكر و تقدير

بسم الله الرحمن الرحيم

الْحَمْدُ لِلَّهِ الَّذِي هَدَانَا لِهَٰذَا وَمَا كُنَّا لِنَهْتَدِيَ لَوْلَا أَنْ هَدَانَا اللَّهُ

أتقدم بجزيل الشكر إلى الأستاذ الهندي هشام على موضوع الأطروحة، ارشاداته، تعاليمه، نصائحه و تشجيعاته الثمينة خلال هذه السنوات التي ساعدتني على إتمام هذا العمل. أشكر كذلك الأستاذ قاسيمي بوعزة على نصائحه، كما أشكر الاساتذة: محمد شريف أحمد ، زبير حنيفي،زعقان عبدالرحيم و بوزير حبيب على تقييمهم للأطروحة و الإرشادات المقدمة. أشكر كذلك كل الاساتذة الذين درسوني و الهموني لطلب العلم.

أهدي هذا العمل و أشكر عائلي بدأ بأمي براك الله فيها و لأبي رحمه الله على مساندتهما و على تضحياتهما. أشكر اخوتي و أخواتي إيمان، عمر، جمال الدين و نبيلة على تشجيعات و الدعم.

دون أن أنسى الأصدقاء: فوزي، بلال، عمر ، عبد الرحمن ، حسين ، مهدي و عائش على دعمهم و تشجيعهم.

و لكل من شجعني و لو بكلمة طيبة.

Contents

Publications

1. Medjadj. A, Elhendi. H, Belarbi. L *Some biharmonic problems on the tangent bundle with a Berger-type deformed Sasaki metric*, journal of the Indian Math. Soc. Vol. 88, Nos. (2021)

2. (submitted) Medjadj. A, Elhendi. H, *Harmonic Maps Between Berger-Type Deformed Sasaki Metric*

3. (submitted) Elhendi. H, Medjadj. A, *Isotropic almost complex structure of Cheeger-Gromoll metrico and harmonic unit vector fields*

4.

Introduction

In this thesis, we study the harmonicity on tangent bundle equipped with different types of structures, for this purpose we are first give some important definitions and well known results that are gonna be used in our study.

The differential geometry of the tangent bundle TM of Riemannian manifold (M, g) was first studied by Sasaki. S in hid paper published in 1958 [30], where he used the Levi-Civita and the metric on the manifold to define the horizontal and vertical lift of vector field noted respectively X^H and X^V by splitting TTM into a horizontal and vertical part, he also defined locally what was later generalized and we now know as the Sasaki metric g^s on TM:

$$\begin{aligned} g^s(X^H, Y^H) =& g^s(X^V, Y^V) = g(X, Y) \circ \pi, \\ g^s(X^V, Y^H) =& 0, \end{aligned}$$

where π is the canonical projection from TM to M.

In 1962 Dombrowski. P [14], gave a definition of the horizontal and vertical part of TTM using the projection map π and the connection map K while proving that they are independent of the connection on M.
In the same paper he also introduced the almost complex structure J on TM defined by $JX^V = -X^H,\ JX^H = X^V$, and showed that it is complex if and only if (M^n, g) is flat.
Dombrowski also calculated the Lie bracket on TM in the same paper.

$$\begin{aligned} [X^H, Y^H] =& [X, Y]^H - (R(X, Y)u)^V, \\ [X^H, Y^V] =& (\nabla_X Y)^V, \\ [X^V, Y^V] =& 0. \end{aligned}$$

In 1962 Tachibana. S studied the Almost-complex structure of tangent bundles of Riemannian spaces [32], he showed that the tangent bundle of any non-flat Riemannian

space admits an almost-K$\ddot{a}$hlerian structure which is not K$\ddot{a}$hlerian.
Starting from there in 1966 Yano. K, Kobayashi. S and Ishihara. S started developing the theory of vertical, complete and horizontal lift [34].

Although the Sasaki metric is naturally defined, in 1988 Musso. E and Tricerri. F [24], have shown that the Sasaki metric has constant scalar curvature if and only if (M, g) is locally Euclidian. In the same paper they gave an explicit expression of a complete metric g^{CG} on TM introduced by Cheeger and Gromoll uniquely determined at the point (p, u) by

$$\begin{aligned} g^{CG}(X^H, Y^H) =& g(X, Y) \circ \pi, \\ g^{CG}(X^V, Y^H) =& 0, \\ g^{CG}(X^V, Y^V) =& \frac{1}{1 + ||u||^2}\big(g_p(X, Y) + g_p(X, u)g_p(Y, u)\big), \end{aligned}$$

for all $X, Y \in \Gamma(TM)$.

In 1995 Cruceanu. V, Fortuny. P and Gadea. P.M [11], gave some properties on paracomplex geometry on a differential manifold.

In 2011 Yampolsky. A studied Geodesics of Tangent Bundle with Fiberwise Deformed Sasaki Metric over K$\ddot{a}$hler Manifold.

In 2012 Salimov. A, Gezer. A and Iscan. M studied para-K$\ddot{a}$hler-Norden structures on the tangent bundles [28].

In 2019 Altunbas. M, Simsek. R, and Gezer. A [3], gave the geometry of the tangent bundle equipped with the Berger type deformed Sasaki metric defined at a point $(p, u) \in TM$ by

$$\begin{aligned} g^{BS}_{(p,u)}(X^H, Y^H) =& g_p(X, Y), \\ g^{BS}_{(p,u)}(X^H, Y^V) =& 0, \\ g^{BS}_{(p,u)}(X^V, Y^V) =& g_p(X, Y) + \delta^2 g_p(X, \phi u)g_p(Y, \phi u), \end{aligned}$$

for all vector fields X, Y on M_{2n} , where δ is some constant and ϕ an almost paracomplex structure compatible with g on M and some almost paracomplex structures with anti-paraHermitian metrics on the tangent bundle.

In 1996 Aguilar. R.M [2], defined the isotropic almost complex structure $J_{\delta,\sigma}$ to be an almost complex structure with respect to a Riemannian metric g on M there are functions $\alpha, \delta, \sigma : TM \to \mathbb{R}$ with $\alpha\delta - \sigma^2 = 1$, such that for all $X \in TM$

$$\begin{aligned} J_{\delta,\sigma}X^H =& \alpha X^V + \sigma X^H, \\ J_{\delta,\sigma}X^V =& -\delta X^H - \sigma X^V. \end{aligned}$$

Aguilar. R.M proved in the same paper that isotropic complex structure exist when M has constant sectional curvature.

The harmonicity of sections on tangent bundle TM equipped withe diagonal metric g^D was first studied in 1979 by Ishihara. T [18], where he proved that the natural projection $\pi : TM \longrightarrow M$ is a total geodesic submersion, he also proved that if M is a compact and orientable manifold then the vector field X witch is a map $X : M \to TM$ is harmonic if and only if the first covariant derivative of X vanishes.

In 1992, Konderak. J [21], gave a simple proof that a vector field $X : M \to TM$ on a compact Riemannian manifold is harmonic with respect to the Sasaki metric on TM if and only if X is parallel.

Using the article by Altunbas. M, Simsek. R, and Gezer. A [3] we studied In 2020 Medjadj. A, Elhendi. H and Belarbi. L studied the harmonicity and the biharmonicity of vector fields $X : (M, \phi, g) \to (TM, \widetilde{\phi}, g^{BS})$.

In 2016 Baghban. A and Abedi. E [5], studied integrability of the isotropic almost complex structures and harmonic unit vector in tangent bundle and unit vector fields.

In 2021 Medjadj. A and Elhendi. H, equipped the tangent bundle with the gradient Sasaki metric and a paracomplex structure to study the integrability of the paracomplex structure, we also proved that a vector field $X : M \to TM$ is harmonic if and only if X is parallel.

In 2021 we constructed a Cheeger-Gromoll isotropic almost complex structure and studied the Harmonicity of vector fields and unit vector fields.

This thesis is divided in four chapter.

In chapter one we give important definitions and result in differential an Riemannian geometry witch are gonna be used (Differential manifold, Riemannian manifold, Tangent bundle, Kähler manifold...) and some property in those spaces. We also add some important definition about harmonic an bi-harmonic maps.

In the second chapter we will first talk about geometry on the tangent bundle and give main definitions and result about lift theory, then we we will give a quick review of the geometry of tangent bundle equipped with Sasaki and Cheeger-Gromoll metric, and give the geometrical structure of the tangent bundle equipped with de Berger type deformed metric g^{BS} and then the gradient Sasaki metric g_f. We also construct a Cheeger-Gromoll isotropic almost complex structure and calculate the Levi-Civita connection.

The third chapter is dedicated to the study of harmonicity an biharmonicity of vector fields for all the structures mentioned in chapter two namely: Tangent bundle equipped with the gradient Sasaki metric and an almost complex structure. Tangent bundle equipped with Berger type deformed Sasaki metric $(TM, \widetilde{\phi}, g^{BS})$. Tangent bundle equipped with Cheeger-Gromoll isotropic almost structure.

Finally in chapter 4 we study the harmonicity and biharmonicity of maps between tangent bundle.

Chapter 1

Riemannian Geometry

1.1 Differentiable Manifolds

We start this chapter with some basics definitions and result concerning structures on a topological manifold in order to define what is known as a differentiable and smooth manifold.

1.1.1 Topological Manifolds

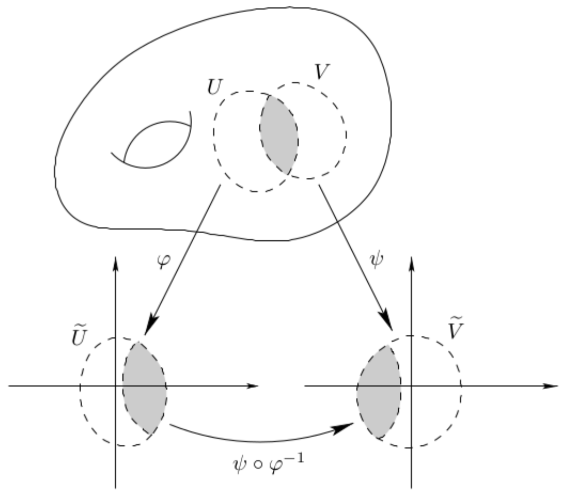
Let M be a topological space. We say that M is a topological n-manifold if it has the following properties:

- M is a Hausdorff space.
- M is second countable: There exists a countable basis for the topology of M.
- M is locally Euclidean of dimension n: Every point has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$.

Definition 1.1.1. *Let M be a topological n-manifold. A chart on M is a pair (U, φ), where U is an open subset of M and $\varphi : U \to \varphi(U) \in \mathbb{R}^n$ a homeomorphism. M have always the same dimension of $\mathbb{R}^n$.*
The map φ is called a local coordinate map, and the component functions of φ are called local coordinates on U.

Let M be a topological n-manifold. If (U, φ), (V, ψ) are two charts such that $U \cap V \neq \emptyset$, then the map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is itself a homeomorphism.

Definition 1.1.2. *Two charts (U, ϕ) and (V, ψ) are said to be smoothly compatible if either $U \cap V = \emptyset$ or the transition map $\psi \circ \varphi^{-1}$ is a diffeomorphism.*



Definition 1.1.3. *An atlas $\mathcal{A}$ for M is a collection of charts whose domains cover M. An atlas $\mathcal{A}$ is called a smooth atlas if any two charts in $\mathcal{A}$ are smoothly compatible with each other.*

Definition 1.1.4. *A smooth atlas $\mathcal{A}$ on M is maximal if it is not contained in any strictly larger smooth atlas. This means that every chart that is smoothly compatible with every chart in $\mathcal{A}$ is already in $\mathcal{A}$.*

Definition 1.1.5. *A smooth structure on a topological n-manifold M is a maximal smooth atlas.*

Definition 1.1.6. *A smooth manifold is a pair $(M, \mathcal{A})$, where M is a topological manifold and $\mathcal{A}$ is a smooth structure on M.*
We usually just say M is a smooth (differentiable) manifold.

Definition 1.1.7. *Let U be an open set of a differentiable manifold M. A map $f : U \to \mathbb{R}^n$ is differentiable if for every $x \in U$ there is a chart $\{U, \varphi\}$ with $x \in U$ such that $f \circ \varphi^{-1}$ is differentiable on $\varphi(U)$.*

Lemma 1.1.1. *Let (U_i, φ_j) be a smooth atlas for M. If $f : M \to \mathbb{R}^n$is a function such that $f \circ \varphi_i^{-1}$ is differentiable for each i, then f is smooth.*

Definition 1.1.8. *A map $f : M \to N$, where M and N are two differential manifold and let (U, φ) and (V, ψ) two chart on M and N respectively, f is said to be differentiable if the map $\psi \circ f \circ \varphi^{-1}$ is differentiable from $\phi(U \cap f^{-1}(V))$ to $\psi(V)$.*
f is a diffeomorphisme if f and f^{-1} are differentiable.

Lemma 1.1.2. *Let M , N be smooth manifolds and let $f : M \to N$ be any map. If $\{(U_i, \phi_i)\}$ and $\{(V_j, \psi_j)\}$ are smooth atlases for M and N, respectively, and if for each i and j, $\psi_j \circ f \circ \varphi_i^{-1}$ is smooth on its domain of definition, then f is smooth.*

Lemma 1.1.3. *Any composition of smooth maps between manifolds is smooth.*

Proof. A differentiable maps $f : M \to N$ and $g : N \to P$, let (U, φ) and (V, ψ) be any charts for M and P respectively.
$\forall p \in U \cap (g \circ f)^{-1}(V)$, there is a chart (W, μ) for N such that $f(p) \in W$.
f and G are differentiable then $\mu \circ f \circ \phi^{-1}$ and $\psi \circ g \circ \mu^{-1}$ are differentiable then $(\psi \circ g \circ \mu^{-1}) \circ (\mu \circ f \circ \phi^{-1}) = \psi \circ g \circ f \circ \phi^{-1}$ is differentiable. □

1.1.2 Oriented manifold

Definition 1.1.9. *An atlas $\mathcal{A} = \{(U_i, \varphi_i)\}$ for a differential manifold M is said to be an orientation atlas if*

$$Jac(\phi_{ij})_p = det(d_{\varphi_j(p)}\phi_{ij}) > 0, \ \forall p \in U_j.$$

Where ϕ_{ij} are the transition maps ($\phi_{ij} = \varphi_i \circ \varphi_j^{-1}$).

Definition 1.1.10. *An oriented manifold is a smooth manifold with maximal oriented atlas.*

1.2 Tangent and cotangent bundle

Vectors are used to talk about direction and calculating distances using its magnitude, and so they are useful to study the behavior of functions. Here we will give their generalization in manifolds, in the sens that a vector at a point associate to every function its derivative in the direction of that vector.

Definition 1.2.1. *Let M be a smooth manifold of dimension n, a tangent vector X_p at a point $p \in M$ is a map which associate to every differentiable function f defined at p a number $X_pf \in \mathbb{R}$, such that for every f and g two differential function at p this map satisfy the flowing:*

1. *If f is constant in the neighborhood of p implies $X_pf = 0$.*
2. *$X_p(f + g) = X_pf + X_pg$.*
3. *$X_p(fg) = (X_pf)g(p) + (X_pg)f(p)$.*

Definition 1.2.2. *The set of all tangent vectors at p are denoted T_pM and is called the tangent space of M at p.*

By defining for every two tangent vectors X_p and Y_p and for every $\alpha \in \mathbb{R}$ the following operation:

$$(X_p + Y_p)f = X_pf + Y_pf,$$
$$(\alpha X_p)f = \alpha(X_pf).$$

Proposition 1.2.1. *Let (U, φ) be a chart on a differential manifold M of dimension n. For $p \in U$ we put $x = \varphi(p)$ ans for every differentiable function f at p the maps X_p^i defined by:*

$$X_p^i f = \frac{\partial(f \circ \varphi^{-1})}{\partial x_i},\ i = 1, ..., n \tag{1.1}$$

are all tangent vectors at p.

Theorem 1.2.1. *Let M be an n-dimensional differential manifold then the tangent space T_pM is an n-dimensional vector space with basis $\left\{(\frac{\partial}{\partial x_1})_p, ...(\frac{\partial}{\partial x_n})_p\right\}$, where x_1, x_2,...,x_n are local coordinates on a chart (U, φ) around $p \in M$, so for $X_p \in T_pM$: $X_p = X_p(x_i)(\frac{\partial}{\partial x_i})_p$.*

Definition 1.2.3. *Let $f : M \to \mathbb{R}$ be a differentiable map. We define the differential df_p by*

$$\begin{aligned} df_p : T_pM &\longrightarrow \mathbb{R} \\ X_p &\longmapsto df_p(X) = X_p(f). \end{aligned}$$

Definition 1.2.4. *Let $f : M \to N$ be a differentiable map then for $p \in M$ the differential df_p is the map $df_p : T_pM \to T_{f(p)}N$ defined by*

$$df_p(X_p)(g) = X_p(g \circ f),$$

for $g \in C^\infty(f(p))$, and $X_p \in T_pM$.

1.2.1 Tangent bundle

Definition 1.2.5. *Let M be a differential manifold, the space TM defined by:*

$$TM = \bigcup_{p \in M} T_pM,$$

is called the tangent bundle of M.

If $V \in TM$ we write $V = (p, u)$ for some $p \in M$, and $u \in T_pM$.

Remark 1.2.1. *Let $(M, \mathcal{A})$ be a differentiable manifold. The tangent bundle (TM, π, M) of M is given by $TM = \{(p, u) | p \in M, u \in T_pM\}$ and the bundle map*

$$\begin{aligned} \pi : TM &\longrightarrow M \\ (p, u) &\longmapsto \pi(p, u) = p, \end{aligned}$$

is called the natural projection of TM.
A local chart $(U, x^i)_{i=1...n}$ on M induces a local chart $(\pi^{-1}(U), x^i, y^i)_{i=1...n}$ on TM.

Theorem 1.2.2. *Let M an n-dimensional differential manifold, then the tangent bundle TM is a $2n$-dimensional differentiable manifold.*

Proof. First we define a smooth charts, given any chart (U, φ) for M , the component functions of φ are $(x^1(p), ..., x^n(p))$.
We also define a map $\Phi : \pi^{-1}(U) \to \mathbb{R}^{2n}$ by $\Phi(v^i \frac{\partial}{\partial x^i}|_p) = (x^1(p), ..., x^n(p), u^1, ..., u^n)$, it's a bijection into $\varphi(U) \times \mathbb{R}^{2n}$.
We take now two chart on M (U, φ) and (V, ψ), where the component functions of φ are $(y^1(p), ..., y^n(p))$. The corresponding chart on TM $(\pi^{-1}(U), \Phi)$ and $(\pi^{-1}(V), \Psi)$.
We then can write the transition map

$$\Psi \circ \Phi^{-1} : \varphi(U \cap V) \times \mathbb{R}^n \to \psi(U \cap V) \times \mathbb{R}^n$$

given by

$$\Psi \circ \Phi^{-1}(x^1, ..., x^n, u^1, ..., u^n) = (\psi \circ \varphi^{-1}(p), \sum_{k=1}^{n} \frac{\partial y^1}{\partial x^k}(\varphi^{-1}(p))u^k, ..., \sum_{k=1}^{n} \frac{\partial y^n}{\partial x^k}(\varphi^{-1}(p))u^k).$$

Since $\psi \circ \varphi^{-1}$ is smooth then $\Psi \circ \Phi^{-1}$ is smooth. Hence we can define a smooth atlas on TM, therefore TM is a differential manifold.
Let now $\{U_i\}$ be a countable cover of M, we obtain a countable cover of TM by coordinate domains $\{\pi^{-1}(U_i)\}$.
Note that any two points in the same fiber of π lie in one chart, while if (p, X) and (q, Y) lie in different fibers there exist disjoint coordinate domains U_i , U_j for M such that $p \in U_i$ and $q \in U_j$, and then the sets $\pi{-}1(U_i)$ and $\pi^{-1}(U_j)$ are disjoint coordinate neighborhoods containing (p, X) and (q, Y), respectively. Then, TM is Hausdorff.
π is smooth, because its coordinate representation with respect to charts (U, φ) for M and $(\pi^{-1}(U), \Phi)$ for TM is $\pi(x, u) = x$. □

1.2.2 Cotangent bundle

Definition 1.2.6. *Let M be an n-dimensional smooth manifold. Then for each $p \in M$ the dual T_p^*M of the tangent space T_pM is called the cotangent space at p that is*

$$T_p^*M = \{\omega_p : T_pM \to \mathbb{R} | \omega_p \textit{ is linear}\}.$$

Definition 1.2.7. *For a differentiable manifold M we define $T^*M = \bigcup_{p \in M} T_p^*M$ and call it the cotangent bundle of M.*

Definition 1.2.8. *A 1-form is a differentiable map $\omega : M \to T^*M$ such that $\omega_p \in T_p^*M$ for every $p \in M$. The set of all 1-form on M is noted $\Gamma(T^*M)$.*

1.3 Vector fields and connection

Definition 1.3.1. *A vector field on M is a map*

$$\begin{aligned} X : & M \longrightarrow TM \\ & p \longmapsto X_p, \end{aligned}$$

such that $\pi \circ X = Id_M$.

Every vector field can be written as follow: $X_p = X_i(p)\dfrac{\partial}{\partial x_i}$.
$\frac{\partial}{\partial x_i}$ being a local coordinate base and X_i are functions into $\mathbb{R}$ called component functions of X.
We note the set of all vector field on M by $\Gamma(TM)$.

Proposition 1.3.1. *X is differentiable if and only if X_i are differentiable.*

Proposition 1.3.2. *Let X and Y two vector fields on M and $f : M \to \mathbb{R}$ a function. $X + Y$ and fX are vector fields defined by:*

$$\begin{aligned} (X+Y)_p =& X_p + Y_p, \\ (fX)_p =& f(p)X_p, \end{aligned}$$

are also vector fields.

Definition 1.3.2. *For every two vector fields X and Y, we define the Lie bracket of X and Y by:*

$$[X, Y] = X \circ Y - Y \circ X.$$

Where $X : C^\infty(M) \to C^\infty(M)$.

Proposition 1.3.3. *The lie bracket of two vector field is also a vector field.*

Proposition 1.3.4. *Let X, Y and Z three vector fields of M, $a, b \in \mathbb{R}$ and f, g two differentiable functions on M, we have:*

$$\begin{aligned} & [X, Y] = -[Y, X], \\ & [aX + bY, Z] = a[X, Z] + b[Y, Z], \\ & [[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0 \quad (identity\ of\ Jacobi), \\ & [fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X, \end{aligned}$$

Definition 1.3.3. *Let X, Y and Z be three differentials vector fields and f ,g two differentiable function on $\mathbb{R}$.*
An affine connection is a map that to each X and Y it give a vector field noted $\nabla_X Y$ that satisfy the following conditions:

1. $\nabla_{fX+gY}Z = f\nabla_X Z + g\nabla_Y Z$.
2. $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$.
3. $\nabla_X fY = f\nabla_X Y + X(f)Y$.

Definition 1.3.4. *Let M be a smooth manifold and ∇ a linear connection on M. then the torsion of ∇ noted T is a $C^\infty(M)$-bilinear map defined by*

$$T : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$$
$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

The connection is said to be torsion free if:

$$\nabla_X Y - \nabla_Y X = [X,Y].$$

1.4 Pull back vector field

Definition 1.4.1. *Let $\psi : M \to N$ be a smooth map between two differentials manifolds, the pull-back bundle is defined by:*

$$\psi^{-1}TN = \{(x,u)\ , x \in M, u \in T_{\psi(x)}N\}.$$

A vector filed X on $\psi^{-1}TN$ is a smooth map between M and TN such that for every $x \in M$ we have $X(x) \in T_{\psi(x)}N$.

Definition 1.4.2. *Let $\psi : M \to N$ be a smooth map between two differentials manifolds. The pull-back connection is a map $\nabla^\psi : \Gamma(TM) \times \Gamma(\psi^{-1}TN) \to \Gamma(\psi^{-1}TN)$ such that:*

$$\nabla^\psi_X(Y \circ \psi) = \nabla^N_{d\psi(X)}Y,$$

for $X \in \Gamma(TM)$, $Y \in \Gamma(TN)$ and ∇^N is a connection on N.

Proposition 1.4.1. *Let ψ be a smooth map between two differentials manifolds and ∇^N a torsion free connection on N then we have*

$$\nabla^\psi_X d\psi(Y) = \nabla^\psi_Y d\psi(X) + d\psi([X,Y]),$$

for every $X, Y \in \Gamma(TM)$.

Proof. Let $X, Y \in \Gamma(TM)$ and $N, M \in \Gamma(TN)$ such that $d\psi(X) = V \circ \psi$ and $d\psi(Y) = W \circ \psi$ then we get that:

$$\begin{aligned}\nabla^\psi_X d\psi(Y) &= \nabla^\psi_X(W \circ \psi) = (\nabla^N_V W) \circ \psi = ([V,W] + \nabla^N_W V) \circ \psi \\ &= d\psi([X,Y]) + (\nabla^N_d \psi(Y) d\psi(X)) = d\psi([X,Y]) + \nabla^\psi_Y d\psi(buX).\end{aligned}$$

□

1.4.1 Second fundamental form

Definition 1.4.3. *Let M, N two differential manifolds and $\psi \in C^\infty(M,N)$, the second fundamental form of ψ is defined by*

$$\nabla d\psi(X,Y) = \nabla^\psi_X d\psi(Y) - d\psi(\nabla_X Y), \tag{1.2}$$

for every $X, Y \in \Gamma(TM)$.

For local coordinate $(x_1, ..., x_n)$ on M and $(y_1, ..., y_m)$ on N we have

$$(\nabla d\psi)_{ij} = \nabla d\psi(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = \psi^k_{ij}\frac{\partial}{\partial y^k}.$$

Where $\dfrac{\partial \psi^k}{\partial x_i \partial x_j}$.

Proposition 1.4.2. *Let $\psi : M \to N$ a differential map.bu The second fundamental form of ψ is linear and symmetric.*

Proof. The linearity is obvious.
For the symmetry we use proposition 1.4.1 and we have:

$$\begin{aligned}
\nabla d\psi(X,Y) &= \nabla^\psi_X d\psi(Y) - d\psi(\nabla^M_X Y) \\
&= \nabla^\psi_Y d\psi(X) + d\psi([X,Y] - \nabla^M_X Y) \\
&= \nabla^\psi_Y d\psi(X) - d\psi(\nabla^M_Y X) \\
&= \nabla d\psi(Y,X).
\end{aligned}$$

□

1.5 Riemannian Manifolds

Definition 1.5.1. *Let M be a smooth manifold. A tensor field T of type (r,s) is a map*

$$T : \underbrace{\Gamma(TM) \otimes ... \otimes \Gamma(TM)}_{r \text{ copies}} \longrightarrow \underbrace{\Gamma(TM) \otimes ... \otimes \Gamma(TM)}_{s \text{ copies}},$$

satisfying
$T(X_1 \otimes ... \otimes (fX_i + gY) \otimes ... \otimes X_r) = fT(X_1 \otimes ... \otimes X_r) + gT(X_1 \otimes ... \otimes Y \otimes ... \otimes X_r)$
For all $X_i, Y \in \Gamma(TM)$, $f, g \in C^\infty(M)$ and $i = 1, ...r$.

From now on $T(X_1 \otimes ... \otimes X_r)$ will be noted $T(X_1, ..., X_r)$.

Definition 1.5.2. *The tensor field T of type (r, s) is said to be smooth if for all $X_1, ..., X_r \in \Gamma(TM)$ the map*

$$\begin{aligned} T(X_1, ..., X_r) : M &\longrightarrow \overbrace{\Gamma(TM) \otimes ... \otimes \Gamma(TM)}^{s \text{ copies}} \\ p &\longmapsto T_p((X_1)_p, ..., (X_r)_p) \end{aligned}$$

is smooth.

Definition 1.5.3. *A Riemannian metric or Reimannian structure on a differential manifold M is a map noted g, defined by*

$$\begin{aligned} g : \Gamma(TM) \times \Gamma(TM) &\longrightarrow C^{\infty}(M) \\ (X, Y) &\longmapsto g(X, Y), \end{aligned}$$

such that g is a symmetric, bi-linear, positive defined form on M.
For any local coordinate system x^i on M the metric g can be written:

$$g = g_{ij} dx^i \otimes dx^j,$$

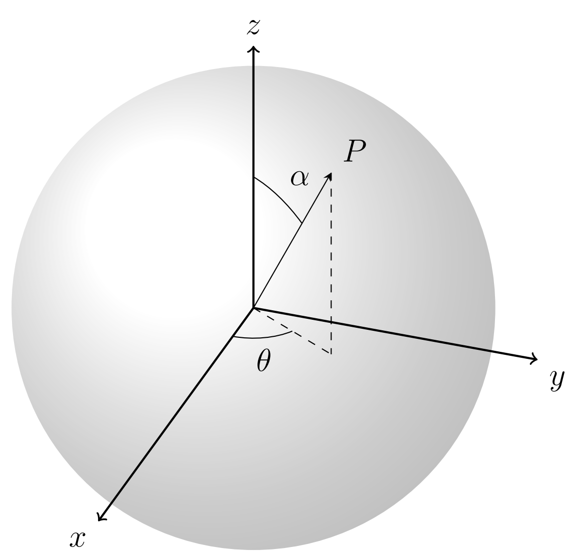
where g_{ij} is a symmetric defined positive matrix of smooth functions.

A Riemannian manifold is a couple (M, g) such that M is a smooth manifold and g is a Riemannian metric.

Example 1.5.1. *In $\mathbb{R}^2$ we have that the Euclidean metric g is given by*

$$g = dx^2 + dy^2.$$

Example 1.5.2. *In the unit sphere $\mathbb{S}^2$ on $\mathbb{R}^3$ we have the following coordinate system for every point $P \in \mathbb{S}^2$ given by $\Big(sin(\alpha)cos(\theta); sin(\alpha)sin(\theta); cos(\alpha)\Big)$ for $\alpha \in [0, \pi]$ and $\theta \in [0, 2\pi]$.*



and we have

$$\frac{\partial P}{\partial \theta} = (-sin(\alpha)sin(\theta); sin(\alpha)cos(\theta), 0),$$
$$\frac{\partial P}{\partial \alpha} = (cos(\alpha)cos(\theta); cos(\alpha)sin(\theta), -sin(\alpha)),$$

and so the metric is given by:

$$g = d\alpha^2 + sin^2(\alpha)d\theta^2.$$

Definition 1.5.4. *The volume measure on v^g on (M^m, g) is defined by:*

$$v^g = \sqrt{det(g_{ij})}dx_1 \wedge dx_2 \wedge ... \wedge dx_m.$$

Example 1.5.3. *From the example 1.5.2, we have that $v^g = |sin(\alpha)|d\alpha \wedge d\theta$.*

Definition 1.5.5. *Let (M, g) be a Riemannian manifold, the connection ∇ is said to be compatible with g if:*

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z),$$

for all $X, Y, Z \in \Gamma(TM)$.

Theorem 1.5.1. *Let (M, g) be an n-dimensional Riemannian manifold and the map $\nabla^g : \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM)$ given by the Koszul formula:*

$$g(\nabla^g_X Y, Z) = \frac{1}{2}(Xg(Y, Z) + Yg(Z, X) - Zg(X, Y)) \quad (1.3)$$
$$+g(Z, [X, Y] + g(Y, [Z, X]) - g(X, [Y, Z])), \quad (1.4)$$

for all $X, Y,$ and $Z \in \Gamma(TM)$, then ∇ is a connection on M.

Definition 1.5.6. *The connection ∇^g on (M, g) defined in the Theorem 1.5.1 is called the Levi-Civita connection of g.*

Remark 1.5.1. *Let $\{\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial_m}\}$ a local base associated to a chart (U, ϕ) such that: $X = X_i \frac{\partial}{\partial x_1}$ and $Y = Y_j \frac{\partial}{\partial x_j}$,*

$$\begin{aligned}
\nabla_X Y =& \nabla_{X_i \frac{\partial}{\partial x_i}} Y_j \frac{\partial}{\partial x_j} \\
=& X_i \nabla_{\frac{\partial}{\partial x_i}} Y_j \frac{\partial}{\partial x_j} \\
=& X_i Y_j \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} + X_i \frac{\partial}{\partial x_i}(Y_i) \frac{\partial}{\partial x_j} \\
=& X_i Y_j (\sum_{k=1}^{m} \Gamma^k_{ij} \frac{\partial}{\partial x_k}) + X(Y_j) \frac{\partial}{\partial x_j}
\end{aligned}$$

$$=\sum_{k=1}^{m}\Big(\sum_{i,j=1}^{m}X_iY_j\Gamma_{ij}^{k}+X(Y_k)\Big)\frac{\partial}{\partial x_k},$$

such that Γ_{ij}^k *are the Christoffel symbols defined as:* $\Gamma_{ij}^k=\frac{1}{2}\sum_{l=1}^{m}g^{kl}\big(\frac{\partial g_{jl}}{\partial x_i}+\frac{\partial g_{il}}{\partial x_j}-\frac{\partial g_{ij}}{\partial x_l}\big)$. *We put* $g^{kl}=(g_{kl})^{-1}$.

Theorem 1.5.2. *(Fundamental theorem of Riemannian Geometry)*
Let (M,g) *be a Riemannian manifold. The Levi-Civita connection is the unique torsion free connection compatible with* g.

1.5.1 Parallel vector field and geodesics

Definition 1.5.7. *Let* (M,g) *be an n-dimensional Riemannian manifold. For a* C^1 *map* $\sigma;[a,b]\subset\mathbb{R}\to M$ *and* $\sigma(t)=(\sigma^1(t),...,\sigma^n(t))$ *be the local expression of* σ. X *is a* C^1 *vector field along* σ *if*

1. $X(t)\in T_{\sigma(t)}M$, *for all* $t\in[a,b]$.
2. *In terms of local coordinate* $(U,(x^1,...,x^n))$ *at each point* $\sigma(t)$, *it hold that*

$$X(t)=X^i(t)\big(\frac{\partial}{\partial x^i}\big)_{\sigma(t)}\in T_{\sigma(t)}M.$$

Such a vector field is parallel with respect to the connection ∇ *if* $\nabla_{\dot\sigma(t)}X=0$, *such that*

$$\dot\sigma(t)=\sum_{i,k=1}^{n}\frac{d\sigma^i(t)}{dt}\big(\frac{\partial}{\partial x^i}\big)_{\sigma(t)}. \tag{1.5}$$

Then using the necessary and sufficient condition to hold $\nabla_{\dot\sigma(t)}X=0$ is

$$\frac{dX^i(t)}{dt}+\sum_{i,k=1}^{n}\Gamma_{jk}^{i}\frac{d\sigma^j(t)}{dt}X^k(t)=0,\qquad i=1,...,n, \tag{1.6}$$

by means of (1.3) and (1.5).

Definition 1.5.8. *A* C^1 *curve* $\sigma:[a,b]\to M$ *in* M *is geodesic if the tangent vector field* $\dot\sigma$ *is parallel, i.e* $\nabla_{\dot\sigma}\dot\sigma=0$.

In term of local coordinate system $(U,(X^1,...,x^2))$, the condition $\nabla_{\dot\sigma}\dot\sigma=0$ holds that

$$\frac{d^2\sigma^i(t)}{dt^2}+\sum_{j,k=1}^{n}\Gamma_{jk}^{i}\frac{d\sigma^j(t)}{dt}\frac{d\sigma^k(t)}{dt}=0, \tag{1.7}$$

given initial conditions $(\sigma^1(a), ..., \sigma(a))$ and $(\frac{d\sigma}{dt}(a), ..., \frac{d\sigma}{dt}(a))$, there exist uniquely solution of (1.7) if t is close enough to a. For every point $p \in M$ and every vector $u \in T_pM$, there exists a unique geodesic $\sigma(t)$, passing through p at the initial t_0 and having u as the initial vector at p if t sufficiently close to 0. Therefore, There exists a unique geodesic satisfying $\sigma(0) = p$ and $\dot{\sigma}(0) = u$.

Definition 1.5.9. *Let (M, g) be a Riemannian manifold and T_pM a tangent space at the point $p \in M$. The exponential map*

$$\begin{aligned} exp_p : T_pM &\longrightarrow M \\ u &\longmapsto \sigma(1) = exp_p(u), \end{aligned}$$

1.5.2 Riemannian curvature

Definition 1.5.10. *Let (M, g) be a Riemannian manifold with the Levi-Civita connection ∇, the Riemannian curvature associated to the Levi-Civita connection ∇ noted R is defined by*

$$\begin{aligned} R : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) &\longrightarrow \Gamma(TM) \\ (X, Y, Z) &\longmapsto R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \end{aligned}$$

for every $X, Y, Z \in \Gamma(TM)$.

Remark 1.5.2.

- *A manifold is flat if the Riemannian curvature is equal to 0.*
- *Locally we have $R(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})\frac{\partial}{\partial x_k} = \sum\limits_{m=1}^{n} R^m_{ijk} \frac{\partial}{\partial x_m}$, by the definition and easy calculation we get:*

  $$R^l_{ijk} = \sum_{m=1}^{n} \Big(\Gamma^m_{jk} \Gamma^l_{im} - \Gamma^m_{ik} \Gamma^l_{jm} \Big) + \frac{\partial \Gamma^l_{jk}}{\partial x_i} - \frac{\partial \Gamma^l_{ik}}{\partial x_j},$$

  for $i, j, k, l = 1, ..., n$.

