الـجـمـهـوريـة الـجـزائـريـة الـديـمـقـراطيـة الـشـعـبيـة

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DEPARTEMENT DE MATHEMATIQUES

جامعة مصطفى اسطمبولي

مـعـسـكـر

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Etude de la stabilité de quelques systèmes élastiques par un feedback frontière et un terme de retard

Présentée par: Hocine MAKHELOUFI

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Study of the stability of some elastic systems by a boundary feedback and a delay term

Presented by: Hocine MAKHELOUFI

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Dedication

To my very dear parents.

To my brother and my sisters.

To everyone who encouraged me.

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Abstract

The present thesis investigates the global well-posedness and the asymptotic behaviour of the global solution for some problems arising in Mathematical elasticity. The first model considered is an abstract viscoelastic equation that includes several PDEs of hyperbolic type such as the wave or plate equation. The second problem is a non-dissipative wave equation with memory-type boundary condition localized on a part of the boundary. The third system is the one-dimensional Timoshenko beam with a linear strong damping and a strong constant delay acting on the transverse displacement of the system. The fourth one is the Porous system subjected to a nonlinear delayed damping acting on the volume fraction equation. The last one is the Bresse system with three control boundary conditions and interior delay in all the three equations. Some well-posedness results are based on the semigroup theory, whereas the others are obtained by combining the Faedo-Galerkin's procedure with some energies estimates. Furthermore, to study the solution's asymptotic behavior we employ the multipiliers method which relies on the construction of a Lyapunov functional satisfying a proper differential inequality that leads to the desired stability estimate. For the first and the fourth problems, we use also some properties of convex functions and some techniques developped in these studies [**22, 23, 102**].

key-words: Evolution equation, Wave-Plate equation, Timoshenko system, Porous system, Bresse system, Global well-posedness, Lyapunov functional, Delay term, Viscolastic damping, Convexity.

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Contents

Introduction

Elasticity is the ability of a body to resist a distorting influence and to return to its original size and shape when that influence or force is removed. Solid objects will deform when adequate loads are applied to them; if the material is elastic, the object will return to its initial shape and size after removal. This is in contrast to plasticity, in which the object fails to do so and instead remains in its deformed state.

In engineering, the elasticity of a material is quantified by the elastic modulus such as the Young's modulus, bulk modulus or shear modulus which measure the amount of stress needed to achieve a unit of strain; a higher modulus indicates that the material is harder to deform. The material's elastic limit or yield strength is the maximum stress that can arise before the onset of plastic deformation.

On the other hand, a viscous fluid in a stress state has a capacity for dissipation energy without storing, and so it flows irreversibly. Material that exhibit both viscous and elastic characteristics when undergoing deformation is called viscoelastic material. Such material is used in a vast range of applications due to its property of dissipation of mechanical energy. Thus one of the importance of this material is to reduce the excessive vibrations which can typically cause the most problems.

The concept of stability of elastic systems is also connected with the concept of stability of motion that my be defined as the ability of physical system to return to equilibrium when slightly distrubed. The classical definition due to Lyapunov states that an equilibrium state is stable if and only if all motions of the system starting close to the equilibrium state remain close to this state for all time.

Time delay is a physical property by which the response to applied forces are delayed in its effect. It appears in many applications, for example, chemical, biological, thermal and economic phenomena. For this reason, it is no surprise that the study of systems with delay effects has been in the focus of attention in the recent years. In fact, it was shown that the delay may be a source of instability. For instance, Datko **[34]** proved that the time delay may destabilize the system

$$\begin{cases} u_{tt}(x,t) = u_{xx}(x,t) - 2u_t(x,t-\tau) & \text{in } [0,1]\times[0,\infty[, \\ u(0,t) = u(1,t) = 0 & \text{in } \quad [-\tau,\infty[, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \quad [0,1]. \end{cases} \tag{0.0.1}$$

The same result was obtained in **[35]** by replacing the internal delay in (0.0.1) by a time delay in the boundary feedback control. Whereas, in the absence of delay, the system is uniformly asymptotically stable (see [**64**]). So, to stabilize a delayed system, adding control terms will be necessary. In this regard, let us consider the following linear wave equation

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0 & \text{in } \Omega \times [0,\infty[, \\ u(x,t) = 0 & \text{in } \Gamma_0 \times [0,\infty[, \\ \partial_\nu u = 0 & \text{in } \Gamma_1 \times [0,\infty[, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & \text{in } \quad \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau), & \text{in } \Omega \times [0,\tau], \end{cases} \tag{0.0.2}$$

where μ_1 is a fixed postive constant, μ_2 is a real number, Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 1$, $\Gamma = \Gamma_0 \cup \Gamma_1$ and Γ_0, Γ_1 are closed subsets of Γ with $\Gamma_0 \cap \Gamma_1 = \emptyset$. It is well-known that the system (0.0.2) is exponentially stable if $\mu_2 = 0$, that is, in the absence of delay. On the contrary, Nicaise and Pignotti [**36**] showed that the exponential stability holds if and only if $|\mu_2| < \mu_1$. This one was obtained by introducing a proper Lyapunov functional and by using appropriate observability inequalities. However, if $|\mu_2| \geq \mu_1$, they constructed a sequence of delays for which the associated solution does not converge to zero. This study was generalized by the same authors [**37**] where they treated the wave equation with a time-varying delay, in which they got an exponential decay result provided that $|\mu_2| < \sqrt{1-d}\mu_1$, where d is a fixed positive constant such that $\tau'(t) \leq d < 1$, for all $t \in \mathbb{R}^+$. Subsequently, Benaissa and Louhibi [**38**] considered the wave equation with a nonlinear damping and a delayed nonlinear internal feedback. Namely, they studied the following problem

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + \mu_1 g_1(u_t(x,t)) + \mu_2 g_2(u_t(x,t-\tau)) = 0 & \text{in } \Omega \times [0,\infty[, \\ u(x,t) = 0 & \text{in } \Gamma_0 \times [0,\infty[, \\ \partial_\nu u = 0 & \text{in } \Gamma_1 \times [0,\infty[, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & \text{in } \quad \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau), & \text{in } \Omega \times [0,\tau], \end{cases} \tag{0.0.3}$$

with $g_1 = g_2$ and proved its stability under a proper relationship between μ_1 and μ_2. The authors of [**61**] examined (0.0.3) with $\tau = \tau(t)$ and $\mu_1 g_1$,$\mu_2 g_2$ are multiplied by a positive decreasing function σ of class $C^1(\mathbb{R}_+)$ satisfying

$$\int_0^\infty \sigma(t)dt = +\infty, \quad \text{and} \quad |\sigma'(t)| \leq c\sigma(t),$$

and gave a general and explicit formula for the decay of solutions in term of σ. Very recently, Messaoudi et al.[**39**] studied a linear wave equation with a strong damping in the presence of a strong constant delay of the form

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) - \mu_1 \Delta u_t(x,t) - \mu_2 \Delta u_t(x,t-\tau) = 0 & \text{in } \Omega \times [0,\infty[, \\ u(x,t) = 0 & \text{in } \Gamma_0 \times [0,\infty[, \\ \partial_\nu u = 0 & \text{in } \Gamma_1 \times [0,\infty[, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & \text{in } \quad \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau), & \text{in } \Omega \times [0,\tau], \end{cases} \tag{0.0.4}$$

and obtained the exponential stability under the assumption $|\mu_2| < \mu_1$. In the same paper, they treated a linear wave equation with a strong damping and a distributed delay by replacing the constant delay by $\displaystyle\int_{\tau_1}^{\tau_2} \mu_2(s)ds$, where $\mu_2 : [\tau_1, \tau_2] \longrightarrow \mathbb{R}$ is a bounded function and $\tau_1 < \tau_2$ are two positive constants. They proved that the uniform stability holds if and only if

$$\int_{\tau_1}^{\tau_2} |\mu_2(s)|ds < \mu_1.$$

On the other hand, Ammari et al.[**65**] investigated a boundary stabilization problem for the wave equation with interior delay. The problem considered is (0.0.2) when the damping is acting in the boundary, that is if $\mu_1 = 0$ and $(0.0.2)_3$ is replaced by

$$\partial_\nu u = -ku_t \qquad \text{in } \Gamma_1 \times [0, +\infty[, \tag{0.0.5}$$

where $k > 0$. Under some Lions geometric condition, they showed that this system is uniformly stable for μ_2 sufficiently small.

Also, we recall two important works of abstract systems with time delay. In [**116**], Nicaise and Pignotti studied the second-order evolution equation:

$$\begin{cases} U_t(t) = \mathcal{A}U(t) + F(U(t)) + K\mathcal{B}U(t-\tau), \\ U(0) = U_0, \quad \mathcal{B}U(t-\tau) = f(t), \end{cases}$$

where $\mathcal{B}$ is a bounded operator. They showed that the operator associated to the part without delay is a generator of a strongly continuous semigroup, which is exponentially stable. Moreover, they obtained, under a smallness condition on the time delay feedback, that the system with delay is also exponentially stable. The same authors (see [**115**]) considered the following second-order evolution equations with time delay

$$\begin{cases} u_{tt} + Au + B_1B_1^*u_t(t) + B_2B_2^*u_t(t-\tau) = 0 & \text{in } [0, \infty[, \\ u(-t) = u_0, \qquad u_t(0) = u_1 & \text{in } [0, \infty[\\ B_2^*u_t(t) = f_0(t) & \text{in } [-\tau, 0], \end{cases}$$

where the bounded operator B_2 is the delay feedback operator. They proved that the system is exponentially stable when $B_2 = 0$ and that the exponential stability is preserved if $\|B_2^*\|$ is sufficiently small.

The main aim of this thesis is to adress some problems related to the global well-posedness and asymptotic stability of systems coming from elasticity in the presence of time delays. Regarding the issue of stabilization, the main purpose is to attenuate the vibrations by viscoelastic feedback or damping which is usually assumed to be viscous or proportional to velocity. So, we are interested in analyzing the decay of the energy (norm of solutions) to zero, i.e.

$$E(t) \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \infty,$$

and then we give the rate of the decay.

This thesis, which consists of six chapters, presents results of global existence and stability behavior of solutions for five evolution systems. The monograph is organized as follows.

Chapter 1 surveys the necessary notations and the main tools needed throughout this thesis. Also, we introduce the stability concepts for abstract problems that include the systems we consider here.

Chapter 2 adresses a second-order viscoelastic equation with a weak internal damping, a time-varying delay term and nonlinear weights together with suitable initial conditions. We first prove the existence of a unique global weak solution by means of the classical Faedo-Galerkin method. Then, we consider finite memory kernels $g : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$g'(t) \leq -\xi(t)H(g(t)), \quad \forall t \geq 0,$$

where H is a positive increasing and convex function and ξ is a positive function which is not necessarily monotone. And, under this general assumption, we establish optimal explicit and general stability estimates by using the well-known multipliers method and some properties of convex functions. This study generalizes and improves many earlier ones in the existing literature.

Chapter 3 presents an important contribution on decay properties of solutions for a wave equation with a viscoelastic boundary damping acting on a part of the boundary. We generalize the work of Messaoudi and Soufyane **[67]** by adding a non-dissipative term and establish a general decay rate result that allows a larger class of relaxation functions and improves previous results.

Chapter 4 deals with a linear Timoshenko system with a strong damping and a strong constant delay acting on the transverse displacement of the beam. Using the semigroup techniques, we first establish the global well-posedness of solutions under a condition on the weight of the delayed feedback and the weight of the non-delayed feedback. By using Prüss's theorem, we obtain that the system is not exponentially stable even in the case of equal-speed wave propagations. In this regard, we prove that the solution decays polynomially with rate $t^{-\frac{1}{2}}$. And in addition, we show the optimality of that rate.

Chapter 5 investigates a nonlinearly damped Porous system with nonlinear delay term together with Dirichlet-Dirichlet boundary conditions in $[0, 1] \times [0, +\infty[$. By the classical Faedo-Galerkin procedure, we first prove the well-posedness of solutions without paying any attention to the weights of feedbacks (delayed or not). This improves many earlier works existing in the literature by removing the usual restrictions imposed on these weights. Furthermore, we establish two general decay estimates with rates that depend on the speeds of wave propagation and the smoothness of the initial data. The result is new and leads to open more research areas into the provided system. The novelty lies also in the study of the nonequal-speeds case, which has never been discussed for nonlinear damped systems even in the absence of delay

Chapter 6 studies the asymptotic behaviour of a Bresse system together with three boundary controls, with delay terms in the first, second and third equation. By using semigroup method, we prove the global well-posedness of solutions. Then, assuming the weights of the delay are small, we establish the exponential decay of energy to the system by using an appropriate Lyapunov functional.

Chapter 1

Basic concepts

This chapter gathers some preliminaries facts and concepts that will be needed throughout this thesis. Also, some methods on which the well-posedness of evolution problems proof is based will be presented.

1.1 Functional Spaces

In this section, we shall introduce the key notions of Sobolev spaces, which can be considered as one of the main tools that made possible the wide development of the theory of partial differential equations in the last several decades.

Let X be a Banach space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and X' its dual space, i.e., the Banach space of all continuous linear forms on X equipped with the norm $\|.\|_{X'}$ given by

$$\|f\|_{X'} = \sup_{x \neq 0} \frac{|\langle f, x \rangle|}{\|x\|},$$

In the same way, we can define the dual space of X' that we denote by X''. The Banach space X'' is also called the bi-dual space of X.

Definition 1.1.1. *A Banach space X is reflexive if $X = X''$.*

Definition 1.1.2. *A Banach space X is called separable if it contains a countable dense subset.*

Theorem 1.1.3. (*Riesz*). *Let $(X; \langle ., . \rangle)$ be a Hilbert space then $X' = X$ in the sense: to each $f \in X'$ there exists a unique $x \in X$ such that $f = \langle x, . \rangle$ and $\|f\|_{X'} = \|x\|_X$.*

Remark 1.1.4. *As a result of the this Theorem we have that $X'' = X$, which means that a Hilbert space is reflexive.*

Corollary 1.1.5. *The following two assertions are equivalent: (i) E is reflexive; (ii) E' is reflexive.*

1.1.1 Lebesgue Spaces

Definition 1.1.6. *For $p <\in [0,\infty[$, we define*

$$L^p(\Omega) = \Big\{ u : \Omega \longrightarrow \mathbb{R} \textit{ is measurable and } \int_\Omega \big|u(x)\big|^p dx < \infty \Big\}.$$

with the following norm

$$\|u\|_{L^p} = \Big(\int_\Omega \big|u(x)\big|^p dx \Big)^{\frac{1}{p}}.$$

In addition, if $p = \infty$, we have

$$L^\infty(\Omega) = \Big\{ u : \Omega \longrightarrow \mathbb{R} \textit{ is measurable and it exists } C > 0 \textit{ such that } \big|u(x)\big| \leq C \textit{ a.e in } \Omega \Big\},$$

and

$$\|u\|_{L^\infty} = inf \Big\{ C, \ \ \big|u(x)\big| \leq C \textit{ a.e in } \Omega \Big\}.$$

Theorem 1.1.7. *The $L^p(\Omega)$ spaces have the following properties:*

i. *$L^p(\Omega)$ is a Banach space.*

ii. *$L^2(\Omega)$ is a Hilbert space with respect to the following scalar product*

$$(u, v)_{L^2(\Omega)} = \int_\Omega u(x)v(x)dx.$$

iii. *$L^p(\Omega)$ is reflexive and separable for $1 < p < \infty$ and its dual is $L^{\frac{p}{p-1}}(\Omega)$.*

iv. *$L^1(\Omega)$ is separable but not reflexive and its dual is $L^\infty(\Omega)$.*

v. *$L^\infty(\Omega)$ is not separable, not reflexive and its dual contains strictly $L^1(\Omega)$.*

Notation 1.1.8. *We denote by $L^p_{loc}(\Omega)$ the space of functions which are L^p on any bounded sub-domain of Ω.*

Theorem 1.1.9. *$C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$.*

The $\mathbf{L^p(0, T; X)}$ Spaces

Definition 1.1.10. *Let X be a Banach space, $1 < p < +\infty$, then $L^p(0, T; X)$ is the space of L^p functions u from $(0, T)$ into X which is a Banach space for the norm*

$$\|u\|_{L^p(0,T;X)} = \Big(\int_0^T \|u(x)\|_X^p \, dt \Big)^{1/p} < +\infty \quad \textit{for} \quad p < +\infty,$$

and for the norm

$$\|u\|_{L^\infty(0,T;X)} = \sup_{t\in(0,T)} \|u(x)\|_X < +\infty \quad \textit{for} \quad p = +\infty.$$

1.1.2 Sobolev Spaces

weak derivative

Definition 1.1.11. *Let $u, v \in L^p(\Omega)$. We say that v is the α^{th}-weak partial derivative of u, written*

$$D^\alpha u = v,$$

provided

$$\int_\Omega u(x) D^\alpha \phi(x) dx = (-1)^{|\alpha|} \int_\Omega v(x)\phi(x) dx, \qquad \forall \phi \in C_0^\infty(\Omega),$$

where

$$D^\alpha \phi(x) = \frac{\partial^{|\alpha|}\phi(x)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_n^{\alpha_n}},$$

and $\alpha = (\alpha_1, ..., \alpha_n)$ is called a multi-index of dimension n and $|\alpha| = \sum_{i=1}^n \alpha_i$ is the length α.

Definition of Sobolev spaces

We now define the Sobolev spaces whose members have weak derivatives of various orders lying in various L^p spaces.

Definition 1.1.12. *Fix $p \in [1, \infty)$ and let k be a nonnegative integer. The Sobolev space $W^{k,p}(\Omega)$ is the set*

$$W^{k,p}(\Omega) = \Big\{ u \in L^p(\Omega) : \ D^\alpha u \in L^p(\Omega). \ \ \forall \alpha; |\alpha| \leq k \Big\}.$$

The space $W^{k,p}(\Omega)$ is a Banack space with respect to the following norm

$$\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{i=0}^k \|D^\alpha u\|_{L^p(\Omega)}^p \right)^{1/p}, \ \text{for } p < +\infty,$$
$$\|u\|_{W^{k,\infty}(\Omega)} = \sum_{i=0}^k \|D^\alpha u\|_{L^\infty(\Omega)} \ , \qquad \text{for } p = +\infty.$$

Notation 1.1.13. *We usually write $W^{k,2}(\Omega)$ as $H^k(\Omega)$.*

Theorem 1.1.14. *The spaces $W^{k,p}(\Omega)$ have the following properties:*

i. For $1 \leq p \leq \infty$, $W^{k,p}(\Omega)$ is separable and it is reflexive space for $1 < p < \infty$.

ii. the $H^k(\Omega)$ is a Hilbert space with a scalar product defined in terms of the L^2 scalar product

$$(u, v)_{H^k(\Omega)} = \sum_{|\alpha| \leq k} (D^\alpha u, D^\alpha v)_{L^2(\Omega)} \ .$$

Notation 1.1.15. *We denote by $W_0^{k,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$.*

1.1.3 The Rellich-Kondrachov Compactness Theorem

The following Theorem will be fundamental in our study of the well-posedness of some non-linear PDEs.

Theorem 1.1.16. *Let $\Omega \subseteq \mathbb{R}^n$ be an open, bounded Lipschitz domain, and let $1 \leq p < n$. Set*

$$p^* := \frac{np}{n-p}.$$

Then, the Sobolev space $W^{1,p}(\Omega)$ is continuously embedded in $L^{p^*}(\Omega)$ and is compactly embedded in $L^q(\Omega)$ for every $1 \leq q < p^*$. In symbols,

$$W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$$

and

$$W^{1,p}(\Omega) \subset\subset L^q(\Omega) \quad \textit{for} \quad 1 \leq q < p^*.$$

Theorem 1.1.17. *Let $p > n$, then $W^{1,p}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$, with continuous imbedding.*

The $\mathbf{W^{k,p}(0,T;X)}$ spaces

Definition 1.1.18. *Let X be a Hilbert space, $1 \leq p \leq +\infty$, then $W^{k,p}(0,T;X)$ is defined by*

$$W^{k,p}(0,T;X) = \Big\{u \in L^p(0,T;X): \quad D^\alpha u \in L^p(0,T;X) \;\; \forall \alpha;\; |\alpha| \leq k\Big\}.$$

Furthermore, $W^{k,p}(0,T;X)$ is a Banach space with respect to the norm

$$\|u\|_{W^{k,p}(0,T;X)} = \left(\sum_{|\alpha|\leq k} \|D^\alpha u\|^p_{L^p(0,T;X)}\right)^{1/p}, \; \textit{for } p < +\infty,$$

$$\|u\|_{W^{k,\infty}(0,T;X)} = \sum_{|\alpha|\leq k} \|D^\alpha u\|_{L^\infty(0,T;X)}, \qquad \textit{for } p = +\infty.$$

And, in particular, the space $W^{k,2}(0,T;X)$, which is noted $H^k(0,T;X)$, is a Hilbert space with the inner product:

$$(u,v)_{H^k(0,T;X)} = \sum_{|\alpha|\leq k} \int_0^T (D^\alpha u, D^\alpha v)_X \, dt \; .$$

1.2 The weak and weak-star topologies

1.2.1 Weak topology

Denote

$$\begin{aligned} \varphi_f : & X \longrightarrow \mathbb{R} \\ & x \longrightarrow \varphi_f(x) = \langle f, x\rangle, \end{aligned}$$

when f cover X', we obtain a family $(\varphi_f)_{f\in X'}$ of applications to X in $\mathbb{R}$.

Definition 1.2.1. *The weak topology on X denoted $\sigma(X,X')$ is the weakest topology for which every $(\varphi_f)_{f\in X'}$ is continous.*

1.2.2 Weak-star topology

For all $x \in X$, denote

$$\begin{aligned} \varphi_x : & X' \longrightarrow \mathbb{R} \\ & x \longrightarrow \varphi_x(f) = \langle f, x \rangle, \end{aligned}$$

when x cover X, we obtain a family $(\varphi_x)_{x\in X}$ of applications to X' in $\mathbb{R}$.

Definition 1.2.2. *The weak-star topology on X' denoted $\sigma(X', X)$ is the weakest topology for which every $(\varphi_x)_{x\in X'}$ is continous.*

1.2.3 Weak and weak-star convergence

• **Strong convergence.** A sequence f_n is said to be strongly convergent if there exitsts $f \in X$ such that

$$\lim_{n\longrightarrow\infty} \|f_n - f\| = 0,$$

written

$$f_n \longrightarrow f.$$

• **Weak convergence.** We say that a sequence $\{x_n\} \subset X$ weakly converges to x in X, written $x_n \rightharpoonup x$ in X, if

$$\langle f, x_n \rangle \to \langle f, x \rangle,$$

for all $f \in X'$.

• **Weak-star convergence.** A sequence $\{f_n\} \subset X'$ is weak-star convergent to $f \in X'$, and we write $f_n \rightharpoonup^*$ in E', if

$$\langle f_n, x \rangle \to \langle f, x \rangle,$$

for all $x \in X$.

Remark 1.2.3. *If $\dim X < \infty$ then strong, weak and weak star convergence are equivalent.*

Remark 1.2.4. *Since $X \subset X''$ we have $f_n \rightharpoonup f$ in X' implies $f_n \rightharpoonup^* f$ in X'. And, if X is reflexive then notions of weak convergence and weak star convergence coincide.*

1.2.4 Weak and weak-star compactness in Lebesgue Spaces

In finite dimension, we have the following Bolzano-Weierstrass's theorem which is a strong compactness theorem.

Theorem 1.2.5. (*Bolzano-Weierstrass*). *If $\dim X < \infty$ and if $\{x_n\} \subset X$ is bounded, then it exist $x \in X$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \longrightarrow x$.*

In Lebesgue Spaces, we have the following two Theorems.

Theorem 1.2.6. (*weak compactness in $L^p(\Omega)$ with $1 < p < \infty$*). *Given $\{f_n\} \subset L^p(\Omega)$, if $\{f_n\}$ is bounded, then there exist $f \in L^p(\Omega)$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_n \rightharpoonup f$ in $L^p(\Omega)$.*

Theorem 1.2.7. (*weak-star compactness in $L^\infty(\Omega)$*).
Given $\{f_n\} \subset L^\infty(\Omega)$, if $\{f_n\}$ is bounded, then it exist $f \in L^\infty(\Omega)$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_n \overset{*}{\rightharpoonup} f$ in $L^\infty(\Omega)$.

1.3 Inequalities

Here in this section we collect some useful inequalities.

• **Poincaré's inequality.** Let Ω be a bounded open domain in $\mathbb{R}^n$, $n \geq 1$. Then, it exists a fixed positive constant $c_* = C\big(\Omega\big)$ such that

$$\|u\|_2 \leq c_* \|\nabla u\|_2 \qquad \forall u \in H_0^1(\Omega).$$

Moreover, if we take $n = 1$ and $\Omega =]0, L[$ we obtain that $c_* = L/\pi$.

• **Young's inequality.** Let a, b and ϵ be fixed positive constants and $m, n \geq 1$, $\frac{1}{m} + \frac{1}{n} = 1$. Then we have the inequality

$$ab \leq \frac{\epsilon^m a^m}{m} + \frac{b^n}{n\epsilon^n}.$$

• **Gronwall's inequality.** Let $T > 0$, $g \in L^1(0,T)$, $g \geq 0$ a.e and c_1, c_2 are positives constants.Let $\varphi \in L^1(0,T)$ $\varphi \geq 0$ a.e such that $g\varphi \in L^1(0,T)$ and

$$\varphi(t) \leq c_1 + c_2 \int_0^t g(s)\varphi(s)ds \qquad a.e \quad \text{in } (0,T),$$

then, we have

$$\varphi(t) \leq c_1 \text{exp}\bigg(c_2 \int_0^t g(s)ds\bigg) \qquad a.e \quad \text{in } (0,T).$$

• **Jensen's inequality** Let (Ω, A, μ) be a measure space, such that $\mu(\Omega) = 1$. If g is a real-valued function that is μ-integrable, and if is a convex function on the real line, then

$$\varphi\left(\int_\Omega g \; d\mu\right) \leq \int_\Omega \varphi \circ g \; d\mu.$$

In real analysis, we may require an estimate on $\varphi\left(\int_a^b g(x)\; dx\right)$ where a, b are real numbers, and g is a non-negative real-valued function that is Lebesgue-integrable. In this case, the Lebesgue measure of $[a, b]$ don't need to be unity. However, by integration by substitution, the interval can be rescaled so that it has measure unity. Then Jensen's inequality can be applied to get

$$\varphi\left(\int_a^b g(x)\; dx\right) \leq \frac{1}{b-a} \int_a^b \varphi((b-a)g(x))\; dx.$$

1.4 The Faedo-Galerkin method

The Faedo-Galerkin approximation is a powerful tool for solving the nonlinear partial differential equations. We explain in this section how the method works through studying a simple example. Consider the initial value problem

$$\text{(P)} \begin{cases} u_{tt}(t) + Au(t) + f(u(t)) = 0 & \text{in } [0,L] \times [0,\infty[, \\ u(0,t) = u(L,t) = 0 & \text{in } \quad [0,\infty[, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) & \text{in } \quad [0,L], \end{cases} \tag{1.4.1}$$

where $A = -\partial/\partial x^2$ with the domain $D(A) = H^2 \cap H_0^1(0, L)$ and $A^{\frac{1}{2}} = \partial/\partial x$ with the domain $D(A^{\frac{1}{2}}) = H_0^1(0, L)$, $\|.\|$ is the norm of $\big(H = L^2(0, L); \langle . \rangle\big)$

To study problem (P) we need the following assumption:
(A_1) f is a C^1 function such that $f(0) = 0$ and that it exists $\beta > 0$ satisfying

$$|f'(s)| \leq \beta, \quad \forall s \in \mathbb{R}.$$

The existence result is ensured by the following Theorem.

Theorem 1.4.1. *Assuming that* (A_1) *holds and that* $(u_0, u_1) \in (D(A), D(A^{\frac{1}{2}}))$*. Then, problem* (P) *has a unique weak solution* u *satisfying*

$$u \in L^{\infty}_{loc}\Big(0, \infty; D(A)\Big), \quad u_t \in L^{\infty}_{loc}\Big(0, \infty; D(A^{\frac{1}{2}})\Big), \quad u_{tt} \in L^{\infty}_{loc}\Big(0, \infty; H\Big).$$

Proof. To prove this result, we will employ the approximation process of Fadeo-Galerkin. For, we consider the following three steps.

i. **Approximate problem.** Let V^m a sub-space of $D(A)$ with the finite dimension d^m, and let $\{\omega^{jm}\}$ one basis of V^m. Define the solution u^m by

$$u^m(t) = \sum_{j=1}^{d_m} \xi^j(t)\omega^{jm}. \tag{1.4.2}$$

We will seek an approximate solution u^m in the form (1.4.2) where $\xi^j(t)$ are determined by

$$\big(P_m\big) \begin{cases} \langle u_{tt}^m(t), \omega^{jm}\rangle + \langle A^{\frac{1}{2}}u^m(t), A^{\frac{1}{2}}\omega^{jm}\rangle + \langle f\big(u(t)\big), \omega^{jm}\rangle = 0, \quad 1 \leq j \leq d^m \\ u^m(0) = u_0^m, \qquad u_t^m(0) = u_1^m. \end{cases} \tag{1.4.3}$$

with

$$\begin{array}{lcc} u_0^m \longrightarrow u_0 & \text{in} & D(A) \\ u_1^m \longrightarrow u_1 & \text{in} & D(A^{\frac{1}{2}}). \end{array} \tag{1.4.4}$$

By virtue of the theory of EDOs, system (P_m) accepts a unique local solution on $[0, t_m[$. In the next step, we obtain a priori estimates for the solution, so that can be extended outside $[0, t_m[$, to obtain one solution defined for all $t > 0$.

ii. **Priori estimates.**

• ***The first priori estimate.*** Replacing ω^{jm} by u^m in $(1.4.3)_1$ and using Young's inequality assumption (A_1) and Poincare's inequality, it follows that

$$\frac{d}{dt}\Big[\|u_t^m\|^2 + \|A^{\frac{1}{2}}u^m\|^2\Big] \leq c\|u_t^m\|^2 + c\|A^{\frac{1}{2}}u^m\|^2,$$

the integration over $[0, t]$, $0 \leq t \leq T$, using (1.4.4), gives

$$\|u_t^m\|^2 + \|A^{\frac{1}{2}}u^m\|^2 \leq c + c\int_0^t \Big[\|u_t^m\|^2 + \|A^{\frac{1}{2}}u^m\|^2\Big]dt,$$

employing then Gronwall's inequality, we can get

$$\|u_t^m\|^2 + \|A^{\frac{1}{2}} u^m\|^2 \leq c, \tag{1.4.5}$$

from which we conclude that u^m exists globally in $[0, +\infty)$ and

$$\begin{aligned} u^m \quad &\text{is uniformly bounded in} \quad L^\infty_{\text{loc}}\Big(0, \infty; D(A^{\frac{1}{2}})\Big), \\ u_t^m \quad &\text{is uniformly bounded in} \quad L^\infty_{\text{loc}}\Big(0, \infty; H\Big). \end{aligned} \tag{1.4.6}$$

• ***The second priori estimate.*** First, we shall estimate $u_{tt}^m(0)$ in the H-norm. For that, let $\omega^{jm} = u_{tt}^m(0)$ in $(1.4.3)_1$ and the exploit Young's inequality, assumption (H) and (1.4.4) in order to have

$$\|u_{tt}^m(0)\|^2 \leq c. \tag{1.4.7}$$

Then, differentiating $(1.4.3)_1$ with respect to t and letting $\omega^{jm} = u_{tt}^m(0)$ in the resulting equation, we obtain that

$$\frac{d}{dt}\Big[\|u_{tt}^m\|^2 + \|A^{\frac{1}{2}} u_t^m\|^2\Big] = \langle u_t^m f'(u^m), \|u_{tt}^m\|\rangle \,,$$

utilizing Young' inequality, the boundedness of f' and (1.4.5), one gets

$$\frac{d}{dt}\Big[\|u_{tt}^m\|^2 + \|A^{\frac{1}{2}} u_t^m\|^2\Big] \leq c + \|u_{tt}^m\|^2.$$

Integrating this latter estimate over $[0, t]$ and using (1.4.7), (1.4.4), we have that

$$\|u_{tt}^m\|^2 + \|A^{\frac{1}{2}} u_t^m\|^2 \leq c + \int_0^t \|u_{tt}^m\|^2 dt,$$

by applying Gronwall's inequality, we arrive at

$$\|u_{tt}^m\|^2 + \|A^{\frac{1}{2}} u_t^m\|^2 \leq c, \tag{1.4.8}$$

we, therefore, deduce that

$$\begin{aligned} u_t^m \quad &\text{is uniformly bounded in} \quad L^\infty_{\text{loc}}\Big(0, \infty; D(A^{\frac{1}{2}})\Big), \\ u_{tt}^m \quad &\text{is uniformly bounded in} \quad L^\infty_{\text{loc}}\Big(0, \infty; H\Big). \end{aligned} \tag{1.4.9}$$

• ***The third priori estimate.*** Let $\omega^j = Au$ in $(1.4.3)_1$, exploit Young's inequality, (A_1), (1.4.5) and (1.4.5) to get

$$\|Au^m\|^2 \leq c. \tag{1.4.10}$$

We thereupon have that

$$u^m \quad \text{is uniformly bounded in} \quad L^\infty_{\text{loc}}\Big(0, \infty; D(A)\Big). \tag{1.4.11}$$

iii. **Passage to the limit.** First, we have form (A_1) that

$$\|f(u^m)\|^2 \leq \beta \|u^m\|^2,$$

and so

$$\|f(u^m)\|^2 \leq c, \tag{1.4.12}$$

we then conclude that

$$f(u^m) \quad \text{is uniformly bounded in} \quad L^2\Big(0,T;H\Big). \tag{1.4.13}$$

It follows from the estimates (1.4.6), (1.4.9), (1.4.11) and (1.4.12) that it exists a sequence $\{u^n\}_{n=1}^{\infty} \subset \{u^m\}_{m=1}^{\infty}$ satisfying

$$\begin{aligned} u^n &\longrightarrow u && \text{weakly-star in} && L^{\infty}_{\text{loc}}\Big(0,\infty;D\big(A\big)\Big),\\ u^n_t &\longrightarrow u_t && \text{weakly-star in} && L^{\infty}_{\text{loc}}\Big(0,\infty;D\big(A^{\frac12}\big)\Big),\\ u^n_{tt} &\longrightarrow u_{tt} && \text{weakly-star in} && L^{\infty}_{\text{loc}}\Big(0,\infty;H\Big),\\ f(u^n) &\longrightarrow \chi && \text{weakly-star in} && L^{2}\Big(0,\infty;H\Big). \end{aligned} \tag{1.4.14}$$

We now want to prove that $\chi = f(u)$. This will be done by applying the following two lemmas.

