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Ministry of Higher Education and Scientific Research

University of MUSTAPHA Stambouli
Mascara
Faculty of exact sciences
Departement of Mathematics

جامعة مصطفى اسطمبولي
معسكر
كلية العلوم الدقيقة
قسم الرياضيات

DOCTORATE THESIS

Speciality : Mathematics
Option : Harmonic Analysis

Entitled

Study of harmonic and biharmonic maps on Thurston geometry

Presented by: BELARBI Mansour
The 22/ 06 /2022

The jury :

President	BENMERIEM Khaled	Professor	Mascara University
Examiner	ZOUBIR Hanifi	Professor	ENPO of Oran
Examiner	MOHAMMED CHERIF Ahmed	Professor	Mascara University
Examiner	BOUZIR Habib	MCA	Mascara University
Supervesor	BELARBI Lakehal	Professor	Mostaganem University

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Ministère de l’enseignement supérieur et de la recherche scientifique

Université MUSTAPHA Stambouli
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Faculté des sciences exactes
Département de Mathématiques

جامعة مصطفى اسطمبولي
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كلية العلوم الدقيقة
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THÈSE de DOCTORAT

Spécialité : **Mathématiques**
Option : **Analyse Harmonique**

Intitulée

Etude des applications harmonique et biharmonique sur les modèles de Thurston

Présentée par : BELARBI Mansour
Le 22/06/2022

Devant le jury :

Président	BENMERIEM Khaled	Professeur	Université de Mascara
Examinateur	ZOUBIR Hanifi	Professeur	ENPO d’Oran
Examinateur	MOHAMMED CHERIF Ahmed	Professeur	Université de Mascara
Examinateur	BOUZIR Habib	MCA	Université de Mascara
Encadreur	BELARBI Lakehal	Professeur	Université de Mostaganem

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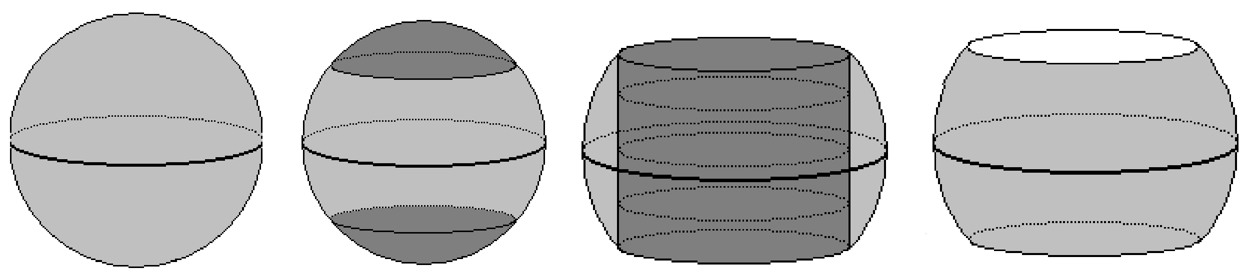
Publications

1. L. Belarbi, M. Belarbi and H. Elhendi, Legendre curves on Lorentzian Heisenberg Space, Bull. Transilv. Univ. Brasov SER. III., **13(62)**(1)(2020), 41–50.
2. M. Belarbi, H. Elhendi and L. Belarbi, Biharmonic curves in Three-Dimensional Generalized Symmetric Spaces, J. Indian Math. Soc., **89**(3-4)(2022), 8–23. (accepted)
3. M. Belarbi, L. Belarbi and H. Elhendi, Biharmonic curves in Thurston geometry of dimension 4. (submitted)

Introduction

In 1976 William Thurston formulated the geometrization conjecture [63]. It says in simplified terms that any compact manifold of dimension 3 can be endowed with a metric which is locally isometric to one of the eight Thurston geometries. William Thurston studied many three-dimensional spaces and found that all can be described by one of these eight geometries. He encourages his students to explore computer databases. His concrete and experimental work is of a very interesting rarity in mathematics. In the geometrization conjecture Thurston used surgery, a method that allow manifolds to be cut out and glued together. For an example we take a simple sphere (bellow a). At first we hollow out two disc-shaped holes as to obtain a surface (b) whose edge consist of two circles (c). A trunk of a cylinder has two circles in its edge, we can therefore sew the two circles of the cylinder along the two circles corresponding to the edges of the hollowed discs. The results is (homeomorphic) to a torus (d). The torus was thus obtained by surgery from a sphere.



(a) (b) (c) (d)

In 2003 Grigori Perelman [59] used the Ricci flow, a technique used by Richard Hamilton in 1982 [36]:

$$\partial_t g_t = -2Ricc(g_t)$$

to prove the geometrization conjecture, and consequently the Poincaré conjecture.

In 1964 J. Eells and J.H. Sampson [21] introduce the harmonic maps. Harmonic maps are solutions to a natural geometrical problem. The map φ between Riemannian manifolds is harmonic if it is a critical point of the energy functional:

$$E(\varphi; D) = \frac{1}{2}\int_D |d\varphi|^2 \, v_g,$$

where $|d\varphi|$ is the Hilbert Schmith norm of the differential $d\varphi$ defined by:

$$|d\varphi|^2 = \sum_{i=1}^{m} h(d\varphi(e_i), d\varphi(e_i)),$$

where $\{e_1, ..., e_m\}$ is a local orthonormal basis on (M^m, g) and $\{\partial_1, ..., \partial_m\}$ is a local vector field basis associated with a map (U, φ) of M. v_g is the volume element of (M^m, g) defined by:

$$v_g = \sqrt{det(g_{ij})} dx_1 \wedge ... \wedge dx_m,$$

and D is a compact domain of M. We also have that φ is harmonic if it satisfies the Euler-Lagrange equation:

$$\tau(\varphi) = trace_g \nabla d\varphi,$$

where $\nabla d\varphi$ is the second fundamental form of φ.

In 1986 G.Y Jiang [40] introduced the concept of biharmonic maps. Biharmonic maps are defined as critical points of the bienergy functional:

$$E_2(\varphi, D) = \frac{1}{2} \int_D |\tau(\varphi)|^2 v_g,$$

and he proved that every biharmonic maps is a solution of the Euler-Lagrange equation:

$$\tau_2(\varphi) = -\text{trace}_g R^N(\tau(\varphi), d\varphi)d\varphi - \text{trace}_g(\nabla^\varphi)^2\tau(\varphi) = 0,$$

$\tau_2(\varphi)$ is called the bitension field of the map φ.

In 2006 Y.L. Ou and Z.P. Wang [57] studied biharmonic maps on Sol_3 and Nil_3 spaces. Two models space of Thurston's 3-dimensional geometries.

In 2020 [8] we classified Legendre curves on three-dimensional Lorentzian Heisenberg space $(\mathbb{H}_3, g)$.

In 2021 [9] we classified the biharmonic maps in three-dimensional generalized symmetric spaces and Sol_3 became a particular consequence.

The principal goal of this work is to study the Biharmonic curves in the Thurston model geometry of dimension three and dimension four. This thesis is organized in five chapters. In the first chapter, we give the definitions of manifolds, differentiable manifolds, tangent spaces, pseudo-Riemannian metrics and we introduce basic concepts of curvature, harmonic and biharmonic maps.

In the second chapter we introduce a Thurston model geometry (G, X). Three-dimensional Thurston model geometries are classified by W. Thurston, this classification has eight geometries, to know, E^3, S^3, H^3, $S^2 \times \mathbb{R}$, $H^2 \times \mathbb{R}$, $\widetilde{Sl_2(\mathbb{R})}$, Nil_3 and Sol_3.

We also study the Thurston geometry of dimension four. R. Filipkiewicz classified the Thurston geometry of dimension four. In this geometry classification we distinguish two categories of spaces, those which are symmetrical: E^4, S^4, H^4, $P^2(\mathbb{C})$, $H^2(\mathbb{C})$, $S^2 \times S^2$, $S^2 \times E^2$, $S^2 \times H^2$, $H^2 \times E^2$, $H^2 \times H^2$, $H^3 \times E^1$ and $H^3 \times E^1$ and those that are not symmetrical: Nil^4, $Sol^4_{m,n}$, Sol^4_0, Sol^4_1, F^4, $\widetilde{Sl_2(\mathbb{R})} \times E^1$ and $Nil_3 \times E^1$.

In the third chapter we show that the Legendre curves on three-dimensional Lorentzian Heisenberg space $(\mathbb{H}_3, g)$ is locally ϕ-symmetric if and only if it is a geodesic. Moreover we prove that the Legendre curves on three-dimensional Lorentzian Heisenberg space is biharmonic if and only if it is a pseudo-helix.

The results obtained in this chapter are published in the paper [8].

In the fourth chapter we study biharmonic curves in three-dimensional generalized symmetric spaces, equipped with a left-invariant pseudo-Riemannian metric. We characterize non-geodesic biharmonic curves in three-dimensional generalized symmetric spaces and prove that there exists no non-geodesic biharmonic spacelike curve helix in three-dimensional generalized symmetric spaces. We also show that a linear map from an Euclidean space in three-dimensional generalized symmetric spaces is biharmonic if and only if it is a harmonic, and we give a complete classification of such maps.

The results obtained in this chapter are published in the paper [9].

In the last chapter we study harmonic and biharmonic applications in Thurston geometry of dimension 4. We introduce the 4-dimensional geometry Nil^4 and we define the metric g_{Nil^4}. We give the Christoffel symbols and the Riemannian curvature to study the biharmonic curves in Nil^4 space.

Chapter 1

Preliminaries

In this first chapter, we give definitions of manifolds, differentiable manifolds, tangent spaces, pseudo-Riemannian metrics and we introduce basic concepts of curvature, harmonic maps and biharmonic maps. [60], [19], [21], [26], [53], [54], [52], [28], [30], [35], [42], [14], [40] and [41].

1.1 Differential geometry

1.1.1 Differential manifold

Let M be a topological space. A topological space M is called a separate space (or a Hausdorff space) if for any two distinct points $p_1, p_2 \in M$ there exists two open sets $U_1, U_2 \in U$ with $p_1 \in U_1, p_2 \in U_2$ and $U_1 \cap U_2 = \emptyset$. M is called a topological manifold if there exist an $n \in \mathbb{N}$ and for every point $x \in M$ an open neighborhood U_x such that U_x is homeomorphic to some open subset V of $\mathbb{R}^n$. The naturel number n is called the dimension of M.

Definition 1. *Let M be a separate topological space. An chart on M is a pair (U, φ) where U is an open subset of M and $\varphi(U)$ is an open subset of $\mathbb{R}^m$ such that $\varphi : U \mapsto \varphi(U)$ is a homeomorphism. m is called the dimension of the chart (U, φ).*

Definition 2. *Let M be a separate topological space. An differentiable atlas $\mathcal{A}$ of dimension n is a collection of open charts $(U_i, \varphi_i)_{i\in I}$ on M where $\varphi_i(U_i)$ is an open subset of $\mathbb{R}^n$ such that $M = \bigcup_{i\in I} U_i$, and for each pair $i, j \in I$ the mapping of all charts transitions:*

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j),$$

are a C^∞-diffeomorphism with $U_i \cap U_j \neq \emptyset$. A differentiable atlas is called a differentiable structure, and a differentiable manifold of dimension n is a manifold of dimension n with a differentiable structure. Two atlas are called compatible if their union is again an atlas. An atlas is called maximal if any compatible with it is already contained in it.

Definition 3. *A differentiable manifold of dimension n is a Hausdorff space provided with a differentiable structure of dimension n.*

Example 1. *Any vector space E of dimension n is a differentiable manifold. In effect let defined the topology $\mathcal{T}_E$:*

$$\phi : E \longrightarrow \mathbb{R}^n, \quad x = \sum_{i=1}^{n} x_i e_i \longmapsto \phi(x) = (x_1, ..., x_n),$$

where $\{e_1, ..., e_n\}$ is a basis of E. Then ϕ is bijective.
$\mathcal{T}_E = \{A \in E \mid \phi(A)$ is an open of $\mathbb{R}^n\}$ is called the reverse image topology, then $(E, \mathcal{T}_E)$ is a topological space.
Let $X, Y \in E$ with $x \neq y$. Like $\mathbb{R}^n$ is a separate topological space, $\exists\, U, V \in \mathbb{R}^n$ such that $\phi(U) \in U$, $\phi(V) \in V$ and $U \cap V = \emptyset$. Let's put $A = \phi^{-1}(U)$ and $B = \phi^{-1}(V)$, then $A \cap B = \emptyset$, so E is separate.
Now let defined the differentiable atlas $\mathcal{A}_E = \{(E, \phi)\}$, and show that $\phi : E \longrightarrow \mathbb{R}^n$ is a homeomorphism.
We have, $\phi : (E, \mathcal{T}_E) \longrightarrow (\mathbb{R}^n, \mathcal{T}_{\mathbb{R}^n})$ is continued, because $\forall U \in \mathcal{T}_{\mathbb{R}^n}, \phi^{-1}(U) = A \in \mathcal{T}_E$.
And $\phi^{-1} : (\mathbb{R}^n, \mathcal{T}_{\mathbb{R}^n}) \longrightarrow (E, \mathcal{T}_E)$ is also continuous, because $\forall A \in \mathcal{T}_E, (\phi^{-1})^{-1}(A) = \phi(A) \in \mathcal{T}_{\mathbb{R}^n}$.
Where ϕ is a homeomorphism, then $(E, \mathcal{A}_E)$ is a differentiable manifold of dimension n.

Example 2. *The Euclidean space $\mathbb{R}^n$ is a differentiable manifold of dimension n with $\mathcal{A} = (\mathbb{R}^n, I_{\mathbb{R}^n})$. The opens $\Omega \subset \mathbb{R}^n$ provided with an atlas $\mathcal{A}_\Omega$ witch contains the only chart (Ω, Id_Ω) are a differentiable manifolds of dimension n.*

Example 3. *The standard sphere $\mathbb{S}^n = \{u \in \mathbb{R}^{n+1} \mid \|u\| = 1\}$ is a differentiable manifold of dimension n. $\mathbb{S}^n$ is a topological space, where $\mathcal{T}_{\mathbb{S}^n}$ is the topology induced by that of $\mathbb{R}^{n+1}$ (its the topology whose openings are of the form $U = \Omega \cap \mathbb{S}^n$ where Ω is an open from $\mathbb{R}^{n+1}$). Let the projections stereographic:*

$$\begin{array}{rcl} \varphi_N : U_N = \mathbb{S}^n - \{N\} & \longrightarrow & \mathbb{R}^n \\ (u_1, ..., u_{n+1}) & \longmapsto & \left(\dfrac{u_1}{1 - u_{n+1}}, ..., \dfrac{u_n}{1 - u_{n+1}} \right). \end{array}$$

$$\begin{array}{rcl} \varphi_S : U_S = \mathbb{S}^n - \{S\} & \longrightarrow & \mathbb{R}^n \\ (u_1, ..., u_{n+1}) & \longmapsto & \left(\dfrac{u_1}{1 + u_{n+1}}, ..., \dfrac{u_n}{1 + u_{n+1}} \right). \end{array}$$

The applications $\varphi_N : U_N \longrightarrow \mathbb{R}^n$ and $\varphi_S : U_S \longrightarrow \mathbb{R}^n$ are homeomorphism. Using $1 - u_{n+1}^2 = u_1^2 + ... + u_n^2$, we find that:

$$\begin{array}{rcl} \varphi_N^{-1} : \mathbb{R}^n & \longrightarrow & U_N \\ (x_1, ..., x_n) & \longmapsto & \left(\dfrac{2x_1}{\|x\|^2 + 1}, ..., \dfrac{2x_n}{\|x\|^2 + 1}, \dfrac{\|x\|^2 - 1}{\|x\|^2 + 1} \right). \end{array}$$

$$\begin{array}{rcl} \varphi_S^{-1} : \mathbb{R}^n & \longrightarrow & U_S \\ (y_1, ..., y_n) & \longmapsto & \left(\dfrac{2y_1}{\|y\|^2 + 1}, ..., \dfrac{2y_n}{\|y\|^2 + 1}, -\dfrac{\|y\|^2 - 1}{\|y\|^2 + 1} \right). \end{array}$$

the mapping of charts transitions are given by:

$$\varphi_S \circ \varphi_N^{-1} = \frac{x}{\|x\|^2}, \quad \varphi_N \circ \varphi_S^{-1} = \frac{y}{\|y\|^2}, \quad \forall x, y \in \mathbb{R}^n - \{0\},$$

which are diffeomorphisms of C^∞. *Therefore* $\mathcal{A}_{\mathbb{S}^n} = \{(U_N, \varphi_N), (U_S, \varphi_S)\}$ *form a differentiable atlas.*

Example 4. *The surfaces* S *of* $\mathbb{R}^3$ *are a differentiable manifolds of dimension* 2. *(If* $X : \Omega \longrightarrow \mathbb{R}^3, (u, v) \longmapsto X(u, v)$ *is a local parametrization of a surface* S *of* $\mathbb{R}^3$*, then* $\varphi = X^{-1} : X(\Omega) \longrightarrow \Omega$ *is a chart of* S*).*

Definition 4. *An atlas for a differentiable manifold* M *is called oriented if all* $\mathcal{A} = \{(U_i, \varphi_i)_{i\in I}\}$ *such that the charts changes mapping* $\psi_{ij} = \varphi_i \circ \varphi_j^{-1}$ *has a positive Jacobian, i.e:*

$$J(\psi_{ij})_x = det(d_{\varphi_j(x)}\psi_{ij}) > 0.$$

Definition 5. *A differentiable manifold is called oriented if it possesses an oriented atlas.*

Remark 1. *If* φ *be a diffeomorphism of* $\mathbb{R}^n$*, its Jacobian is defined by:*

$$J(\varphi)_x = det(d_x\varphi).$$

Example 5. $\mathbb{R}^n$ *is an orientable manifold.*
The Möbius band and the Klein bottle are non-orientable manifold.

Definition 6. *Let* M *be a differentiable manifold,* $f : M \longrightarrow \mathbb{R}$ *is called to be differentiable function at point* $p \in M$*, if there is a chart* (U, φ) *of* M *with* $p \in U$ *such as* $f \circ \varphi^{-1} : \varphi(U) \longrightarrow \mathbb{R}$ *is differentiable. The function* f *is differentiable if it is differentiable in* p *for all* $p \in M$*.*

Definition 7. *Let* M *and* N *two differentiable manifolds, a mapping* $f : M \longrightarrow N$ *is said to be differentiable (or* C^∞*-differentiable), if for every chart* (U_i, φ_i) *of* M *and every chart* (V_j, ψ_j) *of* N *such that* $f(U_i) \subset V_j$*, the mapping* $\psi_j \circ f \circ \varphi_i^{-1} : \varphi_i(U_i) \longrightarrow \psi_i(V_j)$ *is differentiable.*

1.1.2 Tangent space

Definition 8. *[28] Let* M *be a differentiable manifold and* $p \in M$*, then a tangent vector* X_p *at* p *is a map:*

$$\begin{aligned} X_p : C^\infty(M) &\longrightarrow \mathbb{R} \\ f &\longmapsto X_p(f), \end{aligned}$$

such that:

1. $X_p(\lambda f + \mu g) = \lambda X_p(f) + \mu X_p(g)$,
2. $X_p(fg) = X_p(f)g(p) + f(p)X_p(g)$,

for all $\lambda, \mu \in \mathbb{R}$ *and* $f, g \in C^\infty(M)$.
The set of tangent vectors at p *is called the tangent space at* p *and denoted* T_pM*.*
The tangent space T_pM *of* M *at* p *has the structure of a real vector space. The addition* $+$ *and the multiplication* . *by real numbers are simply given by:*

1. $(X_p + Y_p)(f) = X_p(f) + Y_p(f)$,

2. $(\lambda X_p)(f) = \lambda X_p(f)$,

for $X_p, Y_p \in T_pM$, $f \in C^\infty(M)$ *and* $\lambda \in \mathbb{R}$.

Remark 2. $X_p(f)$ *is also called the derivative of* f *by* X_p.

1.1.3 Tangent bundle

Definition 9. *[48] For any smooth manifold* M*, we define the tangent bundle of* M*, denoted by* TM*, to be the disjoint union of the tangent spaces at all points of the tangent spaces at all points of* M*:* $TM = \bigcup_{p\in M} T_pM$*. We consider an element of this disjoint union to be an ordered pair* (p, X)*, where* $p \in M$ *and* $X \in T_pM$*. We will often commit the usual mild sin of identifying* T_pM *with image under the canonical injection* $X \longmapsto (p, X)$*, and depending on context will use any of notations* (p, X)*,* X_p *or* X *for a tangent vector in* T_pM*, depending on how much emphasis we wish to give the point* p*. Define the projection map* $\pi : TM \longrightarrow M$ *by declaring* $\pi(p, X) = p$*.*

Remark 3.

1. *The tangent bundle* TM *to a manifold* M *is an oriented manifold even if* M *is not.*

2. T_x^*M *is the dual space of the tangent space* T_xM *of* M *at* x*.*

3. T_x^*M *is the set of linear form on* T_xM *where* $w_x \in T_x^*M$*:*

$$\begin{aligned} w_x : M &\longrightarrow \mathbb{R} \\ X_x &\longmapsto w_x(X_x). \end{aligned}$$

4. *We call cotangent bundle of* M *the fibre bundle such that:*

$$T_xM = \bigcup_{x\in M} T_x^*M.$$

1.1.4 Vectors fields

Definition 10. *[48] Let* M *be a smooth manifold. A vector field on* M *is a section of* TM*. More concretely, a vector field is a continuous map* $Y : M \longrightarrow TM$*, usually written* $p \longmapsto Y_p$*, with the property that for each* $p \in M$*,* Y_p *is an element of* T_pM*.*

Remark 4.

1. We denote by $\mathfrak{X}(M)$ the set of all differentiable vector fields on M.

2. If f is differentiable function on M, then $X(f)$ is differentiable function on M defined by $X(f))(p) = X(f)$, for all $X \in \mathfrak{X}(M)$ and $p \in M$.

Definition 11. *Let M be an m-dimensional differentiable manifold, (U,φ) be a chart of M and $p \in U$, for $i = 1, ..., m$ we define the map:*

$$\begin{array}{rcl} \frac{\partial}{\partial x_i}\big|_p : C^\infty(M) & \longrightarrow & \mathbb{R} \\ f & \longmapsto & \frac{\partial}{\partial x_i}\big|_p(f) = \frac{\partial (f \circ \varphi^{-1})}{\partial x_i}\big|_{\varphi(f)}. \end{array}$$

$\frac{\partial}{\partial x_i}|_p$ is said derivative associated to the chart (U,φ) and $\{\frac{\partial}{\partial x_1}|_p, ..., \frac{\partial}{\partial x_m}|_p\}$ be a frame for the tangent space T_pM, for all $p \in U$.

Remark 5. *$\{dx_1|_p, ..., dx_m|_p\}$ be a frame for the cotangent space T_p^*M (the dual basis of the basis $\{\frac{\partial}{\partial x_1}|_p, ..., \frac{\partial}{\partial x_m}|_p\}$ for T_pM), for all $p \in U$.*

Definition 12. *Let $T_x^{(r,s)}M = \underbrace{T_xM \otimes ... \otimes T_xM}_{r-\text{once}} \otimes \underbrace{T_x^*M \otimes ... \otimes T_x^*M}_{s-\text{once}}$ be the vectorial space, where $x \in M$ and let $T^{(r,s)}M = \bigcup_{x \in M} T_x^{(r,s)}M$. A element $T \in T_x^{(r,s)}M$ is a tensor of type (r,s) above x. A tensor field of type (r,s) on a manifold M is an assignment section of $T^{(r,s)}M$ i.e. a tensor is a map:*

$$\begin{array}{rcl} T : M & \longrightarrow & T^{(r,s)}M \\ x & \longmapsto & T(x) \in T_x^{(r,s)}M. \end{array}$$

Example 6.

1. A function on a manifold M is a tensor of type $(0,0)$.
2. A vector filed X is a tensor of type $(1,0)$.
3. A differential 1-form w on a manifold M is a tensor of type $(0,1)$.

1.2 Pseudo-Riemannian manifolds

Pseudo-Riemannian geometry involves a particular kind of $(0,2)$ tensor on tangent spaces. To study these in general, let E be a real vector space (finite-dimensional where the context so indicates). A bilinear form on E is an $\mathbb{R}$-bilinear function $g : E \times E \longrightarrow \mathbb{R}$, and we consider only the symmetric case: $g(x,y) = g(y,x)$ for all $x, y \in E$.

1.2.1 Non degenerate bilinear forms

Definition 13. *A symmetric bilinear form g on E is:*

1. *positive [negative] definite provided $x \neq 0$ implies $g(x,x) > 0$ [< 0],*
2. *positive [negative] semi-definite provided $g(x,x) \geq 0$ [≤ 0] for all $x \in E$,*
3. *non degenerate provided $g(x,y) = 0$ for all $y \in E$ implies $x = 0$.*

Remark 6. *If g is a symmetric bilinear form on E then for any subspace F of E the restriction $g/_{(F\times F)}$, denoted merely by $g/_F$, is again symmetric and bilinear. If g is semi-definite, so is $g/_F$.*
A subspace F of E is called non degenerate if $g/_F$ is non degenerate, where:

$$g/_F : F \times F \longrightarrow \mathbb{R},\ g/_{F(x,y)} = g(x,y).$$

Definition 14. *The index ν of a symmetric bilinear form g on E is the largest integer that is the dimension of subspace $F \subset E$ on which $g/_F$ is negative definite.*
Thus $0 \leq \nu \leq dimE$, and $\nu = 0$ if and only if g is positive semi-definite.
If $\{e_1, ..., e_n\}$ is a basis for E, the $n \times n$ matrix $(g_{ij}) = g(e_i, e_j)$ is called the matrix of g relative to $\{e_1, ..., e_n\}$. Since g is symmetric, this matrix is symmetric. Clearly it determines g since:

$$g(\Sigma x_i e_i, \Sigma y_j e_j) = \Sigma g_{ij} x_i y_j.$$

Lemma 1. *A symmetric bilinear form is non degenerate if and only if its matrix relative to one (hence every) basis is invertible.*

Proof. Let $\{e_1, ..., e_n\}$ be a basis for E. If $x \in E$, then $g(x,y) = 0$ for all $y \in E$ if and only if $g(x, e_i) = 0$ for $i = 1, ..., n$. Since (g_{ij}) is symmetric:

$$g(x, e_i) = g(\sum x_j e_j, e_i) = \sum g_{ij} x_j.$$

Thus g is degenerate if and only there exist numbers $x_1, ..., x_n$ not all zero such that $\sum g_{ij} x_j = 0$ for $i = 1, ..., n$. But this is equivalent to the linear dependence of the columns of (g_{ij}), that is, to (g_{ij}) being singular. □

Definition 15. *Let F be a vector subspace of E, the orthogonal of F for g is the subspace of E defined by $F^{\perp} = \{v \in E \mid g(v,w) = 0, \forall w \in F\}$. Thus a symmetric bilinear form g on $E \times E$ is therefore non degenerate if and only if the orthogonal of E is $\{0\}$.*

Definition 16. *A scalar product over E is a bilinear form $g : E \times E \longrightarrow \mathbb{R}$, symmetric and non degenerate.*

Lemma 2. *If F is a subspace of a scalar product space E, then*

1. *$dimF + dimF^{\perp} = n = dimE$,*
2. *$(F^{\perp})^{\perp} = F$.*

Lemma 3. *A subspace F of E is non degenerate if and only if $E = F \oplus F^{\perp}$.*

Lemma 4. *A scalar product space $E \neq 0$ has an orthonormal basis.*

The matrix of g relative to an orthonormal basis $\{e_1, ..., e_n\}$ for E is diagonal, in fact:

$$g(e_i, e_j) = \delta_{ij}\epsilon_j,\ \textit{where}\ \epsilon_j = g(e_i, e_j) = \pm 1.$$

Whenever convenient we shall order the vectors in an orthonormal basis so that the negative signs – if any – come first in the so-called signature $(\epsilon_1, ..., \epsilon_n)$. Taking these signs into account orthonormal expansion is still available.

Lemma 5. *Let $e_1, ..., e_n$ be an orthonormal basis for E, with $\epsilon_i = g(e_i, e_j)$. Then each $x \in E$ has a unique expression:*

$$x = \sum \epsilon_i g(x, e_i) e_i.$$

For the proof it suffices to check that x minus the sum is orthogonal to each e_i, thus by the non degeneracy of g it is zero.
The orthogonal projection π of E into non degenerate subspace F is the linear transformation that sends $F^\perp$ to 0 and leaves any vector of F fixed. An orthonormal basis $\{e_1, ..., e_k\}$ for F can always be enlarged to a basis for E, thus:

$$\pi(x) = \sum_{j=1}^{k} \epsilon_j g(x, e_j) e_j.$$

It is customary to refer to the index ν of the scalar product g of E as the index of E, writing $\nu = IndE$

Lemma 6. *For any orthonormal basis $\{e_1, ..., e_n\}$ for E the number of negative signs in the signature $(\epsilon_1, ..., \epsilon_n)$ is the index Ind of E.*

Lemma 7. *Scalar product spaces E and F have the same dimension and index if and only if there exists a linear isometry from E to F.*

1.2.2 Pseudo-Riemannian metric

Definition 17. *Let M be a manifold of dimension n. A semi-Riemannian metric on M is a tensor field:*

$$g : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow C^\infty(M),$$

such that for each $x \in M$ the restriction of g is a family of applications:

$$g_x = g/_{T_xM \otimes T_xM} : T_xM \otimes T_xM \longrightarrow \mathbb{R} \text{ with,}$$

$$g_x : (X_p, Y_P) \longmapsto g(X, Y)(p)$$

is inner product such that:

1. *For all $x \in M$, g_x is a symmetric bilinear form non degenerate.*
2. *If $X, Y \in \mathfrak{X}(M)$, the function $g(X, Y)(x) = g_x(X_x, Y_x)$ is differentiable.*
3. *The index of g is constant, and noted Ind(M), that is to say:*

$$\exists\ p \in \mathbb{N}, \forall x \in M, Ind(T_xM) = P.$$

Definition 18. *A metric tensor g on a smooth manifold M is a symmetric non degenerate $(0, 2)$ tensor field on M of constant index.*
In order words $g \in \otimes^2 T^*M$ smoothly assigns to each point x of M a scalar product g_x on a tangent space T_xM, and the index of g_x is the same for all x.

Definition 19. *A semi-Riemannian manifold is a pair* (M, g)*, where* M *is a differentiable manifold of dimension n, and* g *is a metric tensor on* M*.*

Remark 7. *Let* (M, g) *a semi-Riemannian manifold, so:*

1. $0 \leq Ind\, M \leq dim\, M$.
2. *If* $Ind\, M = 0$, (M, g) *is said Riemannian manifold.*
3. *If* $p = 1$ *and* $dim\, M \geq 2$, (M, g) *is said Lorentz manifold.*

The metric g associated to a Lorentzian vector space is called pseudo-Riemannian metric. So, when the metric is definite positive or of signature $(-, +, ..., +)$ the group is called pseudo-Riemannian also called semi-Riemannian or Lorentzian.

Definition 20. *A Lorentzian vector space* $(E, \langle , \rangle)$ *is an n-dimensional vector space* E *endowed with a Lorentzian scalar product* $\langle , \rangle$ *that is, a non degenerate symmetric bilinear form of index* 1*. This means that we have a basis* $\{e_1, ..., e_n\}$ *of the space* E*, such that:*

$$\begin{cases} \langle e_i, e_j \rangle = 1 \\ \langle e_i, e_j \rangle = -1 \\ \langle e_i, e_j \rangle = 0, \end{cases}$$

for all $1 \leq i,\ j \leq n$ *and* $i \neq j$*.*

We use $\langle , \rangle$ as an alternative notation for g, writing $g(x, y) = \langle x, y \rangle \in \mathbb{R}$ for tangent vectors, and $g(X, Y) = \langle X, Y \rangle \in C^\infty(M)$ for vector fields.
If $x^1, ..., x^n$ is a coordinate system on $\mathbb{U} \subset M$ ($\mathbb{U}$ is an open set) the components of the metric tensor g on $\mathbb{U}$ are:

$$g_{ij} = \langle \partial_i, \partial_j \rangle \ \ (1 \leq i, j \leq n).$$

Thus for vector fields $X = \sum X^i \partial_i$ and $Y = \sum Y^j \partial_j$,

$$g(X, Y) = \langle X, Y \rangle = \sum g_{ij} X^i Y^j.$$

Since g is non degenerate, at each point p of $\mathbb{U}$ the matrix $(g_{ij}(p))$ is invertible, and its inverse matrix is denoted by $(g^{ij}(p))$. The usual formula for the inverse of a matrix shows that the functions g^{ij} are smooth on $\mathbb{U}$. Since g is symmetric, $g_{ij} = g_{ji}$ and hence $g^{ij} = g^{ji}$ for $1 \leq i, j \leq n$. Finally on u the metric tensor can be written as:

$$g = \sum g_{ij} dx^i \otimes dx^j.$$

Recall from Chapter 1 [53] for each $p \in \mathbb{R}^n$ there is a canonical isomorphism from $\mathbb{R}^n$ to $T_p(\mathbb{R}^n)$ that, in terms of natural coordinates, sends x to $x_p = \sum x^i \partial_i$. Thus the dot product on $\mathbb{R}^n$ gives rise to a metric tensor on $\mathbb{R}^n$ with:

$$\langle x_p, y_p \rangle = x.y = \sum x^i y^i.$$

Henceforth any geometric context $\mathbb{R}^n$ will denote the resulting Riemannian manifold, called Euclidean n-space.