Example 1.5.4. *For the unit sphere using the metric defined in example 1.5.2, we get.*

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & sin^2(\alpha) \end{pmatrix}, g^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{sin^2(\alpha)} \end{pmatrix}.$$

We have the Christoffel symbols as follow
$\Gamma^2_{21} = \Gamma^2_{12} = \frac{1}{2} g^{22} \big(\frac{\partial g_{22}}{\partial \alpha} \big) = \frac{1}{2} \frac{1}{sin^2(\alpha)} (2 sin(\alpha) cos\alpha) = cotan(\alpha)$
and
$\Gamma^1_{22} = \frac{1}{2} g^{11} \big(\frac{\partial g_{22}}{\partial \alpha} \big) = -cos(\alpha) \sin\alpha = -\frac{1}{2} sin(2\alpha)$, *all the others are 0, so we get that* $R^1_{122} = -R^1_{212} = sin^2(\alpha)$ *and the others are 0.*

Proposition 1.5.1. *Let (M, g) be a smooth Riemannian manifold, for every vector fields X, Y, Z and Z, we have*

1. $R(X, Y)Z = -R(Y, X)Z$.

2. $g(R(X, Y)Z, W) = -g(R(X, Y)W, Z)$.

3. $g(R(X, Y)Z, W) + g(R(Z, X)Y, W) + g(R(Y, Z)X, W) = 0$.

4. $g(R(X, Y)Z, W) = g(R(Z, W)X, Y)$.

Definition 1.5.11. *Let (M, g) a Riemannian manifold for every $p \in M$ let X_p, $Y_p \in T_pM$ two linearly independent vectors, then the sectional curvature of M at p is defined by:*

$$k(X_p, Y_p) = \frac{g(R(X_p, Y_p)Y_p, X_p)}{g(X_p, X_p)g(Y_p, Y_p) - g(X_p, Y_p)^2},$$

for a local orthonormal frame $\{e_1, ..., e_n\}$ on M.

Definition 1.5.12. *Let (M, g) be a Riemannian manifold and let $X, Y \in \Gamma(TM)$. The Ricci tensor is on (M, g) is defined by:*

$$Ric(X, Y) = \sum_{i=1}^{m} g(R(X, e_i)e_i, Y),$$

where $\{e_i\}$ is any orthonormal frame on (M, g).

Definition 1.5.13. *The scalar curvature on (M, g) is the functional S defined by:*

$$S = \sum_{i=1}^{m} Ric(e_i, e_i),$$

where $\{e_i\}$ is any orthonormal frame on (M, g).

1.6 Complex and Almost-complex structure on Riemannian manifolds

1.6.1 Almost complex structure

Definition 1.6.1. *Let (M^{2m}, g) be an oriented Riemannian manifold. An almost complex structure at $p \in M$ or on T_pM is a linear transformation $J_p : T_pM \to T_pM$ such that $J_p^2 = -I$.*
A Riemannian metric is said to be Hermitian if $g(JX, JY) = g(X, Y)$.

Definition 1.6.2. *An almost Hermitian structure at p is an almost complex structure at p which is isometric.*

Definition 1.6.3. *[8]The Nijenhuis tensor of an almost complex structure J is:*

$$N(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY], \qquad X,Y \in \Gamma(TM). \quad (1.8)$$

Theorem 1.6.1. *An almost complex structure J is integrable if and only if its Nijenhuis tensor vanishes.*

Definition 1.6.4. *A differentiable manifold equipped with an almost complex (respectively, almost Hermitian) structure is called an almost complex (respectively, almost Hermitian) manifold.*

1.6.2 K$\ddot{a}$hler manifold

Definition 1.6.5. *Let (M, g, J) be an almost Hermitian manifold and Ω the second fundamental form. If $d\omega = 0$ then is called an almost Kähler manifold.*

Theorem 1.6.2. *Let (M, g, J) be an almost Hermitian manifold. If $\nabla J = 0$, then J is automatically integrable. In this case, J is called a Kähler structure on (M, g) and (M, J, g) is called a Kähler manifold.*

1.7 Almost paracomplex manifold

Definition 1.7.1. *An almost paracomplex manifold is an almost product manifold (M, ϕ), $\phi^2 = Id$, such that the two eigenbundles T^+M and T^-M associated to the two eigenvalues $+1$ and -1 of ϕ, respectively, have the same rank.*

1.8 Para-K$\ddot{a}$hler-Norden manifold

Definition 1.8.1. *Let(M, J)be an almost paracomplex manifold of dimension 2n and let g be a pseudo-Riemannian metric on M, if J is a g-symmetric (compatible with g), then g is called para-Norden metric and$(M, J, g)$is called para-Norden manifold.*

Definition 1.8.2. *A para-Kähler-Norden (para-holomorphic Norden) manifold is an almost para-complex Norden manifold (M^{2m}, J, g) such that $\nabla J = 0$, where ∇ is the Levi-Civita connection of g.*

Definition 1.8.3. *Let X be a vector field in an n-dimensional differentiable manifold M The differential transformation L_X is called the Lie derivative with respect to X if*

- $L_X f = Xf$, *for all* $f \in C^\infty(M)$.

- $L_X Y = [X, Y]$.

Definition 1.8.4. *A Tachibana operator Φ_J applied to the pure metric g is given by*

$$\Phi_{Jg}(X, Y, Z) = (JX)(g(Y, Z)) - X(g(JY, Z)) + g((L_Y J)X, Z) + g((L_Z J)X, Y) \quad (1.9)$$

for all $X, Y, Z \in \Gamma(TM)$. It is well known that the theorem ($\nabla J = 0$ is equivalent to $\Phi_{Jg} = 0$), see [32].

1.9 Anti-paraHermitian metric

Definition 1.9.1. *[4] Let (M^{2k}, ϕ) be an almost paracomplex manifold. A Riemannian metric g is said to be an anti-paraHermitian metric if*

$$g(\phi X, \phi Y) = g(X, Y) \quad (1.10)$$

for any vector field X, Y on M^{2k}.
(M^{2k}, ϕ, g) is said to be an almost anti-paraHermitian manifold.

1.10 Harmonic and Bi-harmonic maps

Definition 1.10.1. *Let (M, g) be a Riemannian manifold and a smooth function $f : M \longrightarrow \mathbb{R}$ the gradient of f is a vector field given by $grad(f)$; it is characterized by*

$$g(grad(f), X) = X(f), \quad (p \in M, \; X \in T_pM).$$

Proposition 1.10.1. *Let (M, g) be a Riemannian manifold of dimension n, (U, ϕ) a chart on M and a local coordinate base associated $\frac{\partial}{\partial x^1}, \ldots \frac{\partial}{\partial x^n}$, so for every $f \in C^\infty(M)$ we have:*

$$gradf|_U = g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}.$$

Property 1.10.1. *Let (M, g) be a Riemannian manifold, for every $f, g \in C^\infty(M)$ we have:*

- $grad(f + h) = grad\ f + grad\ h$.
- $grad(fh) = f grad\ h + h grad\ f$.
- $(grad\ f)(h) = (grad\ h)(f)$.

Example 1.10.1. *Let f be a function on $\mathbb{R}^n$, then:*

$$grad\ f = (\frac{\partial\ f}{\partial x^1}, \ldots, \frac{\partial\ f}{\partial x^n}).$$

Definition 1.10.2. *Let (M, g) be a Riemannian manifold and ∇ the Levi-Civita associated to g. The Hessian of a smooth function $f : M \to \mathbb{R}$ is defined as:*

$$Hess_f(X, Y) = g(\nabla_X grad(f), Y) = XY(f) - (\nabla_X Y)(f),$$

such that $X, Y \in \Gamma(TM)$.

Definition 1.10.3. *Let (M, g) a Riemannian manifold of dimension n and ∇ the Levi-Civita connection associated to g then the divergence of a vector field X is a smooth map on M defined by:*

$$div(X) = \sum_{i=1}^{n} g(\nabla_{e_i} X, e_i),$$

where $\{e_1, ..., e_n\}$ is a local orthonormal frame on (M, g).
In local coordinate bases we have:

$$\begin{aligned}
div(X) &= g^{ij} g\Big(\nabla_{\frac{\partial}{\partial x^i}} X, \frac{\partial}{\partial x^j}\Big) \\
&= \sum_{i,j=1}^{n} g^{ij} g\Big(\nabla_{\frac{\partial}{\partial x_i}} X, \frac{\partial}{\partial x_j}\Big) \\
&= \sum_{i,j,k=1}^{n} g^{ij} g\Big(\nabla_{\frac{\partial}{\partial x_i}} X_k \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_j}\Big) \\
&= \sum_{i,j,k=1}^{n} g^{ij} \Big(\frac{\partial X_k}{\partial x_i} g_{kj} + X_k \sum_{l=1}^{m} \Gamma_{ik}^{l} g_{jl}\Big) \\
&= \sum_{i,j,k=1}^{n} \Big(\frac{\partial X_k}{\partial x_i} \delta^{ki} + X_k \sum_{l=1}^{m} \Gamma_{ik}^{l} \delta^{il}\Big) \\
&= \sum_{i=1}^{n} \Big(\frac{\partial X_i}{\partial x_i} + \sum_{k=1}^{m} X_k \Gamma_{ik}^{i}\Big).
\end{aligned}$$

Property 1.10.2. *Let (M, g) a Riemannian manifold of dimension n for every vectors fields X, Y and $f \in C^\infty(M)$ we have:*

- $div(X + Y) = div(X) + div(Y)$.
- $div(fX) = f div(X) + X(f)$.

Theorem 1.10.1. *(Divergence theorem) Let D be a compact domain of a Riemannian manifold (M, g) with smooth boundary. Let ω be a 1-form and X a vector field defined on a neighborhood of D.*

$$\int_D (div\ \omega) v^M = \int_{\partial D} \omega(n) v^{\partial D} \tag{1.11}$$

and

$$\int_D (div\ X)v^M = \int_{\partial D} g(X,n)v^{\partial D}. \tag{1.12}$$

where $n = n(p)$ denotes the outward pointing unit normal at a point $p \in \partial D$.

Corollary 1.10.1. *For any* 1*-form* ω *and a vector field* X *with compact support,*

$$\int_M (div\ \omega)v^M = \int_M (div\ X)v^M = 0. \tag{1.13}$$

Definition 1.10.4. *Let* (M, g) *be a Riemannian manifold, the Laplace operator* Δ *is defined by:*

$$\Delta : C^\infty(M) \longrightarrow C^\infty(M) \tag{1.14}$$
$$f \longmapsto \Delta(f) = div(grad\ f). \tag{1.15}$$

Property 1.10.3. *Let* (M, g) *be a Riemannian manifold, for every* $f, h \in C^\infty(M)$ *we have:*

- $\Delta(f + h) = \Delta(f) + \Delta(h)$.
- $\Delta(fh) = h\Delta(f) + f\Delta(h) + 2g(grad\ f, grad\ h)$.

Example 1.10.2. *The Laplace operator of a differential function* f *on* $\mathbb{R}^n$ *is as follow:*

$$\Delta\ f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}.$$

1.10.1 Harmonic maps

Definition 1.10.5. *Consider a smooth map* $\phi : (M^n, g) \longrightarrow (N^n, h)$ *between two Riemannian manifolds. The energy density of* ϕ *is the smooth function* $e(\phi) : M \to [0, +\infty[$ *such that*

$$e(\phi)_x = \frac{1}{2}|d\phi|^2, \qquad x \in M. \tag{1.16}$$

Let K *be a compact domain of* M *then the energy functional of* ϕ *over* K *is defined by:*

$$E(\phi) = \frac{1}{2}\int_K |d\phi|^2 v_g, \tag{1.17}$$

such that $|d\phi|$ *the Hilbert-Schmidt norm is defined as follow:*

$$|d\phi|^2 = h(d\phi(e_i), d\phi(e_i)) = g^{ij}h_{kl}(\phi)\frac{\partial\phi^k}{\partial x_i}\frac{\partial\phi^l}{\partial x_j}.$$

Example 1.10.3. *In $\mathbb{R}^3$ the energy of $f;\mathbb{R}^3 \to \mathbb{R}$ over L is given by:*

$$E(f) = \int\int\int_K ||\nabla f||^2 dxdydz.$$

Definition 1.10.6. *A smooth map $\psi : (M,g) \to (N,h)$ is called harmonic if it is a critical point of the energy functional E (or $E(K)$ for all compact subsets $K \subset M$).*

Definition 1.10.7. *Let $\psi : (M,g) \longrightarrow (N,h)$ be a differentiable map between two Riemannian manifold (M,g) and (N,h), ψ is totally geodesic if $\nabla d\psi = 0$.*

Remark 1.10.1. *If ψ is totally geodesic then ψ is harmonic.*

1.10.2 First variation of energy

Definition 1.10.8. *[8] Let ϕ be a smooth map from M to N. A smooth variation of the map ϕ is a smooth map*

$$\Phi : M \times (-\epsilon;\epsilon) \longrightarrow N \tag{1.18}$$

$$(p,t) \longmapsto \phi_t(p), \tag{1.19}$$

where $\epsilon > 0$, such that $\phi_0 = \phi$.

Theorem 1.10.2. *Let $\psi : (M,g) \to (N,h)$ be a smooth map, $\{\psi_t\}_{t\in I}$ a smooth variation of ψ, with $\psi_0 = \psi$ and the variation vector field $\mathcal{V} = \frac{d\psi_t}{dt}|_{t=0}$, then*

$$\frac{d}{dt}E(\psi_t)|_{t=0} = -\int_M h(\tau(\psi),\mathcal{V})dv_g, \tag{1.20}$$

where

$$\tau(\psi) = tr_g \nabla d\psi = \sum_{i=1}^{m} \nabla d\psi(e_i,e_i), \tag{1.21}$$

$\tau(\psi) \in \Gamma(\psi^{-1}TN)$ and is called the tension field of ψ.

Theorem 1.10.3. *A smooth map $\psi : (M^m,g) \longrightarrow (N^n,h)$ is harmonic if and only if*

$$\tau(\psi) = 0. \tag{1.22}$$

If $(x^i)_{1\leq i\leq m}$ and $(y^\alpha)_{1\leq\alpha\leq n}$ denote local coordinates on M and N respectively then equation 1.22 takes the form

$$\tau(\psi)^\alpha = (\Delta\psi^\alpha + g^{ij}\, \overset{N}{\Gamma}{}^{\alpha}_{\beta\gamma}\, \frac{\partial\psi^\beta}{\partial x^i}\frac{\partial\psi^\gamma}{\partial x^j}) = 0, \tag{1.23}$$

where

$$\Delta\psi^\alpha = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^i}(\sqrt{|g|}g^{ij}\frac{\partial\phi^\alpha}{\partial x^j}) = g^{ij}\big(\frac{\partial^2\psi^\alpha}{\partial x_i\partial x_j} - \Gamma^k_{ij}\frac{\partial\psi^\alpha}{\partial x_k}\big) \tag{1.24}$$

is the Laplace operator on (M^m,g) and $\overset{N}{\Gamma}{}^{\alpha}_{\beta\gamma}$ are the Christoffel symbols on N.

Example 1.10.4. *For a map* $f : (\mathbb{R}^n, <.,.>_{\mathbb{R}^n}) \to \mathbb{R}$. f *is harmonic if and only if*

$$\Delta f = \sum_{i=1}^{m} \frac{\partial^2 f}{\partial x_i^2} = 0. \tag{1.25}$$

Example 1.10.5. *Let* (M^{2m}, J, g) *and* (N^{2n}, J', h) *be two Riemannian manifolds and a parallel almost complex structures* J *and* J'.
Let a map $\psi : (M^{2m}, J, g) \to (N^{2n}, J', h)$ *assuming that* $d\psi \circ J = J' \circ d\psi$.
For an orthonormal base $\{e_i, Je_i\}_{i=1}^m$ *on* (M^{2m}, g) *we find that:*

$$\begin{aligned}
\nabla d\psi(X, JY) &= \nabla^{\psi}_X d\psi(JY) - d\psi(\nabla_X JY) \\
&= \nabla_{d\psi(X)} d\psi(JY) - d\psi(\nabla_X JY) \\
&= \nabla_{d\psi(X)} d\psi(JY) - d\psi((\nabla_X J)Y + J(\nabla_X Y)), \quad (J \quad \textit{parallel}) \\
&= \nabla_{d\psi(X)} d\psi(JY) - d\psi(J(\nabla_X Y)) \\
&= \nabla_{d\psi(X)} J' d\psi(Y) - J' d\psi(\nabla_X Y) \\
&= (\nabla_{d\psi(X)} J') d\psi(Y) + J'(\nabla_{d\psi(X)} d\psi(Y)) - J' d\psi(\nabla_X Y) \\
&= J'(\nabla_{d\psi(X)} d\psi(Y) - J' d\psi(\nabla_X Y)) \\
&= J'(\nabla d\psi(X, Y)),
\end{aligned}$$

since $\nabla d\psi(X, JY) = \nabla d\psi(JY, X)$, *then* $\nabla d\psi(JX, JY) = J'J'\nabla d\psi(X, Y) = -\nabla d\psi(X, Y)$ *for every* $X, Y \in \Gamma(TM)$. *Hence* $\tau(\psi) = \sum_{i=1}^{m} \nabla d\psi(Je_i, Je_i) + \nabla d\psi(e_i, e_i) = 0$ *and so* ψ *is harmonic.*

1.10.3 Bi-harmonic maps

Definition 1.10.9. *A smooth map* $\phi : (M^m, g) \longrightarrow (N^n, h)$ *between Riemannian manifolds the bienergy functional is defined by*

$$E_2(\phi) = \frac{1}{2} \int_K |\tau(\phi)|^2 v^g, \tag{1.26}$$

we have

$$\frac{d}{dt} E_2(\phi_t)|_{t=0} = -\int_M h(\tau_2(\phi), V) v_g. \tag{1.27}$$

The Euler-Lagrange equation attached to bienergy is given by the vanishing of the bitension field

$$\tau_2(\phi) = -J_\phi(\tau(\phi)) = -(\Delta^\phi \tau(\phi) + tr_g R^N(\tau(\phi), d\phi) d\phi), \tag{1.28}$$

where J_ϕ is the Jacobi operator defined by

$$J_\phi : \Gamma(\varphi^{-1}(TN)) \quad \longrightarrow \quad \Gamma(\varphi^{-1}(TN)) \tag{1.29}$$

$$V \longmapsto \Delta^{\phi}V + tr_g R^N(V, d\phi)d\phi.$$

and

$$\Delta^{\phi}\tau(\phi) = trace_g(\nabla^{\phi})^2\tau(\phi) = \sum_{i=1}^{m}\Big[\nabla^{\phi}_{e_i}\nabla^{\phi}_{e_i}\tau(\phi) - \nabla^{\phi}_{\nabla^M_{e_i}e_i}\tau(\phi)\Big]. \tag{1.30}$$

Definition 1.10.10. *A smooth map $\phi : (M, g) \longrightarrow (N, h)$ between Riemannian manifolds is called biharmonic if it is a critical point of the bienergy functional.*

Example 1.10.6. *In example 1.10.5 since the map ψ is harmonic then it is biharmonic.*

Chapter 2

Geometrical structures on the tangent bundle

In this chapter we introduces some necessary structure on the tangent bundle, we start with what is called vertical and horizontal section, then we define different metrics on the tangent bundle such as the Sasaki metric, the Cheeger-Gromoll metric, the Gradient Sasaki metric, the Berger type deformed Sasaki metric and the isotropic Cheeger-Gromoll metric. We will also give some almost complex and almost paracomplxe structure compatible with those metrics.

2.1 Vertical, complete and horizontal lift

2.1.1 Vertical lift of function

In all the following we consider M to be an n-dimensional differential manifold and (TM, π, M) to be its tangent bundle. A local chart $(U, x^i)_{i=1...n}$ on M induces a local chart $(\pi^{-1}(U), x^i, y^i)_{i=1...n}$ on TM.

Definition 2.1.1. *[34] Let f be a function in differential manifold M, we note f^V the function in TM obtained by the composition of $\pi : TM \longrightarrow M$ and $f : M \longrightarrow \mathbb{R}$, such that*

$$f^V = f \circ \pi.$$

Thus, if a point $(p, u) \in \pi^{-1}(U)$, then

$$f^V((p, u)) = f \circ \pi((p, u)) = f(p).$$

The value $f^V((p, u))$ is constant along each T_pM, we call f^V the vertical lift of f.

Let ω be a 1-form in M, it is regarded in a natural way, as a function in TM. If ω has the local expression $\omega = \omega_i dx_i$ in a coordinate neighborhood and $u = y^i \frac{\partial}{\partial x^i} \in TM$. We define a map,

$$i\omega : TM \longrightarrow \mathbb{R}$$

$$(p, u) \longmapsto \omega_p(u),$$

$i\omega$ has the local expression

$$i\omega = \omega_i(x)y^i,$$

with respect to the induced coordinate in $\pi^{-1}(U)$. If f is a differentiable function in M, then $i(df)$ has the local expression

$$i(df) = y^i \frac{\partial f}{\partial x^i}.$$

Proposition 2.1.1. *[34]. Let $\tilde{X}$ and $\tilde{Y}$ be vector fields in TM such that $\tilde{X}(i(df)) = \tilde{Y}(i(df))$, for an arbitrary function f in M. Then $\tilde{X} = \tilde{Y}$.*

2.1.2 Vertical lift of a vector field

Let $\tilde{X} \in \Gamma(TM)$ such that $\tilde{X} f^V = 0$ for all $f \in C^\infty(M)$. Then we say that $\tilde{X}$ is a vertical vector field. Let $\begin{pmatrix} \tilde{X}^h \\ \tilde{X}^k \end{pmatrix}$ be component of $\tilde{X}$ with respect to the induced coordinates. Then from $\tilde{X} f^V = 0$, we have $\tilde{X}^h \frac{\partial f}{\partial x^i} = 0$ for all $f \in C^\infty(M)$, then $\tilde{X}^h = 0$, then $\begin{pmatrix} \tilde{X}^h \\ \tilde{X}^k \end{pmatrix} = \begin{pmatrix} 0 \\ \tilde{X}^k \end{pmatrix}$.

Definition 2.1.2. *Let X be a vector field in M. We define a vector field X^V in TM by*

$$X^V(i\omega) = (\omega(X))^V. \tag{2.1}$$

ω being an arbitrary 1-form in M. If X^h are components of X. Thus the vertical lift X^V of X has components

$$X^V : \begin{pmatrix} 0 \\ X^h \end{pmatrix}. \tag{2.2}$$

For further details see [34].

The correspondence $X \mapsto X^V$ determines a linear isomorphism of $\Gamma(TM)$ into $\Gamma(TTM)$ with respect to constant coefficients.

From (2.2), we have in each open set $\pi^{-1}(U)$ that $\left(\frac{\partial}{\partial x^i}\right)^V = \frac{\partial}{\partial y^i}$, with respect to the induced coordinates in TM.

Proposition 2.1.2. *For all $X, Y \in \Gamma(TM)$ and $f \in C^\infty(M)$, we have*

$$(X + Y)^V = X^V + Y^V,$$
$$(fX)^V = f^V X^V.$$

2.1.3 Vertical lift of a 1-form

Definition 2.1.3. *Let $\tilde{\omega} \in \Gamma(T^*TM)$ such that $\tilde{\omega}(X^V) = 0$ for all $X \in \Gamma(TM)$. Then we say that $\tilde{\omega}$ is a vertical 1-form in TM.*

Proposition 2.1.3. *Let X^i be the components of X in U and let $(\tilde{\omega}_i, \tilde{\omega}_j)$ be component of $\tilde{\omega}$ with respect to the induced coordinates in $\pi^{-1}(U)$. Then $\tilde{\omega}$ is vertical if and only if $(\tilde{\omega}_i, \tilde{\omega}_j) = (\tilde{\omega}_i, 0)$.*

Proof. Let X^i be the components of X in U and let $(\tilde{\omega}_i, \tilde{\omega}_j)$ be component of $\tilde{\omega}$ with respect to the induced coordinates in $\pi^{-1}(U)$. Then from $\tilde{\omega}(X^V) = 0$, we have $\tilde{\omega}_j(X^i) = 0$. X^i being arbitrary, this implies that $\tilde{\omega}_j = 0$ and

$$(\tilde{\omega}_i, \tilde{\omega}_j) = (\tilde{\omega}_i, 0).$$

□

Definition 2.1.4. *Let $f \in C^\infty(M)$. We define the vertical lift $(df)^V$ of the 1-form df on TM by*

$$(df)^V = d(f^V). \tag{2.3}$$

Proposition 2.1.4. *Let $g, f \in C^\infty(M)$. Then*

$$(gdf)^V = g^V d(f^V). \tag{2.4}$$

Definition 2.1.5. *Let $\omega \in \Gamma(T^*M)$. We define the vertical lift ω^V of the 1-form ω by*

$$\omega^V = (\omega_i)^V (dx^i)^V. \tag{2.5}$$

in each open set $\pi^{-1}(U)$, where (U, x^i) is a coordinate neighborhood in M and ω is given by $\omega = \omega_i dx^i$. The components of ω^V are $(\omega_i, 0)$.

Proposition 2.1.5. *Let $\omega, \theta \in \Gamma(T^*M)$ and $f \in C^\infty(M)$, then*

$$(\omega + \theta)^V = \omega^V + \theta^V,$$
$$(f\omega)^V = f^V \omega^V,$$

and in each open set $\pi^{-1}(U)$,

$$(dx^i)^V = dx^i, \tag{2.6}$$

with respect to the induced coordinates.

2.1.4 Complete lift of a function

Definition 2.1.6. *If f is a function in M, we call f^C the complete lift of the function f in M to the tangent bundle TM defined by*

$$f^C = i(df). \tag{2.7}$$

Proposition 2.1.6. *Let $X \in \Gamma(TM)$ and $f, g \in C^\infty(M)$, we have*

$$X^V f^C = (Xf)^V, \tag{2.8}$$
$$(gf)^C = g^C f^V + g^V f^C. \tag{2.9}$$

2.1.5 Complete lift of a vector fields

Definition 2.1.7. *Let $X \in \Gamma(TM)$, we define a vector field X^C in TM by*

$$X^C f^C = (Xf)^C, \tag{2.10}$$

for all $f \in C^\infty(M)$, we call X^C the complete lift of X to TM.

Proposition 2.1.7. *If X^h are the components of X in U and $\begin{pmatrix} \tilde{X}^h \\ \tilde{X}^k \end{pmatrix}$ the component of X^C with respect to the induced coordinate in $\pi^{-1}(U)$ then*

$$X^C : \begin{pmatrix} X^h \\ y^i \frac{\partial X^h}{\partial x^i} \end{pmatrix}. \tag{2.11}$$

Proof. If X^h are the components of X in U and $\begin{pmatrix} \tilde{X}^h \\ \tilde{X}^k \end{pmatrix}$ the component of X^C with respect to the induced coordinate in $\pi^{-1}(U)$, then by the Definition 2.1.7,

$$\begin{aligned} \tilde{X}^i (\frac{\partial^2 f}{\partial x^i \partial x^j}) y^i + \tilde{X}^j \frac{\partial f}{\partial x^j} &= \big(X^j \frac{\partial f}{\partial x^i}\big)^C = y^i \frac{\partial}{\partial x^i} (X^j \frac{\partial f}{\partial x^i}) \\ &= y^i X^j (\frac{\partial^2 f}{\partial x^i \partial x^j}) + \big(y^i \frac{\partial X^j}{\partial x^i}\big) \frac{\partial f}{\partial x^j}, \end{aligned}$$

then $\tilde{X}^i = X^i$ and $\tilde{X}^j = y^i \frac{\partial X^j}{\partial x^i}$. □

Proposition 2.1.8. *Let $X, Y \in \Gamma(TM)$ and $f \in C^\infty(M)$, then we have.*

- $(X + Y)^C = X^C + Y^C$.
- $(fX)^C = f^C X^V + f^V X^C$.
- $X^C f^V = (Xf)^V$.
- $X^C f^C = (Xf)^C$.
- $[X^V, Y^C] = [X, Y]^V$.
- $[X, Y]^C = [X^C, Y^C]$.

2.1.6 Horizontal lift of a function

Let S be a tensor field on a differentiable manifold M defined by

$$S = S^{j1,\dots,jp}_{i1,\dots,ip} \frac{\partial}{\partial x^{j1}} \otimes \dots \otimes \frac{\partial}{\partial x^{jp}} \otimes dx^{i1} \otimes \dots \otimes dx^{ip}, \tag{2.12}$$

and a vector field $X = X^i \frac{\partial}{\partial x^i}$, we then define a tensor field $\gamma_X S$ in $\pi^{-1}(U)$ by

$$\gamma_X S = \left(X^{i1} S^{j1,...,jp}_{i1,...,ip}\right) \frac{\partial}{\partial y^{j1}} \otimes ... \otimes \frac{\partial}{\partial y^{jp}} \otimes dx^{i2} \otimes ... \otimes dx^{ip}, \tag{2.13}$$

and a tensor field γS in π^{-1} by

$$\gamma S = \left(y^{i1} S^{j1,...,jp}_{i1,...,ip}\right) \frac{\partial}{\partial y^{j1}} \otimes ... \otimes \frac{\partial}{\partial y^{jp}} \otimes dx^{i2} \otimes ... \otimes dx^{ip}, \tag{2.14}$$

with respect to the induced coordinates. If S is a function, then $\gamma_X S = \gamma S = 0$.
Let ∇ be an affine connection in a differentiable manifold M. If f is a function on M, then we have (∇f) the gradient of f in M, and we have $\nabla_\gamma f = \gamma(\nabla f)$.

Definition 2.1.8. *We now define the horizontal lift f^H for f in M to the tangent bundle TM by*

$$f^H = f^C - \nabla_\gamma f = 0. \tag{2.15}$$

2.1.7 Horizontal lift of a vector fields

Definition 2.1.9. *Let $X \in \Gamma(TM)$. Then we define the horizontal lift X^H of X by*

$$X^H = X^C - \nabla_\gamma X. \tag{2.16}$$

If $X = X^i \frac{\partial}{\partial x^i}$, then

$$\nabla X = (\frac{\partial X^i}{\partial x^j} + \Gamma^i_{kj} X^j) \frac{\partial}{\partial x^i} \otimes dx^j,$$
$$\nabla_\gamma X = y^i (\frac{\partial X^i}{\partial x^j} + \Gamma^i_{kj} X^j) \frac{\partial}{\partial y^i},$$

and from Proposition 2.1.7, we get $X^H = X^i \frac{\partial}{\partial x^i} - X^j y^i \Gamma^i_{jk} \frac{\partial}{\partial y^i}$.

Proposition 2.1.9. *Let $X \in \Gamma(TM)$ and $f \in C^\infty$ then,*

$$X^H f^V = (Xf)^V,$$
$$X^H f^C = (Xf)^C - \gamma((df) \circ (\nabla X)).$$

Definition 2.1.10. *Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇ and (TM, π, M) its tangent bundle. A local chart (U, x_i) on M induces a local chart (π^{-1}, x_i, y_i) on TM. We note by π the natural projection of TM into M and we get that $d\pi$ is a smooth map from TTM to TM.*
The vertical subspace of TTM is defined by:

$$\mathcal{V} = ker(d\pi) = \left\{\lambda_i \frac{\partial}{\partial y_i}; \lambda_i \in \mathbb{R}\right\}.$$

Let M be a Riemannian manifold. and U a neighborhood of a point p.

The exponential map exp_p witch is a diffeomorphism from a neighborhood U' of 0 in T_pM into U. let also $\mu : \pi^{-1}(U) \to T_pM$ a C^∞ map that translate every vector Y in a parallel way form $q = \pi(Y)$ to p along the geodesic arc between p and q. For $u \in T_pM$ we define $S_{-u} : T_pM \to T_pM$ such that $S_{-u}(X) = X - u$. Using all the precedents maps we define the connection map

$$K_{(p,u)} : T_{(p,u)}TM \to T_pM,$$

of the Levi-Civita connection ∇ by:

$$K(A) = d(exp_p \circ S_{-u} \circ \mu)(A),$$

for all $A \in T_{(p,u)}TM$.

Definition 2.1.11. *The horizontal subspace of TTM is defined by:*

$$\mathcal{H} = \left\{\lambda_i \frac{\partial}{\partial x_i} - \lambda_i u_j \Gamma_{ij}^k \frac{\partial}{\partial y_k}; \lambda_i \in \mathbb{R}\right\},$$

where $(x,u) \in TM$.

Proposition 2.1.10. *The tangent bundle $T_{(p,u)}TM$ of the tangent bundle TM at the point (p,u) is the direct sum of the horizontal and vertical subspace.i.e*

$$T_{(p,u)}TM = \mathcal{H}_{(p,u)} \oplus \mathcal{V}_{(p,u)}$$

.

Definition 2.1.12. *The horizontal lift of a smooth vector field X is noted X^H of value $X^H_{(p,u)}$ at every point (p,u).*
The vertical lift of a smooth vector field X is noted X^V of value $X^V_{(p,u)}$ at every point (p,u).
We can also define them by

$$d\pi(X^H)_Z = X_{\pi(Z)} \quad \text{and} \quad K(X^H)_Z = 0_{\pi(Z)},$$
$$d\pi(X^V)_Z = 0_{\pi(Z)} \quad \text{and} \quad K(X^V)_Z = X_{\pi(Z)},$$

for every vector field Z.

Proposition 2.1.11. *Let (M,g) be a Riemannian manifold, we note ∇ the Levi-Civita connection associated to g and R the Riemannian curvature tensor of ∇, then*

1. $[X^H, Y^H] = [X,Y]^H - (R(X,Y)u)^V$.
2. $[X^H, Y^V] = (\nabla_X Y)^V$.
3. $[X^V, Y^V] = 0$.

2.1.8 Natural metric on tangent bundle

Definition 2.1.13. *Let (M, g) be a Riemannian manifold. The Riemannian metric g' on the tangent bundle TM is said to be natural with respect to g if*

$$\begin{aligned} g'(X^H, Y^H) &= g(X, Y) \circ \pi, \\ g'(X^H, Y^V) &= 0, \end{aligned}$$

for all vector field X, Y on TM.

2.2 Sasaki metric on tangent bundle

Definition 2.2.1. *Let (M, g) be a Riemannian manifold. Then the Sasaki metric g^s on the tangent bundle TM is the natural metric given by*

$$\begin{aligned} g^s(X^H, Y^H) &= g(X, Y) \circ \pi, \\ g^s(X^H, Y^V) &= 0, \\ g^s(X^V, Y^V) &= g(X, Y) \circ \pi, \end{aligned}$$

for all vector fields $X, Y \in TM$.

Example 2.2.1. *Let the manifold $\mathbb{R}^2$ equipped with the usual metric g defined by*

$$g = dx_1^2 + dx_2^2,$$

the Sasaki metric on its tangent bundle $(T\mathbb{R}^2, g^s)$ is defined by

$$g^s = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2.$$

2.2.1 Levi-Civita connection

Proposition 2.2.1. *Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇ and let $\widehat{\nabla}^s$ be the Levi-Civita connection on (TM, g^s) equipped with the Sasaki metric g^s. Then*

$$\begin{aligned} \widehat{\nabla}^s_{X^H} Y^H &= (\nabla_X Y)^H + \frac{1}{2}(R(Y, X)u)^V, \\ \widehat{\nabla}^s_{X^H} Y^V &= (\nabla_X Y)^V + \frac{1}{2}(R(u, Y)X)^H, \\ \widehat{\nabla}^s_{X^V} Y^H &= \frac{1}{2}(R(u, X)Y)^H, \\ \widehat{\nabla}^s_{X^V} Y^V &= 0, \end{aligned}$$

for all vector field X, Y on TM.

2.2.2 Riemannian curvature

Proposition 2.2.2. *Let (M,g) be a Riemannian manifold, R and $\widetilde{R}$ are the Riemann curvatures tensors of (M,g) and (TM,g^s) respectively. Then, we have*

$$\begin{aligned}
\widetilde{R}(X^H,Y^H)Z^H =& \frac{1}{2}((\nabla_Z R)(X,Y)u)^V + (R(X,Y)Z + \frac{1}{4}R(u,R(Z,Y)u)X \\
&+ \frac{1}{4}R(u,R(X,Z)u)Y + \frac{1}{2}R(u,R(X,Y)u)Z)^H, \\
\widetilde{R}(X^H,Y^H)Z^V =& (R(X,Y)Z + \frac{1}{4}R(R(u,Z)Y,X)u - \frac{1}{4}R(R(u,Z)X,Y)u)^V \\
&+ \frac{1}{2}((\nabla_X R)(u,Z)Y - (\nabla_Y R)(u,Z)X)^H, \\
\widetilde{R}(X^H,Y^V)Z^H =& \frac{1}{4}((R(R(u,Y)Z,X)u) + \frac{1}{2}R(X,Z)Y)^V + \frac{1}{2}((\nabla_X R)(u,Y)Z)^H, \\
\widetilde{R}(X^H,Y^V)Z^V =& -(\frac{1}{2}R(Y,Z)X + \frac{1}{4}R(u,Y)(R(u,Z)X))^H, \\
\widetilde{R}(X^V,Y^V)Z^H =& (R(X,Y)Z + \frac{1}{4}(R(u,X)R(u,Y)Z) - \frac{1}{4}R(u,Y)(R(u,X)Z))^H, \\
\widetilde{R}(X^V,Y^V)Z^V =& 0.
\end{aligned}$$

For all $X,Y,Z \in \Gamma(TM)$.