Lemma 1.4.2. *Let X, X_0, X_1 be three Banack spaces such that $X_0 \subseteq X \subseteq X_1$. Assuming that X_0 is compactly embedded in X and that X is continuously embedded in X_1, also, assume that X_0 and X_1 are reflexive spaces. For $1 < p,\ q < \infty$, let*

$$W = \Big\{u \in L^p(0,T;X_0)\,/\,\dot{u} \in L^q(0,T;X_1)\Big\}.$$

Then, the embedding of $W \hookrightarrow L^p(0,T;X)$ is also compact.

Lemma 1.4.3. *Let $Q = \Omega \times [0,T]$ an open bounded domain in $\mathbb{R}^n \times \mathbb{R}$, and f_μ, f are two functions in $L^q\big(Q\big)$, $1 < q < \infty$ such that*

$$\|f_\mu\|_{L^q(Q)} \leq c, \qquad f_\mu \longrightarrow g \quad a.e \text{ in } Q.$$

Then, $f_\mu \longrightarrow f$ weakly in $L^q(Q)$.

Going back to the proof of Theorem 1.4.1. It follows from (1.4.5) that u^n is bounded in $L^\infty(0,T;D(A^{\frac12}))$ and u^n_t is bounded in $L^\infty(0,T;H)$. Then, the injection by continuity in L^p gives us the boundedness of u^n in $L^2(0,T;D(A^{\frac12}))$ and u^n_t in $L^2(0,T;D(A^{\frac12}))$. It is known that the embedding $D(A^{\frac12}) \hookrightarrow H$ is compact. Then, with

$$W = \Big\{u \in L^2(0,T;D(A^{\frac12}))\,/\,\dot{u} \in L^2(0,T;H)\Big\},$$

it results that the embedding $W \hookrightarrow L^2(0,T;H)$ is compact. And so, we can extract a subsequence of u^n, represented again by u^n, such that

$$u^n \longrightarrow u \quad \text{strongly in} \quad L^2\big(0,T;H\big),$$

which implies

$$u^n \longrightarrow u \qquad \text{a.e. on } Q.$$

By the continuity of f, we have

$$f(u^n) \quad \longrightarrow \quad f(u) \qquad \text{a.e. on } Q. \tag{1.4.15}$$

Combining this with (1.4.12), and using Lemma 1.4.3, we obtain

$$f(u^n) \longrightarrow f(u) \quad \text{weakly-star in} \quad L^2(Q),$$

which implies that $\chi = f(u)$.

Now, we are ready to prove that is u a weak solution of (1.4.1). Consider function $v \in C(0,T;H)$ having the form

$$v(t) = \sum_{i=1}^{N} c^{in}(t)\omega^i, \tag{1.4.16}$$

where $N \geq n$ is a fixed integer.
Then, by multiplying $(1.4.3)_1$ by $c^{in}(t)$ and summing the resultants over i from 1 to N, we find that

$$\int_0^T \big\langle u_{tt}^n + Au^n + f(u^n)\big\rangle v dt = 0. \tag{1.4.17}$$

After passing to the limit in (1.4.17) as $n \longrightarrow +\infty$ and using (1.4.14), we arrive at

$$\int_0^T \big\langle u_{tt} + Au + f(u)\big\rangle v dt = 0.$$

The above equation holds for all $v \in L^2\big(0,T;H\big)$ since the functions of the forms (1.4.16) are dense in $L^2\big(0,T;H\big)$. This ends the proof. □

1.5 Semigroups

Let $\big(X, \langle .,. \rangle, \|.\|\big)$ be a Hilbert space and let $A : D(A) \subset X \longrightarrow X$ be a linear operator. We introduce in this section some basic concepts that will be needed in the study of the initial value problem

$$\begin{cases} u_t(t) = Au(t), & 0 < t \leq T, \\ u(0) = x. \end{cases} \tag{1.5.1}$$

By a strong solution here we mean a function $u : [0,T] \longrightarrow X$ such that it is continuously differentiable, $u(t) \in D(A)$ for $t > 0$ and (1.5.1) is satisfied. And, by a weak solution we mean a function $u \in C\big(0,T;X\big)$ such that for each $v \in D(A^*)$ where A^* is adjoint of A, the function (u,v) is absolutely continous on $[0,T]$ and

$$\frac{d}{dt}\langle u, v\rangle = \langle u, A^* v\rangle \quad a.e \;\text{ on } [0,T].$$

When A is an $n \times n$ constant matrix, it is well-known that the solution of (1.5.1) is given by

$$u(t) = e^{At}x, \tag{1.5.2}$$

where

$$e^{At} = \sum_{n=0}^{\infty} \frac{A^n t^n}{t!}. \tag{1.5.3}$$

The family of matrices e^{At} is called a semigroup. And, it has the following properties, the so-called semigroup properties,

$$e^{A.0} = I, \qquad e^{A(t+s)} = e^{At}e^{As}.$$

The vast majority of evolution equations can be written in the form (1.5.1), and so the solution should be given by

$$u(t) = e^{At}x.$$

However, the definition (1.5.3) of the matrix exponential no longer makes a sense in that case and we have to explain what e^{At} means. This leads to the concept of semigroup in a Banach space.

1.5.1 Definitions and Properties

Definition 1.5.1. *Let X be a Banach space. A family $(S(t))_{t\geq 0}$ of bounded linear operators from X into X is called a semigroup of bounded linear operators on X if*

i. $S(0) = \mathrm{I}d$.

ii. $\forall s, t \geq 0$, $S(t+s) = S(t)S(s)$.

If moreover the semigroup $S(t)$ also fulfills

$$\lim_{t \longrightarrow 0^+} S(t)x = x,$$

then $S(t)$ is called a strongly continuous semigroup (in short, a C_0-semigroup) of bounded linear operators on X.

It is clear that the definition of semigroup is an extension of e^{At} defined in (1.5.3). For the matrix A, one has

$$Au = \lim_{t \longrightarrow 0^+} \frac{e^{At}u - u}{t} = \lim_{t \longrightarrow 0^+} \frac{e^{At}u - e^{A0}u}{t} = \left. \frac{d^+ e^{At}u}{dt} \right|_{t=0}. \tag{1.5.4}$$

We can say that the semigroup is generated by A. The relation (1.5.4) can be generalized to the case of C_0-semigroup. Indeed, the linear operator A defined by

$$D(A) = \left\{ u \in X \ : \ \lim_{t \longrightarrow 0^+} \frac{e^{At}u - u}{t} \text{ exists} \right\}$$

and

$$Au = \lim_{t \longrightarrow 0^+} \frac{S(t)u - u}{t} = \left. \frac{d^+ S(t)u}{dt} \right|_{t=0} \qquad \text{for } u \in X$$

is called the infinitesimal generator of the semigroup $S(t)$. We usually write $S(t) = e^{At}$.

Let us now present some important properties of C_0-semigroups.

Theorem 1.5.2. *Let $S(t) = e^{At}$ be a C_0-semigroup. Then*

i. It exist constants $\omega \geq 0$ and $M \geq 1$ such that

$$\|S(t)\| \leq Me^{\omega t}, \quad \text{for } 0 \leq t \leq \infty. \tag{1.5.5}$$

ii. For each $x \in X$, $t \longrightarrow S(t)x$ is a continuous function from $[0, \infty)$ into X.

iii. For $x \in D(A)$, $S(t)x \in D(A)$ and

$$\frac{d}{dt}S(t)x = AS(t)x = S(t)Ax. \tag{1.5.6}$$

iv. The domain of A is dense in X and A is a closed linear operator.

v. If B is a bounded linear operator on X, then $A + B$ is the infinitesimal generator of a C_0-semigroup $T(t)$ on X satisfying $\|T(t)\| \leq Me^{(w+M\|B\|)t}$.

It is concluded from the property (1.5.6) that $u(t) = S(t)x$ is the solution of the equation

$$u_t(t) = Au,$$

that is why the theory of semigroup is an efficient tool for studying linear partial differential equations.

Theorem 1.5.3. *If A is the infinitesimal generator of a C_0-semigroup $S(t)$ on X, then*

i. for every $x \in D(A)$ the abstract Cauchy problem (1.5.1) *has a unique strong solution given by $u(t) = S(t)x$.*

ii. for all $x \in X$ the abstract Cauchy problem (1.5.1) *has a unique weak solution given by $u(t) = S(t)x$.*

Remark 1.5.4. *If $\omega = 0$ in* (1.5.5) *then the corresponding semigroup is uniformly bounded. If moreover $M = 1$ then $(S(t))_{t\geq 0}$ is called a C_0-semigroup of contractions.*

1.5.2 The Lumer-Phillips Theorem

We shall characterize the infinitesimal generators of C_0-semigroup, that is, we will give conditions on an unbounded operator A to be the infinitesimal generator of C_0-semigroup. For this purpose, we start by recalling the notion of m-dissipative operators.

Definition 1.5.5. *Let $A : D(A) \subset X \longrightarrow X$ be an unbounded linear operator. A is said to be dissipative* (*monotone*) *if $Re\langle Au, u\rangle)_X \leq 0$ $\big(Re\langle Au, u\rangle)_X \geq 0\big)$. The dissipative operator A is m-dissipative if $\lambda I - A$ is surjective for some $\lambda > 0$.*

Let us now present the so-called Lumer-Phyllips Theorem.

Theorem 1.5.6. *A linear operator $A : D(A) \subset X \longrightarrow X$ is the infinitesimal generator of a C_0-semigroup $(S(t))_{t\geq 0}$ if and only if A is m-dissipative.*

1.6 Stability Concepts

Various types of stability have been defined for the solutions of differential equations describing dynamical systems. The most important one is that concerning the stability of solutions near to a point of equilibrium. This may be discussed by the theory of A. Lyapunov. In the simplest of terms, if the solutions that start out near an equilibrium point u_e stay near u_e forever, then u_e is Lyapunov stable. More strongly, if u_e is Lyapunov stable and all solutions that start out near u_e converge to u_e, then u_e is asymptotically stable. Consider in a Hilbert space X, the differential equation

$$\frac{du(t)}{dt} = Au(t), \tag{1.6.1}$$

where A is an unbounded operator with domain $D(A) \subset X$. Suppose that the above differential equation subject to the condition $u(0) = x$ is uniquely solvable and that $u \equiv 0$ is an equilibrium point for (1.6.1). We are interested in the asymptotic stability of the null solution in the sense.

Definition 1.6.1. *The equilibrium of* (1.6.1) *is:*

i. exponentially stable if it exist constants $a, b, \epsilon > 0$ *such that if* $\|x\| < \epsilon$, *then*

$$\|u(t,x)\| \leq ae^{-bt}\|x\|, \qquad \forall t \geq 0.$$

ii. polynomially stable if it exist constants $\alpha, \beta, \epsilon > 0$ *such that if* $\|x\| < \epsilon$, *then*

$$\|u(t,x)\| \leq \beta t^{-\alpha}\|x\|, \qquad \forall t > 0.$$

1.6.1 Stability of semigroup

We present in what follows some definitions about strong, exponential and polynomial stability of a C_0-semigroup.

Definition 1.6.2. *Assume that* A *is the infinitesimal generator of a strongly continuous semigroup of contractions* $(S(t))_{t\geq 0}$ *on* X. *We say that the* C_0 *-semigroup* $(S(t))_{t\geq 0}$ *is:*

i. strongly stable if

$$\lim_{t\to+\infty} \|S(t)u\|_X = 0, \quad \forall u \in X.$$

ii. uniformly stable if

$$\lim_{t\to+\infty} \|S(t)\|_{\mathcal{L}(X)} = 0.$$

iii. exponentially stable if it exist two positive constants M *and* ϵ *such that*

$$\|S(t)u\|_X \leq Me^{-\epsilon t}\|u\|_X, \quad \forall t > 0, \forall u \in X.$$

iv. polynomially stable if it exist two positive constants C *and* α *such that*

$$\|S(t)u\|_X \leq Ct^{-\alpha}\|u\|_X, \quad \forall t > 0, \forall u \in X.$$

Chapter 2

Global well-posedness and stability results for an abstract viscoelastic equation with a non-constant delay term and nonlinear weight

2.1 Introduction

Let $\mu_1 : \mathbb{R}_+ \to]0, +\infty[$ and $\mu_2 : \mathbb{R}_+ \to \mathbb{R}$ be given functions. Let $\mathcal{A} : D(\mathcal{A}) \to \mathcal{H}$ be a self-adjoint linear positive definite operator with dense domain $D(\mathcal{A}) \subset \mathcal{H}$ where $\big(\mathcal{H}, \langle ., . \rangle, \|.\|\big)$ is a real separable Hilbert space. In the present chapter, we deal with the following second-order evolution equation

$$\begin{cases} u_{tt}(t) + \mathcal{A}u(t) - \big(g * \mathcal{A}u\big)(t) + \mu_1(t)u_t(t) + \mu_2(t)u_t(t - \tau(t)) = 0 & \text{in }]0, +\infty[, \\ u(-t) = u_0, \quad u_t(0) = u_1 & \text{in } [0, +\infty[, \\ u(t - \tau(0)) = f_0(t - \tau(0)) & \text{in } [0, \tau(0)], \end{cases} \tag{2.1.1}$$

where $u : \mathbb{R}_+ \longrightarrow \mathcal{H}$ is the displacement vector, $\tau(t) > 0$ is the time-varying delay, the initial data (u_0, u_1, f_0) are given in suitable function spaces, the so-called relaxation function g is positive non-increasing defined on $\mathbb{R}^+$ and $*$ means the usual convolution product in t

$$(\psi * \varphi)(t) = \int_0^t \psi(t - s)\varphi(s)ds.$$

The above general model includes several PDEs of hyperbolic type like the wave and plate equations. So, to illustrate our abstract result, some applications will be given in section 5.

In the absence of delay (i.e. if $\mu_2 \equiv 0$), the issues of well-posedness, stability and long-time dynamics to problem (2.1.1) have been addressed in many papers, see for instance [**1-9**]. In [**1**], Dafermos proved that the solutions converge to 0 as t tends to ∞, but no explicit rate of decay was given. Later on, it has been proved that if the kernel function g decays exponentially, i.e.,

$$\exists k_0 > 0 : \quad g'(t) \leq -k_0 g(t), \quad \forall t \in \mathbb{R}^+ \tag{2.1.2}$$

then the solutions decay exponentially. And, when g decays polynomially then the solutions also decay polynomially with the same rate, see for example [**10, 13, 14**]. Messaoudi [**15**] proposed a general condition on g that gives a general decay result where the usual exponential, polynomial and logarithmic decay rates are only special cases. To be specific, he considered the following viscoelastic wave equation

$$u_{tt}(x,t) - \Delta u(x,t) + \big(g * \Delta u\big)(x,t) = 0 \tag{2.1.3}$$

together with Dirichlet boundary condition in $\Omega \times [0,+\infty[$ where Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 1$. And, for kernel function g satisfying

$$g'(t) \leq -k_0(t)g(t) \quad \forall t \in \mathbb{R}^+, \tag{2.1.4}$$

where $k_0 : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-increasing differentiable function, he established a general energy decay result. This condition has been employed by several authors in their study of the solution's asymptotic stability for systems related to (2.1.3), see for instance [**16-21**]. After that, inspired by the pioneer work of Lasiecka and Tataru [**22**], a more general assumption of the form:

$$g'(t) \leq -H(g(t)) \quad \forall t \in \mathbb{R}^+, \tag{2.1.5}$$

where H is an increasing convex function with $H(0) = 0$, was introduced by Alabau-Boussouira et al.[**23**] and used then in a great number of papers (see [**24, 25**]), in which explicit formulas for the decay rates of the solutions were obtained in terms of H. Very recently, Mustafa [**26**] considered a viscoelastic wave equation with the kernel function g satisfying

$$g'(t) \leq -k_0(t)H(g(t)) \quad \forall t \in \mathbb{R}^+ \tag{2.1.6}$$

and gained a new general energy decay. For some papers employed (2.1.6), we refer the interested readers to [**27-33**].

As is well-known, if the weight of the delayed feedback is smaller than the non-delayed one then we can obtain the stability result for the wave equation (see [**36, 37**]). In this regard, an interesting problem was examined by Benaissa et al.[**40**]. In that work the authors considered the wave equation with time dependent weights $\mu_1(t)$, $\mu_2(t)$ and a constant delay and established a general decay estimate. Analogous result was obtained in [**41**] for the one space dimension and in the case of time-varying delay. We also recall the contribution of Remil and Hakem [**42**], who realized the same results for a viscoelastic wave equation with a constant delay term.

On the other hand, problems similar to (2.1.1) with constant weights μ_1 and μ_2 have been considered in a series of papers, see for instance [**43-51**]. In [**47**], Benaissa et al. discussed a nonlinear wave equation with homogeneous Dirichlet boundary condition of the form

$$u_{tt}(x,t) - \Delta u(x,t) + \big(g * \Delta u\big)(x,t) + \mu_1 F_1(u_t(x,t)) + \mu_2 F_2(u_t(x,t-\tau)) = 0. \tag{2.1.7}$$

in $\Omega\times]0,+\infty[$, where F_1 and F_2 are two real functions. In that paper they got, under a suitable relation between μ_1 and μ_2, a general energy decay result in the case of kernel function satisfying (2.1.4). Also, we would like to mention the contribution of Kirane and Said-Houari [**43**], who investigated (2.1.7) with $F_1(s) = F_2(s) = s$ and obtained, under the assumptions: $0 < \mu_2 \leq \mu_1$ and (2.1.4), a general decay of the total energy. Due to the presence of the internal memory-feedback, they showed that the uniform stability takes place even if $\mu_1 = \mu_2$ in contrast to [**36**] where no memory-feedback was available. It is worth to note that the frictional damping term $\mu_1 u_t$ played a crucial role in the proof of this results that is why the authors indicated that the case when $\mu_1 = 0$ is an open problem. Guesmia [**44**] solved that open problem where he considered the following general model

$$u_{tt}(t) + \mathcal{A}u(t) - \int_0^{+\infty} g(t-s)\mathcal{A}u(s)ds + \mu u_t(t-\tau) = 0 \quad \text{in }]0,+\infty[, \tag{2.1.8}$$

of second-order evolution equation with infinite memory and a time delay term in a real Hilbert space $\mathcal{H}$. Under appropriate assumptions on the operator $\mathcal{A}$ and the weight of the delay μ, he realized the uniform stability of the above mentioned system. Precisely, by assuming g fulfills (2.1.2), he proved that the memory-type damping is enough to stabilize (2.1.8) exponentially.

Equation (2.1.1) is a second-order evolution equation with finite memory and a non-constant delay term and non-linear weight. As far as we know, the general rate of decay for abstract equations of the form (2.1.1) has never been considered.

The aims of this chapter are:

(i) to give the global solvability without the usual restrictions of:

$$\mu_1, \mu_2 > 0, \quad |\mu_2| < \sqrt{1-d}\mu_1, \quad \tau \in W^{2,\infty}(0,T).$$

(ii) to prove, under a wider class of relaxation functions, general decay results which improve many earlier related works with finite and infinite memory.

The remaining parts are written as follows. In the next section, we give the needed assumptions, notations and materials. In section 3, we state and prove the global well-posedness result. In section 4, we study the solution's asymptotic stability by using the multiplier method integrated with some ideas developed in [**22, 23, 26**], taking into account the nature of our problem. Finally, we give in section 5 some applications in order to illustrate our abstract result.

2.2 Preliminaries

In this section, we shall provide some assumptions, preliminary facts and notations which are used in the course of our investigation. We start, as in the work [**37**], by introducing the new variable

$$z(\rho,t) = u_t(t-\rho\tau(t)), \qquad \rho \in [0,1], \; t>0,$$

which satisfies

$$\tau(t)z_t(\rho,t) + \big(1-\rho\tau'(t)\big)z_\rho(\rho,t) = 0 \qquad \text{in } [0,1]\times[0,+\infty].$$

Hence, our problem (2.1.1) is equivalent to

$$\begin{cases} u_{tt}(t) + \mathcal{A}u(t) - \big(g*\mathcal{A}u\big)(t) + \mu_1(t)u_t(t) + \mu_2(t)z(1,t) = 0 & \text{in }]0,\infty[, \\ \tau(t)z_t(\rho,t) + \big(1-\rho\tau'(t)\big)z_\rho(\rho,t) = 0 & \text{in} [0,1]\times[0,\infty[, \\ z(0,t) = u_t & \text{in } [0,\infty[, \\ u(-t) = u_0, \quad u_t(0) = u_1 & \text{in } [0,\infty[, \\ z(\rho,0) = f_0(-\rho\tau(0)) & \text{in } [0,1]. \end{cases} \tag{2.2.1}$$

In order to deal with the new variable z, we consider the following spaces

$$V_1 = \Big\{ z :]0,1[\longrightarrow \mathcal{H}, \qquad \int_0^1 \|z(\rho,t)\|^2 d\rho < \infty \Big\},$$
$$V_2 = \Big\{ z :]0,1[\longrightarrow \mathcal{H}, \quad \int_0^1 \|\mathcal{A}^{\frac{1}{2}} z(\rho,t)\|^2 d\rho < \infty \Big\},$$
$$V_3 = \Big\{ z :]0,1[\longrightarrow \mathcal{H}, \quad \int_0^1 \|\mathcal{A}^{\frac{1}{2}} z_\rho(\rho,t)\|^2 d\rho < \infty \Big\}.$$

We know that V_i, $i = 1,2,3$, are Hilbert spaces and endowed with the following inner products

$$\big\langle z, \widetilde{z} \big\rangle_{V_1} = \int_0^1 \big\langle z(\rho,t), \widetilde{z}(\rho,t) \big\rangle_{\mathcal{H}} d\rho,$$
$$\big\langle z, \widetilde{z} \big\rangle_{V_2} = \int_0^1 \big\langle \mathcal{A}^{\frac{1}{2}} z(\rho,t), \mathcal{A}^{\frac{1}{2}} \widetilde{z}(\rho,t) \big\rangle_{\mathcal{H}} d\rho,$$
$$\big\langle z, \widetilde{z} \big\rangle_{V_3} = \int_0^1 \big\langle \mathcal{A}^{\frac{1}{2}} z_\rho(\rho,t), \mathcal{A}^{\frac{1}{2}} \widetilde{z}_\rho(\rho,t) \big\rangle_{\mathcal{H}} d\rho.$$

For the sake of simplicity, we consider the following notations

$$(\psi \diamond \varphi)(t) = \int_0^t \psi(t-s)\big(\varphi(t) - \varphi(s)\big) ds,$$
$$(\psi \circ \varphi)(t) = \int_0^t \psi(t-s) \|\varphi(t) - \varphi(s)\|^2 ds.$$

To study system (2.2.1), we require the following assumptions:
(A_1) It exists a fixed positive constant γ satisfying

$$\|u\|^2 \leq \gamma \|\mathcal{A}^{\frac{1}{2}} u\|^2, \quad \forall u \in D(\mathcal{A}^{\frac{1}{2}}). \tag{2.2.2}$$

(A_2) τ is a differentiable function such that

$$0 < \tau_1 \leq \tau(t) \leq \tau_2, \qquad \forall t > 0, \tag{2.2.3}$$

$$\tau'(t) \leq d < 1, \qquad \forall t > 0. \tag{2.2.4}$$

(A_3) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a decreasing differentiable function such that

$$g(0) > 0, \qquad \int_0^\infty g(s) ds = \beta < 1. \tag{2.2.5}$$

(A_4) There exist a C^0 function $\xi : \mathbb{R}_+ \to]0,+\infty[$ which is not necessarily monotone and a C^1-function $H : \mathbb{R}_+ \to [0,+\infty$ which is either linear or strictly increasing and strictly convex of class C^2 on $[0,r]$, $r \leq g(0)$, with $H(0) = H'(0) = 0$ such that

$$g'(t) \leq -\xi(t) H\big(g(t)\big), \quad \forall t \geq 0, \tag{2.2.6}$$

and

$$\xi(t) \leq c_1, \quad \forall t \geq 0, \tag{2.2.7}$$

where c_1 is a fixed positive constant.

Remark 2.2.1. *As $\xi(t) > 0$ for all $t \geq 0$, then it exists $c_0 > 0$ such that*

$$c_0 < \xi(t), \quad \forall t \geq 0. \tag{2.2.8}$$

Remark 2.2.2. *[26]* 1. *Assumption (A_3) and (A_4) imply that it exists t_0 such that*

$$g(t_0) = r \quad \text{and} \quad g(t) \leq r, \quad \text{for all } t \geq t_0.$$

As g is decreasing, then we have

$$0 < g(t_0) \leq g(t) \leq g(0).$$

The Combination of this fact with the continuity of H, yield that, for $b, k > 0$,

$$b < H\big(g(t)\big) \leq k, \quad \forall t \in [0, t_0].$$

Then, we can get for any $t \in [0, t_0]$

$$g'(t) \leq -\xi(t)H\big(g(t)\big) \leq -b\xi(t) = -\frac{b}{g(0)}\xi(t)g(0) \leq -\frac{b}{g(0)}\xi(t)g(t),$$

hence, thanks to (2.2.8), we can infer

$$g'(t) \leq -\frac{b\,c_0}{g(0)}g(t), \quad \forall t \in [0, t_0],$$

and so

$$g(t) \leq -\frac{g(0)}{b\,c_0}\,g'(t), \quad \forall t \in [0, t_0]. \tag{2.2.9}$$

2. *If H is a strictly increasing and strictly convex function of class C^2 on $[0, r]$ with*

$$H(0) = H'(0) = 0,$$

then it has an extension $\overline{H}$, which is also strictly increasing and strictly convex C^2 function on $(0, \infty)$. For example, we can define $\overline{H}$ for any $t > r$ as

$$\overline{H}(t) = \frac{H''(r)}{2}t^2 + \big(H'(r) - H''(r)r\big)t + \Big(H(r) + \frac{H''(r)}{2}r^2 - H(r)r\Big).$$

In what follows, we state some essential Lemmas which will be used later.

Lemma 2.2.3. (*[54]*) *For all $\varphi, \psi \in C^1\big(\mathbb{R}_+; \mathbb{R}\big)$, we have*

$$(\psi * \varphi)\varphi_t = -\frac{1}{2}\psi(t)|\varphi|^2 + \frac{1}{2}\left(\psi' \diamond \varphi\right) - \frac{1}{2}\frac{d}{dt}\left[(\psi \diamond \varphi) - \left(\int_0^t \psi(s)ds\right)|\varphi|^2\right].$$

Lemma 2.2.4. (*[26, 55]*) *Assume that (A3) holds, then, with $f(t) = \int_t^\infty g(s)ds$, the functional*

$$\mathcal{R}(t) = \int_0^t f(t-s)\|\mathcal{A}^{\frac{1}{2}}u(s)\|^2 ds$$

fulfills an estimate of the form

$$\mathcal{R}'(t) \leq -\frac{1}{2}\big(g \circ \mathcal{A}^{\frac{1}{2}}u\big)(t) + 3\beta\|\mathcal{A}^{\frac{1}{2}}u\|^2.$$

Lemma 2.2.5. (*Jensen's inequality*) *Let $H : [a,b] \longrightarrow \mathbb{R}$ be a convex function. Assume that $f : \Omega \longrightarrow [a,b]$ and $h : \Omega \longrightarrow \mathbb{R}$ are integrable with $h(s) \geq 0$ for all $x \in \Omega$ and $\int_\Omega h(x)dx = m > 0$. Then,*

$$H\left(\frac{1}{m}\int_\Omega f(x)h(x)dx\right) \leq \frac{1}{m}\int_\Omega H\big(f(x)\big)h(x)dx.$$

We now define the modified energy functional associated with the solution of our problem (2.2.1) as

$$E(t) = \frac{1}{2}\left[\|u_t(t)\|^2 + \|\mathcal{A}^{\frac{1}{2}}u(t)\|^2 + \big(g \circ \mathcal{A}^{\frac{1}{2}}u\big)(t) + \tau(t)\eta(t)\int_0^1 \|z(\rho,t)\|^2 d\rho\right], \tag{2.2.10}$$

where $\eta : \mathbb{R}_+ \longrightarrow]0,+\infty[$ is a non-increasing function of class $C^1(\mathbb{R}_+)$.

Our point of departure will be to provide an explicit upper bound of the derivative of the modified energy functional E.