For integer ν with $1 \leq \nu \leq n$, changing the first ν plus signs above to minus gives a metric tensor:

$$\langle x_p, y_p \rangle = -\sum_{i=1}^{\nu} x^i y^i + \sum_{j=\nu+1}^{n} x^j y^j$$

of index ν. The resulting semi-Euclidean space $\mathbb{R}^n_\nu$ reduces to $\mathbb{R}^n$ if $\nu = 0$. For $n \geq 2$, $\mathbb{R}^n_1$ is called Minkowski n-space, if $n = 4$ it is the simplest example of a relativistic spacetime.
Fix the notation:

$$\epsilon_i = \begin{cases} -1, & for \;\; 1 \leq i \leq \nu, \\ +1, & for \;\; \nu+1 \leq i \leq n. \end{cases}$$

Then the metric tensor of $\mathbb{R}^n_\nu$ can be written:

$$g = \sum \epsilon_i dx^i \otimes dx^i.$$

The geometric significance of the index of a semi-Riemannian manifold derives from the following trichotomy.

Definition 21. *A tangent vector x to M is:*

1. *space-like if $\langle x, x \rangle) > 0$ or $x = 0$,*
2. *null if $\langle x, x \rangle = 0$ and $x \neq 0$,*
3. *time-like if $\langle x, x \rangle < 0$.*

The set of all null vectors in T_pM is called the null-cone at $p \in M$ The category into which a given tangent vector falls is called causal character. Particulary in the Lorentz case, null vectors are also said to be light-like.

Definition 22. *Let $n \geq 2$ and $0 \leq P \leq n$.*

1. *The pseudo-sphere of $\mathbb{R}^{n+1}_p$ is defined by:*

$$\mathbb{S}^n_p = \{(x_1 ... x_{n+1}) \in \mathbb{R}^{n+1} \mid -\sum_{i=1}^{p} x_i^2 + \sum_{i=p+1}^{n+1} x_i^2 = 1\}.$$

2. *The pseudo-hyperbolic of $\mathbb{R}^{n+1}_{p+1}$ is defined by:*

$$\mathbb{H}^n_p = \{(x_1 ... x_{n+1}) \in \mathbb{R}^{n+1} \mid -\sum_{i=1}^{p+1} x_i^2 + \sum_{i=p+2}^{n+1} x_i^2 = -1\}.$$

Example 7. *On the pseudo-sphere:*

$$\mathbb{S}^2_1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid -x_1^2 + x_2^2 + x_3^2 = 1\}.$$

of $\mathbb{R}^3_1$, we consider the parametrization:

$$\begin{cases} x_1 = \sinh \alpha \\ x_2 = \cosh \alpha \sin \beta \\ x_3 = \cosh \alpha \cos \beta. \end{cases}$$

The metric $g = -dx_1^2 + dx_2^2 + dx_3^2$ of $\mathbb{R}_1^3$ induced on $\mathbb{S}_1^2$ a semi-Riemannian metric h, let ∂_α and ∂_β be the base vector fields associated with this parametrization, the components of the metric h is given by:

$$h_{11} = g(\partial_\alpha, \partial_\alpha) = -1, \quad h_{12} = g(\partial_\alpha, \partial_\beta) = 0, \quad h_{22} = g(\partial_\beta, \partial_\beta) = \cosh^2 \alpha,$$

that is to say $h = -d\alpha^2 + \cosh^2 \alpha d\beta^2$.

Example 8. *We consider on the pseudo-sphere:*

$$\mathbb{S}_1^3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid -x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}.$$

of $\mathbb{R}_1^4$, the following parametrization:

$$\begin{cases} x_1 = \sinh \alpha \\ x_2 = \cosh \alpha \sin \beta \\ x_3 = \cosh \alpha \cos \beta \sin \gamma \\ x_4 = \cosh \alpha \cos \beta \cos \gamma. \end{cases}$$

The metric $g = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$ of $\mathbb{R}_1^4$ induced on $\mathbb{S}_1^3$ a semi-Riemannian metric h, let ∂_α, ∂_β and ∂_γ be the base vector fields associated with this parametrization, the components of the metric h is given by:

$$h_{11} = g(\partial_\alpha, \partial_\alpha) = -1, \quad h_{12} = g(\partial_\alpha, \partial_\beta) = 0, \quad h_{13} = g(\partial_\alpha, \partial_\lambda) = 0,$$

$$h_{22} = g(\partial_\beta, \partial_\beta) = \cosh^2 \alpha, \quad h_{23} = g(\partial_\beta, \partial_\gamma) = 0, \quad h_{33} = g(\partial_\gamma, \partial_\gamma) = \cosh^2 \alpha \cos^2 \beta,$$

i.e. $h = -d\alpha^2 + \cosh^2 \alpha d\beta^2 + \cosh^2 \alpha \cos^2 \beta d\gamma^2$.

Example 9. *Let (N, h) a semi-Riemannian manifold, M a differentiable sub-manifold of N, and $i : M \hookrightarrow N$ the canonical inclusion. If $(i^*h)_x$ is non degenerate of constant index for every $x \in M$, then (M, i^*h) is a semi-Riemannian sub-manifold of (N, h) where:*

$$(i^*h)_x(X_x, Y_x) = h(d_x i(X_x), d_x i(Y_x)), \quad x \in M, \quad X_x, Y_x \in T_x M.$$

Example 10. *Let (M, g) a semi-Riemannian manifold, and let γ a differentiable function on M. Then, $(M, e^{2\gamma} g)$ is a semi-Riemannian manifold with the same index of (M, g), said conform to (M, g) of conformity factor $e^{2\gamma} g$.*
Moreover if $\{e_i\}$ is an orthonormal basis on (M, g), then $\{e^{-\gamma} e_i\}$ is an orthonormal basis on $(M, e^{2\gamma} g)$.

1.3 Linear connection

Let X and Y be vector fields on a semi-Riemannian manifold M. The goal of this section is to show how to define a new vector field $\nabla_X Y$ on M whose value at each point p is the vector rate of change of Y in the X_p direction.

Definition 23. *A linear connection on* M *is a map:*

$$\begin{array}{rcl}\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) & \longrightarrow & \mathfrak{X}(M) \\ (X, Y) & \longmapsto & \nabla_X Y\end{array}$$

such that for all $X, Y, Z \in \mathfrak{X}(M)$ *and* $f \in C^\infty(M)$ *we have:*

1. $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$,
2. $\nabla_X(fY) = X(f)Y + f\nabla_X Y$,
3. $\nabla_{X+fY} Z = \nabla_X Z + f\nabla_Y Z$.

We say that $\nabla_X Y$ *is the covariant derivative of* Y *with the direction of* X.

Definition 24. *A section* $y \in \mathfrak{X}(M)$ *is said to be parallel with respect to the connection* ∇ *if:*

$$\nabla_X Y = 0, \ \forall \ X \in \mathfrak{X}(M).$$

Definition 25. *Let* (M, g) *a semi-Riemannian manifold, a linear connection on* M *is said to be compatible with the metric* g *if:*

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(X, \nabla_X Y), \ \forall \ X, Y, Z \in \mathfrak{X}(M).$$

1.3.1 Torsion tensor

Definition 26. *Let* M *be a smooth manifold, and* ∇ *be a connection on the tangent bundle* TM*, then the torsion of* ∇ *is a tensor field of type* $(1, 2)$ *defined by:*

$$\begin{array}{rcl}T : \mathfrak{X}(M) \times \mathfrak{X}(M) & \longrightarrow & \mathfrak{X}(M) \\ (X, Y) & \longmapsto & \nabla_X Y - \nabla_Y X - [X, Y],\end{array}$$

where $[,]$ *is the lie bracket on* $\mathfrak{X}(M)$. *The connection* ∇ *on the tangent bundle* TM *is said to be torsion-free if the corresponding torsion* T *vanishes i.e.:*

$$[X, Y] = \nabla_X Y - \nabla_Y X \ \forall \ X, Y \in \mathfrak{X}(M).$$

Remark 8. $T(X, Y) = -T(Y, X)$, *for all* $X, Y \in \mathfrak{X}(M)$ *(*T *is antisymmetric).*

1.3.2 Levi-Civita connection

Definition 27. *Let* $u^1, ..., u^n$ *be the natural coordinates on* $\mathbb{R}^n_\nu$. *If* X *and* $Y = \sum Y^i \partial_i$ *are vector fields on* $\mathbb{R}^n_\nu$*, the vector field:*

$$\nabla_X Y = \sum X(Y^i)\partial_i$$

is called the natural covariant derivative of Y *with respect to* X.

Definition 28. *A connection* ∇ *on a smooth manifold* M *is a function*

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$$

such that:

1. $\nabla_X Y$ *is* $C^\infty(M)$*-linear in* X*,*
2. $\nabla_X Y$ *is* $\mathbb{R}$*-linear in* Y*,*
3. $\nabla_X(fY) = (Xf)Y + f\nabla_X Y$ *for* $f \in C^\infty(M)$*.*

$\nabla_X Y$ *is called the covariant derivative of* Y *with respect to* X *for the connection* ∇*.*

Proposition 1. *Let* M *be a semi-Riemannian manifold. If* $X \in \mathfrak{X}(M)$ *let* X^* *be the one-form on* M *such that:*

$$X^*(Y) = \langle X, Y\rangle \ \ for \ \ all \ \ Y \in \mathfrak{X}(M).$$

Then the function $X \to X^*$ *is an* $C^\infty(M)$*-linear isomorphism from* $\mathfrak{X}(M)$ *to* $\mathfrak{X}^*(M)$*.*

The following result has been called the miracle of semi-Riemannian geometry:

Theorem 1. *On a semi-Riemannian manifold* M *there is a unique connection* ∇ *such that:*

1. $[X, Y] = \nabla_X Y - \nabla_Y X$*, and*
2. $Z\langle X, Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle$*,*

for all $X, Y, Z \in \mathfrak{X}(M)$*.* ∇ *is called the Levi-Cevita connection of* M*, and characterized by the Koszul formula:*

$$2\langle \nabla_X Y, Z\rangle = X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle + \langle Z, [X, Y]\rangle + \langle Y, [Z, X]\rangle - \langle X, [Y, Z]\rangle.$$

Lemma 8. *The natural connection* ∇ *of Definition* (27) *is the Levi-Cevita connection of the semi-Euclidean space* $\mathbb{R}^n_\nu$*, for every* $\nu = 0, 1, ..., n$*. Relative to natural coordinate on* $\mathbb{R}^n_\nu$

1. $g_{ij} = \delta_{ij}\epsilon_j, \quad where \quad \epsilon_i = \begin{cases} -1, & if \ \ 1 \leq j \leq \nu, \\ +1, & if \ \ \nu + 1 \leq j \leq n. \end{cases}$
2. $\Gamma^k_{ij} = 0, \quad$ *for all* $1 \leq i, j, k \leq n$*.*

Proof. (1) is essentially the definition of the metric tensor of $\mathbb{R}^n_\nu$. To prove that ∇ is the Levi-Cevita connection of $\mathbb{R}^n_\nu$ one must check that is satisfies (1) of (28) and (2) of (1). Take (2) of (1), for example. Since $\langle X, Y\rangle = \sum \epsilon_i X^i Y^i$,

$$\begin{aligned} Z\langle X, Y\rangle &= \sum \epsilon_i Z(X^i)Y^i + \sum \epsilon_i X^i Z(Y^i) \\ &= \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle. \end{aligned}$$

Then (2) follows from Proposition 13(2) [53], since the g_{ij}s are constant.
A vector field X is parallel provided its covariant derivatives $\nabla_Z X$ are zero for all $Z \in \mathfrak{X}(M)$. Thus the vanishing of Christoffel symbols in the lemma means that the natural coordinate vector field on $\mathbb{R}^n_\nu$ are parallel. In general the Christoffel symbols of a coordinate system measure the failure of its coordinate vector fields to be parallel. □

Theorem 2. *Let (M, g) a semi-Riemannian manifold. Then Levi-Civita connection is an unique linear connection compatible with g and torsion free.*

Remark 9. *In a coordinate system (x_i) on M, ∇ is completely defined by the Christoffel symbols Γ^k_{ij} defined by:*

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^{n} \Gamma^k_{ij} \frac{\partial}{\partial x_k}.$$

Let $X = X^i \frac{\partial}{\partial x_i}$, and $Y = Y^j \frac{\partial}{\partial x_j}$, then:

$$\nabla_X Y = \sum_{i,k=1}^{n} X^i \left(\frac{\partial Y^k}{\partial x_i} + \sum_{j=1}^{n} \Gamma^k_{ij} Y^j \right) \frac{\partial}{\partial x_k}.$$

Proposition 2. *Let (M^m, g) a semi-Riemannian manifold with Levi-Civita connection ∇. Further let (U, φ) be a local coordinate on M and put $\partial_i = \frac{\partial}{\partial x_i} \in \mathfrak{X}(U)$. Then $\{\frac{\partial}{\partial x_1} \ldots \frac{\partial}{\partial x_n}\}$ is a local frame of TM on U. We define the Christoffel symbols $\Gamma^k_{ij} : U \to \mathbb{R}$ of the connection ∇ with respect to (U, φ) by:*

$$\Gamma^k_{ij} = \frac{1}{2} \sum_{i=1}^{n} g^{kl} \left\{ \frac{\partial g_{jl}}{\partial x_i} + \frac{\partial g_{il}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l} \right\},$$

where $g_{ij} = g(e_i, e_j) = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ are the components of g, and $g^{ij} = (g_{ij})^{-1}$ is the inverse matrix.

In effect, we put $\partial_i = \frac{\partial}{\partial x_i}$, like $[\partial_i, \partial_j] = 0$, $\forall i, j = 1, ..., m$, we have:

$$\begin{aligned} 2g(\nabla_{\partial_i} \partial_j, \partial_l) &= 2 \sum_{s=1}^{m} g(\Gamma^s_{ij} \partial_j, \partial_l) \\ &= 2 \sum_{s=1}^{m} \Gamma^s_{ij} g_{sl}, \end{aligned}$$

and according the Koszul's formula:

$$2g(\nabla_{\partial_i} \partial_j, \partial_l) = \partial_i(g(\partial_j, \partial_l)) + \partial_j(g(\partial_l, \partial_i)) - \partial_l(g(\partial_i, \partial_j)),$$

then:

$$\sum_{s=1}^{m} \Gamma^s_{ij} g_{sl} = \frac{1}{2} \left\{ \frac{\partial g_{jl}}{\partial x_i} + \frac{\partial g_{il}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l} \right\},$$

from where:

$$\sum_{s=1}^{m} \Gamma^s_{ij} g_{sl} g^{lk} = \frac{1}{2} g^{lk} \left\{ \frac{\partial g_{jl}}{\partial x_i} + \frac{\partial g_{il}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l} \right\},$$

and:

$$\sum_{s,l=1}^{m} \Gamma_{ij}^{s} g_{sl} g^{lk} = \frac{1}{2} \sum_{l=1}^{m} \left\{ \frac{\partial g_{jl}}{\partial x_i} + \frac{\partial g_{il}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l} \right\},$$

as g^{ij} is the inverse matrix of g_{ij} we have:

$$\sum_{l=1}^{m} g_{sl} g^{lk} = \delta_{ks},$$

where δ_{ks} is the Kronecker symbol, we get:

$$\Gamma_{ij}^{k} = \frac{1}{2} \sum_{i=1}^{n} g^{kl} \left\{ \frac{\partial g_{jl}}{\partial x_i} + \frac{\partial g_{il}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l} \right\}.$$

1.3.3 Inverse tangent bundle and connection on tangent bundle

Definition 29. *Let $\varphi : M \longrightarrow N$ be a smooth map between two differentiable manifolds M and N. The inverse tangent bundle is defined by:*

$$\varphi^{-1}TN = \{(x, v) \in M, v \in T_{\varphi(x)}N\}.$$

A section on $\varphi^{-1}TN$ is a smooth map $V : M \longrightarrow TN$ such as $V(x) \in T_{\varphi(x)}N$, $\forall x \in M$. Denote by $\mathfrak{X}(\varphi^{-1}TN)$ the set of sections on $\varphi^{-1}TN$.

Definition 30. *Let $\varphi : M \longrightarrow N$ be a smooth map between two differentiable manifolds M and N, and h a Riemannian metric on N. Then h induce a Riemannian metric on $\mathfrak{X}(\varphi^{-1}TN)$ given by $h(V, W)(x) = h_{\varphi(x)}(V_x, W_x)$, for every $x \in M$ and $V, W \in \mathfrak{X}(\varphi^{-1}TN)$*

Definition 31. *Let $\varphi : M \longrightarrow N$ be a smooth map between two differentiable manifolds M and N and let ∇^N be a linear connection on N, then the Pull-back connection on the tangent bundle $\varphi^{-1}TN$ is defined by:*

$$\begin{aligned} \nabla^{\varphi} : \mathfrak{X}(M) \times \mathfrak{X}(\varphi^{-1}TN) &\longrightarrow \mathfrak{X}(\varphi^{-1}TN) \\ (X, V) &\longmapsto \nabla^{\varphi}_X V = \nabla^N_{d\varphi(X)} \widetilde{V}, \end{aligned} \tag{1.1}$$

where $\widetilde{V} \in \mathfrak{X}(N)$ such that $\widetilde{V} \circ \varphi = V$.

Let $X \in \mathfrak{X}(M)$, and $V \in \mathfrak{X}(\varphi^{-1}TN)$. Locally, we have $X = X^i \frac{\partial}{\partial x_i}$ and $V = V^\alpha (\frac{\partial}{\partial y_\alpha} \circ \varphi)$, where $X^i, V^\alpha \in C^\infty(U)(U$ is a open of $M)$, and $\{\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_m}\}$ (resp. $\{\frac{\partial}{\partial y_1}, ..., \frac{\partial}{\partial y_n}\}$) are the basic fields on M $(resp. N)$. then:

$$\begin{aligned}
\nabla^{\varphi}_X V &= \nabla^{\varphi}_{X^i \frac{\partial}{\partial x_i}} V^{\alpha}(\frac{\partial}{\partial y_\alpha} \circ \varphi) \\
&= X^i \left\{ \frac{\partial V^\alpha}{\partial x_i}(\frac{\partial}{\partial y_\alpha} \circ \varphi) + V^\alpha \nabla^{\varphi}_{\frac{\partial}{\partial x_i}}(\frac{\partial}{\partial y_\alpha} \circ \varphi) \right\} \\
&= X^i \left\{ \frac{\partial V^\alpha}{\partial x_i}(\frac{\partial}{\partial y_\alpha} \circ \varphi) + V^\alpha \nabla^{N}_{d\varphi(\frac{\partial}{\partial x_i})} \frac{\partial}{\partial y_\alpha} \right\} \\
&= X^i \left\{ \frac{\partial V^\alpha}{\partial x_i}(\frac{\partial}{\partial y_\alpha} \circ \varphi) + V^\alpha \frac{\partial \varphi_\beta}{\partial x_i} \left(\nabla^{N}_{\frac{\partial}{\partial x_\beta}} \frac{\partial}{\partial y_\alpha} \right) \circ \varphi \right\} \\
&= X^i \left\{ \frac{\partial V^\alpha}{\partial x_i}(\frac{\partial}{\partial y_\alpha} \circ \varphi) + V^\alpha \frac{\partial \varphi_\beta}{\partial x_i} \left(\Gamma^{\gamma}_{\alpha\beta} \frac{\partial}{\partial y_\gamma} \right) \circ \varphi \right\} \\
&= X^i \left\{ \frac{\partial V^\gamma}{\partial x_i} + V^\alpha \frac{\partial \varphi_\beta}{\partial x_i}(\Gamma^{\gamma}_{\alpha\beta} \circ \varphi) \right\} \left(\frac{\partial}{\partial y_\gamma} \circ \varphi \right).
\end{aligned}$$

Then the relation (1.1) is independent of the choice $\widetilde{V}$ i.e. this connection is well defined.

1.3.4 Second fundamental form

Definition 32. *Let $\varphi : M \longrightarrow N$ be a smooth map between two differentiable manifolds M and N. The second fundamental form of φ is defined by:*

$$\nabla d\varphi(X, Y) = \nabla^{\varphi}_X d\varphi(Y) - d\varphi(\nabla^{M}_X Y), \quad \forall X, Y \in \mathfrak{X}(M).$$

Locally: *Let $\{\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_m}\}$ (resp. $\{\frac{\partial}{\partial y_1}, ..., \frac{\partial}{\partial y_n}\}$) a local basic fields of the vectors on M (resp. N). The second fundamental form in relation to these basis is given by:*

$$\begin{aligned}
(\nabla d\varphi)(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_j}) &= \nabla^{\varphi}_{\frac{\partial}{\partial x_i}} d\varphi(\frac{\partial}{\partial x_j}) - d\varphi(\nabla^{M}_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}) \\
&= \nabla^{\varphi}_{\frac{\partial}{\partial x_i}} \frac{\partial \varphi_\beta}{\partial x_j} \frac{\partial}{\partial y_\beta} \circ \varphi - \frac{\partial \varphi_\gamma}{\partial x_k} {}^{M}\Gamma^{k}_{ij}(\frac{\partial}{\partial y_\gamma} \circ \varphi) \\
&= \frac{\partial^2 \varphi_\beta}{\partial x_i \partial x_j} \frac{\partial}{\partial y_\beta} \circ \varphi + \frac{\partial \varphi_\beta}{\partial x_j} \nabla^{\varphi}_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial y_\beta} \circ \varphi - \frac{\partial \varphi_\gamma}{\partial x_k} {}^{M}\Gamma^{k}_{ij}(\frac{\partial}{\partial y_\gamma} \circ \varphi) \\
&= \frac{\partial^2 \varphi_\beta}{\partial x_i \partial x_j} \frac{\partial}{\partial y_\beta} \circ \varphi + \frac{\partial \varphi_\beta}{\partial x_j} \frac{\partial \varphi_\alpha}{\partial x_i} (\nabla^{N}_{\frac{\partial}{\partial y_\alpha}} \frac{\partial}{\partial y_\beta}) \circ \varphi - \frac{\partial \varphi_\gamma}{\partial x_k} {}^{M}\Gamma^{k}_{ij}(\frac{\partial}{\partial y_\gamma} \circ \varphi) \\
&= \left(\frac{\partial^2 \varphi_\gamma}{\partial x_i \partial x_j} + \frac{\partial \varphi_\alpha}{\partial x_i} \frac{\partial \varphi_\beta}{\partial x_j} {}^{N}\Gamma^{\gamma}_{\alpha\beta} \circ \varphi - \frac{\partial \varphi_\gamma}{\partial x_k} {}^{M}\Gamma^{k}_{ij} \right) \frac{\partial}{\partial y_\gamma} \circ \varphi.
\end{aligned}$$

Proposition 3. *Let $\varphi : M \longrightarrow N$ be a smooth map between two differentiable manifolds M and N. The second fundamental form of φ is a vectorial 1-form $C^\infty(M)$-bilinear symmetric. i.e.*

$$\nabla d\varphi(f_1 X, f_2 Y) = f_1 f_2 \nabla d\varphi(X, Y),$$

for all $X, Y \in \mathfrak{X}(M)$, and $f_1, f_2 \in C^\infty(M)$.

1.3.5 Geodesics

Definition 33. *Let (M, g) be a semi-Riemannian manifold of dimension n, and let $\gamma : I \subset \mathbb{R} \longrightarrow M$ a C^∞ curve on M. A set of the vector fields along γ, is defined by:*

$$\mathfrak{X}(\gamma^{-1}TM) = \{Y : I \longrightarrow TM \mid Y(t) \in T_{\gamma(t)}M, \forall t \in I\}.$$

Remark 10. *Let $X \in \mathfrak{X}(M)$, i.e $X : M \longrightarrow TM$ is an differentiable application, such that $X(x) \in T_xM$, $\forall X \in M$, then $X \circ \gamma \in \mathfrak{X}(\gamma^{-1}TM)$.*

Definition 34. *Let $Y \in \mathfrak{X}(\gamma^{-1}TM)$, the covariant derivative of Y along γ is defined by:*

$$\nabla^\gamma_{\frac{d}{dt}} Y = \nabla^M_{d\gamma(\frac{d}{dt})} \widetilde{Y},$$

where $\widetilde{Y} \in \mathfrak{X}(M)$ such that $\widetilde{Y} \circ \gamma = Y$.

Remark 11. *Let $\{\partial_i\}$ a local basis of vector fields on M, then $\{\partial_i \circ \gamma\}$ is local basis of vector fields along γ. Then, $\forall Y \in \mathfrak{X}(\gamma^{-1}TM)$, $\exists Y_i : I \longrightarrow \mathbb{R}$ $(i = 1, ..., n)$ what $Y(t) = y_i(t)\partial_i|_{\gamma(t)}$. From where:*

$$\begin{aligned}
\nabla^\gamma_{\frac{d}{dt}} Y &= \nabla^\gamma_{\frac{d}{dt}} Y_i(\partial_i \circ \gamma) \\
&= \frac{dY_i}{dt}(\partial_i \circ \gamma) + Y_i \nabla^\gamma_{\frac{d}{dt}}(\partial_i \circ \gamma) \\
&= \frac{dY_i}{dt}(\partial_i \circ \gamma) + Y_i \nabla^M_{d\gamma\frac{d}{dt}}(\partial_i),
\end{aligned}$$

where $d\gamma\frac{d}{dt} \in \mathfrak{X}(\gamma^{-1}TM)$ and locally $d\gamma\frac{d}{dt} = \frac{d\gamma_j}{dt}(\partial_i \circ \gamma)$, where $\gamma_j = x_j \circ \gamma$. So:

$$\begin{aligned}
\nabla^\gamma_{\frac{d}{dt}} Y &= \frac{dY_i}{dt}(\partial_i \circ \gamma) + Y_i \frac{d\gamma_j}{dt}(\nabla^M_{\partial_j}\partial_i) \circ \gamma \\
&= \frac{dY_k}{dt} + Y_i \frac{d\gamma_j}{dt}(\Gamma^k_{ij} \circ \gamma)(\partial_k \circ \gamma).
\end{aligned}$$

So this relation is independent of the chose of $\widetilde{Y}$ i.e. this connection is indeed defined.

Definition 35. *A vector fields $Y(t)$ along a curve $\gamma : I \longrightarrow (M, g)$ is said to be parallel along γ, if $(\nabla^\gamma_{\frac{d}{dt}} Y)|_t = 0$, $\forall t \in I$.*

Proposition 4. *Let $\gamma : I \longrightarrow (M, g)$ a curve, $t_0 \in I$, and $v \in T_{\gamma(t_0)}M$, Then, there is a unique vector field Y_v parallel along γ such that $Y_v(t_0) = v$.*

Definition 36. *Let (M, g) a semi-Riemannian manifold, of dimension n, a curve γ on (M, g) it said to be geodesic if $\nabla^\gamma_{\frac{d}{dt}} d\gamma(\frac{d}{dt}) = 0$, i.e:*

$$\frac{d^2\gamma_k}{dt^2} + \frac{d\gamma_i}{dt}\frac{d\gamma_j}{dt}(\Gamma^k_{ij} \circ \gamma) = 0, \quad \forall k = 1, ..., n.$$

Example 11. *If $M = \mathbb{R}$ and $g = dx^2$, then a curve $\gamma : I \longrightarrow \mathbb{R}$ is a geodesic if and only if $\frac{d^2\gamma_k}{dt^2} = 0$, because $\Gamma^1_{11} = 0$, i.e, $\gamma(t) = at + b$, where $a, b \in \mathbb{R}$.*

Example 12. *If $M = \mathbb{R}^n_p$, then a curve $\gamma : I \longrightarrow \mathbb{R}^n_p$ is a geodesic if and only if $\frac{d^2\gamma_k}{dt^2} = 0$, because $\Gamma^k_{ij} = 0$, i.e, $\gamma(t) = at + b$, where $a, b \in \mathbb{R}^n$.*

Example 13. *We consider the parametrization of the sphere:*

$$\mathbb{S}^n = \{u \in \mathbb{R}^{n+1} \mid \|u\| = 1\},$$

and let the stereographic projection, $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}^{n+1}$ given by:

$$\psi(x) = (\frac{2x_1}{\|x\|^2+1}, ..., \frac{2x_n}{\|x\|^2+1}, \frac{\|x\|^2-1}{\|x\|^2+1}), \quad x \in \mathbb{R}^n,$$

$$\psi^{-1}(u) = (\frac{u_1}{1-u_{n+1}}, ..., \frac{u_n}{1-u_{n+1}}), \quad u \in \mathbb{R}^{n+1}.$$

The components of the metric tensor relatively to ψ are:

$$g_{ij}(x) = \frac{4\delta_{ij}}{(1+\|x\|^2)^2}, \quad x \in \mathbb{R}^n.$$

the Christoffel symbols are:

$$\Gamma^i_{ii}(x) = \Gamma^j_{ij}(x) = \Gamma^i_{ji}(x) = -\Gamma^i_{jj}(x) = \frac{-2x_i}{1+\|x\|^2}, \quad \Gamma^k_{ij}(x) = 0,$$

for $i, j, k = 1, ..., n$ distinct, for the proof using the proposition 1 that is:

$$\gamma(t) = (\cos t, \sin t, 0, ..., 0) \in \mathbb{S}^n, \quad t \in \mathbb{R}.$$

The representation of γ in this map is given by:

$$\begin{aligned}(\psi^{-1} \circ \gamma)(t) &= (\gamma_1(t), ..., \gamma_n(t)) \\ &= (\cos t, \sin t, 0, ..., 0).\end{aligned}$$

According to the geodesic definition and this last equation, γ is a geodesic on $\mathbb{S}^n$.