2.3 Cheeger-Gromoll metric on tangent bundle

Definition 2.3.1. *Let (M,g) be a Riemannian manifold. Then the Cheeger- Gromoll metric g^{CG} is the natural Riemannian metric on the tangent bundle TM such that*

$$\begin{aligned}
g^{CG}_{(p,u)}(X^H,Y^H) =& g_p(X,Y), \\
g^{CG}_{(p,u)}(X^V,Y^H) =& 0, \\
g^{CG}_{(p,u)}(X^V,Y^V) =& \frac{1}{r}\Big(g_p(X,Y) + g_p(X,u)g_p(Y,u)\Big),
\end{aligned}$$

for all $X,\ Y \in TM$, where $u = y^i\frac{\partial}{\partial x^i}$ and $r = 1 + g(u,u)$, we have

$$g^{CG}_{\alpha\beta} = \begin{pmatrix} g_{ij} & 0 \\ 0 & \frac{1}{r}\big(g_{ij} + g_{il}g_{jp}y^l y^p\big) \end{pmatrix}.$$

2.3.1 Levi-Civita connection

Proposition 2.3.1. *Let (M,g) be a Riemannian manifold and let ∇^{CG} be the Levi-Civita connection of (TM,g^{CG}) equipped with the Cheeger-Gromoll metric g^{CG}. Then*

$$\nabla^{CG}_{X^H}Y^H = (\nabla_X Y)^H + \frac{1}{2}(R(Y,X)u)^V,$$

$$
\begin{aligned}
\nabla^{CG}_{X^H} Y^V &= (\nabla_X Y)^V + \frac{1}{2r}(R(u,Y)X)^H, \\
\nabla^{CG}_{X^V} Y^H &= \frac{1}{2r}(R(u,X)Y)^H, \\
\nabla^{CG}_{X^V} Y^V &= -\frac{1}{\alpha}\big(g^{CG}(X^V,U)Y^V + g^{CG}(Y^V,U)X^V\big) + \frac{1+r}{r} g^{CG}(X^V,Y^V)U \\
&\quad - \frac{1}{r} g^{CG}(X^V,U) g^{CG}(Y^V,U)U,
\end{aligned}
$$

for all vector field X, Y on TM. U is the canonical vertical vector at (p,u) defined by

$$U = u^V = Y_i \frac{\partial}{\partial y_i}.$$

2.3.2 Riemannian curvature

Proposition 2.3.2. *Let (M,g) be a Riemannian manifold, R and R^{CG} are the Riemann curvatures tensors of (M,g) and (TM, g^{CG}) respectively. Then, we have*

$$
\begin{aligned}
R^{CG}(X^H,Y^H)Z^H =& \frac{1}{2}((\nabla_Z R)(X,Y)u)^V + (R(X,Y)Z + \frac{1}{4r}R(u,R(Z,Y)u)X \\
&+ \frac{1}{4r}R(u,R(X,Z)u)Y + \frac{1}{2}R(u,R(X,Y)u)Z)^H, \\
R^{CG}(X^H,Y^H)Z^V =& (R(X,Y)Z + \frac{1}{4r}R(R(u,Z)Y,X)u + \frac{1}{4r}R(Y,R(u,Z)X)u)^V \\
&- 4g^{CG}(Z^V,U)(R(X,Y)u)^V + \frac{1+r}{r}g^{CG}((R(X,Y)u)^V,Z^V)U \\
&+ \frac{1}{2r}((\nabla_X R)(u,Z)Y - (\nabla_Y R)(u,Z)X)^H, \\
R^{CG}(X^H,Y^V)Z^H =& \frac{1}{4r}((R(R(u,Y)Z,X)u) + \frac{1}{2}R(X,Z)Y)^V + \frac{1}{2r}((\nabla_X R)(u,Y)Z)^H \\
&- 2g(Y,u)(R(X,Z)u)^V + \frac{1+r}{r}g^{CG}((R(X,Z)u)^V,Y^V)U, \\
R^{CG}(X^H,Y^V)Z^V =& -\Big(\frac{1}{2r}R(Y,Z)X + \frac{1}{4r^2}R(u,Y)(R(u,Z)X)\Big)^H \\
&+ \frac{1}{2r^2}\Big(g(Y,u)(R(u,Z)X)^H - g(Z,u)(R(u,Y)X)^H\Big), \\
R^{CG}(X^V,Y^V)Z^H =& (\frac{1}{2r}R(X,Y)Z + \frac{1}{4r^2}(R(u,X)R(u,Y)Z - R(u,Y)(R(u,X)Z)))^H \\
&+ \frac{1}{r^2}(g(Y,u)(R(u,X)Z) - g(X,u)(R(u,Y)Z))^H, \\
R^{CG}(X^V,Y^V)Z^V =& \frac{r+2}{r}\Big(g^{CG}(X^V,Z^V)g(Y,u)U - g^{CG}(Y^V,Z^V)g(X,u)U\Big) \\
&+ \frac{1+r+r^2}{r^2}\Big(g^{CG}(Y^V,Z^V)X^V - g^{CG}(X^V,Z^V)Y^V\Big)
\end{aligned}
$$

$$+\frac{r+2}{r^2}\Big(g(X,u)g(Z,u)Y^V - g(Y,u)g(Z,u)X^V\Big).$$

For all $X, Y, Z \in \Gamma(TM)$.

2.4 Para-complex Structures of Gradient Sasaki Metric on tangent bundle

2.4.1 Gradient Sasaki metric on tangent bundle

Definition 2.4.1. *Let* (M, g) *be a Riemannian manifold and* $f : M \to]0, +\infty[$. *On the tangent bundle* TM*, we define a gradient Sasaki metric noted* g_f *by*

1. $g_f(X^H, Y^H)_{(p,u)} = g_p(X, Y)$,

2. $g_f(X^H, Y^V)_{(p,u)} = 0$,

3. $g_f(X^V, Y^V)_{(p,u)} = g_p(X, Y) + X_p(f)Y_p(f)$,

where $X, Y \in \Gamma(TM)$*,* $(p, u) \in TM$.

In the following, we consider $\alpha = 1 + \|grad\, f\|^2$, where $\|.\|$ denote the norm with respect to (M, g).

2.4.2 Levi-Civita connection

Theorem 2.4.1. *Let* (M, g) *be a Riemannian manifold and* (TM, g_f) *its tangent bundle equipped with the gradient Sasaki metric. If* ∇ *(resp* ∇^f*) denote the Levi-Civita connection of* (M, g) *(resp* (TM, g_f) *). Then, we have*

$$\begin{aligned}
(\nabla^f_{X^H} Y^H)_p =& (\nabla_X Y)^H_p - \frac{1}{2}(R(X,Y)u)^V_p,\\
(\nabla^f_{X^H} Y^V)_p =& \frac{1}{2}(R(u,Y)X)^H_p + \frac{1}{2}Y(f)(R(u,grad(f))X)^H + (\nabla_X Y)^V\\
&+ \frac{1}{2}Y(f)(\nabla_X grad(f))^V + \frac{1}{2\alpha}\Big[g(Y,\nabla_X grad(f)) - \frac{1}{2}X(\alpha)Y(f)\Big](grad(f))^V,\\
(\nabla^f_{X^V} Y^H)_p =& \frac{1}{2}(R(u,X)Y)^H_p + \frac{1}{2}X(f)(R(u,grad(f))Y)^H + \frac{1}{2}X(f)(\nabla_Y grad(f))^V\\
&+ \frac{1}{2\alpha}\Big[g(X,\nabla_Y grad(f)) - \frac{1}{2}Y(\alpha)X(f)\Big](grad(f))^V,\\
(\nabla^f_{X^V} Y^V)_p =& -\frac{1}{2}X(f)(\nabla_Y grad(f))^H - \frac{1}{2}(Y(f)(\nabla_X grad(f))^H,
\end{aligned}$$

for all $X, Y \in \Gamma(TM)$.

Proof. For any vector fields $X, Y, Z \in \Gamma(TM)$, we have

$$\begin{aligned}
2g_f(\nabla^f_{X^H} Y^H, Z^H) =& X^H g(Y,Z) + Y^H g(X,Z) - Z^H g(X,Y) + g_f(Z^H, [X^H, Y^H]) \\
&+ g_f(Y^H, [Z^H, X^H]) - g_f(X^H, [Y^H, Z^H]) \\
=& g(\nabla_X Y, Z) + g(Y, \nabla_X Z) + g(\nabla_Y X, Z) + g(X, \nabla_Y Z) \\
&- g(\nabla_Z X, Y) - g(X, \nabla_Z Y) + g(Z^H, [X,Y]^H - (R(X,Y)u)^V) \\
&+ g(Y^H, [Z,X]^H - (R(Z,X)u)^V) - g(X^H, [Y,Z]^H - (R(Y,Z)u)^V) \\
=& g(\nabla_X Y, Z) + g(Y, \nabla_X Z) + g(\nabla_Y X, Z) + g(X, \nabla_Y Z) - g(\nabla_Z X, Y) \\
&- g(X, \nabla_Z Y) + g(Z, [X,Y]) + g(Y, [Z,X]) - g(X, [Y,Z]) \\
=& 2g(\nabla_X Y, Z) = g((\nabla_X Y)^H, Z^H).
\end{aligned}$$

$$\begin{aligned}
2g_f(\nabla^f_{X^H} Y^H, Z^V) =& - Z^V g(X,Y) + g_f(Z^V, [X^H, Y^H]) + g_f(Y^H, [Z^V, X^H]) \\
&- g_f(X^H, [Y^H, Z^V]) \\
=& g_f(Z^V, [X,Y]^H - (R(X,Y)u)^V) = g_f(Z^V, -(R(X,Y)u)^V),
\end{aligned}$$

And we have $(\nabla^f_{X^H} Y^H)_p = (\nabla_X Y)^V_p - \frac{1}{2}(R(X,Y)u)^V_p$.

$$\begin{aligned}
2g_f(\nabla^f_{X^H} Y^V, Z^H) =& g_f(Y^V, [Z^H, X^H]) = -g_f(Y^V, (R(Z,X)u)^V) \\
=& - g(Y, R(Z,X)u) - Y(f)(R(Z,X)u)(f) \\
=& g(R(u,Y)X, Z) - Y(f) g(R(Z,X)u, grad(f)) \\
=& g(R(u,Y)X, Z) + Y(f) g(R(grad(f), u)X, Z) \\
=& g_f\Big((R(u,Y)X)^H, Z^H\Big) + Y(f) g_f\Big((R(grad(f), u)X)^H, Z^H\Big),
\end{aligned}$$

$$\begin{aligned}
2g_f(\nabla^f_{X^H} Y^V, Z^V) =& X^H \Big[g(Y,Z) + Y(f)Z(f)\Big] + g_f(Z^V, [X^H, Y^V]) + g_f(Y^V, [Z^V, X^H]) \\
=& g(\nabla_X Y, Z) + g(Y, \nabla_X Z) + \Big(g(\nabla_X grad(f), Y) + g(grad(f), \nabla_X Y)\Big) Z(f) \\
&+ \Big(g(\nabla_X grad(f), Z) + g(grad(f), \nabla_X Z)\Big) Y(f) \\
&+ g_f(Z^V, (\nabla_X Y)^V) - g_f(Y^V, (\nabla_X Z)^V) \\
=& 2g_f(Z^V, (\nabla_X Y)^V) + g(\nabla_X grad(f), Y) Z(f) + g(\nabla_X grad(f), Z) Y(f) \\
&+ g(grad(f), \nabla_X Z) Y(f) + g(Y, \nabla_X Z) - g_f(Y^V, (\nabla_X Z)^V) \\
=& 2g_f(Z^V, (\nabla_X Y)^V) + g(\nabla_X grad(f), Y) Z(f) \\
&+ \Big(g_f((\nabla_X grad(f))^V, Z^V) - (\nabla_X grad(f))(f) Z(f)\Big) Y(f),
\end{aligned}$$

note that

$$Z(f) = g(grad(f), Z) = g_f(grad(f)^V, Z^V) - grad(f)(f) Z(f)$$

$$=g_f(grad(f)^V, Z^V) - ||grad(f)||^2 Z(f),$$

and then

$$Z(f) = \frac{1}{1+||grad(f)||^2} g_f(grad(f)^V, Z^V) = \frac{1}{\alpha} g_f(grad(f)^V, Z^V),$$

we also have

$$\nabla_X grad(f)(f) = g(\nabla_X grad(f), grad(f)) = \frac{1}{2} X(||grad(f)||^2) = \frac{1}{2} X(\alpha),$$

then

$$\begin{aligned}
2g_f(\nabla^f_{X^H} Y^V, Z^V) =& 2g_f(Z^V, (\nabla_X Y)^V) + \frac{1}{\alpha} g(\nabla_X grad(f), Y) g_f(grad(f)^V, Z^V) \\
&+ g_f(Y(f)(\nabla_X grad(f))^V, Z^V) - \frac{1}{2\alpha} X(\alpha) Y(f) g_f(grad(f)^V, Z^V), \\
2g_f(\nabla^f_{X^V} Y^H, Z^H) =& g_f(Z^H, [X^V, Y^H]) + g_f(Y^H, [Z^H, X^V]) - g_f(X^V, [Y^H, Z^H]) \\
=& g_f(X^V, (R(Y,Z)u)^V) = g(R(u,X)Y, Z) + (R(Y,Z)u)(f) X(f) \\
=& g(R(u,X)Y, Z) + g(R(u, grad(f))Y, Z) X(f) \\
=& g_f((R(u,X)Y)^H, Z^H) + g_f(X(f)(R(u, grad(f))Y)^H, Z^H),
\end{aligned}$$

$$\begin{aligned}
2g_f(\nabla^f_{X^V} Y^H, Z^V) =& Y^H g_f(X^V, Z^V) + g_f(Z^V, [X^V, Y^H]) - g_f(X^V, [Y^H, Z^V]) \\
=& Y^H \Big(g(X,Z) + X(f)Z(f) \Big) - g_f(Z^V, (\nabla_Y X)^V) - g_f(X^V, (\nabla_Y Z)^V) \\
=& g(\nabla_Y X, Z) + g(X, \nabla_Y Z) + \Big(g(\nabla_Y grad(f), X) + g(grad(f), \nabla_Y X) \Big) Z(f) \\
&+ \Big(g(\nabla_Y grad(f), Z) + g(grad(f), \nabla_Y Z) \Big) X(f) \\
&- g_f(Z^V, (\nabla_Y X)^V) - g(X, \nabla_Y Z) - X(f)(\nabla_Y Z)(f) \\
=& 2g_f((\nabla_Y X)^V, Z^V) + g(\nabla_Y grad(f), X) Z(f) + g(\nabla_Y grad(f), Z) X(f) \\
=& 2g_f((\nabla_Y X)^V, Z^V) + \frac{1}{\alpha} g(\nabla_Y grad(f), X) g_f(grad(f)^V, Z^V) \\
&+ \Big(g_f((\nabla_Y grad(f))^V, Z^V) - \nabla_Y grad(f)(f) Z(f) \Big) X(f) \\
=& 2g_f((\nabla_Y X)^V, Z^V) + \frac{1}{\alpha} g(\nabla_Y grad(f), X) g_f(grad(f)^V, Z^V) \\
&+ \Big(g_f((\nabla_Y grad(f))^V, Z^V) - \frac{1}{2\alpha} Y(\alpha) g_f(grad(f)^V, Z^V) \Big) X(f),
\end{aligned}$$

$$\begin{aligned}
2g_f(\nabla^f_{X^V} Y^V, Z^H) =& - Z^H \Big(g(X,Y) + X(f)Y(f) \Big) + g_f(Y^V, [Z^H, X^V]) - g_f(X^V, [Y^V, Z^H]) \\
=& - g(\nabla_Z X, Y) - g(X, \nabla_Z Y) - \Big(g(\nabla_Z grad(f), X) + g(grad(f), \nabla_Z X) \Big) Y(f)
\end{aligned}$$

$$
\begin{aligned}
&- \Big(g(\nabla_Z grad(f), Y) + g(grad(f), \nabla_Z Y)\Big)X(f) \\
&+ g(Y, \nabla_Z X) + (\nabla_Z X)(f)Y(f) + g(X, \nabla_Z Y) + (\nabla_Z Y)(f)X(f) \\
=& - g(\nabla_Z grad(f), Y)X(f) - g(\nabla_Z grad(f), X)Y(f),
\end{aligned}
$$

we have that

$$
\begin{aligned}
g(\nabla_Z grad(f), Y) =& Zg(grad(f), Y) - g(grad(f), \nabla_Z Y) = ZY(f) - (\nabla_Z Y)(f) \\
=& ZY(f) - (\nabla_Z Y)(f) - [Z, Y]f + [Z, Y]f,
\end{aligned}
$$

since the Levi-Civita connection is torsion free we get

$$
\begin{aligned}
g(\nabla_Z grad(f), Y) =& ZY(f) - (\nabla_Z Y)(f) - ZY(f) + YZ(f) + (\nabla_Z Y)(f) - (\nabla_Y Z)(f) \\
=& YZ(f) - (\nabla_Y Z)(f) = Yg(grad(f), Z) - g(\nabla_Y Z, grad(f)) \\
g(\nabla_Z grad(f), Y) =& g(\nabla_Y grad(f), Z) = g_f((\nabla_Y grad(f))^H, Z^H),
\end{aligned}
$$

we finally have

$$
2g_f(\nabla^f_{X^V} Y^V, Z^V) = 0.
$$

□

2.4.3 Para-Kăhler-Norden Structures of Gradient Sasaki metric of tangent bundle.

Let (M, g) be a Riemannian manifold and (TM, g_f) be its tangent bundle equipped with the gradient Sasaki metric. Consider an almost para-complex structure J on TM defined by

$$
\begin{cases} JX^H &= -X^H, \\ JX^V &= X^V, \end{cases} \tag{2.17}
$$

for all $X \in \Gamma(TM)$.

Lemma 2.4.1. *Let (M, g) be a Riemannian manifold and (TM, J, g_f) its tangent bundle equipped with the almost para-complex structure J. g_f is pure with respect to almost para-complex structure J defined by (2.17), i.e for all $X, Y \in \Gamma(TM)$ and $k, h \in \{H, V\}$,*

$$
g_f(JX^k, Y^h) = g_f(X^k, JY^h).
$$

Proof.

$$
\begin{aligned}
&g_f(JX^H, Y^H) = g_f(-X^H, Y^H) = g_f(X^H, -Y^H) = g_f(X^H, JY^H), \\
&g_f(JX^H, Y^V) = g_f(-X^H, Y^V) = 0 = g_f(X^H, Y^V) = g_f(X^H, JY^V), \\
&g_f(JX^V, Y^H) = g_f(X^V, Y^H) = 0 = g_f(X^V, -Y^H) = g_f(X^V, JY^H), \\
&g_f(JX^V, Y^V) = g_f(X^V, Y^V) = g_f(X^V, JY^V).
\end{aligned}
$$

□

From Lemme 2.4.1, we have the following theorem.

Theorem 2.4.2. *Let (M, g) be a Riemannian manifold, (TM, g_f) be its tangent bundle equipped with the gradient Sasaki metric and the almost para-complex structure J. Then (TM, J, g_f) is an almost para-complex Norden manifold.*

Proposition 2.4.1. *Let (M, g) be a Riemannian manifold, (TM, g_f) be its tangent bundle equipped with the gradient Sasaki metric and the almost para-complex structure J defined by (2.17). for all $X, Y, Z \in \Gamma(TM)$, then we get*

1. $(\Phi_J g_f)(X^H, Y^H, Z^H) = 0,$

2. $(\Phi_J g_f)(X^V, Y^H, Z^H) = 0,$

3. $(\Phi_J g_f)(X^H, Y^V, Z^H) = 2g(R(Z, X)u, Y) + 2Y(f)g(R(Z, X)u, gradf),$

4. $(\Phi_J g_f)(X^H, Y^H, Z^V) = 2g(R(Y, X)u, Z) + 2Z(f)g(R(Y, X)u, gradf),$

5. $(\Phi_J g_f)(X^V, Y^V, Z^H) = 0,$

6. $(\Phi_J g_f)(X^V, Y^H, Z^V) = 0,$

7. $(\Phi_J g_f)(X^H, Y^V, Z^V) = -2Z(f)g(Y, \nabla_X gradf) - 2Y(f)g(Z, \nabla_X gradf),$

8. $(\Phi_J g_f)(X^V, Y^V, Z^V) = 0,$

where R denote the curvature tensor on (M, g).

Proof. For all $X^k, Y^k, Z^k \in \Gamma(TTM)$ with $k \in \{H, V\}$,

$$\begin{aligned}
\Phi_J g_f(X^H, Y^H, Z^H) &= (JX^H)g_f(Y^H, Z^H) - X^H g_f(JY^H, Z^H) + g_f\big((L_{Y^H}J)X^H, Z^H\big) \\
&\quad + g_f\big(Y^H, (L_{Z^H}J)X^H\big) \\
&= (-X^H)g_f(Y^H, Z^H) - X^H g_f(-Y^H, Z^H) \\
&\quad + g_f\Big(L_{Y^H}(JX^H) - J(L_{Y^H}X^H), Z^H\Big) \\
&\quad + g_f\Big(Y^H, (L_{Z^H}JX^H) - J(L_{Z^H}(X^H))\Big) \\
&= g_f\Big([Y^H, -X^H] - J([Y^H, X^H]), Z^H\Big) \\
&\quad + g_f\Big(Y^H, [Z^H, -X^H] - J([Z^H, X^H])\Big) \\
&= g\Big(-[Y, X]^H + (R(Y, X)u)^V, Z^V\Big) \\
&\quad - g\Big(J([Y, X]^H - (R(Y, X)u)^V), Z^V\Big) \\
&\quad + g\Big(-[Z, X]^H + (R(Z, X)u)^V, Y^V\Big)
\end{aligned}$$

$$
\begin{aligned}
&-g\Big(J([Z,X]^H-(R(Z,X)u)^V),Y^V\Big)\\
&= 0,
\end{aligned}
$$

$$
\begin{aligned}
\Phi_J g_f(X^V,Y^H,Z^H) =&(JX^V)g_f(Y^H,Z^H)-X^V g_f(JY^H,Z^H)+g_f\big((L_{Y^H}J)X^V,Z^H\big)\\
&+g_f\big(Y^H,(L_{Z^H}J)X^V\big)\\
=&(X^V)g_f(Y^H,Z^H)-X^V g_f(-Y^H,Z^H)\\
&+g_f\big((L_{Y^H}JX^V)-J(L_{Y^H}X^V),Z^H\big)\\
&+g_f\big(Y^H,(L_{Z^H}JX^V)-J(L_{Z^H}X^V)\big)\\
=&(X^V)g_f(Y,Z)-X^V g_f(-Y,Z)\\
&+g_f\big((\nabla_Y X)^V-J((\nabla_Y X)^V),Z^H\big)\\
&+g_f\big(Y^H,(\nabla_Z X)^V)-J((\nabla_Z X)^V)\big)=0,\\
\Phi_J g_f(X^H,Y^V,Z^H) =&g_f([Y^V,-X^H],Z^H)-g_f(J([Y^V,X^H]),Z^H)\\
&g_f([Z^H,-X^H],Y^V)-g_f(J([Z^H,X^H]),Y^V)\\
=&g_f((\nabla_X Y)^V,Z^H)-g_f(J(-\nabla_X Y)^V,Z^H)\\
&+g_f(-[Z,X]^H+(R(Z,X)u)^V,Y^V)\\
&-g_f(J[Z,X]^H+J(-R(Z,X)u)^V,Y^V)\\
=&2g_f((R(Z,X)u)^V,Y^V)\\
=&2g((R(Z,X)u),Y)+2Y(f)g(R(Z,X)u,gradf),\\
\Phi_J g_f(X^H,Y^H,Z^V) =&g_f([Y^H,-X^H],Z^V)-g_f(J([Y^H,X^H]),Z^V)\\
=&2g((R(Y,X)u),Z)+2Z(f)g(R(Y,X)u,gradf),\\
\Phi_J g_f(X^V,Y^V,Z^H) =&g([Y^V,X^V],Z^H)-g_f(J([Y^V,X^V]),Z^H)\\
&+g_f([Z^H,X^V],Y^V)-g_f(J([Z^H,X^V]),Y^V)\\
=&g_f(\nabla_Z X,Y)-g_f(\nabla_Z X,Y)=0,\\
\Phi_J g_f(X^V,Y^H,Z^V) =&g_f([Y^H,X^V],Z^V)-g_f(J([Y^H,X^V]),Z^V)=0,\\
\Phi_J g_f(X^V,Y^V,Z^V) =&X^V g_f(Y^V,Z^V)-X^V g_f(Y^V,Z^V)+g_f([Y^V,X^V],Z^V)\\
&-g_f(J[Y^V,X^V],Z^V)+g_f([Z^V,X^V],Y^V)\\
&-g(J[Z^V,X^V],Y^V)=0,\\
\Phi_J g_f(X^H,Y^V,Z^V) =&-X^H g_f(Y^V,Z^V)-X^H g_f(Y^V,Z^V)+g_f([Y^V,-X^H],Z^V)\\
&-g_f(J[Y^V,X^H],Z^V)+g_f([Z^V,-X^H],Y^V)\\
&-g(J[Z^V,X^H],Y^V)\\
=&-2X^H\Big(g(Y,Z)+Y(f)Z(f)\Big)+2g((\nabla_X Y)^V,Z^V)\\
&+2g((\nabla_X Z)^V,Y^V)\\
=&-2\Big[g(\nabla_X Y,Z)+g(Y,\nabla_X Z)+Z(f)\big(g(\nabla_X Y,grad(f))
\end{aligned}
$$

$$
\begin{aligned}
&+ g(Y, \nabla_X grad(f))\big) + Y(f)\big(g(\nabla_X Z, grad(f)) + g(Z, \nabla_X grad(f))\big)\Big] \\
&+ 2\Big[g((\nabla_X Y), Z) + (\nabla_X Y)(f)Z(f) \\
&+ g(\nabla_X Z, Y) + (\nabla_X Z)(f)Y(f)\Big] \\
=& - 2g(Y, \nabla_X grad(f)) - 2g(Z, \nabla_X grad(f)),
\end{aligned}
$$

□

Remark 2.4.1. *Let (M, g) be a Riemannian manifold, (TM, g_f) be its tangent bundle equipped with the gradient Sasaki metric. Another almost para-complex structure $\bar{J}$ on TM is defined by*

$$
\begin{cases} \bar{J}X^H &= X^V, \\ \bar{J}X^V &= X^H. \end{cases} \tag{2.18}
$$

The gradient Sasaki metric g_f is pure with respect to $\bar{J}$ if and only if f is constant function (g_f is the Sasaki metric).

Now we study a generalization of the almost para-complex structure defined by (2.17). Let (M, g) be a Riemannian manifold and (TM, g_f) be its tangent bundle equipped with the gradient Sasaki metric. We define an endomorphism $J : TTM \to TTM$ by,

$$
\begin{cases} JX^H &= -X^H + \eta X(f)(grad\, f)^H, \\ JX^V &= X^V + \mu X(f)(grad\, f)^V, \end{cases} \tag{2.19}
$$

for all $X \in \Gamma(TM)$, where $\eta, \mu : M \to \mathbb{R}$ are smooth functions.

Remark 2.4.2.

1. *If $f = constant$ or $\eta = \mu = 0$, we have the almost para-complex structure defined by (2.17),*

2. $J(grad\, f)^H = (-1 + \eta(\alpha - 1))(grad\, f)^H$,

3. $J(grad\, f)^V = (1 + \mu(\alpha - 1))(grad\, f)^V$,

where $\alpha = 1 + \|grad\, f\|^2$.

In the following, we consider $f \neq constant$ and $\eta \neq 0 \neq \mu$.

Lemma 2.4.2.
Let (M, g) be a Riemannian manifold and (TM, g_f^H) be its tangent bundle equipped with the gradient Sasaki metric. The endomorphism J defined by (2.19) is an almost para-complex structure on TM if and only if

$$
\eta = -\mu = \frac{2}{\alpha - 1}.
$$

Proof.
1) Let $Z \in T_pTM$, then $Z = X^H + Y^V$, $X, Y \in T_xM$ and $p = (x, u) \in TM$.

$$\begin{aligned} J^2Z &= J(J(X^H + Y^V)) = J(JX^H + JY^V) \\ &= J\big(-X^H + \eta X(f)(grad\, f)^H + Y^V + \mu Y(f)(grad\, f)^V\big) \\ &= -(-X^H + \eta X(f)(grad\, f)^H) + \eta X(f)(-1 + \eta(\alpha - 1))(grad\, f)^H \\ &\quad +Y^V + \mu Y(f)(grad\, f)^V + \mu Y(f)(1 + \mu(\alpha - 1))(grad\, f)^V \end{aligned}$$

$$\begin{aligned} J^2Z &= X^H + \eta X(f)(-2 + \eta(\alpha - 1))(grad\, f)^H + Y^V \\ &\quad +\mu Y(f)(2 + \mu(\alpha - 1))(grad\, f)^V \\ &= Z + \eta X(f)(-2 + \eta(\alpha - 1))(grad\, f)^H \\ &\quad +\mu Y(f)(2 + \mu(\alpha - 1))(grad\, f)^V, \end{aligned}$$

J is an almost product structure on TM if and only if

$$-2 + \eta(\alpha - 1) = 2 + \mu(\alpha - 1) = 0,$$

i.e

$$\eta = -\mu = \frac{2}{\alpha - 1},$$

in this case J is written in the following form:

$$\begin{cases} JX^H &= -X^H + \dfrac{2}{\alpha - 1} X(f)(grad\, f)^H, \\ JX^V &= X^V - \dfrac{2}{\alpha - 1} X(f)(grad\, f)^V, \end{cases} \tag{2.20}$$

and

$$\begin{cases} J(grad\, f)^H &= (grad\, f)^H, \\ J(grad\, f)^V &= -(grad\, f)^V. \end{cases}$$

2) Let $\{e_1, \cdots, e_m\}$ be a local basis on M, $A_i = e_i^H - \dfrac{1}{\alpha - 1} e_i(f)(grad\, f)^H$ and $B_i = -e_i^V + \dfrac{1}{\alpha - 1} e_i(f)(grad\, f)^V$, for all $i = \overline{1, m}$.

$$\begin{aligned} J(A_i) &= J\Big(e_i^H - \frac{1}{\alpha - 1} e_i(f)(grad\, f)^H\Big) = J(e_i^H) - \frac{1}{\alpha - 1} e_i(f) J(grad\, f)^H \\ &= -e_i^H + \frac{2}{\alpha - 1} e_i(f)(grad\, f)^H - \frac{1}{\alpha - 1} e_i(f)(grad\, f)^H \\ &= -e_i^H + \frac{1}{\alpha - 1} e_i(f)(grad\, f)^H = -A_i. \end{aligned}$$

Then

$$TTM^- = \{Z \in T_pTM, JZ = -Z\} =< (A_i)_{i=\overline{1,m}} > .$$

Similarly we have:

$$J(B_i) \;=\; B_i.$$

Then

$$TTM^+ = \{Z \in T_pTM, JZ = Z\} =< (B_i)_{i=\overline{1,m}} > .$$

□

Proposition 2.4.2. *Let (M, g) be a Riemannian manifold, (TM, g_f) be its tangent bundle equipped with the gradient Sasaki metric and the almost para-complex structure J defined by (2.20), then g_f is pure with respect to J i.e (TM, J, g_f) is an almost para-complex Norden manifold.*

Proof.

$g_f(JX^V, Y^H) = 0 = g_f(X^V, JY^H)$,

$$\begin{aligned}
g_f(JX^H, Y^H) &= g_f\Big(-X^H + \frac{2}{\alpha-1}X(f)grad(f)^H, Y^H\Big)\\
&= g_f(-X^H, Y^H) + \frac{2}{\alpha-1}X(f)g\Big(grad(f)^H, Y^H\Big)\\
&= g_f(-X^H, Y^H) + \frac{2}{\alpha-1}X(f)g\Big(grad(f), Y\Big)\\
&= g_f(X^H, -Y^H) + \frac{2}{\alpha-1}g(X, grad(f))Y(f)\\
&= g_f(X^H, -Y^H) + \frac{2}{\alpha-1}g_f(X^H, grad(f)^H)Y(f)\\
&= g_f(X^H, -Y^H + \frac{2}{\alpha-1}Y(f)grad(f)^H) = g_f(X^H, JY^H),\\
g_f(JX^V, Y^V) &= g_f\Big(X^V - \frac{2}{\alpha-1}X(f)grad(f)^V, Y^V)\\
&= g_f(X^V, Y^V) - \frac{2}{\alpha-1}X(f)\Big[g(Y, (grad(f))) + Y(f)grad(f)(f)\Big]\\
&= g_f(X^V, Y^V) - \frac{2}{\alpha-1}Y(f)\Big[g(X, grad(f)) + g\big(X, grad(f)\big)grad(f)(f)\Big]\\
&= g_f(X^V, Y^V) - \frac{2}{\alpha-1}Y(f)\Big[g(X, grad(f)) + X(f)grad(f)(f)\Big]\\
&= g_f(X^V, Y^V) - \frac{2}{\alpha-1}Y(f)g(X^V, (grad(f))^V)\\
&= g_f(X^V, Y^V - \frac{2}{\alpha-1}Y(f)(grad(f))^V) = g_f(X^V, JY^V).
\end{aligned}$$

□

Now we study a generalization of the almost para-complex structure defined by 2.18. We define an endomorphism $J : TTM \to TTM$ by, for all $X \in \Gamma(TM)$

$$\begin{cases} JX^H &= X^V + \eta X(f)(grad\, f)^V, \\ JX^V &= X^H + \mu X(f)(grad\, f)^H, \end{cases} \tag{2.21}$$

where $\eta, \mu : M \to \mathbb{R}$ are smooth functions.

Remark 2.4.3. **1.** *If $f = constant$ or $\eta = \mu = 0$, we have the almost para-complex structure defined by (2.18),*

2. $J(grad\, f)^H = (1 + \eta(\alpha - 1))(grad\, f)^V$,

3. $J(grad\, f)^V = (1 + \mu(\alpha - 1))(grad\, f)^H$,

where $\alpha = 1 + \|grad\, f\|^2$.

In the following, we consider $f \neq constant$ and $\eta \neq 0 \neq \mu$.

Lemma 2.4.3. *Let (M, g) be a Riemannian manifold and (TM, g_f) be its tangent bundle equipped with the gradient Sasaki metric. The endomorphism J defined by (2.21) is an almost para-complex structure on TM if and only if*

$$\mu = \frac{-\eta}{1 + \eta(\alpha - 1)}.$$

Proof.
1) Let $Z \in T_pTM$, then $Z = X^H + Y^V$, $X, Y \in T_xM$ and $p = (x, u) \in TM$.

$$\begin{aligned} J^2Z &= J(J(X^H + Y^V)) \\ &= J(JX^H + JY^V) \\ &= J\big(X^V + \eta X(f)(grad\, f)^V + Y^H + \mu Y(f)(grad\, f)^H\big) \\ &= X^H + \mu X(f)(grad\, f)^H) + \eta X(f)(1 + \mu(\alpha - 1))(grad\, f)^H \\ &\quad + Y^V + \eta Y(f)(grad\, f)^V + \mu Y(f)(1 + \eta(\alpha - 1))(grad\, f)^V \\ &= Z + X(f)[\mu + \eta(1 + \mu(\alpha - 1))](grad\, f)^H \\ &\quad + Y(f)[\eta + \mu(1 + \eta(\alpha - 1))](grad\, f)^V, \end{aligned}$$

J is an almost product structure on TM if and only if

$$\mu + \eta(1 + \mu(\alpha - 1)) = \eta + \mu(1 + \eta(\alpha - 1)) = 0,$$

i.e

$$\mu = \frac{-\eta}{1 + \eta(\alpha - 1)},$$

in this case we have

$$\begin{cases} J(grad\, f)^H &= \dfrac{-\eta}{\mu}(grad\, f)^V, \\ \\ J(grad\, f)^V &= \dfrac{-\mu}{\eta}(grad\, f)^H, \end{cases}$$

2) Let $\{e_1, \cdots, e_m\}$ be a local basis on M, and for all $i = \overline{1,m}$ we put,
$A_i = e_i^H + \dfrac{\mu}{2}e_i(f)(grad\, f)^H - e_i^V - \dfrac{\eta}{2}e_i(f)(grad\, f)^V$,
$B_i = e_i^H + \dfrac{\mu}{2}e_i(f)(grad\, f)^H + e_i^V + \dfrac{\eta}{2}e_i(f)(grad\, f)^V$.

$$\begin{aligned} J(A_i) &= J\Big(e_i^H + \frac{\mu}{2}e_i(f)(grad\, f)^H - e_i^V - \frac{\eta}{2}e_i(f)(grad\, f)^V\Big) \\ &= Je_i^H + \frac{\mu}{2}e_i(f)J(grad\, f)^H - Je_i^V - \frac{\eta}{2}e_i(f)J(grad\, f)^V \\ &= e_i^V + \frac{\eta}{2}e_i(f)(grad\, f)^V - e_i^H - \frac{\mu}{2}e_i(f)(grad\, f)^H \\ &= -A_i. \end{aligned}$$

Then

$$TTM^- = \{Z \in T_pTM, JZ = -Z\} =< (A_i)_{i=\overline{1,m}} > .$$

Similarly we have:

$$J(B_i) \;=\; B_i.$$

Then

$$TTM^+ = \{Z \in T_pTM, JZ = Z\} =< (B_i)_{i=\overline{1,m}} > .$$

□

Proposition 2.4.3.
Let (M, g) be a Riemannian manifold, (TM, g_f) be its tangent bundle equipped with the gradient Sasaki metric and the almost para-complex structure J defined by (2.21), then g_f is pure with respect to J if $\eta = 1 + \mu\alpha$.