Lemma 2.2.6. *Let (u,z) be the solution of (2.2.1), then E fulfills for any $t \geq 0$*

$$\begin{aligned} E'(t) \leq &- \Big(\mu_1(t) - \frac{1}{2}\eta(t) - \frac{1}{2\varepsilon_1}|\mu_2(t)|^2\Big)\|u_t\|^2 \\ &- \Big(\frac{1-d}{2}\eta(t) - \frac{\varepsilon_1}{2}\Big)\|z(x,1)\|^2 \\ &+ \frac{1}{2}\big(g' \circ \mathcal{A}^{\frac{1}{2}}u\big)(t) - \frac{1}{2}g(t)\|\mathcal{A}^{\frac{1}{2}}u(t)\|^2. \end{aligned} \tag{2.2.11}$$

Proof. Taking the inner product of Eq.$(2.2.1)_1$ with u_t in $\mathcal{H}$, we obtain the identity

$$\frac{1}{2}\frac{d}{dt}\Big[\|u_t\|^2 + \|\mathcal{A}^{\frac{1}{2}}u\|^2\Big] + \mu_1(t)\|u_t\|^2 + \mu_2(t)\langle u_t, z(x,1)\rangle = \langle\big(g * \mathcal{A}^{\frac{1}{2}}u\big)(t), \mathcal{A}^{\frac{1}{2}}u(t)\rangle, \tag{2.2.12}$$

by Lemma 2.2.3, the latter term rewrites as

$$\begin{aligned} \langle\big(g * \mathcal{A}^{\frac{1}{2}}u\big)(t), \mathcal{A}^{\frac{1}{2}}u(t)\rangle = &\frac{1}{2}\frac{d}{dt}\Big[\int_0^t g(s)ds\|\mathcal{A}^{\frac{1}{2}}u(t)\|^2 - \big(g \circ \mathcal{A}^{\frac{1}{2}}u\big)(t)\Big] \\ &+ \frac{1}{2}\big(g' \circ \mathcal{A}^{\frac{1}{2}}u\big)(t) - \frac{1}{2}g(t)\|\mathcal{A}^{\frac{1}{2}}u(t)\|^2, \end{aligned} \tag{2.2.13}$$

hence, (2.2.12) becomes

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big[\|u_t\|^2 + \Big(1 - \int_0^t g(s)ds\Big)\|\mathcal{A}^{\frac{1}{2}}u\|^2 + \big(g \circ \mathcal{A}^{\frac{1}{2}}u\big)(t)\Big] + \mu_1(t)\|u_t\|^2 \\ &\quad + \mu_2(t)\langle u_t, z(x,1)\rangle = \frac{1}{2}\big(g' \circ \mathcal{A}^{\frac{1}{2}}u\big)(t) - \frac{1}{2}g(t)\|\mathcal{A}^{\frac{1}{2}}u(t)\|^2. \end{aligned} \tag{2.2.14}$$

After taking the inner product of Eq.$(2.2.1)_2$ with $\eta(t)z(x,\rho,t)$ in V_1, we may have

$$\frac{1}{2}\tau(t)\eta(t)\frac{d}{dt}\int_0^1 \|z(\rho,t)\|^2 d\rho + \frac{1}{2}\eta(t)\int_0^1 (1 - \rho\tau'(t))\frac{d}{d\rho}\|z(\rho,t)\|^2 d\rho = 0,$$

using the non-increasing property of η, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1 \tau(t)\eta(t)\|z(\rho,t)\|^2 d\rho \leq & -\frac{1}{2}\eta(t)\int_0^1 (1-\rho\tau'(t))\frac{d}{d\rho}\|z(\rho,t)\|^2 d\rho \\ & +\frac{1}{2}\tau'(t)\eta(t)\int_0^1 \|z(\rho,t)\|^2 d\rho,\end{aligned}$$

that is,

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \tau(t)\eta(t)\|z(\rho,t)\|^2 d\rho \leq -\frac{1}{2}\eta(t)\int_0^1 \frac{d}{d\rho}\Big[(1-\rho\tau'(t))\|z(x,\rho,t)\|^2\Big]d\rho,$$

then, since $z(0,t)=u_t$, it results that

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \tau(t)\eta(t)\|z(\rho,t)\|^2 d\rho \leq -\frac{1}{2}\eta(t)\Big[(1-\tau'(t))\|z(1,t)\|^2-\|u_t\|^2\Big]. \tag{2.2.15}$$

The sum of (2.2.14) and (2.2.15), bearing (2.2.10) in mind, yields

$$\begin{aligned}E'(t)\leq & -\Big(\mu_1(t)-\frac{1}{2}\eta(t)\Big)\|u_t\|^2-\frac{1-d}{2}\eta(t)\|z(x,1)\|^2+\frac{1}{2}\big(g'\circ\mathcal{A}^{\frac{1}{2}}u)(t) \\ & -\frac{1}{2}g(t)\|\mathcal{A}^{\frac{1}{2}}u(t)\|^2-\mu_2(t)\langle u_t,z(1,t)\rangle.\end{aligned} \tag{2.2.16}$$

Applying Cauchy-Schwarz's and Young's inequalities to the latter term of (2.2.16), we obtain

$$\mu_2(t)\langle u_t,z(1,t)\rangle \leq \frac{1}{2\varepsilon_1}|\mu_2(t)|^2.\|u_t\|^2+\frac{\varepsilon_1}{2}\|z(1,t)\|^2.$$

Plugging this estimate into (2.2.16), we get (2.2.11). This concludes the proof. □

We end this section by establishing the following Lemma which will be needed in the proof of global existence.

Lemma 2.2.7. *For any regular solution of system* (2.2.1), *we have*

$$\begin{aligned}&\|z(1,t)\|^2+\int_0^1\|z_t(\rho,t)\|^2 d\rho \\ \leq\ & c\|u_t\|^2+c\int_0^1\|z_\rho(\rho,0)\|^2 d\rho+c\int_0^1\|z(\rho,t)\|^2 d\rho+c\int_0^t\|u_{tt}\|^2 dt.\end{aligned} \tag{2.2.17}$$

Proof. By taking the inner product of Eq.$(2.2.1)_2$ with $z(\rho,t)$ in V_1, it holds that

$$\int_0^1 \tau(t)\big\langle z_t(\rho,t),z(\rho,t)\big\rangle d\rho+\int_0^1\Big[(1-\rho\tau'(t))\frac{d}{d\rho}\|z(\rho,t)\|^2\Big]d\rho=0,$$

that is,

$$\int_0^1\Big[\tau(t)\big\langle z_t(\rho,t),z(\rho,t)\big\rangle+\tau'(t)\|z(\rho,t)\|^2\Big]d\rho+\int_0^1\frac{d}{d\rho}\Big[(1-\rho\tau'(t))\|z(\rho,t)\|^2\Big]d\rho=0,$$

and so

$$(1-\tau'(t))\|z(1,t)\|^2 = \|u_t\|^2 + \int_0^1 \Big[\tau'(t)\|z(\rho,t)\|^2 - \tau(t)\big\langle z_t(\rho,t), z(\rho,t)\big\rangle\Big] d\rho,$$

utilizing then Cauchy-Schwarz's and Young's inequalities, we arrive at

$$\int_0^1 \|z(1,t)\|^2 d\rho \leq c\|u_t\|^2 + c\int_0^1 \|z(\rho,t)\|^2 d\rho + \int_0^1 \|z_t(\rho,t)\|^2 d\rho. \tag{2.2.18}$$

Besides, taking the inner product of Eq.$(2.2.1)_2$ with $z_t(t,\rho)$ in V_1 and proceeding in the same way, we can get

$$2\int_0^1 \|z_t(\rho,t)\|^2 d\rho \leq c\int_0^1 \|z_\rho(\rho,t)\|^2 d\rho. \tag{2.2.19}$$

Moreover, we obtain after taking the inner product of Eq.$(2.2.1)_2$ with $2z_{t\rho}(\rho,t)$ in V_1 that

$$\tau(t)\int_0^1 \frac{d}{d\rho}\Big(\|z_t(\rho,t)\|^2\Big) d\rho + \frac{d}{dt}\int_0^1 \|z_\rho(\rho,t)\|^2 d\rho = 0.$$

This, combined with $z_t(0,t) = u_{tt}$, leads us to

$$\frac{d}{dt}\int_0^1 \|z_\rho(\rho,t)\|^2 d\rho \leq c\|u_{tt}\|^2,$$

the integration on $[0,t]$ gives

$$\int_0^1 \|z_\rho(\rho,t)\|^2 d\rho \leq \int_0^1 \|z_\rho(\rho,0)\|^2 d\rho + c\int_0^t \|u_{tt}\|^2 dt. \tag{2.2.20}$$

Collecting the estimates (2.2.18)-(2.2.20), we end up with (2.2.17). □

2.3 The global well-posedness

This section aims to show the following global well-posedness result:

Theorem 2.3.1. *Assume that* (A_1)*-*(A_3) *hold and that* μ_i*,* $i = 1, 2$*, are bounded. Then, for any* $(u_0, u_1, f_0) \in D\big(\mathcal{A}\big) \times D\big(\mathcal{A}^{\frac{1}{2}}\big) \times V_3$ *satisfying* $f_0(0,.) = u_1$*, problem* (2.1.1) *has only one global weak solution*

$$u \in L^\infty_{loc}\Big(0,\infty; D\big(\mathcal{A}\big)\Big), \quad u_t \in L^\infty_{loc}\Big(0,\infty; D\big(\mathcal{A}^{\frac{1}{2}}\big)\Big), \quad u_{tt} \in L^\infty_{loc}\Big(0,\infty; \mathcal{H}\Big).$$

Proof. In order to prove the result given in Theorem 2.3.1 we will implement the well-known Faedo-Galerkin procedure.

i. **Approximate problem.** First, we assume $D\big(\mathcal{A}^{\frac{1}{2}}\big)$ to be separable. Let $T > 0$ and for every $m \geq 1$, let $\{\Phi^m\}_{m\in\mathcal{N}}$ be an Hilbertian basis of $D\big(\mathcal{A}\big)$, $D\big(\mathcal{A}^{\frac{1}{2}}\big)$ and $\mathcal{H}$. We denote by F^m the space generated by $\Phi^1, ..., \Phi^m$.
Defining, for $1 \leq i \leq m$, the sequence $\Psi^i(\rho)$ as

$$\Psi^i(0) = \Phi^i.$$

Then, we may extend $\Psi^i(0)$ by $\Psi^i(\rho)$ over V_1 and denote Z^m the space generated by $\Psi^1, \Psi^2, ..., \Psi^m$. We shall construct an approximate solution $\big(u^m(t), z^m(\rho,t)\big)$ in the form

$$u^m(t) = \sum_{i=1}^{m} d^{im}(t)\Phi^i, \tag{2.3.1}$$

$$z^m(\rho,t) = \sum_{i=1}^{m} e^{im}(t)\Psi^i(\rho). \tag{2.3.2}$$

So, we are intend to determine the coeffitions d^{im} and e^{im}, $i = 1, ..., m$, to satisfy

$$\begin{cases} \quad \big\langle \mathcal{A}^{\frac{1}{2}}u^m(t) - \big(g * \mathcal{A}^{\frac{1}{2}}u^m\big)(t), \mathcal{A}^{\frac{1}{2}}\Phi^i\big\rangle \\ +\big\langle u^m_{tt}(t) + \mu_1(t)u^m_t(t) + \mu_2(t)z^m(1,t), \Phi^i\big\rangle = 0, \\ \big\langle z^m_t(\rho,t) + (1-\rho\tau'(t))z^m_\rho(\rho,t), \Psi^i(\rho)\big\rangle \quad = 0, \end{cases} \tag{2.3.3}$$

with

$$\begin{cases} u^m(0) = u^m_0 \longrightarrow u_0 & \text{in } D\big(\mathcal{A}\big), \\ u^m_t(0) = u^m_1 \longrightarrow u_1 & \text{in } D\big(\mathcal{A}^{\frac{1}{2}}\big), \\ z^m(.,0) = z^m_0 \longrightarrow f_0(.) & \text{in } V_3, \end{cases} \tag{2.3.4}$$

as $m \longrightarrow +\infty$.

By the standard methods of EDOs, we may show that the system (2.3.3)-(2.3.4) accepts only one solution $\big(u^m(t), z^m(\rho,t)\big)$ on the interval $[0, T_m]$, $0 < T_m < T$. In the next step, we will show that T_m is independent of m, that is, the approximate solution becomes global and defined for all $t > 0$.

ii. **Priori estimates.**

• **The first priori estimate.** In view of Lemma 2.2.6, the functional

$$E^m(t) = \frac{1}{2}\|u^m_t\|^2 + \frac{1}{2}\big(1-g_0\big)\|\mathcal{A}^{\frac{1}{2}}u^m\|^2 + \frac{1}{2}\big(g \circ \mathcal{A}^{\frac{1}{2}}u^m\big) + \frac{1}{2}\tau(t)\eta(t)\int_0^1 \|z^m(\rho,t)\|^2 d\rho,$$

satisfies for any $\varepsilon_1 > 0$

$$\frac{d}{dt}E^m(t) \le c\|u^m_t\|^2 - \Big(\frac{1-d}{2}\eta(t) - \frac{\varepsilon_1}{2}\Big)\|z^m(1,t)\|^2. \tag{2.3.5}$$

For a suitable ε_1, an integration over $(0,t)$ yields that

$$E^m(t) \le E^m(0) + c\int_0^t \|u^m_t\|^2 dt. \tag{2.3.6}$$

Taking the convergences (2.3.4) into account and employing the Gronwall's inequality, we obtain the first estimate below

$$E^m(t) \le L_1, \tag{2.3.7}$$

where $L_1 > 0$ is independent of m. This estimate assures the global existence of (u^m, z^m). And, it is deduced that

$$\begin{array}{lll} u^m & \text{is uniformly bounded in} & L^\infty_{\text{loc}}\Big(0,\infty; D(\mathcal{A}^{\frac{1}{2}})\Big), \\ u^m_t & \text{is uniformly bounded in} & L^\infty_{\text{loc}}\Big(0,\infty; \mathcal{H}\Big), \\ z^m & \text{is uniformly bounded in} & L^\infty_{\text{loc}}\Big(0,\infty; V_1\Big). \end{array} \tag{2.3.8}$$

• **The second priori estimate.** Let $\Phi^i = 2\mathcal{A}u^m_t$ in $(2.3.3)_1$ and exploit lemma 2.2.3 in order to have

$$\begin{aligned} &\frac{d}{dt}\Big[\|\mathcal{A}^{\frac{1}{2}}u^m_t\|^2 + \big(1-g_0\big)\|\mathcal{A}u^m\|^2 + \big(g\circ \mathcal{A}u^m\big)\Big] + g(t)\|\mathcal{A}u^m\|^2 \\ &- \big(g'\circ \mathcal{A}u^m\big)(t) + 2\mu_1(t)\|A^{\frac{1}{2}}u^m_t\|^2 + 2\mu_2(t)\big\langle A^{\frac{1}{2}}z^m(1,t), A^{\frac{1}{2}}u^m_t\big\rangle = 0. \end{aligned} \tag{2.3.9}$$

Next, replacing Ψ^i by $2\mathcal{A}z^m(\rho,t)$ in $(2.3.3)_2$, we find that

$$\frac{d}{dt}\int_0^1 \tau(t)\|\mathcal{A}^{\frac{1}{2}}z^m(\rho,t)\|^2 d\rho = -\int_0^1 \frac{d}{d\rho}\big(1-\rho\tau'(t)\big)\|\mathcal{A}^{\frac{1}{2}}z^m(\rho,t)\|^2 d\rho,$$

which implies

$$\frac{d}{dt}\int_0^1 \tau(t)\|\mathcal{A}^{\frac{1}{2}}z^m(\rho,t)\|^2 d\rho \leq -\big(1-d\big)\|\mathcal{A}^{\frac{1}{2}}z^m(1,t)\|^2 + \|A^{\frac{1}{2}}u^m_t\|^2, \tag{2.3.10}$$

Moreover, with

$$\mathcal{E}^m(t) = \|\mathcal{A}^{\frac{1}{2}}u^m_t\|^2 + \big(1-g_0\big)\|\mathcal{A}u^m\|^2 + \big(g\circ \mathcal{A}u^m\big) + \int_0^1 \tau(t)\|\mathcal{A}^{\frac{1}{2}}z^m(\rho,t)\|^2 d\rho,$$

it follows from the estimates (2.3.9)-(2.3.10) that

$$\frac{d}{dt}\mathcal{E}^m(t) \leq \big(1-2\mu_1(t)\big)\|\mathcal{A}^{\frac{1}{2}}u^m_t\|^2 - \big(1-d\big)\|\mathcal{A}^{\frac{1}{2}}z^m(1,t)\|^2 - 2\mu_2(t)\big\langle \mathcal{A}^{\frac{1}{2}}z^m(1,t), \mathcal{A}^{\frac{1}{2}}u^m_t\big\rangle.$$

Due to Cauchy Schwarz's and Young's inequalities, one has

$$\frac{d}{dt}\mathcal{E}^m(t) \leq c\|\mathcal{A}^{\frac{1}{2}}u^m_t\|^2 - \Big((1-d)-\varepsilon_2\Big)\|\mathcal{A}^{\frac{1}{2}}z^m(1,t)\|^2.$$

Up to fixing ε_2 sufficiently small to get

$$\frac{d}{dt}\mathcal{E}^m(t) \leq c\|\mathcal{A}^{\frac{1}{2}}u^m_t\|^2,$$

the integration over $(0,t)$, bearing (2.3.4) in mind, gives

$$\mathcal{E}^m(t) \leq c + c\int_0^t \|\mathcal{A}^{\frac{1}{2}}u^m_t\|^2 dt.$$

By virtue of Gronwall's inequality, we obtain for $L_2 > 0$, that

$$\mathcal{E}^m(t) \leq L_2. \tag{2.3.11}$$

We, thererfore, conclude that

$$\begin{aligned} u^m \quad & \text{is uniformly bounded in} \quad L^\infty_{\text{loc}}\Big(0,\infty; D(\mathcal{A})\Big), \\ u^m_t \quad & \text{is uniformly bounded in} \quad L^\infty_{\text{loc}}\Big(0,\infty; D(\mathcal{A}^{\frac{1}{2}})\Big), \\ z^m \quad & \text{is uniformly bounded in} \quad L^\infty_{\text{loc}}\Big(0,\infty; V_2\Big). \end{aligned} \tag{2.3.12}$$

- **The third priori estimate:** Let $\Phi^i = u^m_{tt}$ in $(2.3.3)_1$, we have the identity

$$\|u^m_{tt}(t)\|^2 = -\big\langle \mathcal{A}u^m(t) - (g * \mathcal{A}u^m)(t) + \mu_1(t)u^m_t(t) + \mu_2(t)z^m(1,t), u^m_{tt}(t)\big\rangle.$$

The boundedness of μ_i, $i = 1, 2$, Cauchy-Schwarz's inequality and Young's inequality, yield that

$$\|u^m_{tt}\|^2 \leq c\Big(\|\mathcal{A}u^m(t)\|^2 + \|u^m_t(t)\|^2 + \|z^m(1,t)\|^2\Big) + \big\langle (g * \mathcal{A}u^m)(t), u^m_{tt}(t)\big\rangle. \tag{2.3.13}$$

Since $\mathcal{A}u(s) = \big(\mathcal{A}u(s) - \mathcal{A}u(t)\big) + \mathcal{A}u(t)$, we easily show that

$$\begin{aligned} \big\langle \big(g * \mathcal{A}u\big)(t), u^m_{tt}\big\rangle &= \int_0^t g(t-s)\big\langle \mathcal{A}u(s) - \mathcal{A}u(t), u^m_{tt}\big\rangle ds + \big\langle \mathcal{A}u(t), u^m_{tt}\big\rangle \\ &\leq c\big(g \circ \mathcal{A}u\big)(t) + c\|\mathcal{A}u(t)\|^2 + \frac{1}{2}\|u^m_{tt}\|^2. \end{aligned}$$

Substituting this latter estimate into (2.3.13) and using (2.3.11), we get

$$\|u^m_{tt}\|^2 \leq c + c\|z^m(1,t)\|^2,$$

which is

$$\|u^m_{tt}\|^2 + c\int_0^1 \|z^m_t(\rho,t)\|^2 d\rho \leq c + c\Big(\|z^m(1,t)\|^2 + \int_0^1 \|z^m_t(\rho,t)\|^2 d\rho\Big), \tag{2.3.14}$$

Thanks to (2.2.17), we see that (2.3.14) implies

$$\begin{aligned} &\|u^m_{tt}\|^2 + c\int_0^1 \|z^m_t(\rho,t)\|^2 d\rho \\ &\leq c + c\|u^m_t\|^2 + c\int_0^1 \|z^m_\rho(\rho,0)\|^2 d\rho + c\int_0^1 \|z^m(\rho,t)\|^2 d\rho + c\int_0^t \|u^m_{tt}\|^2 dt, \end{aligned} \tag{2.3.15}$$

then, by (2.3.4) and (2.3.7), we have

$$\|u^m_{tt}\|^2 + \int_0^1 \|z^m_t(\rho,t)\|^2 d\rho \leq c + c\int_0^t \|u^m_{tt}\|^2 dt, \tag{2.3.16}$$

and with the help of Gronwall's inequality, we arrive at

$$\|u_{tt}^m(t)\|^2 + \int_0^1 \|z_t^m(\rho,t)\|^2 d\rho \leq L_3, \tag{2.3.17}$$

where L_3 is a fixed postive constant. We hence derive that

$$\begin{array}{lll} u_{tt}^m & \text{is uniformly bounded in} & L^\infty_{\text{loc}}\Big(0,\infty;\mathcal{H}\Big), \\ z_t^m & \text{is uniformly bounded in} & L^\infty_{\text{loc}}\Big(0,\infty;V_1\Big). \end{array} \tag{2.3.18}$$

It follows from the priori estimates (2.3.8), (2.3.12) and (2.3.18) that it exist subsequences $\{u^n\}_{n=1}^\infty \subset \{u^m\}_{m=1}^\infty$ and $\{z^n\}_{n=1}^\infty \subset \{z^m\}_{m=1}^\infty$ such that

$$\begin{array}{lllll} u^n & \longrightarrow & u & \text{weakly-star in} & L^\infty_{\text{loc}}\Big(0,\infty;D\big(\mathcal{A}\big)\Big), \\ u_t^n & \longrightarrow & u_t & \text{weakly-star in} & L^\infty_{\text{loc}}\Big(0,\infty;D\big(\mathcal{A}^{\frac{1}{2}}\big)\Big), \\ u_{tt}^n & \longrightarrow & u_{tt} & \text{weakly-star in} & L^\infty_{\text{loc}}\Big(0,\infty;\mathcal{H}\Big), \\ z^n & \longrightarrow & z & \text{weakly-star in} & L^\infty_{\text{loc}}\Big(0,\infty;V_2\Big), \\ z_t^n & \longrightarrow & z_t & \text{weakly-star in} & L^\infty_{\text{loc}}\Big(0,\infty;V_1\Big). \end{array} \tag{2.3.19}$$

The proof of the existence result can be completed following the same steps of proof of Theoreme 1.4.1.

For the uniqueness, we assume that (u_1, z_1) and (u_2, z_2) are two pairs of weak solutions of (2.2.1). Then, $(u, z) = (u_1, z_1) - (u_2, z_2)$ fulfills the system

$$\begin{cases} u_{tt}(t) + \mathcal{A}u(t) - \big(g * \mathcal{A}u\big)(t) + \mu_1(t)u_t(t) + \mu_2(t)z(1,t) = 0 & \text{in} \quad]0,\infty[, \\ \tau(t)z_t(\rho,t) + \big(1-\rho\tau'(t)\big)z_\rho(\rho,t) \qquad = 0 & \text{in} [0,1]\times[0,\infty[, \\ z(0,t) = u_t & \text{in} \quad [0,\infty[, \\ u(-t) = u_1 = 0 & \text{in} \quad [0,\infty[, \\ z(\rho,0) = 0 & \text{in} \quad [0,1]. \end{cases} \tag{2.3.20}$$

To get the uniquness result, it is sufficient to show that $(0, 0)$ is the only weak solution of (2.3.20). For that, invoking (2.3.6), and noting that $E(0) = 0$, we obtain

$$E(t) \leq c \int_0^t E(s)ds.$$

As $E > 0$, the reserved Gronwall's inequality implies that $E(t) = 0$ for all $t > 0$ and so $(u, z) \equiv (0, 0)$. Consequently, (2.2.1) has only one global weak solution. □

2.4 Stability

We will divide this section into three subsections: in the first part, we investigate the decay property in the case of $|\mu_2| < \sqrt{1-d}\mu_1$, in the second one, we discuss the situation when

$|\mu_2| = \sqrt{1-d}\mu_1$. And, in the last one, we give some examples to illustrate our new general decay results. As a starting point, letting $\varepsilon_1 = \frac{1}{\sqrt{1-d}}|\mu_2(t)|$ in (2.2.11), we immediately get

$$
\begin{aligned}
E'(t) \le &-\Big(\mu_1(t) - \frac{1}{2}\eta(t) - \frac{1}{2\sqrt{1-d}}|\mu_2(t)|\Big)\|u_t\|^2 \\
&-\Big(\frac{1-d}{2}\eta(t) - \frac{\sqrt{1-d}}{2}|\mu_2(t)|\Big)\|z(x,1)\|^2 \\
&+\frac{1}{2}\big(g' \circ \mathcal{A}^{\frac{1}{2}}u\big)(t) - \frac{1}{2}g(t)\|\mathcal{A}^{\frac{1}{2}}u(t)\|^2.
\end{aligned}
\tag{2.4.1}
$$

Then, we assume that the non-increasing function η satisfies

$$
\begin{cases}
\frac{1}{\sqrt{1-d}}\mu_2(t) < \eta(t) < 2\mu_1(t) - \frac{1}{\sqrt{1-d}}\mu_2(t), & \text{if } |\mu_2(t)| < \sqrt{1-d}\mu_1(t), \\
\eta(t) = \mu_1(t) = \frac{1}{\sqrt{1-d}}|\mu_2(t)|, & \text{if } |\mu_2(t)| = \sqrt{1-d}\,\mu_1(t).
\end{cases}
\tag{2.4.2}
$$

2.4.1 General decay for $|\mu_2(t)| < \sqrt{1-d}\mu_1(t)$

In this subsection, we prove our new general decay result in the case of $|\mu_2| < \sqrt{1-d}\mu_1$. Recalling (2.4.2), then (2.4.1) implies

$$
E'(t) \le -C_1\|u_t\|^2 - C_2\|z(x,1)\|^2 + \frac{1}{2}\big(g' \circ A^{\frac{1}{2}}u\big)(t) - \frac{1}{2}g(t)\|A^{\frac{1}{2}}u(t)\|^2, \tag{2.4.3}
$$

where

$$
\begin{aligned}
C_1 &= \mu_1(t) - \frac{1}{2}\eta(t) - \frac{1}{2\sqrt{1-d}}|\mu_2(t)| > 0, \\
C_2 &= \frac{1-d}{2}\eta(t) - \frac{\sqrt{1-d}}{2}|\mu_2(t)| > 0.
\end{aligned}
$$

The main result of this part is ensured by the following Theorem.

Theorem 2.4.1. *Let (u,z) be the solution of* (2.2.1). *Assuming that (A_1)-(A_4) are fulfilled, $|\mu_2| < \sqrt{1-d}\,\mu_1$ and that μ_1 is a bounded function. Then, it exist two positive constants a and a_1 such that the solution of* (2.2.1) *satisfies*

$$
E(t) \le a\,H_1^{-1}\left(a_1\int_{t_0}^{t}\zeta(s)ds\right), \quad \forall t > t_0 \tag{2.4.4}
$$

where

$$
t_0 = g^{-1}(r), \qquad \zeta(t) = \min\big(\xi(t), \mu_1(t)\big), \qquad H_1(t) = \int_t^r \frac{1}{H_2(s)}ds, \quad \text{and} \quad H_2(s) = sH'(\epsilon_0 s).
$$

Proof. To derive the stability result stated in Theorem 2.4.1, we shall establish some Lemmas given for all regular solution of (2.2.1).