Theorem 3. *Let (M, g) be a semi-Riemannian manifold. For all $x \in M$ and any vector $v \in T_xM$, there exists an open interval I of $\mathbb{R}$ with $0 \in I$, and a unique geodesic $\gamma : I \longrightarrow (M, g)$ such that $\gamma(0) = x$ and $\dot{\gamma}(0) = v$.*

1.4 Curvatures

1.4.1 curvature tensor

Definition 37. *Let (M, g) be a semi-Riemannian manifold of dimension m, and ∇ a Levi-Civita connection. Then the function:*

$$R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M) \ defined \quad by :$$

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \quad \forall X, Y, Z \in \mathfrak{X}(M),$$

is a tensor of type $(1,3)$ *on* M*, called a curvature tensor. The curvature tensor type* $(1,4)$ *is given by:*

$$R(X,Y,Z,W) = g(R(X,Y)Z,W).$$

The curvature tensor R *is expressed as a function of the Christoffel symbols:*

$$R(\partial_i,\partial_j)\partial_k = \sum_{s=1}^{m} R_{ijk}^{s}\partial_s.$$

where $\{\partial_i\}$ *is a local basis of the vector fields on* M*. Like* $[\partial_i,\partial_j]=0$ *we obtain:*

$$\begin{aligned}
R(\partial_i,\partial_j)\partial_k &= \nabla_{\partial_i}\nabla_{\partial_j}\partial_k - \nabla_{\partial_j}\nabla_{\partial_i}\partial_k \\
&= \nabla_{\partial_i}(\Gamma_{jk}^{l}\partial_l) - \nabla_{\partial_j}(\Gamma_{ik}^{l}\partial_l) \\
&= \frac{\partial\Gamma_{jk}^{l}}{\partial x_i}\partial_l + \Gamma_{jk}^{l}\nabla_{\partial_i}\partial_l - \frac{\partial\Gamma_{ik}^{l}}{\partial x_j}\partial_l + \Gamma_{ik}^{l}\nabla_{\partial_j}\partial_l \\
&= \frac{\partial\Gamma_{jk}^{l}}{\partial x_i}\partial_l + \Gamma_{jk}^{l}\Gamma_{il}^{s}\partial_s - \frac{\partial\Gamma_{ik}^{l}}{\partial x_j}\partial_l + \Gamma_{ik}^{l}\Gamma_{jl}^{s}\partial_s \\
&= \left\{\frac{\partial\Gamma_{jk}^{s}}{\partial x_i} - \frac{\partial\Gamma_{ik}^{s}}{\partial x_j} + \Gamma_{jk}^{l}\Gamma_{il}^{s} - \Gamma_{ik}^{l}\Gamma_{jl}^{s}\right\}\partial_s.
\end{aligned}$$

Therefore the components of the curvature tensor R *is given by:*

$$R_{ijk}^{s} = \Gamma_{jk}^{l}\Gamma_{il}^{s} - \Gamma_{ik}^{l}\Gamma_{jl}^{s} + \frac{\partial\Gamma_{jk}^{s}}{\partial x_i} - \frac{\partial\Gamma_{ik}^{s}}{\partial x_j}.$$

Proposition 5. *Let* (M,g) *be a semi-Riemannian manifold. For all* $X,Y,Z,W \in \mathfrak{X}(M)$ *we have:*

1. $R(X,Y)Z = -R(Y,X)Z$ *(antisymmetric).*
2. $g(R(X,Y)Z,W) = -g(R(X,Y)W,Z)$.
3. $g(R(X,Y)Z,W) = g(R(Z,W)X,Y)$.
4. *R verified Bianchi's identity algebraic:*
$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0.$$
5. *R verified Bianchi's identity differential:*
$$(\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) + (\nabla_Z R)(X,Y) = 0.$$

1.4.2 Sectional curvature

Definition 38. *Let* (M,g) *be a semi-Riemannian manifold, of dimension* n*, with* $n \geq 2$*,* $x \in M$ *and* π *a 2-plane of* T_xM *of basic* $\{X,Y\}$*.*

1. π *is said to be non-degenerate if* $Q(X,Y) = g(X,X)g(Y,Y) - g(X,Y)^2$.

2. *If π is non-degenerate, we defined the Sectional curvature of π as follows:*

$$K(\pi) = K(X, Y) = \frac{g(R(X, Y)Y, X)}{Q(X, Y)}.$$

3. *We say that (M, g) is of constant curvature if $K(\pi) = k$ (for any 2-plane π).*

Definition 39. *Let (M, g) be a semi-Riemannian manifold. We define the smooth tensor field $R_1 : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ of type $(1, 3)$ by:*

$$R_1(X, Y)Z = g(Y, Z)X - g(X, Z)Y, \quad \forall X, Y, Z \in \mathfrak{X}(M).$$

Corollary 1. *A semi-Riemannian manifold (M, g) is of constant curvature k if and only if the The curvature tensor verifies the equation:*

$$R(X, Y)Z = k[R_1(X, Y)Z], \quad \forall X, Y, Z \in \mathfrak{X}(M).$$

1.4.3 Ricci curvature

Definition 40. *The Ricci curvature of a semi-Riemannian manifold (M, g), of dimension n is a tensor of type $(0, 2)$ defined by:*

$$\begin{aligned} Ric(X, Y) &= trace(Z \longmapsto R(Z, X)Y) \\ &= \sum_{i=1}^{n} \epsilon_i g(R(e_i, X)Y, e_i), \end{aligned}$$

for all $X, Y \in \mathfrak{X}(M)$, where $\{e_i\}$ is an orthonormal frame on M ($\epsilon_i = g(e_i, e_i)$).

Proposition 6. *The Ricci curvature is symmetrical. Indeed:*

$$\begin{aligned} Ric(X, Y) &= \sum_{i=1}^{n} \epsilon_i g(R(e_i, X)Y, e_i) \\ &= \sum_{i=1}^{n} \epsilon_i g(R(Y, e_i)e_i, X) \\ &= \sum_{i=1}^{n} \epsilon_i g(R(e_i, Y)X, e_i) \\ &= Ric(Y, X). \end{aligned}$$

Definition 41. *The Ricci tensor of a semi-Riemannian manifold (M, g), of dimension n is a tensor of type $(1, 1)$ defined by:*

$$Ricci(X) = \sum_{i=1}^{n} \epsilon_i R(X, e_i)e_i, \quad \forall X \in \mathfrak{X}(M).$$

Remark 12. *For all $\forall X, Y \in \mathfrak{X}(M)$ we have:*

$$Ric(X, Y) = g(Ricci(X), Y).$$

Definition 42. *We call scalar curvature of a semi-Riemannian manifold (M, g), of dimension n, the function defined on M by:*

$$S = trace_g Ric = \sum_{i,j=1}^{n} \epsilon_i \epsilon_j g(R(e_i, e_j)e_j, e_i).$$

Corollary 2. *Let (M, g) be a semi-Riemannian manifold of dimension n and of constant curvature k, then:*

1. $Ricci(X) = (n-1)kX$.
2. $Ric(X, Y) = (n-1)kg(X, Y)$.
3. $S = n(n-1)k$.

Example 14.

1. The sphere $\mathbb{S}^n$ has constant sectional curvature $+1$.
2. The space $\mathbb{R}^n$ has curvature 0.
3. $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2, \, y > 0\}$ the hyperbolic space with the metric $g = \frac{dx^2+dy^2}{y^2}$, has constant sectional curvature -1.

1.5 Operators on Pseudo-Riemannian manifolds

1.5.1 Gradient operator

Let (M, g) be a semi-Riemannian manifold, of dimension n, and $X \in \mathfrak{X}(M)$. We put:

$$X^\flat(Y) = g(X, Y),$$

for all $Y \in \mathfrak{X}(M)$, then the application:

$$\begin{array}{rcl} \flat : \mathfrak{X}(M) & \longrightarrow & \mathfrak{X}^*(M) \\ X & \longmapsto & X^\flat, \end{array}$$

is $C^\infty(M)$-isomorphism. Moreover, $\flat^{-1} = \sharp$, and:

$$\begin{array}{rcl} \sharp : \mathfrak{X}^*(M) & \longrightarrow & \mathfrak{X}(M) \\ w & \longmapsto & w^\sharp \end{array}$$

is a isomorphism map between the cotangent bundle and the tangent bundle given by:

$$\forall X \in \mathfrak{X}(M), \quad g(w^\sharp, X) = w(X).$$

Definition 43. *Let (M, g) be a semi-Riemannian manifold, of dimension n, we defines the gradient operator by:*

$$\begin{array}{rcl} grad : C^{\infty}(M) & \longrightarrow & \mathfrak{X}(M) \\ f & \longmapsto & grad\, f = (df)^{\sharp}. \end{array}$$

So that for all $X \in \mathfrak{X}(M)$ we have:

$$g(grad\, f, X) = X(f) = df(X).$$

Locally:

$$grad\, f = \sum_{i=1}^{n} g^{ij} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j},$$

where $\{\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\}$ is a local coordinate. Let $\{e_1, ..., e_n\}$ be an orthonormal frame on (M, g). Then:

$$grad\, f = \sum_{i=1}^{n} \epsilon_i e_i(f) e_i.$$

Proposition 7. *Let (M, g) be a semi-Riemannian manifold, then:*

1. *grad(f+h)=gradf+gradh.*
2. *grad(fh)=hgradf+fgradh.*
3. *(gradf)(h)=(gradh)(f).*

1.5.2 Hessian operator

Definition 44. *Let (M, g) be a semi-Riemannian manifold, of dimension n and $f \in C^{\infty}(M)$. The Hessian of the function f denoted by $Hess\, f$ is a $C^{\infty}(M)$-bilinear map, defined by:*

$$\begin{array}{rcl} Hess\, f : \mathfrak{X}(M) \times \mathfrak{X}(M) & \longrightarrow & C^{\infty}(M) \\ (X, Y) & \longmapsto & (Hess\, f)(X, Y) = g(\nabla_X grad\, f, Y). \end{array}$$

Proposition 8. *Let (M, g) be a semi-Riemannian manifold, of dimension n and $f \in C^{\infty}(M)$, then:*

1. *Hessf be a tensor of type (0,2).*
2. *Hessf is symmetric.*

Locally:

$$Hess\, f = \sum_{i,j=1}^{n} (Hess\, f)_{ij} dx_i \otimes dx_j,$$

where:

$$\begin{aligned} (Hess\, f)_{ij} &= g(\nabla_{\partial_i} grad\, f, \partial_j) \\ &= \frac{\partial^2 f}{\partial x_i \partial x_j} - \sum_{i=1}^{n} \Gamma_{ij}^{k} \frac{\partial f}{\partial x_k}). \end{aligned}$$

1.5.3 Divergence operator

Let X be a vector field on a semi-Riemannian manifold (M, g), then:

$$\begin{aligned} \nabla X : \mathfrak{X}(M) &\longrightarrow \mathfrak{X}(M) \\ Y &\longmapsto \nabla_Y X, \end{aligned}$$

is a $C^\infty(M)$-linear mapping.

Definition 45. *The divergence of the vector field $X \in \mathfrak{X}(M)$, denoted $div\, X$ is defined by:*

$$div\, X = trace \nabla X.$$

Locally:

$$\begin{aligned} div\, X &= dx_i(\nabla_{\frac{\partial}{\partial x_i}} X) \\ &= g^{ij} g(\nabla_{\frac{\partial}{\partial x_i}} X, \frac{\partial}{\partial x_j}). \end{aligned}$$

Let $\{e_1, ..., e_m\}$ be an orthonormal frame on M, then:

$$div\, X = \sum_{i=1}^{n} \epsilon_i g(\nabla_{e_i} X, e_i).$$

The divergence of 1-form w on M such that $w \in \mathfrak{X}^*(M)$ is defined by:

$$\begin{aligned} div^M w &= trace(Y \longmapsto \nabla_Y w) \\ &= \sum_{i=1}^{n} \epsilon_i (\nabla_{e_i} w)(e_i) \\ &= \sum_{i=1}^{n} \epsilon_i (e_i(w(e_i)) - w(\nabla^M_{e_i} e_i)) \\ &= g^{ij} g(\nabla_{\frac{\partial}{\partial x_i}} w)(\frac{\partial}{\partial x_j}). \end{aligned}$$

In the definition of $div\, X$ we can also define the divergence of $(1, r)$-tensor T to be $(0, r)$-tensor:

$$(div\, T)(X_1, ..., X_r) = trace(Y \longmapsto (\nabla_Y T)(X_1, ..., X_r)).$$

1.5.4 First expression of the divergence in local coordinates

Proposition 9. *Let (M, g) be a semi-Riemannian manifold, of dimension n, then:*

$$div\, X = \sum_{i,j=1}^{n} (\frac{\partial X_i}{\partial x_i} + X_j \Gamma^k_{ij}),$$

with $X = \sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i} \in \mathfrak{X}(M)$.

Proposition 10. *Let (M, g) be a semi-Riemannian manifold, then:*

1. $div(X + Y) = div\, X + div Y$.
2. $div(fX) = f div\, X + X(f)$.

for all $X, Y \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$.

1.5.5 Second expression of the divergence in local coordinates

Lemma 9. *On a Riemannian manifold (M, g), we have:*

$$\frac{\partial}{\partial x_k}\left(\sqrt{det(g_{ij})}\right) = \sqrt{det(g_{ij})}\sum_{i=1}^{n}\Gamma_{ij}^{k}.$$

Proposition 11. *Let (M, g) be a Riemannian manifold, then:*

$$div\, X = \frac{1}{\sqrt{det(g_{ij})}}\frac{\partial}{\partial x_k}\left(\sqrt{det(g_{ij})}X_k\right),$$

for all $X \in \mathfrak{X}(M)$.

1.5.6 Laplacian operator

Let (M, g) be a semi-Riemannian manifold, we define the Laplacian operator note $\triangle$, on M by:

$$\begin{array}{rcl} \triangle : C^\infty(M) & \longrightarrow & C^\infty(M) \\ f & \longmapsto & \triangle(f) = div(grad\, f). \end{array}$$

Proposition 12. *Let (M, g) be a semi-Riemannian manifold, then:*

1. $\triangle(f + h) = \triangle(f) + \triangle(h)$,
2. $\triangle(fh) = h\triangle(f) + f\triangle(h) + 2g(grad\, f, grad\, h)$,

for all $f, h \in C^\infty(M)$.

Proposition 13. *Let (M, g) be a semi-Riemannian manifold, then the expression of the Laplacian in local coordinates is given by:*

$$\triangle(f) = g^{ij}(\frac{\partial^2 f}{\partial x_i \partial x_j} - \Gamma_{ij}^{k}\frac{\partial f}{\partial x_k}), \quad for\ all\ f \in C^\infty(M).$$

In effect, let $f \in C^{\infty}(M)$, then:

$$\begin{aligned}\triangle(f) &= div(grad\, f)\\ &= g^{ij} g(\nabla_{\frac{\partial}{\partial x_i}} grad\, f, \frac{\partial}{\partial x_j})\\ &= g^{ij}(\frac{\partial}{\partial x_i} g(grad\, f, \frac{\partial}{\partial x_j}) - g(grad\, f, \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}))\\ &= g^{ij}(\frac{\partial}{\partial x_i}(\frac{\partial}{\partial x_j}) - \Gamma_{ij}^k g(grad\, f, \frac{\partial}{\partial x_k}))\\ &= g^{ij}(\frac{\partial^2 f}{\partial x_i \partial x_j} - \Gamma_{ij}^k \frac{\partial f}{\partial x_k}).\end{aligned}$$

Example 15. *Let $\mathbb{R}_p^n$ provided with the product scalar $g = -dx_1^2 - ... - dx_p^2 + dx_{p+1}^2 + ... + dx_n^2$, like $g_{ij} = \delta_{ij}\epsilon_j$, then for every differentiable function f on $\mathbb{R}_p^n$ and $X = (X_1, ..., X_n)$ a vector field on $\mathbb{R}^n$, we have:*

$$grad\, f = \sum_{i=1}^{n} \epsilon_i \frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_i}, \quad div\, X = \sum_{i=1}^{n} \epsilon_i \frac{\partial X_i}{\partial x_i}, \quad \triangle(f) = \sum_{i=1}^{n} \epsilon_i \frac{\partial^2 f}{\partial x_i^2}.$$

1.6 Pseudo-Riemannian sub-manifolds

1.6.1 Sub-manifolds

Definition 46. *Let (N^n, h) a semi-Riemannian manifold, M^m a sub-manifold of N, and $i : M \hookrightarrow N$ the canonical inclusion. If h is non degenerate on M (i.e if $h(X_x, Y_x) = 0$, $\forall\, Y_x \in T_xM$, then $X_x = 0$, where $x \in M$), and $IndM$ is constant, then M is a semi-Riemannian manifold called a semi-Riemannian sub-manifold, endowed with the induce semi-Riemannian metric:*

$$g(X, Y)_x = h_x(X_x, Y_x),\ \forall\, X, Y \in \mathfrak{X}(M) \text{ and } x \in M.$$

where $g : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow C^{\infty}(M)$ is the tensor field on M.

Definition 47. *Let (N^n, h) a semi-Riemannian manifold and let (M^m, g) be a semi-Riemannian sub-manifold of (N^n, h). We define the normal space $T_xM^{\perp}$ by:*

$$T_xM^{\perp} = \{v \in T_xN \mid h_x(v, w) = 0,\ \forall w \in T_xM\}.$$

For all $x \in M$ we have the orthogonal decomposition:

$$T_xN = T_xM \oplus T_xM^{\perp}.$$

The normal bundle of M in N is defined by:

$$TM^{\perp} = \{(x, v) \mid x \in M, v \in T_xM^{\perp}\}.$$

For all $v \in T_xN$, $\exists! v^{\top} \in T_xM$, $\exists! v^{\perp} \in T_xM^{\perp}$ such that $v = v^{\top} + v^{\perp}$.
The maps $\top : T_xN \longrightarrow T_xM$, $v \longmapsto v^{\top}$ and $\perp : T_xN \longrightarrow T_xM^{\perp}$, $v \longmapsto v^{\perp}$ are $\mathbb{R}$-linear.
A vector field X of N is said to be normal, if $X_x \in T_xM^{\perp}$ for all $x \in M$.
$\mathfrak{X}(M)^{\perp}$ is the set of normal vector fields.

Definition 48. *Let (M,g) be a semi-Riemannian sub-manifold of (N,h), and let ∇^N the Levi-Civita connection of (N,h). Then we define:*

$$\nabla^M : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$$

by:

$$\nabla^M_X Y = (\nabla^N_X Y)^\top,$$

∇^M is the Levi-Civita connection of the sub-manifold of (M,g). Further more let:

$$B : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)^\perp$$

be given by:

$$B(X,Y) = (\nabla^N_X Y)^\perp,$$

the operator B is called the second fundamental form of (N,h).

Proposition 14. *For all $X, Y \in \mathfrak{X}(M)$, we have:*

1. *$B(X,Y) = B(Y,X)$, (B is symmetric).*
2. *B is $C^\infty(M)$-bilinear.*

Proposition 15. *Let (M,g) be a semi-Riemannian sub-manifold of (N,h), and let R^M(resp.R^N) the curvature tensor of (M,g) (resp. of (N,h)). Then:*

$$g(R^M(V,W)X,Y) = h(R^N(V,W)X,Y) - h(B(V,X),B(W,Y)) \\ +h(B(V,Y),B(W,X)), \quad \forall\, X,Y,V,W \in \mathfrak{X}(M).$$

Corollary 3. *Let (M,g) be a semi-Riemannian sub-manifold of (N,h), and let R^M(resp.R^N) the sectional curvature of (M,g) (resp. of (N,h)). Then:*

$$K^M(v,w) = K^M(v,w) + \frac{h(B(v,v),B(w,w)) - h(B(v,w),B(v,w))}{g(v,w)g(w,w) - g(v,w)^2},$$

where $\{v,w\}$ is a basis of $\pi \subset T_xM$ ($x \in M$).

1.6.2 Pseudo-Riemannian Hypersurfaces

Definition 49. *Let (N,h) a semi-Riemannian manifold, of dimension n. a semi-Riemannian hypersurface of (N,h) is a semi-Riemannian sub-manifold (M,g) of (N,h), of dimension $m = n-1$.*

Definition 50. *Let (M,g) a semi-Riemannian hypersurface of (N,h). Then:*

$$Sign(M) = \begin{cases} +1, & if \;\; h(z,z) > 0, \forall z \in T_xM^\perp - \{0\}, \\ -1, & if \;\; h(z,z) < 0, \forall z \in T_xM^\perp - \{0\}. \end{cases}$$

Remark 13.

1. If $Sign(M) = +1$, we have: $Ind(M) = Ind(N)$.

2. If $Sign(M) = -1$, we have: $Ind(M) = Ind(N) - 1$.

Proposition 16. *Let (N, h) a semi-Riemannian manifold, f a differentiable function on N, $y_0 = f(x_0)$ $(x_O \in \mathbb{N})$. Then $M = f^{-1}(\{y_0\})$ is a semi-Riemannian hypersurface if and only if $h(grad^N f, grad^N f) > 0$ or < 0 on M, and:*

$$Sign(M) = Signh(grad^N f, grad^N f).$$

Example 16. *Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$, $f(x_1, ..., x_{n+1}) = x_1^2 + ... + x_{n+1}^2 - r^2$, then the sphere $\mathbb{S}^n(r) = f^{-1}(\{0\})$ is a semi-Riemannian hypersurface of $\mathbb{R}^{n+1}$. In effect:*

$$grad^{\mathbb{R}^{n+1}} f = 2 \sum_{k=1}^{n+1} x_i \frac{\partial}{\partial x_i},$$

$$< grad^{\mathbb{R}^{n+1}} f, grad^{\mathbb{R}^{n+1}} f >_{\mathbb{R}^{n+1}} = 4 \sum_{k=1}^{n+1} x_i^2,$$

from where $\forall\, (x_1, ..., x_{n+1}) \in \mathbb{S}^n(r)$, we have $< grad^{\mathbb{R}^{n+1}} f, grad^{\mathbb{R}^{n+1}} f >_{\mathbb{R}^{n+1}} = 4r^2 > 0$.

Definition 51. *Let (M, g) a semi-Riemannian hypersurface of (N, h), and let U the unit vector field normal to M. The operator*

$$\begin{array}{rcl} A : \mathfrak{X}(M) & \longrightarrow & \mathfrak{X}(M) \\ X & \longmapsto & AX = -\nabla^N_X U \end{array}$$

is called a chape operator.

Proposition 17. *$\forall\, X, Y \in \mathfrak{X}(M)$, we have $g(AX, Y) = \epsilon_i h(B(X, Y), U)$, where $\epsilon_i = \pm 1$.*

Corollary 4. *$\forall\, X, Y \in \mathfrak{X}(M)$, we have $B(X, Y) = \epsilon_i g(AX, Y)U$, where $\epsilon_i = h(U, U)$.*

Definition 52. *Let (M, g) be a semi-Riemannian sub-manifold of (N, h).*

$$\begin{array}{rcl} \nabla^\perp : \mathfrak{X}(M) \times \mathfrak{X}(M)^\perp & \longrightarrow & \mathfrak{X}(M)^\perp \\ (X, Y) & \longmapsto & \nabla^\perp_X Y = (\nabla^N_X Y)^\perp \end{array}$$

is called the normal connection of M.

Proposition 18. *Let (M, g) be a semi-Riemannian sub-manifold of (N, h).*

1. *$\nabla^\perp_X Y$ is $C^\infty(M)$-linear with respect to X and $\mathbb{R}$-linear with respect to Y.*
2. *$\nabla^\perp_X fY = X(f)Y + f\nabla^\perp_X Y$, $\forall X \in \mathfrak{X}(M)$, $\forall Y \in \mathfrak{X}(M)^\perp$ and $\forall f \in C^\infty(M)$.*
3. *$X(h(Y, Z)) = h(\nabla^\perp_X Y, Z) + h(Y, \nabla^\perp_X Z)$, $\forall\, X \in \mathfrak{X}(M)$ and $\forall\, Y, Z \in \mathfrak{X}(M)^\perp$.*

Definition 53. *Let (M, g) be a semi-Riemannian sub-manifold of (N, h). A curve $\gamma : I \subset \mathbb{R} \longrightarrow M$ is said to be of timelike if $d\gamma(\frac{\partial}{\partial t})|_t$ is a tangent vector of timelike, that is:*

$$g(d\gamma(\frac{\partial}{\partial t}), d\gamma(\frac{\partial}{\partial t}))|_t < 0, \ \ \forall t \in I.$$

Definition 54. *Let (M, g) a semi-Riemannian hypersurface of (N, h). M is said to be of spacelike if all the tangent vectors to M are of spacelike, that is :*

$$g(v, v) > 0 \; or \; v = 0 \; (v \in T_x M).$$

1.7 Harmonic hypersurface of pseudo-Riemannian manifolds

1.7.1 Harmonic maps

Definition 55. *Let $\varphi : (M^m, g) \to (N^n, h)$ be a smooth map between two semi-Riemannian manifolds, for any compact domain D of M the energy functional of φ is defined by :*

$$E(\varphi; D) = \frac{1}{2}\int_D \epsilon_i |d\varphi|^2 \, v_g, \tag{1.2}$$

where $|d\varphi|$ is the Hilbert Schmidt norm of differential of the map φ given by:

$$|d\varphi|^2 = \sum_{i=1}^{m} h(d\varphi(e_i), d\varphi(e_i))$$

and $\{e_1, \ldots, e_m\}$ be an orthonormal frame on M.

Definition 56. *A variation of φ to support in a compact domain $D \subset M$, is a smooth family maps $(\varphi_t)_{t\in(-\epsilon,\epsilon)} : M \to N$, such that $\varphi_0 = \varphi$ and $\varphi_t = \varphi$ on $M \setminus \operatorname{int}(D)$.*

Definition 57. *A map is called harmonic if it is a critical point of the energy functional over any compact subset D of M. i.e.*

$$\frac{d}{dt}E(\varphi_t; D)\Big|_{t=0} = 0.$$

1.7.2 First variation of energy

Theorem 4. *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map and let $(\varphi_t)_{t\in(-\epsilon,\epsilon)}$ be a smooth variation of φ supported in D. Then:*

$$\frac{d}{dt}E(\varphi_t; D)\Big|_{t=0} = -\int_D h(v, \tau(\varphi)) \, v_g,$$

where $v = \frac{d\varphi_t}{dt}\Big|_{t=0}$ denotes the variation vector field of $\{\varphi_t\}$,

$$\tau(\varphi) = \operatorname{trace}_g \nabla d\varphi = \sum_{i=1}^{m} \epsilon_i \left\{ \nabla^{\varphi}_{e_i} d\varphi(e_i) - d\varphi(\nabla^{M}_{e_i} e_i) \right\}, \tag{1.3}$$

is called tension field of φ where $\{e_1, ..., e_m\}$ is an orthonormal frame on (M^m, g) and $\epsilon_i = g(e_i, e_i) = \pm 1$.

Proof. Defined $\phi : M \times (-\epsilon, \epsilon) \to N$ by $\phi(x, t) = \varphi_t(x)$, let ∇^{ϕ} denote the pull-back connection on $\phi^{-1}TN$. Note that, for any vector field X on M considered as a vector field on $M \times (-\epsilon, \epsilon)$, we have $[\partial_t, X] = 0$. Using (1.2) we obtain:

$$\begin{aligned}
\frac{d}{dt}E(\varphi_t; D)\Big|_{t=0} &= \frac{1}{2}\frac{d}{dt}\int_D \sum_{i=1}^{m} h(d\varphi_t(e_i), d\varphi_t(e_i))\, v_g\Big|_{t=0} \\
&= \frac{1}{2}\frac{d}{dt}\int_D \sum_{i=1}^{m} h(d\phi(e_i,0), d\phi(e_i,0))\, v_g\Big|_{t=0} \\
&= \frac{1}{2}\int_D \frac{\partial}{\partial t}\sum_{i=1}^{m} h(d\phi(e_i,0), d\phi(e_i,0))\, v_g\Big|_{t=0} \\
&= \int_D \sum_{i=1}^{m} h(\nabla^{\phi}_{(0,\frac{d}{dt})} d\phi(e_i,0), d\phi(e_i,0))\, v_g\Big|_{t=0} \\
&= \int_D \sum_{i=1}^{m} h(\nabla^{\phi}_{(e_i,0)} d\phi(0,\frac{d}{dt}), d\phi(e_i,0))\, v_g\Big|_{t=0} \\
&= \int_D \sum_{i=1}^{m} h(\nabla^{N}_{d\varphi(e_i)} v, d\varphi(e_i))\, v_g \\
&= \int_D \sum_{i=1}^{m} h(\nabla^{\varphi}_{e_i} v, d\varphi(e_i))\, v_g. \qquad (1.4)
\end{aligned}$$

Define an 1-form on M by:

$$\omega(X) = h(v, d\varphi(X)), \quad X \in \mathfrak{X}(M).$$

We have:

$$\begin{aligned}
\mathrm{div}^M\omega &= \epsilon_i(\nabla_{e_i}\omega)(e_i) \\
&= \sum_{i=1}^{m} \epsilon_i \left\{ e_i(\omega(e_i)) - \omega(\nabla^M_{e_i} e_i) \right\} \\
&= \sum_{i=1}^{m} \left\{ h(\nabla^{\varphi}_{e_i} v, d\varphi(e_i)) + h(v, \nabla^{\varphi}_{e_i} d\varphi(e_i)) - h(v, d\varphi(\nabla^M_{e_i} e_i)) \right\} \\
&= \sum_{i=1}^{m} h(\nabla^{\varphi}_{e_i} v, d\varphi(e_i)) + h(v, \tau(\varphi)), \qquad (1.5)
\end{aligned}$$

according to formulas (3.22), (4.3), and the divergence theorem we obtain:

$$\frac{d}{dt}E(\varphi_t; D)\Big|_{t=0} = -\int_D h(v, \tau(\varphi)) v_g.$$

□

Theorem 5. *A smooth map $\varphi : (M^m, g) \to (N^n, h)$ between two semi-Riemannian manifolds is harmonic if and only if:*

$$\tau(\varphi) = trace_g \nabla d\varphi = 0.$$

If $(x^i)_{1\leq i\leq m}$ and $(y^\alpha)_{1\leq\alpha\leq n}$ denote local coordinates on M and N respectively, then respectively the equation $\tau(\varphi)=0$ takes the form:

$$\tau(\varphi)^\gamma=\epsilon_i(\Delta\varphi^\gamma+g^{ij\,N}\Gamma^\gamma_{\alpha\beta}\frac{\partial\varphi^\alpha}{\partial x^i}\frac{\partial\varphi^\beta}{\partial x^j})=0. \tag{1.6}$$

where $\Delta\varphi^\gamma=\dfrac{1}{\sqrt{|g|}}\dfrac{\partial}{\partial x^i}(\sqrt{|g|}g^{ij}\dfrac{\partial\varphi^\gamma}{\partial x^j})$ is the Laplace operator on (M^m,g) and ${}^N\Gamma^\gamma_{\alpha\beta}$ are the Christoffel symbols on N.

Example 17. *Every constant map:*
$\varphi:(M^m,g)\longrightarrow(N^n,h)$, $x\longmapsto y_0$ is harmonic (i.e. $d_x\varphi=0\ \forall x\in M$).

Example 18. *The identity mapping:*
$\mathrm{Id_M}:(M^m,g)\longrightarrow(M^m,g)$, $x\longmapsto x$ is totally geodesic (i.e. $\nabla dId_M=0$). Therefore $\mathrm{Id_M}$ is harmonic.

Example 19. *Let (M^m,g) be a semi-Riemannian manifold and let $f:(M^m,g)\longrightarrow\mathbb{R}$ be a smooth function, then:*

$$\begin{aligned}\tau(f)&=\epsilon_i\mathrm{trace}_g\nabla df\\&=\epsilon_i\nabla df(e_i,e_i)\\&=\epsilon_i(\nabla^f_{e_i}df(e_i)-df(\nabla^M_{e_i}e_i))\\&=\epsilon_i(e_i(e_i(f))-(\nabla^M_{e_i}e_i)(f))\\&=g(\nabla_{e_i}\mathrm{grad}\,f,e_i)\\&=\mathrm{div}\,\mathrm{grad}f\\&=\triangle(f),\end{aligned}$$

where $\{e_i\}$ is an orthonormal frame on (M^m,g).

Remark 14. *The composition of two harmonic maps is not in general a harmonic application. In particular if φ is harmonic and ψ is totally geodesic (i.e. $\nabla\psi=0$), then $\psi\circ\varphi$ is harmonic.*

Example 20. *Let the map:*

$$\begin{aligned}\varphi:(\mathbb{R},dx^2)&\longrightarrow(\mathbb{R}^2,dx^2+dy^2)\\x&\longmapsto(x,0),\end{aligned}$$

we have:

$$\begin{aligned}\tau(\varphi)&=\Big(\frac{\partial^2x}{dx^2},\frac{\partial^2 0}{dx^2}\Big)\\&=0,\end{aligned}$$

and let the map:

$$\begin{aligned}\psi:(\mathbb{R}^2,dx^2+dy^2)&\longrightarrow(\mathbb{R},dz^2)\\(x,y)&\longmapsto\frac{x^2-y^2}{2},\end{aligned}$$

we have:

$$\begin{aligned}\tau(\psi) &= \Delta(\psi)\\ &= \frac{\partial^2\psi}{dx^2}+\frac{\partial^2\psi}{dy^2}\\ &= 1-1\\ &= 0,\end{aligned}$$

then the two maps φ and ψ are harmonic, but the compound:

$$\begin{aligned}\psi\circ\varphi:(\mathbb{R},dx^2) &\longrightarrow (\mathbb{R},dz^2)\\ x &\longmapsto \frac{x^2}{2},\end{aligned}$$

is not harmonic, $\tau(\psi\circ\varphi)=1$.