Proof.

$$\begin{aligned} g_f(JX^H, Y^H) = g_f(X^V + \eta X(f)(grad(f))^V, Y^H) = 0 = g_f(X^H, JY^H), \\ g_f(JX^V, Y^V) = g_f(X^V + \mu X(f)(grad(f))^V, Y^H) = 0 = g_f(X^V, JY^V), \end{aligned}$$

we have

$$\begin{aligned} g_f(JX^H, Y^V) &= g_f(X^V + \eta X(f)((grad(f))^V, Y^V) \\ &= g(X, Y) + X(f)Y(f) + \eta X(f)g_f(Y^V, (grad(f))^V) \end{aligned}$$

$$= g(X,Y) + X(f)Y(f) + \eta X(f)\Big[g(Y, grad(f)) + Y(f)grad(f)(f)\Big]$$
$$= g(X,Y) + X(f)Y(f) + \eta X(f)\Big[g(Y(f) + Y(f)grad(f)(f)\Big]$$
$$= g(X,Y) + X(f)Y(f)\big(1 + \eta + (\alpha - 1)\big),$$
$$g_f(X^H, JY^V) = g_f(X^H, Y^H + \mu X(f)((grad(f))^H)) = g(X,Y) + X(f)Y(f)\eta,$$

in order for g_f to be pure with respect to J we need that $\eta = 1 + \alpha$. □

2.4.4 Almost product connection symmetric of gradient Sasaki metric

Let (M,g) be a Riemannian manifold, (TM, g_f) be its tangent bundle equipped with the gradient Sasaki metric and the almost product structure J defined by (2.17).
∇^f denote the Levi-Civita connection of (TM, g_f). We define a tensor field S^f of type $(1,2)$ and a linear connection $\overline{\nabla}$ on TM, for all $\widetilde{X}, \widetilde{Y} \in \Gamma(TTM)$ by,

$$S^f(\widetilde{X}, \widetilde{Y}) = \frac{1}{2}\Big[(\nabla^f_{J\widetilde{Y}} J)\widetilde{X} + J(\nabla^f_{\widetilde{Y}} J)\widetilde{X} - J(\nabla^f_{\widetilde{X}} J)\widetilde{Y}\Big], \tag{2.22}$$

$$\overline{\nabla}_{\widetilde{X}} \widetilde{Y} = \nabla^f_{\widetilde{X}} \widetilde{Y} - S(\widetilde{X}, \widetilde{Y}). \tag{2.23}$$

$\overline{\nabla}$ is an almost product connection on TM,

Then, we have the following lemma.

Lemma 2.4.4. *Let (M,g) be a Riemannian manifold, (TM, g_f) be its tangent bundle equipped with the gradient Sasaki metric and the almost product structure J defined by (2.17). Then, tensor field S is given by*

$$S^f(X^H, Y^H) = -\frac{1}{2}(R(X,Y)u)^V,$$
$$S^f(X^H, Y^V) = -Y(f)(\nabla_X grad(f))^V - \frac{1}{\alpha}\Big[g(Y, \nabla_X grad(f)) - \frac{1}{2}X(\alpha)Y(f))\Big](grad(f))^V + \frac{1}{2}(R(u,Y)X)^H + \frac{1}{2}Y(f)(R(u, grad(f))X)^H,$$
$$S^f(X^V, Y^H) = -(R(u,X)Y)^H_p - X(f)(R(u, grad(f))Y)^H + \frac{1}{2}X(f)(\nabla_Y grad(f))^V + \frac{1}{2\alpha}\big[g(X, \nabla_Y grad(f)) - \frac{1}{2}Y(\alpha)X(f)\big](grad(f))^V,$$
$$S^f(X^V, Y^V) = -\frac{1}{2}X(f)(\nabla_Y grad(f))^H - \frac{1}{2}(Y(f)(\nabla_X grad(f))^H.$$

For all $X, Y \in \Gamma(TM)$.

Proof. Using the Theorem 2.4.1, we have for all $X, Y \in \Gamma(TM)$

$$\begin{aligned}
S^f(X^V,Y^V) =& \frac{1}{2}\Big[(\nabla^f{}_{JY^V}J)X^V + J(\nabla^f_{Y^V}J)X^V - J(\nabla^f_{X^V}J)Y^V\Big]\\
=&\frac{1}{2}\Big[\nabla^f_{JY^V}JX^V - J(\nabla^f_{JY^V}X^V) + J(\nabla^f_{Y^V}JX^V - J(\nabla^f_{Y^V}X^V))\\
&- J(\nabla^f_{X^V}JY^V - J(\nabla^f_{X^V}Y^V))\Big],\\
S^f(X^V,Y^V) =& \frac{1}{2}\Big[\nabla^f_{Y^V}X^V - J(\nabla^f_{Y^V}X^V) + J(\nabla^f_{Y^V}JX^V) - (\nabla^f_{Y^V}X^V)\\
&- J(\nabla^f_{X^V}Y^V) + (\nabla^f_{X^V}Y^V)\Big],
\end{aligned}$$

from Theorem 2.4.1 we have that $\nabla^f_{X^V}Y^V$ is horizontal, then we have $J(\nabla^f_{X^V}Y^V) = -\nabla^f_{X^V}Y^V$ and we get

$$\begin{aligned}
S^f(X^V,Y^V) =& \frac{1}{2}\Big[\nabla^f_{Y^V}X^V + (\nabla^f_{Y^V}X^V) - (\nabla^f_{Y^V}JX^V) - (\nabla^f_{Y^V}X^V)\\
&+ (\nabla^f_{X^V}Y^V) + (\nabla^f_{X^V}Y^V)\Big] = \Big[(\nabla^f_{X^V}Y^V)\Big]\\
=& -\frac{1}{2}X(f)(\nabla_Y grad(f))^H - \frac{1}{2}(Y(f)(\nabla_X grad(f))^H,\\
S^f(X^V,Y^H) =& \frac{1}{2}\Big[(\nabla^f{}_{JY^H}J)X^V + J(\nabla^f_{Y^H}J)X^V - J(\nabla^f_{X^V}J)Y^H\Big]\\
=&\frac{1}{2}\Big[\nabla^f_{JY^H}JX^V - J(\nabla^f_{JY^H}X^V) + J(\nabla^f_{Y^H}JX^V) - JJ(\nabla^f_{Y^H}X^V))\\
&- J(\nabla^f_{X^V}JY^H) + JJ(\nabla^f_{X^V}Y^H)\Big]\\
=&\frac{1}{2}\Big[-\nabla^f_{Y^H}X^V + J(\nabla^f_{Y^H}X^V) + J(\nabla^f_{Y^H}X^V) - (\nabla^f_{Y^H}X^V))\\
&+ J(\nabla^f_{X^V}Y^H) + (\nabla^f_{X^V}Y^H)\Big]\\
=&\frac{1}{2}\Big[-2\nabla^f_{Y^H}X^V + 2J(\nabla^f_{Y^H}X^V) + J(\nabla^f_{X^V}Y^H) + (\nabla^f_{X^V}Y^H)\Big]\\
=&\frac{1}{2}\Big[-2(R(u,X)Y)^H_p - 2X(f)(R(u,grad(f))Y)^H\\
&+ X(f)(\nabla_Y grad(f))^V + \frac{1}{\alpha}\big[g(X,\nabla_Y grad(f)) - \frac{1}{2}Y(\alpha)X(f)\big](grad(f))^V\Big]\\
=& -(R(u,X)Y)^H_p - X(f)(R(u,grad(f))Y)^H\\
&+ \frac{1}{2}X(f)(\nabla_Y grad(f))^V + \frac{1}{2\alpha}\big[g(X,\nabla_Y grad(f)) - \frac{1}{2}Y(\alpha)X(f)\big](grad(f))^V,
\end{aligned}$$

$$\begin{aligned}
S^f(X^H,Y^H) =& \frac{1}{2}[\nabla^f_{-Y^H}(-X^H) - J(\nabla^f_{-Y^H}X^H) + J(\nabla^f_{Y^H} - X^H) - \nabla^f_{Y^H}X^H\\
&- J(\nabla^f_{X^H} - Y^H) + \nabla^f_{X^H}Y^H]
\end{aligned}$$

$$=\frac{1}{2}[J(\nabla^f_{X^H}Y^H)+\nabla^f_{X^H}Y^H]=-\frac{1}{2}(R(X,Y)u)^V,$$

$$\begin{aligned}
S^f(X^H,Y^V)=&\frac{1}{2}\Big[(\nabla^f{}_{JY^V}J)X^H+J(\nabla^f_{Y^V}J)X^H-J(\nabla^f_{X^H}J)Y^V\Big]\\
=&\frac{1}{2}\Big[\nabla^f{}_{Y^V}(-X^H)-J(\nabla^f{}_{Y^V}(X^H))+J(\nabla^f_{Y^V}(-X^H))\\
&-JJ(\nabla^f_{Y^V}(X^H))-J(\nabla^f_{X^H}Y^V)+JJ(\nabla^f_{X^H}Y^V)\Big]\\
=&\frac{1}{2}\Big[-2\nabla^f{}_{Y^V}X^H-2J(\nabla^f{}_{Y^V}X^H)-J(\nabla^f_{X^H}Y^V)+(\nabla^f_{X^H}Y^V)\Big]\\
=&-\nabla^f{}_{Y^V}X^H-J(\nabla^f{}_{Y^V}X^H)-\frac{1}{2}J(\nabla^f_{X^H}Y^V)+\frac{1}{2}(\nabla^f_{X^H}Y^V)\\
=&-Y(f)(\nabla_X grad(f))^V-\frac{1}{\alpha}\Big[g(Y,\nabla_X grad(f))-\frac{1}{2}X(\alpha)Y(f))\Big](grad(f))^V\\
&+\frac{1}{2}(R(u,Y)X)^H+\frac{1}{2}Y(f)(R(u,grad(f))X)^H.
\end{aligned}$$

□

Theorem 2.4.3. *Let (M,g) be a Riemannian manifold, (TM,g_f) be its tangent bundle equipped with the gradient Sasaki metric and the almost product structure J defined by (2.17). Then the almost product connection $\overline{\nabla}$ defined by (2.23) is given by*

$$\begin{aligned}
\overline{\nabla}_{X^H}Y^H=&(\nabla_X Y)^H,\\
\overline{\nabla}_{X^H}Y^V=&(\nabla_X Y)^V+\frac{3}{2}Y(f)(\nabla_X grad(f))^V\\
&+\frac{3}{2\alpha}\Big[g(Y,\nabla_X grad(f))-\frac{1}{2}X(\alpha)Y(f)\Big](grad(f))^V,\\
\overline{\nabla}_{X^V}Y^H=&\frac{3}{2}(R(u,X)Y)^H+\frac{3}{2}X(f)(R(u,grad(f))Y)^V,\\
\overline{\nabla}_{X^V}Y^V=&0,
\end{aligned}$$

for all vector fields $X,Y\in\Gamma(TM),(x,u)\in TM$.

Lemma 2.4.5. *Let (M,g) be a Riemannian manifold, (TM,g_f) be its tangent bundle equipped with the gradient Sasaki metric and the almost product structure J defined by (2.17). Then the torsion tensor $\overline{T}$ of $\overline{\nabla}$, is given by*

- $\overline{T}(X^H,Y^H)=(R(X,Y)u)^V,$
- $\overline{T}(X^H,Y^V)=\frac{3}{2}\Big[-(R(u,Y)X)^H-Y(f)(R(u,grad(f))X)^H+Y(f)(\nabla_X grad(f))^V + \frac{3}{2\alpha}\big[g(Y,\nabla_X grad(f))-\frac{1}{2}X(\alpha)Y(f)\big](grad(f))^V\Big],$

- $\overline{T}(X^V, Y^H) = -\frac{3}{2}\Big[-(R(u,X)Y)^H - X(f)(R(u,grad(f))Y)^H + X(f)(\nabla_Y grad(f))^V$
$+ \frac{3}{2\alpha}\big[g(X, \nabla_Y grad(f)) - \frac{1}{2}Y(\alpha)X(f)\big](grad(f))^V\Big],$

- $\overline{T}(X^V, Y^V) = 0,$

for all $X, Y \in \Gamma(TM)$.

2.5 Fiberwise deformed Sasaki metric on tangent bundle

2.5.1 Almost complex structure on tangent bundle

We now introduce another structure on the tangent bundle induced by the vertical and horizontal lift, this structure is defined by :

Definition 2.5.1. *Let* $J : TTM \to TTM$ *be the linear endomorphism of the tangent bundle characterized by*

$$d\pi(JX) = -K(X) \text{ and } K(JA) = d\pi(X),$$

using definition 2.1.12 we get

$$J(X^H) = X^V \text{ and } J(X^V) = -X^H,$$

and so we have $J^2 = -Id_{TTM}$ *and that it define an almost complex structure on* TM.

2.6 Berger-type deformed Sasaki metric on the tangent bundle

Let (M, g, J) be a Hermitian manifold of dimension $2n$ with an almost paracomplex structure J, i.e. the $(1,1)$-tensor field J satisfying $J^2 = id$. Denote by TM_0 a slashed tangent bundle, i.e. the tangent bundle with zero section deleted. Define a fiber-wise Berger-type deformation of the Sasaki metric on TM defined by

Definition 2.6.1. *Let* (M^{2k}, ϕ, g) *be an almost anti-paraHermitian manifold and* TM *be its tangent bundle.The Berger type deformed Sasaki metric on* TM *is defined by*

1. $g^{BS}(X^H, Y^H) = g(X, Y)$
2. $g^{BS}(X^H, Y^V) = 0$
3. $g^{BS}(X^V, Y^V) = g(X, Y) + \delta^2 g(X, \phi u)g(Y, \phi u)$

for all vector fields $X, Y \in \Gamma(TM)$, where δ is some constant. If $\delta = 0$ then g^{BS} is called the Sasaki metric.

and $\widetilde{\phi}$ in TM is defined by

$$\widetilde{\phi}(X^H) = X^H, \quad \widetilde{\phi}(X^V) = -X^V \tag{2.24}$$

for all vector field $X \in \Gamma(TM)$.
The almost paracomplex structure $\widetilde{\phi}$ on TM satisfy the purity condition. i.e

$$g^{BS}(\widetilde{\phi}\widetilde{X}, \widetilde{Y}) = g^{BS}(\widetilde{X}, \widetilde{\phi}\widetilde{Y})$$

for all vector fields $\widetilde{X}$, $\widetilde{Y}$

Proof. Let $X, Y \in \Gamma(TM))$ and g^{BS} the Berger Type deformed Sasaki metric on TM then

$$\begin{aligned}
g^{BS}(\widetilde{\phi}X^H, Y^H) =& g^{BS}(X^H, Y^H) = g^{BS}(X^H, \widetilde{\phi}Y^H) \\
g^{BS}(\widetilde{\phi}X^V, Y^H) =& g^{BS}(-X^V, Y^H) = 0 = g^{BS}(X^V, \widetilde{\phi}Y^H) \\
g^{BS}(\widetilde{\phi}X^V, Y^V) =& g^{BS}(-X^V, Y^V) = g^{BS}(X^V, -Y^V) = g^{BS}(X^V, \widetilde{\phi}Y^V)
\end{aligned}$$

□

Lemma 2.6.1. *Let (M, g) be a Riemannian manifold. Then we have*

- $X^H g(Y, u) = g(\nabla_X Y, u)$.
- $X^V g(Y, u) = g(X, Y)$,

for all $X, Y \in \Gamma(TM)$.

Proof. Locally, we have $u = y^i \frac{\partial}{\partial x^i}$, $X = X^i \frac{\partial}{\partial x^i}$ et $Y = Y^i \frac{\partial}{\partial x^i}$, then

$$\begin{aligned}
X^H g(Y, u) =& \Big(X^i \frac{\partial}{\partial x^i} - X^i y^j \Gamma_{ij}^k \frac{\partial}{\partial y^k}\Big) g(Y, u) \\
=& X^i \frac{\partial}{\partial x^i} g(Y, u) - X^i y^j \Gamma_{ij}^k \frac{\partial}{\partial y^k}(g_{lm} Y^l y^m) \\
=& g(\nabla_X Y, u) + g(Y, \nabla_X u) - X^i y^j \Gamma_{ij}^k (g_{lm} Y^l) \\
=& g(\nabla_X Y, u). \\
X^V g(Y, u) =& X^i \frac{\partial}{\partial y^i} g_{lm} Y^l y^m = X^i g_{li} Y^l = g(X, Y).
\end{aligned}$$

□

Lemma 2.6.2. *Let (M_{2k}, ϕ, g) be an anti-paraKähler, we have the following*

- $X^H g(Y, \phi u) = g(\nabla_X Y, \phi u)$,
- $X^V g(Y, \phi u) = g(Y, \phi X)$,

for all $X, Y \in \Gamma(TM)$.

Proof. The proof come directly from Lemma 2.6.1. □

Theorem 2.6.1. *Let* (M_{2k}, ϕ, g) *be an anti-paraKähler manifold and* (TM, g^{BS}) *be the tangent bundle equipped with the Berger type deformed Sasaki metric* g^{BS} *and the paracomplex structure* $\widetilde{\phi}$. *The triple* $(TM, \widetilde{\phi}, g^{BS})$ *is an anti-paraKähler manifold if and only if* M_{2k} *is flat.*

Proof.

$$\begin{aligned}
\Phi(X^H, Y^H, Z^H) =& (\widetilde{\phi} X^H)(g^{BS}(Y^H, Z^H)) - X^H(g^{BS}(\widetilde{\phi} Y^H, Z^H)) + g^{BS}((L_{Y^H}\widetilde{\phi})X^H, Z^H) \\
&+ g^{BS}((L_{Z^H}\widetilde{\phi})X^H, Y^H) \\
=& X^H(g^{BS}(Y^H, Z^H)) - X^H(g^{BS}(Y^H, Z^H)) \\
&+ g^{BS}\Big(L_{Y^H}\widetilde{\phi}(X^H) - \widetilde{\phi}(L_{Y^H}X^H), Z^H\Big) \\
&+ g^{BS}\Big(L_{Z^H}\widetilde{\phi}(X^H) - \widetilde{\phi}(L_{Z^H}X^H), Y^H\Big) \\
=& g^{BS}\Big([Y, X]^H, Z^H\Big) - g^{BS}\Big([Y, X]^H, Z^H\Big) \\
&+ g^{BS}\Big([Z, X]^H, Y^H\Big) - g^{BS}\Big([Z, X]^H, Y^H\Big) = 0,
\end{aligned}$$

$$\begin{aligned}
\Phi(X^V, Y^H, Z^H) =& (\widetilde{\phi} X^V)(g^{BS}(Y^H, Z^H)) - X^V(g^{BS}(\widetilde{\phi} Y^H, Z^H)) + g^{BS}((L_{Y^H}\widetilde{\phi})X^V, Z^H) \\
&+ g^{BS}((L_{Z^H}\widetilde{\phi})X^V, Y^H) \\
=& -2X^V g^{BS}(Y^H, Z^H) + g^{BS}([Y^H, -X^V], Z^H) - g^{BS}(\widetilde{\phi}([Y^H, X^V]), Z^H) \\
&+ g^{BS}([Z^H, -X^V], Y^H) - g^{BS}(\widetilde{\phi}([Z^H, X^V]), Y^H) \\
=& g^{BS}(-(\nabla_Y X)^V, Z^H) - g^{BS}(-((\nabla_Y X)^V), Z^H) \\
&+ g^{BS}(-(\nabla_Z X)^V, Y^H) - g^{BS}(-(\nabla_Z X)^V, Y^H) = 0,
\end{aligned}$$

$$\begin{aligned}
\Phi(X^H, Y^V, Z^H) =& (X^H)(g^{BS}(Y^V, Z^H)) - X^H(g^{BS}(-Y^V, Z^H)) + g^{BS}((L_{Y^V}\widetilde{\phi})X^H, Z^H) \\
&+ g^{BS}((L_{Z^H}\widetilde{\phi})X^H, Y^V) \\
=& g^{BS}(([Y^V, X^H], Z^H) - g^{BS}(\widetilde{\phi}([Y^V, X^H]), Z^H) \\
&+ g^{BS}(([Z^H, X^H], Y^V) - g^{BS}(\widetilde{\phi}([Z^H, X^H]), Y^V) \\
=& -2g^{BS}((R(Z, X)u)^V, Y^V) \\
=& -2g(R(Z, X)u, Y) - 2\delta^2 g(R(Z, X)u, \phi u)g(Y, \phi u),
\end{aligned}$$

$$\begin{aligned}\Phi(X^H,Y^H,Z^V)=&(X^H)(g^{BS}(Z^V,Y^H))-X^H(g^{BS}(Y^H,Z^V))+g^{BS}((L_{Z^V}\widetilde{\phi})X^H,Y^H)\\&+g^{BS}((L_{Y^H}\widetilde{\phi})X^H,Z^V)\\=&g^{BS}(([Z^V,X^H],Y^H)-g^{BS}(\widetilde{\phi}([Z^V,X^H]),Y^H)\\&+g^{BS}(([Y^H,X^H],Z^V)-g^{BS}(\widetilde{\phi}([Y^H,X^H]),Z^V)\\=&-2g(R(Y,X)u,Z)-2\delta^2g(R(Y,X)u,\phi u)g(Z,\phi u),\end{aligned}$$

$$\begin{aligned}\Phi(X^V,Y^V,Z^H)=&-(X^V)(g^{BS}(Z^H,Y^V))-X^V(g^{BS}(-Y^V,Z^H))+g^{BS}((L_{Z^H}\widetilde{\phi})X^V,Y^V)\\&+g^{BS}((L_{Y^V}\widetilde{\phi})X^V,Z^H)\\=&g^{BS}((L_{Z^H}\widetilde{\phi})X^V,Y^V)=g^{BS}([Z^H,-X^V],Y^V)-g^{BS}(\widetilde{\phi}([Z^H,X^V],Y^V))\\=&g^{BS}(-(\nabla_ZX)^V,Y^V)-g^{BS}(\widetilde{\phi}((\nabla_ZX)^V),Y^V)=0,\end{aligned}$$

$$\begin{aligned}\Phi(X^V,Y^H,Z^V)=&-(X^V)(g^{BS}(Y^H,Z^V))-X^V(g^{BS}(Y^H,Z^V))+g^{BS}((L_{Y^H}\widetilde{\phi})X^V,Z^V)\\&+g^{BS}((L_{Z^V}\widetilde{\phi})X^V,Y^H)\\=&g^{BS}(-(\nabla_YX)^V,Z^V)-g^{BS}(\widetilde{\phi}((\nabla_YX)^V),Z^V)=0,\end{aligned}$$

$$\begin{aligned}\Phi(X^H,Y^V,Z^V)=&(\widetilde{\phi}X^H)(g^{BS}(Y^V,Z^V))-X^H(g^{BS}(\widetilde{\phi}Y^V,Z^V))+g^{BS}((L_{Y^V}\widetilde{\phi})X^H,Z^V)\\&+g^{BS}((L_{Z^V}\widetilde{\phi})X^H,Y^V)\\=&2(X^H)(g^{BS}(Y^V,Z^V))+g^{BS}([Y^V,X^H],Z^V)-g^{BS}(\widetilde{\phi}([Y^V,X^H]),Z^V)\\&+g^{BS}([Z^V,X^H],Y^V)-g^{BS}(\widetilde{\phi}([Z^V,X^H]),Y^V)\\=&2(X^H)(g^{BS}(Y^V,Z^V))+g^{BS}(-(\nabla_XY)^V,Z^V)-g^{BS}(\widetilde{\phi}(-(\nabla_XY)^V),Z^V)\\&+g^{BS}((-(\nabla_XZ)^V,Y^V)-g^{BS}(\widetilde{\phi}(-(\nabla_XZ)^V),Y^V)\\=&2(X^H)(g^{BS}(Y^V,Z^V))-2g^{BS}((\nabla_XY)^V,Z^V)-2g^{BS}(\nabla_XZ)^V,Y^V),\end{aligned}$$

using lemme 2.6.2 we have

$$\begin{aligned}\Phi(X^H,Y^V,Z^V)=&2\Big[g(\nabla_XY,Z)+g(\nabla_XZ,Y)+g(\nabla_XY,\phi u)g(Z,\phi u)+g(\nabla_XZ,\phi u)g(Y,\phi u)\Big]\\&-2\Big[g(\nabla_XY,Z)+g(\nabla_XY,\phi u)g(Z,\phi u)\Big]\\&-2\Big[g(\nabla_XZ,Y)+g(\nabla_XZ,\phi u)g(Y,\phi u)\Big]=0,\end{aligned}$$

$$\begin{aligned}\Phi(X^V,Y^V,Z^V)=&(\widetilde{\phi}X^V)(g^{BS}(Y^V,Z^V))-X^V(g^{BS}(\widetilde{\phi}Y^V,Z^V))+g^{BS}((L_{Y^V}\widetilde{\phi})X^V,Z^V)\\&+g^{BS}((L_{Z^V}\widetilde{\phi})X^V,Y^V)\\=&(-X^V)(g^{BS}(Y^V,Z^V))-X^V(g^{BS}(-Y^V,Z^V))=0.\end{aligned}$$

□

2.6.1 Levi-Civita connection

A direct result of usual calculations using the Koszul formula gives the following result

Proposition 2.6.1. *Let (M_{2k},ϕ,g) be an anti-paraKähler manifold and TM be its tangent bundle. The Levi-Civita connection of the Berger type deformed Sasaki metric g^{BS} on TM is given by*

$$\begin{aligned}
(\widetilde{\nabla}_{X^H}Y^H)_{(p,u)} &= (\nabla_X Y)^H_{(p,u)} - \frac{1}{2}(R_x(X,Y)u)^V,\\
(\widetilde{\nabla}_{X^H}Y^V)_{(p,u)} &= (\nabla_X Y)^V_{(p,u)} + \frac{1}{2}(R_x(u,Y)X)^H,\\
(\widetilde{\nabla}_{X^V}Y^H)_{(p,u)} &= \frac{1}{2}(R_x(u,X)Y)^H_{(p,u)},\\
(\widetilde{\nabla}_{X^V}Y^V)_{(p,u)} &= \frac{\delta^2}{1+\delta^2\alpha}g(X,\phi Y)(\phi u)^V,
\end{aligned}$$

where ∇ is the Levi-Civita connection on (M,ϕ,g), R is its Riemannian curvature tensor and $\alpha=g(u,u)$.

Lemma 2.6.3. *Let (M,ϕ,g) be an an anti-paraKähler manifold and the tangent bundle TM be equipped with a metric g^* which is natural with respect to g on M. If $F:TM\to TM$ is a smooth bundle endomorphism of the tangent bundle, then*

$$\begin{aligned}
(\nabla^*_{X^V}VF)_{(p,u)} &= F(X)^V_{(p,u)} + \sum_{i=1}^{m} u(x_i)(\nabla^*_{X^V}F(\partial_i)^V)_{(p,u)}.\\
(\nabla^*_{X^V}HF)_{(p,u)} &= F(X)^H_{(p,u)} + \sum_{i=1}^{m} u(x_i)(\nabla^*_{X^V}F(\partial_i)^H)_{(p,u)}.\\
(\nabla^*_{X^H}VF)_{(p,u)} &= (\nabla^*_{X^H}F(\partial_i)^V)_{(p,u)}.\\
(\nabla^*_{X^H}HF)_{(p,u)} &= (\nabla^*_{X^H}F(\partial_i)^H)_{(p,u)}.
\end{aligned}$$

for any $X\in\Gamma(TM)$, $(p,u)\in TM$.

Proof. Let $(x_1,...,x_n)$ be local coordinates on M in a neighborhood V of p. Then, we have

$$\begin{aligned}
(\nabla^*_{X^V}VF)_{(p,u)} =& \nabla^*_{X^V}(y^i)F(\frac{\partial}{\partial x^i})^V = X^V(y^i)F(\frac{\partial}{\partial x^i})^V + y^i\nabla^*_{X^V}F(\frac{\partial}{\partial x^i})^V\\
=& X^iF(\frac{\partial}{\partial x^i})^V + y^i\nabla^*_{X^V}F(\frac{\partial}{\partial x^i})^V\\
=& F(X)^V + y^i\nabla^*_{X^V}F(\frac{\partial}{\partial x^i})^V,\\
(\nabla^*_{X^V}HF)_{(p,u)} =& \nabla^*_{X^V}(y^i)F(\frac{\partial}{\partial x^i})^H = X^V(y^i)F(\frac{\partial}{\partial x^i})^H + y^i\nabla^*_{X^V}F(\frac{\partial}{\partial x^i})^H
\end{aligned}$$

$$= X^i F(\frac{\partial}{\partial x^i})^H + y^i \nabla^*_{X^V} F(\frac{\partial}{\partial x^i})^H,$$

we prove the remaining two in the same way. □

Proposition 2.6.2. *Let (M_{2k}, ϕ, g) be an anti-paraKähler manifold and TM be its tangent bundle. If $F \in \mathfrak{T}^1_1(M)$ is a tensor of type (1,1) then*

$$(\widetilde{\nabla}_{X^H} HF)_{(x,u)} = (\nabla_X F)^H_{(x,u)} - \frac{1}{2}(R_x(X_x, F_x(u))u)^V.$$
$$(\widetilde{\nabla}_{X^H} VF)_{(x,u)} = (\nabla_X F)^V_{(x,u)} + \frac{1}{2}(R_x(u, F_x(u))X_x)^H.$$
$$(\widetilde{\nabla}_{X^V} HF)_{(x,u)} = (F(X))^H_{(x,u)} + \frac{1}{2}(R_x(u, X_x)F(u))^H.$$
$$(\widetilde{\nabla}_{X^V} VF)_{(x,u)} = (F(X))^V_{(x,u)} + \frac{\delta^2}{1+\delta^2\alpha} g(X, \phi F(u))(\phi u)^V.$$

where $(x,u) \in TM$ and $X \in \Gamma(TM)$.

Theorem 2.6.2. *Let (M_{2k}, ϕ, g) be an anti-paraKähler manifold and TM be its tangent bundle equipped with the Berger type deformed Sasaki metric g^{BS}. Then the corresponding Riemannian curvature tensor $\widetilde{R}$ is given by*

$$\begin{aligned}
\widetilde{R}(X^H, Y^H)Z^H =& \Big[R(X,Y)Z + \frac{1}{4}R(u, R(Z,Y)u)X + \frac{1}{4}R(u, R(X,Z)u)Y \\
&+ \frac{1}{2}R(u, R(X,Y)u)Z\Big]^H + \frac{1}{2}((\nabla_Z R)(X,Y)u)^V \\
\widetilde{R}(X^H, Y^H)Z^V =& \frac{1}{2}\Big[(\nabla_X R)(u,Z)Y - (\nabla_Y R)(u,Z)X\Big]^H + \Big[R(X,Y)Z + \frac{1}{4}R(R(u,Z)Y, X)u \\
&- \frac{1}{4}R(R(u,Z)X, Y)u\Big]^V + \frac{\delta^2}{1+\delta^2\alpha} g(R(X,Y)u, \phi Z)(\phi u)^V, \\
\widetilde{R}(X^H, Y^V)Z^H =& \frac{1}{2}((\nabla_X R)(u,Y)Z)^H + \Big[\frac{1}{4}R(R(u,Y)Z, X)u + \frac{1}{2}R(X,Z)Y)\Big]^V, \\
&+ \frac{\delta^2}{1+\delta^2\alpha} g(R(X,Z)u, \phi Y)(\phi u)^V \\
\widetilde{R}(X^H, Y^V)Z^V =& \Big[-\frac{1}{2}R(Y,Z)X - \frac{1}{4}R(u,Y)R(u,Z)X\Big]^H, \\
\widetilde{R}(X^V, Y^V)Z^H =& \Big[R(X,Y)Z + \frac{1}{4}R(u,X)R(u,Y)Z - \frac{1}{4}R(u,Y)R(u,X)Z\Big]^H \\
\widetilde{R}(X^V, Y^V)Z^V =& \frac{\delta^4}{(1+\delta^2\alpha)^2}\big(g(Y,u)g(X,\phi Z) - g(X,u)g(Y,\phi Z)\big)(\phi u)^V.
\end{aligned}$$

2.6.2 Almost product connection symmetric

Let ∇ be an arbitrary linear connection on M and $\widetilde{S}$ be the $(1,2)$-tensor field defined by

$$\widetilde{S}(\widetilde{X},\widetilde{Y}) = \frac{1}{2}\Big[(\widetilde{\nabla}_{\widetilde{\phi}\widetilde{Y}}\widetilde{\phi})\widetilde{X} + \widetilde{\phi}(\widetilde{\nabla}_{\widetilde{Y}}\widetilde{\phi})\widetilde{X} - \widetilde{\phi}(\widetilde{\nabla}_{\widetilde{X}}\widetilde{\phi})\widetilde{Y}\Big]. \tag{2.25}$$

Proposition 2.6.3. *Let (M_{2k},ϕ,g) be an anti-paraKähler manifold, TM be its tangent bundle equipped with the Berger type deformed Sasaki metric g^{BS} and the paracomplex structure $\widetilde{\phi}$. Then tensor field $\widetilde{S}$ is as follows,*

$$\begin{aligned}
\widetilde{S}(X^V,Y^V) =& 0,\\
\widetilde{S}(X^V,Y^H) =& -(R(u,X)Y)^H,\\
\widetilde{S}(X^H,Y^V) =& \frac{1}{2}(R(u,Y)X)^H,\\
\widetilde{S}(X^H,Y^H) =& -\frac{1}{2}(R(X,Y)u)^V,
\end{aligned}$$

for all $X,Y\in\Gamma(TM)$.