Lemma 2.4.2. *The functional K defined by*

$$K(t) = \langle u_t, u \rangle$$

fulfills, along the solution of (2.2.1)*, the estimate*

$$\begin{aligned} K'(t) \leq & \Big(1 + \frac{\gamma}{1-g_0}|\mu_1(t)|^2\Big)\|u_t\|^2 - \frac{1}{4}\Big(1-g_0\Big)\|\mathcal{A}^{\frac{1}{2}}u\|^2 \\ & + \frac{C_\nu}{4(1-g_0)}\big(h \circ u\big)(t) + \frac{\gamma}{1-g_0}|\mu_2(t)|^2.\|z(1,t)\|^2, \end{aligned} \tag{2.4.5}$$

for all $0 < \nu < 1$*, where*

$$C_\nu = \int_0^\infty \frac{g^2(s)}{\nu g(s) - g'(s)} ds \qquad and \qquad h(t) = \nu g(t) - g'(t).$$

Proof. Taking the derivative of K and using Eq.$(2.2.1)_1$, we can get

$$\begin{aligned} K'(t) = & \|u_t\|^2 + \|\mathcal{A}^{\frac{1}{2}}u\|^2 + \big\langle (g * \mathcal{A}^{\frac{1}{2}}u)(t), \mathcal{A}^{\frac{1}{2}}u \big\rangle \\ & - \mu_1(t)\langle u_t, u\rangle - \mu_2(t)\langle z(1,t), u\rangle, \end{aligned}$$

which is,

$$\begin{aligned} K'(t) = & \|u_t\|^2 - \big(1-g_0\big)\|\mathcal{A}^{\frac{1}{2}}u\|^2 - \big\langle (g \diamond \mathcal{A}^{\frac{1}{2}}u)(t), \mathcal{A}^{\frac{1}{2}}u(t) \big\rangle \\ & + \mu_1(t)\langle u_t, u(t)\rangle + \mu_2(t)\langle z(1,t), u(t)\rangle . \end{aligned}$$

In view of the assumption (A1), Cauchy Schwarz's inequality and Young's inequality, we obtain that

$$\begin{aligned} K'(t) \leq & \Big(1 + \frac{\gamma}{1-g_0}|\mu_1(t)|^2\Big)\|u_t\|^2 - \frac{1}{4}\Big(1-g_0\Big)\|\mathcal{A}^{\frac{1}{2}}u\|^2 \\ & + \frac{\gamma}{1-g_0}|\mu_2(t)|^2.\|z(1,t)\|^2 + \frac{1}{4(1-g_0)}\|(g \diamond u)(t)\|^2. \end{aligned}$$

Now, using Cauchy Schwarz's inequality, the latter term of the above inequality can be estimated as

$$\begin{aligned} \|(g \diamond u)(t)\|^2 \leq & \Big(\int_0^t g(t-s)\|\mathcal{A}^{\frac{1}{2}}u(s) - \mathcal{A}^{\frac{1}{2}}u(t)\|ds\Big)^2 \\ = & \Big(\int_0^t \frac{g(t-s)}{\sqrt{h(t-s)}}\sqrt{h(t-s)}\|\mathcal{A}^{\frac{1}{2}}u(s) - \mathcal{A}^{\frac{1}{2}}u(t)\|ds\Big)^2 \\ \leq & \Big(\int_0^t \frac{g^2(s)}{\nu g(s) - g'(s)}ds\Big)\int_0^t h(t-s)\|\mathcal{A}^{\frac{1}{2}}u(s) - \mathcal{A}^{\frac{1}{2}}u(t)\|^2 ds \\ \leq & C_\nu (h \circ u)(t). \end{aligned} \tag{2.4.6}$$

Collecting all above equations establishes (2.4.5). □

Lemma 2.4.3. *The functional*

$$F(t) = \tau(t)\int_0^1 e^{-\rho\tau(t)}\|z(\rho,t)\|^2 d\rho$$

has the following property

$$F'(t) \le -\tau_1 e^{-\tau_2} \int_0^1 \|z(\rho,t)\|^2 d\rho - (1-d)e^{-\tau_2}\|z(1,t)\|^2 + \|u_t\|^2. \tag{2.4.7}$$

Proof. It is easy to see that

$$F'(t) = \int_0^1 \Big[\tau'(t)e^{-\rho\tau(t)}\|z(\rho,t)\|^2\Big]d\rho + \int_0^1 \Big[\tau(t)e^{-\rho\tau(t)}\frac{d}{dt}\Big(\|z(\rho,t)\|^2\Big)\Big]d\rho - \int_0^1 \Big[\rho\tau'(t)\tau(t)e^{-\rho\tau(t)}\|z(\rho,t)\|^2\Big]d\rho,$$

then, by $(2.2.1)_2$, we can get

$$F'(t) = -\int_0^1 \Big[e^{-\rho\tau(t)}\frac{d}{d\rho}\Big(\big(1-\rho\tau'(t)\big)\|z(\rho,t)\|^2\Big)\Big]d\rho - \int_0^1 \Big[\rho\tau(t)\tau'(t)e^{-\rho\tau(t)}\|z(\rho,t)\|^2\Big]d\rho,$$

that is,

$$F'(t) = -\int_0^1 \frac{d}{d\rho}\Big[e^{-\rho\tau(t)}(1-\rho\tau'(t))\|z(\rho,t)\|^2\Big]d\rho - \tau(t)\int_0^1 e^{-\rho\tau(t)}\|z(\rho,t)\|^2 d\rho,$$

this, together with (2.2.4), implies

$$F'(t) \le -\tau(t)\int_0^1 e^{-\rho\tau(t)}\|z(\rho,t)\|^2 d\rho - (1-d)e^{-\tau(t)}\|z(1,t)\|^2 + \|u_t\|^2.$$

As $e^{-\tau(t)} \le e^{-\rho\tau(t)}$ for all $\rho \in [0,1]$, we have

$$F'(t) \le -\tau(t)e^{-\tau(t)}\int_0^1 \|z(\rho,t)\|^2 d\rho - (1-d)e^{-\tau(t)}\|z(1,t)\|^2 + \|u_t\|^2,$$

using then (2.2.3), we end up with (2.4.7). □

Lemma 2.4.4. *For a suitable choice of* N *and* N_i, $i = 1,2$, *the functional defined by*

$$\mathcal{L}(t) = NE(t) + N_1K(t) + N_2F(t),$$

satisfies

$$\mathcal{L} \sim E, \tag{2.4.8}$$

and

$$\mathcal{L}'(t) \le -c\|u_t\|^2 - 4\beta\|\mathcal{A}^{\frac{1}{2}}u\|^2 + \frac{1}{4}\big(g\circ\mathcal{A}^{\frac{1}{2}}u\big)(t) - c\int_0^1 \|z(\rho,t)\|^2 d\rho. \tag{2.4.9}$$

Proof. It is not hard to establish that $\mathcal{L} \sim E$. Then, combining the estimates (2.4.3), (2.4.5), (2.4.7), we immediately get

$$\begin{aligned}\mathcal{L}'(t) \le& -\Big[C_1N - \Big(1+\frac{\gamma}{1-g_0}|\mu_1(t)|^2\Big)N_1 - N_2\Big)\Big]\|u_t\|^2\\ &- \frac{N_1}{4}\Big(1-g_0\Big)\|\mathcal{A}^{\frac{1}{2}}u\|^2 + \frac{N}{2}\big(g'\circ\mathcal{A}^{\frac{1}{2}}u\big)(t)\\ &+ \frac{N_1}{1-g_0}C_\nu\big(h\circ\mathcal{A}^{\frac{1}{2}}u\big)(t) - \tau_1e^{-\tau_2}N_2\int_0^1 \|z(\rho,t)\|^2 d\rho\\ &- \Big[C_2N + (1-d)e^{-\tau_2}N_2 - \frac{\gamma N_1}{1-g_0}|\mu_2(t)|^2\Big]\|z(1,t)\|^2.\end{aligned}$$

As μ_1 is bounded, we get that there exists a fixed positive constant α such that

$$\mu_1(t) \leq \alpha, \qquad \text{for all } \ t \geq 0, \tag{2.4.10}$$

This, together with $|\mu_2| < \sqrt{1-d}\,\mu_1$ and $g' = \nu g - h$, gives

$$\begin{aligned}
\mathcal{L}'(t) \leq &- \left[C_1 N - \Big(1 + \frac{\gamma\alpha^2}{1-g_0}\Big) N_1 - N_2\right] \|u_t\|^2 - \frac{N_1}{4}\Big(1 - g_0\Big)\|\mathcal{A}^{\frac{1}{2}} u\|^2 \\
&+ \frac{N\nu}{2}\big(g \circ \mathcal{A}^{\frac{1}{2}} u\big)(t) - \left[\frac{N}{2} - \frac{N_1}{4(1-g_0)} C_\nu\right]\big(h \circ \mathcal{A}^{\frac{1}{2}} u\big)(t) \\
&- \tau_1 e^{-\tau_2} N_2 \int_0^1 \|z(\rho, t)\|^2 d\rho \\
&- \left[C_2 N + (1-d)e^{-\tau_2} N_2 - \frac{\gamma\alpha^2(1-d)}{1-g_0} N_1\right] \|z(1,t)\|^2.
\end{aligned}$$

Furthermore, the choices

$$N_1 = \frac{16\beta}{1-g_0}, \qquad N_2 = \frac{C_1}{2} N,$$

give

$$\begin{aligned}
\mathcal{L}'(t) \leq &- \left[\frac{C_1}{2} N - \Big(1 + \frac{\gamma\alpha^2}{1-g_0}\Big) \frac{16\beta}{1-g_0}\right] \|u_t\|^2 - 4\beta\|\mathcal{A}^{\frac{1}{2}} u\|^2 \\
&+ \frac{N\nu}{2}\big(g \circ \mathcal{A}^{\frac{1}{2}} u\big)(t) - \left[\frac{N}{2} - \frac{4\beta}{(1-g_0)^2} C_\nu\right]\big(h \circ \mathcal{A}^{\frac{1}{2}} u\big)(t) \\
&- \frac{C_1 \tau_1 e^{-\tau_2}}{2} N \int_0^1 \|z(\rho, t)\|^2 d\rho \\
&- \left[C_2 N + \frac{C_1(1-d)e^{-\tau_2}}{2} N - \frac{16\beta\gamma\alpha^2(1-d)}{(1-g_0)^2}\right] \|z(1,t)\|^2.
\end{aligned}$$

Since $\frac{\nu g^2(s)}{\nu g(s) - g'(s)} < g(s)$, then it is readily seen, by the Lebesgue dominated convergence theorem, that

$$\lim_{\nu \to \infty} \nu C_\nu = \lim_{\nu \to \infty} \int_0^\infty \frac{\nu g^2(s)}{\nu g(s) - g'(s)} ds = 0. \tag{2.4.11}$$

Consequently, it exist some ν_0 $(0 < \nu_0 < 1)$ such that if $\nu_0 < \nu$ then

$$\nu C_\nu < \frac{1}{8\left[\frac{4\beta}{(1-g_0)^2}\right]}.$$

Choosing N sufficiently large and take ν satisfying

$$\frac{C_1}{2} N - \Big(1 + \frac{\gamma\alpha^2}{1-g_0}\Big)\frac{8\beta}{1-g_0} > 0,$$

$$C_2 N + \frac{C_1(1-d)e^{-\tau_2}}{2} N - \frac{16\beta\gamma\alpha^2(1-d)}{(1-g_0)^2} \geq 0,$$

$$\nu = \frac{1}{2N} < \nu_0,$$

and then

$$\frac{N}{2} - \frac{16\beta}{(1-g_0)^2} C_\nu > 0.$$

Hence, (2.4.9) is established. □

Going back to our proof of Theorem 2.4.1 . By (2.4.9) and (2.2.10), one has

$$\mathcal{L}'(t) \le -m_0 E(t) + c\big(g \circ \mathcal{A}^{\frac{1}{2}} u\big)(t),$$

tha is,

$$\mathcal{L}'(t) \le -m_0 E(t) + c\big(g \circ \mathcal{A}^{\frac{1}{2}} u\big)(t_0) + c \int_{t_0}^{t} g(s) \|\mathcal{A}^{\frac{1}{2}} u(t) - \mathcal{A}^{\frac{1}{2}} u(t-s)\|^2 ds. \tag{2.4.12}$$

It follows from (2.2.8), that

$$\big(g \circ \mathcal{A}^{\frac{1}{2}} u\big)(t_0) \le -\frac{g(0)}{bc_0} \big(g' \circ \mathcal{A}^{\frac{1}{2}} u\big)(t_0). \tag{2.4.13}$$

Simple substitution of this latter estimate into (2.4.12), using (2.4.3), leads to

$$\mathcal{L}'(t) \le -\, m_0 E(t) - cE'(t) + c \int_{t_0}^{t} g(s) \|\mathcal{A}^{\frac{1}{2}} u(t) - \mathcal{A}^{\frac{1}{2}} u(t-s)\|^2 ds. \tag{2.4.14}$$

Hence, with $\mathcal{L}_0 = \mathcal{L} + cE$, we clearly have $\mathcal{L}_0 \sim E$ and

$$\mathcal{L}_0'(t) \le -\, m_0 E(t) + c \int_{t_0}^{t} g(s) \|\mathcal{A}^{\frac{1}{2}} u(t) - \mathcal{A}^{\frac{1}{2}} u(t-s)\|^2 ds. \tag{2.4.15}$$

Now, the main task is to estimate the last term of (2.4.15). For, we distinguish two cases.
(I). H is linear: Making use of (2.2.7), (2.4.10) and (2.4.3), one obtains

$$\begin{aligned}
\mathcal{L}_0'(t) &\le -\Big(\frac{m_0}{2c_1} c_1 + \frac{m_0}{2\alpha} \alpha\Big) E(t) + c \int_{t_0}^{t} \frac{c_0}{c_0} g(s) \|\mathcal{A}^{\frac{1}{2}} u(t) - \mathcal{A}^{\frac{1}{2}} u(t-s)\|^2 ds \\
&\le -m_1 \Big(\xi(t) + \mu_1(t)\Big) E(t) + c \int_{t_0}^{t} \xi(s) g(s) \|\mathcal{A}^{\frac{1}{2}} u(t) - \mathcal{A}^{\frac{1}{2}} u(t-s)\|^2 ds \\
&\le -m_1 \zeta(t) E(t) - c \int_{t_0}^{t} g'(s) \|\mathcal{A}^{\frac{1}{2}} u(t) - \mathcal{A}^{\frac{1}{2}} u(t-s)\|^2 ds \\
&\le -m_1 \zeta(t) E(t) - cE'(t),
\end{aligned} \tag{2.4.16}$$

where $m_1 = \min\Big(\frac{m_0}{2c_1}, \frac{m_0}{2\alpha}\Big)$. Obviously, the function $\mathcal{L}_1 = \mathcal{L}_0 + cE$ satisfies

$$\mathcal{L}_1 \sim E, \tag{2.4.17}$$

and

$$\mathcal{L}_1'(t) \le -c\zeta(t) \mathcal{L}_1(t), \tag{2.4.18}$$

the integration over $[t_0, t]$ gives

$$\mathcal{L}_1(t) \leq \mathcal{L}_1(t_0)\exp\Big(- c\int_{t_0}^{t} \zeta(s)ds\Big), \tag{2.4.19}$$

and so

$$E(t) \leq cE(t_0)\exp\Big(- c\int_{t_0}^{t} \zeta(s)ds\Big). \tag{2.4.20}$$

(II). <u>H is non-linear</u>: We first define the functional

$$\mathcal{G}(t) = \mathcal{L}(t) + \mathcal{R}(t),$$

where

$$\mathcal{R}(t) = \int_0^t f(t-s)\|\mathcal{A}^{\frac{1}{2}}u(s)\|^2 ds \quad \text{and} \quad f(t) = \int_t^\infty g(s)ds.$$

It is clear that $\mathcal{G}$ is positive. Then, using Lemma 2.2.4 and Lemma 2.4.4, we conclude that it exists $\alpha_0 > 0$ such that

$$\mathcal{G}'(t) \leq -c\|u_t\|^2 - \beta\|\mathcal{A}^{\frac{1}{2}}u\|^2 - \frac{1}{4}\big(g \circ \mathcal{A}^{\frac{1}{2}}u\big)(t) - c\int_0^1 \|z(\rho,t)\|^2 d\rho \leq -\alpha_0 E(t).$$

Therefore,

$$\alpha_0 \int_0^t E(s)ds \leq \mathcal{G}(0) - \mathcal{G}(t) \leq \mathcal{G}(0),$$

this guarantees that

$$\int_0^\infty E(s)ds < \infty. \tag{2.4.21}$$

We now define θ by

$$\theta(t) = \delta\int_{t_0}^{t} \|\mathcal{A}^{\frac{1}{2}}(t) - \mathcal{A}^{\frac{1}{2}}(t-s)\|^2 ds, \tag{2.4.22}$$

from which, thanks to (2.4.21), we have

$$\begin{aligned}\int_{t_0}^{t} \|\mathcal{A}^{\frac{1}{2}}(t) - \mathcal{A}^{\frac{1}{2}}(t-s)\|^2 ds &\leq 2\int_{t_0}^{t} \Big[\|\mathcal{A}^{\frac{1}{2}}(t)\|^2 + \|\mathcal{A}^{\frac{1}{2}}(t-s)\|^2\Big] ds \\ &\leq 4\int_{t_0}^{t} \Big(E(t) + E(t-s)\Big) ds \\ &< \infty,\end{aligned} \tag{2.4.23}$$

this property enables us to take $0 < \delta < 1$ so that,

$$\theta(t) < 1, \quad \forall t \geq t_0. \tag{2.4.24}$$

Assuming without any loss of generality that $\theta(t) > 0$ for all $t \geq t_0$; otherwise, (2.4.15) leads to (2.4.20). Also, we define

$$\vartheta(t) = -\int_{t_0}^{t} g'(s)\|\mathcal{A}^{\frac{1}{2}}(t)\|^2 - \|\mathcal{A}^{\frac{1}{2}}(t-s)\|^2 ds,$$

which obviously satisfies

$$\vartheta(t) \leq -cE'(t). \tag{2.4.25}$$

Moreover, the strict convexity of H on $[0, r]$ and $H(0) = 0$ entail that

$$H(\varsigma s) \leq \varsigma H(s), \qquad \text{for all} \quad 0 \leq \varsigma \leq 1, \quad s \in [0, r].$$

Using this fact, Jensen's inequality, (2.2.6), (2.2.7), (2.2.8) and (2.4.24), it follows that

$$\begin{aligned}
\frac{\vartheta(t)}{\xi(t)} &= -\frac{1}{\xi(t)} \int_{t_0}^{t} g'(s) \|\mathcal{A}^{\frac{1}{2}} u(t) - \mathcal{A}^{\frac{1}{2}} u(t-s)\|^2 ds \\
&= \frac{1}{\xi(t)\theta(t)} \int_{t_0}^{t} \theta(t) \left(-g'(s)\right) \|\mathcal{A}^{\frac{1}{2}} u(t) - \mathcal{A}^{\frac{1}{2}} u(t-s)\|^2 ds \\
&\geq \frac{1}{c_1 \theta(t)} \int_{t_0}^{t} \theta(t) \xi(s) H\big(g(s)\big) \|\mathcal{A}^{\frac{1}{2}} u(t) - \mathcal{A}^{\frac{1}{2}} u(t-s)\|^2 ds \\
&\geq \frac{c_0}{c_1 \theta(t)} \int_{t_0}^{t} H\big(\theta(t) g(s)\big) \|\mathcal{A}^{\frac{1}{2}} u(t) - \mathcal{A}^{\frac{1}{2}} u(t-s)\|^2 ds \qquad (2.4.26) \\
&\geq \frac{c_0}{c_1} H\left(\frac{1}{\theta(t)} \int_{t_0}^{t} \theta(t) g(s) \|\mathcal{A}^{\frac{1}{2}} u(t) - \mathcal{A}^{\frac{1}{2}} u(t-s)\|^2 ds\right) \\
&= \frac{c_0}{c_1} H\left(\int_{t_0}^{t} g(s) \|\mathcal{A}^{\frac{1}{2}} u(t) - \mathcal{A}^{\frac{1}{2}} u(t-s)\|^2 ds\right) \\
&= \frac{c_0}{c_1} \overline{H}\left(\int_{t_0}^{t} g(s) \|\mathcal{A}^{\frac{1}{2}} u(t) - \mathcal{A}^{\frac{1}{2}} u(t-s)\|^2 ds\right),
\end{aligned}$$

where $\overline{H}$ is a C^2 extension of H which is also strictly convex and strictly increasing of class C^2 on $(0, \infty)$. This gives that

$$\int_{t_0}^{t} g(s) \|\mathcal{A}^{\frac{1}{2}} u(t) - \mathcal{A}^{\frac{1}{2}} u(t-s)\|^2 ds \leq c\overline{H}^{-1}\left(\frac{\vartheta(t)}{\xi(t)}\right). \tag{2.4.27}$$

We hence derive from (2.4.15) that, for any $t \geq t_1$

$$\mathcal{L}_0'(t) \leq -m_0 E(t) + c\overline{H}^{-1}\left(\frac{\vartheta(t)}{\xi(t)}\right). \tag{2.4.28}$$

Let $0 < \epsilon_0 < r$ and $\lambda > 0$, then the functional given by

$$L(t) = \overline{H}'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) \mathcal{L}_0(t) + \lambda E(t),$$

fulfills, for some k_0 and k_1,

$$k_0 L(t) \leq E(t) \leq k_1 L(t), \tag{2.4.29}$$

and

$$L'(t) = \epsilon_0 \frac{E'(t)}{E(0)} \overline{H}''\left(\epsilon_0 \frac{E(t)}{E(0)}\right) \mathcal{L}_0(t) + \overline{H}'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) \mathcal{L}_0'(t) + \lambda E'(t).$$

As $\overline{H}$ is an increasing-convex function, we have that $\overline{H}' > 0$, $\overline{H}'' > 0$. Using these facts with (2.4.28) and $E' < 0$, one gets

$$L'(t) \leq -\, m_0 \overline{H}'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) E(t) + c\,\overline{H}'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) \overline{H}^{-1}\left(\frac{\vartheta(t)}{\xi(t)}\right) + \lambda E'(t). \tag{2.4.30}$$

Let $\overline{H}^*$ be the convex conjugate of the differential convex function $\overline{H}$, i.e.

$$\overline{H}^*(s) = \sup_{t\in\mathbb{R}_+} \big(st - \overline{H}(t)\big),$$

then, $\overline{H}^*$ is the Legendre transform of $\overline{H}$, which satisfies (see Arnold [57], pp.61-64)

$$XY \leq \overline{H}^*(X) + \overline{H}(Y) \tag{2.4.31}$$

and

$$\overline{H}^*(s) = s(\overline{H}')^{-1}(s) - \overline{H}\big[(\overline{H}')^{-1}(s)\big]. \tag{2.4.32}$$

By taking

$$X = \overline{H}'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) \quad \text{and} \quad Y = \overline{H}^{-1}\left(\frac{\vartheta(t)}{\xi(t)}\right),$$

and using (2.4.31)-(2.4.32) with the fact that $\overline{H}$ is non-negative, we can obtain

$$\begin{aligned} \overline{H}'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) \overline{H}^{-1}\left(\frac{\vartheta(t)}{\zeta(t)}\right) &\leq \overline{H}^*\left(\overline{H}'\left(\epsilon_0 \frac{E(t)}{E(0)}\right)\right) + \frac{\vartheta(t)}{\xi(t)} \\ &\leq \epsilon_0 \frac{E(t)}{E(0)} \overline{H}'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) + \frac{\vartheta(t)}{\xi(t)}. \end{aligned}$$

So, owing to (2.4.25), we end up with

$$L'(t) \leq -\,\big(m_0 E(0) - c\epsilon_0\big) \frac{E(t)}{E(0)} \overline{H}'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) + \big(\lambda - c\big) E'(t). \tag{2.4.33}$$

Since $E' < 0$, putting $\epsilon_0 = \frac{m_0 E(0)}{2c}$ and $\lambda = 2c$, we can derive

$$\begin{aligned} L'(t) &\leq -\frac{m_0 E(0)}{2} \frac{E(t)}{E(0)} H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) \\ &= -\left(\frac{m_0 E(0)}{4c_1} c_1 + \frac{m_0 E(0)}{4\alpha} \alpha\right) \frac{E(t)}{E(0)} H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right), \end{aligned}$$

then, by (2.2.7) and (2.4.10), we get

$$\begin{aligned} L'(t) &\leq -\left(\frac{m_0 E(0)}{4c_1} \xi(t) + \frac{m_0 E(0)}{4\alpha} \mu_1(t)\right) \frac{E(t)}{E(0)} H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) \\ &\leq -a_0 \zeta(t) \frac{E(t)}{E(0)} H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right). \end{aligned}$$

where $a_0 = \min\left(\frac{m_0 E(0)}{4\alpha}, \frac{m_0 E(0)}{4c_1}\right)$, moreover, with $H_2(t) = tH'(\epsilon_0 t)$ one has

$$L'(t) \leq -a_0 \zeta(t) H_2\left(\frac{E(t)}{E(0)}\right). \tag{2.4.34}$$

Defining

$$L_1(t) = \frac{k_0 L(t)}{E(0)},$$

it then follows from (2.4.29) that $L_1 \sim E$. Hence, (2.4.34) may be transformed into

$$L_1'(t) \leq -a_1 \zeta(t) H_2\big(L_1(t)\big). \tag{2.4.35}$$

By the definition of H_1, we know that

$$H_1'(t) = -\frac{1}{H_2(t)} < 0, \quad \forall t \geq 0,$$

then, (2.4.35) rewrites as

$$L_1'(t) \leq \frac{a_1 \zeta(t)}{H_1'\big(L_1(t)\big)},$$

that is,

$$\big[H_1(L_1(t))\big]' \geq a_1 \zeta(t). \tag{2.4.36}$$

Integrating Eq.(2.4.36) on $[t_0, t]$, we yield that

$$H_1\big(L_1(t)\big) \geq H_1\big(L_1(t_0)\big) + a_1 \int_{t_0}^{t} \zeta(s) ds,$$

we then use the non-increasing property of H_1^{-1}, we infer

$$L_1(t) \leq H_1^{-1}\left(H_1\big(L_1(t_0)\big) + a_1 \int_{t_0}^{t} \zeta(s) ds\right),$$

and so

$$L_1(t) \leq H_1^{-1}\left(a_1 \int_{t_0}^{t} \zeta(s) ds\right).$$

This gives us the required result in Theorem 2.4.1 when combined with $L_1 \sim E$. □

2.4.2 General decay for $|\mu_2(t)| = \sqrt{1-d}\,\mu_1(t)$

In this current subsection, we will show that the result given in Theorem 2.4.1 remains valid even if $|\mu_2(t)| = \sqrt{1-d}\,\mu_1(t)$. Firstly, in light of (2.4.2), we have

$$\eta(t) = \mu_1(t) = \frac{1}{\sqrt{1-d}} |\mu_2(t)|,$$

hence, (2.4.3) takes the form

$$E'(t) \leq \frac{1}{2}\big(g' \circ \mathcal{A}^{\frac{1}{2}} u\big)(t) - \frac{1}{2} g(t) \|\mathcal{A}^{\frac{1}{2}} u(t)\|^2. \tag{2.4.37}$$

The main result of this part is the following.

Theorem 2.4.5. *Let (u, z) be the solution of (2.2.1). Assume that (A_1)-(A_4) are satisfied and $|\mu_2| = \sqrt{1-d}\,\mu_1$. Then, there exist two positive constants w and w_1 such that the solution of (2.2.1) satisfies*

$$E(t) \leq w\, H_1^{-1}\Big(w_1 \int_{t_0}^{t} \zeta(s) ds \Big), \quad \forall t > t_0 \tag{2.4.38}$$

Proof. As usual, our argument is based on the construction of a proper Lyapunov functional $\Lambda(t)$ which it is equivalent to $E(t)$ and satisfies

$$\Lambda'(t) \leq -c\|u_t\|^2 - 4\beta\|\mathcal{A}^{\frac{1}{2}}u\|^2 + \frac{1}{4}\big(g \circ \mathcal{A}^{\frac{1}{2}}u\big)(t) - c\int_0^1 \|z(\rho,t)\|^2 d\rho. \tag{2.4.39}$$

To this purpose, we introduce the functional

$$J(t) = -\big\langle u_t, \big(g \diamond u\big)(t)\big\rangle.$$

Then, we have the following estimate.

Lemma 2.4.6. *The functional J satisfies for any positive constants δ_1, δ_2 and δ_3, the estimate*

$$\begin{aligned} J'(t) \leq & -\big(g_0 - 2\delta_1\big)\|u_t\|^2 + \delta_2\|\mathcal{A}^{\frac{1}{2}}u\|^2 + \delta_3\|z(1,t)\|^2 \\ & + \Big(\frac{c|\mu_1(t)|^2}{\delta_1} + \frac{c|\mu_2(t)|^2}{\delta_3} + 1\Big) C_\nu \big(h \circ \mathcal{A}^{\frac{1}{2}}u\big)(t). \end{aligned} \tag{2.4.40}$$

Proof. A straightforward computation, using Eq.$(2.2.1)_1$, leads to

$$\begin{aligned} J'(t) = & - g_0\|u_t\|^2 - \big\langle u_t, \big(g' \diamond u\big)(t)\big\rangle + \big\langle \mathcal{A}^{\frac{1}{2}}u, \big(g \diamond \mathcal{A}^{\frac{1}{2}}u\big)(t)\big\rangle \\ & - \big\langle \big(g * \mathcal{A}^{\frac{1}{2}}u\big)(t), \big(g \diamond \mathcal{A}^{\frac{1}{2}}u\big)(t)\big\rangle + \mu_1(t)\big\langle u_t(t), \big(g \diamond u\big)(t)\big\rangle \\ & + \mu_2(t)\big\langle z(1,t), \big(g \diamond u\big)(t)\big\rangle, \end{aligned}$$

which is,

$$\begin{aligned} J'(t) = & -g_0\|u_t\|^2 - \underbrace{\big\langle u_t, \big(g' \diamond u\big)(t)\big\rangle}_{I_1} + \underbrace{\big(1-g_0\big)\big\langle \mathcal{A}^{\frac{1}{2}}u, \big(g \diamond \mathcal{A}^{\frac{1}{2}}u\big)(t)\big\rangle}_{I_2} \\ & + \underbrace{\|\big(g \diamond \mathcal{A}^{\frac{1}{2}}u\big)(t)\|^2}_{I_3} + \underbrace{\mu_1(t)\big\langle u_t(t), \big(g \diamond u\big)(t)\big\rangle}_{I_4} + \underbrace{\mu_2(t)\big\langle z(1,t), \big(g \diamond u\big)(t)\big\rangle}_{I_5}. \end{aligned}$$

In what follows, we will estimate the terms $I_1, ..., I_5$, using Cauchy-Schwarz's inequality, Young's inequality, (2.2.2) and similar computations in (2.4.6). So, for any $\delta_1 > 0$, one has

$$\begin{aligned} -I_1 = & \big\langle u_t, \big(h \diamond u\big)(t)\big\rangle - \big\langle u_t, \nu\big(g \diamond u\big)(t)\big\rangle \\ \leq & \delta_1\|u_t\|^2 + \frac{1}{4\delta_1}\Big(\int_0^t \sqrt{h(t-s)}\sqrt{h(t-s)}\|u(s) - u(t)\|\Big)^2 \\ & + \frac{\nu^2}{4\delta_1}\Big(\int_0^t g(t-s)\|u(s) - u(t)\|\Big)^2 \\ \leq & \delta_1\|u_t\|^2 + \frac{\int_0^t g(s)ds}{4\delta_1}\big(h \circ \mathcal{A}u\big)(t) + \frac{cC_\nu}{4\delta_1}\big(h \circ \mathcal{A}u\big)(t) \\ \leq & \delta_1\|u_t\|^2 + \frac{c}{\delta_1}\big(h \circ \mathcal{A}^{\frac{1}{2}}u\big)(t) + \frac{cC_\nu}{\delta_1}\big(h \circ \mathcal{A}^{\frac{1}{2}}u\big)(t). \end{aligned}$$

Analogously,

$$I_2 \le \delta_2 \|\mathcal{A}^{\frac{1}{2}} u\|^2 + \frac{cC_\nu}{\delta_2}\big(h \circ \mathcal{A}^{\frac{1}{2}} u\big)(t),$$

$$I_3 \le C_\nu \big(h \circ \mathcal{A}^{\frac{1}{2}} u\big)(t),$$

$$I_4 \le \delta_1 \|u_t\|^2 + \frac{cC_\nu}{\delta_1}|\mu_1(t)|^2\big(h \circ \mathcal{A}^{\frac{1}{2}} u\big)(t),$$

$$I_5 \le \delta_3 \|z(1,t)\|^2 + \frac{cC_\nu}{\delta_3}|\mu_2(t)|^2\big(h \circ \mathcal{A}^{\frac{1}{2}} u\big)(t),$$

Adding all above estimates, we obtain (2.4.40). □

Let us then define the Lyapunov functional Λ by

$$\Lambda(t) = ME(t) + M_1 K(t) + M_2 F(t) + M_3 J(t), \tag{2.4.41}$$

where M and M_i are fixed positive constants to be selected posteriori. It is straightforward to show that $E(t)$ and $\Lambda(t)$ are equivalent (i.e.$E \sim \Lambda$). Then, gathering the estimates (2.4.37), (2.4.5), (2.4.7) and (2.4.40), we have

$$\begin{aligned} \Lambda'(t) \le &- \left[\big(g_0 - 2\delta_1\big)M_3 - \Big(1 + \frac{\gamma}{1-g_0}|\mu_1(t)|^2\Big)M_1 - M_2\right]\|u_t\|^2 \\ &- \left[\frac{M_1}{4}\big(1-g_0\big) - \delta_2 M_3\right]\|\mathcal{A}^{\frac{1}{2}} u\|^2 + \frac{M}{2}\big(g' \circ \mathcal{A}^{\frac{1}{2}} u\big)(t) \\ &+ \left[\frac{M_1}{4(1-g_0)} + \Big(\frac{c|\mu_1(t)|^2}{\delta_1} + \frac{c|\mu_2(t)|^2}{\delta_3} + 1\Big)M_3\right]C_\nu\big(h \circ \mathcal{A}^{\frac{1}{2}} u\big)(t) \\ &- \left[(1-d)e^{-\tau_2}M_2 - \frac{\gamma M_1}{1-g_0}|\mu_2(t)|^2 - \delta_3 M_3\right]\|z(1,t)\|^2 \\ &- \tau_1 e^{-\tau_2} M_2 \int_0^1 \|z(\rho,t)\|^2 d\rho. \end{aligned} \tag{2.4.42}$$