Example 21. *If $M=]a,b[$ be an interval of $\mathbb{R}$, then a curve $\gamma:(a,b)\longrightarrow(N^n,h)$ is harmonic if:*

$$\frac{d^2\gamma^\alpha}{dt^2}+^N\Gamma^\alpha_{\beta\delta}\frac{d\gamma^\beta}{dt}\frac{d\gamma^\delta}{dt}=0,$$

therefore, γ is harmonic if and only if it is a geodesic.

1.7.3 Second variation of energy

Theorem 6. *Let $\varphi:(M^m,g)\longrightarrow(N^n,h)$ be a harmonic map between Riemannian manifolds, and $\{\varphi_{t,s}\}$ be a two parameter variation with compact support in D. We set:*

$$v=\frac{\partial\varphi_{t,s}}{\partial t}\big|_{(t,s)=(0,0)}\quad \text{and}\quad w=\frac{\partial\varphi_{t,s}}{\partial s}\big|_{(t,s)=(0,0)}$$

denotes the variation vector fields of φ.
Under the notation above we have the following:

$$\frac{\partial^2}{\partial t\partial s}E(\varphi_{t,s},D)\big|_{(t,s)=(0,0)}=\int_D h(J_\varphi(v),w)v_g,$$

where $J_\varphi(v)\in\Gamma(\varphi^{-1}TN)$ given by:

$$J_\varphi(v)=-traceR^N(v,d\varphi)d\varphi-trace(\nabla^\varphi)^2v.$$

R^N is the curvature tensor on (N,h), and

$$trace(\nabla^\varphi)^2v=\sum_{i=1}^m[\nabla^\varphi_{e_i}\nabla^\varphi_{e_i}v-\nabla^\varphi_{\nabla^M_{e_i}e_i}v].$$

Proof. Defined $\phi : M \times (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \to N$ by $\phi(x,t,s) = \varphi_{t,s}(x)$. Let ∇^ϕ denote the pull-back connection on $\phi^{-1}TN$. Note that, for any vector field X on M considered as a vector field on $M \times (-\epsilon, \epsilon) \times (-\epsilon, \epsilon)$, we have:

$$[\partial_t, X] = 0, \quad [\partial_s, X] = 0, \quad [\partial_t, \partial_s] = 0.$$

We put $E_i = (e_i, 0, 0)$, $\frac{\partial}{\partial t} = (0, \frac{d}{dt}, 0)$ and $\frac{\partial}{\partial t} = (0, 0, \frac{d}{dt})$. Then, by (1.2) we obtain:

$$\frac{\partial^2}{\partial t \partial s} E(\varphi_{t,s}, D)\big|_{(t,s)=(0,0)} \;=\; \frac{1}{2} \int_D \sum_{i=1}^m \frac{\partial^2}{\partial t \partial s} h(d\phi(E_i), d\phi(E_i)) v_g, \tag{1.7}$$

first, note that:

$$\begin{aligned} \frac{1}{2} \frac{\partial^2}{\partial t \partial s} h(d\phi(E_i), d\phi(E_i) &= \frac{\partial}{\partial t} h(\nabla^\phi_{\frac{\partial}{\partial s}} d\phi(E_i), d\phi(E_i) \\ &= h(\nabla^\phi_{\frac{\partial}{\partial t}} \nabla^\phi_{\frac{\partial}{\partial s}} d\phi(E_i), d\phi(E_i) \\ &+ h(\nabla^\phi_{\frac{\partial}{\partial s}} d\phi(E_i), \nabla^\phi_{\frac{\partial}{\partial t}} d\phi(E_i), \end{aligned} \tag{1.8}$$

the first term on the left-hand side of (1.8) is:

$$\begin{aligned} h(\nabla^\phi_{\frac{\partial}{\partial t}} \nabla^\phi_{\frac{\partial}{\partial s}} d\phi(E_i), d\phi(E_i) &= h(\nabla^\phi_{\frac{\partial}{\partial s}} d\phi(E_i), \nabla^\phi_{\frac{\partial}{\partial t}} d\phi(E_i) \\ &= h(R^N(d\phi(\frac{\partial}{\partial t}), d\phi(E_i)) d\phi(\frac{\partial}{\partial s}), d\phi(E_i)) \\ &+ h(\nabla^\phi_{E_i} \nabla^\phi_{\frac{\partial}{\partial t}} d\phi(\frac{\partial}{\partial s}), d\phi(E_i)) \\ &+ h(\nabla^\phi_{[\frac{\partial}{\partial t}, E_i]} d\phi(\frac{\partial}{\partial s}), d\phi(E_i)). \end{aligned} \tag{1.9}$$

Define an 1-form on M by:

$$w(X) = h\big(\nabla^\phi_{\frac{\partial}{\partial t}} d\phi(\frac{\partial}{\partial s})\big|_{(t,s)=(0,0)}, d\varphi(X)\big), \quad X \in \mathfrak{X}(M).$$

We calculate the divergence of w.

$$\begin{aligned} div^M\omega &= \sum_{i=1}^m \epsilon_i \left\{ e_i(\omega(e_i)) - \omega(\nabla^M_{e_i} e_i) \right\} \\ &= \sum_{i=1}^m \{ e_i(h(\big(\nabla^\phi_{\frac{\partial}{\partial t}} d\phi(\frac{\partial}{\partial s})\big|_{(t,s)=(0,0)}, d\varphi(e_i))) \\ &- h\big(\nabla^\phi_{\frac{\partial}{\partial t}} d\phi(\frac{\partial}{\partial s})\big|_{(t,s)=(0,0)}, d\varphi(\nabla^M_{e_i} e_i))\} \\ &= \sum_{i=1}^m \{ h(\nabla^\phi_{E_i} \nabla^\phi_{\frac{\partial}{\partial t}} d\phi(\frac{\partial}{\partial s})\big|_{(t,s)=(0,0)}, d\varphi(e_i)) \\ &+ h(\big(\nabla^\phi_{\frac{\partial}{\partial t}} d\phi(\frac{\partial}{\partial s})\big|_{(t,s)=(0,0)}, \nabla^\varphi_{e_i} d\varphi(e_i)) \\ &- h\big(\nabla^\phi_{\frac{\partial}{\partial t}} d\phi(\frac{\partial}{\partial s})\big|_{(t,s)=(0,0)}, d\varphi(\nabla^M_{e_i} e_i))\} \\ &= \sum_{i=1}^m \{ h(\nabla^\phi_{E_i} \nabla^\phi_{\frac{\partial}{\partial t}} d\phi(\frac{\partial}{\partial s})\big|_{(t,s)=(0,0)}, d\varphi(e_i)) \\ &+ h\big(\nabla^\phi_{\frac{\partial}{\partial t}} d\phi(\frac{\partial}{\partial s})\big|_{(t,s)=(0,0)}, \tau(\varphi))\}. \end{aligned}$$

According to the harmonicity of φ we obtain:

$$div^M\omega = \sum_{i=1}^m \{ h(\nabla^\phi_{E_i} \nabla^\phi_{\frac{\partial}{\partial t}} d\phi(\frac{\partial}{\partial s})\big|_{(t,s)=(0,0)}, d\varphi(e_i))\}. \tag{1.10}$$

From the formulas (1.9) and (1.10), with $[\frac{\partial}{\partial t}, E_i] = 0$, we get:

$$\begin{aligned} h(\nabla^\phi_{\frac{\partial}{\partial t}} \nabla^\phi_{\frac{\partial}{\partial s}} d\phi(E_i), d\phi(E_i))\big|_{(t,s)=(0,0)} &= \sum_{i=1}^m h(R^N(v, d\phi(e_i))w, d\phi(e_i)) \\ &+ div^M\omega. \end{aligned} \tag{1.11}$$

The second term on the left-hand side of (1.8) is:

$$\begin{aligned} h(\nabla^\phi_{\frac{\partial}{\partial s}} d\phi(E_i), \nabla^\phi_{\frac{\partial}{\partial t}} d\phi(E_i)) &= h(\nabla^\phi_{E_i} d\phi(\frac{\partial}{\partial s}), \nabla^\phi_{E_i} d\phi(\frac{\partial}{\partial t})) \\ &+ E_i\big(h(d\phi(\frac{\partial}{\partial s}), \nabla^\phi_{E_i} d\phi(\frac{\partial}{\partial t}))\big) \\ &- h(d\phi(\frac{\partial}{\partial s}), \nabla^\phi_{E_i} \nabla^\phi_{E_i} d\phi(\frac{\partial}{\partial t})). \end{aligned} \tag{1.12}$$

Define an 1-form on M by:

$$\eta(X) = h(w, \nabla^\varphi_X v), \quad X \in \mathfrak{X}(M).$$

Then:

$$\begin{aligned} div^M\eta &= \sum_{i=1}^{m}\{e_i(\eta(e_i)) - \eta(\nabla^M_{e_i}e_i)\} \\ &= \sum_{i=1}^{m}\left\{e_i(h(w,\nabla^\varphi_{e_i}v)) - h(w,\nabla^\varphi_{\nabla^M_{e_i}e_i}v)\right\}. \end{aligned} \tag{1.13}$$

According to formulas (1.12) and (1.13), we obtain:

$$\begin{aligned} \sum_{i=1}^{m} h(\nabla^\phi_{\frac{\partial}{\partial t}}d\phi(E_i),\nabla^\phi_{\frac{\partial}{\partial s}}d\phi(E_i))\big|_{(t,s)=(0,0)} &= div^M\eta + \sum_{i=1}^{m} h(w,\nabla^\varphi_{\nabla^M_{e_i}e_i}v). \\ &- \sum_{i=1}^{m} h(w,\nabla^\varphi_{e_i}\nabla^\varphi_{e_i}v). \end{aligned} \tag{1.14}$$

From the formulas (1.7), (1.8), (1.11), (1.14) and the divergence theorem, the theorem (6) follows. □

1.7.4 Biharmonic maps

The bi-energy functional of a smooth map $\varphi : (M^m, g) \longrightarrow (N^n, h)$ is defined by:

$$E_2(\varphi, D) = \frac{1}{2}\int_D |\tau(\varphi)|^2\, v_g. \tag{1.15}$$

Definition 58. *A map is called biharmonic if it is a critical point of the bi-energy functional over any compact subset D of M.*

1.7.5 First variation of bi-energy

Theorem 7. *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map between Riemannian manifolds, D a compact subset of M and let $\{\varphi_t\}_{t\in(-\epsilon,\epsilon)}$ be a smooth variation with compact support in D. Then:*

$$\frac{d}{dt}E_2(\varphi_t; D)|_{t=0} = -\int_D h(v,\tau_2(\varphi))v_g,$$

where $v = \frac{d\varphi_t}{dt}|_{t=0}$ denotes the variation vector field of φ and in locale frame at $x \in M$, we have:

$$\begin{aligned} \tau_2(\varphi) &= -\text{trace}_g R^N(\tau(\varphi), d\varphi)d\varphi - \text{trace}_g(\nabla^\varphi)^2\tau(\varphi) \\ &= -\sum_{i=1}^{m} R^N(\tau(\varphi), d\varphi(e_i))d\varphi(e_i) - \sum_{i=1}^{m}\{\nabla^\varphi_{e_i}\nabla^\varphi_{e_i}\tau(\varphi) - \nabla^\varphi_{\nabla^M_{e_i}e_i}\tau(\varphi)\}. \end{aligned} \tag{1.16}$$

$\tau_2(\varphi)$ is called the bi-tension field of φ.

Proof. Define $\phi : M \times (-\epsilon, \epsilon) \longrightarrow N$ by $\phi(x,t) = \varphi_t(x)$.
First note that:

$$\frac{d}{dt}E_2(\varphi_t; D)|_{t=0} = \int_D \sum_{i=1}^{m} h\left(\nabla^{\phi}_{(0,\frac{d}{dt})}\nabla d\phi((e_i,0),(e_i,0)), \nabla d\phi((e_i,0),(e_i,0))\right) v_g\Big|_{t=0}. \quad (1.17)$$

Calculating in a normal frame at $x \in M$ we have:

$$\begin{aligned}
\nabla^{\phi}_{(0,\frac{d}{dt})} d\phi(e_i,0) &= \nabla^{\phi}_{(e_i,0)} d\phi(0,\frac{d}{dt}) + d\phi([(0,\frac{d}{dt}),(e_i,0)]) \\
&= \nabla^{\phi}_{(e_i,0)} d\phi(0,\frac{d}{dt}). && (1.18)\\
\nabla^{\phi}_{(0,\frac{d}{dt})} d\phi(\nabla^M_{e_i}e_i,0) &= \nabla^{\phi}_{(\nabla^M_{e_i}e_i,0)} d\phi(0,\frac{d}{dt}). && (1.19)\\
\nabla^{\phi}_{(0,\frac{d}{dt})}\nabla d\phi((e_i,0),(e_i,0)) &= \nabla^{\phi}_{(0,\frac{d}{dt})}\nabla^{\phi}_{(e_i,0)} d\phi(e_i,0) - \nabla^{\phi}_{(0,\frac{d}{dt})} d\phi\left(\nabla^{M\times(-\epsilon,\epsilon)}_{(e_i,0)}(e_i,0)\right) \\
&= R^N(d\phi(0,\frac{d}{dt}), d\phi(e_i,0))d\phi(e_i,0) + \nabla^{\phi}_{(e_i,0)}\nabla^{\phi}_{(0,\frac{d}{dt})} d\phi(e_i,0) \\
&+ \nabla^{\phi}_{[(0,\frac{d}{dt}),(e_i,0)]} d\phi(e_i,0) - \nabla^{\phi}_{(0,\frac{d}{dt})} d\phi(\nabla^M_{e_i}e_i,0). \\
&= R^N(d\phi(0,\frac{d}{dt}), d\phi(e_i,0))d\phi(e_i,0) \\
&+ \nabla^{\phi}_{(e_i,0)}\nabla^{\phi}_{(e_i,0)} d\phi(0,\frac{d}{dt}) \\
&- \nabla^{\phi}_{(\nabla^M_{e_i}e_i,0)} d\phi(0,\frac{d}{dt}). && (1.20)
\end{aligned}$$

From where:

$$\begin{aligned}
h(\nabla^{\phi}_{(0,\frac{d}{dt})}\nabla d\phi((e_i,0),(e_i,0)),\ \nabla d\phi((e_i,0),(e_i,0)))\Big|_{t=0} &= h(R^N(v, d\varphi(e_i))d\varphi(e_i), \tau(\varphi)) \\
&+ h(\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i} v, \tau(\varphi)) \\
&- h(\nabla^{\varphi}_{\nabla^M_{e_i}e_i} v, \tau(\varphi)). && (1.21)
\end{aligned}$$

Let $\omega \in \mathfrak{X}^*(M)$, be a 1-form to support in D, defined by:

$$\omega(X) = h(\nabla^{\varphi}_X v, \tau(\varphi)), \qquad X \in \mathfrak{X}(M).$$

We calculate the divergence of ω:

$$\begin{aligned}
\mathrm{div}^M \omega &= \sum_{i=1}^{m}\{e_i(\omega(e_i)) - \omega(\nabla^M_{e_i}e_i)\} \\
&= \sum_{i=1}^{m}\{e_i(h(\nabla^{\varphi}_{e_i}v, \tau(\varphi))) - h(\nabla^{\varphi}_{\nabla^M_{e_i}e_i} v, \tau(\varphi))\} \\
&= \sum_{i=1}^{m}\{h(\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}v, \tau(\varphi)) + h(\nabla^{\varphi}_{e_i}v, \nabla^{\varphi}_{e_i}\tau(\varphi)) - h(\nabla^{\varphi}_{\nabla^M_{e_i}e_i} v, \tau(\varphi))\}. && (1.22)
\end{aligned}$$

From the formulas (1.21) and (1.22), we obtain:

$$\begin{aligned}\sum_{i=1}^{m} h(\nabla^{\phi}_{(0,\frac{d}{dt})}\nabla d\phi((e_i,0),(e_i,0)),\ \ \nabla d\phi((e_i,0),(e_i,0)))|_{t=0} &= \sum_{i=1}^{m} h(R^N(v,d\varphi(e_i))d\varphi(e_i),\tau(\varphi)) \\ &- \sum_{i=1}^{m} h(\nabla^{\varphi}_{e_i}v,\nabla^{\varphi}_{e_i}\tau(\varphi)) \\ &+ \operatorname{div}^M\omega. \qquad (1.23)\end{aligned}$$

Let $\eta \in \Gamma(T^*M)$, be an 1-form to support in D, given by:

$$\eta(X) = h(v,\nabla^{\varphi}_{X}\tau(\varphi)), \qquad X \in \mathfrak{X}(M).$$

We calculate the divergence of η:

$$\begin{aligned}\operatorname{div}^M\eta &= \sum_{i=1}^{m}\{e_i(\eta(e_i)) - \eta(\nabla^M_{e_i}e_i)\} \\ &= \sum_{i=1}^{m}\{e_i(h(v,\nabla^{\varphi}_{e_i}\tau(\varphi))) - h(v,\nabla^{\varphi}_{\nabla^M_{e_i}e_i}\tau(\varphi))\} \\ &= \sum_{i=1}^{m}\{h(\nabla^{\varphi}_{e_i}v,\nabla^{\varphi}_{e_i}\tau(\varphi)) + h(v,\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}\tau(\varphi)) - h(v,\nabla^{\varphi}_{\nabla^M_{e_i}e_i}\tau(\varphi))\}. \qquad (1.24)\end{aligned}$$

Substituting (1.24) in (1.23), we obtain:

$$\begin{aligned}\sum_{i=1}^{m} h(\nabla^{\phi}_{(0,\frac{d}{dt})}\nabla d\phi((e_i,0),(e_i,0)),\ \ \nabla d\phi((e_i,0),(e_i,0)))|_{t=0} &= \sum_{i=1}^{m} h(R^N(\tau(\varphi),d\varphi(e_i))d\varphi(e_i),v) \\ &+ \sum_{i=1}^{M} h(v,\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}\tau(\varphi)) \\ &- \sum_{i=1}^{M} h(v,\nabla^{\varphi}_{\nabla^M_{e_i}e_i}\tau(\varphi)) \\ &+ \operatorname{div}^M\omega - \operatorname{div}^M\eta. \qquad (1.25)\end{aligned}$$

From the formulas (1.17), (1.25) and according and if:

$$\int_D \operatorname{div}(\omega)v_g = 0, \qquad (1.26)$$

we obtain:

$$\frac{d}{dt}E_2(\varphi_t;D)|_{t=0} = -\int_D \sum_{i=1}^{m} h\left(-R^N(\tau(\varphi),d\varphi(e_i))d\varphi(e_i) - \nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}\tau(\varphi) + \nabla^{\varphi}_{\nabla^M_{e_i}e_i}\tau(\varphi),v\right)v_g.$$

□

Theorem 8. *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map between two Riemannian manifolds, then φ is said biharmonic if and only if:*

$$\tau_2(\varphi) = -\text{trace}_g R^N(\tau(\varphi), d\varphi)d\varphi - \text{trace}_g(\nabla^\varphi)^2\tau(\varphi) = 0. \tag{1.27}$$

1. The equation (1.27) is called the Euler-Lagrange equation.
2. Let M and N be two Riemanian manifolds with the coordinates $(x^i)$and (y^α) respectively, then, in the neighborhood of the points $x \in M$ and $\varphi(x) \in N$ we have

$$\begin{aligned}\tau_2(\varphi) &= g^{ij}\{\frac{\partial^2\tau^\sigma}{\partial x^i\partial x^j} + 2\frac{\partial\tau^\sigma\partial\tau^\beta}{\partial x^j\partial x^j}{}^N\Gamma^\sigma_{\alpha\beta} + \tau^\alpha\frac{\partial^2\varphi^\beta}{\partial x^i\partial x^j}{}^N\Gamma^\sigma_{\alpha\beta}\\ &+\tau^\alpha\frac{\partial\varphi^\beta}{\partial x^i}\frac{\partial^N\Gamma^\sigma_{\alpha\beta}}{\partial x^j} + \tau^\alpha\frac{\partial\varphi^\beta}{\partial x^i}\frac{\partial\varphi^\rho}{\partial x^j}{}^N\Gamma^\upsilon_{\alpha\beta}{}^N\Gamma^\sigma_{\upsilon\rho}\\ &-{}^M\Gamma^k_{ij}(\frac{\partial\tau^\sigma}{\partial x^k} + \tau^\alpha\frac{\partial\varphi^\beta}{\partial x^k}{}^N\Gamma^\sigma_{\alpha\beta}) - \tau^\upsilon\frac{\partial\varphi^\alpha}{\partial x^i}\frac{\partial\varphi^\beta}{\partial x^j}{}^N R^\sigma_{\beta\alpha\upsilon}\}\frac{\partial}{\partial y^\sigma}\circ\varphi,\end{aligned}$$

   where $\tau^\gamma = g^{ij}\left(\frac{\partial^2\varphi^\gamma}{\partial x^i\partial x^j} + \frac{\partial\varphi^\alpha}{\partial x^i}\frac{\partial\varphi^\beta}{\partial x^j}{}^N\Gamma^\gamma_{\alpha\beta}\circ\varphi - \frac{\partial\varphi^\gamma}{\partial x^k}{}^M\Gamma^k_{ij}\right)$ and ${}^N R^\sigma_{\beta\alpha\upsilon}$ designate the components of the curvature tensor of (N^n, h).
3. Any harmonic map is a biharmonic.
4. Biharomnic maps are not generally harmonic maps.

Example 22.

1. The polynomials of degrees 3 on $\mathbb{R}$ are biharmonic non-harmonic maps.
2. The identity map $Id : (M^m, g) \longrightarrow (M^m, g)$ is biharmonic.
3. A smooth map $\varphi : (M^m, g) \longrightarrow (\mathbb{R}^n, <, >_{\mathbb{R}^n})$, is biharmonic if and only if $\Delta^M(\Delta^M\varphi^\sigma) = 0$, for all $\sigma = 1, \ldots, n$.

Example 23. *Let $f : \mathbb{R}^n \to \mathbb{R}$ a harmonic function (i.e. $\Delta(f) = 0$). Then the function $\varphi(x) = r^2(x)f(x)$ is a biharmonic function non-harmonic, where $r(x) = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$ for every $x = \{x_1, x_2, ..., x_n\} \in \mathbb{R}^n$ is the distance function.*
In effect:

$$\begin{aligned} r^2(x) &= x_1^2 + x_2^2 + ... + x_n^2 = \sum_{j=1}^n x_j^2\\ \frac{\partial r^2}{\partial x_i} &= \sum_{j=1}^n \frac{\partial x_j^2}{\partial x_i} = 2x_i\\ \frac{\partial\varphi}{\partial x_i} &= \frac{\partial r^2}{\partial x_i}f + r^2\frac{\partial f}{\partial x_i} = 2x_i f + r^2\frac{\partial f}{\partial x_i}\\ \frac{\partial\varphi^2}{\partial x_i^2} &= 2f + 2x_i\frac{\partial f}{\partial x_i} + 2x_i\frac{\partial f}{\partial x_i} + r^2\frac{\partial^2 f}{\partial^2 x_i}.\end{aligned}$$

Then the Laplacian of a function φ is given by:

$$\begin{aligned}\Delta\varphi &= \sum_{i=1}^{n}\frac{\partial^2\varphi}{\partial^2 x_i} = \sum_{i=1}^{n}2f + \sum_{i=1}^{n}4x_i\frac{\partial f}{\partial x_i} + \sum_{i=1}^{n}r^2\frac{\partial^2 f}{\partial^2 x_i}\\ &= 2nf + 4\sum_{i=1}^{n}x_i\frac{\partial f}{\partial x_i} + r^2\underbrace{\Delta f}_{=0} \neq 0.\end{aligned}$$

Thus φ is non-harmonic. For j fixed, we have:

$$\begin{aligned}\frac{\partial\Delta\varphi}{\partial x_j} &= 2n\frac{\partial f}{\partial x_j} + 4\sum_{i=1}^{n}\underbrace{\frac{\partial x_i}{\partial x_j}}_{=\delta_{ij}}\frac{\partial f}{\partial x_i} + 4\sum_{i=1}^{n}x_i\frac{\partial^2 f}{\partial x_j\partial x_i}\\ &= 2n\frac{\partial f}{\partial x_j} + 4\frac{\partial f}{\partial x_j} + 4\sum_{i=1}^{n}x_i\frac{\partial^2 f}{\partial x_j\partial x_i}.\end{aligned}$$

Therefore:

$$\frac{\partial^2\Delta\varphi}{\partial x_j^2} = 2n\frac{\partial^2 f}{\partial x_j^2} + 4\frac{\partial^2 f}{\partial x_j^2} + 4\sum_{i=1}^{n}\underbrace{\frac{\partial x_i}{\partial x_j}}_{=\delta_{ij}}\frac{\partial^2 f}{\partial x_i\partial x_j} + 4\sum_{i=1}^{n}x_i\frac{\partial^3 f}{\partial x_i\partial x_j^2},$$

$$\tau_2(f) = -\Delta(\Delta\varphi) = -\sum_{j=1}^{n}\frac{\partial^2\Delta\varphi}{\partial x_j^2} = -(2n+4)\Delta f - 4\Delta f - 4\sum_{i=1}^{n}x_i\frac{\partial}{\partial x_i}(\Delta f) = 0.$$

Example 24. *The inversion $\varphi : \mathbb{R}^n - \{0\} \to \mathbb{R}^n - \{0\}$, $x \mapsto \varphi(x) = \frac{x}{\|x\|^2}$ is a biharmonic application non-harmonic if and only if $n = 4$.*
In effect. We pose $\varphi^\alpha(x) = \frac{x_\alpha}{\|x\|^2}$ for every $\alpha \in \{1, 2, ..., n\}$. For i fixed, we have:

$$\begin{aligned}\frac{\partial\varphi^n}{\partial x_i} &= \delta_{i\alpha}\|x\|^{-2} - 2x_\alpha x_i\|x\|^{-4}\\ \frac{\partial^2\varphi^n}{\partial x_i^2} &= -4\delta_{i\alpha}\|x\|^{-2} - 2x_\alpha\|x\|^{-4} + 8x_\alpha x_i^2\|x\|^{-6}.\end{aligned}$$

Then the Laplacian of a function φ^α is given by:

$$\begin{aligned}\Delta\varphi^n &= -4\sum_{i=1}^{n}\delta_{i\alpha}\|x\|^{-2} - 2\sum_{i=1}^{n}x_\alpha\|x\|^{-4} + 8\sum_{i=1}^{n}x_\alpha x_i^2\|x\|^{-6}\\ &= 4x_\alpha\|x\|^{-4} - 2nx_\alpha\|x\|^{-4}\\ &= 2(2-n)x_\alpha\|x\|^{-4}.\end{aligned}$$

Thus $\tau(\varphi) = 2(2-n)\|x\|^{-4}x$. From the same technique, we obtain:

$$\tau_2(\varphi) = -8(2-n)(4-n)\|x\|^{-6}x.$$

Therefore, φ is biharmonic non-harmonic if and only if n=4.

Chapter 2

Thurston Geometry

In this chapter we introduce a Thurston model geometry (G, X). Three-dimensional Thurston model geometries are classified by W. Thurston, this classification has eight geometries, to know, E^3, S^3, H^3, $S^2 \times \mathbb{R}$, $H^2 \times \mathbb{R}$, $\widetilde{Sl_2(\mathbb{R})}$, Nil_3 and Sol_3.
R. Filipkiewicz classified the Thurston geometry of dimension four. In this classification we distinguish two categories of spaces, those which are symmetrical, to know, E^4, S^4, H^4, $P^2(\mathbb{C})$, $H^2(\mathbb{C})$, $S^2 \times S^2$, $S^2 \times E^2$, $S^2 \times H^2$, $H^2 \times E^2$, $H^2 \times H^2$, $H^3 \times E^1$ and $H^3 \times E^1$ and those that are not symmetrical, to know, Nil^4, $Sol^4_{m,n}$, Sol^4_0, Sol^4_1, F^4, $\widetilde{Sl_2(\mathbb{R})} \times E^1$ and $Nil_3 \times E^1$. [50], [65], [66], [37], [61], [63], [64], [27] and [49].

Definition 59. *Let (M, g) and (N, h) be Riemannian manifolds. An isometry is a diffeomorphism $f : M \longrightarrow N$ such that $g = f^{-1}h$ where $g = f^{-1}h$ denotes the pullback of the metric tensor h by f. If f is a local diffeomorphism then f is local isometry. We say that M and N are isometric, $M \simeq N$, if there is such isometry between them. The set of isometries from M to itself forms a group under composition, and is denoted $Isom(M)$.*

Definition 60. *Let G be a group and M a set. A left action of G on M is a map:*

$$\begin{array}{rcl} G \times M & \longrightarrow & M \\ (g, m) & \longrightarrow & g.m, \end{array}$$

such that $g_1.(g_2.m) = (g_1.g_2).m$ for all $g_1, g_2 \in M$ and $m \in M$, and $e.m = m$ for e the identity element of G and all $m \in M$. A right group action can be defined similarly.

Definition 61. *Given a set with a left action of a group G and $m \in M$, the orbit of m under the action G is the set $orb(m) = \{g.m \mid g \in G\}$, that is, the set of all images of m under the action of elements of the group G.*

Definition 62. *An action $G \times M \to M$ of a group G on a set M is called transitive if it has a single orbit, i.e. for any two elements $m, n \in M$, there exist $g \in G$ such that $n = g.m$.*

Example 25. *The modular group $PSL_2(\mathbb{Z})$ acts transitively on the rational projective line $P^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$. The projective general linear group $PGL_2(\mathbb{C})$ acts 3-transitively on the Riemann sphere $P^1(\mathbb{C})$.*

Definition 63. *Given an action $G \times M \to M$ of a group G on a set M, for every element $m \in M$, the stabilizer subgroup of m (also called the isotropy group of m) is the set of all elements in G that leave m fixed:*

$$stab(m) = \{g \in G \mid g.m = m\}.$$

Definition 64. *A Lie group is a group with a differentiable manifold structure compatible with its group structure, i.e. the map:*

$$\begin{array}{rcl} \phi : G \times G & \longrightarrow & G \\ (g,h) & \longmapsto & g.h^{-1}, \end{array}$$

is differentiable.

Definition 65. *Let H and N be groups and let $Aut(N)$ the automorphism group of N for the law $\circ$. The direct product $N \times H$ of N and H is the group whose underlying set is the product set $N \times H$, with the law $(n_1, h_1)(n_2, h_2) = (n_1 n_2, h_1 h_2)$ for all $n_1, n_2 \in N$ and $h_1, h_2 \in H$.*
The semi-direct product is a generalization of this notion. Let $\phi : H \to AutN$ a group morphism which in particular defines an action $h_1.n_1 = \phi(h)(n)$ of N on H.

Proposition 19. *We define a group law on the product set $N \times H$ in posing:*

$$(n_1.h_1).(n_2.h_2) = (n_1(h_1.n_2), h_1 h_2).$$

This group is called the semi-direct product of N by H relative to the action ϕ, it is denoted $H \rtimes_\phi N$ (or simply $H \rtimes N$).

2.1 Thurston Geometries of dimension 3

Definition 66. *Let X be one of E^2, S^2 or H^2, where E^2 is Euclidean two-space, S^2 is the two-sphere and H^2 is hyperbolic two-space. Let Γ be a subgroup of $Isom(X)$. If F is a two-manifold such that $F \simeq X/\Gamma$ and the projection $X \to X/\Gamma$ is a covering map, we say that F has a geometric structure modeled on X.*

Definition 67. *A geometry is a simply connected, complete, homogeneous Riemannian manifold X together with its isometry group. A manifold M has a geometric structure modeled on X if $M \simeq X/\Gamma$ where Γ is a subgroup of the isometry group of X and $\simeq$ indicate that M is isometric to X/Γ.*

Definition 68. *Two geometries (X_1, G_1) and (X_2, G_2) are equivalent if G_1 is isomorphic to G_2 and there exist a map $\varphi : X_1 \to X_2$. That is, $\phi(g_1.x) = g_2.\phi(x)$ where g_2 is the isomorphic image of g_1 in G_2.*
A geometry (X, G_1) is maximal if there is no geometry (X, G_2) with $G_1 \subsetneqq G_2$.
This definition is needed to identify all possible geometric structures in a given dimension. Thurston has identified the eight geometric structures in dimension three.