Proof. Let X and Y be vectors field on TM, then using (2.25) and 2.24 we get,

$$\begin{aligned}
\widetilde{S}(X^V,Y^V) =& \frac{1}{2}\Big[(\widetilde{\nabla}_{\widetilde{\phi}Y^V}\widetilde{\phi})X^V + \widetilde{\phi}(\widetilde{\nabla}_{Y^V}\widetilde{\phi})X^V - \widetilde{\phi}(\widetilde{\nabla}_{X^V}\widetilde{\phi})Y^V\Big]\\
=& \frac{1}{2}\Big[\widetilde{\nabla}_{-Y^V}(-X^V) - \widetilde{\phi}(\widetilde{\nabla}_{-Y^V}X^V) + \widetilde{\phi}\Big(\widetilde{\nabla}_{Y^V}(-X^V)\\
&- \widetilde{\phi}(\widetilde{\nabla}_{Y^V}X^V)\Big) - \widetilde{\phi}\Big(\widetilde{\nabla}_{X^V}(-Y^V) + \widetilde{\phi}\widetilde{\nabla}_{X^V}Y^V\Big))\Big] = 0,
\end{aligned}$$

$$\begin{aligned}
\widetilde{S}(X^V,Y^H) =& \frac{1}{2}\Big[(\widetilde{\nabla}_{\widetilde{\phi}Y^H}\widetilde{\phi})X^V + \widetilde{\phi}(\widetilde{\nabla}_{Y^H}\widetilde{\phi})X^V - \widetilde{\phi}(\widetilde{\nabla}_{X^V}\widetilde{\phi})Y^H\Big]\\
=& \frac{1}{2}\Big[\widetilde{\nabla}_{Y^H}(-X^V) - \widetilde{\phi}(\widetilde{\nabla}_{Y^H}X^V) + \widetilde{\phi}(\widetilde{\nabla}_{Y^H}(-X^V)\\
&- \widetilde{\phi}(\widetilde{\nabla}_{Y^H}X^V)) - \widetilde{\phi}(\widetilde{\nabla}_{X^V}(Y^H) + \widetilde{\phi}(\widetilde{\nabla}_{X^V}Y^H))\Big]\\
=& \frac{1}{2}\Big[-(\nabla_Y X)^V - \frac{1}{2}(R(u,X)Y)^H + (\nabla_Y X)^V - \frac{1}{2}(R(u,X)Y)^H\\
&+ (\nabla_Y X)^V - \frac{1}{2}(R(u,X)Y)^H - (\nabla_Y X)^V - \frac{1}{2}(R(u,X)Y)^H\\
&- \frac{1}{2}(R(u,X)Y)^H + \frac{1}{2}(R(u,X)Y)^H\Big]\\
=& -(R(u,X)Y)^H.
\end{aligned}$$

$$\widetilde{S}(X^H,Y^V) = \frac{1}{2}\Big[\widetilde{\nabla}_{-Y^V}(X^H) - \widetilde{\phi}(\widetilde{\nabla}_{-Y^V}X^H) + \widetilde{\phi}(\widetilde{\nabla}_{Y^V}(X^H)$$

$$
\begin{aligned}
& -\widetilde{\phi}(\widetilde{\nabla}_{Y^V}X^H)) - \widetilde{\phi}(\widetilde{\nabla}_{X^H}(-Y^V) + \widetilde{\phi}(\widetilde{\nabla}_{X^H}Y^V))\Big] \\
=& \frac{1}{2}\Big[-\frac{1}{2}(R(u,Y)X)^H + \frac{1}{2}(R(u,Y)X)^H + \frac{1}{2}(R(u,Y)X)^H \\
& -\frac{1}{2}(R(u,Y)X)^H - (\nabla_X Y)^V + \frac{1}{2}(R(u,Y)X)^H + (\nabla_X Y)^V \\
& +\frac{1}{2}(R(u,Y)X)^H\Big] = \frac{1}{2}(R(u,Y)X)^H,
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{S}(X^H,Y^H) =& \frac{1}{2}\Big[\widetilde{\nabla}_{Y^H}(X^H) - \widetilde{\phi}(\widetilde{\nabla}_{Y^H}X^H) + \widetilde{\phi}(\widetilde{\nabla}_{Y^H}(X^H) \\
& -\widetilde{\phi}(\widetilde{\nabla}_{Y^H}X^H)) - \widetilde{\phi}(\widetilde{\nabla}_{X^H}(Y^H) + \widetilde{\phi}(\widetilde{\nabla}_{X^H}Y^H))\Big] \\
=& \frac{1}{2}\Big[(\nabla_Y X)^H - \frac{1}{2}(R(Y,X)u)^V - (\nabla_Y X)^H - \frac{1}{2}(R(Y,X)u)^V \\
& + (\nabla_Y X)^H + \frac{1}{2}(R(Y,X)u)^V - (\nabla_Y X)^H + \frac{1}{2}(R(Y,X)u)^V \\
& - (\nabla_X Y)^H - \frac{1}{2}(R(X,Y)u)^V + (\nabla_X Y)^H - \frac{1}{2}(R(X,Y)u)^V\Big] \\
=& -\frac{1}{2}(R(X,Y)u)^V
\end{aligned}
$$

□

Then we can construct any almost paracomplex connection on TM by

$$\overline{\nabla}_{\widetilde{X}}\widetilde{Y} = \widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} - \widetilde{S}(\widetilde{X},\widetilde{Y}) \tag{2.26}$$

Theorem 2.6.3. *Let (M_{2k},ϕ,g) be an anti-paraKähler manifold and TM be its tangent bundle equipped with the Berger type deformed Sasaki metric g^{BS} and the paracomplex structure $\widetilde{\phi}$ defined by (2.24). Then the almost paracomplex connection $\overline{\nabla}$ constructed by the Levi-Civita connection $\widetilde{\nabla}$ is as follows:*

$$
\begin{aligned}
\overline{\nabla}_{X^H}Y^H &= (\nabla_X Y)^H, \\
\overline{\nabla}_{X^H}Y^V &= (\nabla_X Y)^V, \\
\overline{\nabla}_{X^V}Y^H &= \frac{3}{2}(R(u,X)Y)^H, \\
\overline{\nabla}_{X^V}Y^V &= \frac{\delta^2}{1+\delta^2\alpha}g(X,\phi Y)(\phi u)^V.
\end{aligned}
$$

For all X, Y on TM.

Proposition 2.6.4. *Let (M_{2k},ϕ,g) be an anti-paraKähler manifold and TM be its tangent bundle equipped with the Berger type deformed Sasaki metric g^{BS} and the paracomplex structure $\widetilde{\phi}$. If $F \in \mathfrak{T}^1_1(M)$ is a tensor of type (1,1), then we have*

$$(\overline{\nabla}_{X^H}HF)_{(x,u)} = (\nabla_X F)^H_{(x,u)}.$$

$$
\begin{aligned}
(\overline{\nabla}_{X^H} VF)_{(x,u)} &= (\nabla_X F)^V_{(x,u)}. \\
(\overline{\nabla}_{X^V} HF)_{(x,u)} &= (F(X))^H_{(x,u)} + \frac{3}{2}(R_x(u, X_x)F(u))^H. \\
(\overline{\nabla}_{X^V} VF)_{(x,u)} &= (F(X))^V_{(x,u)} + \frac{\delta^2}{1+\delta^2\alpha} g(X, \phi F(u))(\phi u)^V.
\end{aligned}
$$

where $(x, u) \in TM$ *and* $X \in \Gamma(TM)$.

Using Theorem 2.6.3 and Proposition 4.8.1 and formula of curvature, we have

Theorem 2.6.4. *Let* (M_{2k}, ϕ, g) *be an anti-paraKähler manifold and* TM *be its tangent bundle equipped with the Berger type deformed Sasaki metric* g^{BS} *and the paracomplex structure* $\widetilde{\phi}$. *Then the corresponding Riemannian curvature tensor* $\overline{R}$ *is given by*

$$
\begin{aligned}
\overline{R}(X^H, Y^H)Z^H &= \Big[R(X,Y)Z + \frac{3}{2}R(u, R(X,Y)u)Z\Big]^H, \\
\overline{R}(X^H, Y^H)Z^V &= \Big[R(X,Y))Z + \frac{\delta^2}{1+\alpha\delta^2} g(R(X,Y)u, \phi Z)(\phi u)\Big]^V, \\
\overline{R}(X^H, Y^V)Z^H &= \Big[\frac{3}{2}(\nabla_X R)(u, Y)Z\Big]^H, \\
\overline{R}(X^H, Y^V)Z^V &= 0, \\
\overline{R}(X^V, Y^V)Z^H &= \frac{3}{4}\Big[4R(X,Y)Z + 3R(u,X)R(u,Y)Z - 3R(u,Y)R(u,X)Z\Big]^V, \\
\overline{R}(X^V, Y^V)Z^V &= \Big(\frac{\delta^2}{1+\delta^2\alpha}\Big)^2 \big[g(X,u)g(Y,\phi Z) - g(Y,u)g(X,\phi Z)\big](\phi u)^V.
\end{aligned}
$$

Chapter 3

Harmonicity on tangent bundle structure

In this chapter, we study the harmonicity of vector fields considered to be maps from the Riemannian manifold M into its tangent bundle TM witch is equipped with an almost complex structure or an almost paracomplex. We have studied here three cases. First, the tangent bundle equipped with the gradient Sasaki metric g_f and an almost paracomplex structure compatible with g_f. Then we studied the harmonicity and bi-harmonicity on the tangent bundle of Kähler manifold (M, ϕ, g) equipped with the Berger type deformed Sasaki metric g^{BS}.
Finally, we study the harmonicity of vector field on the tangent bundle equipped with an isotropic almost complex structure $J_{\delta,0}$ and the isotropic Cheeger-Gromoll metric $g^{CG}_{\delta,0}$.

3.1 Harmonic vector field on tangent bundle equipped with gradient Sasaki metric

Lemma 3.1.1. *[12]. Let (M, g) be a Riemannian manifold. If $X, Y \in \Gamma(TM)$ are vector fields on M and $(p, u) \in TM$ such that $X_p = u$, then we have:*

$$d_pX(Y_p) = Y_p^H + (\nabla_X Y)^V_{(p,u)}.$$

Proof. Let (U, φ) be a local chart on M and $(\pi^{-1}(U), x^i, y^j)$ the induced chart on TM, if $X_p = X^i(p)\frac{\partial}{\partial x^i}|_p$ and $Y_p = Y^i(p)\frac{\partial}{\partial x^i}|_p$, then

$$\begin{aligned} d_pX(Y_p) =& Y^i(p)\frac{\partial}{\partial x^i}|_{(p,u)} + Y^i(p)\frac{\partial X^k}{\partial x^i}(p)\frac{\partial}{\partial y^k}|_{(p,u)} \\ =& Y^i(p)\frac{\partial}{\partial x^i}|_{(p,u)} + Y^i(p)\frac{\partial X^k}{\partial x^i}(p)\frac{\partial}{\partial y^k}|_{(p,u)} - Y^i(p)X^j(p)\Gamma^k_{ij}(p)\frac{\partial}{\partial y^k}|_{(p,u)} \end{aligned}$$

$$+ Y^i(p)X^j(p)\Gamma^k_{ij}(p)\frac{\partial}{\partial y^k}|_{(p,u)} = Y^H_p + (\nabla_Y X)^V_{(p,u)}.$$

□

3.1.1 Harmonic vector field $X : (M, g) \to (TM, J, g_f)$

Theorem 3.1.1. *Let (M, g) be a Riemannian manifold and (TM, J, g_f) its tangent bundle equipped with the almost paracomplex structure J and the gradient Sasaki g_f. Then, the tension field associated to $X \in \Gamma(TM)$ is given by*

$$\begin{aligned}\tau(X) =& \frac{3}{2}tr_g(R(X,\nabla_* X)*)^H + tr_g\Big[(\nabla^2_* X) + \frac{3}{2}(\nabla_* X)(f)(R(X, grad(f))*)\\ &+ \frac{3}{2}g(\nabla_* X, grad(f))(\nabla_* grad(f)) + \frac{3}{2\alpha}g(\nabla_* X, \nabla_* grad(f))grad(f)\\ &- \frac{3}{4\alpha}(\nabla_{grad(\alpha)}X)(f)(grad(f))\Big]^V.\end{aligned}$$

Proof. Let $(x, u) \in TM$ and $\{e_i\}_{i=1}^{2n}$ be a local orthonormal frame on M such that $(\nabla_{e_i} e_j) = 0$ and $X_x = u$, then by summing over i, we have

$$\begin{aligned}\tau(X) &= \nabla^X_{ei} d(X(e_i))\\ &= \overline{\nabla}_{e_i^H + (\nabla_{e_i}X)^V} e_i^H + (\nabla_{e_i}X)^V\\ &= \overline{\nabla}_{e_i^H} e_i^H + \overline{\nabla}_{e_i^H}(\nabla_{e_i}X)^V\\ &+ \overline{\nabla}_{(\nabla_{e_i}X)^V} e_i^H + \overline{\nabla}_{(\nabla_{e_i}X)^V}(\nabla_{e_i}X)^V.\end{aligned}$$

By using Theorem 2.4.3, we obtain

$$\begin{aligned}\tau(X) &= (\nabla_{e_i}e_i)^H + (\nabla_{e_i}\nabla_{e_i}X)^V + \frac{3}{2}g(\nabla_{e_i}X, grad(f))(\nabla_{e_i}grad(f))^V\\ &+ \frac{3}{2\alpha}\Big[g(\nabla_{e_i}X, \nabla_{e_i}grad(f)) - \frac{1}{2}e_i(\alpha)(\nabla_{e_i}X)(f)\Big](grad(f))^V\\ &+ \frac{3}{2}(R(u, \nabla_{e_i}X)e_i)^H + \frac{3}{2}(\nabla_{e_i}X)(f)(R(u, grad(f))e_i)^V.\end{aligned}$$

□

Theorem 3.1.2. *Let (M, g) be a Riemannian manifold and (TM, J, g_f) its tangent bundle equipped with the almost paracomplex structure J and gradient Sasaki metric g_f. Then, the vector field $X : (M, g) \longrightarrow (TM, J, g_f)$ is harmonic if and only if*

$$tr_g(R(u, \nabla_* X)*) = 0 \tag{3.1}$$

and

$$\begin{aligned}&tr_g\Big[(\nabla^2_* X) + \frac{3}{2}(\nabla_* X)(f)(R(X, grad(f))*) - \frac{3}{4\alpha}(\nabla_{grad(\alpha)}X)(f)(grad(f))\\ &+ \frac{3}{2}g(\nabla_* X, grad(f))(\nabla_* grad(f)) + \frac{3}{2\alpha}g(\nabla_* X, \nabla_* grad(f))grad(f)\Big] = 0.\end{aligned} \tag{3.2}$$

Corollary 3.1.1. *Let (M, g) be a Riemannian manifold and (TM, J, g_f) its tangent bundle equipped with the almost paracomplex structure J and gradient Sasaki metric g_f. If X parallel then, X is harmonic.*

Lemma 3.1.2. *Let (M, g) be a Riemannian manifold and (TM, J, g_f) its tangent bundle equipped with the almost paracomplex structure J defined in (2.17) and the gradient Sasaki metric g_f. If $X \in \Gamma(TM)$, then the energy density associated to X is given by*

$$e(X) = \frac{n}{2} + \frac{1}{2}\Big[tr_g||\nabla_* X||^2 + tr_g((\nabla_* X)f)^2\Big].$$

Proof. Let $\{e_1, e_2, ..., e_{2n}\}$ be a local orthonormal frame on M, then

$$e(X) = \frac{1}{2}\sum_{i=1}^{2k} g_f(dX(e_i), dX(e_i)).$$

Using Lemma 3.1.1 , we obtain

$$\begin{aligned} &= \frac{1}{2}\sum_{i=1}^{2k} g_f(dX(e_i), dX(e_i)) = \frac{1}{2}\sum_{i=1}^{2k} g_f(e_i^H + (\nabla_{e_i}X)^V, e_i^H + (\nabla_{e_i}X)^V) \\ &= \frac{1}{2}\sum_{i=1}^{2k}\Big[g_f(e_i^H, e_i^H) + g_f((\nabla_{e_i}X)^V, (\nabla_{e_i}X)^V)\Big]. \end{aligned}$$

□

Theorem 3.1.3. *Let (M, g) be a Riemannian manifold and (TM, J, g_f) its tangent bundle equipped with the almost paracomplex structure J and the gradient Sasaki metric g_f. Then $X \in \Gamma(TTM)$ is harmonic if and only if X is parallel.*

Proof. From theorem 3.1.2, we have that if $\nabla X = 0$, then X is harmonic.
Inversely, let X_t be a variation of X defined by $X_t = (1+t)X$, then

$$\begin{aligned} \frac{d}{dt}E(X_t)_{t=0} =& \frac{1}{2}\frac{d}{dt}\int_M (1+t)^2\Big[||\nabla X||^2 + g(grad(f), \nabla X)^2\Big]v_g. \\ =& \int_M ||\nabla X||^2 v_g + \int_M g(grad(f), \nabla X)^2 v_g \end{aligned}$$

If X is a critical point we have

$$0 = \int_M ||\nabla X||^2 v_g + \int_M g(grad(f), \nabla X)^2 v_g \tag{3.3}$$

then $\nabla X = 0$. □

3.1.2 Geodesics on tangent bundle equipped with gradient Sasaki metric

Definition 3.1.1. *Let (M, g) be a Riemannian manifold and (TM, J, gf) be its tangent bundle equipped with almost paracomplex structure J and the gradient Sasaki metric g_f. If $x(t)$ is a curve on (M, g), then the curve $C(t) = (x(t), \dot{x}(t))$ is called the natural lift of the curve $x(t)$.*

Definition 3.1.2. *Let (M, g) be a Riemannian manifold and (TM, J, g_f) be its tangent bundle equipped with almost paracomplex structure J and the gradient Sasaki metric g_f. A curve $C(t) = (x(t), y(t))$ is said to be a horizontal lift of the curve $x(t)$ if and only if $\nabla_{\dot{x}} y = 0$.*

Lemma 3.1.3. *Let (M, g) be a Riemannian manifold and $x : I \to M$ be a curve on M. If $C : t \in I \mapsto C(t) = (x(t); y(t)) \in TM$ is a curve in TM such $y(t) \in T_{x(t)}M$ (i.e. $y(t)$ is a vector field along $x(t)$), then*

$$\dot{C} = \dot{x}^H + (\nabla_{\dot{x}} y)^V.$$

Proof. Locally, if $Y \in \Gamma(TM)$ is a vector field such $Y(x(t)) = y(t)$, then we have

$$\dot{C}(t) = dC(t) = dY(x(t)).$$

Using Lemma 3.1.1, we obtain

$$\dot{C}(t) = dY(x(t)) = \dot{x}^H + (\nabla_{\dot{x}} y)^V.$$

□

Theorem 3.1.4. *Let (M, g) be a Riemannian manifold and (TM, J, g_f) be its tangent bundle equipped with the almost paracomplex structure J and Sasaki gradient metric g_f. If $x(t)$ is a curve on M and $C(t) = (x(t), y(t))$ is a curve on TM such that $y(t)$ is a vector fields on along $x(t)$, then*

$$\begin{aligned}\overline{\nabla}_{\dot{C}}\dot{C} =& \Big[\nabla_{\dot{x}}\dot{x} + \frac{3}{2}(R(u, \nabla_{\dot{x}}y)\dot{x})\Big]^H + \Big[\nabla^2_{\dot{x}}y + \frac{3}{2}(\nabla_{\dot{x}}y)(f)\Big(R(u, grad(f))\dot{x} + \nabla_{\dot{x}}grad(f)\Big) \\ &+ \frac{3}{2\alpha}g(\nabla_{\dot{x}}y, \nabla_{\dot{x}}grad(f))grad(f) - \frac{3}{4\alpha}\dot{x}(\alpha)(\nabla_{\dot{x}}y)(f)grad(f)\Big]^V.\end{aligned}$$

Proof. Let $\overline{\nabla}$ be the connection on (TM, J, g_f), then we have

$$\begin{aligned}\overline{\nabla}_{\dot{C}}\dot{C} =& \overline{\nabla}_{\dot{x}^H+(\nabla_{\dot{x}}y)^V}\left(\dot{x}^H + (\nabla_{\dot{x}}y)^V\right) \\ =& (\nabla_{\dot{x}}\dot{x})^H + (\nabla_{\dot{x}}\nabla_{\dot{x}}y)^V + \frac{3}{2}(\nabla_{\dot{x}}y)(f)(\nabla_{\dot{x}}grad(f))^V \\ &+ \frac{3}{2\alpha}g(\nabla_{\dot{x}}y, \nabla_{\dot{x}}grad(f))grad(f)^V - \frac{3}{4\alpha}\dot{x}(\alpha)(\nabla_{\dot{x}}y)(f)grad(f)^V \\ &+ \frac{3}{2}(R(u, \nabla_{\dot{x}}y)\dot{x})^H + \frac{3}{2}(\nabla_{\dot{x}}y)(f)(R(u, grad(f))\dot{x})^V.\end{aligned}$$

□

Theorem 3.1.5. *Let (M,g) be a Riemannian manifold and (TM,J,g_f) be its tangent bundle equipped with the almost paracomplex structure J and gradient Sasaki metric g_f, and let $C(t)=(x(t),y(t))$ be a curve on TM, then C is geodesic if and only if*

$$\nabla_{\dot{x}}\dot{x} = -\frac{3}{2}(R(y,\nabla_{\dot{x}}y)\dot{x}) \tag{3.4}$$

and

$$\begin{aligned}\nabla^2_{\dot{x}}y =& \frac{3}{2}(\nabla_{\dot{x}}y)(f)\Big(R(y,grad(f))\dot{x}+\nabla_{\dot{x}}grad(f)\Big)\\ &+\frac{3}{2\alpha}g(\nabla_{\dot{x}}y,\nabla_{\dot{x}}grad(f))grad(f)-\frac{3}{4\alpha}\dot{x}(\alpha)(\nabla_{\dot{x}}y)(f)grad(f).\end{aligned} \tag{3.5}$$

Corollary 3.1.2. *Let (M,g) be a Riemannian manifold and (TM,J,g_f) be its tangent bundle equipped with the almost paracomplex structure J and gradient Sasaki metric g_f, and let $\tilde{C}(t)=(x(t),y(t))$ be the horizontal lift of $x(t)$, then $\tilde{C}$ is geodesic if and only if $x(t)$ is geodesic i.e*

$$\nabla_{\dot{x}}\dot{x}=0.$$

Example 3.1.1. *et $\mathbb{R}$ equipped with the metric $g=e^x dx^2$. It's tangent bundle $(T\mathbb{R},g^f,J)$ where*

$$\begin{cases} JX^H &= -X^H,\\ JX^V &= X^V.\end{cases}$$

The Christoffel symbols of the Levi-Civita connection are given by

$$\Gamma^1_{11}=\frac{1}{2}g^{11}(\frac{\partial g_{11}}{\partial x^1}+\frac{\partial g_{11}}{\partial x^1}-\frac{\partial g_{11}}{\partial x^1})=\frac{1}{2},$$

the geodesics $x(t)$ of g, such that $x(0)=a, x'(0)=b$ where $a,b\in\mathbb{R}$, verify the equation

$$\nabla_{\dot{x}}\dot{x}=x''+\frac{1}{2}(x')^2=0,$$

where $x'(t)=\frac{2b}{2+bt}$ so $x(t)=a+2ln(1+\frac{1}{2}bt)$, then $C(t)=(a+2ln(1+\frac{1}{2}bt),\frac{2b}{2+bt}\frac{d}{dx})$ is the natural lift of $x(t)$ on $T\mathbb{R}$.
Now to find the horizontal lift of the curve $x(t)$, we need to find $y(t)$ such that $\nabla_{\dot{x}}y=0$. We have $\nabla_{\dot{x}}y=\frac{dy}{dt}+y\Gamma^1_{11}\frac{dx}{dt}=y'+\frac{1}{2}yx'=0$, then we have $y'=-y\frac{b}{2+bt}$, we get a particular solution is given by that $y(t)=k(2+bt)\frac{d}{dx}$, $k\in\mathbb{R}^+$. Then the horizontal lift of $x(t)$ is given by $\tilde{C}(t)=(a+2ln(1+\frac{1}{2}bt),k(2+bt)\frac{d}{dx})$. By Corollary 3.1.2 we have that $\tilde{C}(t)$ is geodesic.

3.2 Harmonic and bi-harmonic vector field on tangent bundle equipped with Berger type deformed Sasaki metric

3.2.1 Harmonic vector field $X : (M^{2k}, \phi, g) \to (TM, \widetilde{\phi}, g^{BS})$

Theorem 3.2.1. *Let (M^{2k}, ϕ, g) be an anti-paraKähler manifold and TM its tangent bundle equipped with the Berger type deformed Sasaki metric g^{BS} and the paracomplex structure $\widetilde{\phi}$. Then the tension field associated to $X \in \Gamma(TM)$ is given by*

$$\tau(X) = \frac{3}{2}\Big[tr_g(R(X, \nabla_* X)*)\Big]^H + \Big[(tr_g \nabla_*^2 X) + \frac{\delta^2}{1+\delta^2\alpha} tr_g\big[g(\nabla_* X, \phi\nabla_* X)(\phi X)\big]\Big]^V. \tag{3.6}$$

Proof. Let $(x,u) \in TM$ and $\{e_i\}_{i=1}^{2k}$ be a local orthonormal frame on M such that $(\nabla_{e_i} e_j) = 0$ and $X_x = u$, then by summing over i, we have

$$\begin{aligned}
\tau(X) &= \nabla^X_{ei} d(X(e_i)) \\
&= \widetilde{\nabla}_{e_i^H + (\nabla_{e_i}X)^V}\Big(e_i^H + (\nabla_{e_i}X)^V\Big) \\
&= \widetilde{\nabla}_{e_i^H} e_i^H + \widetilde{\nabla}_{e_i^H}(\nabla_{e_i}X)^V \\
&\quad + \widetilde{\nabla}_{(\nabla_{e_i}X)^V} e_i^H + \widetilde{\nabla}_{(\nabla_{e_i}X)^V}(\nabla_{e_i}X)^V.
\end{aligned}$$

By using Theorem 2.6.3, we obtain

$$\tau(X) = \frac{3}{2} tr_g(R(X, \nabla_* X)*)^H + (tr_g \nabla_*^2 X)^V + \frac{\delta^2}{1+\delta^2\alpha} tr_g\big[g(\nabla_* X, \phi\nabla_* X)(\phi X)\big]^V.$$

□

Theorem 3.2.2. *Let (M_{2k}, ϕ, g) be an anti-paraKähler manifold and TM its tangent bundle equipped with the Berger type deformed Sasaki metric g^{BS}. Then the vector field $X : (M_{2k}, \phi, g) \to TM$ is harmonic if and only if the following conditions are verified*

$$\tau^h(X) = tr_g(R(X, \nabla_* X)*) = 0, \tag{3.7}$$

$$\tau^v(X) = tr_g \nabla^2 X + \frac{\delta^2}{1+\delta^2\alpha} tr_g\big[g(\nabla_* X, \phi\nabla_* X)(\phi X)\big] = 0. \tag{3.8}$$

Example 3.2.1. *Let $(M = \mathbb{R}^2, \phi, dx^2 + dy^2)$ be a Riemannian manifold, the orthonormal basis on $T\mathbb{R}^2$ is given by $\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}$, the almost paracomplex structure ϕ satisfy $\phi\Big(\frac{\partial}{\partial x}\Big) = \frac{\partial}{\partial y}$ and $\phi\Big(\frac{\partial}{\partial y}\Big) = \frac{\partial}{\partial x}$. We have that a vector field $X \in T\mathbb{R}^2$ can be written*

$$X = f(x,y)\frac{\partial}{\partial x} + g(x,y)\frac{\partial}{\partial y},$$

where $f, g \in C^\infty(\mathbb{R})$.
We have

$$\begin{aligned}
tr_g \nabla_*^2 X =& \Big(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\Big)\frac{\partial}{\partial x} + \Big(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2}\Big)\frac{\partial}{\partial y},\\
tr_g g(\nabla_* X, \phi\nabla_* X)(\phi X) =& \Big[g\Big(\frac{\partial f}{\partial x}\frac{\partial}{\partial x} + \frac{\partial g}{\partial x}\frac{\partial}{\partial y}, \phi(\frac{\partial f}{\partial x}\frac{\partial}{\partial x} + \frac{\partial g}{\partial x}\frac{\partial}{\partial y})\Big)\\
&+ g\Big(\frac{\partial f}{\partial y}\frac{\partial}{\partial x} + \frac{\partial g}{\partial y}\frac{\partial}{\partial y}, \phi(\frac{\partial f}{\partial y}\frac{\partial}{\partial x} + \frac{\partial g}{\partial y}\frac{\partial}{\partial y})\Big)\Big](f(x,y)\frac{\partial}{\partial y} + g(x,y)\frac{\partial}{\partial x})\\
=& \Big[g\Big(\frac{\partial f}{\partial x}\frac{\partial}{\partial x} + \frac{\partial g}{\partial x}\frac{\partial}{\partial y}, \frac{\partial f}{\partial x}\frac{\partial}{\partial y} + \frac{\partial g}{\partial x}\frac{\partial}{\partial x}\Big)\\
&+ g\Big(\frac{\partial f}{\partial y}\frac{\partial}{\partial x} + \frac{\partial g}{\partial y}\frac{\partial}{\partial y}, \frac{\partial f}{\partial y}\frac{\partial}{\partial y} + \frac{\partial g}{\partial y}\frac{\partial}{\partial x}\Big)\Big](f(x,y)\frac{\partial}{\partial y} + g(x,y)\frac{\partial}{\partial x})\\
=& \Big[2\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + 2\frac{\partial f}{\partial y}\frac{\partial g}{\partial y}\Big](f(x,y)\frac{\partial}{\partial y} + g(x,y)\frac{\partial}{\partial x}).
\end{aligned}$$

Then, X is harmonic if and only if

$$\begin{cases}
0 = & \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{2\delta^2}{1+\delta^2\alpha}\Big(\dfrac{\partial f}{\partial x}\dfrac{\partial g}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{\partial g}{\partial y}\Big)g(x,y)\\
\text{and} & \\
0 = & \dfrac{\partial^2 g}{\partial x^2} + \dfrac{\partial^2 g}{\partial y^2} + \dfrac{2\delta^2}{1+\delta^2\alpha}\Big(\dfrac{\partial f}{\partial x}\dfrac{\partial g}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{\partial g}{\partial y}\Big)f(x,y).
\end{cases}$$

For $X = f(x,y)\frac{\partial}{\partial x} + K\frac{\partial}{\partial y}$, *where K constant. X is harmonic if and only if f is harmonic.*

Corollary 3.2.1. *Let* (M_{2k}, ϕ, g) *be an anti-paraKähler manifold and* TM *its tangent bundle equipped with the Berger type deformed Sasaki metric* g^{BS} *and the paracomplex structure* $\widetilde{\phi}$. *If* $X \in \Gamma(TM)$ *is a parallel vector field (i.e* $\nabla X = 0$*), then X is harmonic.*

Lemma 3.2.1. *Let* (M_{2k}, ϕ, g) *be an anti-paraKähler manifold and* TM *its tangent bundle equipped with the Berger type deformed Sasaki metric* g^{BS} *and the paracomplex structure* $\widetilde{\phi}$. *If* $X \in \Gamma(TM)$, *then the energy density associated to X is given by*

$$e(X) = k + \frac{1}{2}\Big[||\nabla X||^2 + \delta^2(g(\nabla X, \phi u))^2\Big].$$

Proof. Let $\{e_1, e_2, ..., e_{2k}\}$ be a local orthonormal frame on M, then

$$e(X) = \frac{1}{2}\sum_{i=1}^{2k} g^{BS}(dX(e_i), dX(e_i)).$$

Using Lemma 3.1.1 , we obtain

$$= \frac{1}{2}\sum_{i=1}^{2k} g^{BS}(dX(e_i), dX(e_i)) = \frac{1}{2}\sum_{i=1}^{2k} g^{BS}(e_i^H + (\nabla_{e_i}X)^V, e_i^H + (\nabla_{e_i}X)^V)$$

$$= \frac{1}{2}\sum_{i=1}^{2k} g^{BS}(e_i^H, e_i^H) + \frac{1}{2}g^{BS}((\nabla_{e_i}X)^V, (\nabla_{e_i}X)^V))$$
$$= \frac{1}{2}\big[2k + ||\nabla X||^2 + \delta^2(g(\nabla X, \phi u))^2\big].$$

□

Theorem 3.2.3. *Let (M_{2k}, ϕ, g) be a compact anti-paraKähler manifold and TM its tangent bundle equipped with the Berger type deformed Sasaki metric g^{BS} and the para-complex structure $\widetilde{\phi}$. Then $X \in \Gamma(TM)$ is harmonic if and only if X is parallel.*

Proof. If X is parallel, from Corollary 3.2.1, we deduce that X is harmonic vector field. Inversely: Let X_t be a compactly supported variation of X defined by $X_t = (1+t)X$. From Lemma 3.2.1 we have

$$\frac{d}{dt}E(X_t)_{t=0} = \frac{d}{dt}\int_M (1+t)^2\Big[||\nabla X||^2 + \delta^2(g(\nabla X, \phi u))^2\Big]v_g$$
$$= \int_M ||\nabla X||^2 v_g + \int_M \delta^2(g(\nabla X, \phi u))^2 v_g.$$

If X is a critical point of the energy functional, then we have :

$$\frac{d}{dt}E(X_t)_{t=0} = 0,$$
$$\int_M ||\nabla X||^2 v_g + \int_M \delta^2(g(\nabla X, \phi u))^2 v_g = 0.$$

then $\nabla X = 0$. □

Theorem 3.2.4. *Let $(TM, \widetilde{\phi}, g^{BS})$ be a anti-paraKähler manifold. Then $X \in \Gamma(TM)$ is harmonic if and only if*

$$tr_g \nabla^2 X = -\frac{\delta^2}{1+\delta^2\alpha} tr_g g(\nabla_* X, \phi\nabla_* X)(\phi u). \tag{3.9}$$

Proof. Let $(TM, \widetilde{\phi}, g^{BS})$ be a anti-paraKähler manifold, from Theorem 2.6.1, we have M is flat manifold, thus the Riemannian curvature $R = 0$. By using the Theorem 3.2.2, we have X is harmonic if and only if

$$tr_g \nabla^2 X + \frac{\delta^2}{1+\delta^2\alpha} tr_g\big[g(\nabla_* X, \phi\nabla_* X)(\phi u)\big] = 0.$$

□

3.2.2 Biharmonic vector field $X : (M_{2k}, \phi, g) \to (TM, \widetilde{\phi}, g^{BS})$

Theorem 3.2.5. *Let (M_{2k}, ϕ, g) be a compact anti-paraKähler manifold and TM its tangent bundle equipped with the Berger type deformed Sasaki metric g^{BS} and the para-complex structure $\widetilde{\phi}$. Then $X \in \Gamma(TM)$ is biharmonic if and only if X is harmonic.*

Proof. Let X_t be a compactly supported variation of X defined by $X_t = (1+t)X$. From the formulas 3.7 and 3.8, we have

$$\begin{aligned}
\tau^h(X_t) &= (1+t)^2\tau^h(X),\\
\tau^v(X_t) &= (1+t)tr_g\nabla^2 X + (1+t)^2\Big[\frac{\delta^2}{1+\delta^2\alpha}tr_g\big[g(\nabla_* X, \phi\nabla_* X)(\phi u)\big]\Big]\\
&= (1+t)\Delta X + (1+t)^2 A(X),
\end{aligned}$$

$$\begin{aligned}
E_2(X_t) =& \frac{1}{2}\int |\tau(X_t)|^2_{g^{BS}} v_g\\
=& \frac{1}{2}\int |\tau^h(X_t)|^2_{g^{BS}} v_g + \frac{1}{2}\int |\tau^v(X_t)|^2_{g^{BS}} v_g\\
=& \frac{(1+t)^4}{2}\int g(\tau^h(X), \tau^h(X))v_g + \frac{1}{2}\int g(\tau^v(X), \tau^v(X))v_g\\
&+ \frac{\delta^2}{2}\int g^2(\tau^v(X_t)), \phi(u))v_g\\
=& \frac{(1+t)^4}{2}\int g(\tau^h(X), \tau^h(X))v_g + \frac{(1+t)^2}{2}\int g(\Delta X, \Delta X)v_g\\
&+ \frac{(1+t)^3}{2}\int g(\Delta X, A(X))v_g + \frac{(1+t)^4}{2}\int g(A(X), A(X))v_g\\
&+ \frac{\delta^2(1+t)^2}{2}\int g^2(\Delta X, \phi(u))v_g + \frac{\delta^2(1+t)^4}{2}\int g^2(A(X), \phi(u))v_g,
\end{aligned}$$

then

$$\begin{aligned}
0 = \frac{d}{dt}E_2(X_t)_{t=0} =& 2\int g(\tau^h(X), \tau^h(X))v_g + \int g(\Delta X, \Delta X)v_g\\
&+ \frac{3}{2}\int g(\Delta X, A(X))v_g + 2\int g(A(X), A(X))v_g\\
&+ \delta^2\int g^2(\Delta X, \phi(u))v_g + 2\delta^2\int g^2(A(X), \phi(u))v_g.
\end{aligned}$$

Since both functions $g(\tau^h(X), \tau^h(X))$ and $g(\Delta X, \Delta X), g(A(X), A(X))$ are positive, we easily conclude that

$$\tau^h(X) = \Delta X = A(X) = 0$$

everywhere on M. Equivalently, X is an harmonic map. □

Corollary 3.2.2. *Let (M_{2k}, ϕ, g) be a compact anti-paraKähler manifold and TM its tangent bundle equipped with the Berger type deformed Sasaki metric g^{BS} and the para-complex structure $\widetilde{\phi}$. Then $X \in \Gamma(TM)$ is biharmonic if and only if X is parallel.*

Lemma 3.2.2. *Let (M_{2k}, ϕ, g) be an anti-paraKähler manifold and TM its tangent bundle equipped with the Berger type deformed Sasaki metric g^{BS} and the paracomplex structure $\widetilde{\phi}$. If $X : M \to TM$ is a smooth vector field then the Jacobi tensor $J_X(\tau^v(X)^V)$ is given by*

$$\begin{aligned} J_X(\tau^v(X)^V)_{(x,u)} =& \Big[-\frac{3}{2} tr_g(\nabla_* R)(u, \tau^v(X)) *\Big]^H_{(x,u)} \\ &+ \Big[tr_g\Big(\nabla^2_* \tau^v(X) + 3R(\tau^v(X), \nabla_* X)* \\ &+ \frac{9}{4} R(u, \tau^v(X)) R(u, \nabla_* X)* \\ &- \frac{9}{4} R(u, \nabla_* X) R(u, \tau^v(X)) * \Big)\Big]^V_{(x,u)} \\ &+ \frac{\delta^2}{1+\alpha\delta^2}\Big[tr_g\Big((g(\nabla^2 X, \phi\tau^v(X))\phi u \\ &+ 2g(\nabla_* X, \phi\nabla_*\tau^v(X))\phi u \\ &+ \frac{\delta^2}{1+\alpha\delta^2} g(\tau^v(X), u) g(\nabla_* X, \phi\nabla_* X)\phi u\Big)\Big]^V_{(x,u)}. \end{aligned}$$