As η is non-increasing function, it then results that $\eta(t) \le \eta(0)$ for all $t \ge 0$. Hence, thanks to $\eta = \mu_1 = \dfrac{1}{\sqrt{1-d}}|\mu_2|$ and $g' = \nu g - h$, we obtain

$$\begin{aligned} \Lambda'(t) \le &-\left[\big(g_0 - 2\delta_1\big)M_3 - \Big(1 + \frac{\gamma|\eta(0)|^2}{1-g_0}\Big)M_1 - M_2\right]\|u_t\|^2 \\ &- \left[\frac{M_1\big(1-g_0\big)}{4} - \delta_2 M_3\right]\|\mathcal{A}^{\frac{1}{2}} u\|^2 + \frac{M\nu}{2}\big(g \circ \mathcal{A}^{\frac{1}{2}} u\big)(t) \\ &- \left[\frac{M}{2} - \left(\frac{M_1}{4(1-g_0)} + \Big(\frac{c|\eta(0)|^2}{\delta_1} + \frac{c(1-d)|\eta(0)|^2}{\delta_3} + 1\Big)M_3\right)C_\nu\right]\big(h \circ \mathcal{A}^{\frac{1}{2}} u\big)(t) \\ &- \left[(1-d)e^{-\tau_2}M_2 - \frac{\gamma(1-d)|\eta(0)|^2}{1-g_0}M_1 - \delta_3 M_3\right]\|z(1,t)\|^2 \\ &- \tau_1 e^{-\tau_2} M_2 \int_0^1 \|z(\rho,t)\|^2 d\rho. \end{aligned}$$

Moreover, by setting

$$M_1 = \frac{20\beta}{1-g_0}, \quad M_2 = \frac{21\beta\gamma|\eta(0)|^2}{(1-g_0)^2 e^{-\tau_2}}, \quad \delta_1 = \frac{g_0}{4}, \quad \delta_2 = \frac{\beta}{M_3},$$

we immediately get

$$\begin{aligned}\Lambda'(t) \leq &-\Big[\frac{g_0}{2}M_3 - \frac{20\beta}{1-g_0}\Big(1+\frac{\gamma|\eta(0)|^2}{1-g_0}\Big) - \frac{21\beta\gamma|\eta(0)|^2}{(1-g_0)^2 e^{-\tau_2}}\Big]\|u_t\|^2 \\ &- 4\beta\|\mathcal{A}^{\frac{1}{2}}u\|^2 + \frac{M\nu}{2}\big(g\circ\mathcal{A}^{\frac{1}{2}}u\big)(t) \\ &- \Big[\frac{M}{2} - \Big(\frac{5\beta}{(1-g_0)^2} + \Big(\frac{4c|\eta(0)|^2}{g_0} + \frac{c(1-d)|\eta(0)|^2}{\delta_3} + 1\Big)M_3\Big)C_\nu\Big]\big(h\circ\mathcal{A}^{\frac{1}{2}}u\big)(t) \\ &- \Big[\frac{\beta\gamma|\eta(0)|^2}{1-g_0} - \delta_3 M_3\Big]\|z(1,t)\|^2 - \frac{21\beta\gamma|\eta(0)|^2}{(1-g_0)^2}\int_0^1 \|z(\rho,t)\|^2 d\rho.\end{aligned}$$

Now, we will select our constants M, M_3 and δ_3 very carefully. At the first, we take M_3 sufficiently large so that

$$\frac{g_0}{2}M_3 - \frac{20\beta}{1-g_0}\Big(1+\frac{\gamma|\eta(0)|^2}{1-g_0}\Big) - \frac{21\beta\gamma|\eta(0)|^2}{(1-g_0)^2 e^{-\tau_2}} > 0.$$

Then, for any fixed M_3, we pick δ_3 small enough so that

$$\frac{\beta\gamma^2|\eta(0)|^2}{1-g_0} - \delta_3 M_3 \geq 0.$$

Since $\lim_{\nu\to\infty}\nu C_\nu = 0$ (for the same raison given in (2.4.11)), it then follows that it exist some ν_1 $(0<\nu_1<1)$ such that if $\nu_1 < \nu$ then

$$\nu C_\nu < \frac{1}{8\big[\frac{5\beta}{(1-g_0)^2} + \Big(\frac{4c|\eta(0)|^2}{g_0} + \frac{c(1-d)|\eta(0)|^2}{\delta_3} + 1\Big)M_3\big]}. \tag{2.4.43}$$

Choosing M sufficiently large and take ν satisfying

$$\nu = \frac{1}{2M} < \nu_1,$$

and so

$$\frac{M}{2} - \Big(\frac{5\beta}{(1-g_0)^2} + \Big(\frac{4c|\eta(0)|^2}{g_0} + \frac{c(1-d)|\eta(0)|^2}{\delta_3} + 1\Big)M_3\Big)C_\nu > 0.$$

Consequently, we end up with

$$\Lambda'(t) \leq -c\|u_t\|^2 - 4\beta\|\mathcal{A}^{\frac{1}{2}}u\|^2 + \frac{1}{4}\big(g\circ\mathcal{A}^{\frac{1}{2}}u\big)(t) - c\int_0^1 \|z(\rho,t)\|^2 d\rho. \tag{2.4.44}$$

Therefore, following the same steps as in the proof of (2.4.4), we obtain the claim (2.4.38).

□

2.4.3 Examples

Along this subsection we assume that $\xi(t) < \mu_1(t)$ for all $t \in \mathbb{R}^+$. So our general decay result takes the form:

$$E(t) \le cH_1^{-1}\Big(c\int_{t_0}^{t}\xi(s)ds\Big).$$

In order to illustrate this new general decay result we shall give here some examples.
Example 1: Consider

$$H(s) = s \quad \text{and} \quad g(t) = b\exp\Big(-pt - v\Big(\ln\big(2+qt\big)\Big)^v - \sigma\ln\big(2+\ln(2+t)\big)\Big),$$

with $v, p, q, \sigma \ge 0$ and $b > 0$ is chosen so that (A_3) is satisfied, then

$$g'(t) = -b\xi(t)g(t),$$

where

$$\xi(t) = p + \frac{qv^2}{2+qt}\Big(\ln\big(2+qt\big)\Big)^{v-1} + \frac{\sigma}{(2+t)\big(2+\ln(2+t)\big)}.$$

It is clear that the function $\xi : \mathbb{R}^+ \to]0, \infty[$ is bounded and not necessarily monotone. Then, we have for all $t \ge 0$

$$E(t) \le c\exp\Big(-pt - v\Big(\ln\big(2+qt\big)\Big)^v - \sigma\ln\big(2+\ln(2+t)\big)\Big), \quad \text{if } v, p, q, \sigma > 0,$$

and

$$E(t) \le \begin{cases} c\exp(-ct), & \text{if } p > 0 \quad \text{and} \quad q = \sigma = 0, \\ \dfrac{c}{t}, & \text{if } q > 0, \ v = 1 \quad \text{and} \quad p = \sigma = 0, \\ \dfrac{c}{\ln(2+t)}, & \text{if } \sigma > 0 \quad \text{and} \quad p = q = 0. \end{cases}$$

Example 2: Assume that (A1) and (A2) are satisfied and that

$$H(s) = s^p, \quad \text{for} \quad 1 \le p \le 2.$$

Then,

$$E(t) \le \begin{cases} c\exp\Big(-c\displaystyle\int_{t_0}^{t}\xi(s)ds\Big) & \text{if} \quad p = 1, \\ c\Big(1 + \displaystyle\int_{t_0}^{t}\xi(s)ds\Big)^{-\frac{1}{p-1}} & \text{if } 1 < p < 2. \end{cases}$$

For more examples, we refer readers to these studies [**24, 55, 58, 60**].

2.5 Applications

Our abstract results are valid in many problems. In this section, we give only four illustrative applications.

2.5.1 Infinite memory

By adopting the method in **[59]**, our stability results can be extended to the case which the memory is infinite.

2.5.2 A more general model

Our results hold for the following more general form

$$\begin{cases} u_{tt}(t) + \mathcal{A}u(t) - \big(g * \mathcal{B}u\big)(t) + \mu_1(t)u_t(t) + \mu_2(t)u_t(t-\tau(t)) = 0 & \text{in }]0,+\infty[, \\ u(-t) = u_0, \quad u_t(0) = u_1 & \text{in } [0,+\infty[, \\ u(t-\tau(0)) = f_0(t-\tau(0)) & \text{in } [0,\tau(0)], \end{cases}$$

where $\mathcal{B} : D(\mathcal{A}) \to \mathcal{H}$ is a self-adjoint linear positive definite operator having domain $D(\mathcal{A}) \subset D(\mathcal{B}) \subset \mathcal{H}$ with dense embeddings such that,

$$\|u\|^2 \le \gamma_1 \|\mathcal{B}^{\frac{1}{2}}u\|^2 \le \gamma_2 \|\mathcal{A}^{\frac{1}{2}}u\|^2 \le \gamma_3 \|\mathcal{B}^{\frac{1}{2}}u\|^2, \quad \forall u \in D(\mathcal{A}^{\frac{1}{2}}),$$

where γ_i are fixed positive constants and $\beta \in]0, 1/\gamma_2[$.

2.5.3 Abstract system

Consider the following problem:

$$\begin{cases} u_{tt}(t) + \mathcal{A}u(t) - \big(g * \mathcal{A}u\big)(t) + \mu_1(t)u_t(t) + \mu_2(t)u_t(t-\tau(t)) = 0 & \text{in }]0,+\infty[, \\ v_{tt}(t) + \mathcal{A}v(t) - \big(g * \mathcal{A}^{\alpha}v\big)(t) + \widetilde{\mu}_1(t)v_t(t) + \widetilde{\mu}_2(t)v_t(t-\tau(t)) = 0 & \text{in }]0,+\infty[\\ u(-t) = u_0, \quad u_t(0) = u_1 & \text{in } [0,+\infty[, \\ v(-t) = v_0, \quad v_t(0) = v_1 & \text{in } [0,+\infty[, \\ u(t-\tau(0)) = f_0(t-\tau(0)) & \text{in } [0,\tau(0)], \\ v(t-\tau(0)) = \widetilde{f}_0(t-\tau(0)) & \text{in } [0,\tau(0)], \end{cases}$$

where $\alpha \in [0,1]$. The energy functional $\mathcal{E}$ associated with the solution of this problem is defined as

$$\begin{aligned} \mathcal{E}(t) =& \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\|v_t(t)\|^2 + \frac{1}{2}\|\mathcal{A}^{\frac{1}{2}}u(t)\|^2 + \frac{1}{2}\|\mathcal{A}^{\frac{1}{2}}v(t)\|^2 + \frac{1}{2}\big(g \circ \mathcal{A}^{\frac{1}{2}}u\big)(t) + \frac{1}{2}\big(g \circ \mathcal{A}^{\frac{\alpha}{2}}v\big)(t) \\ &+ \frac{1}{2}\tau(t)\eta(t)\int_0^1 \|z(\rho,t)\|^2 d\rho + \frac{1}{2}\tau(t)\widetilde{\eta}(t)\int_0^1 \|\widetilde{z}(\rho,t)\|^2 d\rho \end{aligned}$$

Then, with $\zeta(t) = \min\big(\xi(t), \mu_1(t), \widetilde{\mu}_1(t)\big)$, we have the following decay properties

$$\mathcal{E}(t) \le \begin{cases} cH_1^{-1}\left(c\displaystyle\int_{t_0}^t \zeta(s)ds\right) & \text{if } \alpha = 1, \\ cH_2^{-1}\left(c\left(\displaystyle\int_{t_0}^t \zeta(s)ds\right)^{-1}\right) & \text{if } \alpha \ne 1. \end{cases}$$

In particularly, if we consider $H(s) = s$, we obtain that

$$\mathcal{E}(t) \leq \begin{cases} c\exp\Big(- c\displaystyle\int_{t_0}^{t} \zeta(s)ds \Big) & \text{if } \alpha = 1, \\ c\Big(\displaystyle\int_{t_0}^{t} \zeta(s)ds \Big)^{-1} & \text{if } \alpha \neq 1. \end{cases}$$

2.5.4 Wave-Petrovsky equation

Let Ω be an open bounded domain in $\mathbb{R}^n$, $n \geq 1$, with smooth boundary Γ. Our results are valid for the following wave equation with Dirichlet boundary condition:

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + \big(g * \Delta u\big)(x,t) & \\ +\mu_1(t)u_t(x,t) + \mu_2(t)u_t(x,t-\tau(t)) = 0 & \text{in } \Omega\times]0,+\infty[, \\ u(x,t) = 0 & \text{in } \Gamma\times]0,+\infty[, \\ u(x,0) = u_0, \quad u_t(x,0) = u_1 & \text{in } \quad \Omega, \\ u(x,t-\tau(0)) = f_0(x,t-\tau(0)) & \text{in}\, \Omega \times [0,\tau(0)], \end{cases}$$

which is (2.1.1) with $\mathcal{A} = -\Delta$, $D(\mathcal{A}) = H^2 \cap H_0^1(\Omega)$ and $\mathcal{H} = L^2(\Omega)$.

Also, one could obtain the same results for the following Petrovsky equation with Dirichlet and Neumann boundary conditions:

$$\begin{cases} u_{tt}(x,t) + \Delta^2 u(x,t) - \big(g * \Delta^2 u\big)(x,t) & \\ +\mu_1(t)u_t(x,t) + \mu_2(t)u_t(x,t-\tau(t)) = 0 & \text{in } \Omega\times]0,+\infty[, \\ u(x,t) = \partial_\nu u(x,t) = 0 & \text{in } \Gamma\times]0,+\infty[, \\ u(x,0) = u_0, \quad u_t(x,0) = u_1 & \text{in } \quad \Omega, \\ u(x,t-\tau(0)) = f_0(x,t-\tau(0)) & \text{in}\, \Omega \times [0,\tau(0)], \end{cases}$$

which is (2.1.1) with $\mathcal{A} = \Delta^2$, $D(\mathcal{A}) = H^4 \cap H_0^2(\Omega)$ and $\mathcal{H} = L^2(\Omega)$.

Chapter 3

The control of a non-dissipative wave equation by memory-type condition on the boundary

3.1 Introduction

Let Ω be an open bounded domain of $\mathbb{R}^n$, $n \geq 2$, with a smooth boundary $\partial\Omega$. We assume that $\partial\Omega = \Gamma_0 \cup \Gamma_1$, where partition Γ_0, Γ_1 are closed and disjoint with $meas(\Gamma_0) > 0$. In this work, we are concerned with the following initial boundary value problem of wave equation:

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + g(\nabla u(x,t)) = 0 & \text{in } \Omega \times]0,+\infty[, \\ u(x,t) = 0 & \text{in } \Gamma_0 \times]0,+\infty[, \\ u(x,t) = -(h * \partial_\nu u)(x,t) & \text{in } \Gamma_1 \times]0,+\infty[, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \quad \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau) & \text{in } \Omega \times]0,\tau[, \end{cases} \tag{3.1.1}$$

where ν is the unit normal vector and $\partial_\nu u$ is the normal derivative. Moreover, h is a positive non-increasing function defined on $\mathbb{R}^+$, and g is $C^1(\mathbb{R})$ function, the initial data (u_0, u_1, f_0) are taken in a suitable Sobolev space.

Stabilization of wave equations or wave systems by memory-feedback on the boundary has been widely considered in the literature, see for example [**11, 12, 53, 66**] and so on. It has been shown that if k is the resolvent kernel of $-h/h(0)$, then the solutions decay at the same decay rate as h and h', that is, the energy decays exponentially when the resolvent kernels decay exponentially and decay polynomially when the resolvent kernels decay polynomially. Motivated by the important paper [**23**] of Alabau-Boussouira and Cannarsa, Mustafa [**24**] considered (3.1.1) with $g \equiv 0$. Under the general assumption: $k'(t) \leq -H(k(t))$, where H is strictly convex and increasing function such that $H(0) = 0$, he established an explicit and general decay result.

In [**67**], Messaoudi and Soufyane studied (3.1.1) with $g \equiv 0$ and established a general decay estimate. In fact, they assumed that the resolvent kernel $k : [0,\infty[\longrightarrow [0,\infty[$ is a C^2 non-increasing function satisfying the following conditions

$$\lim_{t \longrightarrow \infty} k(t) = 0, \qquad k(0) > 0, \qquad k'(t) \leq -\zeta(t)k(t), \qquad k''(t) \geq \zeta(t)(-k'(t)), \tag{3.1.2}$$

where $\zeta : \mathbb{R}_+ \to]0, +\infty[$ is a C^0 non-increasing function. And, they showed that the energy of solutions E has the following decay property

$$E(t) \leq c\exp\bigg(- c\int_0^t \zeta(s)ds\bigg), \quad \text{if } u_0 = 0 \text{ on } \Gamma_1.$$

Otherwise,

$$E(t) \leq c\bigg[E(0) + \Big(\int_{\Gamma_1} |u_0|^2 d\Gamma_1\Big)\int_0^t k^2(s)\Big[1 + \exp\bigg(c\int_{t_0}^s \zeta(\sigma)d\sigma\bigg)\Big]\bigg)ds\bigg]\exp\bigg(- c\int_0^t \zeta(s)ds\bigg).$$

Noting that the problem considered is dissipative where the fact that $E' \leq 0$ played an important role in the proof of the case when $u_0 \neq 0$ on Γ_1. Here in this work we discuss the situation when the problem is not necessary dissipative in the sense that E' is not negative in general in which we introduce a new Lemma that gives us a general energy decay where the exponential, polynomial and logarithm decay rates are only special cases.

This chapter is planned as follows. In section 2, we give some assumptions and materials that will be needed in the course of our investigation. In the same section, we state, without proof, the well-posedness result of the system. In the last, we establish a general decay of solutions by the use of the multiplier method.

3.2 Preliminaries

Due to the condition $(3.1.1)_3$, we introduce the following space

$$H^1_\star(\Omega) = \Big\{f \,:\, f \in H^1(\Omega) \quad \text{and} \quad f = 0 \text{ on } \Gamma_0\Big\}.$$

Now, to estimate the term $\frac{\partial u}{\partial \nu}$ on Γ_1 we shall use the equation $(3.1.1)_3$. For that, by taking the derivative of $(3.1.1)_3$ we obtain the following Volterr'a equation

$$\frac{\partial u}{\partial \nu} = -\frac{1}{h(0)}u_t - \frac{1}{h(0)}h' * \frac{\partial u}{\partial \nu}.$$

Utilizing the Volterra's inverse operator, we can get

$$\frac{\partial u}{\partial \nu} = -\frac{1}{h(0)}\left(u_t + h' * \frac{\partial u}{\partial \nu}\right).$$

Assuming that $h(0) > 0$ and we denote by k the resolvant kernel of $-\frac{h'}{h(0)}$, which satisfies

$$k(t) + \frac{1}{h(0)}(h' * k)(t) = -\frac{1}{h(0)}h'(t), \qquad t \geq 0.$$

Then, denoting by $\gamma = \frac{1}{h(0)}$, we end up with

$$\frac{\partial u}{\partial \nu} = -\gamma\big(u_t + k(0)u - k(t)u_0 + k' * u\big). \tag{3.2.1}$$

Reciprocally, one can show that (3.2.1) imply $(3.1.1)_3$ by taking the intial data such that $u_0 = 0$ on Γ_1. So, we will use (3.2.1) instead of $(3.1.1)_3$.

We now consider the following assumptions:

(A_1) It exists $x_0 \in \mathbb{R}^n$ such that, for $\omega(x) = x - x_0$, we have

$$\omega.\nu \le 0 \quad \text{on } \Gamma_0, \tag{3.2.2}$$

$$\omega.\nu > 0 \quad \text{on } \Gamma_1. \tag{3.2.3}$$

(A_2) $k : [0,\infty[\longrightarrow [0,\infty[$ is a C^2 non-increasing function satisfying the following conditions

$$\lim_{t\longrightarrow\infty} k(t) = 0, \qquad k(0) > 0, \qquad k'(t) \le -\zeta(t)k(t), \qquad k''(t) \ge \zeta(t)(-k'(t)), \tag{3.2.4}$$

where $\zeta : \mathbb{R}_+ \rightarrow]0,+\infty[$ is a C^0 function which is not necessarily monotone such that it exists a fixed positive constant c_1 satisfying

$$\zeta(t) \le c_1, \quad \forall t \ge 0. \tag{3.2.5}$$

(A_3) $g : \mathbb{R} \longrightarrow \mathbb{R}$ is a C^1 function such that $g(0) = 0$ and

$$|g'(s)| \le \beta. \tag{3.2.6}$$

Remark 3.2.1. • *Assumption (A_1) implies that*

$$\omega.\nu \ge \delta_0 > 0 \quad \textit{on} \quad \Gamma_1 \quad \textit{and} \quad |\omega(x)| \le r, \quad \forall x \in \Omega, \tag{3.2.7}$$

where δ_0 and r are two fixed positive constants.

• *As $\zeta(t) > 0$ for all $t \ge 0$, then it exists $c_0 > 0$ such that*

$$c_0 < \zeta(t), \quad \forall t \ge 0. \tag{3.2.8}$$

• *Assumption (A_3) implies that*

$$|g(s)| \le \beta|s|, \quad \forall s \in \mathbb{R}. \tag{3.2.9}$$

Motivated by **[66]**, we introduce the following Lemma:

Lemma 3.2.2. *Let $\mathcal{L} : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a C^1 function. Assuming that there exist positive constants λ_0, λ_1, λ_2 and a continous and bounded function $\zeta : \mathbb{R}_+ \rightarrow]0,+\infty[$ which is not necessarily monotone such that*

$$\mathcal{L}'(t) \le -\lambda_0\mathcal{L}(t) + \lambda_1 exp\bigg(-\lambda_2\int_0^t \zeta(s)ds\bigg),$$

then

$$\mathcal{L}(t) \le C(\mathcal{L}(0))exp\bigg(-c\int_0^t \zeta(s)ds\bigg).$$

Proof. Let $N > 0$, defining a function F by

$$F(t) = \mathcal{L}(t) + N\text{exp}\Big(- \lambda_2 \int_0^t \zeta(s)ds\Big).$$

Then

$$\begin{aligned} F'(t) &= \mathcal{L}'(t) - N\lambda_2\zeta(t)\text{exp}\Big(- \lambda_2 \int_0^t \zeta(s)ds\Big) \\ &\leq -\lambda_0\mathcal{L}(t) - \Big(N\lambda_2\zeta(t) - \lambda_1\Big)\text{exp}\Big(- \lambda_2 \int_0^t \zeta(s)ds\Big). \end{aligned}$$

Since $\zeta(t) > 0$ for all $t \geq 0$, we can find a fixed positive constant ζ_0 such that $\zeta(t) \geq \zeta_0$ for all $t \geq 0$. And so the latter inequality becomes

$$F'(t) \leq -\lambda_0\mathcal{L}(t) - \Big(N\lambda_2\zeta_0 - \lambda_1\Big)\text{exp}\Big(- \lambda_2 \int_0^t \zeta(s)ds\Big).$$

Choosing N large enough so that $\lambda_3 = N\lambda_2\zeta_0 - \lambda_1 > 0$ and making use of the definition of $F(t)$, we can get

$$F'(t) \leq -\lambda_4 F(t).$$

As ζ is bounded then it exists $\zeta_1 > 0$ such that $\zeta(t) \leq \zeta_1$ for all $t \geq 0$. This leads us to

$$F'(t) \leq -\frac{\lambda_4}{\zeta_1}\zeta(t)F(t).$$

A simple integration over $(0, t)$ gives

$$F(t) \leq F(0)\text{exp}\Big(- c\int_0^t \zeta(s)ds\Big) \quad \Rightarrow \quad \mathcal{L}(t) \leq \big(\mathcal{L}(0) + N\big)\text{exp}\Big(- c\int_0^t \zeta(s)ds\Big).$$

This ends the proof. □

For completeness, we state the global well-posedness result in the following Theorem.

Theorem 3.2.3. *Assume that* (A_2)*-*(A_3) *hold. Then, for any* $(u_0, u_1) \in H^2 \cap H^1_\star(\Omega) \times H^1_\star(\Omega)$ *satisfying the compatibility condition*

$$\frac{\partial u_0}{\partial \nu} + \gamma u_1 = 0 \quad \text{on } \Gamma_1,$$

problem (3.1.1) *has only one global weak solution*

$$u \in L^\infty_{loc}\Big(0, \infty; H^2 \cap H^1_\star(\Omega)\Big), \quad u_t \in L^\infty_{loc}\Big(0, \infty; H^1_\star(\Omega)\Big), \quad u_{tt} \in L^\infty_{loc}\Big(0, \infty; L^2(\Omega)\Big).$$

Remark 3.2.4. *The proof of the existence result given in Theorem* 3.2.3 *can be done by using the Faedo-Galerkin method, see, for example* ***[66]***. *And, the uniqueness of this solution is a consequence of the assumption* (A_3).

3.3 Asymptotic Stability

In this section, we investigate the asymptotic stability of our problem by the use of the energy method. At the first, we define the modified energy functional of the problem (3.1.1) as

$$E(t)=\frac{1}{2}\int_{\Omega}\Big[u_t^2+|\nabla u|^2\Big]dx+\frac{\gamma}{2}\int_{\Gamma_1}\Big[k(t)|u|^2-(k'\circ u)(t)\Big]d\Gamma. \tag{3.3.1}$$

Then, the following Lemma holds true.

Lemma 3.3.1. *The energy functional E satisfies along the solutions of* (3.1.1)*, the following estimate*

$$E'(t)\leq-\frac{\gamma}{2}\int_{\Gamma_1}\Big[u_t^2+(k''\circ u)(t)-k'(t).u^2-k^2(t)u_0^2\Big]d\Gamma+\frac{\beta}{2}\int_{\Omega}\Big[u_t^2+|\nabla u|^2\Big]dx. \tag{3.3.2}$$

Proof. By multiplying $(3.1.1)_1$ by u_t and using integration by parts over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}\Big[u_t^2+|\nabla u|^2\Big]dx=-\int_{\Omega}u_t g(\nabla u)dx=\int_{\Gamma_1}\frac{\partial u}{\partial\nu}u_t d\Gamma. \tag{3.3.3}$$

Substituting the boundary term by (3.2.1) and using Lemma 2.2.3, we find that

$$E'(t)=-\int_{\Omega}u_t g(\nabla u)dx-\frac{\gamma}{2}\int_{\Gamma_1}\Big[|u_t|^2+(k''\circ u)(t)-k'(t).u^2-k^2(t).u_0^2\Big]d\Gamma, \tag{3.3.4}$$

using then Young's inequality and (3.2.9), we obtain (3.3.2). That concludes the proof. □

The main result of this chapter is:

Theorem 3.3.2. *Let u be the solution of* (3.1.1)*. Assuming that* (A_1)*-*(A_3) *hold with β small enough. Then it exists two positive constants a_1 and a_2 such that the solution of* (3.1.1) *satisfies the following decay property*

$$E(t)\leq a_1 exp\Big(-a_2\int_0^t\zeta(s)ds\Big),\quad\forall t\geq 0. \tag{3.3.5}$$

To prove the stability result stated in Theorem 3.3.2 we need the following Lemma.

Lemma 3.3.3. *The functional*

$$I(t)=\int_{\Omega}\Big[2\omega.\nabla u+(n-1)u\Big]u_t dx, \tag{3.3.6}$$

satisfies, along the solution of (3.1.1)*,*

$$\begin{aligned}I'(t)\leq\ &-\int_{\Omega}|u_t|^2dx+\Big[\|\omega\|_\infty+4\gamma^2\Big(\frac{2\|\omega\|_\infty^2}{\delta_0}+2(n-1)^2c_*\Big)\Big]\int_{\Gamma_1}|u_t|^2d\Gamma\\ &-\Big[\frac{1}{2}-\Big(1+c_*+\|\omega\|_\infty^2+\frac{(1-n)^2}{4}\Big)\beta-4\gamma^2c_*\Big(\frac{2\|\omega\|_\infty^2}{\delta_0}+2(n-1)^2c_*\Big)k^2(t)\Big]\\ &\times\int_{\Omega}|\nabla u|^2dx+\Big[\frac{2\|\omega\|_\infty^2}{\delta_0}+2(n-1)^2c_*\Big]c\int_{\Gamma_1}(-k'\circ u)(t)d\Gamma+ck^2(t)\int_{\Gamma_1}u_0^2d\Gamma.\end{aligned} \tag{3.3.7}$$

Proof. A simple differentiation with respect to t yields

$$I'(t) = \int_\Omega (2\omega.\nabla u_t)u_t dx + (n-1)\int_\Omega |u_t|^2 dx + \int_\Omega (2\omega.\nabla u + (n-1)u)u_{tt} dx,$$

by $(3.1.1)_1$, one has

$$\begin{aligned} I'(t) = &- \int_\Omega |u_t|^2 dx + \int_{\Gamma_1} (\omega.\nu)|u_t|^2 d\Gamma + \int_\Omega \big(2\omega.\nabla u + (n-1)u\big)\Delta u dx \\ &- \int_\Omega (2\omega.\nabla u + (n-1)u) g(\nabla u) dx. \end{aligned} \tag{3.3.8}$$

Using the identity $2\nabla u.\nabla(\omega.\nabla u) = 2|\nabla u|^2 + \omega\nabla(|\nabla u|^2)$, we can get

$$\begin{aligned} \int_\Omega (2\omega.\nabla u)\Delta u dx &= -\int_\Omega \nabla(2\omega.\nabla u).\nabla u dx + \int_{\partial\Omega} (2\omega.\nabla u)\frac{\partial u}{\partial \nu} d\Gamma \\ &= -\int_\Omega \big(2|\nabla u|^2 + \omega\nabla(|\nabla u|^2)\big) + \int_{\partial\Omega} (2\omega.\nabla u)\frac{\partial u}{\partial \nu} d\Gamma \\ &= (n-2)\int_\Omega |\nabla u|^2 dx - \int_{\partial\Omega} (\omega.\nu)|\nabla u|^2 d\Gamma + \int_{\partial\Omega} (2\omega.\nabla u)\frac{\partial u}{\partial \nu} d\Gamma. \end{aligned}$$

By the fact that

$$\nabla u = \left(\frac{\partial u}{\partial \nu}\right)\nu \quad \text{on} \quad \Gamma_0,$$

one gets

$$\begin{aligned} \int_\Omega (2\omega.\nabla u)\Delta u dx = &(n-2)\int_\Omega |\nabla u|^2 dx - \int_{\Gamma_0} (\omega.\nu)|\nabla u|^2 d\Gamma \\ &+ \int_{\Gamma_1} (\omega.\nu)|\nabla u|^2 d\Gamma + \int_{\Gamma_1} (2\omega.\nabla u)\frac{\partial u}{\partial \nu} d\Gamma. \end{aligned}$$

Then

$$\begin{aligned} \int_\Omega \Big[2\omega.\nabla u + (n-1)u\Big]\Delta u dx \le &-\int_\Omega |\nabla u|^2 dx - \delta_0 \int_{\Gamma_1} |\nabla u|^2 d\Gamma \\ &+ \int_{\Gamma_1} \Big[(2\omega.\nabla u) + (n-1)u\Big]\frac{\partial u}{\partial \nu} d\Gamma, \end{aligned} \tag{3.3.9}$$

where we used $m.\nu \ge \delta_0 > 0$ on Γ_1, then, (3.3.8) becomes

$$\begin{aligned} I'(t) \le &- \int_\Omega |u_t|^2 dx + \|\omega\|_\infty \int_{\Gamma_1} |u_t|^2 d\Gamma - \int_\Omega |\nabla u|^2 dx - \delta_0 \int_{\Gamma_1} |\nabla u|^2 d\Gamma \\ &+ \int_{\Gamma_1} \Big[(2\omega.\nabla u) + (n-1)u\Big]\frac{\partial u}{\partial \nu} d\Gamma - 2\int_\Omega (\omega.\nabla u) g(\nabla u) dx \\ &- (n-1)\int_\Omega u g(\nabla u) dx. \end{aligned} \tag{3.3.10}$$