2.1.1 Model geometries

Definition 69. *A model geometry (G, X) is a manifold X together with a Lie group G of diffeomorphisms of X such that:*

1. *X is connected and simply connected.*
2. *G acts transitively on X, with compact point stabilizers.*
3. *G is not contained in any larger group of diffeomorphisms of X with compact point stabilizers of points, and*
4. *there exists at least one compact manifold modeled (G, X)-manifold.*

Theorem 9. *The only Thurston model geometry of dimension 1 is $E^1 \simeq \mathcal{E}(1)/O(1)$.*

Theorem 10. *[64] page:(181) (two-dimensional model geometries). there are precisely three two-dimensional model geometries: the Euclidean two-space $E^2 \simeq \mathcal{E}(2)/O(2)$, The two-sphere $S^2 \simeq O(2)/O(2)$ and the hyperbolic two-space $H^2 = PGL_2(\mathbb{R})/PO(2)$.*

Proof. Since G acts transitively on X, it follows that any G-invariant Riemannian metric on X has constant Gaussian curvature. When a metric is multiplied by a constant k, the Gaussian curvature is multiplied by k^2, so we can find a metric whose curvature is either 0, 1 or -1. It is a standard fact from Riemannian geometry that the only simply connected complete Riemannian n-manifolds with constant sectional curvature 0, 1 and -1 are E^n, S^n and H^n. □

In enumerating three-dimensional model geometries (G, X), we will first look at the connected component of the identity of G call it G'. The action of G' is still transitive, and the stabilizers G'_x of points $x \in X$ are connected. This is because the quotients $G'_x/(G'_x)_o$, where $(G'_x)_o$ is the component of the identity of G'_x, from a covering space of X. Since X is simply connected, the covering is trivial.
Therefore G'_x is a connected closed subgroup of $SO(3)$. Using the fact that a closed subgroup of a Lie group is also a Lie group, and therefore a manifold, it is easy to see that there are only three possibilities: $SO(3)$, $SO(2)$ and the trivial group. The stabilizer G_x is a Lie group of the same dimension.

Theorem 11. *[64] page:(181-184) There are eight three-dimensional model geometries (G, X), as follows:*

1. *If the point stabilizers are three-dimensional, X is S^3, E^3 or H^3.*
2. *If the point stabilizers are one-dimensional, X fibers over one of the two-dimensional model geometries, in a way that is invariant under G. There a G-invariant Riemannian metric on X such that the connection orthogonal to the fibers has curvature 0 or 1.*
   (a) If the curvature is zero, X is $S^2 \times E^1$ or $H^2 \times E^1$.
   (b) If the curvature is one, we have nilgeometry (which fibers over E^2) or the geometry of $\widetilde{Sl_2(\mathbb{R})}$ (which fibers over H^2).

3. *The only geometry with zero-dimensional stabilizers is solvegeometry, which fibers over the line.*

The geometry in (1) we have already discussed extensively, and those in (a) are self-explanatory. The remanning ones will be described in more detail in the course of the proof. We start by giving X a G-invariant Riemannian metric.

Proof. For (1). If G' acts with stabilizer $SO(2)$, there is a non-zero, G'-invariant vector field V on X whose direction at each point gives the axis of rotation of the elements of G' that fix that point. The trajectories of V form a G'-invariant one-dimensional foliation $\mathcal{F}$. Also, the flow of V call it ϕ_t at time t commutes with the action of G', so if an element of G' fixes some points on a leaf F of $\mathcal{F}$, it fixes any other point on F: all points on the same leaf have the same stabilizer. This also implies that if an element of G' takes a point $x \in F$ to another point $y \in F$, it commutes with any element of the stabilizer $G'_x = G'_y$.
Now fix a leaf F and a point $x \in F$, and let g_t be an element of $\mathcal{G}$ taking $\phi_t(x)$ back to x. Then $g_t \circ \phi_t$ fixes x, and its derivative at x is a linear automorphism of T_xM. The derivative is the identity along the axis of the action of G'_x. It commutes with rotations around this axis, that is, with elements of G'_x. Then it must be itself a rotation around this axis, possibly composed with an expansion or contraction. But an expansion or contraction is ruled out, because the assumption that there is a compact manifold modeled on (G, X) implies that V must preserve volume.
The divergence of a vector field V on a manifold X with a volume form w is a measure of how much V expands or contracts volume. More precisely, $divV$ is a Lie derivative $L_V w$, expressed in units of w.
Now suppose that X is a manifold on which is Lie group G acts transitively, and that V and w are a vector field and a volume form on X, both invariant under G. Show that $divV$ is constant over X.
In the situation of the proof, if there is a compact manifold modeled on (G, X), this manifold inherits the vector field and the volume form from X. The vector field must preserve the total volume, and so much preserves volume at every point. Therefore V has divergence zero. Show that this implies that $g_t \circ \phi_t$ acts as a rotation on T_xM.
We conclude that the derivative of ϕ_t maps T_xM to $T_{\phi_t(x)}M$ isometrically. Since x was arbitrary, the flow of the vector field V is by isometries.
By considering a neighborhood of a point on a leaf and the fact that the leaf is invariant under a invariant under a subgroup G'_x isomorphic to $SO(2)$, we conclude that the leaf does not accumulate on itself, but is an embedded image of either S^1 or $\mathbb{R}$. In fact, it is easy to see that distinct leaves have disjoint neighborhoods. Therefore the quotient space $X/\mathcal{F}$ is a two-dimensional manifold Y. Since V acts by isometries, Y inherits a Riemannian metric from X, and a transitive action of G' by isometries. Also, Y is connected and simply connected because X is. By proof of Theorem 1, Y must be one of the two-dimensional model geometries: E^3, S^2 or H^2. In addition, X is a principal fiber bundle over Y, with fiber and structure group equal to S^1 or $\mathbb{R}$.
The plane field $\mathcal{T}$ orthogonal to $\mathcal{F}$ is a connection for this bundle. Since the group of isometries of X acts transitively, $\mathcal{T}$ has constant curvature.
(a) If the curvature is zero, $\mathcal{T}$ defines a foliation. Since Y is simply connected, the bundle is

trivial. There are three possibilities, depending on Y (an open circle indicates that no new geometry arises possibility):

- If $Y = S^2$, we obtain the model geometry $S^2 \times E^1$. As a compact manifold modeled on this geometry, we can take $S^2 \times S^1$.
- If $Y = E^2$, then $X = E^2 \times E^1 = E^3$. Thus G' (and hence G) is contained in a bigger group of isometries, and we don't get a new model geometry.
- If $Y = H^2$, we obtain the model geometry $H^2 \times E^1$. Any compact hyperbolic surface cross a circle is an example.

In each of these two geometries, the full group of isometries G contains G' with index 4, since we can reverse the orientation of either factor independently.
(b) If the curvature of $\mathcal{T}$ is non-zero, $\mathcal{T}$ defines a contact structure. After rescaling our metric in the direction of the fibers and choosing appropriate orientations for the base and the fiber, we can assume that the curvature is 1, expressed in terms of the standard bases for $\bigwedge^2 TY$ and TF. This, together with the condition that X is simply connected, essentially determines the geometry. If Y has no-zero curvature, X can be taken as the tangent circle bundle of Y (or rather, its universal cover) with the Levi-Civita connection. The group is made of derivatives of isometries of Y, together with rotation of unit tangent vectors keeping the base point fixed.

- If $Y = S^2$, the tangent circle bundle is $SO(3)$, whose universal cover is S^3. For G, we get the group of isometries of S^3 that preserve the Hopf fibration. This is not a maximal group acting with compact stabilizers, so it not a model geometry.
- If $Y = E^2$, we obtain nilgeometry. This can be defined in terms of our model contact structure $\mathcal{T}$ as the group of contact automorphisms that are lifts of isometries of the xy-plane.
- If $Y = H^2$, the unit tangent bundle is $PSL_2(\mathbb{R})$, the group of orientation preserving isometries of H^2. Passing to the universal cover, we get $X = \widetilde{Sl_2(\mathbb{R})}$. The unit tangent bundle of a compact hyperbolic surface is an example of a three-manifold with this geometry.

For $\widetilde{Sl_2(\mathbb{R})}$ and nilgeometry, the contact structure determines an orientation of the geometry which cannot be reversed. However the orientation of the base two-dimensional geometry can be reversed simultaneously with the orientation of the fiber, so the index of G' in G is 2. □

2.2 The Eight Geometries

2.2.1 E^3

Euclidean 3-space, E^3, is the space $\mathbb{R}^3$ with the metric $ds^2 = dx^2 + dy^2 + dz^2$. As in E^2 any isometry of E^3 can be written as $x \to Ax + b$, but now A is a real orthogonal 3×3 matrix and b is a translation vector in $\mathbb{R}^3$. Thus there is a group homeomorphism $\phi : Isom(E^3) \to O(3)$, and the kernel of ϕ is the translation subgroup of $Isom(E^3)$. In the case where $n = 3$, it can be show that the translation subgroup of G is of finite index in G, or G is a finite extension of $\mathbb{Z}$, where $\mathbb{Z} \cong \{g : g\, is\, a\, translation\}$.

2.2.2 S^3

The spherical geometry is the three-sphere and its isometry group. S^3 can be embedded in $\mathbb{R}^4$ and thus the metric on S^3 is the one induced from $\mathbb{R}^4$, that is, $ds^2 = dx^2 + dy^2 + dz^2 + dw^2$. The isometry group of S^3 is $O(3)$, the group of orthogonal 3×3 matrices. It is interesting to note that any orientation reversing isometry of S^3 has a fixed point, which limits the discrete subgroups of $Isom(S^3)$ to subgroups of $SO(3)$.

2.2.3 H^3

The basic properties of hyperbolic space H^3 can be developed exactly along the lines which I used for H^2 in Chapter 1 [61]. One starts with upper half 3-space $\mathbb{R}^3_+ = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}$ and the metric $ds^2 = \frac{1}{z^2}(dx^2 + dy^2 + dz^2)$ to assign a length to any smooth path in $\mathbb{R}^3_+$ and hence define a metric on $\mathbb{R}^3_+$. One checks that vertical straight lines are geodesics in this new metric. One also checks that inversion of $\mathbb{R}^3$ in sphere with center on the xy-plane defines an isometry of H^3. Now one can show that the geodesics of H^3 are exactly the vertical straight lines and arcs of circles which meet the xy-plane orthogonally. One can also show that the full isometry group of H^3 is generated by reflections, which are simply the above inversions (including the reflections in vertical planes). Clearly an isometry of H^3 is determined by its restriction to the "2-sphere at infinity" consisting of $\mathbb{C} \cup \{\infty\}$, where we identify the xy-plane with $\mathbb{C}$. The group of orientation preserving isometries of H^3 can be identified with the group of Möebius transformation of $\mathbb{C} \cup \{\infty\}$. Recall that a Möebius transformation of $\mathbb{C} \cup \{\infty\}$ is a map of the form $z \to \frac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. The group of these transformation is naturally isomorphic to $PSL_2(\mathbb{C})$. We identify the point $\{x, y, z\}$ of $\mathbb{R}^3_+$ with the quaternion $x + yi + zj$. The 2×2 complex matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts on $\mathbb{R}^3_+$, extending its natural action on $\mathbb{C} \cup \{\infty\}$, by the formula $w \to (aw + b)(cw + d)^{-1}$, where w is a quaternion of the form $x + yi + zj$, $z > 0$. One can check that this yields all orientation preserving isometries of H^3.

2.2.4 $S^2 \times \mathbb{R}$

The space $S^2 \times \mathbb{R}$ is precisely the Cartesian cross product of the unit two-sphere and the real line with the product metric. The isometry group of $S^2 \times \mathbb{R}$ is identified with the product of $Isom(S^2)$ and $Isom(\mathbb{R})$. That is, $Isom(S^2 \times \mathbb{R}) \cong Isom(S^2) \times Isom(\mathbb{R})$. This geometry is relatively simple. In fact, there are exactly seven manifolds without boundary which have a geometric structure modeled on $S^2 \times \mathbb{R}$.

2.2.5 $H^2 \times \mathbb{R}$

The space $H^2 \times \mathbb{R}$ is the Cartesian cross product of hyperbolic two-space and the real line with the product metric. It has isometry group $Isom(H^2 \times \mathbb{R}) \cong Isom(H^2) \times Isom(\mathbb{R})$. There are infinitely many manifolds with a geometric structure modeled on $H^2 \times \mathbb{R}$. Let H^2 be represented by the upper half-plane model $\{(x, y) \in \mathbb{R}^2 \mid y > 0\}$, therefore the space $H^2 \times \mathbb{R}$ is a Lie group with respect to the operation $(x, y, t)(x', y', z') = (x'y + x, yy', z + z')$ and the left-invariant metric:

$$ds^2 = \frac{1}{y^2}(dx^2 + dy^2) + dz^2.$$

2.2.6 $\widetilde{Sl_2(\mathbb{R})}$

The 3-dimensional Lie group of all 2×2 real matrices with determinant 1 is denoted $Sl_2(\mathbb{R})$ and $\widetilde{Sl_2(\mathbb{R})}$ denotes its universal covering. The unit tangent bundle of H^2 can be identified with $PSL_2(\mathbb{R})$, which is covered by $Sl_2(\mathbb{R})$. The metric on H^2 can then be pulled back to induce a metric on $\widetilde{Sl_2(\mathbb{R})}$. It is well-know that $\widetilde{Sl_2(\mathbb{R})}$ can, as a Riemannian manifold, be modeled as $\mathbb{R}^3$ equipped with the following metric:

$$ds^2 = \frac{1}{y^2}(dx^2 + dy^2) + (dz + \frac{dx}{y})^2.$$

2.2.7 Nil_3

The space Nil_3 on Thurston's list can be presented as the 3-dimensional nilpotent Lie subgroup

$$Nil_3 = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \mid x, y, z \in \mathbb{R} \right\}$$

of $SL_3(\mathbb{R})$ equipped with its standard left-invariant Riemannian metric. The restriction of this metric to Nil^3 is determined by orthonormal basis $\{X, Y, Z\}$ of its Lie algebra $\mathfrak{nil}$ given by:

$$X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \ Z = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It is well-know that in the global coordinate $\{x, y, z\}$ on Nil^3 the left-invariant Riemannian metric satisfies:

$$ds^2 = dx^2 + dy^2 + (dz + xdy)^2.$$

This geometry is called Nil_3 because the Lie group is nilpotent.

2.2.8 Sol_3

The model space Sol_3 on Thurston's list can be seen as the 3-dimensional solvable Lie group

$$Sol_3 = \left\{ \begin{bmatrix} e^z & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{bmatrix} \mid x, y, z \in \mathbb{R} \right\}$$

of $SL_3(\mathbb{R})$. The metric on Sol_3 is determined by orthonormal basis $\{X, Y, Z\}$ of its Lie algebra $\mathfrak{sol}_3$ given by:

$$X = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We can identify Sol_3 with $\mathbb{R}^3$ with the multiplication given by $(x, y, t)(x', y', z') = (x + e^{-z}x', y + e^z y', z + z')$. In the global coordinates $\{x, y, z\}$ on Sol^3 this takes the following well-known metric form:

$$ds^2 = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2.$$

This group is called Sol_3 because it is a solvable group.

Remark 15. *Except Sol_3 all models Thurston admit canonical normal almost contact metric structures.*

2.3 Thurston Geometries of dimension 4

The definition of model geometries of dimension 4 is the same for the dimension 3, except that in condition 4 of definition (69) the word compact is replaced by finite volume.
For a geometry (X, G) and any point x of the n-manifold X, the stabilizer G_x of the transitive, effective and isometric G-action on X is compact. In fact, G_x is isomorphic to a closed subgroup of $O(n)$ since the action is isometric. By classifying all the possible closed subgroups of $SO(4)$, R. Filipkiewicz proved that there are 19 classes of maximal geometries in 4-dimension.
Table. Nineteen classes of 4-dimensional geometries of Filipkiewicz:

type	geometries
solvable type	E^4, $Nil_3 \times E^1$, Nil^4, $\underbrace{Sol^4_{m,n}}$, Sol^4_0, Sol^4_1
product type	$H^2 \times S^2$, $S^3 \times E^1$, $S^2 \times E^2$, $H^3 \times E^1$, $Sl_2(\mathbb{R}) \times E^1$, $H^2 \times E^2$, $H^2 \times H^2$
hyperbolic	H^4, $H^2(\mathbb{C})$
finite group	S^4, $P^2(\mathbb{C})$, $S^2 \times S^2$
$\mathscr{T}H^2$	F^4

Suppose M is closed 4-manifold with $\chi(M) = 0$, then [37] M can be seen as a quotient X/Γ, where X is a 1-connected solvable Lie group and Γ is a closed torsion-free subgroup of $X \rtimes Aut(X)$.

2.3.1 Nil^4

The geometry Nil^4 is the semi-direct product $\mathbb{R}^3 \rtimes_\theta \mathbb{R}$ where $\theta = [t, t, \frac{t^2}{2}]$ and the component of its isometry group with identity is Nil^4 itself as left translation. It has abelianization $\mathbb{R}^2$ and central series $\zeta Nil^4 \cong \mathbb{R} < \zeta_2 Nil^4 = Nil^{4'} \cong \mathbb{R}^2$.
These Lie groups have natural left invariant metrics, and the isometry groups are generated by left translations and the stabilizer of the identity. For Nil^3 this stabilizer is $O(2)$, and $Isom(Nil^3)$ is an extension of $\mathcal{E}(2)$ by $\mathbb{R}$. Hence $Isom(Nil_3 \times E^1) = Isom(Nil_3) \times \mathcal{E}(1)$. For Nil^4 the stabilizer is $(\mathbb{Z}/2\mathbb{Z})^2$, and is generated by two involution, which send $((x, y, z), t)$ to $(-(x, y, z), t)$ and $((-x, y, z), -t)$, respectively.

2.3.2 $Sol^4_{m,n}$

The geometry $Sol^4_{m,n}$ represents the semi-direct product $\mathbb{R}^3 \rtimes_{\theta_{m,n}} \mathbb{R}$, where m and n are integers such that the polynomial $f_{m,n} = X^3 - mX^2 + nX - 1$ has distinct roots e^a, e^b and e^c (with $a < b < c$ real) and $\theta_{m,n}(t)$ is the diagonal matrix $\mathrm{diag}[e^{at}, e^{bt}, e^{ct}]$. Since $\theta_{m,n}(t) = \theta_{n,m}(-t)$ we may assume that $m \le n$, the condition on the roots then holds if and only if $2\sqrt{n} \le m \le n$. The metric is given by $ds^2 = e^{-2at}dx^2 + e^{-2bt}dy^2 + e^{-2ct}dz^2 + dt^2$ (in the obvious global coordinates) is left invariant, and the automorphism of $Sol^4_{m,n}$ which sends (x, y, z, t) to (px, qy, rz, t) is an isometry if and only if $p^2 = q^2 = r^2 = 1$. Let G be a subgroup of $GL(4, \mathbb{R})$ of bordered matrices $\begin{pmatrix} D & \xi \\ 0 & 1 \end{pmatrix}$, where $D = diag[\pm e^{at}, \pm e^{bt}, \pm e^{ct}]$ and $\xi \in \mathbb{R}^3$. Then $Sol^4_{m,n}$ is a subgroup of G with positive diagonal entries, and $G = Isom(Sol^4_{m,n})$ if $m \ne n$. If $m = n$ then $b = 0$ and $Sol^4_{m,n} = Sol^3 \times E^1$, which admits the additional isometry sending (x, y, z, t) to $(x, y, z, -t)$, and G has index 2 in $Isom(Sol^3 \times E^1)$. The stabilizer of the identity in the full isometry group is $(\mathbb{Z}/2\mathbb{Z})^3$ for $Sol^4_{m,n}$ if $m \ne n$ and $D_8 \times (\mathbb{Z}/2\mathbb{Z})$ for $Sol^3 \times \mathbb{R}$. In all cases $Isom(Sol^4_{m,n}) \le (Sol^4_{m,n}) \rtimes Aut(Sol^4_{m,n})$. In general $Sol^4_{m,n} = Sol^4_{m',n'}$ if and only if $(a, b, c) = \lambda(a', b', c')$ for some $\lambda \ne 0$.

2.3.3 Sol^4_0

The geometry Sol^4_0 is the semi-direct product $\mathbb{R}^3 \rtimes_\xi \mathbb{R}$, where $\xi(t)$ denotes the diagonal matrix $\mathrm{diag}\,[e^t, e^t, e^{-2t}]$. Note that if $\xi(t)$ preserves a lattice in $\mathbb{R}^3$ then its characteristic polynomial has integral coefficients and constant term -1. Since it has e^t as a repeated root we must have $\xi(t) = I$. Therefore Sol^4_0 does not admits any lattices. The metric given by the expression $ds^2 = e^{-2t}(dx^2 + dy^2) + e^{4t}dz^2 + dt^2$ is left invariant, and $O(2) \times O(1)$ acts via rotations and reflections in the (x, y)-coordinates and reflection in the z-coordinate, to give the stabilizer of the identity. These actions are automorphisms of Sol^4_0, so $Isom(Sol^4_0) = Sol^4_0 \rtimes (O(2) \times O(1)) \le (Sol^4_0) \rtimes Aut(Sol^4_0)$. The identity component of $Isom(Sol^4_0)$ is not triangular.

2.3.4 Sol^4_1

The Sol^4_1 geometry is the group of real matrices: $\left\{ \begin{pmatrix} 1 & x & z \\ 0 & t & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z, t \in \mathbb{R},\ t > 0 \right\}$.

The metric given by $ds^2 = t^{-2}((1+x^2)(dt^2+dy^2) + t^2(dx^2+dz^2) - 2tx(dtdx+dydz))$ is left invariant, and the stabilizer of the identity is D_8, generated by the isometries which send (t,x,y,z) to $(t,-x,y,-z)$ and to $t^{-1}(1,-y,-x,xy-tz)$. These are automorphisms. (The latter one is the restriction of the involution Ω of $GL(3,\mathbb{R})$ which sends A to $J(A^{tr})^{-1}J$, where J reverses the order of the standard basis of $\mathbb{R}^3$.) Thus $Isom(Sol_1^4) \cong Sol_1^4 \rtimes D_8 \leq (Sol_1^4) \rtimes Aut(Sol_1^4)$. The orientation-preserving subgroup is isomorphic to the subgroup $\mathfrak{G}$ of $GL(3,\mathbb{R})$ generated by Sol_1^4 and the diagonal matrices $diag[-1,1,1]$ and $diag[1,1,-1]$. (Note that these diagonal matrices act by conjugation on Sol_1^4.)

2.3.5 F^4

The geometry F^4 is the tangent bundle $\mathscr{T}H^2$ of the hyperbolic plane H^2, which we may identify with $\mathbb{R}^2 \times H^2$. Its isometry group is the semidirect product $\mathbb{R}^2 \rtimes_\alpha Sl_2^{\pm}(\mathbb{R})$, where $Sl_2^{\pm}(\mathbb{R}) = \{A \in GL(2,\mathbb{R}) \mid detA = \pm 1\}$, and α is the natural action of $Sl_2^{\pm}(\mathbb{R})$ on $\mathbb{R}^2$. The identity component acts on $\mathbb{R}^2 \times H^2$ as follows: if $u \in \mathbb{R}^2$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sl_2(\mathbb{R})$ then $u(w,z) = (u+w,z)$ and $A(w,z) = (Aw, \frac{az+b}{cz+d})$ for all $(w,z) \in \mathbb{R}^2 \times H^2$. The matrix $D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ acts via $D(w,z) = (Dw, -\bar{z})$. All $\mathbb{H}^2(\mathbb{C})$ and F^4-manifolds are orientable.

2.3.6 Stabilizers of Thurston geometries

The stabilizers of these geometries are:

stabilizer	geometries	nature of geometry
So_4	S^4, E^4, H^4	constant curvature
U_2	$P^2(\mathbb{C})$, $H^2(\mathbb{C})$	symmetric
$So_2 \times So_2$	$S^2 \times S^2$, $S^2 \times E^2$, $S^2 \times H^2$, $E^2 \times H^2$, $H^2 \times H^2$	
So_3	$S^3 \times E^1$, $H^3 \times E^1$	
So_2	$\widetilde{Sl_2(\mathbb{R})} \times E$, $Nil_3 \times E^1$, Sol_0^4, F^4	not symmetric
$\{1\}$	Nil^4, $Sol_{m,n}^4$, Sol_1^4	

Chapter 3

Legendre curve on Lorentzian Heisenberg space

The Legendre curves play a fundamental role in $3-$dimensional contact geometry. Legendre curves on contact manifolds have been studied by C. Baikoussis and D. E. Blair in the paper [1]. M. Belkhelfa, I. E. Hirică, R. Rosca and L. Verstraelen [11] have investigated Legendre curves in Riemannian and Lorentzian manifolds. Heisenberg group is a unimodular Lie group with left invariant Sasakian structure.
The concept of local ϕ- symmetric was introduced by T. Takahashi [62]. According to Takahashi a differentiable manifold is called locally if it satisfies:

$$\phi^2(\nabla_W R)(X,Y)Z = 0. \tag{3.1}$$

In this chapter, we show that the Legendre curves on three-dimensional Lorentzian Heisenberg space $(\mathbb{H}_3, g)$ is locally ϕ- symmetric if and only if is a geodesic. Moreover we prove that the Legendre curves on three-dimensional Lorentzian Heisenberg space is biharmonic if and only if is a pseudo-helix.

3.1 Contact Lorentzian manifold

Let M be a $(2n+1)$-dimensional differentiable manifold. M has an almost contact structure (ϕ, ξ, η) if it admits a $(1,1)$ tensor field ϕ, a vector field ξ and a 1-form η satisfying:

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1. \tag{3.2}$$

Suppose M has an almost contact structure (ϕ, ξ, η). Then $\phi\xi = 0$ and $\eta \circ \phi = 0$. Moreover, the endomorphism ϕ has rank $2n$.
If a $(2n+1)$-dimensional smooth manifold M with almost contact structure (ϕ, ξ, η) admits a compatible Lorentzian metric such that:

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{3.3}$$

then we say M has an almost contact Lorentzian structure (η, ξ, ϕ, g). Setting $Y = \xi$ we have:

$$\eta(X) = -g(X, \xi). \tag{3.4}$$

Next, if the compatible Lorentzian metric g satisfies:

$$d\eta(X,Y) = g(X,\phi Y), \tag{3.5}$$

then η is a contact form on M, ξ the associated Reeb vector field, g an associated metric and (M,ϕ,ξ,η,g) is called a contact Lorentzian manifold.
An almost contact Lorentzian manifold (M,ϕ,ξ,η,g) is Sasakian if and only if:

$$(\nabla_X\phi)Y = g(X,Y)\xi + \eta(Y)X. \tag{3.6}$$

Let (M,ϕ,ξ,η,g) be a contact Lorentzian manifold. Then we have:

$$\nabla_X\xi = \phi X - \phi hX, \quad h = \frac{1}{2}L_\xi\phi. \tag{3.7}$$

If ξ is a killing vector field with respect to the Lorentzian metric, then we have:

$$\nabla_X\xi = \phi X. \tag{3.8}$$

An arbitrary curve $\gamma : I \longrightarrow M^3, \gamma = \gamma(s)$ in Lorentzian 3-manifolds is called spacelike, timelike or null (lightlike), if all of its velocity vectors $\gamma'(s)$ are respectively spacelike, timelike or null (lightlike). If γ is a spacelike or timelike curve, we can reparametrize it such that $g(\dot\gamma(s),\dot\gamma(s)) = \epsilon$, where $\epsilon = 1$ if γ is spacelike and $\epsilon = -1$ if γ is timelike, respectively. In this case $\gamma(s)$ is said to be unit speed or arclength parametrization. Then the Frenet-Serret equations are following:

$$\begin{aligned}&\nabla_T T = \epsilon_2\kappa N\\ &\nabla_T N = -\epsilon_1\kappa T + \epsilon_3\tau B\\ &\nabla_T B = -\epsilon_2\tau N\end{aligned}$$

where $\kappa = |\nabla_T T|$ is the geodesic curvature of γ and τ is the geodesic torsion.
A Frenet curve is a geodesic if and only if $\kappa = 0$. A Frenet curve γ with constant geodesic curvature and zero geodesic torsion is called a pseudo-circle. A pseudo-helix is a Frenet curve γ whose geodesic curvature and torsion are constants.
The constant $\epsilon_1, \epsilon_2, \epsilon_3$ defined by $g(T,T) = \epsilon_1$, $g(N,N) = \epsilon_2$, $g(B,B) = \epsilon_3$, and called second causal character and third causal character of γ, respectively. Thus it satisfied $\epsilon_1\epsilon_2 = -\epsilon_3$.

Proposition 20. *Let $\{T,N,B\}$ are orthonomal Frame field in a Lorentzian 3-manifold. Then*

$$T\wedge N = \epsilon_3 B, \quad N\wedge B = \epsilon_1 T, \quad B\wedge T = \epsilon_2 N. \tag{3.9}$$

3.2 Legendre curve on Lorentzian Heisenberg space

Definition 70. *A Frenet curve γ in a Riemannian manifold is said to be a Legendre curve if it is an integral curve of the contact distribution $\mathcal{D} = Ker(\eta)$, i.e., if $\eta(\dot\gamma) = 0$.*

Let us consider the three-dimensional Heisenberg group

$$\mathbb{H}_3 = \begin{pmatrix} 1 & x & z + \frac{xy}{2}\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix}$$

Now, we take the contact form:

$$\eta = dz + (ydx - xdy).$$

Then the characteristic vector field of η is $\xi = \frac{\partial}{\partial z}$.
Now, we equip the Lorentzian metric as following:

$$g = dx^2 + dy^2 - (dz + (ydx - xdy))^2.$$

We take a left-invariant Lorentzian orthonormal frame field (e_1, e_2, e_3) on $(\mathbb{H}_3, g)$:

$$e_1 = \frac{\partial}{\partial x} - y\frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial z},$$

and the commutative relations are derived as follows:

$$[e_1, e_2] = 2e_3, \quad [e_2, e_3] = [e_3, e_1] = 0.$$

Then the endomorphism field:

$$\phi e_1 = e_2, \quad \phi e_2 = -e_1, \quad \phi e_3 = 0.$$

The Levi-Civita connection ∇ of $(\mathbb{H}_3, g)$ is described as:

$$\begin{array}{lll} \nabla_{e_1} e_3 = -e_2, & \nabla_{e_1} e_2 = e_3, & \nabla_{e_1} e_1 = 0, \\ \nabla_{e_2} e_3 = e_1, & \nabla_{e_2} e_2 = 0, & \nabla_{e_2} e_1 = -e_3, \\ \nabla_{e_3} e_3 = 0, & \nabla_{e_3} e_2 = e_1, & \nabla_{e_3} e_1 = -e_2. \end{array}$$

The contact form η satisfies $d\eta(X, Y) = g(X, \phi Y)$. Moreover the structure (η, ξ, ϕ, g) is Sasakian. The Riemannian curvature tensor R of $(\mathbb{H}_3, g)$ is given:

$$\begin{array}{ll} R(e_1, e_2)e_1 = 3e_2, & R(e_1, e_2)e_2 = -3e_1, \\ R(e_2, e_3)e_2 = -e_3, & R(e_2, e_3)e_3 = -e_2, \\ R(e_3, e_1)e_3 = e_1, & R(e_3, e_1)e_1 = e_3, \end{array}$$

the others are zero.
The sectional curvature is given by:

$$K(\xi, e_i) = -1, \quad \text{for} \quad i = 1, 2,$$

and

$$K(e_1, e_2) = 3.$$

Hence Lorentzian Heisenberg space $(\mathbb{H}_3, g)$ is the Lorentzian Sasakian space forms with constant holomorphic sectional curvature $\mu = 3$.

Definition 71. *A 1-dimensional integral submanifold of a contact manifold is called a Legendre curve.*

Theorem 12. *[11] Let M be a 3-dimensional contact metric manifold. Then M is Sasakian if and only if the torsion of its Legendre curves is equal to* 1.

3.2.1 Locally ϕ-symmetric Legendre curves on Lorentzian Heisenberg space

Definition 72. *A Legendre curves γ on Lorentzian Heisenberg space will be called locally ϕ-symmetric if it satisfies:*

$$\phi^2(\nabla_T R)(\nabla_T T, T)T = 0 \tag{3.10}$$

where $T = \dot{\gamma}$.