Proof. Let $(x,u) \in TM$ and $\{e_i\}_{i=1}^{2k}$ be a local orthonormal frame on M such that $(\nabla_{e_i} e_i)_x = 0$. Then by summing over i and by using the Theorem 2.6.3, we have

$$\begin{aligned} \nabla^X_{e_i}(\tau^v(X))^V|_{(x,u)} &= \overline{\nabla}_{e_i^H + (\nabla_{e_i} X)^V}(\tau^v(X))^V \\ &= (\nabla_{e_i}\tau^v(X))^V + \frac{\delta^2}{1+\alpha\delta^2} g(\nabla_{e_i} X, \phi\tau^v(X))(\phi u)^V, \end{aligned}$$

$$\begin{aligned} tr_g(\nabla^X)^2(\tau^v(X))^V =& \nabla^X_{e_i}\nabla^X_{e_i}(\tau^v(X))^V = \overline{\nabla}_{e_i^H + (\nabla_{e_i} X)^V}\Big[(\nabla_{e_i}\tau^v(X))^V \\ &+ \frac{\delta^2}{1+\alpha\delta^2} g(\nabla_{e_i} X, \phi\tau^v(X))(\phi u)^V\Big] \\ =& (\nabla^2_{e_i}\tau^v(X))^V + \frac{\delta^2}{1+\alpha\delta^2}\Big[\nabla_{e_i}(g(\nabla_{e_i} X, \phi\tau^v(X))\phi u) \\ &+ g(\nabla_{e_i} X, \phi\nabla_{e_i}\tau^v(X))(\phi u) \\ &+ \frac{\delta^2}{1+\alpha\delta^2} g(\nabla_{e_i} X, \phi\tau^v(X)) g(\nabla_{e_i} X, u)(\phi u)\Big]^V, \end{aligned}$$

from Theorem 2.6.4 and Lemma 3.1.1, we have

$$tr_g\overline{R}(\tau^v(X), dX)dX = \sum_{i=1}^{2k}\Big(\overline{R}((\tau^v(X))^V, e_i^H)e_i^H + \overline{R}((\tau^v(X))^V, (\nabla_{e_i} X)^V)e_i^H$$

$$+ \overline{R}((\tau^v(X))^V, e_i^H)(\nabla_{e_i}X)^V$$
$$+ \overline{R}((\tau^v(X))^V, (\nabla_{e_i}X)^V)(\nabla_{e_i}X)^V\Big).$$

By calculating at (x,u), we obtain

$$\begin{aligned} tr(\overline{R}(\tau^v(X)^V, dX)dX) =& \Big[-\frac{3}{2}(\nabla_{e_i}R)(u,\tau^v(X))e_i\Big]^H + \frac{3}{4}\Big[4R(\tau^v(X),\nabla_{e_i}X)e_i \\ &+ 3R(u,\tau^v(X))R(u,\nabla_{e_i}X)e_i \\ &- 3R(u,\nabla_{e_i}X)R(u,\tau^v(X))e_i\Big]^V \\ &+ \frac{\delta^4}{(1+\delta^2\alpha)^2}\Big[g(\tau^v(X),u)g(\nabla_{e_i}X,\phi\nabla_{e_i}X)\phi u \\ &- g(\nabla_{e_i}X,u)g(\tau^v(X),\phi\nabla_{e_i}X)\phi u\Big]^V. \end{aligned}$$

Considering the formula 1.29, we deduce the result. □

Lemma 3.2.3. *Let (M_{2k},ϕ,g) be an anti-paraKähler manifold and TM its tangent bundle equipped with the Berger type deformed Sasaki metric g^{BS} and the paracomplex structure $\widetilde{\phi}$. If $X : M \to TM$ is a smooth vector field then the Jacobi tensor $J_X(\tau^h(X)^H)$ is given by*

$$\begin{aligned} J_X(\tau^h(X)^H) =& \Big[tr_g\Big(R(\tau^h(X),*)*+\frac{3}{2}R(u,R(\tau^h(X),*)u)* \\ &+ \frac{3}{2}(\nabla_{\tau^h(X)}R)(u,\nabla_*X)*+(\nabla_*^2\tau^h(X)) \\ &+ \frac{3}{2}\nabla_*R(u,\nabla_*)\tau^h(X) + \frac{3}{2}R(u,\nabla_*X)\nabla_*\tau^h(X) \\ &+ \frac{9}{4}R(u,\nabla_*X)R(u,\nabla_*X)\tau^h(X)\Big]^H + \Big[R(\tau^h(X),*)\nabla_*X \\ &+ \frac{\delta^2}{1+\delta^2\alpha}g(R(\tau^h(X),*)u,\phi\nabla_*X)\phi u\Big)\Big]^V. \end{aligned}$$

Proof. Let $(x,u)\in TM$ and $\{e_i\}_{i=1}^{2k}$ be a local orthonormal frame on M such that $(\nabla_{e_i}e_i)_x = 0$. Then by summing over i, we have:

$$\nabla_{e_i}^X(\tau^h(X))^H|_{(x,u)} = (\nabla_{e_i}\tau^h(X))^H + \frac{3}{2}(R(u,\nabla_{e_i}X)\tau^h(X))^H.$$

From Proposition 2.6.4, we have

$$\begin{aligned} \nabla_{e_i}^X\nabla_{e_i}^X(\tau^h(X))_{(x,u)}^H =& (\nabla_{e_i}^2\tau^h(X))^H + \frac{3}{2}\Big[\nabla_{e_i}R(u,\nabla_{e_i})\tau^h(X) + R(u,\nabla_{e_i}X)\nabla_{e_i}\tau^h(X) \\ &+ \frac{3}{2}R(u,\nabla_{e_i}X)R(u,\nabla_{e_i}X)\tau^h(X)\Big]^H. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} tr(\overline{R}(\tau^h(X)^H, dX)dX) =& \Big[R(\tau^h(X), e_i)e_i + \frac{3}{2}R(u, R(\tau^h(X), e_i)u)e_i \\ &+ \frac{3}{2}(\nabla_{\tau^h(X)}R)(u, \nabla_{e_i}X)e_i\Big]^H + \Big[R(\tau^h(X), e_i)\nabla_{e_i}X\Big]^V \\ &+ \frac{\delta^2}{1+\delta^2\alpha}\Big[g(R(\tau^h(X), e_i)u, \phi\nabla_{e_i}X)\phi u\Big]^V. \end{aligned}$$

Considering the formula 1.29, we deduce the result. □

From Lemma 3.2.2 and Lemma 3.2.3, we deduce the next theorem.

Theorem 3.2.6. *Let (M_{2k}, ϕ, g) be an anti-paraKähler manifold and TM its tangent bundle equipped with the Berger type deformed Sasaki metric g^{BS} and the paracomplex structure $\widetilde{\phi}$, if $X : M \to TM$ is a smooth vector field then the bitension field of X is given by*

$$\begin{aligned} \tau_2(X) =& \Big[R(\tau^h(X), e_i)e_i + \frac{3}{2}R(u, R(\tau^h(X), e_i)u)e_i \\ &+ \frac{3}{2}(\nabla_{\tau^h(X)}R)(u, \nabla_{e_i}X)e_i + (\nabla^2_{e_i}\tau^h(X)) \\ &+ \frac{3}{2}\nabla_{e_i}R(u, \nabla_{e_i})\tau^h(X) + \frac{3}{2}R(u, \nabla_{e_i}X)\nabla_{e_i}\tau^h(X) \\ &+ \frac{9}{4}R(u, \nabla_{e_i}X)R(u, \nabla_{e_i}X)\tau^h(X) - \frac{3}{2}(\nabla_{e_i}R)(u, \tau^v(X))e_i\Big]^H \\ &+ \Big[R(\tau^h(X), e_i)\nabla_{e_i}X + \nabla^2_{e_i}\tau^v(X) + 3R(\tau^v(X), \nabla_{e_i}X)e_i \\ &+ \frac{9}{4}R(u, \tau^v(X))R(u, \nabla_{e_i}X)e_i - \frac{9}{4}R(u, \nabla_{e_i}X)R(u, \tau^v(X))e_i\Big]^V \\ &+ \frac{\delta^2}{1+\alpha\delta^2}\Big[\nabla_{e_i}(g(\nabla_{e_i}X, \phi\tau^v(X))\phi u) \\ &+ g(\nabla_{e_i}X, \phi\nabla_{e_i}\tau^v(X))(\phi u) + g(R(\tau^h(X), e_i)u, \phi\nabla_{e_i}X)\phi u \\ &+ \frac{\delta^2}{1+\delta^2\alpha}\Big(g(\tau^v(X), u)g(\nabla_{e_i}X, \phi\nabla_{e_i}X)\phi u\Big)\Big]\Big]^V. \end{aligned}$$

Theorem 3.2.7. *Let (M_{2k}, ϕ, g) be an anti-paraKähler manifold and TM its tangent bundle equipped with the Berger type deformed Sasaki metric g^{BS} and the paracomplex structure $\widetilde{\phi}$. A vector field $X : M \to TM$ is biharmonic if and only if the following conditions are verified.*

$$\begin{aligned} 0 =& tr_g\Big[-\frac{3}{2}(\nabla_* R)(u, \tau^v(X)) * + \nabla^2_*\tau^h(X) + \frac{3}{2}\nabla_* R(u, \nabla_*)\tau^h(X) \\ &+ \frac{3}{2}R(u, \nabla_* X)\nabla_*\tau^h(X) + \frac{9}{4}R(u, \nabla_* X)R(u, \nabla_* X)\tau^h(X) \\ &+ R(\tau^h(X), *) * - \frac{3}{2}R(u, R(\tau^h(X), *)u) * + \frac{3}{2}(\nabla_{\tau^h(X)}R)(u, \nabla_* X) * \Big] \end{aligned}$$

and

$$
\begin{aligned}
0 =& tr_g \Big[R(\tau^h(X), *)\nabla_{e_i} X + \nabla_*^2 \tau^v(X) + 3R(\tau^v(X), \nabla_* X)e_i \\
&+ \frac{9}{4} R(u, \tau^v(X))R(u, \nabla_* X)e_i - \frac{9}{4} R(u, \nabla_* X)R(u, \tau^v(X))* \\
&+ \frac{\delta^2}{1+\alpha\delta^2} \big[\nabla_*(g(\nabla_* X, \phi\tau^v(X))\phi u) \\
&+ g(\nabla_* X, \phi\nabla_* \tau^v(X))(\phi u) + g(R(\tau^h(X), *)u, \phi\nabla_* X)\phi u \\
&+ \frac{\delta^2}{1+\delta^2\alpha} \Big(g(\nabla_* X, \phi\nabla_* \tau^v(X))g(\nabla_* X, u)(\phi u) \\
&+ g(\tau^v(X), u)g(\nabla_* X, \phi\nabla_* X)\phi u \\
&- g(\nabla_* X, u)g(\tau^v(X), \phi\nabla_* X)\phi u \Big) \big] \Big].
\end{aligned}
$$

From Theorem 3.2.4 and the Lemma 3.2.2, we have

Theorem 3.2.8. *Let $(TM, \widetilde{\phi}, g^{BS})$ be a anti-paraKähler manifold. Then $X \in \Gamma(TM)$ is biharmonic if and only if*

$$
\begin{cases}
tr_g \Big((g(\nabla_*^2 X, \phi\tau^v(X))\phi X + 2g(\nabla_* X, \phi\nabla_* \tau^v(X))\phi X \\
+ \frac{\delta^2}{1+\alpha\delta^2} g(\tau^v(X), X)g(\nabla_* X, \phi\nabla_* X)\phi X \Big) = 0, \\
\textit{and} \\
tr_g \nabla^2 \tau^v(X) = 0.
\end{cases}
$$

Chapter 4

Harmonic and bi-harmonic maps between tangent bundle

In this final chapter we study the harmonicity of some maps in tangent bundle. Those tangent bundle being equipped withe almost complex or almost paracomplex structures. We took the example of the projection map π and the identity map. And in the case of the tangent bundle equipped with the Berger type deformed Sasaki metric we also studied the harmonicity of a map between tangent bundle with one or both of them equipped with the Berger type deformed Sasaki metric.
We start this chapter with an important notion and lemma about map between two tangent bundles.

4.1 Maps between tangent bundles

Let (M^m, g) and (N^n, h) two Riemannian manifolds, and let $\psi : (M^m, g) \longrightarrow (N^n, h)$ a smooth map. The map ψ induce a map

$$\begin{aligned} \Psi = d\psi : TM &\longrightarrow TN \\ (x, y) &\longmapsto (\psi(x), d_x\psi(y)). \end{aligned}$$

Let x^i and (x^i, y^i) be local coordinates on M and TM respectively, the local frames of vector fields on M and TM are $\left\{\frac{\partial}{\partial x^i}\right\}$, $\left\{\left(\frac{\partial}{\partial x^i}\right)^V = \frac{\partial}{\partial y^i}, \left(\frac{\partial}{\partial x^i}\right)^H = \frac{\partial}{\partial x^i} - \Gamma^k_{i,j} y^j \frac{\partial}{\partial y^k}\right\}$, where $i, j, k = 1, ..., m$.
Let v^α and (v^α, w^α) be local coordinates on N and TN respectively, the local frames of vector fields on N and TN are $\left\{\frac{\partial}{\partial v^\alpha}\right\}$, $\left\{\left(\frac{\partial}{\partial v^\alpha}\right)^V = \frac{\partial}{\partial w^\alpha}, \left(\frac{\partial}{\partial v^\alpha}\right)^H = \frac{\partial}{\partial v^\alpha} - \Gamma^\gamma_{\alpha,\beta} w^\beta \frac{\partial}{\partial w^\gamma}\right\}$, where $\alpha, \beta, \gamma = 1, ..., n$.
We have that

$$\psi = (\psi^1, ..., \psi^n),$$

$$d_x\psi(y) = \frac{\partial\psi^\alpha}{\partial x^j} y^j \frac{\partial}{\partial u^\alpha} \in T_{\psi(x)}N.$$

Then, we have

$$(d\psi)_i^\alpha = \Big(\frac{\partial\psi^\alpha}{\partial x^i}\Big)_i^\alpha = \begin{pmatrix} \frac{\partial\psi^1}{\partial x^1} & \cdots & \frac{\partial\psi^1}{\partial x^m} \\ . & \cdots & . \\ . & \cdots & . \\ . & \cdots & . \\ \frac{\partial\psi^n}{\partial x^1} & \cdots & \frac{\partial\psi^n}{\partial x^m} \end{pmatrix},$$

$$d\psi(\frac{\partial}{\partial x^i}) = \frac{\partial\psi^\alpha}{\partial x^i}\frac{\partial}{\partial u^\alpha}.$$

Lemma 4.1.1. . *Let $\psi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map between Riemannian manifolds. The map ψ induces the tangent map*

$$\begin{aligned} \Psi = d\psi : TM &\longrightarrow TN \\ (x, y) &\longmapsto (\psi(x), d\psi(y)). \end{aligned}$$

For all vector fields $X \in \Gamma(TM)$, we have

$$\begin{aligned} d\Psi\Big((X)^V\Big) &= \Big(d\psi(X)\Big)^V, \\ d\Psi\Big((X)^H\Big) &= \Big(d\psi(X)\Big)^H + \Big(\nabla d\psi(y, X)\Big)^V. \end{aligned}$$

Proof. Let x^i and (x^i, y^i) be local coordinates on M and TM respectively, then the local frames of vector fields on M and TM are
$\Big\{\frac{\partial}{\partial x^i}\Big\}$, $\Big\{\Big(\frac{\partial}{\partial x^i}\Big)^V = \frac{\partial}{\partial y^i}, \Big(\frac{\partial}{\partial x^i}\Big)^H = \frac{\partial}{\partial x^i} - \Gamma_{i,j}^k y^j \frac{\partial}{\partial y^k}\Big\}$, $i, j, k = 1, ..., m$.

Let now v^α and (v^α, w^α) be local coordinates on N and TN respectively, then the local frames of vector fields on N and TN are
$\Big\{\frac{\partial}{\partial v^\alpha}\Big\}$, $\Big\{\Big(\frac{\partial}{\partial v^\alpha}\Big)^V = \frac{\partial}{\partial w^\alpha}, \Big(\frac{\partial}{\partial v^\alpha}\Big)^H = \frac{\partial}{\partial v^\alpha} - \Gamma_{\alpha,\beta}^\gamma w^\beta \frac{\partial}{\partial w^\gamma}\Big\}$,$\alpha, \beta, \gamma = 1, ..., n$, where $w^\alpha = y^j \frac{\partial\psi^\alpha}{\partial x^j}$.
We have,

$$d\Psi\Big((\frac{\partial}{\partial x^i})^V\Big) = d\Psi\Big(\frac{\partial}{\partial y^i}\Big) = \frac{\partial\psi^\alpha}{\partial x^i}\frac{\partial}{\partial w^\alpha} = \frac{\partial\psi^\alpha}{\partial x^i}\Big(\frac{\partial}{\partial v^\alpha}\Big)^V = \Big(\frac{\partial\psi^\alpha}{\partial x^i}\frac{\partial}{\partial v^\alpha}\Big)^V = \Big(d\psi(\frac{\partial}{\partial x^i})\Big)^V$$

we also have

$$\begin{aligned} d\Psi\Big((\frac{\partial}{\partial x^i})^H\Big) =& d\Psi\Big(\frac{\partial}{\partial x^i} - \overset{M}{\Gamma}{}_{ij}^k y^j \frac{\partial}{\partial y^k}\Big) = \frac{\partial\psi^\alpha}{\partial x^i}\frac{\partial}{\partial v^\alpha} + y^i \frac{\partial^2\psi^\alpha}{\partial x^i x^j}\frac{\partial}{\partial w^\alpha} - \overset{M}{\Gamma}{}_{ij}^k y^i \frac{\partial\psi^\alpha}{\partial x^k}\frac{\partial}{\partial w^\gamma} \\ =& \frac{\partial\psi^\alpha}{\partial x^i}\Big(\frac{\partial}{\partial v^\alpha}\Big)^H + \frac{\partial\psi^\alpha}{\partial x^i}\overset{N}{\Gamma}{}_{\alpha\beta}^\gamma w^\beta \frac{\partial}{\partial w^\gamma} + y^i \frac{\partial^2\psi^\alpha}{\partial x^i x^j}\frac{\partial}{\partial w^\alpha} - \overset{M}{\Gamma}{}_{ij}^k y^i \frac{\partial\psi^\alpha}{\partial x^k}\frac{\partial}{\partial w^\gamma} \end{aligned}$$

$$
\begin{aligned}
&= \Big(d\psi(\frac{\partial}{\partial x^i})\Big)^H + \frac{\partial \psi^\alpha}{\partial x^i}\overset{N}{\Gamma}{}^\gamma_{\alpha\beta} w^\beta \frac{\partial}{\partial w^\gamma} + y^i \frac{\partial^2\psi^\alpha}{\partial x^i x^j}\frac{\partial}{\partial w^\alpha} - \overset{M}{\Gamma}{}^k_{ij} y^i \frac{\partial \psi^\alpha}{\partial x^k}\frac{\partial}{\partial w^\gamma},\\
\nabla d\psi(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}) &= \nabla^\psi_{\frac{\partial}{\partial x^i}} d\psi\Big(\frac{\partial}{\partial x^j}\Big) - d\psi\Big(\overset{M}{\nabla}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}\Big) = \nabla^\psi_{\frac{\partial}{\partial x^i}}\frac{\partial \psi^\alpha}{\partial x^j}\Big(\frac{\partial}{\partial v^\alpha}\Big) - d\psi\Big(\overset{M}{\Gamma}{}^k_{ij}\frac{\partial}{\partial x^k}\Big)\\
&= \frac{\partial^2\psi^\alpha}{\partial x^j\partial x^i}\Big(\frac{\partial}{\partial v^\alpha}\Big) + \frac{\partial\psi^\alpha}{\partial x^j}\nabla_{\frac{\partial\psi^\alpha}{\partial x^i}\frac{\partial}{\partial v^\beta}}\Big(\frac{\partial}{\partial v^\alpha}\Big) - \Big(\overset{M}{\Gamma}{}^k_{ij}\frac{\partial\psi^\alpha}{\partial x^k}\frac{\partial}{\partial v^\alpha}\Big)\\
&= \frac{\partial^2\psi^\alpha}{\partial x^j\partial x^i}\Big(\frac{\partial}{\partial v^\alpha}\Big) + \frac{\partial\psi^\alpha}{\partial x^j}\frac{\partial\psi^\alpha}{\partial x^i}\nabla_{\frac{\partial}{\partial v^\beta}}\Big(\frac{\partial}{\partial v^\alpha}\Big) - \Big(\overset{M}{\Gamma}{}^k_{ij}\frac{\partial\psi^\alpha}{\partial x^k}\frac{\partial}{\partial v^\alpha}\Big)\\
&= \frac{\partial^2\psi^\alpha}{\partial x^j\partial x^i}\Big(\frac{\partial}{\partial v^\alpha}\Big) + \frac{\partial\psi^\alpha}{\partial x^j}\frac{\partial\psi^\alpha}{\partial x^i}\Big(\overset{N}{\Gamma}{}^\gamma_{\alpha\beta}\frac{\partial}{\partial v^\gamma}\Big) - \Big(\overset{M}{\Gamma}{}^k_{ij}\frac{\partial\psi^\alpha}{\partial x^k}\frac{\partial}{\partial v^\alpha}\Big).
\end{aligned}
$$

Then, we have $y^i\Big(\nabla d\psi(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j})\Big)^V + \Big(d\psi(\frac{\partial}{\partial x^i})\Big)^H = d\Psi\Big((\frac{\partial}{\partial x^i})^H\Big)$. □

4.2 Harmonicity of the map $\pi : (TM, J, g_f) \longrightarrow (M, g)$

Theorem 4.2.1. *Let (M,g) be an Riemannian manifold and TM its tangent bundle equipped with the almost paracomplex structure J and the gradient Sasaki metric g_f. The Riemannian submersion $\pi : (TM, J, g_f) \longrightarrow M$. Then*

$$\tau(\pi) = 0.$$

Proof. Let $(x,u)\in TM$ and let $\{e_i\}_{i=1}^n$ is an orthonormal basis of TM at x. We put $e_1 = \dfrac{gradf}{||gradf||^2}$ Then $\Big\{e_i^H, \frac{1}{\sqrt{\alpha}}e_1^V, e_j^V, j = 2,...,n\Big\}$ is an orthonormal basis of $T_{(x,u)}TM$ at (x,u) such that $(\nabla_{e_i}e_i)_x = 0$. Then by summing over i, we have using Theorem 2.4.3 for the Levi-Civita connection on the tangent bundle equipped with the gradient Sasaki metric and the almost paracomplex structure J.

$$
\begin{aligned}
\nabla d\pi(e_i^H, e_i^H) &= \nabla^\pi_{e_i^H} d\pi(e_i^H) - d\pi(\nabla^f_{e_i^H} e_i^H)\\
&= \nabla_{d\pi(e_i^H)} d\pi(e_i^H) - d\pi\Big(\overline{\nabla}_{e_i^H} e_i^H\Big) = -d\pi\Big(\overline{\nabla}_{e_i^H} e_i^H\Big) = 0\\
\nabla d\pi(\frac{1}{\sqrt{\alpha}}e_1^V, \frac{1}{\sqrt{\alpha}}e_1^V) &= \nabla^\pi_{\frac{1}{\sqrt{\alpha}}e_1^V} d\pi(\frac{1}{\sqrt{\alpha}}e_1^V) - d\pi(\overline{\nabla}_{\frac{1}{\sqrt{\alpha}}e_1^V}\frac{1}{\sqrt{\alpha}}e_1^V)\\
&= \frac{1}{\alpha}\nabla_{d\pi e_1^V} d\pi(e_1^V) - \frac{1}{\alpha}d\pi(\overline{\nabla}_{e_1^V} e_1^V) = -\frac{1}{\alpha}d\pi(\overline{\nabla}_{e_1^V} e_1^V) = 0.\\
\nabla d\pi(e_j^V, e_j^V) &= \nabla^\pi_{e_j^V} d\pi(e_j^V) - d\pi(\overline{\nabla}_{e_j^V} e_j^V)\\
&= \nabla_{d\pi e_j^V} d\pi(e_j^V) - d\pi(\overline{\nabla}_{e_j^V} e_j^V) = -d\pi(\overline{\nabla}_{e_j^V} e_j^V) = 0.
\end{aligned}
$$

□

4.3 Harmonic identity map $I : (TM, J, g_f) \longrightarrow (TM, g^s)$

Proposition 4.3.1. *Let $(M, , g)$ be a Riemannian manifold and TM its tangent bundle equipped with the gradient Sasaki metric g_f and the almost paracomplex structure J. Suppose that $I : (TM, J, g_f) \longrightarrow (TM, g^S)$ is the identity map. Then the tension field $\tau(I)$ of I is given by*

$$\tau(I) = 0. \tag{4.1}$$

Proof. Let $(x, u) \in TM$ and let $\{e_i\}_{i=1}^n$ be an orthonormal basis of TM at x. Then $\left\{e_i^H, \frac{1}{\sqrt{\alpha}}e_1^V, e_j^V, j = 2, ..., n\right\}$ is an orthonormal basis of $T_{(x,u)}TM$ at (x, u) such that $(\nabla_{e_i}e_i)_x = 0$. Then by summing over i, we have

$$\tau(I) =\nabla^I_{e_i^H} dI(e_i^H) + \frac{1}{\alpha}\nabla^I_{e_1^V} dI(e_1^V) + \nabla^I_{e_j^V} dI(e_j^V) - dI\Big(\overline{\nabla}_{e_i^H} e_i^H + \frac{1}{\alpha}\overline{\nabla}_{e_1^V} e_1^V + \overline{\nabla}_{e_j^V}(e_j^V)\Big) = 0.$$

□

4.4 Harmonicity of the map $\pi : (TM, \widetilde{\phi}, g^{BS}) \longrightarrow M_{2k}$

Theorem 4.4.1. *Let (M_{2k}, ϕ, g) be an anti-paraKähler manifold and TM its tangent bundle equipped with the Berger type deformed Sasaki metric g^{BS} and the paracomplex structure $\widetilde{\phi}$. The Riemannian submersion $\pi : (TM, \widetilde{\phi}, g^{BS}) \longrightarrow M_{2k}$ is totally geodesic. Moreover π is a biharmonic map.*

Proof. Let $(x, u) \in TM$ and let $\{e_i\}_{i=1}^{2k}$, such that $e_1 = \dfrac{u}{\|u\|}$ is an orthonormal basis of TM at x. Then $\left\{e_i^H, \dfrac{1}{\sqrt{1 + \alpha\delta^2}}(\phi(e_1))^V, (\phi(e_j))^V, j = 2...n\right\}$ is an orthonormal basis of $T_{(x,u)}TM$ at (x, u) such that $(\nabla_{e_i}e_i)_x = 0$. Then by summing over i, we have

$$\begin{aligned}
&\nabla d\pi(e_i^H, e_i^H) = 0,\\
&\nabla d\pi\Big(\frac{1}{\sqrt{1 + \alpha\delta^2}}(\phi(e_1))^V, \frac{1}{\sqrt{1 + \alpha\delta^2}}(\phi(e_1))^V\Big) = 0,\\
&\nabla d\pi\Big((\phi(e_j))^V, (\phi(e_j))^V\Big) = 0,\\
&\nabla d\pi\Big(e_i^H, (\phi(e_j))^V\Big) = 0,\\
&\nabla d\pi\Big(e_i^H, \frac{1}{\sqrt{1 + \alpha\delta^2}}(\phi(e_1))^V\Big) = 0.
\end{aligned}$$

□

Let h be another anti-paraHermitian metric on M with respect to an almost paracomplex structure ϕ_1. We take in consideration the projection
$\pi : (TM, \widetilde{\phi}, g^{BS}) \longrightarrow (M_{2k}, \phi_1, h)$. Then we have

$$\nabla d\pi(e_i^H, e_i^H) = \nabla^h_{e_i} e_i,$$

$$\begin{aligned}
&\nabla d\pi\Big(\frac{1}{\sqrt{1+\alpha\delta^2}}(\phi(e_1))^V,\frac{1}{\sqrt{1+\alpha\delta^2}}(\phi(e_1))^V\Big)=0,\\
&\nabla d\pi\Big((\phi(e_j))^V,(\phi(e_j))^V\Big)=0,\\
&\nabla d\pi\Big(e_i^H,(\phi(e_j))^V\Big)=0,\\
&\nabla d\pi\Big(e_i^H,\frac{1}{\sqrt{1+\alpha\delta^2}}(\phi(e_1))^V\Big)=0,
\end{aligned}$$

where ∇^h the Levi-Civita connection of the metric h. Hence we get the proposition below.

Proposition 4.4.1. *Let (M_{2k},ϕ,g) be an anti-paraKähler manifold and TM its tangent bundle equipped with the Berger type deformed Sasaki metric g^{BS} and the paracomplex structure $\widetilde{\phi}$. Then $\pi:(TM,\widetilde{\phi},g^{BS})\longrightarrow(M_{2k},\phi_1,h)$ is a biharmonic if and only if $I:(M_{2k},\phi,g)\longrightarrow(M_{2k},\phi_1,h)$ is totally geodesic.*

4.5 Harmonicity of the map $\Psi:(TM,\widetilde{\phi},g^{BS})\to(TN,h^S)$

In the section, we denote (M_{2k},ϕ,g) be an anti-paraKähler manifold and (TM,ϕ,g^{BS}) its tangent bundle equipped with the Berger type deformed Sasaki metric g^{BS} and the paracomplex structure $\widetilde{\phi}$, (N^n,g) be an n-dimensional Riemannian manifold and (TM,h^S) its tangent bundle equipped with the Sasaki metric h^S.

Theorem 4.5.1. *Let $\Psi:(TM,\widetilde{\phi},g^{BS})\longrightarrow(TN,h^S)$ be a the tangent map of the map $\psi:(M_{2k},\phi,g)\longrightarrow(N^n,h)$, then the tension field $\tau(\Psi)$ of ψ is given by*

$$\begin{aligned}
\tau(\Psi) \;=\; &\Big[\tau(\psi)+tr_hR^N(d\psi(u),\nabla d\psi(u,*))d\psi(*)\Big]^H+\Big[div(\nabla d\psi)(u)\\
&-\frac{\delta^2}{1+\alpha\delta^2}\Big(tr_g g(*,\phi(*))-\frac{\delta^2}{1+\alpha\delta^2}g(u,\phi u)\Big)d\psi(\phi(u))\Big]^V.
\end{aligned}$$

Proof. Let $(\psi(x),d\psi(u))\in TN$ and let $\{e_i\}_{i=1}^{2k}$ is an orthonormal basis of TM at x. Then $\Big\{e_i^H,\frac{1}{\sqrt{1+\alpha\delta^2}}(\phi(e_1))^V,(\phi(e_j))^V,j=2...n\Big\}$ is an orthonormal basis of $T_{(x,u)}TM$ at (x,u) such that $e_1=\dfrac{u}{\|u\|}$ and $(\nabla_{e_i}e_i)_x=0$. Then by summing over i, we have:

$$\begin{aligned}
\tau(\Psi)=&\widetilde{\nabla}^\Psi_{e_i^H}d\Psi(e_i^H)-d\Psi(\widetilde{\nabla}_{e_i^H}e_i^H)+\widetilde{\nabla}^\Psi_{\frac{1}{\sqrt{1+\alpha\delta^2}}\phi(e_1)^V}d\Psi(\frac{1}{\sqrt{1+\alpha\delta^2}}\phi(e_1)^V)\\
&-d\Psi(\widetilde{\nabla}_{\frac{1}{\sqrt{1+\alpha\delta^2}}\phi(e_1)^V}\frac{1}{\sqrt{1+\alpha\delta^2}}\phi(e_1)^V)+\widetilde{\nabla}^\Psi_{\phi(e_j)^V}d\Psi\phi(e_j)^V\\
&-d\Psi(\widetilde{\nabla}_{\phi(e_j)^V}\phi(e_j)^V),
\end{aligned}$$

using Lemma 4.1.1, we have

$$
\begin{aligned}
=&\widehat{\nabla}_{d\psi(e_i)^H + \nabla d\psi(u,e_i)^V}(d\psi(e_i)^H + \nabla d\psi(u,e_i)^V) - d\psi(\widetilde{\nabla}_{e_i^H} e_i^H)\\
&+ \widehat{\nabla}_{\frac{1}{\sqrt{1+\alpha\delta^2}} d\Psi\phi(e_1)^V} \frac{1}{\sqrt{1+\alpha\delta^2}} d\Psi(\phi(e_1)^V) + \widehat{\nabla}_{d\Psi(\phi(e_j)^V)} d\Psi\phi(e_j)^V\\
&- d\Psi(\widetilde{\nabla}_{\frac{1}{\sqrt{1+\alpha\delta^2}}\phi(e_1)^V} \frac{1}{\sqrt{1+\alpha\delta^2}}\phi(e_1)^V) - d\Psi(\widetilde{\nabla}_{\phi(e_j)^V}\phi(e_j)^V),
\end{aligned}
$$

from Proposition 2.6.1, we have

$$
\begin{aligned}
\tau(\Psi) =&(\nabla_{d\psi(e_i)} d\psi(e_i))^H + (\nabla_{d\psi(e_i)} \nabla d\psi(u,e_i))^V\\
&+ (R(d\psi(u), \nabla d\psi(u,e_i))d\psi(e_i))^H - d\psi(\nabla_{e_i} e_i)^H\\
&- \Big(\frac{\delta}{1+\alpha\delta^2}\Big)^2 g(\phi(e_1), \phi\phi(e_1)) d\psi(\phi(u))^V - g(\phi(e_j), \phi\phi(e_j)) d\psi(\phi(u))^V.
\end{aligned}
$$

□

Theorem 4.5.2. *Let $\Psi : (TM, \widetilde{\phi}, g^{BS}) \longrightarrow (TN, h^S)$ be a the tangent map of the map $\psi : (M_{2k}, \phi, g) \longrightarrow (N^n, h)$, then Ψ is harmonic if and only if the following conditions are verified*

$$
\begin{aligned}
&\tau(\psi) + tr_h R^N(d\psi(u), \nabla d\psi(u,*))d\psi(*) = 0,\\
&div(\nabla d\psi)(u) - \frac{\delta^2}{1+\alpha\delta^2}\Big(tr_g g(*, \phi(*)) - \frac{\delta^2}{1+\alpha\delta^2} g(u, \phi u)\Big) d\psi(\phi(u)) = 0.
\end{aligned}
$$

Corollary 4.5.1. *Let $\Psi : (TM, \widetilde{\phi}, g^{BS}) \longrightarrow (TN, h^S)$ be a the tangent map of the map $\psi : (M_{2k}, \phi, g) \longrightarrow (N^n, h)$, if ψ is totally geodesic then Ψ is harmonic if and only if*

$$
tr_g g(*, \phi(*)) = \frac{\delta^2}{1+\alpha\delta^2} g(u, \phi u).
$$

Lemma 4.5.1. *Let $\Psi : (TM, \widetilde{\phi}, g^{BS}) \longrightarrow (TN, h^S)$ be a the tangent map of the map $\psi : (M_{2k}, \phi, g) \longrightarrow (N^n, h)$, then the energy density associated to Ψ is given by*

$$
e(\Psi) = 2e(\psi) + \frac{1}{2} tr_h |(\nabla d\psi(u,*))|^2 - \frac{\delta^2}{2(1+\alpha\delta^2)} |d\psi(\phi(u))|^2. \tag{4.2}
$$

Proof. Let $(x,u) \in TM$ and let $\{e_i\}_{i=1}^{2k}$, such that $e_1 = \dfrac{u}{\|u\|}$ is an orthonormal basis of TM at x. Then $\Big\{e_i^H, \dfrac{1}{\sqrt{1+\alpha\delta^2}}(\phi(e_1))^V, (\phi(e_j))^V, j = 2...n\Big\}$ is an orthonormal basis of $T_{(x,u)}TM$ at (x,u) such that $(\nabla_{e_i} e_i)_x = 0$. Then by summing over i, we have:

$$
e(\Psi)_{(\psi(x), d\psi(x))} = \frac{1}{2}\Big[\big(h^S_{(\psi(x), d\psi(x))}(d\Psi(e_i^H), d\Psi(e_i^H))\big)
$$

$$
\begin{aligned}
&+\frac{1}{1+\alpha\delta^2}h^S_{(\psi(x),d\psi(x))}(d\Psi(\phi(e_1)^V),d\Psi(\phi(e_1)^V))\\
&+h^S_{(\psi(x),d\psi(x))}(d\Psi(\phi(e_j)^V),d\Psi(\phi(e_j)^V))\Big]\\
=&\frac{1}{2}\Big[\Big(h^S(d\psi(e_i)^H,d\psi(e_i)^H)+h^S(\nabla d\psi(u,e_i)^V,\nabla d\psi(u,e_i)^V)\Big)\\
&+\frac{1}{1+\alpha\delta^2}h(d\psi(\phi(e_1))^V,d\psi(\phi(e_1))^V)\\
&+h^S(d\psi(\phi(e_j))^V,d\psi(\phi(e_j))^V)\Big]\\
=&\frac{1}{2}\Big[2e(\psi)+tr_h|\nabla d\psi(u,*)|^2+\frac{1}{1+\alpha\delta^2}h(d\psi(\phi(e_1)),d\psi(\phi(e_1)))\\
&+h^S(d\psi\phi(e_i)),d\psi(\phi(e_i)))-h^S(d\psi(\phi(e_1)),d\psi(\phi(e_1)))\Big]\\
=&\frac{1}{2}\Big[4e(\psi)+tr_h||\nabla d\psi(u,*)||^2+\frac{1}{\alpha(1+\alpha\delta^2)}h(d\psi(\phi(u)),d\psi(\phi(u)))\\
&-\frac{1}{\alpha}h(d\psi(\phi(u)),d\psi(\phi(u)))\Big].
\end{aligned}
$$

□

Theorem 4.5.3. *Let TM be a compact tangent bundle and $\Psi : (TM,\widetilde{\phi},g^{BS}) \longrightarrow (TN,h^S)$ be a the tangent map of the map $\psi : (M_{2k},\phi,g) \longrightarrow (N^n,h)$, then Ψ is harmonic if and only if ψ is totally geodesic and*

$$tr_g g(*,\phi(*)) = \frac{\delta^2}{1+\alpha\delta^2}g(u,\phi u).$$

Proof. If ψ is totally geodesic and $tr_g g(*,\phi(*)) = \frac{\delta^2}{1+\alpha\delta^2}g(u,\phi u)$, from Corollary1.27, we deduce that Ψ is harmonic. Inversely.
Let $\omega : I\times M \longrightarrow N$ be a smooth map satisfying for all $t\in I=(-\epsilon,\epsilon),\epsilon>0$ and all $x\in M$

$$\omega(t,x)=\psi_t(x)=(1+t)\psi(x)$$

and

$$\omega(0,x)=\psi(x).$$

The variation vector field $v\in\Gamma(\psi^{-1}TN)$ associated to the variation $\{\psi_t\}_{t\in I}$ is given for all $x\in M$ by

$$v(x)=d_{(0,x)}\omega(\frac{d}{dt}),$$

from Lemma 4.5.1, we have

$$e(\Psi_t)=2(1+t)^2e(\psi_t)+\frac{(1+t)^2}{2}tr_h|(\nabla d\psi_t(u,*))|^2-\frac{\delta^2(1+t)^2}{2(1+\alpha\delta^2)}|d\psi_t(\phi(u))|^2.$$

If Ψ is a critical point of the energy functional, from equation 1.20, we have

$$\begin{aligned}\frac{d}{dt}E(\phi_t)_{t=0} &= 0\\ &= \int_{TM} 4e(\psi) + tr_h|(\nabla d\psi(u,*))|^2 - \frac{\delta^2}{(1+\alpha\delta^2)}|d\psi(\phi(u))|^2 dv_{g^{BS}} = 0.\end{aligned}$$

If Ψ is harmonic hence $\nabla d\psi = 0$. □

Example 4.5.1. *Let the map*

$$\begin{aligned}\psi : (\mathbb{R}^2, \phi, dx^2 + dy^2) &\longrightarrow (\mathbb{R}, dt^2)\\ (x, y) &\longmapsto \psi(x, y) = x^2 - y^2,\end{aligned}$$

let the basis on $\mathbb{R}$ be ∂t, the orthonormal basis on $\mathbb{R}^2$ is $\{e_1 = \frac{\partial}{\partial x}, e_2 = \frac{\partial}{\partial y}\}$ and $u = u_1\frac{\partial}{\partial x} + u_2\frac{\partial}{\partial y}$, let us take the almost paracomplex structure on $\mathbb{R}^2$ that verify

$$\phi(\frac{\partial}{\partial x}) = \frac{\partial}{\partial y} \text{ and } \phi(\frac{\partial}{\partial y}) = \frac{\partial}{\partial x}.$$

then we have $\Psi = d\psi : (T\mathbb{R}^2, \widetilde{\phi}, g^{BS}) \longrightarrow (T\mathbb{R}, h^s)$.
Then, $\tau(\Psi) = 2u_1u_2$

$$\begin{cases}\tau(\psi) = \Delta\psi = 0\\ div(\nabla d\psi)(u) = 0\\ g(\frac{\partial}{\partial x}, \phi(\frac{\partial}{\partial x})) + g(\frac{\partial}{\partial y}, \phi(\frac{\partial}{\partial y})) = g(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) + g(\frac{\partial}{\partial y}, \frac{\partial}{\partial x}) = 0\\ g(u, \phi(u)) = g(u_1\frac{\partial}{\partial x} + u_2\frac{\partial}{\partial y}, \phi(u_1\frac{\partial}{\partial x} + u_2\frac{\partial}{\partial y})) = 2u_1u_2\end{cases}$$

and Ψ is harmonic if and only if $u_1u_2 = 0$.