Applying the Young's and Poincaré's inequalities, we have that

$$\int_{\Gamma_1} (2\omega.\nabla u)\frac{\partial u}{\partial \nu} d\Gamma \le \frac{\delta_0}{2}\int_{\Gamma_1} |\nabla u|^2 dx + \frac{2\|\omega\|_\infty^2}{\delta_0}\int_{\Gamma_1} \left|\frac{\partial u}{\partial \nu}\right|^2 d\Gamma, \tag{3.3.11}$$

$$\int_{\Gamma_1}(n-1)u\frac{\partial u}{\partial\nu}d\Gamma \leq \frac{1}{2}\int_{\Omega}|\nabla u|^2dx + 2(n-1)^2c_*\int_{\Gamma_1}\left|\frac{\partial u}{\partial\nu}\right|^2d\Gamma, \tag{3.3.12}$$

$$\int_{\Omega}(2\omega.\nabla u)g(\nabla u)dx \leq \beta\int_{\Omega}|\nabla u|^2dx + \frac{\|\omega\|_\infty^2}{\beta}\int_{\Omega}|g(\nabla u)|^2dx, \tag{3.3.13}$$

$$(n-1)\int_{\Omega}ug(\nabla u)dx \leq \beta c_*\int_{\Omega}|\nabla u|^2dx + \frac{(n-1)^2}{4\beta}\int_{\Omega}|g(\nabla u)|^2dx. \tag{3.3.14}$$

A simple substituation of (3.3.11)-(3.3.13) into (3.3.10), using (3.2.9), gives

$$\begin{aligned} I'(t) \leq\ & \|\omega\|_\infty\int_{\Gamma_1}|u_t|^2d\Gamma - \left[\frac{1}{2} - \Big(1 + c_* + \|\omega\|_\infty^2 + 2(1-n)^2c_*\Big)\beta\right]\int_{\Omega}|\nabla u|^2dx \\ & - \int_{\Omega}|u_t|^2dx + \left[\frac{2\|\omega\|_\infty^2}{\delta_0} + \frac{(n-1)^2}{4\delta_1}\right]\int_{\Gamma_1}\left|\frac{\partial u}{\partial\nu}\right|^2d\Gamma. \end{aligned} \tag{3.3.15}$$

Noting that the boundary condition (3.2.1) can be rewritten as

$$\frac{\partial u}{\partial\nu} = -\gamma\left(u_t + k(t)u - k(t)u_0 - k'\diamond u\right),$$

and

$$\begin{aligned} |(k'\diamond u)(t)|^2 &\leq \left(\int_0^t -k'(s)ds\right)(-k'\circ u)(t) \\ &= \Big[k(t) - k(0)\Big](-k'\circ u)(t). \end{aligned}$$

Hence

$$\left|\frac{\partial u}{\partial\nu}\right|^2 \leq 4\gamma^2\Big[u_t^2 + k^2(t)u^2 + k^2(t)u_0^2 - c(-k'\circ u)(t)\Big]. \tag{3.3.16}$$

Inseting (3.3.16) into (3.3.15) and using Poincaré's inequality, we obtain (3.3.7). □

We now define a Lyapunov functional $\mathcal{L}$ as follows:

$$\mathcal{L}(t) = NE(t) + I(t), \tag{3.3.17}$$

where N is positive real numbers that we will be choosen later.
It is clear that $\mathcal{L}$ is equivalent to E for N sufficiently large. Then, combining (3.3.2),(3.3.7), we have that

$$\begin{aligned} \mathcal{L}'(t) \leq\ & -\left[1 - \frac{\beta}{2}\right]\int_{\Omega}|u_t|^2dx - \left[\frac{\gamma}{2}\|\omega\|_\infty N - 4\gamma^2\Big(\frac{2\|\omega\|_\infty^2}{\delta_0} + 2(n-1)^2c_*\Big)\right]\int_{\Gamma_1}|u_t|^2d\Gamma \\ & - \left[\frac{1}{2} - \Big(\frac{3}{2} + c_* + \|\omega\|_\infty^2 + \frac{(1-n)^2}{4}\Big)\beta - 4\gamma^2c_*\Big(\frac{2\|\omega\|_\infty^2}{\delta_0} + 2(n-1)^2c_*\Big)k^2(t)\right] \\ & \times\int_{\Omega}|\nabla u|^2dx + \left[\frac{\gamma c_0}{2}N - \Big(\frac{2\|\omega\|_\infty^2}{\delta_0} + 2(n-1)^2c_*\Big)c\right]\int_{\Gamma_1}(-k'\circ u)(t)d\Gamma \\ & + ck^2(t)\int_{\Gamma_1}u_0^2d\Gamma - \frac{\gamma c}{2}N\int_{\Gamma_1}k(t)|u|^2d\Gamma. \end{aligned} \tag{3.3.18}$$

At this point, for a fixed γ we want to choose our constants N and β very carefully in order to get

$$\mathcal{L}'(t) \leq -cE(t) + ck^2(t) \tag{3.3.19}$$

First, we take N large enough so that

$$\frac{\gamma}{2}\|\omega\|_\infty N - 4\gamma^2\Big(\frac{2\|\omega\|_\infty^2}{\delta_0} + 2(n-1)^2 c_*\Big) \geq 0,$$

$$\frac{\gamma c_0}{2}N - \Big(\frac{2\|\omega\|_\infty^2}{\delta_0} + 2(n-1)^2 c_*\Big)c > 0.$$

Second, using the fact that $\lim_{t\longrightarrow\infty} k(t) = 0$ and we choose β sufficiently small so that

$$1 - \frac{\beta}{2} > 0,$$

$$\frac{1}{2} - \Big(\frac{3}{2} + c_* + \|\omega\|_\infty^2 + \frac{(1-n)^2}{4}\Big)\beta - 4\gamma^2 c_*\Big(\frac{2\|\omega\|_\infty^2}{\delta_0} + 2(n-1)^2 c_*\Big)k^2(t) > 0.$$

Thus, using the definition of $E(t)$ we end up with

$$\mathcal{L}'(t) \leq -cE(t) + ck^2(t) \tag{3.3.20}$$

Noting that the assumption $k'(t) \leq -\zeta(t)k(t)$ implies that

$$k(t) \leq c\exp\Big(-c\int_0^t \zeta(s)ds\Big), \quad \forall t \geq 0.$$

Then, (3.3.20) becomes

$$\mathcal{L}'(t) \leq -c\mathcal{L}(t) + c\exp\Big(-c\int_0^t \zeta(s)ds\Big). \tag{3.3.21}$$

Applying Lemma (3.2.2), we obtain that

$$\mathcal{L}(t) \leq c\exp\Big(-c\int_0^t \zeta(s)ds\Big). \tag{3.3.22}$$

The use of $\mathcal{L} \sim E$ leads us to the above mentioned stability result.

Chapter 4

Optimal polynomial decay for a Timoshenko system with a strong damping and a strong delay

4.1 Introduction

This chapter is devoted to the study of the well-posedness and the stability of the Timoshenko system which was introduced in **[73]** as a simple model describing the transverse vibration of a beam. The system is governed by two hyperbolic equations where the main variables $\varphi(x,t)$ and $\psi(x,t)$ denote, respectively, the transverse displacement of the beam and the rotation angle of the filament of the beam. The system is represented as

$$\begin{cases} \rho_1\varphi_{tt}(x,t) = S_x(x,t) & \text{in }]0,L[\times]0,+\infty[, \\ \rho_2\psi_{tt}(x,t) = M_x(x,t) - S(x,t) & \text{in }]0,L[\times]0,+\infty[. \end{cases} \tag{4.1.1}$$

Here $S = \kappa(\varphi_x + \psi)$ and $M = b\psi_x$, x is the space variable along the beam of length L, t is the time variable and

$$\rho_1 = \rho, \quad \rho_2 = I_\rho, \quad \kappa = K, \quad b = EI,$$

where ρ, I_ρ, K, E are positive constants for the elastic properties. More precisely, ρ for density (the mass per unit length), I_ρ for the polar moment of inertia of a cross section, E for the modulus of elasticity (the Young's coefficient), I for the moment of inertia of a cross section and K for the shear modulus. To understand our motivation, we appeal to keep in mind that the system (4.1.1) is purely conservative. More specifically, by taking any suitable boundary condition into consideration, the energy of the beam defined by

$$E(t) = \frac{1}{2}\int_0^L \left[\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \kappa(\varphi_x+\psi)^2 + b\psi_x^2\right]dx \tag{4.1.2}$$

satisfies the so-called energy conservation property, that is $E(t) = E(0)$ for all $t \geq 0$. So, to attenuate that vibrations, control terms, such as: frictional damping, thermal dissipation and viscoelastic damping, will be necessary. For instance, Kim and Renardy **[79]** considered (4.1.1) with two boundary controls and they established the exponential decay of the energy

$E(t)$ by using a multiplier method. Soufyane and Wehbe **[81]** established an exponential stability result by employing the unique frictional damping $\alpha(x)\psi_t$. This result was obtained in the case of the equal-speeds, i.e.,

$$\frac{\rho_1}{\kappa} = \frac{\rho_2}{b}. \tag{4.1.3}$$

Raposo et al. **[114]** used two frictional dissipative terms φ_t and ψ_t and proved that the solution decays exponentially without imposing any conditions on the coefficients ρ_1, ρ_2, κ, b. Alabau-Boussouira **[83]** generalized the result in **[81]** by employing the unique non-linear damping $g(\psi_t)$. In that work, she gave a semi-explicit and general formula for the decay rate of solution at infinity provided (4.1.3) holds true. Mustafa and Messaoudi **[85]** improved this result when they considered (4.1.1) with $\rho_1 = \rho_2 = \kappa = b = 1$. They used the weak non-linear damping $\alpha(t)g(\psi_t)$ and established a general and explicit decay result. Motivated by **[86]**, Park and Kang **[87]** examined (4.1.1) with two weak non-linear damping $\alpha(t)g(\varphi_t)$ and $\alpha(t)g(\psi_t)$ and established the stability result without assuming equal speeds of propagation of waves.

Now, we concentrate on the stability problem for the Timoshenko system with delay which is the subject of the present chapter. Consider the following model:

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - \kappa(\varphi_x + \psi)_x(x,t) + a_1\varphi_t(x,t) + a_2\varphi_t(x,t-\tau) = 0 & \text{in }]0,1[\times]0,\infty[, \\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + \kappa(\varphi_x + \psi)(x,t) + \mu_1\psi_t(x,t) + \mu_2\psi_t(x,t-\tau) = 0 & \text{in }]0,1[\times]0,\infty[, \end{cases} \tag{4.1.4}$$

where $a_i, \mu_i > 0$ for $i = 1, 2$. If $a_i = 0$ and $\mu_2 < \mu_1$ then the exponential stability has been established by Said-Houari and Laskri **[76]** in the case of equal-speeds of propagation. Apalara **[69]** examined (4.1.4) when $\mu_i = 0$ and $a_2 < a_1$ and realized an exponential stability result in the case $\frac{\rho_1}{\kappa} = \frac{\rho_2}{b}$. And, in the opposite case, only the polynomial stability was given.

As a consequence of the results cited above, if the frictional damping is acting in only one equation of the Timoshenko system then we can prove the uniform (exponential) stability for weak solutions in the case of equal-speeds of propagation. For the opposite case, a slow (polynomial) decay rate result is achieved for strong solutions. For Timoshenko system with weak delay term, if the weight of delay term is small and satisfies some conditions between the weights of delay term and the weights of frictional damping, we can get the same results, see **[76, 69]** and so on.

According to this remarks, one question naturally arise: is it possible to consider the Timoshenko system with a strong damping in the presence of a constant delay in the strong internal feedback and get the same result as in **[69]**?

This chapter aims to answer this question by investigating the following system:

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - \kappa(\varphi_x + \psi)_x(x,t) - \mu_1\varphi_{xxt}(x,t) - \mu_2\varphi_{xxt}(x,t-\tau) = 0 & \text{in }]0,1[\times]0,\infty[, \\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + \kappa(\varphi_x + \psi)(x,t) = 0 & \text{in }]0,1[\times]0,\infty[, \\ \varphi(0,t) = \varphi(1,t) = \psi_x(0,t) = \psi_x(1,t) = 0 & \text{in }]0,\infty[, \\ \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x) & \text{in }]0,1[, \\ \psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x) & \text{in }]0,1[, \\ \varphi_t(x,t-\tau) = f_0(x,t-\tau) & \text{in }]0,1[\times]0,\tau[, \end{cases} \tag{4.1.5}$$

where $\mu_1 > 0$, μ_2 is a real number, $\tau > 0$ is the time of delay and $(\varphi_0, \varphi_1, \psi_0, \psi_1, f_0)$ are in a suitable Sobolev space.

The remaining parts of this chapter are as follows. In section 2, we provide the needed assumptions and materials. In section 3, we study the well-posedness by the semi-group techniques. In section 4, we prove the lack of exponential decay even if $\frac{\rho_1}{\kappa} = \frac{\rho_2}{b}$. In the last section, to establish the polynomial decay of solution, we introduce a suitable Lyapunov functional. We use c throughout the paper to denote a fixed positive number which may be different at different estimates.

4.2 Preliminaries

In this section, we will present some materials and notations that will be needed to prove our main results. First, we introduce as in Nicaise and Pignotti **[36]** the new variable

$$z(x,\rho,t) = \varphi_t(x, t-\rho\tau), \qquad x \in [0,1],\ \rho \in [0,1],\ t > 0.$$

It is easy to show that

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0 \qquad \text{in } ([0,1])^2 \times [0,\infty].$$

Thus, problem (4.1.5) becomes

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - \kappa(\varphi_x+\psi)_x(x,t) - \mu_1\varphi_{xxt}(x,t) - \mu_2 z_{xx}(x,1,t) = 0 & \text{in }]0,1[\ \times\ ]0,\infty[, \\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + \kappa(\varphi_x+\psi)(x,t) = 0 & \text{in }]0,1[\ \times\ ]0,\infty[, \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0 & \text{in } (]0,1[)^2 \times\ ]0,\infty[, \\ \varphi(0,t) = \varphi(1,t) = \psi_x(0,t) = \psi_x(1,t) = 0 & \text{in }]0,\infty[, \\ z(x,0,t) = \varphi_t(x,t) & \text{in }]0,\infty[, \\ \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x) & \text{in }]0,1[, \\ \psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x) & \text{in }]0,1[, \\ z(x,\rho,0) = f_0(x,-\rho\tau) & \text{in } (]0,1[)^2. \end{cases} \tag{4.2.1}$$

We will show that the assumption

$$|\mu_2| < \mu_1 \tag{4.2.2}$$

guarantees the global well-posedness as well as the uniform decay of the energy E, given by

$$E(t) = \frac{1}{2}\int_0^1 \left[\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \kappa(\varphi_x+\psi)^2 + b\psi_x^2 + \tau\gamma\int_0^1 z_x^2(x,\rho)d\rho\right]dx, \tag{4.2.3}$$

where γ is a fixed positive constant satisfying

$$|\mu_2| < \gamma < 2\mu_1 - |\mu_2|. \tag{4.2.4}$$

Remark 4.2.1. *By using Eq.*$(4.2.1)_2$ *and the boundary conditions, we find that*

$$\frac{d^2}{dt^2}\int_0^1 \psi(x,t)dx + \frac{\kappa}{\rho_2}\int_0^1 \psi(x,t)dx = 0,$$

which directly gives

$$\int_0^1 \psi(x,t)dx = \left(\int_0^1 \psi_0(x)dx\right)\cos\left(\sqrt{\frac{\kappa}{\rho_2}}t\right) + \sqrt{\frac{\rho_2}{\kappa}}\left(\int_0^1 \psi_1(x)dx\right)\sin\left(\sqrt{\frac{\kappa}{\rho_2}}t\right).$$

Then, if we pose

$$\overline{\psi}(x,t) = \psi(x,t) - \left(\int_0^1 \psi_0(x)dx\right)\cos\left(\sqrt{\frac{\kappa}{\rho_2}}t\right) - \sqrt{\frac{\rho_2}{\kappa}}\left(\int_0^1 \psi_1(x)dx\right)\sin\left(\sqrt{\frac{\kappa}{\rho_2}}t\right),$$

we can easily show that $(\varphi, \overline{\psi}, z)$ *satisfies problem* (4.2.1) *together with its boundary conditions and with initial conditions for* $\overline{\psi}$ *given as*

$$\overline{\psi}(x,0) = \psi_0(x) - \int_0^1 \psi_0(x)dx, \quad \overline{\psi}_t(x,0) = \psi_1(x) - \int_0^1 \psi_1(x)dx.$$

In addition, one has

$$\int_0^1 \overline{\psi}(x,t)dx = 0.$$

Thus, we will work with $\overline{\psi}$ *but we write* ψ *for simplicity.*

Our starting point will be to show that the energy functional E is non-increasing.

Lemma 4.2.2. *Assuming that* (4.2.2) *holds. Then, the energy functional* E *defined by* (4.2.3) *is non-increasing and satisfies, for all* $t \geq 0$, *the following estimate*

$$E'(t) \leq -\beta_1 \int_0^1 \varphi_{xt}^2 dx - \beta_2 \int_0^1 z_x^2(x,1)dx, \tag{4.2.5}$$

where

$$\beta_1 = \mu_1 - \frac{\gamma}{2} - \frac{|\mu_2|}{2} > 0 \quad \textit{and} \quad \beta_2 = \frac{\gamma}{2} - \frac{|\mu_2|}{2} > 0.$$

Proof. Multiplyting Eq.$(4.2.1)_1$ by φ_t, Eq.$(4.2.1)_2$ by ψ_t and integrating the products by parts over $(0,1)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \left[\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \kappa(\varphi_x + \psi)^2 + b\psi_x^2\right]dx = -\mu_1 \int_0^1 \varphi_{xt}^2 dx - \mu_2 \int_0^1 \varphi_{xt} z_x(x,1)dx. \tag{4.2.6}$$

Multiplying then Eq.$(4.2.1)_3$ by $-\gamma z_{xx}(x,\rho)$, integrating over $(0,1)\times(0,1)$ and using Eq.$(4.2.1)_5$, we get

$$\frac{\tau\gamma}{2}\frac{d}{dt}\int_0^1\int_0^1 z_x^2(x,\rho)d\rho dx = -\frac{\gamma}{2}\int_0^1 z_x^2(x,1)dx + \frac{\gamma}{2}\int_0^1 \varphi_{xt}^2 dx. \tag{4.2.7}$$

The Combination of (4.2.6) and (4.2.7), bearing (4.2.3) in mind, gives

$$E'(t) = -\left(\mu_1 - \frac{\gamma}{2}\right)\int_0^1 \varphi_{xt}^2 dx - \frac{\gamma}{2}\int_0^1 z_x^2(x,1)dx - \mu_2 \int_0^1 \varphi_{xt} z_x(x,1)dx,$$

applying then Young's inequality, we obtain the desired result (4.2.5). That completes the proof. □

4.3 The well-posedness

In this section, we study the existence and uniqueness of solutions for system (4.2.1) by using the theory of semigroup. For this aim, let $U = U(t) = \big(\varphi, u, \psi, v, z\big)^T$, where $u = \varphi_t$ and $v = \psi_t$. Then, because of boundary conditions we consider the following spaces

$$\begin{aligned}
&L^2_\star(0,1) = \left\{ u \in L^2(0,1) : \quad \int_0^1 u(x)dx = 0 \quad \right\}, \\
&H^1_\star(0,1) = L^2_\star(0,1) \cap H^1(0,1), \\
&H^2_\star(0,1) = \left\{ u \in H^2(0,1) : u_x(0) = u_x(1) = 0 \right\}.
\end{aligned}$$

Next, we define the energy space as

$$\mathcal{H} = H^1_0(0,1) \times H^1_\star(0,1) \times L^2(0,1) \times L^2_\star(0,1) \times L^2\big(0,1; H^1_0(0,1)\big), ?$$

which endowed with the inner product

$$\langle U, \overline{U} \rangle_{\mathcal{H}} = \int_0^1 \left[\rho_1 u\overline{u} + \rho_2 v\overline{v} + \kappa(\varphi_x + \psi)(\overline{\varphi_x} + \overline{\psi}) + b\psi_x\overline{\psi_x} + \tau\gamma \int_0^1 z_x(x,\rho)\overline{z}_x(x,\rho) \right] dx.$$

Therefore, our system (4.2.1) rewrites as

$$\begin{cases} U' = \mathcal{A}U, \\ U_0 = U(0) = \big(\varphi_0, \varphi_1, \psi_0, \psi_1, f_0(., . - \tau)\big), \end{cases}$$

where the domain $D(\mathcal{A}) \subset \mathcal{H}$ of the linear operator $\mathcal{A}$ is given by

$$D(\mathcal{A}) = \begin{Bmatrix} U \in \mathcal{H} \;/\; \varphi + \mu_1 u + \mu_2 z(.,1) \in H^2 \cap H^1_0(0,1), \quad \psi \in H^2_\star \cap H^1_\star(0,1) \\ v \in H^1_\star(0,1), \quad z \in L^2(0,1; H^1_0(0,1)), \quad z(.,0,.) = u \end{Bmatrix}$$

and

$$\mathcal{A}U = \begin{pmatrix} u \\ \dfrac{\kappa}{\rho_1}(\varphi_x + \psi)_x + \dfrac{\mu_1}{\rho_1} u_{xx} + \dfrac{\mu_2}{\rho_1} z_{xx}(.,1) \\ v \\ \dfrac{b}{\rho_2}\psi_{xx} - \dfrac{\kappa}{\rho_2}(\varphi_x + \psi) \\ -\tau^{-1} z_\rho \end{pmatrix}.$$

Our first main result is given by the following Theorem.

Theorem 4.3.1. *Assume that* (4.2.2) *holds. Then, for any $U_0 \in \mathcal{H}$, it exists a unique weak solution $U \in C\big([0,+\infty); \mathcal{H}\big)$ of system* (4.2.1)*. Moreover, if $U_0 \in D(\mathcal{A})$ then* (4.2.1) *admits a unique classical solution $U \in C\big([0,+\infty); D(\mathcal{A})\big) \cap C^1\big([0,+\infty); \mathcal{H}\big)$.*

Proof. To prove the result given in Theorem 4.3.1, we use the semigroup arguments, that is, we show that the linear operator $\mathcal{A}$ generates a C_0-semigroup on $\mathcal{H}$. For that, we need the following two lemmas.

Lemma 4.3.2. *The operator $\mathcal{A}$ is dissipative and satisfies for all $U \in D(\mathcal{A})$,*

$$\langle \mathcal{A}U, U\rangle_{\mathcal{H}} \leq -\beta_1 \int_0^1 u_x^2 dx - \beta_2 \int_0^1 z_x^2(x,1)dx \leq 0. \tag{4.3.1}$$

Proof. As $E(t) = \frac{1}{2}\|U\|_{\mathcal{H}}^2, \ \forall U \in D(\mathcal{A})$, a simple differentiation gives

$$\langle U', U\rangle_{\mathcal{H}} = E'(t),$$

and so

$$\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = E'(t),$$

then by (4.2.5), we obtain (4.3.1). Hence, $\mathcal{A}$ is dissipative. □

Lemma 4.3.3. *The operator $\lambda I - \mathcal{A}$ is surjective.*

Proof. It suffices to prove that, for all $F = (f_1, f_2, f_3, f_4, f_5)^T \in \mathcal{H}$, there exists $U \in D(\mathcal{A})$ satisfying

$$(\lambda I - \mathcal{A})U = F, \tag{4.3.2}$$

which is

$$\begin{cases} \lambda\varphi - u & = f_1, \\ \lambda\rho_1 u - \kappa(\varphi_x + \psi)_x - \mu_1 u_{xx} - \mu_2 z_{xx}(.,1) & = \rho_1 f_2, \\ \lambda\psi - v & = f_3, \\ \lambda\rho_2 v - b\psi_{xx} + \kappa(\varphi_x + \psi) & = \rho_2 f_4, \\ \lambda z + \tau^{-1} z_\rho & = f_5. \end{cases} \tag{4.3.3}$$

Following the same method as in **[36]** by using Eqs.$(4.3.3)_5$-$(4.2.1)_5$, we obtain that

$$z(x,\rho) = e^{-\lambda\tau\rho}u(x) + \tau e^{-\lambda\tau\rho} \int_0^\rho e^{\lambda\tau s} f_5(x,s)ds,$$

then by $(4.3.3)_1$, we have

$$z(x,\rho) = \lambda e^{-\lambda\tau\rho}\varphi(x) - e^{-\lambda\tau\rho} f_1(x) + \tau e^{-\lambda\tau\rho} \int_0^\rho e^{\lambda\tau s} f_5(x,s)ds,$$

therefore,

$$z(x,1) = \lambda e^{-\lambda\tau}\varphi(x) - e^{-\lambda\tau} f_1(x) + \tau e^{-\lambda\tau} \int_0^1 e^{\lambda\tau s} f_5(x,s)ds. \tag{4.3.4}$$

Plugging $u = \lambda\varphi - f_1$, $v = \lambda\psi - f_3$ and (4.3.4) into $(4.3.3)_2$ and $(4.3.3)_4$ to get

$$\begin{cases} \lambda^2\rho_1\varphi - \kappa(\varphi_x + \psi)_x - \lambda(\mu_1 + \mu_2 e^{-\lambda\tau})\varphi_{xx} = \rho_1(\lambda f_1 + f_2) + h_{xx}, \\ \lambda^2\rho_2\psi - b\psi_{xx} + \kappa(\varphi_x + \psi) = \rho_2(\lambda f_3 + f_4), \end{cases} \tag{4.3.5}$$

where

$$h_{xx} = -(\mu_1 + \mu_2 e^{-\lambda\tau}) f_{1xx} + \mu_2 \tau e^{-\lambda\tau} \int_0^1 e^{\lambda\tau s} f_{5xx}(x,s)ds.$$

Solve (4.3.5) is equivalent to find $(\varphi,\psi)\in H_0^1\times H_\star^1$ such that

$$\begin{cases}\displaystyle\int_0^1\Big[\lambda^2\rho_1\varphi\omega+\kappa(\varphi_x+\psi)\omega_x+\lambda(\mu_1+\mu_2e^{-\lambda\tau})\varphi_x\omega_x\Big]dx=\int_0^1\Big[\rho_1(\lambda f_1+f_2)\omega-h_x\omega_x\Big]dx,\\ \displaystyle\int_0^1\Big[\lambda^2\rho_2\psi\varpi+b\psi_x\varpi_x+\kappa(\varphi_x+\psi)\varpi\Big]dx=\int_0^1\rho_2(\lambda f_3+f_4)\varpi dx.\end{cases} \tag{4.3.6}$$

Combining $(4.3.6)_1$ and $(4.3.6)_2$, we find the following variational formulation of (4.3.2)

$$B\big((\varphi,\psi),(\omega,\varpi)\big)=L(\omega,\varpi) \tag{4.3.7}$$

where the bilinear form $B:\left[H_0^1(0,1)\times H_\star^1(0,1)\right]^2\longrightarrow\mathbb{R}$ is defined by

$$\begin{aligned}B\big((\varphi,\psi),(\omega,\varpi)\big)=\int_0^1\Big[&\lambda^2\rho_1\varphi\omega+\lambda^2\rho_2\psi\varpi+\kappa(\varphi_x+\psi)(\omega_x+\varpi)\\&+b\psi_x\varpi_x+\lambda(\mu_1+\mu_2e^{-\lambda\tau})\varphi_x\omega_x\Big]dx\end{aligned}$$

and the linear form $L:H_0^1(0,1)\times H_\star^1(0,1)\longrightarrow\mathbb{R}$ is given by

$$\begin{aligned}L(\omega,\varpi)=\int_0^1\Big[&\rho_1(\lambda f_1+f_2)\omega+\rho_2(\lambda f_3+f_4)\varpi+(\mu_1+\mu_2e^{-\lambda\tau})f_{1x}\omega_x\\&-\mu_2\tau e^{-\lambda\tau}\omega_x\int_0^1e^{\lambda\tau s}f_{5x}(x,s)ds\Big]dx.\end{aligned}$$

Next, for $V=H_0^1(0,1)\times H_\star^1(0,1)$ endowed with the norm

$$\|(\varphi,\psi)\|_V^2=\|\varphi_x+\psi\|_2^2+\|\varphi\|_2^2+\|\psi_x\|_2^2,$$

it is readily seen that B and L are bounded. On the other hand, we have

$$B\big((\varphi,\psi),(\varphi,\psi)\big)\geq c\|(\varphi,\psi)\|_V^2,$$

which implies that B is coercive. Applying then Lax-Miligram theorem, we deduce that (4.3.7) admits only one solution

$$(\varphi,\psi)\in H_0^1(0,1)\times H_\star^1(0,1).$$

By the classical elliptic regularity, we conclude that the solution (φ,ψ) belongs into $H^2\cap H_0^1(0,1)\times H_\star^2\cap H_\star^1(0,1)$. Consequently, Eq.(4.3.2) has a unique solution $U\in D(\mathcal{A})$. This shows that the operator $\lambda I-\mathcal{A}$ is surjective. □

Finally, Lemma 4.3.2 and Lemma 4.3.3 imply that $-\mathcal{A}$ is maximal monotone operator. Thanks to Lummer-Phillips Theorem, we conclude that the operator $\mathcal{A}$ generates a linear C_0-semigroup in $\mathcal{H}$ and hence (4.2.1) is well-posed (see Pazy [**94**]). □

4.4 The lack of exponential stability

In this section, using the following Gearhart-Herbst-Prüss-Huang Theorem (see [**89, 90, 91**]), we prove that the semigroup associated to the system (4.2.1) is not exponentially stable.

Theorem 4.4.1. *Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup of contractions on Hilbert space. Then $S(t)$ is exponentially stable if and only if*

$$i\mathbb{R} \subset \varrho(\mathcal{A})$$

and

$$\overline{\lim_{|\lambda|\to\infty}} \|\left(i\lambda I - \mathcal{A}\right)^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty,$$

where $\varrho(\mathcal{A})$ is the resolvent set of $\mathcal{A}$.

To demonstrate the lack of exponential stability, we will employ the above Theorem integrated with some techniques used in [**80, 88, 92, 93, 96**] taking into account the nature of our problem. So, we will prove that it exists a subsequence $(\lambda_\nu)_{\nu\in\mathbb{N}} \subset \mathbb{R}$ such that

$$\left\|\left(i\lambda_\nu I - \mathcal{A}\right)^{-1}\right\|_{\mathcal{L}(\mathcal{H})} \longrightarrow \infty \qquad \text{as} \qquad \nu \longrightarrow \infty.$$

Which is equivalent to show that it exist $(\lambda_\nu)_{\nu\in\mathbb{N}} \subset \mathbb{R}$ and $(F_\nu)_{\nu\in\mathbb{N}} \subset \mathcal{H}$, with $\|F_\nu\|_{\mathcal{H}} \leq 1$, such that

$$\left\|(i\lambda_\nu I - \mathcal{A})^{-1} F_\nu\right\|_{\mathcal{H}} = \|U_\nu\|_{\mathcal{H}} \to \infty \qquad \text{as} \qquad \nu \longrightarrow \infty.$$

We, therefore, consider the following spectral equation

$$i\lambda_\nu U_\nu - \mathcal{A}U_\nu = F_\nu, \tag{4.4.1}$$

and we shall prove that the corresponding solution U_ν is not bounded when F_ν is bounded in $\mathcal{H}$. Rewriting (4.4.1) in term of its components, we obtain that

$$\begin{aligned}
&i\lambda\varphi - u && = f_1,\\
&i\lambda u - \frac{\kappa}{\rho_1}(\varphi_x + \psi)_x - \frac{\mu_1}{\rho_1}u_{xx} - \frac{\mu_2}{\rho_1}z_{xx}(.,1) && = f_2,\\
&i\lambda\psi - v && = f_3,\\
&i\lambda v - \frac{b}{\rho_2}\psi_{xx} + \frac{\kappa}{\rho_2}(\varphi_x + \psi) && = f_4,\\
&i\lambda\tau z(x,\rho) + z_\rho(x,\rho) && = \tau f_5,
\end{aligned} \tag{4.4.2}$$

where $\lambda \in \mathbb{R}$ and $F = (f_1, f_2, f_3, f_4, f_5)^T \in \mathcal{H}$.