Theorem 13. *[8] A Legendre curves on Lorentzian Heisenberg space is a locally ϕ-symmetric if and only if is a geodesic.*

Proof. Let consider a loccaly ϕ-symmetric Legendre curves on Lorentzian Heisenberg space. Let T, ϕT, ξ be a Frenet frame on Legendre curve. To maintain oriontation let $\phi T = N$ and $\phi N = -T$. Also we take $B = \xi$. Now using Serret Frenet formula, we get:

$$R(\nabla_T T, T)T = R(\epsilon_2 \kappa \phi T, T)T = \epsilon_2 \kappa R(N, T)T. \tag{3.11}$$

Since T and N are orthogonal to $\xi = e_3$, we can take $T = t_1 e_1 + t_2 e_2$ and $N = n_1 e_1 + n_2 e_2$. Here t_1, t_2, n_1, n_2 are scalars.
Using the definition of curvature tensor R the expression of T and N and (3.2) we gate after straight forward calculation:

$$R(N, T)T = 3t_1(-n_2 t_1 e_2 + n_1 t_2 e_2) + 3t_2(-n_1 t_2 e_1 + n_2 t_1 e_1). \tag{3.12}$$

Since T, ϕT and $\xi = e_3$ forms a right handed system. We have $t_1 n_2 - t_2 n_1 = \epsilon_3$, then:

$$R(N, T)T = 3\epsilon_3 t_2 e_1 - 3\epsilon_3 t_1 e_2. \tag{3.13}$$

Combining (3.11) and (3.12), we obtain:

$$\begin{aligned} R(\nabla_T T, T)T &= 3\kappa\epsilon_2\epsilon_3 t_2 e_1 - 3\kappa\epsilon_2\epsilon_3 t_1 e_2 \\ &= -3\kappa\epsilon_1 t_2 e_1 + 3\kappa\epsilon_1 t_1 e_2. \end{aligned} \tag{3.14}$$

Now

$$\begin{aligned} (\nabla_T R)(\nabla_T T, T)T &= \nabla_T R(\nabla_T T, T)T - R(\nabla_T^2 T, T)T - R(\nabla_T T, \nabla_T T)T \\ &\quad - R(\nabla_T T, T)\nabla_T T \\ &= \nabla_T R(\epsilon_2 \kappa N, T)T - \epsilon_2 \kappa' R(N, T)T - \epsilon_3 \kappa^2 R(T, T)T \\ &\quad + \epsilon_1 \kappa\tau R(B, T)T - \kappa^2 R(N, T)N. \end{aligned} \tag{3.15}$$

Now

$$\begin{aligned} R(B, T)T &= R(\xi, t_1 e_1 + t_2 e_2)(t_1 e_1 + t_2 e_2) \\ &= t_1 t_1 R(e_1, \xi)e_1 - t_1 t_2 R(e_2, \xi)e_1 - t_1 t_2 R(e_1, \xi)e_2 + t_2 t_2 R(e_2, \xi)e_2. \end{aligned} \tag{3.16}$$

Using (3.2) in (3.16), we get:

$$R(B, T)T = (t_1^2 + t_2^2)e_3, \tag{3.17}$$

and

$$R(N,T)N = -3\epsilon_3 n_1 e_2 + 3\epsilon_3 n_2 e_1.$$

Again

$$\begin{aligned}
\nabla_T R(\epsilon_2 \kappa N, T)T &= \nabla_T R(\nabla_T T, T)T \\
&= \nabla_{t_1 e_1 + t_2 e_2} - 3\kappa\epsilon_1 t_2 e_1 + 3\kappa\epsilon_1 t_1 e_2 \\
&= (t_1 e_1)(-3\kappa\epsilon_1 t_2)e_1 - 3\kappa\epsilon_1 t_1 t_2 \nabla_{e_1} e_1 \\
&\quad + (t_1 e_1)(3\kappa\epsilon_1 t_1)e_2 + 3\kappa\epsilon_1 t_1 t_1 \nabla_{e_1} e_2 \\
&\quad + (t_2 e_2)(-3\kappa\epsilon_1 t_2)e_1 - 3\kappa\epsilon_1 t_2 t_2 \nabla_{e_2} e_1 \\
&\quad + (t_2 e_2)(3\kappa\epsilon_1 t_1)e_2 + 3\kappa\epsilon_1 t_1 t_2 \nabla_{e_2} e_2 \\
&= 3\epsilon_1 \kappa' t_1 e_2 - 3\epsilon_1 \kappa' t_2 e_1 + 3\kappa\epsilon_1 t_1 t_1 e_3 + 3\kappa\epsilon_1 t_2 t_2 e_3.
\end{aligned} \tag{3.18}$$

Using (3.17), (3.18) in (3.15), we have:

$$\begin{aligned}
(\nabla_T R)(\nabla_T T, T)T = & 3\epsilon_1 \kappa' t_1 e_2 - 3\epsilon_1 \kappa' t_2 e_1 - 3\kappa\epsilon_1 t_1 t_1 e_3 - 3\kappa\epsilon_1 t_2 t_2 e_3 \\
& - \epsilon_2 \kappa'(3\epsilon_3 t_2 e_1 - 3\epsilon_3 t_1 e_2) \\
& + \epsilon_1 \kappa (t_1^2 + t_2^2) e_3 \\
& - \kappa^2(-3\epsilon_3 n_1 e_2 + 3\epsilon_3 n_2 e_1) \\
= & - 3\epsilon_1 \kappa t_1 t_1 e_3 - 3\epsilon_1 \kappa t_2 t_2 e_3 + \epsilon_1 \kappa (t_1^2 + t_2^2) e_3 \\
& + 3\kappa^2 \epsilon_3 n_1 e_2 - 3\kappa^2 \epsilon_3 n_2 e_1.
\end{aligned}$$

By (3.2) and (3.3), the above equation yields:

$$\phi^2 (\nabla_T R)(\nabla_T T, T)T = + 3\kappa^2 \epsilon_3 n_1 e_2 - 3\kappa^2 \epsilon_3 n_2 e_1. \tag{3.19}$$

Let the Legendre curve be locallyϕ-symmetric. Then by definition:

$$-3\kappa^2 \epsilon_3 (n_2 e_1 - n_1 e_2) = 0. \tag{3.20}$$

In both sides of (3.20) taking inner product with e_1, we get:

$$\kappa = 0. \tag{3.21}$$

□

3.2.2 Biharmonic Legendre curves on Lorentzian Heisenberg Space

Definition 73. *[46] A Legendre curve on a three-dimensional Heisenberg group will be called biharmonic if it satisfies the biharmonic equation*

$$\nabla_T^3 T + R(\nabla_T T, T)T = 0, \tag{3.22}$$

where $T = \dot{\gamma}$.

Theorem 14. *[8] A Legendre curves on Lorentzian Heisenberg space is biharmonic if and only if is a pseudo-helix.*

Proof. Using Serret-Frenet formula, by direct computations, we have:

$$\begin{aligned}
\nabla_T^3 T &= \nabla_T(\nabla_T(\nabla_T T)) \\
&= \nabla_T(\nabla_T \epsilon_2 \kappa N) \\
&= \epsilon_2(\nabla_T(\nabla_T \kappa N)) \\
&= \epsilon_2(\nabla_T(\kappa' N + \kappa \nabla_T N)) \\
&= \epsilon_2(\nabla_T(\kappa' N - \kappa^2 \epsilon_1 T + \epsilon_3 \kappa \tau B)) \\
&= \epsilon_2(\kappa'' N - 2\kappa\kappa' \epsilon_1 T + \epsilon_3 \kappa' \tau B + \epsilon_3 \kappa \tau' B \\
&\quad + \kappa' \nabla_T N - \kappa^2 \epsilon_1 \nabla_T T + \epsilon_3 \kappa \tau \nabla_T B) \\
&= 3\epsilon_3 \kappa\kappa' T + \epsilon_2(\kappa'' - \epsilon_3 \kappa^3 - \epsilon_1 \kappa \tau^2) N - \epsilon_1(2\tau\kappa' + \kappa\tau') B.
\end{aligned}$$

Using Theorem 1, we have:

$$\begin{aligned}
\nabla_T^3 T &= 3\epsilon_3 \kappa\kappa'(t_1 e_1 + t_2 e_2) + \epsilon_2(\kappa'' - \epsilon_3 \kappa^3 - \epsilon_1 \kappa)(n_1 e_1 + n_2 e_2) \\
&\quad - 2\epsilon_1 \kappa' e_3.
\end{aligned}$$

In view of (3.14) and (3.22), it follows that:

$$\begin{aligned}
\nabla_T^3 T + R(\nabla_T T, T)T &= 3\epsilon_3 \kappa\kappa'(t_1 e_1 + t_2 e_2) + \epsilon_2(\kappa'' - \epsilon_3 \kappa^3 - \epsilon_1 \kappa)(n_1 e_1 + n_2 e_2) \\
&\quad - 2\epsilon_1 \kappa' e_3 - 3\epsilon_1 \kappa t_2 e_1 + 3\epsilon_1 \kappa t_1 e_2. \qquad (3.23)
\end{aligned}$$

Consider that the Legendre curve is biharmonic. Then by definition:

$$\begin{aligned}
0 &= 3\epsilon_3 \kappa\kappa'(t_1 e_1 + t_2 e_2) + \epsilon_2(\kappa'' + \epsilon_3 \kappa^3 + \epsilon_1 \kappa)(n_1 e_1 + n_2 e_2) \\
&\quad - 2\epsilon_1 \kappa' e_3 - 3\epsilon_1 \kappa t_2 e_1 + 3\epsilon_1 \kappa t_1 e_2. \qquad (3.24)
\end{aligned}$$

In both sides of (3.24) taking inner product with e_3, we obtain:

$$2\epsilon_1 \kappa' = 0,$$

which gives κ an arbitrary constant. □

Chapter 4

Biharmonic curves in 3-Dimensional Generalized Symmetric Spaces

In 1967, A. J. Ledger [46] initiated the study of generalized Riemannian symmetric spaces. These spaces are geometrically characterized by the fact that the (local) geodesic symmetries are isometries. A generalized symmetric space is a pseudo-Riemannian manifold which admits at least a regular s-structure. Kowalski showed that all generalized symmetric spaces are necessarily homogeneous and classify them in dimension ≤ 5 [43]. While the only three-dimensional (Riemannian or Lorentzian) generalized symmetric space is the Lie group Sol_3. In this chapter, we study bi-harmonic curves in three-dimensional generalized symmetric spaces, equipped with a left-invariant pseudo-Riemannian metric. We characterize non-geodesic biharmonic curves in three-dimensional generalized symmetric spaces and prove that there exists no non-geodesic biharmonic spacelike curve helix in three-dimensional generalized symmetric spaces. We also show that a linear map from a Euclidean space in three-dimensional generalized symmetric spaces is biharmonic if and only if it is a harmonic map, and give a complete classification of such maps.

4.1 Three-dimensional generalized symmetric spaces

Let (M, g) be a connected pseudo-Riemannian and x a point of M. A symmetry at x is an isometry s_x of M, having x as an isolated fixed point. When (M, g) is a symmetric space, each point x admits a symmetry s_x reversing geodesics through the point. Hence, s_x is involutive for all x. This property was generalized by A.J. Ledger, who defined a regular $s-$structure as a family $\{s_x : x \in M\}$ of symmetries of (M, g) satisfying:

$$s_x \circ s_y = s_z \circ s_x, \;\; z = s_x(y),$$

for all x, y of M. The order of an $s-$structure is the least integer $k \geq 2$, such that $(s_x)^k = id_M$ for all x (it may happen that $k = \infty$). A generalized symmetric spaces is a connected pseudo-Riemannian (M, g) admitting a regular $s-$structure. The order of a generalized symmetric spaces is the minimum of all integers $k \geq 2$ such that M admits a regular $s-$structure of order k.

Following [13], any proper (that is, non-symmetric) three-dimensional generalized symmetric spaces (M, g) is of order 4. Moreover, it is given by the space $\mathbb{R}^3(x, y, t)$ with the pseudo-Riemannian metric:

$$g_{\epsilon,\lambda} = \epsilon(e^{2t}dx^2 + e^{-2t}dy^2) + \lambda dt^2, \tag{4.1}$$

where $\epsilon = \pm 1$ and $\lambda \neq 0$ is a real constant. Depending on the values of ϵ and λ, these metrics attain any possible signature: $(3, 0)$, $(0, 3)$, $(2, 1)$, $(1, 2)$.

Let (M, g) be a three-dimensional generalized symmetric space which is the space $\mathbb{R}^3(x, y, t)$, and denote by ∇, R and Ric the Levi-Civita connection, the Riemann curvature tensor and the Ricci tensor of M, respectively.

A left-invariant orthonormal frame $\{E_1, E_2, E_3\}$ in the low-three-dimensional generalized symmetric space is given by:

$$E_1 = e^{-t}\frac{\partial}{\partial x}, \ E_2 = e^{t}\frac{\partial}{\partial y}, \ E_3 = \frac{1}{\sqrt{|\lambda|}}\frac{\partial}{\partial t}. \tag{4.2}$$

With respect to this orthonormal basis, the Levi-Civita connection can be easily computed as:

$$\begin{array}{lll} \nabla_{E_1}E_1 = -\frac{\epsilon\epsilon_1}{\sqrt{|\lambda|}}E_3, & \nabla_{E_1}E_2 = 0, & \nabla_{E_1}E_3 = \frac{1}{\sqrt{|\lambda|}}E_1, \\ \nabla_{E_2}E_1 = 0, & \nabla_{E_2}E_2 = \frac{\epsilon\epsilon_1}{\sqrt{|\lambda|}}E_3, & \nabla_{E_2}E_3 = -\frac{1}{\sqrt{|\lambda|}}E_2, \\ \nabla_{E_3}E_1 = 0, & \nabla_{E_3}E_2 = 0, & \nabla_{E_3}E_3 = 0, \end{array} \tag{4.3}$$

where $\epsilon_1 = \frac{\lambda}{|\lambda|}$.
The Lie brackets can be easily computed as:

$$[E_1, E_2] = 0, \quad [E_2, E_3] = \frac{-1}{\sqrt{|\lambda|}}E_2, \quad [E_1, E_3] = \frac{1}{\sqrt{|\lambda|}}E_1, \tag{4.4}$$

We adopt the following notation and sign convention for Riemannian curvature operator:

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \tag{4.5}$$

The Riemannian curvature tensor is given by:

$$R(X, Y, Z, W) = -g(R(X, Y)Z, W). \tag{4.6}$$

Moreover we put:

$$R_{abc} = R(E_a, E_b)E_c, \qquad R_{abcd} = R(E_a, E_b, E_c, E_d). \tag{4.7}$$

A direct computation using (4.3), (4.4), (4.5), (4.6) and (4.7) gives the following non zero components of Riemannian curvature of the three-dimensional generalized symmetric space with respect to the orthonomal basis $\{E_1, E_2, E_3\}$ (we do not list those that can be obtained by symmetric properties of curvature):

$$\begin{array}{ll} R(E_1, E_2)E_1 = -\frac{\epsilon}{\lambda}E_2, & R(E_1, E_2)E_2 = \frac{\epsilon}{\lambda}E_1, \\ R(E_1, E_3)E_1 = \frac{\epsilon}{\lambda}E_3 & R(E_1, E_3)E_3 = -\frac{1}{|\lambda|}E_1, \\ R(E_2, E_3)E_2 = \frac{\epsilon}{\lambda}E_3, & R(E_2, E_3)E_3 = -\frac{1}{|\lambda|}E_2, \end{array} \tag{4.8}$$

and:

$$\begin{array}{ll} R_{1212} = \frac{1}{\lambda}, & R_{1221} = -\frac{1}{\lambda}, \\ R_{1313} = -\frac{\epsilon}{|\lambda|}, & R_{1331} = \frac{\epsilon}{|\lambda|}, \\ R_{2323} = -\frac{\epsilon}{|\lambda|}, & R_{2332} = \frac{\epsilon}{|\lambda|}. \end{array} \tag{4.9}$$

The Ricci curvature components $\{Ric_{ij}\}$ are computed as:

$$Ric_{11} = Ric_{12} = Ric_{13} = Ric_{23} = Ric_{22} = 0, \quad Ric_{33} = -\frac{2}{|\lambda|}. \tag{4.10}$$

The scalar curvature τ of the three-dimensional generalized symmetric spaces is given by:

$$\tau = tr Ric = \sum_{i=1}^{3} g(E_i, E_i) Ric(E_i, E_i) = -\frac{2}{|\lambda|}. \tag{4.11}$$

4.2 Biharmonic curves in 3-dimensional generalized symmetric spaces

An arbitrary curve $\gamma : I \longrightarrow M, \gamma = \gamma(s)$ in three-dimensional generalized symmetric spaces is called spacelike, timelike or null (lightlike), if all of its velocity vectors $\dot{\gamma}(s)$ are respectively spacelike, timelike or null (lightlike). If γ is a spacelike or timelike curve, we can reparameterize it such that $g(\dot{\gamma}(s), \dot{\gamma}(s)) = \epsilon$, where $\epsilon = 1$ if γ is spacelike and $\epsilon = -1$ if γ is timelike, respectively. In this case $\gamma(s)$ is said to be unit speed or arclength parametrization.

Let $\{T, N, B\}$ be the orthonormal frame field tangent to M along γ and defined as follows: T is the unit vector field tangent to γ, N is the unit vector field in the direction of $\nabla_T T$ normal to γ and $B = T \times_M N$.
The pseudo-vector product operation $\times_M$ is related to the determinant function by:

$$det(u, v, w) = g(u \times_M v, w).$$

With respect to the orthonormal basis $\{E_1, E_2, E_3\}$ we can write:

$$\begin{aligned} T &= T_1 E_1 + T_2 E_2 + T_3 E_3 \\ N &= N_1 E_1 + N_2 E_2 + N_3 E_3 \\ B &= B_1 E_1 + B_2 E_2 + B_3 E_3. \end{aligned}$$

The following Frenet formulas hold:

$$\begin{aligned} \nabla_T T &= \epsilon \kappa N \\ \nabla_T N &= -\epsilon \kappa T + \epsilon_1 \tau B \\ \nabla_T B &= -\epsilon \tau N, \end{aligned} \tag{4.12}$$

where $g(T, T) = \epsilon, g(N, N) = \epsilon, g(B, B) = \epsilon_1$. Here $\kappa = |\nabla_T T|$ is the geodesic curvature of γ and τ is the geodesic torsion.

Using Serret-Frenet formulas (4.12), by direct computations, we have:

$$\begin{aligned}
\nabla_T^3 T &= \nabla_T(\nabla_T(\nabla_T T)) \\
&= \nabla_T(\nabla_T \epsilon\kappa N) \\
&= \epsilon(\nabla_T(\nabla_T \kappa N)) \\
&= \epsilon(\nabla_T(\kappa' N + \kappa\nabla_T N)) \\
&= \epsilon(\nabla_T(\kappa' N - \epsilon\kappa^2 T + \epsilon_1\kappa\tau B)) \\
&= \epsilon(\kappa'' N - 2\epsilon\kappa\kappa' T + \epsilon_1\kappa'\tau B + \epsilon_1\kappa\tau' B \\
&+ \kappa'\nabla_T N - \epsilon\kappa^2\nabla_T T + \epsilon_1\kappa\tau\nabla_T B) \\
&= -3\kappa\kappa' T + (\epsilon\kappa'' - \epsilon\kappa^3 - \epsilon_1\kappa\tau^2)N + 2\epsilon\epsilon_1(\tau\kappa' + \kappa\tau')B.
\end{aligned}$$

Then the biharmonic equation (3.22) reduces to the system:

$$\begin{cases} \kappa\kappa' = 0 \\ \epsilon\kappa'' - \epsilon\kappa^3 - \epsilon_1\kappa\tau^2 + \kappa R(T,N,T,N) = 0 \\ 2\kappa'\tau + \kappa\tau' + \kappa R(T,N,T,B) = 0, \end{cases} \tag{4.13}$$

which is equivalent to:

$$\begin{cases} \kappa = \text{constant} \neq 0 \\ \epsilon\kappa^2 + \epsilon_1\tau^2 = R(T,N,T,N) \\ \tau' = -R(T,N,T,B). \end{cases}$$

Theorem 15. *Let γ be a non-null curve parameterized by arclength of three-dimensional generalized symmetric spaces. Then γ is a proper non-geodesic biharmonic curve if and only if:*

$$\begin{cases} \kappa = \textit{constant} \neq 0 \\ \epsilon\kappa^2 + \epsilon_1\tau^2 = \frac{1}{\lambda}\left(2B_3^2 - 1\right) \\ \tau' = \frac{2}{\lambda}N_3B_3. \end{cases} \tag{4.14}$$

Proof. By direct calculation, using (4.9), we obtain:

$$\begin{aligned}
R(T,N,T,N) &= \textstyle\sum_{i,j,l,p=1}^3 T_l N_p T_i B_j R_{lpij} \\
&= T_1N_2T_1N_2R_{1212} + T_1N_2T_2N_1R_{1221} \\
&+ T_2N_1T_2N_1R_{2121} + T_2N_1T_1N_2R_{2112} \\
&+ T_1N_3T_1N_3R_{1313} + T_1N_3T_3N_1R_{1331} \\
&+ T_3N_1T_3N_1R_{3131} + T_3N_1T_1N_3R_{3113} \\
&+ T_2N_3T_2N_3R_{2323} + T_2N_3T_3N_2R_{2332} \\
&+ T_3N_2T_3N_2R_{3232} + T_3N_2T_2N_3R_{3223} \\
&= \tfrac{1}{\lambda}\left(B_3^2 - \epsilon\epsilon_1\left(B_1^2 + B_2^2\right)\right) \\
&= \tfrac{1}{\lambda}\left(2B_3^2 - 1\right).
\end{aligned}$$

$$\left(\textit{by}\quad T\times N = B, T\times B = -N, \epsilon B_1^2 + \epsilon B_2^2 + \epsilon_1 B_3^2 = \epsilon_1\right).$$

$$
\begin{aligned}
R(T,N,T,B) &= \textstyle\sum_{i,j,l,p=1}^{3} T_l N_p T_i B_j R_{lpij} \\
&= T_1N_2T_1B_2R_{1212} + T_1N_2T_2B_1R_{1221} \\
&+ T_2N_1T_2B_1R_{2121} + T_2N_1T_1B_2R_{2112} \\
&+ T_1N_3T_1B_3R_{1313} + T_1N_3T_3B_1R_{1331} \\
&+ T_3N_1T_3B_1R_{3131} + T_3N_1T_1B_3R_{3113} \\
&+ T_2N_3T_2B_3R_{2323} + T_2N_3T_3B_2R_{2332} \\
&+ T_3N_2T_3B_2R_{3232} + T_3N_2T_2B_3R_{3223} \\
&= \tfrac{1}{\lambda}T_1^2N_2B_2 - \tfrac{1}{\lambda}T_1T_2N_1B_2 \\
&+ \tfrac{1}{\lambda}T_2^2N_1B_1 - \tfrac{1}{\lambda}T_1T_2N_2B_1 \\
&- \tfrac{\epsilon}{|\lambda|}T_1^2N_3B_3 + \tfrac{\epsilon}{|\lambda|}T_1T_3N_1B_3 \\
&- \tfrac{\epsilon}{|\lambda|}T_3^2N_1B_1 + \tfrac{\epsilon}{|\lambda|}T_1T_3B_1N_3 \\
&- \tfrac{\epsilon}{|\lambda|}T_2^2N_3B_3 + \tfrac{\epsilon}{|\lambda|}T_2T_3N_2B_3 \\
&- \tfrac{\epsilon}{|\lambda|}T_3^2N_2B_2 + \tfrac{\epsilon}{|\lambda|}T_2T_3B_2N_3 \\
&= \tfrac{1}{\lambda}\Big(T_1B_2 - T_2B_1\Big)\Big(T_1N2 - T_2N_1\Big) \\
&- \tfrac{\epsilon}{|\lambda|}\Big(T_3N_1 - T_1N_3\Big)\Big(T_3B_1 - T_1B_3\Big) \\
&- \tfrac{\epsilon}{|\lambda|}\Big(T_3N_2 - T_2N_3\Big)\Big(T_3B_2 - T_2B_3\Big) \\
&= \tfrac{\epsilon_1}{\lambda}\Big(-\epsilon_1N_3B_3 + \epsilon N_2B_2 + \epsilon N_1B_1\Big) \\
&= -\tfrac{2}{\lambda}N_3B_3.
\end{aligned}
$$

$$\Big(by \quad T\times N = B, T\times B = -N, \epsilon N_1B_1 + \epsilon N_2B_2 + \epsilon_1 N_3B_3 = 0\Big).$$

These, together with equation (4.13), complete the proof of the theorem. □

Corollary 5. *If $\kappa = constant \neq 0$ and $\tau = 0$ for a non-null curve $\gamma : I \longrightarrow M$ then γ is a non-geodesic biharmonic curve if and only if $N_3B_3 = 0$ and $\kappa^2 = \frac{\epsilon}{\lambda}(2B_3^2 - 1)$.*

Corollary 6. *Let γ be a non-geodesic curve parameterized by arclength of three-dimensional generalized symmetric spaces. If $B_3 = 0$ and $\epsilon = \epsilon_1 = 1$, then γ is not biharmonic.*

Corollary 7. *Let γ be a non-geodesic curve parameterized by arclength of three-dimensional generalized symmetric spaces. If B_3 is constant and $N_3B_3 \neq o$, then γ is not biharmonic.*

Definition 74. *A differentiable curve of three-dimensional generalized symmetric spaces having constant both geodesic curvature and geodesic torsion is called a **helix**.*

Corollary 8. *Let γ be be a non-geodesic curve parameterized by arclength of three-dimensional generalized symmetric spaces. If γ is biharmonic helix, then:*

$$\begin{cases} B_3 = constant \neq 0 \\ N_3 = 0 \\ \epsilon\kappa^2 + \epsilon_1\tau^2 = \frac{1}{\lambda}(2B_3^2 - 1) \end{cases} \tag{4.15}$$

Theorem 16. *Let γ be a non-null spacelike curve parameterized by arclength. Then γ is non-geodesic biharmonic helix in three-dimensional generalized symmetric spaces.*

Proof. Suppose that $\gamma : I \longrightarrow M$ is a non-geodesic biharmonic helix parameterized by arclength. We shall derive a contradiction by showing that γ must be a geodesic. We can use (4.3) to compute the covariant derivatives of the vector fields T, N, B as:

$$\begin{cases} \nabla_T T &= \left(T_1' + \frac{1}{\sqrt{|\lambda|}}T_1T_3\right)E_1 + \left(T_2' - \frac{1}{\sqrt{|\lambda|}}T_2T_3\right)E_2 \\ &+ \left(\frac{\epsilon_1}{\sqrt{|\lambda|}}T_2^2 - \frac{\epsilon_1}{\sqrt{|\lambda|}}T_1^2 + T_3'\right)E_3 \\ \nabla_T N &= \left(N_1' + \frac{1}{\sqrt{|\lambda|}}T_1N_3\right)E_1 + \left(N_2' - \frac{1}{\sqrt{|\lambda|}}T_2N_3\right)E_2 \\ &+ \left(\frac{\epsilon_1}{\sqrt{|\lambda|}}T_2N_2 - \frac{\epsilon_1}{\sqrt{|\lambda|}}T_1N_1 + N_3'\right)E_3 \\ \nabla_T B &= \left(B_1' + \frac{1}{\sqrt{|\lambda|}}T_1B_3\right)E_1 + \left(B_2' - \frac{1}{\sqrt{|\lambda|}}T_2B_3\right)E_2 \\ &+ \left(\frac{\epsilon_1}{\sqrt{|\lambda|}}T_2B_2 - \frac{\epsilon_1}{\sqrt{|\lambda|}}T_1B_1 + B_3'\right)E_3. \end{cases} \tag{4.16}$$

It follows that the third components of these vectors are given by:

$$\begin{cases} \langle \nabla_T T, E_3\rangle = \left(\frac{1}{\sqrt{|\lambda|}}T_2^2 - \frac{1}{\sqrt{|\lambda|}}T_1^2 + \epsilon_1 T_3'\right) \\ \langle \nabla_T N, E_3\rangle = \left(\frac{1}{\sqrt{|\lambda|}}T_2N_2 - \frac{1}{\sqrt{|\lambda|}}T_1N_1 + \epsilon_1 N_3'\right) \\ \langle \nabla_T B, E_3\rangle = \left(\frac{1}{\sqrt{|\lambda|}}T_2B_2 - \frac{1}{\sqrt{|\lambda|}}T_1B_1 + \epsilon_1 B_3'\right). \end{cases} \tag{4.17}$$

On the other hand, using Frenet formulas (4.12):

$$\begin{cases} \langle \nabla_T T, E_3\rangle = \epsilon_1 \kappa N_3 \\ \langle \nabla_T N, E_3\rangle = -\epsilon_1\kappa T_3 + \tau B_3 \\ \langle \nabla_T B, E_3\rangle = -\epsilon_1 \tau N_3. \end{cases} \tag{4.18}$$

Since γ is assumed to be a non-geodesic biharmonic helix, we have, by Corollary (8), $N_3 = 0$, $B_3 =$ constant These, together with Equations (4.17), (4.18), give:

$$\begin{cases} \frac{1}{\sqrt{|\lambda|}}T_2^2 - \frac{1}{\sqrt{|\lambda|}}T_1^2 + \epsilon_1 T_3' = 0 \\ \frac{1}{\sqrt{|\lambda|}}T_2N_2 - \frac{1}{\sqrt{|\lambda|}}T_1N_1 = -\epsilon_1\kappa T_3 + \tau B_3 \\ \frac{1}{\sqrt{|\lambda|}}T_2B_2 - \frac{1}{\sqrt{|\lambda|}}T_1B_1 = 0. \end{cases} \tag{4.19}$$

Noting that $T \times B = -N$, we also have:

$$T_2B_1 - T_1B_2 = N_3. \tag{4.20}$$

Thus, we have:

$$\begin{cases} \frac{1}{\sqrt{|\lambda|}}T_2N_2 - \frac{1}{\sqrt{|\lambda|}}T_1N_1 = -\epsilon_1\kappa T_3 + \tau B_3 & \langle 1\rangle \\ \frac{1}{\sqrt{|\lambda|}}T_2^2 - \frac{1}{\sqrt{|\lambda|}}T_1^2 + \epsilon_1 T_3' = 0 & \langle 2\rangle \\ \frac{1}{\sqrt{|\lambda|}}T_2B_2 - \frac{1}{\sqrt{|\lambda|}}T_1B_1 = 0 & \langle 3\rangle \\ T_2B_1 - T_1B_2 = 0 & \langle 4\rangle \end{cases} \tag{4.21}$$

Case A: $T_1^2 \neq T_2^2$. In this case, Equations $\langle 3 \rangle$ and $\langle 4 \rangle$ in system (4.21) viewed as equations in B_1 and B_2 has a unique solution $B_1 = B_2 = 0$. This implies that $T_3 = 0$. Substitute this into $\langle 2 \rangle$ of system (4.21), we have $T_1^2 = T_2^2$, a contradiction. Thus, we must have.
Case B: $T_1^2 = T_2^2$. In this case, equation $\langle 2 \rangle$ of system (4.21) implies that $T_3 =$ constant. To understand the meaning of this, if $\lambda > 0$ or $\epsilon_1 = 1$ we represent the unit tangent vector T as:

$$T = \sin\alpha\cos\beta E_1 + \sin\alpha\sin\beta E_2 + \cos\alpha E_3. \tag{4.22}$$

where $\alpha = \alpha(s), \beta = \beta(s)$. With this representation, $T_3 =$ constant implies that $\cos\alpha =$ constant and hence $\alpha(s) = \alpha_0$, a constant. This, together with $T_1^2 = T_2^2$, gives:

$$\sin\alpha_0(\cos\beta \pm \sin\beta) = 0. \tag{4.23}$$

If $\sin\alpha_0 = 0$, then we have $T_1 = T_2 = 0$, and it follows from the first equation of (4.16) that $\nabla_T T = 0$ which means that γ is a geodesic, a contradiction. Thus, we must have $\sin\alpha_0 \neq 0$, which, together with (4.23), implies that:

$$\cos\beta = \pm\sin\beta = \pm\frac{\sqrt{2}}{2},$$

and hence,

$$T_1 = \pm T_2 = \pm\frac{\sqrt{2}}{2}\sin\alpha_0. \tag{4.24}$$

We use the first equation of (4.16) again to get:

$$\nabla_T T = \sin\alpha_0\cos\alpha_0\Big(\pm\frac{\sqrt{2}}{2\sqrt{\lambda}}E_1 \pm \frac{\sqrt{2}}{2\sqrt{\lambda}}E_2\Big) = \kappa N,$$

which yields:

$$N_1 = N_2 = \pm\frac{\sqrt{2}}{2}, \tag{4.25}$$

since $|\nabla_T T| = |\frac{\sin\alpha_0\cos\alpha_0}{\sqrt{\lambda}}|$. By the assumption that γ is non-geodesic, we may assume, without loss of generality, that $\sin\alpha_0\cos\alpha_0 > 0$ so:

$$\kappa = \frac{\sin\alpha_0\cos\alpha_0}{\sqrt{\lambda}}. \tag{4.26}$$

Using equations (4.24), (4.25) and the fact that $B = T \times N$, we have:

$$\begin{cases} B_1 = \text{constant} \\ B_2 = \text{constant} \\ B_3 = T_1N_2 - T_2N_1 = \pm\sin\alpha_0. \end{cases} \tag{4.27}$$

It follows from (4.27), $\langle 3 \rangle$ of (4.21), and and the third equation of (4.16), that:

$$\tau^2 = |\nabla_T B|^2 = \frac{1}{\lambda}\sin^4\alpha_0. \tag{4.28}$$

Substituting (4.26), (4.27), (4.28) into the third equation in (4.15), we have:

$$\frac{1}{\lambda}\sin^2\alpha_0\cos^2\alpha_0 + \frac{1}{\lambda}\sin^4\alpha_0 = \frac{1}{\lambda}(2\sin^2\alpha_0 - 1), \tag{4.29}$$

which implies:

$$\sin^2\alpha_0 = 1,$$

and hence:

$$\cos\alpha_0^2 = 0.$$

It follows that $\cos\alpha_0^2 = 0$ from which and (4.26), we conclude that: $\kappa = 0$, .i.e., γ is a geodesic, a contradiction.