4.5.1 Harmonic identity map $I : (TM, \widetilde{\phi}, g^{BS}) \longrightarrow (TM, g^S)$

Proposition 4.5.1. *Let (M_{2k}, ϕ, g) be an anti-paraKähler manifold and TM its tangent bundle equipped with the Berger type deformed Sasaki metric g^{BS} and the paracomplex structure $\widetilde{\phi}$. Suppose that $I : (TM, \widetilde{\phi}, g^{BS}) \longrightarrow (TM, g^S)$ is the identity map. Then the tension field $\tau(I)$ of I is given by*

$$\tau(I) = \frac{\delta^4}{(1+\alpha\delta^2)^2}g(u, \phi(u))(\phi(u))^V - \frac{\delta^2}{1+\alpha\delta^2}tr_g\Big(g(*, \phi(*))(\phi(u))^V\Big). \tag{4.3}$$

Proof. Let $(x, u) \in TM$ and let $\{e_i\}_{i=1}^{2k}$, such that $e_1 = \dfrac{u}{\|u\|}$ is an orthonormal basis of TM at x. Then $\left\{e_i^H, \dfrac{1}{\sqrt{1+\alpha\delta^2}}(\phi(e_1))^V, (\phi(e_j))^V, j = 2...n\right\}$ is an orthonormal basis of $T_{(x,u)}TM$ at (x, u) such that $(\nabla_{e_i}e_i)_x = 0$. Then by summing over i, we have:

$$\tau(I) = \nabla^I_{e_i^H} dI(e_i^H) + \nabla^I_{(\phi(e_1))^V} dI((\phi(e_1))^V) + \nabla^I_{(\phi(e_j))^V} dI((\phi(e_j))^V)$$

$$- dI(\overline{\nabla}_{e_i^H} e_i^H + \frac{1}{1+\alpha\delta^2}\overline{\nabla}_{(\phi(e_1))^V}(\phi(e_1))^V + \overline{\nabla}_{(\phi(e_j))^V}(\phi(e_j))^V).$$

From Theorem 2.6.3, we have

$$\begin{aligned}\tau(I) =& \frac{-\delta^2}{1+\alpha\delta^2}\Big(\frac{1}{1+\alpha\delta^2}g(e_1,\phi(e_1)) + g(e_j,\phi(e_j))\Big)(\phi(u))^V \\ =& \frac{-\delta^2}{1+\alpha\delta^2}\Big(\frac{-\delta^2}{1+\alpha\delta^2}g(u,\phi(u)) + g(e_i,\phi(e_i))\Big)(\phi(u))^V.\end{aligned}$$

□

Theorem 4.5.4. *Let (M_{2k},ϕ,g) be an anti-paraKähler manifold and TM its tangent bundle equipped with the Berger type deformed Sasaki metric g^{BS} and the paracomplex structure $\widetilde{\phi}$. Suppose that $I : (TM,\widetilde{\phi},g^{BS}) \longrightarrow (TM,g^S)$ is the identity map. Then the bitension field $\tau_2(I)$ of I is given by*

$$\tau_2(I)_{(x,u)} = \Big(\Delta(\tau(I))\Big)^V_{(x,u)} + tr_g\Big(R(u,\nabla_*\tau(I)) *\Big)^H_{(x,u)},$$

where $\Delta(\tau(I)) = tr_g(\nabla^2_*\tau(I))$.

Proof. Let $(x,u) \in TM$ and let $\{e_i\}_{i=1}^{2k}$, such that $e_1 = \dfrac{u}{\|u\|}$ is an orthonormal basis of TM at x. Then $\Big\{e_i^H, \dfrac{1}{\sqrt{1+\alpha\delta^2}}(\phi(e_1))^V, (\phi(e_j))^V, j = 2...n\Big\}$ is an orthonormal basis of $T_{(x,u)}TM$ at (x,u) such that $(\nabla_{e_i}e_i)_x = 0$. Then by summing over i, we have:

$$\begin{aligned}tr_{g^S}(\nabla^2\tau(I))_{(x,u)} =& \Big(\nabla^I_{e_i^H}\nabla^I_{e_i^H}\tau(I)\Big)_{(x,u)} + \frac{1}{1+\alpha\delta^2}\Big(\nabla^I_{(\phi(e_1))^V}\nabla^I_{(\phi(e_1))^V}\tau(I)\Big)_{(x,u)} \\ &+ \Big(\nabla^I_{(\phi(e_j))^V}\nabla^I_{(\phi(e_j))^V}\tau(I)\Big)_{(x,u)} - \Big(\nabla^I_{\overline{\nabla}_{e_i^H}e_i^H}\tau(I)\Big)_{(x,u)} \\ &- \frac{1}{1+\gamma\delta^2}\Big(\nabla^I_{\overline{\nabla}_{(\phi(e_1))^V}(\phi(e_1))^V}\tau(I)\Big)_{(x,u)} \\ &- \Big(\nabla^I_{\overline{\nabla}_{(\phi(e_j))^V}(\phi(e_j))^V}\tau(I)\Big)_{(x,u)}.\end{aligned}$$

By using the Levi-Civita connection of Sasaki metric, we have

$$\begin{aligned}tr_{g^S}(\nabla^2\tau(I))_{(x,u)} =& \Big(\nabla_{e_i}\nabla_{e_i}\tau(I) - \frac{1}{4}R(e_i,R(u,\tau(I))e_i)u\Big)^V_{(x,u)} \\ &+ \frac{1}{2}\Big(R(u,\nabla_{e_i}\tau(I))e_i + \nabla_{e_i}R(u,\tau(I))e_i\Big)^H_{(x,u)}.\end{aligned}$$

By using the Riemannian curvature tensor of Sasaki metric, we have

$$tr_{g^S}(R(\tau(I),dI)dI)_{(x,u)} = (R(\tau(I),e_i^H)e_i^H)_{(x,u)} = -(R(e_i^H,\tau(I))e_i^H)_{(x,u)}$$

$$
\begin{aligned}
=&\Big(-\frac{1}{4}R(R(u,\tau(I))e_i,e_i)u-\frac{1}{2}R(e_i,e_i)\tau(I)\Big)^V_{(x,u)}\\
&\Big(-\frac{1}{2}(\nabla_{e_i}R)(u,\tau(I))e_i\Big)^H_{(x,u)}.
\end{aligned}
$$

Considering the formula 1.29, we deduce

$$
\tau_2(I)_{(x,u)} = \Big(\nabla_{e_i}\nabla_{e_i}\tau(I)\Big)^V_{(x,u)} + \Big(R(u,\nabla_{e_i}\tau(I))e_i\Big)^H_{(x,u)}.
$$

□

Theorem 4.5.5. *Let (M_{2k}, ϕ, g) be an anti-paraKähler manifold and TM its tangent bundle equipped with the Berger type deformed Sasaki metric g^{BS} and the paracomplex structure $\widetilde{\phi}$. Suppose that $I : (TM, \widetilde{\phi}, g^{BS}) \longrightarrow (TM, g^S)$ is the identity map. Then I is biharmonic if and only if*

$$
\Delta(\tau(I)) = 0 \qquad \textit{and} \quad tr_g(R(u,\nabla_*\tau(I))*) = 0.
$$

Corollary 4.5.2. *Let (M_{2k}, ϕ, g) be an anti-paraKähler manifold and TM its tangent bundle equipped with the Berger type deformed Sasaki metric g^{BS} and the paracomplex structure $\widetilde{\phi}$. If $\tau(I)$ is a parallel tension field then I is biharmonic.*

4.6 Harmonicity of the map $\Psi : (TN, h^S) \to (TM, \widetilde{\phi}, g^{BS})$

In the section, we denote (N^n, g) be an n-dimensional Riemannian manifold and (TM, h^S) its tangent bundle equipped with the Sasaki metric h^S, (M_{2k}, ϕ, g) be an anti-paraKähler manifold and (TM, ϕ, g^{BS}) its tangent bundle equipped with the Berger type deformed Sasaki metric g^{BS} and the paracomplex structure $\widetilde{\phi}$.

Theorem 4.6.1. *Let $\Psi : (TN, h^S) \to (TM, \widetilde{\phi}, g^{BS})$ be a the tangent map of the map $\psi : (N^n, h) \longrightarrow (M_{2k}, \phi, g)$, then the tension field $\tau(\Psi)$ of ψ is given by*

$$
\begin{aligned}
\tau(\Psi) \;=\;& \Big[\tau(\psi) + tr_g R^M(d\psi(u), \nabla d\psi(u,*))d\psi(*)\Big]^H + \Big[div(\nabla d\psi)(u)\\
&+\frac{\delta^2}{1+\alpha\delta^2} tr_g\Big(g(\nabla d\psi(u,*), \phi\nabla d\psi(u,*)) + g(d\psi(*), \phi d\psi(*)))\Big)\phi d\psi(u)\Big]^V.
\end{aligned}
$$

Proof. Let $(\psi(x), d\psi(y)) \in TN$, and $\{e_i^H, f_i^V\}_{i=1}^m$ be a local orthonormal frame on TM such that $(\nabla_{e_i}e_i)_x = 0$ then by summing over i, we have

$$
\begin{aligned}
\tau(\Psi) =& \nabla^\Psi_{e_i^H} d\Psi(e_i^H) - d\Psi(\widehat{\nabla}_{e_i^H} e_i^H) + \nabla^\Psi_{e_i^V} d\Psi(e_i^V) - d\Psi(\widehat{\nabla}_{e_i^V} e_i^V)\\
=& \widetilde{\nabla}_{d\Psi(e_i^H)} d\Psi(e_i^H) - d\Psi(\widehat{\nabla}_{e_i^H} e_i^H) + \widetilde{\nabla}_{d\Psi(e_i^V)} d\Psi(e_i^V) - d\Psi(\widehat{\nabla}_{e_i^V} e_i^V),
\end{aligned}
$$

From Proposition 2.6.1 and Proposition 2.2.1 , we have

$$\begin{aligned}
\tau(\Psi) =&\widetilde{\nabla}_{d\psi(e_i)^H+(\nabla d\psi(u,e_i))^V}[d\psi(e_i)^H + (\nabla d\psi(u,e_i))^V] + \widetilde{\nabla}_{d\psi(e_i)^V} d\psi(e_i)^V \\
=&(\nabla_{d\psi(e_i)} d\psi(e_i))^H + (R(d\psi(u), \nabla d\psi(u,e_i))d\psi(e_i))^H - (d\psi(\nabla_{e_i} e_i))^H \\
&+ (\nabla_{d\psi(e_i)} \nabla d\psi(u,e_i))^V - (\nabla d\psi(u, \nabla_{e_i} e_i))^V \\
&+ \frac{\delta^2}{1+\delta^2\alpha}\big[g(\nabla d\psi(u,e_i), \phi\nabla d\psi(u,e_i))\phi d\psi(u) + g(d\psi(e_i), \phi d\psi(e_i))\phi d\psi(u)\big]^V \\
=&\Big[\tau(\psi) + (R(d\psi(u), \nabla d\psi(u,e_i))d\psi(e_i))\Big]^H - \Big[div(\nabla d\psi) \\
&+ \frac{\delta^2}{1+\delta^2\alpha}\big(g(\nabla d\psi(u,e_i), \phi\nabla d\psi(u,e_i)) + g(d\psi(e_i), \phi d\psi(e_i))\big)\phi d\psi(u)\Big]^V.
\end{aligned}$$

□

Theorem 4.6.2. *Let $\Psi : (TN, h^S) \to (TM, \widetilde{\phi}, g^{BS})$ be a the tangent map of the map $\psi : (N^n, h) \longrightarrow (M_{2k}, \phi, g)$, then Ψ is harmonic if and only if*

$$0 = \tau(\psi) + tr_h R^M(d\psi(u), \nabla d\psi(u,*))d\psi(*),$$

and

$$0 = div(\nabla d\psi)(u) + \frac{\delta^2}{1+\delta^2\alpha} tr_g\Big(g(\nabla d\psi(u,*), \phi\nabla d\psi(u,*)) + g(d\psi(*), \phi d\psi(*)))\Big)\phi d\psi(u).$$

Corollary 4.6.1. *Let $\Psi : (TN, h^S) \to (TM, \widetilde{\phi}, g^{BS})$ be a the tangent map of the map $\psi : (N^n, h) \longrightarrow (M_{2k}, \phi, g)$, if ψ is totally geodesic then Ψ is harmonic if and only if*

$$tr_g g(d\psi(*), \phi d\psi(*)) = 0.$$

Lemma 4.6.1. *Let $\Psi : (TN, h^S) \to (TM, \widetilde{\phi}, g^{BS})$ be a the tangent map of the map $\psi : (N^n, h) \longrightarrow (M_{2k}, \phi, g)$, then the energy density associated to Ψ is given by*

$$e(\Psi) = 2e(\psi) + \frac{1}{2} tr_g |\nabla d\psi(u,*)|^2 + \frac{\delta^2}{2}(g^2(\nabla d\psi(u,*), \phi d\psi(u)) + g^2(d\psi(*), \phi d\psi(u))). \tag{4.4}$$

Proof. Let $(x,u) \in TM$ and let $\{e_i\}_{i=1}^{2k}$, such that $e_1 = \frac{u}{\|u\|}$ is an orthonormal basis of TM at x. Then $\Big\{e_i^H, e_i^V, i = 1...n\Big\}$ is an orthonormal basis of $T_{(x,u)}TN$ at (x,u) such that $(\nabla_{e_i} e_i)_x = 0$. Then by summing over i, we have:

$$\begin{aligned}
e(\Psi) =&\frac{1}{2}\Big[g^{BS}(d\Psi(e_i^H), d\Psi(e_i^H)) + g^{BS}(d\Psi(e_i^V), d\Psi(e_i^V))\Big] \\
=&\frac{1}{2}\Big[g^{BS}(d\psi(e_i)^H, d\psi(e_i)^H) + g^{BS}(\nabla d\psi(u,e_i)^V, \nabla d\psi(u,e_i)^V) \\
&+ g^{BS}(d\psi(e_i)^V, d\psi(e_i)^V)\Big].
\end{aligned}$$

From Definition 3.2.1, we have

$$e(\Psi) = 2e(\psi) + \frac{1}{2} tr_g |\nabla d\psi(u,*)|^2 + \frac{\delta^2}{2}(g^2(\nabla d\psi(u,e_i), \phi d\psi(u)) + g^2(d\psi(e_i), \phi d\psi(u))).$$

□

Theorem 4.6.3. *Let TN be a compact tangent bundle and $\Psi : (TN, h^S) \to (TM, \widetilde{\phi}, g^{BS})$ be a the tangent map of the map $\psi : (N^n, h) \longrightarrow: (M_{2k}, \phi, g)$, then Ψ is harmonic if and only if ψ is totally geodesic and*

$$tr_g g(d\psi(*), \phi d\psi(*)) = 0.$$

Proof. If ψ is totally geodesic and $tr_g g(d\psi(*), \phi d\psi(*)) = 0$ from Corollary4.6.1, we deduce that Ψ is harmonic. Inversely.
Let $\omega : I \times N \longrightarrow M$ be a smooth map satisfying for all $t \in I = (-\epsilon, \epsilon), \epsilon > 0$ and all $x \in N$

$$\omega(t,x) = \psi_t(x) = (1+t)\psi(x),$$

and

$$\omega(0,x) = \psi(x).$$

The variation vector field $v \in \Gamma(\psi^{-1}TM)$ associated to the variation $\{\psi_t\}_{t\in I}$ is given for all $x \in N$ by

$$v(x) = d_{(0,x)}\omega\Big(\frac{d}{dt}\Big),$$

From Lemma 4.6.1, we have

$$\begin{aligned} e(\Psi_t) =& 2e(\psi) + \frac{(1+t)^2}{2} tr_g |\nabla d\psi(u,*)|^2 + \frac{\delta^2(1+t)^2}{2}\Big(g^2(\nabla d\psi(u,e_i), \phi d\psi(u)) \\ &+ (1+t)^2 g^2(d\psi(e_i), \phi d\psi(u))\Big). \end{aligned}$$

If Ψ is a critical point of the energy functional, from equation 1.20, we have

$$\begin{aligned} \frac{d}{dt}E(\phi_t)_{t=0} &= 0 \\ &= \int_{TN} \Big[2e(\psi) + tr_g|\nabla d\psi(u,*)|^2 + \frac{\delta^2}{2}\Big(g^2(\nabla d\psi(u,e_i), \phi d\psi(u)) \\ &\quad + g^2(d\psi(e_i), \phi d\psi(u))\Big)\Big] v_g = 0. \end{aligned}$$

If Ψ is harmonic hence $\nabla d\psi = 0$. □

Example 4.6.1. *Let the map*

$$\psi : (\mathbb{R}, dt^2) \longrightarrow (\mathbb{R}^2, \phi, dx^2 + dy^2)$$

$$t \longmapsto \psi(t) = (x(t), y(t)),$$

let $\{e_1 = \frac{d}{dt}\}$ *be the basis on* $\mathbb{R}$ *and* $u = u_0\frac{d}{dt}$, *the orthonormal basis on* $\mathbb{R}^2$ *is defined by* $\{f_1 = \frac{\partial}{\partial x}, f_2 = \frac{\partial}{\partial y}\}$, *let us take the almost paracomplex structure on* $\mathbb{R}^2$ *that verify*

$$\phi(\frac{\partial}{\partial x}) = -\frac{\partial}{\partial x} \ \ and \ \ \phi(\frac{\partial}{\partial y}) = \frac{\partial}{\partial y},$$

then for $\Psi = d\psi : (T\mathbb{R}, h^s) \longrightarrow (T\mathbb{R}^2, \widetilde{\phi}, g^{BS})$, *we have*

$$\begin{aligned}
\tau(\psi) =& \nabla^{\psi}_{\partial t} d\psi(\partial t) - d\psi(\nabla_{\partial t}\partial t) = \nabla^{\psi}_{\partial t}\frac{d\psi^1}{dt}\frac{\partial}{\partial x} + \nabla^{\psi}_{\partial t}\frac{d\psi^2}{dt}\frac{\partial}{\partial y} \\
=& \frac{d^2\psi^1}{dt^2}\frac{\partial}{\partial x} + \frac{d^2\psi^2}{dt^2}\frac{\partial}{\partial y} = x''\frac{\partial}{\partial x} + y''\frac{\partial}{\partial y} \\
\nabla^{\psi}_{\partial t}\nabla d\psi(\partial t, u) =& \nabla^{\psi}_{\partial t}\big(\nabla^{\psi}_{\partial t} d\psi(u)\big) = u_0\nabla^{\psi}_{\partial t}\Big(\frac{d^2\psi^1}{dt^2}\frac{\partial}{\partial x} + \frac{d^2\psi^2}{dt^2}\frac{\partial}{\partial y}\Big) \\
=& u_0\frac{d^3\psi^1}{dt^3}\frac{\partial}{\partial x} + u_0\frac{d^3\psi^2}{dt^3}\frac{\partial}{\partial y} = u_0(x^{(3)}\frac{\partial}{\partial x} + y^{(3)}\frac{\partial}{\partial y})
\end{aligned}$$

then we get

$$\begin{aligned}
g(\nabla d\psi(u, \partial t), \phi\nabla d\psi(u, \partial t)) =& u_0 g\Big(\frac{d^3\psi^1}{dt^3}\frac{\partial}{\partial x} + \frac{d^3\psi^2}{dt^3}\frac{\partial}{\partial y}, \phi\big(\frac{d^3\psi^1}{dt^3}\frac{\partial}{\partial x} + \frac{d^3\psi^2}{dt^3}\frac{\partial}{\partial y}\big)\Big) \\
=& 2u_0\frac{d^3\psi^1}{dt^3}\frac{d^3\psi^2}{dt^3} = 2u_0 x^{(3)} y^{(3)} \\
g(d\psi(\partial t), \phi d\psi(\partial t)) =& g\Big(\frac{d\psi^1}{dt}\frac{\partial}{\partial x} + \frac{d\psi^2}{dt}\frac{\partial}{\partial y}, \phi\big(\frac{d\psi^1}{dt}\frac{\partial}{\partial x} + \frac{d\psi^2}{dt}\frac{\partial}{\partial y}\big)\Big) \\
=& 2\frac{d\psi^1}{dt}\frac{d\psi^2}{dt} = 2x'y' \\
\phi(d\psi(u)) =& u_0\frac{d\psi^1}{dt}\frac{\partial}{\partial x} + u_0\frac{d\psi^2}{dt}\frac{\partial}{\partial y} = u_0(x'\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y})
\end{aligned}$$

then Ψ *to be harmonic if*

$$\begin{cases} x''\frac{\partial}{\partial x} + y''\frac{\partial}{\partial y} = 0 \\ and \\ (x^{(3)}\frac{\partial}{\partial x} + y^{(3)}\frac{\partial}{\partial y}) + \frac{\delta^2}{1+\delta^2\alpha}\big(2u_0 x^{(3)} y^{(3)} + 2x'y'\big)(x'\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y}) = 0. \end{cases}$$

4.7 Bi-harmonic identity map $I : (TM, g^S) \longrightarrow (TM, \widetilde{\phi}, g^{BS})$

Now we investigate the harmonicity of the Berger type deformed Sasaki metric g^{BS} and the Sasaki metric g^S with respect to each other. By using the Levi-Civita connection of these metrics we state the following two propositions (for the Levi-Civita connection of the Sasaki metric.

Proposition 4.7.1. *Let (M_{2k},ϕ,g) be an anti-paraKähler manifold and TM its tangent bundle equipped with the Berger type deformed Sasaki metric g^{BS} and the paracomplex structure $\widetilde{\phi}$. Suppose that $I:(TM,g^S)\longrightarrow(TM,\widetilde{\phi},g^{BS})$ is the identity map. Then the tension field $\tau(I)$ of I is given by*

$$\tau(I)=tr_g\Big(\frac{\delta^2}{1+\alpha\delta^2}g(*,\phi(*))(\phi(u))^V\Big). \tag{4.5}$$

Theorem 4.7.1. *Let (M_{2k},ϕ,g) be an anti-paraKähler manifold and TM its tangent bundle equipped with the Berger type deformed Sasaki metric g^{BS} and the paracomplex structure $\widetilde{\phi}$. Suppose that TM is a compact tangent bundle, then the identity map $I:(TM,g^S)\longrightarrow(TM,\widetilde{\phi},g^{BS})$ is biharmonic if and only if is hrarmonic.*

Proof. Let I_t be a compactly supported variation of $\tau(I)$ defined by $\tau(I_t)=(1+t)\tau(I)$.

$$\begin{aligned}E_2(\tau(I)_t)=&\frac{1}{2}\int|\tau(I_t)|^2_{g^{BS}}v_g\\=&\frac{1}{2}\int g(\tau(I_t),\tau(I_t))v_g+\frac{\delta^2}{2}\int g(\tau(I_t),\phi(u))v_g\\=&\frac{(1+t)^2}{2}\int g(\tau(I),\tau(I)))v_g+\frac{\delta^2(1+t)^2}{2}\int(g(\tau(I),\phi(u))^2v_g,\end{aligned}$$

then

$$0=\frac{d}{dt}E_2(\tau(I)_t)_{t=0}=\int g(\tau(I),\tau(I)))v_g+\int(g(\tau(I),\phi(u))^2v_g,$$

we now have

$$\tau(I)=0.$$

□

Theorem 4.7.2. *Let (M_{2k},ϕ,g) be an anti-paraKähler manifold and TM its tangent bundle equipped with the Berger type deformed Sasaki metric g^{BS} and the paracomplex structure $\widetilde{\phi}$. Then the bitension field $\tau_2(I)$ of I is given by*

$$\begin{aligned}\tau_2(I)_{(x,u)}=&\Big\{tr_g\Big(-\frac{3}{2}(\nabla_*R)(u,\tau(I))*\Big)\Big\}^H_{(x,u)}+\Big\{tr_g\Big(\triangle(\tau(I))\\&+\Big(\frac{\delta^2}{1+\alpha\delta^2}\Big)^2g(\tau(I),u)g(*,\phi(*))\phi(u)\Big)\Big\}^V_{(x,u)},\end{aligned}$$

where $\Delta(\tau(I))=tr_g(\nabla^2_*\tau(I))$.

Proof. Let $(x,u)\in TM$ and $\{e_i^H,e_i^H\}_{i=1}^{2k}$ be a local orthonormal frame on TM such that $(\nabla_{e_i}e_i)_x=0$. Then by summing over i, we have:

$$tr_{g^S}(\nabla^2\tau(I))_{(x,u)}=\Big(\nabla^I_{e_i^H}\nabla^I_{e_i^H}\tau(I)\Big)_{(x,u)}+\Big(\nabla^I_{e_i^V}\nabla^I_{e_i^V}\tau(I)\Big)_{(x,u)}$$

$$-\Big(\nabla^I_{\nabla^S_{e_i^H} e_i^H}\tau(I)\Big)_{(x,u)} - \Big(\nabla^I_{\nabla^S_{e_i^V} e_i^V}\tau(I)\Big)_{(x,u)}.$$

From Theorem 4.8.1, we have

$$tr_{g^S}(\nabla^2\tau(I))_{(x,u)} = \Big(\nabla_{e_i}\nabla_{e_i}\tau(I)\Big)^V_{(x,u)} + \Big(\frac{\delta^2}{1+\alpha\delta^2}\Big)^2\Big(g(e_i,\phi(\tau(I))g(e_i,u)\phi(u)\Big)^V_{(x,u)}.$$

On the other hand, we have

$$tr_{g^S}(\overline{R}(\tau(I),dI)dI)_{(x,u)} = \overline{R}(\tau(I),e_i^H)e_i^H)_{(x,u)} + \overline{R}(\tau(I),e_i^V)e_i^V)_{(x,u)}$$

From Theorem 2.6.4, we have

$$\begin{aligned} tr_{g^S}(\overline{R}(\tau(I),dI)dI)_{(x,u)} =& \Big(-\frac{3}{2}(\nabla_{e_i}R)(u,\tau(I))e_i\Big)^H_{(x,u)} + \Big(\frac{\delta^2}{1+\alpha\delta^2}\Big)^2\Big((g(\tau(I),u)g(e_i,\phi(e_i)) \\ & - g(e_i,u)g(\tau(I),\phi(e_i)))\phi(u)\Big)^V_{(x,u)} \end{aligned}$$

Considering the formula 1.29, we deduce

$$\begin{aligned} \tau_2(I)_{(x,u)} =& \Big(-\frac{3}{2}(\nabla_{e_i}R)(u,\tau(I))e_i\Big)^H_{(x,u)} + \Big(\triangle(\tau(I)) \\ & + \Big(\Big(\frac{\delta^2}{1+\alpha\delta^2}\Big)^2 g(\tau(I),u)g(e_i,\phi(e_i))\phi(u)\Big)^V_{(x,u)} \end{aligned}$$

□

From Theorem 2.6.1, we have

Theorem 4.7.3. *Let $(TM,\widetilde{\phi},g^{BS})$ be a anti-paraKähler manifold. Then the identity map $I:(TM,g^S)\longrightarrow(TM,\widetilde{\phi},g^{BS})$ is biharmonic if and only if*

$$\triangle(\tau(I)) + \Big(\frac{\delta^2}{1+\alpha\delta^2}\Big)^2 g(\tau(I),u)g(*,\phi(*))\phi(u) = 0.$$

Example 4.7.1. *Let $(\mathbb{R}^2,\phi,g)$ be an anti-paraKähler manifold such that*

$$g = e^{2x}dx^2 + e^{2y}dy^2,$$

and

$$\phi(\frac{\partial}{\partial x}) = \frac{e^x}{e^y}\frac{\partial}{\partial y} \ \ and\ \ \phi(\frac{\partial}{\partial y}) = \frac{e^y}{e^x}\frac{\partial}{\partial x}.$$

The orthonormal basis on $(\mathbb{R}^2,g)$ is $\{e_1,e_2\}$ where $e_1 = e^{-x}\frac{\partial}{\partial x}$ and $e_2 = e^{-y}\frac{\partial}{\partial y}$. we have

$$g(e_1,\phi(e_1)) = g(\phi(e_1),e_1),$$

$$\begin{aligned}
g(e_2,\phi(e_1)) =& g(\phi(e_2),e_2),\\
g(e_1,\phi(e_2)) =& e^{-x}e^{-y}\frac{e^y}{e^x}g(\frac{\partial}{\partial x},\frac{\partial}{\partial x}) = e^{-x}e^{-y}\frac{e^y}{e^x}e^{2x} = 1,\\
g(\phi(e_1),e_2) =& e^{-x}e^{-y}\frac{e^x}{e^y}g(\frac{\partial}{\partial y},\frac{\partial}{\partial y}) = e^{-x}e^{-y}\frac{e^x}{e^y}e^{2y} = 1,
\end{aligned}$$

then ϕ is pure with respect to the metric g.
Now let the identity map $I:(T\mathbb{R}^2,g^s)\to(T\mathbb{R}^2,\widetilde{\phi},g^{BS})$
now using proposition 4.7.1, we get that

$$\begin{aligned}
\tau(I) =& \Big(\frac{\delta^2}{1+\alpha\delta^2}g(e_1,\phi(e_1))(\phi(u))^V\Big) + \Big(\frac{\delta^2}{1+\alpha\delta^2}g(e_2,\phi(e_2))(\phi(u))^V\Big)\\
=& \Big(\frac{\delta^2}{1+\alpha\delta^2}g(e^{-x}\frac{\partial}{\partial x},\phi(e^{-x}\frac{\partial}{\partial x}))(\phi(u))^V\Big) + \Big(\frac{\delta^2}{1+\alpha\delta^2}g(e^{-y}\frac{\partial}{\partial y},\phi(e^{-y}\frac{\partial}{\partial y}))(\phi(u))^V\Big)\\
=& \Big(\frac{e^y}{e^x}e^{-2x}\frac{\delta^2}{1+\alpha\delta^2}g(\frac{\partial}{\partial x},\frac{\partial}{\partial y})(\phi(u))^V\Big) + \Big(e^{-2y}\frac{e^y}{e^x}\frac{\delta^2}{1+\alpha\delta^2}g(\frac{\partial}{\partial y},\frac{\partial}{\partial x})(\phi(u))^V\Big) = 0.
\end{aligned}$$

Example 4.7.2. *Let $(\mathbb{R}^2,\phi,g)$ be an anti-paraKähler manifold such that*

$$g = e^{2x}dx^2 + e^{2y}dy^2$$

and

$$\phi_1(\frac{\partial}{\partial x}) = -\frac{\partial}{\partial x} \;\; and \;\; \phi_1(\frac{\partial}{\partial y}) = \frac{\partial}{\partial y},$$

then fo the identity map $I:(T\mathbb{R}^2,g^s)\to(T\mathbb{R}^2,\widetilde{\phi}_1,g^{BS})$, we have

$$\begin{aligned}
\tau(I) =& \Big(\frac{\delta^2}{1+\alpha\delta^2}g(e_1,\phi_1(e_1))(\phi(u))^V\Big) + \Big(\frac{\delta^2}{1+\alpha\delta^2}g(e_2,\phi_1(e_2))(\phi(u))^V\Big)\\
=& \Big(\frac{\delta^2}{1+\alpha\delta^2}g(e^{-x}\frac{\partial}{\partial x},\phi_1(e^{-x}\frac{\partial}{\partial x}))(\phi_1(u))^V\Big) + \Big(\frac{\delta^2}{1+\alpha\delta^2}g(e^{-y}\frac{\partial}{\partial y},\phi_1(e^{-y}\frac{\partial}{\partial y}))(\phi_1(u))^V\Big)\\
=& \Big(\frac{\delta^2}{1+\alpha\delta^2}g(e^{-x}\frac{\partial}{\partial x},\phi_1(e^{-x}\frac{\partial}{\partial x}))(\phi_1(u))^V\Big) + \Big(\frac{\delta^2}{1+\alpha\delta^2}g(e^{-y}\frac{\partial}{\partial y},\phi_1(e^{-y}\frac{\partial}{\partial y}))(\phi_1(u))^V\Big)\\
=& \frac{\delta^2}{1+\alpha\delta^2}(-1+1)(\phi_1 u)^V = 0.
\end{aligned}$$

4.8 Harmonicity of the map $\Psi:(TM,\widetilde{\phi},g^{BS})\to(TN,\widetilde{\phi}',h^{BS})$

In this section, we denote (M_{2k},ϕ,g) be an anti-paraKähler manifold and $(TM,\widetilde{\phi},g^{BS})$ its tangent bundle equipped with the Berger type deformed Sasaki metric g^{BS} and the paracomplex structure $\widetilde{\phi}$, $(N_{2k'},\phi',h)$ be an anti-paraKähler manifold and $(TN,\widetilde{\phi}',h^{BS})$ its tangent bundle equipped with the Berger type deformed Sasaki metric h^{BS} and the paracomplex structure $\widetilde{\phi}'$.