The main result of this section is the following.

Theorem 4.4.2. *The semigroup associated to the problem* (4.2.1) *is not exponentially stable even if $\frac{\rho_1}{\kappa} = \frac{\rho_2}{b}$.*

Proof. We will prove that it exists a sequence of real numbers $(\lambda_\nu)_{\nu\in\mathcal{N}}$ and functions $(F_\nu)_{\nu\in\mathcal{N}} \subset \mathcal{H}$, with $\|F_\nu\|_{\mathcal{H}} \leq 1$ verifying (4.4.2). For that, we take $f_1 = f_2 = f_3 = f_5 = 0$. So, one has

$$\begin{aligned} u &= i\lambda\varphi, \\ v &= i\lambda\psi. \end{aligned} \tag{4.4.3}$$

The substitution of $(4.4.3)_1$ and $(4.4.3)_2$ into $(4.4.2)_2$ and $(4.4.2)_4$, respectively, leads to

$$\begin{aligned} -\lambda^2\varphi - \frac{\kappa + i\mu_1\lambda}{\rho_1}\varphi_{xx} - \frac{\kappa}{\rho_1}\psi_x - \frac{\mu_2}{\rho_1}z_{xx}(.,1) &= 0, \\ -\lambda^2\psi - \frac{b}{\rho_1}\psi_{xx} + \frac{\kappa}{\rho_2}(\varphi_x + \psi) &= f_4, \\ i\lambda\tau z(x,\rho) + z_\rho(x,\rho) &= 0, \end{aligned} \tag{4.4.4}$$

Next, choosing f_4 as follows

$$f_4(x) = \cos(\nu\pi x).$$

And, due to the boundary conditions, we let

$$\varphi(x) = A\sin(\nu\pi x), \qquad \psi(x) = B\cos(\nu\pi x), \qquad z(x,\rho) = \delta(\rho)\sin(\nu\pi x),$$

where A, B and $\delta(\rho)$ depend on λ and will be determined explicitly in what follows. Hence, system (4.4.4) is equivalent to

$$\begin{aligned} \left[-\lambda^2 + \frac{\kappa + i\lambda\mu_1}{\rho_1}\big(\nu\pi\big)^2\right]A + \frac{\kappa}{\rho_1}(\nu\pi)B + \frac{\mu_2}{\rho_1}\delta(1)\big(\nu\pi\big)^2 &= 0, \\ \left[-\lambda^2 + \frac{b}{\rho_2}\big(\nu\pi\big)^2\right]B + \frac{\kappa}{\rho_2}\big(\nu\pi\big)A &= 1, \\ \delta'(\rho) + i\lambda\tau\delta(\rho) &= 0. \end{aligned} \tag{4.4.5}$$

Solving $(4.4.5)_3$ and using the fact that $\delta(0) = i\lambda A$, we immediately get

$$\delta(\rho) = i\lambda A e^{-i\lambda\tau\rho}.$$

Consequently, (4.4.5) becomes

$$\begin{aligned} \left[-\lambda^2 + \frac{1}{\rho_1}\left(\kappa + i\lambda\mu_1 + i\lambda\mu_2 e^{-i\lambda\tau}\right)(\nu\pi)^2\right]A + \frac{\kappa}{\rho_1}(\nu\pi)B &= 0, \\ \left[-\lambda^2 + \frac{b}{\rho_2}(\nu\pi)^2\right]B + \frac{\kappa}{\rho_2}(\nu\pi)A &= 1. \end{aligned}$$

Now, we select $\lambda = \lambda_\nu$ such that

$$|\lambda_\nu| = \nu\pi\sqrt{\frac{b}{\rho_2}}.$$

Then, straightforward computations give

$$A = \frac{\rho_2}{\kappa\big(\nu\pi\big)},$$

$$B = \frac{\rho_1\rho_2}{\kappa^2}\left(\frac{b}{\rho_2} - \frac{\kappa}{\rho_1}\right) - \frac{i\mu_1\rho_2}{\kappa^2}\lambda_\nu - \frac{i\mu_2\rho_2}{\kappa^2}\lambda_\nu e^{-i\tau\lambda_\nu}.$$

It is obvious that $|B| \longrightarrow \infty$ as $\nu \longrightarrow \infty$ in both cases $\frac{\rho_1}{\kappa} = \frac{\rho_2}{b}$ and $\frac{\rho_1}{\kappa} \neq \frac{\rho_2}{b}$. Then, since

$$\|U_\nu\|_{\mathcal{H}}^2 \geq \rho_2\|v_\nu\|_2^2 = \rho_2\lambda_\nu^2|B|^2 \int_0^1 |\cos(\nu\pi x)|^2\, dx,$$

it results that

$$\|U_\nu\|_{\mathcal{H}} \longrightarrow \infty \quad \text{as} \quad \nu \longrightarrow \infty,$$

and so the lack of exponential stability follows. □

4.5 Optimal polynomial decay

In this section, using the multiplier method, we prove that the solution decays polynomially to zero as t tends to infinity with rate $t^{-\frac{1}{2}}$. And, that rate is optimal.

The main result of this section reads as follows.

Theorem 4.5.1. *Assuming that* (4.2.2) *is fulfilled. Then, for any $U_0 \in D(\mathcal{A})$, it exists a positive constant ω_0 such that the solution of* (4.2.1) *satisfies*

$$\|U\|_{\mathcal{H}} \leq \frac{\omega_0}{\sqrt{t}}, \qquad \forall t > 0. \tag{4.5.1}$$

In addition, this rate of decay is optimal.

Proof. The problem of proving the polynomial decay of the semigroup $S(t) = e^{\mathcal{A}t}$

$$\|U\|_{\mathcal{H}} = \|S(t)U_0\|_{\mathcal{H}} \leq \frac{\omega_0}{\sqrt{t}}, \qquad \forall t > 0,$$

is equivalently, proving the polynomial decay of the energy E, that is,

$$E(t) \leq \frac{\omega_1}{t}, \qquad \forall t > 0.$$

To this end, we introduce some functionals which permit us to obtain the desired estimate.

Lemma 4.5.2. *Let (φ, ψ, z) be a solution of* (4.2.1). *Then, the functional*

$$F_1(t) = -\rho_2 \int_0^1 \psi_t\psi dx$$

satisfies

$$F_1'(t) \leq -\rho_2 \int_0^1 \psi_t^2 dx + c\int_0^1 \psi_x^2 dx + c\int_0^1 (\varphi_x + \psi)^2 dx. \tag{4.5.2}$$

Proof. A simple differentiation yields that

$$F_1'(t) = -\rho_2 \int_0^1 \psi_t^2 dx + b\int_0^1 \psi_x^2 dx + \kappa\int_0^1 (\varphi_x + \psi)\psi dx.$$

Estimate (4.5.2) follows by exploiting Young's and Poincaré's inequalities. □

Lemma 4.5.3. *let* (φ, ψ, z) *be a solution of* (4.2.1). *Then, the functional defined by*

$$F_2(t) = -\rho_2 \int_0^1 \psi_t \varphi_x dx - \frac{b\rho_1}{\kappa} \int_0^1 \varphi_t \psi_x dx$$

satisfies, for any $\varepsilon_1 > 0$, *the following estimate*

$$\begin{aligned} F_2'(t) \leq & -\frac{b}{2} \int_0^1 \psi_x^2 dx + \varepsilon_1 \int_0^1 \psi_t^2 dx + c_{\varepsilon_1} \Big(\rho_2 - \frac{b\rho_1}{\kappa}\Big)^2 \int_0^1 \varphi_{xt}^2 dx \\ & + c \int_0^1 (\varphi_x + \psi)^2 dx + c \int_0^1 \varphi_{xxt}^2 dx + c \int_0^1 z_{xx}^2(x,1) dx. \end{aligned} \tag{4.5.3}$$

Proof. Taking the derivative of F_2 with respect to t, we get

$$\begin{aligned} F_2'(t) = & - b \int_0^1 \psi_{xx} \varphi_x dx - b \int_0^1 (\varphi_x + \psi)_x \psi_x dx + \kappa \int_0^1 (\varphi_x + \psi) \varphi_x dx \\ & + \left(\rho_2 - \frac{b\rho_1}{\kappa}\right) \int_0^1 \psi_t \varphi_{xt} dx - \frac{b}{\kappa}\mu_1 \int_0^1 \varphi_{xxt} \psi_x dx - \frac{b}{\kappa}\mu_2 \int_0^1 z_{xx}(x,1) \psi_x dx. \end{aligned}$$

An integration by parts leads to

$$\begin{aligned} F_2'(t) = & - b \int_0^1 \psi_x^2 dx + \kappa \int_0^1 (\varphi_x + \psi) \varphi_x dx + \left(\rho_2 - \frac{b\rho_1}{\kappa}\right) \int_0^1 \psi_t \varphi_{xt} dx \\ & - \frac{b}{\kappa}\mu_1 \int_0^1 \varphi_{xxt} \psi_x dx - \frac{b}{\kappa}\mu_2 \int_0^1 z_{xx}(x,1) \psi_x dx, \end{aligned}$$

using then Young's inequality with the fact that

$$\int_0^1 \varphi_x^2 dx \leq 2 \int_0^1 (\varphi_x + \psi)^2 dx + 2 \int_0^1 \psi_x^2 dx,$$

we obtain (4.5.3). □

Lemma 4.5.4. *let* (φ, ψ, z) *be a solution of* (4.2.1). *Then the functional*

$$F_3(t) = \rho_1 \int_0^1 \varphi_t \Big(\varphi + \int_0^x \psi(y,t) dy\Big) dx$$

satisfies, for any $\varepsilon_2 > 0$,

$$F_3'(t) \leq -\frac{\kappa}{2} \int_0^1 (\varphi_x + \psi)^2 dx + \varepsilon_2 \int_0^1 \psi_t^2 dx + c \int_0^1 \varphi_{xt}^2 dx + c \int_0^1 z_x^2(x,1) dx. \tag{4.5.4}$$

Proof. Differentiating F_3 and exploiting Eq.$(4.2.1)_1$, we can get

$$\begin{aligned} F_3'(t) = \kappa & \int_0^1 (\varphi_x + \psi)_x \Big(\varphi + \int_0^x \psi(y,t) dy\Big) dx + \rho_1 \int_0^1 \varphi_t^2 dx \\ & + \rho_1 \int_0^1 \varphi_t \int_0^x \psi_t(y,t) dy dx + \int_0^1 \Big(\mu_1 \varphi_{xxt} + \mu_2 z_{xx}(x,1)\Big) \varphi dx \\ & + \int_0^1 \Big(\mu_1 \varphi_{xxt} + \mu_2 z_{xx}(x,1)\Big) \int_0^x \psi(y,t) dy dx. \end{aligned}$$

Observing

$$\varphi(x,t)=\int_0^x \psi(y,t)dy=0, \quad \text{for} \quad x=0=1,$$

then, integrating by parts, we find that

$$\begin{aligned} F_3'(t) = &-\kappa\int_0^1(\varphi_x+\psi)^2dx+\rho_1\int_0^1\varphi_t^2dx+\rho_1\int_0^1\varphi_t\int_0^x\psi_t(y,t)dydx \\ &-\mu_1\int_0^1(\varphi_x+\psi)\varphi_{xt}dx-\mu_2\int_0^1(\varphi_x+\psi)z_x(x,1). \end{aligned} \tag{4.5.5}$$

For all $\varepsilon_2>0$, using Young's, Poincaré's and Cauchy-Schwarz inequalities, we can estimate the last three terms in the right-hand side of (4.5.5) as follows

$$\rho_1\int_0^1\varphi_t\int_0^x\psi_t(y,t)dydx\le\varepsilon_2\int_0^1\psi_t^2dx+c\int_0^1\varphi_{xt}^2dx, \tag{4.5.6}$$

$$-\mu_1\int_0^1(\varphi_x+\psi)\varphi_{xt}dx\le\frac{\kappa}{4}\int_0^1(\varphi_x+\psi)^2dx+c\int_0^1\varphi_{xt}^2dx. \tag{4.5.7}$$

$$-\mu_2\int_0^1(\varphi_x+\psi)z_x(x,1)dx\le\frac{\kappa}{4}\int_0^1(\varphi_x+\psi)^2dx+c\int_0^1 z_x^2(x,1)dx. \tag{4.5.8}$$

By inserting the estimates (4.5.6)-(4.5.8) into (4.5.5), we obtain (4.5.4). □

Lemma 4.5.5. *The functional*

$$F_4(t)=\tau\int_0^1\int_0^1 e^{-\tau\rho}z_x^2(x,\rho)d\rho dx$$

satisfies, along the solution of (4.2.1) *an estimate of the form*

$$F_4'(t)\le -e^{-\tau}\int_0^1 z_x^2(x,1)dx-\tau e^{-\tau}\int_0^1\int_0^1 z_x^2(x,\rho)d\rho dx+\int_0^1\varphi_{xt}^2dx. \tag{4.5.9}$$

Proof. By exploiting Eq.$(4.2.1)_3$, we can obtain

$$\begin{aligned} F_4'(t) &= -2\int_0^1\int_0^1 e^{-\tau\rho}z_x(x,\rho)z_{x\rho}(x,\rho)d\rho dx \\ &= -\int_0^1\int_0^1\left[\frac{d}{d\rho}\Big(e^{-\tau\rho}z_x^2(x,\rho)\Big)\right]d\rho dx-\tau\int_0^1\int_0^1 e^{-\tau\rho}z_x^2(x,\rho)d\rho dx \\ &= -e^{-\tau}\int_0^1 z_x^2(x,1)dx+\int_0^1 z_x^2(x,0,t)dx-\tau\int_0^1\int_0^1 e^{-\tau\rho}z_x^2(x,\rho)d\rho dx. \end{aligned}$$

Estimate (4.5.9) follows by using Eq.$(4.2.1)_5$ and the fact that $e^{-\tau\rho}\le e^{-\tau}$, for all $\rho\in[0,1]$. □

We now define, for any strong solution, the second-order energy functional to our problem (4.2.1) as

$$\mathcal{E}(t)=\frac{1}{2}\int_0^1\left[\rho_1\varphi_{xt}^2+\rho_2\psi_{xt}^2+b\psi_{xx}^2+\kappa(\varphi_{xx}+\psi_x)^2+\tau\gamma\int_0^1 z_{xx}^2(x,\rho)d\rho\right]dx.$$

Lemma 4.5.6. *Assuming that* (4.2.2) *holds. Then, the second-order energy functional* $\mathcal{E}$ *is non-increasing and satisfies, for any* $t \geq 0$,

$$\mathcal{E}'(t) \leq -\beta_1 \int_0^1 \varphi_{xxt}^2 dx - \beta_2 \int_0^1 z_{xx}^2(x,1)dx. \tag{4.5.10}$$

Proof. Multiplying Eq.(4.2.1)$_1$ by $-\varphi_{xxt}$ and Eq.(4.2.1)$_2$ by $-\psi_{xxt}$, then, integrating the products over $(0,1)$, we get

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \left[\rho_1\varphi_{xt}^2 + \rho_2\psi_{xt}^2 + \kappa(\varphi_{xx}+\psi_x)^2 + b\psi_{xx}^2\right]dx = -\mu_1 \int_0^1 \varphi_{xxt}^2 dx - \mu_2 \int_0^1 \varphi_{xxt} z_{xx}(x,1)dx. \tag{4.5.11}$$

From Eq.(4.2.1)$_3$, one has

$$\tau z_{xxt}(x,\rho) + z_{xx\rho}(x,\rho) = 0,$$

then a multiplication by $\gamma z_{xx}(x,\rho)$ yields that

$$\frac{\tau\gamma}{2}\frac{d}{dt}\int_0^1\int_0^1 z_{xx}^2(x,\rho)d\rho dx = -\frac{\gamma}{2}\int_0^1 z_{xx}^2(x,1)dx + \frac{\gamma}{2}\int_0^1 \varphi_{xxt}^2 dx. \tag{4.5.12}$$

Combining (4.5.11)-(4.5.12) and using the definition of $\mathcal{E}(t)$ and Young's inequality, we obtain (4.5.10). □

Lemma 4.5.7. *Let* (φ, ψ, z) *be a solution of* (4.2.1), *then for a suitable choice of* N *and* N_i, $(i = 1, ..., 4)$, *the functional* $\mathcal{L}$ *defined by*

$$\mathcal{L}(t) = N\big(E(t) + \mathcal{E}(t)\big) + \sum_{i=1}^{4} N_i F_i(t)$$

satisfies the estimate

$$\mathcal{L}'(t) \leq -m_1 E(t), \qquad \forall t \geq 0, \tag{4.5.13}$$

where m_1 *is a fixed positive number.*

Proof. It should be noticed that $\mathcal{L}$ is not equivalent to E. Then, gathering the estimates (4.2.5), (4.5.2), (4.5.3), (4.5.4), (4.5.9) and (4.5.10) and using the facts

$$-\int_0^1 \varphi_{xt}^2 dx \leq -\int_0^1 \varphi_t^2 dx,$$
$$\int_0^1 \varphi_{xt}^2 dx \leq \int_0^1 \varphi_{xxt}^2 dx,$$
$$\int_0^1 z_x^2(x,1)dx \leq \int_0^1 z_{xx}^2(x,1)dx,$$

we obtain

$$\begin{aligned}\mathcal{L}'(t) \leq &-\left[\beta_1 N - cN_3 - \left(c_{\varepsilon_1}\left(\rho_2 - \frac{b\rho_1}{\kappa}\right)^2 + c\right)N_2 - N_4\right]\int_0^1 \varphi_{xxt}^2 dx\\ &- \beta_1 N \int_0^1 \varphi_t^2 dx - \left[\frac{\kappa}{2}N_3 - c(N_1 + N_2)\right]\int_0^1 (\varphi_x + \psi)^2 dx\\ &- \left[\rho_2 N_1 - \varepsilon_1 N_2 - \varepsilon_2 N_3\right]\int_0^1 \psi_t^2 dx - \left[\frac{b}{2}N_2 - cN_1\right]\int_0^1 \psi_x^2 dx\\ &- \tau e^{-\tau} N_4 \int_0^1\int_0^1 z_x^2(x,\rho)d\rho dx - \left[\beta_2 N - c\big(N_2 + N_3\big)\right]\int_0^1 z_{xx}^2(x,1)dx.\end{aligned}$$

Furthermore, the choices

$$N_1 = 3\varepsilon_1, \qquad N_2 = \rho_2, \qquad \varepsilon_2 = \frac{\varepsilon_1\rho_2}{N_3}, \qquad N_4 = 1,$$

yield that

$$\begin{aligned}\mathcal{L}'(t) \leq &-\left[\beta_1 N - cN_3 - \left(c_{\varepsilon_1}\left(\rho_2 - \frac{b\rho_1}{\kappa}\right)^2 + c\right)\rho_2 - 1\right]\int_0^1 \varphi_{xxt}^2 dx\\ &- \beta_1 N \int_0^1 \varphi_t^2 dx - \left[\frac{\kappa}{2}N_3 - 3c\varepsilon_1 - \rho_2\right]\int_0^1 (\varphi_x + \psi)^2 dx\\ &- \varepsilon_1\rho_2 \int_0^1 \psi_t^2 dx - \left[\frac{b\rho_2}{2} - 3c\varepsilon_1\right]\int_0^1 \psi_x^2 dx\\ &- \tau e^{-\tau} \int_0^1\int_0^1 z_x^2(x,\rho)d\rho dx - \left[\beta_2 N - c\big(\rho_2 + N_3\big)\right]\int_0^1 z_{xx}^2(x,1)dx.\end{aligned}$$

At this point, we have to select our constants ε_1, N_3 and N very carefully. Choosing first ε_1 small enough so that

$$\frac{b\rho_2}{2} - 3c\varepsilon_1 > 0.$$

Then, we take N_3 large enough such that

$$\frac{\kappa}{2}N_3 - 3c\varepsilon_1 - \rho_2 > 0.$$

As long as N_3 and ε_1 are fixed, we pick N large enough so that

$$\begin{gathered}\beta_2 N - c\big(\rho_2 + N_3\big) \geq 0,\\ \beta_1 N - cN_3 - \left(c_{\varepsilon_1}\left(\rho_2 - \frac{b\rho_1}{\kappa}\right)^2 + c\right)\rho_2 - 1 \geq 0.\end{gathered}$$

Thus, we can find a fixed positive constant m_0 such that

$$\mathcal{L}'(t) \leq -m_0 \int_0^1 \left[\varphi_t^2 + \psi_t^2 + (\varphi_x + \psi)^2 + \psi_x^2 + \int_0^1 z_x^2(x,\rho)d\rho\right]dx,$$

which, together with (4.2.3), leads us to (4.5.13). □

Now, going back to the proof of Theorem 4.5.1, by integrating (4.5.13) over $(0,t)$, we yield that

$$\int_0^t E(s)ds \leq \frac{1}{\omega}\mathcal{L}(0).$$

Then, using the fact that $E' \leq 0$, we get

$$tE(t) \leq \frac{1}{\omega}\mathcal{L}(0),$$

which gives us the above mentioned decay result.

Next, we will show that the rate $t^{-\frac{1}{2}}$ is optimal. For, we use the following Theorem.

Theorem 4.5.8. *(**[95]**) Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup of contractions on Hilbert space such that $i\mathbb{R} \subset \varrho(\mathcal{A})$. If*

$$\|U\|_{\mathcal{H}} \leq \frac{c}{t^{\alpha}}\|U_0\|_{D(\mathcal{A})}.$$

Then, for any $\eta > 0$, it exists a $c_\eta > 0$ such that

$$\frac{1}{\lambda^{\eta+1/\alpha}}\|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{H}} \leq c_\eta.$$

First, by (4.5.1), we have $\lim\limits_{t\to\infty}\|U(t)\|_{\mathcal{H}} = 0$, which means that $i\mathbb{R} \subset \varrho(\mathcal{A})$. Now, we are in the position to prove the optimality of the rate $t^{-\frac{1}{2}}$ by applying the above Theorem. Suppose that the rate can be better than $t^{-\frac{1}{2}}$, for instance, the rate is $t^{-\frac{1}{2-\eta_0}}$, for $0 < \eta_0 < 2$. And, we prove that it exists $\eta > 0$ such that the operator

$$|\lambda|^{-\eta-(2-\eta_0)}\|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{H}},$$

is illimited. Choosing $\eta = \frac{\eta_0}{2}$, then we will show that it exist a subsequence $(\lambda_\nu)_{\nu\in\mathbb{N}} \subset \mathbb{R}$ with $\lim\limits_{\nu\to\infty}|\lambda_\nu| = \infty$ and $(U_\nu)_{\nu\in\mathbb{N}} \subset D(\mathcal{A})$ and $(F_\nu)_{\nu\in\mathbb{N}} \subset \mathcal{H}$ such that $(i\lambda_\nu I - \mathcal{A})U_\nu = F_\nu$ is bounded in $\mathcal{H}$ and $\lim\limits_{\lambda_\nu\to\infty}|\lambda_\nu|^{-2+\frac{\eta_0}{2}}\|U_\nu\|_{\mathcal{H}} = \infty$.

For each $\nu \in \mathbb{N}$, we can consider,

$$F_\nu = \big(0,0,0,\cos(\nu\pi x),0\big)^T \quad \text{and} \quad U_\nu = \big(\varphi_\nu, u_\nu, \psi_\nu, v_\nu, z_\nu\big)^T,$$

where $\varphi_\nu = A\sin(\nu\pi x)$, $\psi_\nu = B\cos(\nu\pi x)$ and $z_\nu = \delta(\rho)\sin(\nu\pi x)$. Then, following the same arguments as in the proof of Theorem 4.4.2, we find that

$$|\lambda_\nu|^{-2+\frac{\eta_0}{2}}\|U_\nu\|_{\mathcal{H}} \geq \mathrm{O}(\nu^{\frac{\eta_0}{2}}) \longrightarrow \infty \quad \text{as} \quad \nu \longrightarrow \infty.$$

Thus, the rate cannot be better than $t^{-\frac{1}{2}}$. The proof of Theorem 4.5.1 is hence complete. □

Chapter 5

On the decay rates of solutions for a nonlinearly damped Porous system with a delay

5.1 Introduction

In the present chapter, we study the global well-posedness and asymptotic behavior of solution of the following Porous system

$$\begin{cases} \rho_1 u_{tt} - \kappa u_{xx} - b\phi_x = 0 & \text{in }]0,1[\times]0,\infty[, \\ \rho_2\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \mu_1 g_1(\phi_t) + \mu_2 g_2(\phi_t(x,t-\tau)) = 0 & \text{in }]0,1[\times]0,\infty[, \\ u(0,t) = u(1,t) = \phi(0,t) = \phi(1,t) = 0 & \text{in } \quad]0,\infty[, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \quad]0,1[, \\ \phi(x,0) = \phi_1(x), \quad \phi_t(x,0) = \phi_1(x) & \text{in } \quad]0,1[, \\ \phi_t(x,t-\tau) = f_0(x,t-\tau) & \text{in } [0,\tau]\times]0,1[, \end{cases} \tag{5.1.1}$$

where $\mu_1 > 0$, μ_2 is a real number and $\tau > 0$ is a time delay. The function $u = u(x,t)$ represents the displacement of the solid elastic material, $\phi = \phi(x,t)$ is the volume fraction and the initial data $(u_0, u_1, \phi_0, \phi_1, f_0)$ are in suitable functional spaces. The original Porous system is governed by the following evolution equations

$$\begin{cases} \rho_1 u_{tt} = T_x, \\ \rho_2 \phi_{tt} = H_x + G. \end{cases}$$

Here T denotes the stress, H is the equilibrated stress and G is the equilibrated body force such that

$$\begin{cases} T = \kappa u_x + b\phi, \\ H = \delta\phi_x, \\ G = -bu_x - \xi\phi, \end{cases} \tag{5.1.2}$$

where ρ_1, ρ_2, κ, b, δ and ξ are positive constants satisfying the following condition

$$\kappa\xi > b^2.$$

There are a number of publications concerning the stabilization of the Porous system with frictional dampings. Let us mention some well-known papers which discussed the stability of (5.1.2) by frictional dampings. Quintanilla [**108**] proved that the damping $\mu\phi_t$ is not strong enough to obtain the exponential stabiliy result. However, Apalara [**105**] got the exponential decay of the solutions for the same problem provided that

$$\frac{\rho_1}{\kappa} = \frac{\rho_2}{\delta}. \tag{5.1.3}$$

Furthermore, he [**106**] established a general decay result when he used the weak non-linear damping $\mu(t)g(\phi_t)$. Related to the Porous system with delay term, we can cite the works [**112, 113, 103, 111**]. For instance, the authors of [**113**] proved that, under the assumption $|\mu_2| < \mu_1$, the system

$$\begin{cases} \rho_1 u_{tt} - \kappa u_{xx} - b\phi_x = 0 & \text{in }]0,1[\times]0,\infty[, \\ \rho_2\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \mu_1\phi_t + \mu_2\phi_t(x,t-\tau) + \alpha(t)g(\phi_t) = 0 & \text{in }]0,1[\times]0,\infty[\end{cases}$$

is uniformly stable if and only if the wave speeds of the two equations are the same.

If we consider $\kappa = b = \xi$ in (5.1.1) we obtain the following standard Timoshenko system with delay

$$\begin{cases} \rho_1 u_{tt} - \kappa(u_x + \phi)_x = 0 & \text{in }]0,1[\times]0,\infty[, \\ \rho_2\phi_{tt} - \delta\phi_{xx} + \kappa(u_x + \phi) + \mu_1 g_1(\phi_t) + \mu_2 g_2(\phi_t(x,t-\tau)) = 0 & \text{in }]0,1[\times]0,\infty[, \\ u(0,t) = u(1,t) = \phi(0,t) = \phi(1,t) = 0 & \text{in }]0,\infty[, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in }]0,1[, \\ \phi(x,0) = \phi_1(x), \quad \phi_t(x,0) = \phi_1(x) & \text{in }]0,1[, \\ \phi_t(x,t-\tau) = f_0(x,t-\tau) & \text{in } [0,\tau]\times]0,1[, \end{cases} \tag{5.1.4}$$

which was studied by Benaissa and Bahlil [**97**]. In that work the authors considered only the equal-speeds case where they obtained an explicit decay estimate under a proper relation between μ_1 and μ_2 and some assumptions on the functions g_i.

As is known, if only one equation of a Timoshenko system is damped then the uniform stability can be obtained for weak solutions in the case $\frac{\rho_1}{\kappa} = \frac{\rho_2}{\delta}$. However, in the situation when $\frac{\rho_1}{\kappa} \neq \frac{\rho_2}{\delta}$, a weaker decay rate result is achieved for stronger solutions. According to this results, three questions can be asked:

1. Is it possible to consider the Porous system with a nonlinear damping term and a constant delay in a non-linear internal feedback acting only in the second equation and get the same result as in the Timoshenko system?

2. In the equal-speeds case, is it possible to get the stability result with same hypotheses on μ_1, μ_2, g_1 and g_2 as in the Timoshenko system?

3. As we have mentioned above, the nonequal-speeds case is not considered for the Timoshenko system with a nonlinear delay term (see [**97, 101**]). So, is it possible to obtain the slow decay result under the same conditions imposed for the equal-speeds case?

The main objectives of this chapter are twofold. Firstly, using the Faedo-Galerkin scheme (see [**56, 100**]) together with some energy estimates, the global solvability will be given without any limitation on μ_1 and μ_2. Secondly, we shall give positive answers to the above three questions. To do so, we use the well-known multiplier method and some ideas developed in **[22]** and **[102]**, taking into account the nature of Porous systems.

The outline of this chapter is as follows. In the next section, we give the needed materials and assumptions. In section 3, we prove the existence and the uniqueness results. In the last section, we study the solution's asymptotic behavior in the equal-speeds case as well as in the opposite case.