Similarly if $\lambda < 0$ or $\epsilon_1 = -1$, we represent the unit tangent vector T as:

$$T = \cosh\alpha\cos\beta E_1 + \cosh\alpha\sin\beta E_2 + \sinh\alpha E_3. \tag{4.30}$$

Then we have:

$$\kappa = |\nabla_T T| = \frac{\cosh\alpha_0\sinh\alpha_0}{\sqrt{-\lambda}}, \tag{4.31}$$

and:

$$\tau^2 = |\nabla_T B|^2 = \frac{-1}{\lambda}\cosh^4\alpha_0, \tag{4.32}$$

$$B_3 = \pm\cosh\alpha_0. \tag{4.33}$$

Substituting (4.31), (4.32), (4.33) into the third equation in (4.15), we have:

$$\frac{-1}{\lambda}\cosh^2\alpha_0\sinh^2\alpha_0 + \frac{1}{\lambda}\cosh^4\alpha_0 = \frac{1}{\lambda}(2\cosh^2\alpha_0 - 1), \tag{4.34}$$

which implies:

$$\cosh^2\alpha_0 = 1,$$

and hence:

$$\sinh\alpha_0^2 = 0.$$

It follows that $\cosh\alpha_0^2 = 0$ from which and (4.26), we conclude that $\kappa = 0$, .i.e., γ is a geodesic, a contradiction. □

4.3 Linear biharmonic maps in 3-dimensional generalized symmetric spaces

In [58] Ye-Lin Ou and Ze-Ping Wang study linear biharmonic maps from a Euclidian space into Sol_3, Nil_3 and Heisenberg spaces. They show that a linear map from a Euclidian space into Sol_3, Nil_3 or Heisenberg space is biharmonic if and only if it is a harmonic map. In this section we study linear biharmonic maps from a Euclidian space in 3-dimensional generalized symmetric spaces and we show that a linear map from a Euclidian space into this space is biharmonic if and only if it is a harmonic map, and we give classification of such maps.

4.3.1 Biharmonic map equation in local coordinates

Lemma 10. *[57] Let $\phi : (M^m, g) \longrightarrow (N^n, h)$ with $\phi(x_1, ...x_m) = (\phi_1(x), ..., \phi_m(x))$ be a map between Riemannian manifolds. With respect to to local coordinates (x_i) in M and (y_α) in N, ϕ is biharmonic if and only if it is a solution of the following system of PDE's:*

$$g^{ij}\Big(\tau^\sigma_{ij} + \tau^\alpha_j\phi^\beta_i\Gamma^\sigma_{\alpha\beta} + \frac{\partial}{\partial x_i}(\tau^\alpha\phi^\beta_j\Gamma^\sigma_{\alpha\beta}) + \tau^\alpha\phi^\beta_j\phi^\rho_i\Gamma^\nu_{\alpha\beta}\Gamma^\sigma_{\nu\rho}$$
$$-\Gamma^k_{ij}(\tau^\sigma_k + \tau^\alpha\phi^\beta_k\Gamma^\sigma_{\alpha\beta}) - \tau^\nu\phi^\alpha_i\phi^\beta_j R^\sigma_{\beta\alpha\nu}\Big) = 0, \quad \sigma = 1, 2, ..., n.$$

Corollary 9. *Let $\phi : \mathbb{R}^m \longrightarrow (N^n, h)$ with $\phi(x_1, ..., x_n) = (\phi_1(x), ..., \phi_n(x))$ be a map from a Euclidean space into a Riemannian manifold. Then ϕ is biharmonic if and only if it is a solution of the following system of PDE's:*

$$\Delta\tau^\sigma + \langle\nabla\tau^\alpha, \nabla\phi^\beta\rangle\Gamma^\sigma_{\alpha\beta} + \langle\nabla\phi^\beta, \nabla(\tau^\alpha\Gamma^\sigma_{\alpha\beta})\rangle$$
$$+\langle\nabla\varphi^\beta, \nabla\phi^\rho\rangle\tau^\alpha\Gamma^\nu_{\alpha\beta}\Gamma^\sigma_{\nu\rho} - \tau^\nu\langle\nabla\phi^\alpha, \nabla\phi^\beta\rangle R^\sigma_{\beta\alpha\nu} = 0, \quad \sigma = 1, 2, ..., n.$$

4.3.2 Linear biharmonic maps in 3-dimensional generalized symmetric spaces

Let $(\mathbb{R}^3, g)$ denote three-dimensional generalized symmetric spaces where the metric can be written as $g_{\epsilon,\lambda} = \epsilon(e^{2t}dx^2 + e^{-2t}dy^2) + \lambda dt^2$ with respect to the standard coordinates (y_1, y_2, y_3) in $\mathbb{R}^3$. Then a direct computation gives the following components of metric and the coefficients of the connection:

$$\begin{array}{llll} g_{11} = \epsilon e^{2t}, & g_{22} = \epsilon e^{-2t}, & g_{33} = \lambda, & \text{all other } \ g_{ij} = 0, \\ g^{11} = \frac{1}{\epsilon}e^{-2t}, & g^{22} = \frac{1}{\epsilon}e^{2t}, & g^{33} = \frac{1}{\lambda}, & \text{all other } \ g^{ij} = 0. \end{array}$$

$$\begin{array}{lll} \Gamma^1_{11} = 0, & \Gamma^2_{11} = 0, & \Gamma^3_{11} = \frac{-\epsilon}{\lambda}e^{2t} \\ \Gamma^1_{12} = 0, & \Gamma^2_{12} = 0, & \Gamma^3_{12} = 0 \\ \Gamma^1_{13} = 1, & \Gamma^2_{13} = 0, & \Gamma^3_{13} = 0 \\ \Gamma^1_{21} = 0, & \Gamma^2_{21} = 0, & \Gamma^3_{21} = 0 \\ \Gamma^1_{22} = 0, & \Gamma^2_{22} = 0, & \Gamma^3_{22} = \frac{\epsilon}{\lambda}e^{-2t} \\ \Gamma^1_{23} = 0, & \Gamma^2_{23} = -1, & \Gamma^3_{23} = 0 \\ \Gamma^1_{31} = 1, & \Gamma^2_{31} = 0, & \Gamma^3_{31} = 0 \\ \Gamma^1_{32} = 0, & \Gamma^2_{32} = -1, & \Gamma^3_{32} = 0 \\ \Gamma^1_{33} = 0, & \Gamma^2_{33} = 0, & \Gamma^3_{33} = 0. \end{array} \tag{4.35}$$

By our convention of curvature operator and the following notation for the components of the Riemannian curvature:

$$R(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j})\frac{\partial}{\partial y_k} = R^l_{kij}\frac{\partial}{\partial y_l} \tag{4.36}$$

we have:

$$R^l_{kij} = \frac{\partial}{\partial y_i}\Gamma^l_{kj} - \frac{\partial}{\partial y_j}\Gamma^l_{ki} + \Gamma^l_{it}\Gamma^t_{kj} - \Gamma^l_{jt}\Gamma^t_{ki} \tag{4.37}$$

A straightforward computation using (4.36) and (4.37) gives the following components of the Riemannian curvature of three-dimensional generalized symmetric spaces:

$$\begin{cases} R^1_{221} = \frac{-\epsilon}{\lambda}e^{-2t}, & R^1_{331} = 1, & R^1_{212} = \frac{\epsilon}{\lambda}e^{-2t}, & R^1_{313} = -1, \\ R^2_{121} = \frac{\epsilon}{\lambda}e^{2t}, & R^2_{112} = \frac{-\epsilon}{\lambda}e^{2t}, & R^2_{332} = 1, & R^2_{323} = -1, \\ R^3_{131} = \frac{-\epsilon}{\lambda}e^{2t}, & R^3_{232} = \frac{-\epsilon}{\lambda}e^{-2t}, & R^3_{113} = \frac{\epsilon}{\lambda}e^{2t}, & R^3_{223} = \frac{\epsilon}{\lambda}e^{-2t}. \end{cases}$$

Theorem 17. *Let $\varphi : \mathbb{R}^m \longrightarrow (\mathbb{R}^3, g_{\epsilon,\lambda})$ with*

$$\varphi(x) = \begin{pmatrix} a_{11} & a_{11} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ a_{31} & a_{32} & \cdots & a_{3m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$$

i.e., $\phi(X) = (A_1X^t, A_2X^t, A_3X^t)$ be a linear map of three-dimensional generalized symmetric spaces, where A_i denotes the row vectors of the representation matrix. Then, ϕ is a biharmonic map if and if it is a harmonic map, which is equivalent to either

$$\begin{cases} A_3 = 0, |A_1|^2 = |A_2|^2, \\ or, \\ A_3 \neq 0, A_1 = A_2 = 0, \end{cases}$$

Proof. With respect to the standard Cartesian coordinates x_i in $\mathbb{R}^m$ and (y^α) in $\mathbb{R}^3$, the tension field of ϕ is given by

$$\begin{aligned} \tau(\phi) &= tr_g\nabla d\phi \in (\Gamma\phi^{-1}TN) \\ &= \tau^\sigma\frac{\partial}{\partial y_\sigma} \\ &= g^{ij}(\phi^\sigma_{ij} - \Gamma^k_{ij}\phi^\sigma_k + \Gamma^\sigma_{\alpha\beta}\phi^\alpha_i\phi^\beta_j)\frac{\partial}{\partial y_\sigma} \\ &= (\sum_{i=1}^m \Gamma^\sigma_{\alpha\beta}\phi^\alpha_i\phi^\beta_i)\frac{\partial}{\partial y_\sigma} \\ &= \Gamma^\sigma_{\alpha\beta}A^\alpha A^B\frac{\partial}{\partial y_\sigma} \end{aligned} \tag{4.38}$$

where $A^\alpha = \phi^\beta_i, A^\beta = \phi^\beta_i$ denotes the inner product and $|A^\alpha|$ the norm of the vectors in Euclidean space.
Putting $\tau(\phi) = \tau^\sigma\frac{\partial}{\partial y_\sigma}$ and substituting (4.35) into Equation (4.38), we find the following components of the tension field of ϕ:

$$\begin{aligned} \tau^1 &= 2A_3.A_1 \\ \tau^2 &= -2A_3.A_2 \\ \tau^3 &= \frac{\epsilon}{\lambda}\Big(|A_2|^2e^{-2t} + |A_1|^2e^{2t}\Big) \end{aligned} \tag{4.39}$$

where and in the sequel $y_3 = A_3X^t$.
A further computation gives:

$$\begin{aligned}\Big(\triangle\tau^\sigma + \langle\nabla\tau^\alpha,\nabla\phi^\beta\rangle\Gamma^\sigma_{\alpha\beta}\Big)\frac{\partial}{\partial y_\sigma} &= (\triangle\tau^\sigma + A^\beta.\nabla\tau^\alpha\Gamma^\sigma_{\alpha\beta})\frac{\partial}{\partial y_\sigma}\\ &= \Big(-\frac{2\epsilon}{\lambda}A_1A_3(|A_2|^2e^{-2t}+|A_1|^2e^{2t}\Big)\frac{\partial}{\partial y_1}\\ &+ \Big(\frac{2\epsilon}{\lambda}A_2A_3(|A_2|^2e^{-2t}+|A_1|^2e^{2t}\Big)\frac{\partial}{\partial y_2}\\ &+ \Big(\frac{4\epsilon}{\lambda}|A_3|^2(|A_2|^2e^{-2t}-|A_1|^2e^{2t}\Big)\frac{\partial}{\partial y_3}\end{aligned}\tag{4.40}$$

$$\begin{aligned}\langle\nabla\phi^\beta,\nabla\tau^\alpha\Gamma^\sigma_{\alpha\beta}\rangle\frac{\partial}{\partial y_\sigma} &= \Big(A^\beta.\nabla\tau^\alpha\Gamma^\sigma_{\alpha\beta} + \tau^\alpha A^\beta.\nabla\Gamma^\sigma_{\alpha\beta}\Big)\frac{\partial}{\partial y_\sigma}\\ &= \Big(-\frac{2\epsilon}{\lambda}A_1.A_3(|A_2|^2e^{-2t}+|A_1|^2e^{2t})\Big)\frac{\partial}{\partial y_1}\\ &+ \Big(\frac{2\epsilon}{\lambda}A_2.A_3(|A_2|^2e^{-2t}+|A_1|^2e^{2t}\Big)\frac{\partial}{\partial y_2}\\ &+ \Big(\frac{\epsilon}{\lambda}(-4(A_1.A_3)^2e^{2t}+4(A_2.A_3)^2e^{-2t})\Big)\frac{\partial}{\partial y_3}\end{aligned}\tag{4.41}$$

$$\begin{aligned}\langle\nabla\phi^\beta,\nabla\phi^\rho\rangle\tau^\alpha\Gamma^\nu_{\alpha\beta}\Gamma^\sigma_{\nu\rho}\frac{\partial}{\partial y_\sigma} &= \tau^\alpha A^\beta.A^\rho\Gamma^\nu_{\alpha\beta}\Gamma^\sigma_{\nu\rho}\frac{\partial}{\partial y_\sigma}\\ &= \Big(2A_1A_3|A_3|^2 - \frac{3\epsilon}{\lambda}A_1A_3|A_1|^2e^{2t}\\ &- \frac{\epsilon}{\lambda}A_1A_3|A_2|^2e^{-2t}\Big)\frac{\partial}{\partial y_1}\\ &+ \Big(-2A_2A_3|A_3|^2 + \frac{\epsilon}{\lambda}A_2A_3|A_1|^2e^{2t}\\ &+ \frac{3\epsilon}{\lambda}A_3A_2|A_2|^2e^{-2t}\Big)\frac{\partial}{\partial y_2}\\ &+ \Big(\frac{-2\epsilon}{\lambda}(A_1A_3)^2e^{2t} + \frac{1}{\lambda^2}|A_1|^4e^{4t}\\ &+ \frac{2\epsilon}{\lambda}(A_2A_3)^2e^{-2t} - \frac{1}{\lambda^2}|A_2|^4e^{-4t}\Big)\frac{\partial}{\partial y_3}\end{aligned}\tag{4.42}$$

$$\begin{aligned}
\tau^{\nu}\langle\nabla\phi^{\alpha},\nabla\phi^{\beta}\rangle R^{\sigma}_{\beta\alpha\nu}\frac{\partial}{\partial y_{\sigma}} &= A^{\beta}.A^{\alpha}\tau^{\nu}R^{\sigma}_{\beta\alpha\nu}\frac{\partial}{\partial y_{\sigma}} \\
&= \Big(\frac{-3\epsilon}{\lambda}A_1A_3|A_2|^2e^{-2t}-\frac{2\epsilon}{\lambda}A_2.A_3A_1A_2e^{-2t} \\
&+ \frac{\epsilon}{\lambda}A_1A_3|A_1|^2e^{2t}+2A_1A_3|A_3|^2\Big)\frac{\partial}{\partial y_1} \\
&+ \Big(\frac{2\epsilon}{\lambda}A_1^2A_3A_2e^{2t}+\frac{3\epsilon}{\lambda}|A_1|^2A_3A_2e^{2t} \\
&- \frac{\epsilon}{\lambda}A_2A_3|A_2|^2e^{-2t}-2A_2A_3A_3^2\Big)\frac{\partial}{\partial y_2} \\
&+ \Big(-\frac{1}{\lambda^2}|A_1|^4e^{4t}+\frac{1}{\lambda^2}|A_2|^4e^{-4t} \\
&- \frac{2\epsilon}{\lambda}(A_1A_3)^2e^{2t}+\frac{2\epsilon}{\lambda}(A_2A_3)^2e^{-2t}\Big)\frac{\partial}{\partial y_3}.
\end{aligned} \tag{4.43}$$

It follows from equations (4.40), (4.41), (4.42), (4.43) and Corollary (9) that the linear map ϕ is a biharmonic map if and only if:

$$\begin{cases}
\frac{-8\epsilon}{\lambda}A_1A_3|A_1|^2e^{2t}=0 \\
\frac{8\epsilon}{\lambda}A_2A_3|A_2|^2e^{-2t}=0 \\
\frac{4\epsilon}{\lambda}(|A_2|^2|A_3|^2+(A_2A_3)^2)e^{-2t}-\frac{4\epsilon}{\lambda}(|A_1|^2|A_3|^2+(A_1A_3)^2)e^{2t} \\
+\frac{2}{\lambda^2}|A_1|^4e^{4t}-\frac{2}{\lambda^2}|A_2|^4e^{-4t}=0.
\end{cases} \tag{4.44}$$

Solving System of equations (4.44), we have either $(i)\quad A_3=0, |A_1|^2=|A_2|^2$, or $(ii)\quad A_3\neq 0,\quad A_1=A_2=0$. It follows from equation (4.39) that in both cases the tension field vanishes identically, i.e., ϕ is also harmonic. Therefore, we obtain the theorem. □

Remark 16. *If we put $\epsilon=1$ and $\lambda=1$ the three-dimensional (Riemannian or Lorentzian) generalized symmetric space is the Lie group Sol_3.*

Chapter 5

Biharmonic curves in Thurston geometry of dimension 4

In this chapter we study harmonic and biharmonic applications in Thurston geometry of dimension 4. We introduce the The 4-dimensional geometry Nil^4 and we define the metric g_{Nil^4}. We give the Christofell symbols, the Riemannian curvature and we study the biharmonic curves in Nil^4 space. [16], [2], [37], [49], [50] [27], [8], [9] and [64].

5.1 The 4-dimensional geometry Nil^4

The geometry Nil^4 can be identified with $\mathbb{R}^4$ endowed with the metric:

$$g_{Nil^4} = ds^2 = dx_1^2 + dx_3^2 + (dx_2 + x_1 dx_3)^2 + (dx_4 + x_1 dx_2 + \frac{x_1^2}{2} dx_3)^2, \tag{5.1}$$

where (x_1, x_2, x_3, x_4) are the standard coordinates in $\mathbb{R}^4$. This can be calculated from its characterization as a left-invariant metric with respect to the group structure of Nil^4. There is a natural harmonic Riemannian submersion $(x_1, x_2, x_3, x_4) \to (x_1, x_2, x_3)$ onto $(\mathbb{R}^3, h = dx_1^2 + dx_3^2 + (dx_2 + x_1 dx_3)^2) = Nil^3$.
The components of the matrix g_{ij} are given by:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 + x_1^2 & x_1 + \frac{x_1^3}{2} & x_1 \\ 0 & x_1 + \frac{x_1^3}{2} & 1 + x_1^2 + \frac{x_1^4}{4} & \frac{x_1^2}{2} \\ 0 & x_1 & \frac{x_1^2}{2} & 1 \end{pmatrix} \tag{5.2}$$

Note that the Nil^4 metric can be also written as:

$$ds^2 = \sum_{i=1}^{4} w^i \otimes w^j,$$

where:

$$w^1 = dx_1, \;\; w^2 = dx_3, \;\; w^3 = dx_2 + x_1 dx_3, \;\; w^4 = dx_4 + x_1 dx_2 + \frac{x_1^2}{2} dx_3,$$

and the orthonormal basis dual to the 1-forms is:

$$E_1 = \frac{\partial}{\partial x_1},\ E_2 = -x_1\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + \frac{x_1^2}{2}\frac{\partial}{\partial x_4},\ E_3 = \frac{\partial}{\partial x_2} - x_1\frac{\partial}{\partial x_4},\quad E_4 = \frac{\partial}{\partial x_4}. \tag{5.3}$$

With respect to this orthonormal basis, the non-zero Christoffel symbols and the non-zero Lie brackets can be easily computed as:

$$\begin{array}{llll} \Gamma^1_{22} = -x_1, & \Gamma^1_{23} = -\frac{1}{2}(1+\frac{3x_1^2}{2}), & \Gamma^1_{24} = -\frac{1}{2}, & \Gamma^1_{33} = -x_1(1+\frac{3x_1^2}{2}), \\ \Gamma^1_{34} = -\frac{x_1}{2}, & \Gamma^2_{13} = \frac{1}{2}(1-\frac{x_1^2}{2}), & \Gamma^2_{14} = \frac{1}{2}, & \Gamma^3_{12} = \frac{1}{2}, \\ \Gamma^3_{13} = \frac{x_1}{2}, & \Gamma^4_{12} = \frac{1}{2}(1-\frac{x_1^2}{2}), & \Gamma^4_{14} = -\frac{x_1}{2}. & \end{array} \tag{5.4}$$

$$[E_1, E_2] = -E_3, \quad [E_1, E_3] = -E_4. \tag{5.5}$$

We adopt the following notation and sign convention for Riemannian curvature operator:

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \tag{5.6}$$

The Riemannian curvature tensor is given by:

$$R(X,Y,Z,W) = -g(R(X,Y)Z,W). \tag{5.7}$$

Moreover we put:

$$R_{abc} = R(E_a,E_b)E_c, \qquad R_{abcd} = R(E_a,E_b,E_c,E_d), \tag{5.8}$$

where the indices a,b,c,d take the values $1,2,3,4$. A direct computation using (5.4), (5.5), (5.6), (5.7), and (5.8) gives the following non zero components of Riemannian curvature of Nil^4 space with respect to the orthonormal basis $\{E_1,E_2,E_3,E_4\}$ (we do not list those that can be obtained by symmetric properties of curvature):

$$\begin{array}{lll} R_{1212} = -\frac{1}{2}+\frac{1}{4}x_1^2, & R_{1213} = -\frac{3}{4}x_1+\frac{1}{8}x_1^3, & R_{1214} = \frac{1}{4}x_1, \\ R_{1313} = -\frac{3}{4}-\frac{3}{4}x_1^2+\frac{1}{16}x_1^4, & R_{1314} = -\frac{1}{4}+\frac{1}{8}x_1^2, & R_{1414} = \frac{1}{4}, \\ R_{2323} = \frac{1}{4}-\frac{1}{4}x_1^2+\frac{1}{16}x_1^4, & R_{2324} = \frac{1}{4}-\frac{1}{8}x_1^2, & R_{2334} = \frac{1}{4}x_1-\frac{1}{8}x_1^3, \\ R_{2424} = \frac{1}{4}, & R_{2434} = \frac{1}{4}x_1, & R_{3434} = \frac{1}{4}x_1^2. \end{array} \tag{5.9}$$

The Ricci curvature non zero components $\{Ric_{ij}\}$ are computed as:

$$\begin{array}{llll} Ric_{11} = 1, & Ric_{22} = -\frac{1}{2}x_1^2, & Ric_{23} = -\frac{1}{4}x_1^3, & Ric_{24} = -\frac{1}{2}x_1, \\ Ric_{33} = \frac{1}{2}-\frac{1}{8}x_1^4, & Ric_{34} = -\frac{1}{4}x_1^2, & Ric_{44} = -\frac{1}{2}. & \end{array} \tag{5.10}$$

The scalar curvature σ of the Nil^4 space is given by:

$$\sigma = tr\,Ric = \sum_{i=1}^{4} g(E_i,E_i)Ric(E_i,E_i) = 1. \tag{5.11}$$

5.2 Biharmonic curves in Nil^4 space

Let $\gamma : I \longrightarrow (Nil^4, g)$ be a differentiable curve parameterized by arc length and let $\{T, N, B, D\}$ be the Frenet frame fields tangent to Nil^4 space along γ and defined as follows:
T is the unit vector field tangent to γ, N is the unit vector field in the direction of $\nabla_T T$ normal to γ, $B = T \times_{Nil^4} N$ is the first binormal vector fields and $D = N \times_{Nil^4} B$ is the second binormal vector fields.
With respect to the orthonormal basis $\{E_1, E_2, E_3, E_4\}$ we can write:

$$\begin{aligned} T &= T_1E_1 + T_2E_2 + T_3E_3 + T_4E_4 \\ N &= N_1E_1 + N_2E_2 + N_3E_3 + N_4E_4 \\ B &= B_1E_1 + B_2E_2 + B_3E_3 + B_4E_4 \\ D &= D_1E_1 + D_2E_2 + D_3E_3 + D_4E_4. \end{aligned} \tag{5.12}$$

Denote by $T = \dot{\gamma}$, the tangent unit field along γ.
We have the following result:

Lemma 11. *There are vectors fields N, B, D along the curve γ and some functions κ, τ and ρ defined on $\gamma(I) \subset Nil^4$ such that:*

$$\begin{aligned} \nabla_T T &= \kappa N \\ \nabla_T N &= -\kappa T + \tau B \\ \nabla_T B &= -\tau N + \rho D \\ \nabla_T D &= -\rho B. \end{aligned} \tag{5.13}$$

Where T, N, B and D are mutually orthogonal vectors satisfying the equations:

$$\begin{aligned} &g(T,T) = 1,\ g(N,N) = 1,\ g(B,B) = 1,\ g(D,D) = 1, \\ &g(N,T) = g(B,T) = g(D,T) = g(B,N) = g(D,N) = g(D,B) = 0. \end{aligned} \tag{5.14}$$

Proof. When $g(T,T) = 1$ we have $g(\nabla_T T, T) = 0$. Then there exist a function $\kappa \in C^\infty(I)$ and a unitary vector field $N \in \gamma^{-1}(TNil^4)$, orthogonal to T such that $\nabla_T T = \kappa N$.
Next $g(N,N) = 1$ implies $g(\nabla_T N, N) = 0$, and from the equality $g(N,T) = 0$, we derive the relation:

$$\begin{aligned} g(\nabla_T N, T) + g(N, \nabla_T T) &= g(\nabla_T N, T) + \kappa g(N,N) \\ &= g(\nabla_T N + \kappa T, T) \\ &= 0. \end{aligned}$$

Hence $\nabla_T N + \kappa T \in (span\{T, N\})^\perp$, and its exists a smooth function τ on $\gamma(I)$ and a unitary vector field $B \in \gamma^{-1}(TNil^4)$ such that the system $\{T, N, B\}$ is orthogonal and $\nabla_T N + \kappa T = \tau B$. this gives the second relation of the system.
Similarly, $g(B,B) = 1$ leads to $g(\nabla_T B, B) = 0$ and from the relation $g(B,T) = 0$, we get:

$$\begin{aligned} g(\nabla_T B, T) + g(B, \nabla_T T) &= g(\nabla_T B, T) + \kappa g(B,N) \\ &= g(\nabla_T B + \kappa T, T) \\ &= 0. \end{aligned}$$

From $g(B, N) = 0$, we have:

$$\begin{aligned} g(\nabla_T B, N) + g(B, \nabla_T N) &= g(\nabla_T B, N) - \kappa g(B, N) + \tau g(B, B) \\ &= g(\nabla_T B + \tau g(N, N)) \\ &= 0. \end{aligned}$$

Hence $\nabla_T B + \tau N \in (span\{T, N, B\})^{\perp}$, and its exists a smooth function ρ on $\gamma(I)$ and a unitary vector field $D \in \gamma^{-1}(TNil^4)$ such that the system $\{T, N, B, D\}$ is orthogonal and $\nabla_T B + \tau N = \rho D$. So we obtain the third equation of the system.
Furthermore, from $g(D, D) = 1$, we have $g(\nabla_T D, D) = 0$ and the relation $g(D, T) = 0$ leads to:

$$\begin{aligned} g(\nabla_T D, T) + g(D, \nabla_T T) &= g(\nabla_T D, T) + \kappa g(D, T) \\ &= 0. \end{aligned}$$

Also from $g(D, N) = 0$, it follows that:

$$\begin{aligned} g(\nabla_T D, N) + g(D, \nabla_T N) &= g(\nabla_T D, N) - \kappa g(D, T) + \tau g(D, N) \\ &= 0. \end{aligned}$$

Similarly the relation $g(D, B) = 0$ gives:

$$\begin{aligned} g(\nabla_T D, B) + g(D, \nabla_T B) &= g(\nabla_T D, B) - \tau g(D, N) + \rho g(D, D) \\ &= g(\nabla_T D, B) + \rho \\ &= 0. \end{aligned}$$

We get then $g(\nabla_T D, B) = -\rho$ and the last relation of the system follows.
The functions κ, τ and ρ defined in the Lemma (11) are the curvature, the torsion and the bitorsion along the curve γ, respectively.
The quadruplet (T, N, B, D) is called four-dimentional Frenet frame. □

The tension field $\tau_1(\gamma)$ of the curve γ is then given in the four-dimentional Frenet frame (T, N, B, D) by:

$$\tau_1(\gamma) = \kappa N, \tag{5.15}$$

in effect, from the definition of the tension field, we get directly:

$$\tau_1(\gamma) = \nabla_T T. \tag{5.16}$$

The biharmonic equation:

$$\tau_2(\gamma) = -(\Delta^{\gamma}\tau(\gamma) + trace_g R^N(\tau(\gamma), d\gamma)d\gamma) = 0 \tag{5.17}$$

in the case of a curve $\gamma : I \to (N, g)$ from an open interval $I \subset \mathbb{R}$ to a Riemannian manifold (N, g) parameterized by arc length transforms $(d\gamma = T, \tau(\gamma) = \nabla^{\gamma}_{\frac{\partial}{\partial s}} d\gamma(\frac{\partial}{\partial s}) = \nabla_T T)$ to the differential equation:

$$\tau_2(\gamma) = \nabla^3_T T - R(T, \nabla_T T)T = 0. \tag{5.18}$$

Using the Frenet formulas (5.13), biharmonic equation (5.18) reduces to the system:

$$\begin{array}{rcl} \kappa\kappa' & = & 0 \\ \kappa'' + \kappa^3 - \kappa\tau^2 + \kappa R(T,N,T,N) & = & 0 \\ 2\kappa'\tau + \kappa\tau' + \kappa R(T,N,T,B) & = & 0 \\ \kappa\tau\rho + \kappa R(T,N,T,D) & = & 0. \end{array} \tag{5.19}$$