Theorem 4.8.1. *Let* $\Psi : (TM, \widetilde{\phi}, g^{BS}) \to (TN, \widetilde{\phi'}, h^{BS})$ *be a the tangent map of the map*
$\psi : (M_{2k}, \phi, g) \longrightarrow: (N_{2k'}, \phi', h)$*, then the tension field* $\tau(\Psi)$ *of* ψ *is given by*

$$\begin{aligned}\tau(\Psi) =& \Big[\tau(\psi) + tr_h R^N(d\psi(u), \nabla d\psi(u,*))d\psi(*)\Big]^H \\ &+ \Big(div(\nabla d\psi)(u) + \frac{\delta^2}{(1+\alpha\delta^2)}\Big[\frac{\delta^2}{(1+\alpha\delta^2)}g(u,\phi(u)) - trg(*,\phi(*))\Big]d\psi(\phi u)^V\Big)^V \\ &+ \frac{\delta'^2}{1+\alpha'\delta'^2}\Big[tr_h h(\nabla d\psi(u,*), \phi'(\nabla d\psi(u,*))) + tr_h h(d\psi(\phi(*)), \phi'(d\psi(\phi(*)))) \\ &- h(d\psi(\phi(u)), \phi'(d\psi(\phi(u))))\Big]\phi'(d\psi(u))^V.\end{aligned}$$

Proof. Let $(\psi(x), d\psi(u)) \in TN$ and let $\{e_i\}_{i=1}^{2k}$, such that $e_1 = \dfrac{u}{\|u\|}$ is an orthonormal basis of TM at x. Then $\Big\{e_i^H, \dfrac{1}{\sqrt{1+\alpha\delta^2}}(\phi(e_1))^V, (\phi(e_j))^V, j = 2...n\Big\}$ is an orthonormal basis of $T_{(x,u)}TM$ at (x,u) such that $(\nabla_{e_i}e_i)_x = 0$. Then by summing over i, we have:

$$\begin{aligned}\tau(\Psi) =& \nabla^\Psi_{e_i^H} d\Psi(e_i^H) - d\Psi(\widetilde{\nabla}_{e_i^H}e_i^H) + \nabla^\Psi_{\frac{1}{\sqrt{1+\alpha\delta^2}}\phi(e_1)^V} d\Psi(\frac{1}{\sqrt{1+\alpha\delta^2}}\phi(e_1)^V) \\ &- d\Psi(\widetilde{\nabla}_{\frac{1}{\sqrt{1+\alpha\delta^2}}\phi(e_1)^V}\frac{1}{\sqrt{1+\alpha\delta^2}}\phi(e_1)^V) + \nabla^\Psi_{\phi(e_j)^V} d\Psi\phi(e_j)^V) \\ &- d\Psi(\widetilde{\nabla}_{\phi(e_j)^V}\phi(e_j)^V).\end{aligned}$$

From lemme 4.1.1 Proposition 2.6.1, we have

$$\begin{aligned}\tau(\Psi) =& \Big[\nabla^\psi_{e_i} d\psi(e_i) + R(d\psi(u), \nabla d\psi(u,e_i))d\psi(e_i)\Big]^H + \Big[div(\nabla d\psi)(u)\Big]^V \\ &+ \frac{\delta'^2}{1+\alpha'\delta'^2} tr_h h(\nabla d\psi(u,e_i), \phi'(\nabla d\psi(u,e_i)))\phi'(d\psi(u))^V \\ &- \frac{\delta^2}{(1+\alpha\delta^2)^2} g(e_1,\phi(e_1))d\psi(\phi(u))^V - \frac{\delta^2}{(1+\alpha\delta^2)} g(e_j,\phi(e_j))d\psi(\phi(u))^V \\ &+ \frac{\delta'^2}{1+\alpha'\delta'^2}\frac{1}{1+\alpha\delta^2} h(d\psi(\phi(e_1)), \phi'(d\psi(\phi(e_1))))\phi'(d\psi(u))^V \\ &+ \frac{\delta'^2}{1+\alpha'\delta'^2} h(d\psi(\phi(e_j)), \phi'(d\psi(\phi(e_j))))\phi'(d\psi(u))^V\end{aligned}$$

since we have $\sum_{i=1}^{2k} e_i = e_1 + \sum_{j=2}^{2k} e_j$ and $e_1 = \frac{u}{||u||}$ then

$$\begin{aligned}\tau(\Psi) =& \Big[\nabla^\psi_{e_i} d\psi(e_i) + R(d\psi(u), \nabla d\psi(u,e_i))d\psi(e_i)\Big]^H + \Big[div(\nabla d\psi)(u)\Big]^V \\ &+ \frac{\delta'^2}{1+\alpha'\delta'^2} tr_h h(\nabla d\psi(u,e_i), \phi'(\nabla d\psi(u,e_i)))\phi'(d\psi(u))^V\end{aligned}$$

$$
\begin{aligned}
&- \frac{\delta^2}{(1+\alpha\delta^2)^2} g(e_1, \phi(e_1)) d\psi(\phi(u))^V - \frac{\delta^2}{(1+\alpha\delta^2)} g(e_i, \phi(e_i)) d\psi(\phi(u))^V \\
&+ \frac{\delta^2}{(1+\alpha\delta^2)} g(e_1, \phi(e_1)) d\psi(\phi(u))^V \\
&+ \frac{\delta'^2}{1+\alpha'\delta'^2} \frac{1}{1+\alpha\delta^2} h(d\psi(\phi(e_1)), \phi'(d\psi(\phi(e_1)))) \phi'(d\psi(u))^V \\
&+ \frac{\delta'^2}{1+\alpha'\delta'^2} h(d\psi(\phi(e_i)), \phi'(d\psi(\phi(e_i)))) \phi'(d\psi(u))^V \\
&- \frac{\delta'^2}{1+\alpha'\delta'^2} h(d\psi(\phi(e_1)), \phi'(d\psi(\phi(e_1)))) \phi'(d\psi(u))^V \\
=&\Big[\nabla^{\psi}_{e_i} d\psi(e_i) + R(d\psi(u), \nabla d\psi(u, e_i)) d\psi(e_i)\Big]^H + \Big[div(\nabla d\psi)(u)\Big]^V \\
&+ \frac{\delta'^2}{1+\alpha'\delta'^2} tr_h h(\nabla d\psi(u, e_i), \phi'(\nabla d\psi(u, e_i))) \phi'(d\psi(u))^V \\
&- \frac{\delta^2}{(1+\alpha\delta^2)} g(e_i, \phi(e_i)) d\psi(\phi(u))^V + \frac{\delta'^2}{1+\alpha'\delta'^2} h(d\psi(\phi(e_i)), \phi'(d\psi(\phi(e_i)))) \phi'(d\psi(u))^V \\
&+ \frac{\delta^2}{(1+\alpha\delta^2)} \Big[\frac{\delta^2}{(1+\alpha\delta^2)} g(u, \phi(u)) d\psi(\phi u) - \frac{\delta'^2}{(1+\alpha'\delta'^2)} h(d\psi(\phi u), \phi' d\psi(\phi u)) \phi' d\psi(\phi u)\Big]^V.
\end{aligned}
$$

□

Theorem 4.8.2. *Let $\Psi : (TM, \widetilde{\phi}, g^{BS}) \to (TN, \widetilde{\phi'}, h^{BS})$ be a the tangent map of the map*
$\psi : (M_{2k}, \phi, g) \longrightarrow: (N_{2k'}, \phi', h)$, then Ψ is harmonic if and only if the following conditions are verified

$$
\begin{aligned}
0 =& \tau(\psi) + tr_h R^N(d\psi(u), \nabla d\psi(u, *)) d\psi(*), \\
0 =& div(\nabla d\psi)(u) + \frac{\delta^2}{(1+\alpha\delta^2)} \Big[\frac{\delta^2}{(1+\alpha\delta^2)} g(u, \phi(u)) - trg(*, \phi(*))\Big] d\psi(\phi u)^V, \\
0 =& \Big[tr_h h(\nabla d\psi(u, *), \phi'(\nabla d\psi(u, *))) + tr_h h(d\psi(\phi(*)), \phi'(d\psi(\phi(*)))) \\
&- h(d\psi(\phi(u)), \phi'(d\psi(\phi(u))))\Big] \phi'(d\psi(u))^V.
\end{aligned}
$$

Corollary 4.8.1. *Let $\Psi : (TM, \widetilde{\phi}, g^{BS}) \to (TN, \widetilde{\phi'}, h^{BS})$ be a the tangent map of the map*
$\psi : (M_{2k}, \phi, g) \longrightarrow: (N_{2k'}, \phi', h)$, if ψ is totally geodesic then Ψ is harmonic if and only if

$$
\begin{aligned}
\Big[\frac{\delta^2}{(1+\alpha\delta^2)} g(u, \phi(u)) - trg(*, \phi(*))\Big] d\psi(\phi u)^V =& 0, \\
\Big[tr_h h(d\psi(\phi(*)), \phi'(d\psi(\phi(*)))) - h(d\psi(\phi(u)), \phi'(d\psi(\phi(u))))\Big] \phi'(d\psi(u))^V =& 0.
\end{aligned}
$$

Lemma 4.8.1. *Let* $(TM, \widetilde{\phi}, g^{BS}) \to (TN, \widetilde{\phi'}, h^{BS})$ *be a the tangent map of the map* $\psi : (N_{2k'}, \phi', h) \longrightarrow (M_{2k}, \phi, g)$, *then the energy density associated to* Ψ *is given by*

$$\begin{aligned} e(\Psi) =& \frac{1}{2}\Big[2e(\psi) + tr_g||\nabla d\psi(u,*)||^2 + (\delta tr_g h(\nabla d\psi(u,*), \phi' d\psi(u)))^2 \\ & + tr_g||d\psi(\phi(*))||^2 + \delta^2 tr_h h^2(d\psi(\phi(*)), \phi' d\psi(u)) \\ & - \frac{\delta^2}{\alpha(1+\alpha\delta^2)}\Big(||d\psi(\phi(u))||^2 + h^2(d\psi(\phi(u)), \phi' d\psi(u))\Big). \end{aligned}$$

Proof. Let $(x,u) \in TM$ and let $\{e_i\}_{i=1}^{2k}$, such that $e_1 = \dfrac{u}{\|u\|}$ is an orthonormal basis of TM at x. Then $\Big\{e_i^H, \dfrac{1}{\sqrt{1+\alpha\delta^2}}(\phi(e_1))^V, (\phi(e_j))^V, j = 2...n\Big\}$ is an orthonormal basis of $T_{(x,u)}TM$ at (x,u) such that $(\nabla_{e_i} e_i)_x = 0$. Then by summing over i, we have:

$$\begin{aligned} e(\Psi)_{(\psi(x), d\psi(x))} =& \frac{1}{2}\Big[\big(h(d\Psi(e_i^H), d\Psi(e_i^H))\big) + \frac{1}{1+\alpha\delta^2} h(d\Psi(\phi(e_1)^V), d\Psi(\phi(e_1)^V)) \\ & + \sum_{j=2}^{n} h(d\Psi(\phi(e_j)^V), d\Psi(\phi(e_j)^V))\Big] \\ =& \frac{1}{2}\Big[\Big(\sum h(d\psi(e_i)^H, d\psi(e_i)^H) + h(\nabla d\psi(u,e_i)^V, \nabla d\psi(u,e_i)^V)\Big) \\ & + \frac{1}{1+\alpha\delta^2} h(d\psi(\phi(e_1))^V, d\psi(\phi(e_1))^V) + \sum_{j=2}^{n} h(d\psi(\phi(e_j))^V, d\psi(\phi(e_j))^V)\Big] \\ =& \frac{1}{2}\Big[\Big(h(d\psi(e_i), d\psi(e_i)) + h(\nabla d\psi(u,e_i), \nabla d\psi(u,e_i))\Big) \\ & + (\delta h(\nabla d\psi(u,e_i), \phi' d\psi(u)))^2 + \frac{1}{1+\alpha\delta^2} h(d\psi(\phi(e_1)), d\psi(\phi(e_1))) \\ & + \frac{\delta^2}{1+\alpha\delta^2} h^2(d\psi(\phi(e_1)), \phi' d\psi(u)) + \sum_{j=2}^{n} h(d\psi(\phi(e_j)), d\psi(\phi(e_j))) \\ & + (\delta h(d\psi(\phi(e_j)), \phi' d\psi(u)))^2\Big] \\ =& \frac{1}{2}\Big[2e(\psi) + tr_h||\nabla d\psi(u,*)||^2 + (\delta h(\nabla d\psi(u,e_i), \phi' d\psi(u)))^2 \\ & + \frac{1}{1+\alpha\delta^2}\Big(h(d\psi(\phi(e_1)), d\psi(\phi(e_1))) + (\delta h(d\psi(\phi(e_1)), \phi' d\psi(u)))^2\Big) \\ & + h(d\psi(\phi(e_i)), d\psi(\phi(e_i))) - h(d\psi(\phi(e_1)), d\psi(\phi(e_1))) \\ & + \delta^2 h^2(d\psi(\phi(e_i)), \phi' d\psi(u)) - \delta^2 h^2(d\psi(\phi(e_1)), \phi' d\psi(u))\Big] \\ =& \frac{1}{2}\Big[2e(\psi) + tr_g||\nabla d\psi(u,*)||^2 + (\delta h(\nabla d\psi(u,e_i), \phi' d\psi(u)))^2 \\ & + ||d\psi(\phi(e_i))||^2 + \delta^2 h^2(d\psi(\phi(e_i)), \phi' d\psi(u)) \end{aligned}$$

$$- \frac{\delta^2}{\alpha(1+\alpha\delta^2)}\Big(||d\psi(\phi(u))||^2 + h^2(d\psi(\phi(u)), \phi' d\psi(u))\Big).$$

□

Theorem 4.8.3. *Let TN be a compact tangent bundle and $\Psi : (TM, \widetilde{\phi}, g^{BS}) \to (TN, \widetilde{\phi'}, h^{BS})$ be a the tangent map of the map $\psi : (M_{2k}, \phi, g) \longrightarrow (N_{2n}, \phi', h)$, then Ψ is harmonic if and only if ψ is totally geodesic and*

$$tr_g g(d\psi(*), \phi d\psi(*)) = tr_h h(d\psi(\phi(*)), \phi'(d\psi(*))) = 0.$$

Proof. If ψ is totally geodesic and $tr_g g(d\psi(*), \phi d\psi(*)) = 0$ from Corollary4.8.1, we deduce that Ψ is harmonic. Inversely:
Let $\omega : I \times M \longrightarrow N$ be a smooth map satisfying for all $t \in I = (-\epsilon, \epsilon), \epsilon > 0$ and all $x \in M$

$$\omega(t, x) = \psi_t(x) = (1 + t)\psi(x)$$

and

$$\omega(0, x) = \psi(x).$$

The variation vector field $v \in \Gamma(\psi^{-1}TN)$ associated to the variation $\{\psi_t\}_{t \in I}$ is given for all $x \in N$ by

$$v(x) = d_{(0,x)}\omega(\frac{d}{dt}),$$

From Lemma 4.8.1, we have

$$\begin{aligned} e(\Psi) =& \frac{1}{2}\Big[2e(\psi) + (1+t)^2 tr_g||\nabla d\psi(u,*)||^2 + (1+t)^2(\delta tr_g h(\nabla d\psi(u,*), \phi' d\psi(u)))^2 \\ &+ (1+t)^2 tr_g||d\psi(\phi(*))||^2 + \delta^2(1+t)^2 tr_g h^2(d\psi(\phi(*)), \phi' d\psi(u)) \\ &- \frac{\delta^2(1+t)^2}{\alpha(1+\alpha\delta^2)}\Big(||d\psi(\phi(u))||^2 + h^2(d\psi(\phi(u)), \phi' d\psi(u))\Big). \end{aligned}$$

If Ψ is a critical point of the energy functional, from equation 1.20, we have

$$\begin{aligned} \frac{d}{dt}E(\phi_t)_{t=0} &= 0 \\ &= \int_{TM} \frac{1}{2}\Big[2e(\psi) + 2tr_g||\nabla d\psi(u,*)||^2 + 2tr_g(\delta h(\nabla d\psi(u,*), \phi' d\psi(u)))^2 \\ &+ 2tr_g||d\psi(\phi(*))||^2 + \delta^2 2tr_g h^2(d\psi(\phi(*)), \phi' d\psi(u)) \\ &- \frac{\delta^2 2}{\alpha(1+\alpha\delta^2)}\Big(||d\psi(\phi(u))||^2 + 2h^2(d\psi(\phi(u)), \phi' d\psi(u))\Big)\Big] dv_{g^{BS}} = 0. \end{aligned}$$

If Ψ is harmonic hence $\nabla d\psi = 0$. □

4.9 Harmonicity of the map $\pi : (TM, J_{\delta,0}, g^{CG}_{\delta,0}) \longrightarrow (M, g)$

Theorem 4.9.1. *Let (M,g) be an Riemannian manifold and TM its tangent bundle equipped with the isotropic almost complex structure $J_{\delta,0}$ and the isotropic Cheeger-Gromoll metric $g^{CG}_{\delta,0}$. The Riemannian submersion $\pi : (TM, J_{\delta,0}, g^{CG}_{\delta,0}) \longrightarrow M$. Then*

$$\tau(\pi) = -\Big(\frac{1}{\sqrt{\alpha}}\nabla\Big(\frac{1}{\sqrt{\alpha}}\Big)\circ X\Big) + \frac{1-n\alpha}{\alpha^2}\Big(\nabla(\alpha)\circ X\Big) + \frac{n}{\delta}\nabla\Big(\delta\Big)\circ X.$$

Proof. Let $(x,u) \in TM$ and let $\{e_i\}_{i=1}^n$, such that $e_1 = \dfrac{u}{\|u\|}$ is an orthonormal basis of TM at x. Then $\Big\{\dfrac{1}{\sqrt{\alpha}}e_i^H, \dfrac{1}{\sqrt{\delta}}e_1^V, \sqrt{\dfrac{r}{\delta}}e_j^V, j = 2...n\Big\}$ is an orthonormal basis of $T_{(x,u)}TM$ at (x,u) such that $(\nabla_{e_i}e_i)_x = 0$. Then by summing over i, we have

$$\begin{aligned}
\nabla d\pi\Big(\frac{1}{\sqrt{\alpha}}e_i^H, \frac{1}{\sqrt{\alpha}}e_i^H\Big) =& \nabla^{\pi}_{\frac{1}{\sqrt{\alpha}}e_i^H} d\pi\Big(\frac{1}{\sqrt{\alpha}}e_i^H\Big) - d\pi\Big(\widetilde{\nabla}^{GC}_{\frac{1}{\sqrt{\alpha}}e_i^H}\frac{1}{\sqrt{\alpha}}e_i^H\Big)\\
=& \frac{1}{\sqrt{\alpha}}\nabla_{d\pi(e_i^H)}\frac{1}{\sqrt{\alpha}}d\pi(e_i^H) - d\pi\Big(\frac{1}{\sqrt{\alpha}}\nabla^{GC}_{e_i^H}\frac{1}{\sqrt{\alpha}}e_i^H\Big)\\
=& \frac{1}{\sqrt{\alpha}}\nabla_{d\pi(e_i^H)}d\pi(\frac{1}{\sqrt{\alpha}}e_i^H) - d\pi\Big[\frac{1}{\sqrt{\alpha}}e_i^H\Big(\frac{1}{\sqrt{\alpha}}\Big)e_i^H\\
&+ \frac{1}{\alpha^2}\Big(e_i^H(\alpha)e_i^H\Big) - \frac{n}{\alpha}\nabla(\alpha)\Big],\\
\nabla d\pi\Big(\frac{1}{\sqrt{\delta}}e_1^V, \frac{1}{\sqrt{\delta}}e_1^V\Big) =& \nabla^{\pi}_{\frac{1}{\sqrt{\delta}}e_1^V} d\pi\Big(\frac{1}{\sqrt{\delta}}e_1^V\Big) - d\pi\Big(\widetilde{\nabla}^{GC}_{\frac{1}{\sqrt{\delta}}e_1^V}\frac{1}{\sqrt{\delta}}e_1^V\Big)\\
=& \nabla_{d\pi\frac{1}{\sqrt{\delta}}e_1^V} d\pi\Big(\frac{1}{\sqrt{\delta}}e_1^V\Big) - d\pi\Big(\widetilde{\nabla}^{GC}_{\frac{1}{\sqrt{\delta}}e_1^V}\frac{1}{\sqrt{\delta}}e_1^V\Big),\\
\nabla d\pi\Big(\sqrt{\frac{r}{\delta}}e_j^V, \sqrt{\frac{r}{\delta}}e_j^V\Big) =& \nabla^{\pi}_{\sqrt{\frac{r}{\delta}}e_j^V} d\pi\Big(\sqrt{\frac{r}{\delta}}e_j^V\Big) - d\pi\Big(\widetilde{\nabla}^{GC}_{\sqrt{\frac{r}{\delta}}e_j^V}\sqrt{\frac{r}{\delta}}e_j^V\Big)\\
=& \nabla_{d\pi\sqrt{\frac{r}{\delta}}e_j^V} d\pi\Big(\sqrt{\frac{r}{\delta}}e_j^V\Big) - d\pi\Big(\widetilde{\nabla}^{GC}_{\sqrt{\frac{r}{\delta}}e_j^V}\sqrt{\frac{r}{\delta}}e_j^V\Big).
\end{aligned}$$

By definition we have $d\pi(X^H) = X \circ \pi$ and $d\pi(X^V) = 0$, then

$$\nabla d\pi\Big(\frac{1}{\sqrt{\alpha}}e_i^H, \frac{1}{\sqrt{\alpha}}e_i^H\Big) = \frac{1}{\alpha}\nabla_{e_i\circ\pi}e_i\circ\pi - d\pi\Big[\frac{1}{\sqrt{\alpha}}e_i^H\Big(\frac{1}{\sqrt{\alpha}}\Big)e_i^H$$

$$
\begin{aligned}
&+ \frac{1}{\alpha^2}\Big(e_i^H(\alpha)e_i^H\Big) - \frac{n}{\alpha}\nabla(\alpha)\Big] \\
&= -\Big(\frac{1}{\sqrt{\alpha}}\nabla\Big(\frac{1}{\sqrt{\alpha}}\Big)\circ X\Big) + \frac{1-n\alpha}{\alpha^2}\Big(\nabla(\alpha)\circ X\Big)
\end{aligned}
$$

$$
\begin{aligned}
\nabla d\pi\Big(\frac{1}{\sqrt{\delta}}e_1^V, \frac{1}{\sqrt{\delta}}e_1^V\Big) &= - d\pi\Big(\widetilde{\nabla}^{GC}_{\frac{1}{\sqrt{\delta}}e_1^V}\frac{1}{\sqrt{\delta}}e_1^V\Big) = -\frac{1}{\sqrt{\delta}}d\pi\Big(\widetilde{\nabla}^{GC}_{e_1^V}\frac{1}{\sqrt{\delta}}e_1^V\Big) \\
&= -\frac{1}{\sqrt{\delta}}d\pi\Big(e_1^V\Big(\frac{1}{\sqrt{\delta}}\Big)e_1^V + \frac{1}{\sqrt{\delta}}\widetilde{\nabla}^{GC}_{e_1^V}e_1^V\Big) = \frac{1}{\delta}\Big(1+||u||^2\Big)\nabla(\frac{\delta}{r}) \\
&= \frac{r}{\delta}\nabla(\frac{\delta}{r})\circ X, \\
\nabla d\pi\Big(\sqrt{\frac{r}{\delta}}e_j^V, \sqrt{\frac{r}{\delta}}e_j^V\Big) &= - d\pi\Big(\widetilde{\nabla}^{GC}_{\sqrt{\frac{r}{\delta}}e_j^V}\sqrt{\frac{r}{\delta}}e_j^V\Big) = -\sqrt{\frac{r}{\delta}}d\pi\Big(\widetilde{\nabla}^{GC}_{e_j^V}\sqrt{\frac{r}{\delta}}e_j^V\Big) \\
&= \frac{(n-1)r}{\delta}\nabla\Big(\frac{\delta}{r}\Big)\circ X,
\end{aligned}
$$

□

4.10 Harmonic identity map $I : (TM, J_{\delta,0}, g^{CG}_{\delta,0}) \longrightarrow (TM, g^s)$

Proposition 4.10.1. *Let $(M, , g)$ be a Riemannian manifold and TM its tangent bundle equipped with the Isotropic Cheeger-Gromoll metric $g^{CG}_{\delta,0}$ and the isotropic almost complex structure $J_{\delta,0}$. Suppose that $I : (TM, J_{\delta,0}, g^{CG}_{\delta,0}) \longrightarrow (TM, g^S)$ is the identity map. Then the tension field $\tau(I)$ of I is given by*

$$
\begin{aligned}
\tau(I) = &-\Big[\frac{1}{\alpha^2}d\pi(\nabla(\alpha)\circ X) + \frac{n}{\alpha}d\pi(\nabla(\alpha)\circ X) + \frac{n}{\delta}d\pi(\nabla(\delta)\circ X)\Big]^H \\
&+ \Big[\frac{n}{\alpha}K(\nabla(\alpha)\circ X) + \Big(\frac{1}{\delta^2}U\Big(\delta\Big) - \frac{n}{r\delta}\Big)U + \frac{n\delta - r}{\delta^2}K((\delta)\circ X)\Big]^V
\end{aligned}
$$

Proof. Let $(x, u) \in TM$ and let $\{e_i\}_{i=1}^n$, such that $e_1 = \dfrac{u}{\|u\|}$ is an orthonormal basis of TM at x. Then $\Big\{\frac{1}{\sqrt{\alpha}}e_i^H, \frac{1}{\sqrt{\delta}}e_1^V, \sqrt{\frac{r}{\delta}}e_j^V, j = 2...n\Big\}$ is an orthonormal basis of $T_{(x,u)}TM$ at (x, u) such that $(\nabla_{e_i}e_i)_x = 0$. Then by summing over i, we have:

$$
\begin{aligned}
\tau(I) = &\nabla^I_{\frac{1}{\sqrt{\alpha}}e_i^H}dI(\frac{1}{\sqrt{\alpha}}e_i^H) + \nabla^I_{\frac{1}{\sqrt{\delta}}e_1^V}dI(\frac{1}{\sqrt{\delta}}e_1^V) + \nabla^I_{\sqrt{\frac{r}{\delta}}e_j^V}dI(\sqrt{\frac{r}{\delta}}e_j^V) \\
&- dI\Big(\widetilde{\nabla}^{GC}_{\frac{1}{\sqrt{\alpha}}e_i^H}\frac{1}{\sqrt{\alpha}}e_i^H + \widetilde{\nabla}^{GC}_{\frac{1}{\sqrt{\delta}}e_1^V}(\frac{1}{\sqrt{\delta}}e_1^V) + \widetilde{\nabla}^{GC}_{\sqrt{\frac{r}{\delta}}e_j^V}(\sqrt{\frac{r}{\delta}}e_j^V)\Big)
\end{aligned}
$$

$$\begin{aligned}\tau(I) =& \frac{1}{\sqrt{\alpha}}\widetilde{\nabla}^{GC}_{dI(e_i^H)}\frac{1}{\sqrt{\alpha}}dI(e_i^H) + \frac{1}{\sqrt{\delta}}\widetilde{\nabla}^{GC}_{dI(e_1^V)}\frac{1}{\sqrt{\delta}}dI(e_1^V) + \sqrt{\frac{r}{\delta}}\widetilde{\nabla}^{GC}_{dI(e_j^V)}dI(e_j^V)\\ & - dI\Big(\frac{1}{\sqrt{\alpha}}\widetilde{\nabla}^{GC}_{e_i^H}\Big(\frac{1}{\sqrt{\alpha}}e_i^H\Big) + \frac{1}{\sqrt{\delta}}\widetilde{\nabla}^{GC}_{e_1^V}\Big(\frac{1}{\sqrt{\delta}}e_1^V\Big) + \sqrt{\frac{r}{\delta}}\widetilde{\nabla}^{GC}_{e_j^V}\Big(\sqrt{\frac{r}{\delta}}e_j^V\Big)\Big).\end{aligned}$$

From Lemme 4.1.1, we have

$$\begin{aligned}\tau(I) =& \frac{1}{\sqrt{\alpha}}\widehat{\nabla}^s_{e_i^H+\nabla I(u,e_i)^V}\frac{1}{\sqrt{\alpha}}(e_i^H + \nabla I(u,e_i)^V) + \frac{1}{\sqrt{\delta}}\widehat{\nabla}^s_{e_1^V}\frac{1}{\sqrt{\delta}}(e_1^V) + \sqrt{\frac{r}{\delta}}\widehat{\nabla}^s_{e_j^V}\sqrt{\frac{r}{\delta}}(e_j^V)\\ & - dI\Big(\frac{1}{\sqrt{\alpha}}\widetilde{\nabla}^{GC}_{e_i^H}\Big(\frac{1}{\sqrt{\alpha}}e_i^H\Big) + \frac{1}{\sqrt{\delta}}\widetilde{\nabla}^{GC}_{e_1^V}\Big(\frac{1}{\sqrt{\delta}}e_1^V\Big) + \sqrt{\frac{r}{\delta}}\widetilde{\nabla}^{GC}_{e_j^V}\Big(\sqrt{\frac{r}{\delta}}e_j^V\Big)\Big)\\ =& \frac{1}{\sqrt{\alpha}}\widehat{\nabla}^s_{e_i^H}\frac{1}{\sqrt{\alpha}}(e_i^H) + \frac{1}{\sqrt{\delta}}\widehat{\nabla}^s_{e_1^V}\frac{1}{\sqrt{\delta}}(e_1^V) + \sqrt{\frac{r}{\delta}}\widehat{\nabla}^s_{e_j^V}\sqrt{\frac{r}{\delta}}(e_j^V)\\ & - dI\Big(\frac{1}{\sqrt{\alpha}}\widetilde{\nabla}^{GC}_{e_i^H}\Big(\frac{1}{\sqrt{\alpha}}e_i^H\Big) + \frac{1}{\sqrt{\delta}}\widetilde{\nabla}^{GC}_{e_1^V}\Big(\frac{1}{\sqrt{\delta}}e_1^V\Big) + \sqrt{\frac{r}{\delta}}\widetilde{\nabla}^{GC}_{e_j^V}\Big(\sqrt{\frac{r}{\delta}}e_j^V\Big)\Big)\\ =& \frac{1}{\sqrt{\alpha}}e_i^H(\frac{1}{\sqrt{\alpha}})e_i^H + \frac{1}{\sqrt{\delta}}e_1^V(\frac{1}{\sqrt{\delta}})e_1^V + \sqrt{\frac{r}{\delta}}e_j^V\Big(\sqrt{\frac{r}{\delta}}\Big)e_j^V\\ & - dI\Big[\frac{1}{\sqrt{\alpha}}e_i^H(\frac{1}{\sqrt{\alpha}})e_i^H + \frac{1}{\alpha^2}e_i^H(\alpha)e_i^H - \frac{n}{\alpha}\nabla(\alpha) + \frac{1}{\sqrt{\delta}}e_1^V(\frac{1}{\sqrt{\delta}})e_1^V + \frac{r}{\delta^2}e_1^V\Big(\frac{\delta}{r}\Big)e_1^V\\ & + \frac{1}{r\delta}U - \frac{r}{\delta}\nabla(\frac{\delta}{r}) + \sqrt{\frac{r}{\delta}}e_j^V\Big(\sqrt{\frac{r}{\delta}}\Big)e_j^V + \frac{r^2}{\delta^2}e_j^V\Big(\frac{\delta}{r}\Big)e_j^V + \frac{n-1}{\delta}U - \frac{(n-1)r}{\delta}\nabla\Big(\frac{\delta}{r}\Big)\Big]\\ =& -\frac{1}{\alpha^2}e_i^H(\alpha)e_i^H + \frac{n}{\alpha}\nabla(\alpha) - \frac{r}{\delta^2}e_1^V\Big(\frac{\delta}{r}\Big)e_1^V - \frac{n}{r\delta}U + \frac{nr}{\delta}\nabla(\frac{\delta}{r}) - \frac{r^2}{\delta^2}e_j^V\Big(\frac{\delta}{r}\Big)e_j^V.\end{aligned}$$

Then we have

$$\begin{aligned}\tau(I) =& -\frac{1}{\alpha^2}e_i^H(\alpha)e_i^H + \frac{n}{\alpha}\nabla(\alpha) - \frac{r}{\delta^2}e_1^V\Big(\frac{\delta}{r}\Big)e_1^V - \frac{n}{r\delta}U + \frac{nr}{\delta}\nabla(\frac{\delta}{r}) - \frac{r^2}{\delta^2}e_j^V\Big(\frac{\delta}{r}\Big)e_j^V\\ =& -\frac{1}{\alpha^2}e_i^H(\alpha)e_i^H + \frac{n}{\alpha}\nabla(\alpha) - \frac{r}{\delta^2}e_1^V\Big(\frac{\delta}{r}\Big)e_1^V - \frac{n}{r\delta}U + \frac{nr}{\delta}\nabla(\frac{\delta}{r})\\ & - \frac{r^2}{\delta^2}\Big(e_i^V\Big(\frac{\delta}{r}\Big)e_i^V - e_1^V\Big(\frac{\delta}{r}\Big)e_1^V\Big)\\ =& -\frac{1}{\alpha^2}e_i^H(\alpha)e_i^H + \frac{n}{\alpha}\nabla(\alpha) - \frac{r}{\delta^2}e_1^V\Big(\frac{\delta}{r}\Big)e_1^V - \frac{n}{r\delta}U + \frac{nr}{\delta}\nabla(\frac{\delta}{r})\\ & - \frac{r^2}{\delta^2}e_i^V\Big(\frac{\delta}{r}\Big)e_i^V + \frac{r^2}{\delta^2}\Big(e_1^V\Big(\frac{\delta}{r}\Big)e_1^V\Big)\\ =& -\frac{1}{\alpha^2}e_i^H(\alpha)e_i^H + \frac{n}{\alpha}\nabla(\alpha) - \frac{n}{r\delta}U + \frac{nr}{\delta}\nabla(\frac{\delta}{r}) - \frac{r^2}{\delta^2}e_i^V\Big(\frac{\delta}{r}\Big)e_i^V + \Big[\frac{r}{\delta^2}\Big]U\Big(\frac{\delta}{r}\Big)U.\end{aligned}$$

□

Conclusion and perspective

In this thesis we studied the harmonicity of vectors fields from a Riemannian manifold (M, g) into its tangent bundle TM equipped with an almost complex or paracomplex structure, giving in the process necessary and sufficient condition in order for a vector field to be harmonic.
In the future we may look into other type of structure like the golden structure which satisfy for a $(1,1)$ tensor φ on M the relation $\varphi^2 = \varphi + Id$ and is compatible withe the metric on g. We also want to study the the Tangent bundle of a Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, where ϕ is a $(1,1)$ tensor, η a 1-form and g a Riemannian metric on M verifying

$$\begin{aligned} \phi(\xi) =& 0,\ \eta(\phi(X)) = 0,\ \eta(\xi) = 1, \\ \phi^2(X) =& -X + \eta(X)\xi, \\ g(X,\xi) =& \eta(X) \end{aligned}$$

and

$$g(\phi(X), \phi(Y)) = g(X,Y) - \eta(X)\eta(Y).$$

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ملخص

في هذه الأطروحة سنتطرق الى دراسة توافق الحقول الشعاعية X و التي يمكن اعتبارها تطبيقات بين المنوعة (M, g) و حزمتها المماسية TM المزود ببعض البُنيَات الشبه مركبة J. بحيث تكون متناسقة مع أحدا القياسات: مقياس شعاع التدرج لسسكي g_f, تشويه بارجر لمقياس سسكي g^{BS} و المقياس الإزوتروبي لشيقر قرومور $g^{CG}_{\delta,0}$ التي ستزود في TM. أخيرا سنعطي بعض الشروط لكي تكون التطبيقات بين الحزم المماسية متوافقة.

كلمات مفتاحية: حزمة مماسية،رفع عمودي و رفع افقي، بنيات شبه مركبة، توافقية.

Abstract

In this thesis we will study the harmonicity of vector fields X whom can be seen as maps from M to its tangent bundle TM endowed with a structures that are almost complex or almost paracomplex, such that those structures are compatible with one of the three metrics: Gradient sasaki metric g_f, Berger type deformed Sasaki metric g^{BS} and the isotropic Cheeger-Gromoll metric $g^{CG}_{\delta,0}$. We will also give conditions under witch maps between two tangents bundle are harmonic and biharmonic.

Keywords; Tangent bundle, vertical and horizontal lift, almost complex structures, almost paracomplex structures, Harmonicity.

Résumé

Dans cette thèse nous nous intéressant à l'harmonicité des champs de vecteurs X que l'on peut considérer comme des applications entre la variété Riemannien (M, g) et le fibré tangent TM équipé de structures presque complexe ou presque paracomplexe qui sont compatible avec l'une des trois métriques: $Gradient\ sasaki\ g_f$, deformation de Barger Sasaki g^{BS} et la métrique isotropique de $Cheeger - Gromoll\ g^{CG}_{\delta,0}$ sur TM. Nous donnerons finalement des conditions d'harmonicité et de biharminicité des applications entre deux fibrés tangents.

mots clés:fibré tangent, relèvement vertical et horizontal, structures presque complexe et presque paracomplexe, Harmonicité.