Theorem 18. *$\gamma : I \longrightarrow Nil^4$ is a biharmonic curve if and only if:*

$$\begin{array}{l} \kappa = constant \neq 0 \\ \kappa^2 - \tau^2 = R(T,N,T,N) \\ \tau' = R(T,N,T,B) \\ \tau\rho = R(T,N,T,D), \end{array} \tag{5.20}$$

with:

$$\begin{array}{rcl} R(T,N,T,N) & = & (-\frac{1}{2} + \frac{1}{4}x_1^2)(T_1N_2 - T_2N_1)^2 + (-\frac{3}{4} - \frac{3}{4}x_1^2 + \frac{1}{16}x_1^4)(T_1N_3 - T_3N_1)^2 \\ & + & (\frac{1}{4} - \frac{1}{4}x_1^2 + \frac{1}{16}x_1^4)(T_2N_3 - T_3N_2)^2 + \frac{1}{4}(T_1N_4 - T_4N_1)^2 \\ & + & \frac{1}{4}x_1^2(T_3N_4 - T_4N_3)^2 + \frac{1}{4}(T_2N_4 - T_4N_2)^2 \\ & + & 2(-\frac{3}{4}x_1 + \frac{1}{8}x_1^3)(T_1N_3 - T_3N_1)(T_1N_2 - T_2N_1) \\ & + & 2(\frac{1}{4}x_1)(T_1N_2 - T_2N_1)(T_1N_4 - T_4N_1) \\ & + & 2(-\frac{1}{4} + \frac{1}{8}x_1^2)(T_1N_3 - T_3N_1)(T_1N_4 - T_4N_1) \\ & + & 2(\frac{1}{4} - \frac{1}{8}x_1^2)(T_2N_3 - T_3N_2)(T_2N_4 - T_4N_2) \\ & + & 2(\frac{1}{4}x_1)(T_4N_2 - T_2N_4)(T_4N_3 - T_3N_4) \\ & + & 2(-\frac{1}{4}x_1 + \frac{1}{8}x_1^3)(T_3N_2 - T_2N_3)(T_3N_4 - T_4N_3), \end{array}$$

and:

$$\begin{array}{rcl} R(T,N,T,B) & = & (-\frac{1}{2} + \frac{1}{4}x_1^2)(T_1N_2 - T_2N_1)(T_1B_2 - T_2B_1) \\ & + & (-\frac{3}{4} - \frac{3}{4}x_1^2 + \frac{1}{16}x_1^4)(T_1N_3 - T_3N_1)(T_1B_3 - T_3B_1) \\ & + & (\frac{1}{4} - \frac{1}{4}x_1^2 + \frac{1}{16}x_1^4)(T_2N_3 - T_3N_2)(T_2B_3 - T_3B_2) \\ & + & (\frac{1}{4})[(T_1N_4 - T_4N_1)(T_1B_4 - T_4B_1) + (T_2N_4 - T_4N_2)(T_2B_4 - T_4B_2)] \\ & + & (-\frac{3}{4}x_1 + \frac{1}{8}x_1^3)[(T_1N_2 - T_2N_1)(T_1B_3 - T_3B_1) \\ & + & (T_1N_3 - T_3N_1)(T_1B_2 - T_2B_1)] + (\frac{1}{4}x_1^2)(T_3N_4 - T_4N_3)(T_3B_4 - T_4B_3) \\ & + & (-\frac{1}{4} + \frac{1}{8}x_1^2)[(T_1N_3 - T_3N_1)(T_1B_4 - T_4B_1) \\ & + & (T_1N_4 - T_4N_1)(T_1B_3 - T_3B_1)] \\ & + & (\frac{1}{4} - \frac{1}{8}x_1^2)[(T_2N_3 - T_3N_2)(T_2B_4 - T_4B_2) \\ & + & (T_2N_4 - T_4N_2)(T_2B_3 - T_3B_2)] \\ & + & (\frac{1}{4}x_1)[(T_1N_2 - T_2N_1)(T_1B_4 - T_4B_1) + (T_1N_4 - T_4N_1)(T_1B_2 - T_2B_1)] \\ & + & (\frac{1}{4}x_1)[(T_4N_2 - T_2N_4)(T_4B_3 - T_3B_4) + (T_4N_3 - T_3N_4)(T_4B_2 - T_2B_4)] \\ & + & (\frac{1}{4}x_1 - \frac{1}{8}x_1^3)[(T_2N_3 - T_3N_2)(T_3B_4 - T_4B_3) \\ & + & (T_4N_3 - T_3N_4)(T_3B_2 - T_2B_3)]. \end{array}$$

and:

$$\begin{aligned}
R(T,N,T,D) &= (-\tfrac{1}{2}+\tfrac{1}{4}x_1^2)(T_1N_2-T_2N_1)(T_1D_2-T_2D_1)\\
&+ (-\tfrac{3}{4}-\tfrac{3}{4}x_1^2+\tfrac{1}{16}x_1^4)(T_1N_3-T_3N_1)(T_1D_3-T_3D_1)\\
&+ (\tfrac{1}{4}-\tfrac{1}{4}x_1^2+\tfrac{1}{16}x_1^4)(T_2N_3-T_3N_2)(T_2D_3-T_3D_2)\\
&+ (\tfrac{1}{4})[(T_1N_4-T_4N_1)(T_1D_4-T_4D_1)+(T_2N_4-T_4N_2)(T_2D_4-T_4D_2)]\\
&+ (-\tfrac{3}{4}x_1+\tfrac{1}{8}x_1^3)[(T_1N_2-T_2N_1)(T_1D_3-T_3D_1)\\
&+ (T_1N_3-T_3N_1)(T_1D_2-T_2D_1)]+(\tfrac{1}{4}x_1^2)(T_3N_4-T_4N_3)(T_3D_4-T_4D_3)\\
&+ (-\tfrac{1}{4}+\tfrac{1}{8}x_1^2)[(T_1N_3-T_3N_1)(T_1D_4-T_4D_1)\\
&+ (T_1N_4-T_4N_1)(T_1D_3-T_3D_1)]\\
&+ (\tfrac{1}{4}-\tfrac{1}{8}x_1^2)[(T_2N_3-T_3N_2)(T_2D_4-T_4D_2)\\
&+ (T_2N_4-T_4N_2)(T_2D_3-T_3D_2)]\\
&+ (\tfrac{1}{4}x_1)[(T_1N_2-T_2N_1)(T_1D_4-T_4D_1)+(T_1N_4-T_4N_1)(T_1D_2-T_2D_1)]\\
&+ (\tfrac{1}{4}x_1)[(T_4N_2-T_2N_4)(T_4D_3-T_3D_4)+(T_4N_3-T_3N_4)(T_4D_2-T_2D_4)]\\
&+ (\tfrac{1}{4}x_1-\tfrac{1}{8}x_1^3)[(T_2N_3-T_3N_2)(T_3D_4-T_4D_3)\\
&+ (T_4N_3-T_3N_4)(T_3D_2-T_2D_3)].
\end{aligned}$$

Proof. Using the fact that:

$$det(T,N,B,D)=1,$$

and:

$$det(T,N,B,D)=\begin{vmatrix} +T_1 & -T_2 & +T_3 & -T_4\\ -N_1 & +N_2 & -N_3 & +N_4\\ +B_1 & -B_2 & +B_3 & -B_4\\ -D_1 & +D_2 & -D_3 & +D_4 \end{vmatrix}.$$

Using Serret-Frennet formula, by direct computations, we have:

$$\begin{aligned}
\nabla_T^3 T &= \nabla_T(\nabla_T(\nabla_T T))\\
&= \nabla_T(\nabla_T \kappa N)\\
&= \nabla_T(\kappa' N+\kappa\nabla_T N)\\
&= \nabla_T(\kappa' N-\kappa(-\kappa T+\tau B))\\
&= \nabla_T(\kappa' N-\kappa^2 T+\kappa\tau B)\\
&= \nabla_T\kappa' N-\nabla_T\kappa^2 T+\nabla_T\kappa\tau B)\\
&= \kappa'' N-2\kappa\kappa' T+\kappa'\tau B+\kappa\tau' B\\
&\quad +\kappa'\nabla_T N-\kappa^2\nabla_T T+\kappa\tau\nabla_T B)\\
&= \kappa'' N-2\kappa\kappa' T+(\kappa'\tau+\kappa\tau')B\\
&\quad +\kappa'(-\kappa T+\tau B)-\kappa^2(\kappa N)+\kappa\tau(-\tau N+\rho D)\\
&= -3\kappa\kappa' T+(\kappa''-\kappa^3-\kappa\tau^2)N+(2\kappa'\tau+\kappa\tau')B+\kappa\tau\rho D.
\end{aligned}$$

By direct calculation using (5.9), we obtain:

$$
\begin{aligned}
R(T,N,T,N) &= \textstyle\sum_{i,j,k,l=1}^{4} T_iN_jT_kN_lR_{ijkl} \\
&= T_1N_2T_1N_2R_{1212} + T_1N_2T_2N_1R_{1221} + T_2N_1T_2N_1R_{2121} + T_2N_1T_1N_2R_{2112} \\
&+ T_1N_3T_1N_3R_{1313} + T_1N_3T_3N_1R_{1331} + T_3N_1T_3N_1R_{3131} + T_3N_1T_1N_3R_{3113} \\
&+ T_2N_3T_2N_3R_{2323} + T_2N_3T_3N_2R_{2332} + T_3N_2T_3N_2R_{3232} + T_3N_2T_2N_3R_{3223} \\
&+ T_1N_4T_1N_4R_{1414} + T_1N_4T_4N_1R_{1441} + T_4N_1T_4N_1R_{4141} + T_4N_1T_1N_4R_{4114} \\
&+ T_3N_4T_3N_4R_{3434} + T_3N_4T_4N_3R_{3443} + T_4N_3T_4N_3R_{4343} + T_4N_3T_3N_4R_{4334} \\
&+ T_2N_4T_2N_4R_{2424} + T_2N_4T_4N_2R_{2442} + T_4N_2T_4N_2R_{4242} + T_4N_2T_2N_4R_{4224} \\
&+ T_1N_2T_1N_3R_{1213} + T_1N_2T_3N_1R_{1231} + T_2N_1T_1N_3R_{2113} + T_3N_1T_2N_1R_{3121} \\
&+ T_1N_3T_1N_2R_{1312} + T_1N_3T_2N_1R_{1321} + T_3N_1T_1N_2R_{3112} + T_2N_1T_3N_1R_{2131} \\
&+ T_1N_2T_1N_4R_{1214} + T_1N_2T_4N_1R_{1241} + T_2N_1T_1N_4R_{2114} + T_4N_1T_2N_1R_{4121} \\
&+ T_1N_4T_1N_2R_{1412} + T_1N_4T_2N_1R_{1421} + T_4N_1T_1N_2R_{4112} + T_2N_1T_4N_1R_{2141} \\
&+ T_1N_3T_1N_4R_{1314} + T_1N_3T_4N_1R_{1341} + T_3N_1T_1N_4R_{3114} + T_4N_1T_3N_1R_{4131} \\
&+ T_1N_4T_1N_3R_{1413} + T_1N_4T_3N_1R_{1431} + T_4N_1T_1N_3R_{4113} + T_3N_1T_4N_1R_{3141} \\
&+ T_2N_3T_2N_4R_{2324} + T_2N_3T_4N_2R_{2342} + T_3N_2T_2N_4R_{3224} + T_4N_2T_3N_2R_{4232} \\
&+ T_2N_4T_2N_3R_{2423} + T_2N_4T_3N_2R_{2432} + T_4N_2T_2N_3R_{4223} + T_3N_2T_4N_2R_{3242} \\
&+ T_4N_2T_4N_3R_{4243} + T_4N_2T_3N_4R_{4234} + T_2N_4T_4N_3R_{2443} + T_3N_4T_2N_4R_{3424} \\
&+ T_4N_3T_4N_2R_{4342} + T_4N_3T_2N_4R_{4324} + T_3N_4T_4N_2R_{3442} + T_2N_4T_3N_4R_{2434} \\
&+ T_3N_2T_3N_4R_{3234} + T_3N_2T_4N_3R_{3243} + T_2N_3T_3N_4R_{2334} + T_4N_3T_2N_3R_{4323} \\
&+ T_3N_4T_3N_2R_{3432} + T_3N_4T_2N_3R_{3423} + T_4N_3T_3N_2R_{4332} + T_2N_3T_4N_3R_{2343} \\
&= (-\tfrac{1}{2} + \tfrac{1}{4}x_1^2)(T_1N_2 - T_2N_1)^2 + (-\tfrac{3}{4} - \tfrac{3}{4}x_1^2 + \tfrac{1}{16}x_1^4)(T_1N_3 - T_3N_1)^2 \\
&+ (\tfrac{1}{4} - \tfrac{1}{4}x_1^2 + \tfrac{1}{16}x_1^4)(T_2N_3 - T_3N_2)^2 + \tfrac{1}{4}(T_1N_4 - T_4N_1)^2 \\
&+ \tfrac{1}{4}x_1^2(T_3N_4 - T_4N_3)^2 + \tfrac{1}{4}(T_2N_4 - T_4N_2)^2 \\
&+ 2(-\tfrac{3}{4}x_1 + \tfrac{1}{8}x_1^3)(T_1N_3 - T_3N_1)(T_1N_2 - T_2N_1) \\
&+ 2(\tfrac{1}{4}x_1)(T_1N_2 - T_2N_1)(T_1N_4 - T_4N_1) \\
&+ 2(-\tfrac{1}{4} + \tfrac{1}{8}x_1^2)(T_1N_3 - T_3N_1)(T_1N_4 - T_4N_1) \\
&+ 2(\tfrac{1}{4} - \tfrac{1}{8}x_1^2)(T_2N_3 - T_3N_2)(T_2N_4 - T_4N_2) \\
&+ 2(\tfrac{1}{4}x_1)(T_4N_2 - T_2N_4)(T_4N_3 - T_3N_4) \\
&+ 2(-\tfrac{1}{4}x_1 + \tfrac{1}{8}x_1^3)(T_3N_2 - T_2N_3)(T_3N_4 - T_4N_3).
\end{aligned}
$$

$$
\begin{aligned}
R(T,N,T,B) &= \textstyle\sum_{i,j,k,l=1}^{4} T_iN_jT_kB_lR_{ijkl} \\
&= T_1N_2T_1B_2R_{1212} + T_1N_2T_2B_1R_{1221} + T_2N_1T_2B_1R_{2121} + T_2N_1T_1B_2R_{2112} \\
&+ T_1N_3T_1B_3R_{1313} + T_1N_3T_3B_1R_{1331} + T_3N_1T_3B_1R_{3131} + T_3N_1T_1B_3R_{3113} \\
&+ T_2N_3T_2B_3R_{2323} + T_2N_3T_3B_2R_{2332} + T_3N_2T_3B_2R_{3232} + T_3N_2T_2B_3R_{3223} \\
&+ T_1N_4T_1B_4R_{1414} + T_1N_4T_4B_1R_{1441} + T_4N_1T_4B_1R_{4141} + T_4N_1T_1B_4R_{4114} \\
&+ T_3N_4T_3B_4R_{3434} + T_3N_4T_4B_3R_{3443} + T_4N_3T_4B_3R_{4343} + T_4N_3T_3B_4R_{4334} \\
&+ T_2N_4T_2B_4R_{2424} + T_2N_4T_4B_2R_{2442} + T_4N_2T_4B_2R_{4242} + T_4N_2T_2B_4R_{4224} \\
&+ T_1N_2T_1B_3R_{1213} + T_1N_2T_3B_1R_{1231} + T_2N_1T_1B_3R_{2113} + T_3N_1T_2B_1R_{3121} \\
&+ T_1N_3T_1B_2R_{1312} + T_1N_3T_2B_1R_{1321} + T_3N_1T_1B_2R_{3112} + T_2N_1T_3B_1R_{2131} \\
&+ T_1N_2T_1B_4R_{1214} + T_1N_2T_4B_1R_{1241} + T_2N_1T_1B_4R_{2114} + T_4N_1T_2B_1R_{4121} \\
&+ T_1N_4T_1B_2R_{1412} + T_1N_4T_2B_1R_{1421} + T_4N_1T_1B_2R_{4112} + T_2N_1T_4B_1R_{2141} \\
&+ T_1N_3T_1B_4R_{1314} + T_1N_3T_4B_1R_{1341} + T_3N_1T_1B_4R_{3114} + T_4N_1T_3B_1R_{4131} \\
&+ T_1N_4T_1B_3R_{1413} + T_1N_4T_3B_1R_{1431} + T_4N_1T_1B_3R_{4113} + T_3N_1T_4B_1R_{3141} \\
&+ T_2N_3T_2B_4R_{2324} + T_2N_3T_4B_2R_{2342} + T_3N_2T_2B_4R_{3224} + T_4N_2T_3B_2R_{4232} \\
&+ T_2N_4T_2B_3R_{2423} + T_2N_4T_3B_2R_{2432} + T_4N_2T_2B_3R_{4223} + T_3N_2T_4B_2R_{3242} \\
&+ T_4N_2T_4B_3R_{4243} + T_4N_2T_3B_4R_{4234} + T_2N_4T_4B_3R_{2443} + T_3N_4T_2B_4R_{3424} \\
&+ T_4N_3T_4B_2R_{4342} + T_4N_3T_2B_4R_{4324} + T_3N_4T_4B_2R_{3442} + T_2N_4T_3B_4R_{2434} \\
&+ T_3N_2T_3B_4R_{3234} + T_3N_2T_4B_3R_{3243} + T_2N_3T_3B_4R_{2334} + T_4N_3T_2B_3R_{4323} \\
&+ T_3N_4T_3B_2R_{3432} + T_3N_4T_2B_3R_{3423} + T_4N_3T_3B_2R_{4332} + T_2N_3T_4B_3R_{2343} \\
&= (-\tfrac{1}{2} + \tfrac{1}{4}x_1^2)(T_1N_2 - T_2N_1)(T_1B_2 - T_2B_1) \\
&+ (-\tfrac{3}{4} - \tfrac{3}{4}x_1^2 + \tfrac{1}{16}x_1^4)(T_1N_3 - T_3N_1)(T_1B_3 - T_3B_1) \\
&+ (\tfrac{1}{4} - \tfrac{1}{4}x_1^2 + \tfrac{1}{16}x_1^4)(T_2N_3 - T_3N_2)(T_2B_3 - T_3B_2) \\
&+ (\tfrac{1}{4})[(T_1N_4 - T_4N_1)(T_1B_4 - T_4B_1) + (T_2N_4 - T_4N_2)(T_2B_4 - T_4B_2)] \\
&+ (-\tfrac{3}{4}x_1 + \tfrac{1}{8}x_1^3)[(T_1N_2 - T_2N_1)(T_1B_3 - T_3B_1) \\
&+ (T_1N_3 - T_3N_1)(T_1B_2 - T_2B_1)] + (\tfrac{1}{4}x_1^2)(T_3N_4 - T_4N_3)(T_3B_4 - T_4B_3) \\
&+ (-\tfrac{1}{4} + \tfrac{1}{8}x_1^2)[(T_1N_3 - T_3N_1)(T_1B_4 - T_4B_1) \\
&+ (T_1N_4 - T_4N_1)(T_1B_3 - T_3B_1)] \\
&+ (\tfrac{1}{4} - \tfrac{1}{8}x_1^2)[(T_2N_3 - T_3N_2)(T_2B_4 - T_4B_2) \\
&+ (T_2N_4 - T_4N_2)(T_2B_3 - T_3B_2)] \\
&+ (\tfrac{1}{4}x_1)[(T_1N_2 - T_2N_1)(T_1B_4 - T_4B_1) + (T_1N_4 - T_4N_1)(T_1B_2 - T_2B_1)] \\
&+ (\tfrac{1}{4}x_1)[(T_4N_2 - T_2N_4)(T_4B_3 - T_3B_4) + (T_4N_3 - T_3N_4)(T_4B_2 - T_2B_4)] \\
&+ (\tfrac{1}{4}x_1 - \tfrac{1}{8}x_1^3)[(T_2N_3 - T_3N_2)(T_3B_4 - T_4B_3) \\
&+ (T_4N_3 - T_3N_4)(T_3B_2 - T_2B_3)].
\end{aligned}
$$

$$
\begin{aligned}
R(T,N,T,D) &= \textstyle\sum_{i,j,k,l=1}^{4} T_iN_jT_kD_lR_{ijkl} \\
&= T_1N_2T_1D_2R_{1212} + T_1N_2T_2D_1R_{1221} + T_2N_1T_2D_1R_{2121} + T_2N_1T_1D_2R_{2112} \\
&+ T_1N_3T_1D_3R_{1313} + T_1N_3T_3D_1R_{1331} + T_3N_1T_3D_1R_{3131} + T_3N_1T_1D_3R_{3113} \\
&+ T_2N_3T_2D_3R_{2323} + T_2N_3T_3D_2R_{2332} + T_3N_2T_3D_2R_{3232} + T_3N_2T_2D_3R_{3223} \\
&+ T_1N_4T_1D_4R_{1414} + T_1N_4T_4D_1R_{1441} + T_4N_1T_4D_1R_{4141} + T_4N_1T_1D_4R_{4114} \\
&+ T_3N_4T_3D_4R_{3434} + T_3N_4T_4D_3R_{3443} + T_4N_3T_4D_3R_{4343} + T_4N_3T_3D_4R_{4334} \\
&+ T_2N_4T_2D_4R_{2424} + T_2N_4T_4D_2R_{2442} + T_4N_2T_4D_2R_{4242} + T_4N_2T_2D_4R_{4224} \\
&+ T_1N_2T_1D_3R_{1213} + T_1N_2T_3D_1R_{1231} + T_2N_1T_1D_3R_{2113} + T_3N_1T_2D_1R_{3121} \\
&+ T_1N_3T_1D_2R_{1312} + T_1N_3T_2D_1R_{1321} + T_3N_1T_1D_2R_{3112} + T_2N_1T_3D_1R_{2131} \\
&+ T_1N_2T_1D_4R_{1214} + T_1N_2T_4D_1R_{1241} + T_2N_1T_1D_4R_{2114} + T_4N_1T_2D_1R_{4121} \\
&+ T_1N_4T_1D_2R_{1412} + T_1N_4T_2D_1R_{1421} + T_4N_1T_1D_2R_{4112} + T_2N_1T_4D_1R_{2141} \\
&+ T_1N_3T_1D_4R_{1314} + T_1N_3T_4D_1R_{1341} + T_3N_1T_1D_4R_{3114} + T_4N_1T_3D_1R_{4131} \\
&+ T_1N_4T_1D_3R_{1413} + T_1N_4T_3D_1R_{1431} + T_4N_1T_1D_3R_{4113} + T_3N_1T_4D_1R_{3141} \\
&+ T_2N_3T_2D_4R_{2324} + T_2N_3T_4D_2R_{2342} + T_3N_2T_2D_4R_{3224} + T_4N_2T_3D_2R_{4232} \\
&+ T_2N_4T_2D_3R_{2423} + T_2N_4T_3D_2R_{2432} + T_4N_2T_2D_3R_{4223} + T_3N_2T_4D_2R_{3242} \\
&+ T_4N_2T_4D_3R_{4243} + T_4N_2T_3D_4R_{4234} + T_2N_4T_4D_3R_{2443} + T_3N_4T_2D_4R_{3424} \\
&+ T_4N_3T_4D_2R_{4342} + T_4N_3T_2D_4R_{4324} + T_3N_4T_4D_2R_{3442} + T_2N_4T_3D_4R_{2434} \\
&+ T_3N_2T_3D_4R_{3234} + T_3N_2T_4D_3R_{3243} + T_2N_3T_3D_4R_{2334} + T_4N_3T_2D_3R_{4323} \\
&+ T_3N_4T_3D_2R_{3432} + T_3N_4T_2D_3R_{3423} + T_4N_3T_3D_2R_{4332} + T_2N_3T_4D_3R_{2343} \\
&= (-\tfrac{1}{2} + \tfrac{1}{4}x_1^2)(T_1N_2 - T_2N_1)(T_1D_2 - T_2D_1) \\
&+ (-\tfrac{3}{4} - \tfrac{3}{4}x_1^2 + \tfrac{1}{16}x_1^4)(T_1N_3 - T_3N_1)(T_1D_3 - T_3D_1) \\
&+ (\tfrac{1}{4} - \tfrac{1}{4}x_1^2 + \tfrac{1}{16}x_1^4)(T_2N_3 - T_3N_2)(T_2D_3 - T_3D_2) \\
&+ (\tfrac{1}{4})[(T_1N_4 - T_4N_1)(T_1D_4 - T_4D_1) + (T_2N_4 - T_4N_2)(T_2D_4 - T_4D_2)] \\
&+ (-\tfrac{3}{4}x_1 + \tfrac{1}{8}x_1^3)[(T_1N_2 - T_2N_1)(T_1D_3 - T_3D_1) \\
&+ (T_1N_3 - T_3N_1)(T_1D_2 - T_2D_1)] + (\tfrac{1}{4}x_1^2)(T_3N_4 - T_4N_3)(T_3D_4 - T_4D_3) \\
&+ (-\tfrac{1}{4} + \tfrac{1}{8}x_1^2)[(T_1N_3 - T_3N_1)(T_1D_4 - T_4D_1) \\
&+ (T_1N_4 - T_4N_1)(T_1D_3 - T_3D_1)] \\
&+ (\tfrac{1}{4} - \tfrac{1}{8}x_1^2)[(T_2N_3 - T_3N_2)(T_2D_4 - T_4D_2) \\
&+ (T_2N_4 - T_4N_2)(T_2D_3 - T_3D_2)] \\
&+ (\tfrac{1}{4}x_1)[(T_1N_2 - T_2N_1)(T_1D_4 - T_4D_1) + (T_1N_4 - T_4N_1)(T_1D_2 - T_2D_1)] \\
&+ (\tfrac{1}{4}x_1)[(T_4N_2 - T_2N_4)(T_4D_3 - T_3D_4) + (T_4N_3 - T_3N_4)(T_4D_2 - T_2D_4)] \\
&+ (\tfrac{1}{4}x_1 - \tfrac{1}{8}x_1^3)[(T_2N_3 - T_3N_2)(T_3D_4 - T_4D_3) \\
&+ (T_4N_3 - T_3N_4)(T_3D_2 - T_2D_3)].
\end{aligned}
$$

These, together with equation (5.19), complete the proof of the theorem. □

Corollary 10. *$\gamma : I \longrightarrow Nil^4$ is a biharmonic curve if and only if:*

$$\begin{array}{l} N_3 = B_3 = D_3 = 0 \\ \kappa = constant \neq 0 \\ \tau = constant \neq 0 \\ \kappa^2 - \tau^2 = 0 \\ \rho = 0. \end{array} \tag{5.21}$$

Abstract

This thesis deals with the study of Harmonic and Biharmonic maps on Thurston geometry. The aim of this thesis is to classify harmonic and biharmonic applications in Thurston model geometries of dimension 3. Three-dimensional Thurston model geometries are classified by W. Thurston, this classification has eight Three-dimensional model geometries, to know, E^3, S^3, H^3, $S^2 \times \mathbb{R}$, $H^2 \times \mathbb{R}$, $\widetilde{Sl_2(\mathbb{R})}$, Nil_3 and Sol_3.

Also we classify harmonic and biharmonic applications in Thurston model geometries of dimension 4. R. Filipkiewicz classified the Thurston geometry of dimension four and he proved that there are 19 classes of maximal geometries in 4-dimension, to know, E^4, S^4, H^4, $P^2(\mathbb{C})$, $H^2(\mathbb{C})$, $S^2 \times S^2$, $S^2 \times E^2$, $S^2 \times H^2$, $H^2 \times E^2$, $H^2 \times H^2$, $H^3 \times E^1$, $H^3 \times E^1$, Nil^4, $Sol^4_{m,n}$, Sol^4_0, Sol^4_1, F^4, $\widetilde{Sl_2(\mathbb{R})} \times E^1$ and $Nil_3 \times E^1$.

In dimension 3 we study biharmonic Legendre curves on three-dimensional Lorentzian Heisenberg space $(\mathbb{H}_3, g)$ and we study biharmonic curves in three-dimensional generalized symmetric spaces.

We also show that a linear map from an Euclidean space in three-dimensional generalized symmetric spaces is biharmonic, and we give a complete classification of such maps.

In dimension 4 we study harmonic and biharmonic applications in Thurston geometry of dimension 4. We introduce the 4-dimensional geometry Nil^4 and we define the metric g_{Nil^4}. We give the Christoffel symbols and the Riemannian curvature to study the biharmonic curves in Nil^4 space.

Keywords: harmonic applications, biharmonic applications, Lgendre curves, Generalized symmetric spaces, Thurston geometry.

Résumé

Cette thèse porte sur l'étude des applications harmonique et biharmonique sur les modèles de Thurston. Le but de cette thèse est de classifier les applications harmoniques et biharmoniques dans les modèles Thurston de dimension 3. Les géometries tridimensionnel de Thurston sont classifier par W. Thurston, Cette classification a huit géometries, à savoir, E^3, S^3, H^3, $S^2 \times \mathbb{R}$, $H^2 \times \mathbb{R}$, $\widetilde{Sl_2(\mathbb{R})}$, Nil_3 et Sol_3.
Nous classifions également les applications harmoniques et biharmoniques dans les modèles de Thurston de dimension 4. R. Filipkiewicz a classifier les geometries de Thurston de dimension quatre et il a prouvé qu'il existe 19 classes de géometries maximales en dimension 4, à savoir, E^4, S^4, H^4, $P^2(\mathbb{C})$,

$H^2(\mathbb{C})$, $S^2 \times S^2$, $S^2 \times E^2$, $S^2 \times H^2$, $H^2 \times E^2$, $H^2 \times H^2$, $H^3 \times E^1$, $H^3 \times E^1$, Nil^4, $Sol^4_{m,n}$, Sol^4_0, Sol^4_1, F^4, $\widetilde{Sl_2(\mathbb{R})} \times E^1$ et $Nil_3 \times E^1$.
En dimension 3 nous étudions les courbes biharmoniques de Legendre sur l'espace Lorentzian Heisenberg tridimentionnel $(\mathbb{H}_3, g)$ et nous étudions les courbes biharmoniques sur l'espace symétrique généralisé tridimensionnel.
Nous montrons égualement q'une application linéaire à partir d'un espace Euclidien sur l'espace symétrique généralisé tridimensionnel est biharmonique, et nous donnons une classification complète pour chaque application.
En dimension 4 nous étudions les applications harmoniques et biharmoniques sur les geometries de Thurston de dimension 4. Nous introduisons la geometrie de dimension 4 Nil^4 et nous definissons la metrique g_{Nil^4}. Nous donnons les symboles de Christoffel et la courbure de Riemann pour étudier les applications biharmoniques dans l'espace Nil^4.

Mots clés: applications harmonique, applications biharmonic, courbes de Legendre, espaces symétrique généralisées, géométrie de Thurston.

ملخص

تتناول هذه الأطروحة دراسة التطبيقات التوافقية والبيهارمونكية على هندسة Thurston الهدف من هذه الأطروحة هو تصنيف التطبيقات التوافقية والبيهارمونيكية في فضاءات Thurston ذات 3 ابعاد. هذه الفضائات مصنفة من طرف Thurston، هذا تصنيف يحتوي على 8 نماذج هندسية و هي E^3, S^3, H^3, $S^2 \times \mathbb{R}$, $H^2 \times \mathbb{R}$, $\widetilde{Sl_2(\mathbb{R})}$, Nil^3 و Sol^3.

نصنف كذلك التطبيقات الهرمونكية و البيهارمونكية في نماذج Thurston ذات 4 أبعاد. R. Filipkiewicz صنف هندسة Thurston ذات 4 أبعاد و برهن أنه يوجد 19 صنف أقصى للهندسة في اربع ابعاد و هي E^4, S^4, H^4, $P^2(\mathbb{C})$, $H^2(\mathbb{C})$, $S^2 \times S^2$, $S^2 \times E^2$, $S^2 \times H^2$, $H^2 \times E^2$, $H^2 \times H^2$, $H^3 \times E^1$, $H^3 \times E^1$, Nil^4, $Sol^4_{m,n}$, Sol^4_0, Sol^4_1, F^4, $\widetilde{Sl_2(\mathbb{R})} \times E^1$ و $Nil^3 \times E^1$.

في بعد 3 ندرس منحنيات Legendre البيهارمونكية في فضاء Heisenberg Lorentzian ذات بعد 3، $(\mathbb{H}_3, g)$ و ندرس المنحنيات البيهرمونكية في الفضاء المتماثل العام بعد 3.

نبين كذلك انا تطبيق خطي من فضاء إقليدي ثلاثي الابعاد متماثل هو هرمونيكي, نعطي تصنيف تام لمثل هذه التطبيقات.

في بعد 4 ندرس هرمونكية و بيهارمونكية التطبيقات في هندسة Thurston ذات بعد 4. نعرف كذلك الهندسة ذات بعد 4 Nil^4 و القياس g_{Nil^4}. نعطي رموز Christofell و إنحناء Riemann لدراسة بيهرمونكية المنحنيات في الفضاء Nil^4.

كلمات مفتاحية: منحنيات هرمونكية، منحنيات بيهرمونكية، منحنيات Legendre ، فضاء معمم متماثل، هندسة Thurston .

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