5.2 Preliminaries

We present here some assumptions, materials and notations that we shall use to prove our results. We begin by introducing, as in the work **[36]**, the new variable

$$z(x,\rho,t)=\phi_t(x,t-\rho\tau), \qquad x\in[0,1],\ \rho\in[0,1],\ t>0,$$

which satisfies

$$\tau z_t(x,\rho,t)+z_\rho(x,\rho,t)=0 \qquad \text{in } ([0,1])^2\times[0,\infty].$$

Hence, our problem (5.1.1) becomes

$$\begin{cases}\rho_1 u_{tt}-\kappa u_{xx}-b\phi_x =0 & \text{in }]0,1[\ \times]0,\infty[,\\ \rho_2\phi_{tt}-\delta\phi_{xx}+bu_x+\xi\phi+\mu_1 g_1(\phi_t)+\mu_2 g_2(z(x,1))=0 & \text{in }]0,1[\ \times]0,\infty[,\\ \tau z_t(x,\rho,t)+z_\rho(x,\rho,t) =0 & \text{in } (]0,1[)^2\times]0,\infty[,\\ u(0,t)=u(1,t)=\phi(0,t)=\phi(1,t)=0 & \text{in }]0,\infty[,\\ u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x) & \text{in }]0,1[,\\ \phi(x,0)=\phi_1(x),\quad \phi_t(x,0)=\phi_1(x) & \text{in }]0,1[,\\ z(x,\rho,0)=f_0(x,-\rho\tau) & \text{in } (]0,1[)^2.\end{cases} \tag{5.2.1}$$

In order to deal with the new variable z, we define the following space

$$L_z^2(0,1)=L^2\Big(0,1;L^2(0,1)\Big)=\left\{z:]0,1[\longrightarrow L^2(0,1),\ \int_0^1\int_0^1 z^2(x,\rho)d\rho dx<\infty\right\},$$

which is Hilbert space and endowed with the inner product

$$(z,\widetilde{z})=\int_0^1\int_0^1 z(x,\rho,t)\widetilde{z}(x,\rho,t)d\rho dx.$$

To study system (5.2.1), we need the following assumptions:
(A_1) $g_1:\mathbb{R}\longrightarrow\mathbb{R}$ is a strictly increasing function of class C^1 such that it exist $\epsilon<1$, c_0,c_1 and a C^1-function $H:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ which is linear on $[0,\epsilon]$ or non-decreasing and convex function of class C^2 with $H(0)=H'(0)=0$ such that

$$\begin{cases}c_0|s|\le|g_1(s)|\le c_1|s| & \text{if } |s|>\epsilon,\\ s^2+g_1^2(s)\le H^{-1}\big(sg_1(s)\big) & \text{if } |s|\le\epsilon,\end{cases} \tag{5.2.2}$$

where H^{-1} is the inverse function of H.
(A$_2$) $g_2 : \mathbb{R} \longrightarrow \mathbb{R}$ is a non-decreasing function of class C^1 such that $g_2(0) = 0$ and that it exist three constants $\lambda_2, \alpha_1 > 0$ and $\alpha_2 \leq 1$ satisfying

$$|g_2'(s)| \leq \lambda_2 \tag{5.2.3}$$

and

$$\alpha_1 s g_2(s) \leq G(s) \leq \alpha_2 s g_1(s), \tag{5.2.4}$$

where $G(s) = \displaystyle\int_0^s g_2(\sigma)d\sigma$.

We now define the total energy associated with the solution of (5.2.1) as

$$E(t) = \frac{1}{2}\int_0^1 \left[\rho_1 u_t^2 + \rho_2 \phi_t^2 + \kappa u_x^2 + \delta\phi_x^2 + \xi\phi^2 + 2bu_x\phi + 2\tau\gamma\int_0^1 G(z(x,\rho))d\rho\right] dx, \tag{5.2.5}$$

where γ is a fixed positive constant to be selected posteriori.

Remark 5.2.1. *The energy functional $E(t)$ defined in* (5.2.5) *is positive. Indeed, direct calculations show that*

$$\kappa u_x^2 + 2bu_x\phi + \xi\phi^2 = \frac{1}{2}\left[\kappa\Big(u_x + \frac{b}{\kappa}\phi\Big)^2 + \xi\Big(\phi + \frac{b}{\xi}u_x\Big)^2 + 2\kappa_1 u_x^2 + 2\xi_1\phi^2\right],$$

where $2\kappa_1 = \kappa - \frac{b^2}{\xi}$ and $2\xi_1 = \xi - \frac{b^2}{\kappa}$ are positives due to $\kappa\xi > b^2$. Thus,

$$\kappa u_x^2 + 2bu_x\phi + \xi\phi^2 > \frac{1}{2}\left[\kappa\Big(u_x + \frac{b}{\kappa}\phi\Big)^2 + \xi\Big(\phi + \frac{b}{\xi}u_x\Big)^2\right] > 0,$$

this implies that $E(t) > 0$ and

$$E(t) > \frac{1}{2}\int_0^1 \left[\rho_1 u_t^2 + \rho_2\phi_t^2 + \kappa_1 u_x^2 + \delta\phi_x^2 + \xi_1\phi^2 + 2\gamma\tau\int_0^1 G(z(x,\rho))d\rho\right] dx. \tag{5.2.6}$$

Remark 5.2.2. • *As g_1 is a strictly increasing function, then we can find a positive constant λ_1 satisfying*

$$\lambda_1 < g_1'(s), \quad \forall s \in \mathbb{R}.$$

• *By the mean value Theorem for integrals and the monotonicity of g_2, we have that*

$$G(s) = \int_0^s g_2(\sigma)d\sigma \leq s g_2(s),$$

so, $\alpha_1 \leq 1$.

Remark 5.2.3. (*[57]*) *Let ω^* be the conjugate function of the differential convex function ω, i.e.*

$$\omega^*(s) = \sup_{t\in\mathbb{R}_+}\big(st - \omega(t)\big),$$

then, ω^* is the Legendre transform of ω, which satisfies

$$AB \leq \omega^*(A) + \omega(B), \qquad \text{if} \quad A \in [0, \omega'(r)] \quad \text{and} \quad B \in [0, r]. \tag{5.2.7}$$

and

$$\omega^*(s) = s(\omega')^{-1}(s) - \omega\big[(\omega')^{-1}(s)\big], \qquad \text{if} \quad s \in [0, \omega'(r)],$$

Our starting point will be to provide a derivative's upper bounded of the functional E_1 defined, for $0 \leq a_0 < 1$ and $a_1 \geq 0$, as

$$\begin{aligned} E_1(t) =& \frac{1}{2} \int_0^1 \Big[\rho_1 u_t^2 + \rho_2 \phi_t^2 + \kappa u_x^2 + \delta \phi_x^2 + \xi \phi^2 + 2b u_x \phi \\ &+ 2\tau \int_0^1 \Big(\gamma\big(1-a_0\big) G(z(x,\rho)) + a_1 z^2(x,\rho) \Big) d\rho \Big] dx. \end{aligned} \tag{5.2.8}$$

Lemma 5.2.4. *The functional E_1 satisfies along the solution of system* (2.2.1), *the following estimate*

$$\begin{aligned} E_1'(t) \leq -a_1 \int_0^1 z^2(x,1)dx &- \Big(\gamma\big(1-a_0\big)\alpha_1 - (1-\alpha_1)|\mu_2|\Big) \int_0^1 z(x,1) g_2(z(x,1))dx \\ + a_1 \int_0^1 \phi_t^2 dx &- \Big(\mu_1 - \gamma\big(1-a_0\big)\alpha_2 - \alpha_2|\mu_2|\Big) \int_0^1 \phi_t g_1(\phi_t)dx. \end{aligned} \tag{5.2.9}$$

Proof. Multiplying $(5.2.1)_1$ and $(5.2.1)_2$ by u_t and ϕ_t, respectively, and using integration by parts over $[0,1]$, we obtain the identity

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt} \int_0^1 \Big[\rho_1 u_t^2 + \rho_2 \phi_t^2 + \kappa u_x^2 + \delta \phi_x^2 + \xi \phi^2 + 2b u_x \phi \Big] dx \\ &+ \mu_1 \int_0^1 \phi_t g_1(\phi_t)dx + \mu_2 \int_0^1 \phi_t g_2(z(x,1))dx = 0. \end{aligned} \tag{5.2.10}$$

Multiplying $(5.2.1)_3$ by $\gamma\big(1-a_0\big) g_2(z(x,\rho))$ and integrating over $([0,1])^2$, we get

$$\gamma\big(1-a_0\big) \int_0^1 \int_0^1 \Big[\tau z_t(x,\rho) g_2(z(x,\rho)) + z_\rho(x,\rho) g_2(z(x,\rho)) \Big] d\rho dx = 0,$$

that is,

$$\gamma\big(1-a_0\big)\tau \frac{d}{dt} \int_0^1 \int_0^1 G(z(x,\rho)) d\rho dx + \gamma\big(1-a_0\big) \int_0^1 \int_0^1 \frac{\partial}{\partial \rho} G(z(x,\rho)) d\rho dx = 0.$$

Consequently, using the fact that $z_t(x,0,t) = \phi_t$, we infer

$$\gamma\big(1-a_0\big)\tau \frac{d}{dt} \int_0^1 \int_0^1 G(z(x,\rho) d\rho dx = -\gamma\big(1-a_0\big) \int_0^1 \Big[G(z(x,1)) - G(\phi_t) \Big] dx. \tag{5.2.11}$$

Also, for $a_1 > 0$ one has

$$\tau a_1 \frac{d}{dt} \int_0^1 \int_0^1 z^2(x,\rho) d\rho dx = -a_1 \int_0^1 \Big[z^2(x,1)) - \phi_t^2 \Big] dx. \tag{5.2.12}$$

This last equality has been obtained by applying the same previous arguments and after multiplying $(5.2.1)_3$ by $2a_1 z(x,\rho)$. Combining the estimates (5.2.10)-(5.2.12) and making use of (5.2.4), we can get

$$\begin{aligned} E_1'(t) \leq& - a_1 \int_0^1 z^2(x,1)dx + a_1 \int_0^1 \phi_t^2 dx - \Big(\mu_1 - \gamma\big(1-a_0\big)\alpha_2\Big) \int_0^1 \phi_t g_1(\phi_t)dx \\ &- \gamma\big(1-a_0\big)\alpha_1 \int_0^1 z(x,1) g_2(z(x,1))dx - \mu_2 \int_0^1 \phi_t g_2(z(x,1))dx. \end{aligned} \tag{5.2.13}$$

From Remark 5.2.3, we have

$$G^*(s) = sg_2^{-1}(s) - G(g_2^{-1}(s)), \qquad \forall s \geq 0,$$

and so

$$G^*\big(g_2(z(x,1))\big) = z(x,1)g_2(z(x,1)) - G\big(z(x,1)\big).$$

Taking (5.2.7) with $A = g_2(z(x,1))$ and $B = \phi_t$, and using (5.2.4) once more, we deduce that

$$\mu_2\phi_t g_2(z(x,1)) \leq \alpha_2|\mu_2|.\phi_t g_1(\phi_t) + \big(1-\alpha_1\big)|\mu_2|.z(x,1)g_2(z(x,1)). \tag{5.2.14}$$

By inserting (5.2.14) into (5.2.13), we arrive at the desired inequality (5.2.9). This finishes the proof. □

Next, we will give a bound of the derivative of the second-order energy functional $\mathcal{F}$ which is defined as

$$\mathcal{F}(t) = \frac{1}{2}\int_0^1 \left[\rho_1 u_{xt}^2 + \rho_2\phi_{xt}^2 + \kappa u_{xx}^2 + \delta\phi_{xx}^2 + \xi\phi_x^2 + 2bu_{xx}\phi_x + \tau\lambda_2|\mu_2|\int_0^1 z_x^2(x,\rho)d\rho\right]dx.$$

Lemma 5.2.5. *The second-order energy functional $\mathcal{F}$ satisfies an estimate of the form*

$$\mathcal{F}'(t) \leq -\big(\lambda_1\mu_1 - \lambda_2|\mu_2|\big)\int_0^1 \phi_{xt}^2 dx. \tag{5.2.15}$$

Proof. Multiplying $(5.2.1)_1$ and $(5.2.1)_2$ by $-u_{xxt}$ and $-\phi_{xxt}$, respectively, and integrating by parts over $[0,1]$, we can get

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \left[\rho_1 u_{xt}^2 + \rho_2\phi_{xt}^2 + \kappa u_{xx}^2 + \delta\phi_{xx}^2 + \xi\phi_x^2 + 2bu_{xx}\phi_x\right]dx$$
$$+ \mu_1\int_0^1 \phi_{xt}^2 g_1'(\phi_t)dx + \mu_2\int_0^1 \phi_{xt}z_x(x,1)g_2'(z(x,1))dx = 0.$$

And, after multiplying $(5.2.1)_3$ by $-\lambda_2|\mu_2|z_{xx}(x,\rho,t)$, we obtain

$$\frac{\tau\lambda_2|\mu_2|}{2}\frac{d}{dt}\int_0^1\int_0^1 z_x^2(x,\rho,t)d\rho dx = -\frac{\lambda_2|\mu_2|}{2}\int_0^1 z_x^2(x,1)dx + \frac{\lambda_2|\mu_2|}{2}\int_0^1 \phi_{xt}^2 dx.$$

Adding the two identities above and using the fact that $\lambda_1 < g_1'(s)$, we yield that

$$\mathcal{F}'(t) \leq -\left(\lambda_1\mu_1 - \frac{\lambda_2|\mu_2|}{2}\right)\int_0^1 \phi_{xt}^2 dx - \frac{\lambda_2|\mu_2|}{2}\int_0^1 z_x^2(x,1)dx - \mu_2\int_0^1 \phi_{xt}z_x(x,1)g_2'(z(x,1))dx. \tag{5.2.16}$$

Therefore, since $|g_2'(s)| < \lambda_2$ for all $s \in \mathbb{R}$, one obtains by the Young's inequality

$$\mu_2\int_0^1 \phi_{xt}z_x(x,1)g_2'(z(x,1))dx \leq \frac{\lambda_2|\mu_2|}{2}\int_0^1 \phi_{xt}^2 dx + \frac{\lambda_2|\mu_2|}{2}\int_0^1 z_x^2(x,1)dx. \tag{5.2.17}$$

A substitution of (5.2.17) into (5.2.16) gives (5.2.15). This concludes the proof of this Lemma. □

5.3 The well-posedness of the problem

In this current section, we shall establish, for arbitrary real numbers μ_1 and μ_2, the global well-posedness of the system (5.2.1). For that, we let $U = U(t) = \big(u, u_t, \phi, \phi_t, z\big)^T$ and $U_0 = U(0) = \big(u_0, u_1, \phi_0, \phi_1, f_0(., -.\tau)\big)^T$ and we then consider the following spaces

$$\mathcal{H} = H_0^1(0,1) \times L^2(0,1) \times H_0^1(0,1) \times L^2(0,1) \times L_z^2(0,1)$$

and

$$\mathcal{H}_0 = \big(H^2 \cap H_0^1(0,1)\big) \times H_0^1(0,1) \times \big(H^2 \cap H_0^1(0,1)\big) \times H_0^1(0,1) \times H_0^1\big(0,1; H^1(0,1)\big).$$

Our global well-posedness result is:

Theorem 5.3.1. *Assuming that the assumptions (A_1)-(A_2) are satisfied and that $\kappa\xi > b^2$. Then, for all $U_0 \in \mathcal{H}_0$ satisfying the compatibility condition $f_0(., 0) = \phi_1$, problem* (5.2.1) *has only one global weak solution*

$$U \in \Big(L^\infty_{loc}\Big((0,\infty); H^2 \cap H_0^1(0,1)\Big) \times L^\infty_{loc}\big((0,\infty); H_0^1(0,1)\big)\Big)^2 \times L^\infty_{loc}\big((0,\infty); L_z^2(0,1)\big),$$

$$U_t \in \Big(L^\infty_{loc}\big((0,\infty); H_0^1(0,1)\big) \times L^\infty_{loc}\big((0,\infty); L^2(0,1)\big)\Big)^2 \times L^\infty_{loc}\big((0,\infty); L_z^2(0,1)\big).$$

Proof. To prove the existence result, we will implement the classical Faedo-Galerkin method. For, we divide the arguments into three steps.

i. ***Approximated problem.*** Assuming first that $U_0 \in \mathcal{H}_0$. Then, let $T > 0$ be fixed and for $m = 1, 2...$, we denote by $\{\Phi^m\}_{m\in\mathbb{N}}$ the Hilbertian basis of $H^2 \cap H_0^1(0,1)$ and F^m the vector space generated by $\Phi^1, \Phi^2, ..., \Phi^m$. Defining, for $1 \le i \le m$, the sequence $\Psi^i(x,\rho)$ as

$$\Psi^i(x,0) = \Phi^i(x).$$

Then, we may extend $\Psi^i(x,0)$ by $\Psi^i(x,\rho)$ over $L_z^2(0,1)$ and denote Z^m the space generated by $\Psi^1, \Psi^2, ..., \Psi^m$.
We aim to construct an approximate solution (u^m, ϕ^m, z^m), $i = 1, 2, ...$, in the form

$$\big(u^m(x,t), \phi^m(x,t)\big) = \Big(\sum_{i=1}^m c^{im}(t), \sum_{i=1}^m d^{im}(t)\Big)\Phi^i(x),$$
$$z^m(x,\rho) = \sum_{i=1}^m e^{im}(t)\Psi^i(x,\rho),$$

where c^{im}, d^{im} and e^{im}, $(i = 1, 2, ..., m)$, are determined by the following finite dimensional problem

$$\begin{cases} \big(\kappa u_x^m + b\phi^m, \Phi_x^i\big) + \big(\rho_1 u_{tt}^m, \Phi^i\big) & = 0, \\ \big(\delta\phi_x^m, \Phi_x^i\big) + \big(\rho_2\phi_{tt}^m + bu_x^m + \xi\phi^m + \mu_1 g_1(\phi_t^m) + \mu_2 g_2(z^m(x,1)), \Phi^i\big) & = 0, \\ \big(\tau\, z_t^m(x,\rho) + z_\rho^m(x,\rho), \Psi^j(x,\rho)\big) & = 0, \end{cases} \tag{5.3.1}$$

with

$$\begin{cases} u^m(.,0) = u_0^m = \sum_{i=1}^{m}(u_0, \Phi^i)\Phi^i \longrightarrow u_0 & \text{in} \quad H^2 \cap H_0^1(0,1), \\ u_t^m(.,0) = u_1^m = \sum_{i=1}^{m}(u_1, \Phi^i)\Phi^i \longrightarrow u_1 & \text{in} \quad H_0^1(0,1), \\ \phi^m(.,0) = \phi_0^m = \sum_{i=1}^{m}(\phi_0, \Phi^i)\Phi^i \longrightarrow \phi_0 & \text{in} \quad H^2 \cap H_0^1(0,1), \\ \phi_t^m(.,0) = \phi_1^m = \sum_{i=1}^{m}(\phi_1, \Phi^i)\Phi^i \longrightarrow \phi_1 & \text{in} \quad H_0^1(0,1), \\ z^m(.,.,0) = z_0^m = \sum_{i=1}^{m}(f_0, \Psi^i)\Psi^i \longrightarrow f_0 & \text{in} \quad H_0^1\big(0,1; H^1(0,1)\big) \end{cases} \tag{5.3.2}$$

as $m \longrightarrow +\infty$.

The standard methods of ODEs assures the existence of a unique solution of (5.3.1)-(5.3.2) on the inertval $[0, T_m]$, $0 < T_m < T$. In the next step, we will prove that T_m is independent of m. In other words, the approximate solution becomes global and defined for all $t > 0$.

ii. ***Priori estimates***.

• ***The first priori estimate.*** As for Lemma 5.2.4, the functional

$$\begin{aligned} E_1^m(t) =& \frac{1}{2}\int_0^1 \Big[\rho_1|u_t^m|^2 + \rho_2|\phi_t^m|^2 + \kappa|u_x^m|^2 + \delta|\phi_x^m|^2 + \xi|\phi^m|^2 + 2bu_x^m\phi^m \\ & + 2\tau\int_0^1 \Big(\gamma\big(1-a_0\big)G(z^m(x,\rho)) + 2a_1|z^m(x,\rho)|^2\Big)d\rho\Big]dx \end{aligned}$$

satisfies for all $0 \le a_0 < 1$, $0 \le a_1$,

$$\begin{aligned} & \frac{dE_1^m(t)}{dt} + \gamma a_0 \alpha_2 \int_0^1 \phi_t^m g_1(\phi_t^m)dx + \gamma\alpha_1 \int_0^1 z^m(x,1)g_2(z^m(x,1))dx \\ \le & -a_1\int_0^1 |z^m(x,1)|^2dx + \Big(\gamma a_0\alpha_1 + \big(1-\alpha_1\big)|\mu_2|\Big)\int_0^1 z^m(x,1)g_2(z^m(x,1))dx \\ & + a_1\int_0^1 |\phi_t^m|^2dx - \Big(\mu_1 - \gamma\alpha_2 - \alpha_2|\mu_2|\Big)\int_0^1 \phi_t^m g_1(\phi_t^m)dx. \end{aligned}$$

Choosing $a_0, a_1 > 0$, then by Young's inequality, we may have

$$\begin{aligned} & \frac{dE_1^m(t)}{dt} + \gamma a_0 \alpha_2 \int_0^1 \phi_t^m g_1(\phi_t^m)dx + \gamma\alpha_1 \int_0^1 z^m(x,1)g_2(z^m(x,1))dx \\ \le & -\big(a_1 - c_{a_2}\big)\int_0^1 |z^m(x,1)|^2dx + \big(a_1 + c_{a_3}\big)\int_0^1 |\phi_t^m|^2dx + a_2\int_0^1 g_2^2(z^m(x,1))dx \\ & + a_3\int_0^1 g_1^2(\phi_t^m)dx. \end{aligned} \tag{5.3.3}$$

Let us estimate the last two term in the right-hand side of (5.3.3). We firstly observe that, owing to (5.2.2),

$$\begin{aligned}\int_0^1 g_1^2(\phi_t^m)dx &\le \int_{|\phi_t^m|\le\epsilon} g_1^2(\phi_t^m)dx + \int_{|\phi_t^m|>\epsilon} g_1^2(\phi_t^m)dx \\ &\le \int_{|\phi_t^m|\le\epsilon} H^{-1}\Big(\phi_t^m g_1(\phi_t^m)\Big)dx + \int_{|\phi_t^m|>\epsilon} \phi_t^m g_1(\phi_t^m)dx.\end{aligned} \tag{5.3.4}$$

By the Jensen's inequality and the concavity of H^{-1}, we assert that

$$\int_0^1 H^{-1}\Big(\phi_t^m g_1(\phi_t^m)\Big)dx \le H^{-1}\left(\int_0^1 \phi_t^m g_1(\phi_t^m)dx\right),$$

taking then (5.2.7) with

$$A = 1 \quad \text{and} \quad B = H^{-1}\Big(\int_0^1 \phi_t^m g_1(\phi_t^m)dx\Big),$$

we can get

$$H^{-1}\Big(\int_0^1 \phi_t^m g_1(\phi_t^m)dx\Big) \le H^*(1) + \int_0^1 \phi_t^m g_1(\phi_t^m)dx,$$

where H^* is the conjugate function of H. Then, collecting the above estimates, the inequality (5.3.4) becomes

$$\int_0^1 g_1^2(\phi_t^m)dx \le H^*(1) + \int_0^1 \phi_t^m g_1(\phi_t^m)dx. \tag{5.3.5}$$

From the assumption (A_2), that is $|g_2(s)| \le \lambda_2|s|\ \forall s \in \mathbb{R}$, one has

$$\int_0^1 g_2^2(z^m(x,1))dx \le c\int_0^1 z^m(x,1)g_2(z^m(x,1))dx. \tag{5.3.6}$$

Plugging (5.3.5)-(5.3.6) into (5.3.3), we obtain that

$$\begin{aligned}\frac{dE_1^m(t)}{dt} &+ \big(\gamma a_0\alpha_2 - a_3c\big)\int_0^1 \phi_t^m g_1(\phi_t^m)dx + \big(\gamma\alpha_1 - a_2c\big)\int_0^1 z^m(x,1)g_2(z^m(x,1))dx \\ &\le cH^*(1) - \big(a_1 - c_{a_2}\big)\int_0^1 |z^m(x,1)|^2dx + \big(a_1 + c_{a_3}\big)\int_0^1 |\phi_t^m|^2dx.\end{aligned}$$

At this point, we select a_2 and a_3 small enough such that

$$\gamma\alpha_1 - a_2c > 0, \qquad \gamma a_0\alpha_2 - a_3c > 0.$$

Once a_2 is fixed, we then pick a_1 sufficiently large so that

$$a_1 - c_{a_2} \ge 0.$$

It thus results that

$$\frac{dE_1^m(t)}{dt} + \int_0^1 \phi_t^m g_1(\phi_t^m)dx + \int_0^1 z^m(x,1)g_2(z^m(x,1))dx \le c + c\int_0^1 |\phi_t^m|^2dx,$$

the integration with respect to $t < T$ on $[0,t]$, using (5.2.6), (5.2.8) and (5.3.2), gives

$$E_1^m(t) + \int_0^t \int_0^1 \phi_t^m g_1(\phi_t^m) dxdt + \int_0^t \int_0^1 z^m(x,1) g_2(z^m(x,1)) dxdt \le c + c\int_0^t E_1^m(t)dt.$$

Applying then Gronwall's inequality, we arrive at

$$E_1^m(t) + \int_0^t \int_0^1 \phi_t^m g_1(\phi_t^m) dxdt + \int_0^t \int_0^1 z^m(x,1) g_2(z^m(x,1)) dxdt \le c. \tag{5.3.7}$$

This bound gives us the global existence of (u^m, ϕ^m, z^m) in $[0,+\infty)$ and

$$\begin{array}{lll} z^m & \text{is uniformly bounded in} & L^\infty_{\text{loc}}\Big(0,\infty; L^2_z(0,1)\Big), \\ u^m, \phi^m & \text{are uniformly bounded in} & L^\infty_{\text{loc}}\Big(0,\infty; H^1_0(0,1)\Big), \\ u_t^m, \phi_t^m & \text{are uniformly bounded in} & L^\infty_{\text{loc}}\Big(0,\infty, L^2(0,1)\Big), \\ \phi_t^m g_1(\phi_t^m) & \text{is uniformly bounded in} & L^1\Big((0,T)\times(0,1)\Big), \\ z^m(x,1)g_2(z^m(x,1)) & \text{is uniformly bounded in} & L^1\Big((0,T)\times(0,1)\Big). \end{array} \tag{5.3.8}$$

- ***The second priori estimate.*** In view of Lemma 5.2.5, one has for all $t \ge 0$,

$$\frac{d\mathcal{F}^m(t)}{dt} \le c\int_0^1 |\phi_{xt}^m|^2 dx, \tag{5.3.9}$$

where

$$\mathcal{F}^m(t) = \frac{1}{2}\int_0^1 \Big[\rho_1|u_{xt}^m|^2 + \rho_2|\phi_{xt}^m|^2 + \kappa|u_{xx}^m|^2 + \delta|\phi_{xx}^m|^2 + \xi|\phi_x^m|^2 + 2bu_{xx}^m\phi_x^m + \tau\,\lambda_2\,|\mu_2|\int_0^1 |z_x^m(x,\rho)|^2 d\rho\Big]dx.$$

Integrating (5.3.9) over $[0,t]$ and taking the convergences (5.3.2) into account, we get

$$\mathcal{F}^m(t) \le c + c\int_0^t\int_0^1 |\phi_{xt}^m|^2 dxdt.$$

The Gronwall's inequality provides the second priori estimate below

$$\mathcal{F}^m(t) \le c. \tag{5.3.10}$$

We thereupon conclude that

$$\begin{array}{lll} u^m, \phi^m & \text{are uniformly bounded in} & L^\infty_{\text{loc}}\Big(0,\infty; H^2\cap H^1_0(0,1)\Big), \\ u_t^m, \phi_t^m & \text{are uniformly bounded in} & L^\infty_{\text{loc}}\Big(0,\infty; H^1_0(0,1)\Big). \end{array} \tag{5.3.11}$$

• ***The third priori estimate.*** Firstly, we are going to estimate $u_{tt}^m(0)$ and $\phi_{tt}^m(0)$ in the L^2-norm. Also, we need to estimate $z_t^m(x,\rho,0)$ in the L_z^2-norm. For that, we replace Φ^i in $(5.3.1)_1$ by u_{tt}^m, Φ^i in $(5.3.1)_2$ by ϕ_{tt}^m and using Young's inequality, we can obtain

$$\int_0^1 \Big[|u_{tt}^m(0)|^2 + |\phi_{tt}^m(0)|^2\Big]dx \le c\int_0^1 \Big[|u_{xx}^m(0)|^2 + |u_x^m(0)|^2 + |\phi_{xx}^m(0)|^2 + |\phi_x^m(0)|^2 + |\phi^m(0)|^2 + g_1^2(\phi_t^m(0)) + g_2^2(z^m(x,1,0))\Big]dx. \tag{5.3.12}$$

Let $\Psi^i = z_t^m(x,\rho,t)$ in $(5.3.1)_3$, then exploit Cauchy-Schwarz and Young's inequalities to get

$$\int_0^1\int_0^1 |z_t^m(x,\rho,0)|^2 d\rho dx \le c\int_0^1\int_0^1 |z_\rho^m(x,\rho,0)|^2 d\rho dx. \tag{5.3.13}$$

The sum of (5.3.12)-(5.3.13), using (5.3.2), yields that

$$\int_0^1 \Big[|u_{tt}^m(0)|^2 + |\phi_{tt}^m(0)|^2 + \int_0^1 |z_t^m(x,\rho,0)|^2 d\rho\Big]dx \le c. \tag{5.3.14}$$

Now, differentiating $(5.3.1)_1$ and $(5.3.1)_2$ with respect to t. Then, we set $\Phi^i = 2u_{tt}^m$ and $\Phi^i = 2\phi_{tt}^m$, respectively, in the first and the second resulting equations and using the non-decreasing property of g_1, we find

$$\frac{d}{dt}\int_0^1 \Big[\rho_1|u_{tt}^m|^2 + \rho_2|\phi_{tt}^m|^2 + \kappa|u_{xt}^m|^2 + \delta|\phi_{xt}^m|^2 + \xi|\phi_t^m|^2 + 2bu_{xt}^m\phi_t^m\Big]dx \le -\mu_2\int_0^1 z_t^m(x,1)g_2'(z^m(x,1))\phi_{tt}^m dx.$$

The boundedness of g_2' and the Young's inequality imply that

$$\frac{d}{dt}\int_0^1 \Big[\rho_1|u_{tt}^m|^2 + \rho_2|\phi_{tt}^m|^2 + \kappa|u_{xt}^m|^2 + \delta|\phi_{xt}^m|^2 + \xi|\phi_t^m|^2 + 2bu_{xt}^m\phi_t^m\Big]dx \le \varepsilon_2\int_0^1 |z_t^m(x,1)|^2 dx + c\int_0^1 |\phi_{tt}^m|^2 dx. \tag{5.3.15}$$

In the other hand, taking the derivative of $(5.3.1)_3$ with respect to t and setting $\Psi^i = 2z_t^m(x,\rho,t)$ in the resulting equation, it then follows that

$$\tau\frac{d}{dt}\int_0^1\int_0^1 |z_t^m(x,\rho,t)|^2 d\rho dx + \int_0^1\int_0^1 \frac{d}{d\rho}|z_t^m(x,\rho,t)|^2 d\rho dx = 0.$$

As $z_t^m(x,0,t) = \phi_{tt}^m(x,t)$, it comes

$$\tau\frac{d}{dt}\int_0^1\int_0^1 |z_t^m(x,\rho,t)|^2 d\rho dx = -\int_0^1 |z_t^m(x,1,t)|^2 dx + \int_0^1 |\phi_{tt}^m|^2 dx. \tag{5.3.16}$$

Moreover, defining

$$\mathcal{I}^m(t) = \int_0^1 \Big[\rho_1|u_{tt}^m|^2 + \rho_2|\phi_{tt}^m|^2 + \kappa|u_{xt}^m|^2 + \delta|\phi_{xt}^m|^2 + \xi|\phi_t^m|^2 + 2bu_{xt}^m\phi_t^m + \tau\int_0^1 |z_t^m(x,\rho)|^2 d\rho\Big]dx,$$

one obtains from (5.3.15) and (5.3.16), that

$$\frac{d\mathcal{I}^m(t)}{dt} \leq -\big(1-\varepsilon_2\big)\int_0^1 |z_t^m(x,1))|^2dx + c\int_0^1 |\phi_{tt}^m|^2dx.$$

For a suitable ε_2, an integration over $[0,t]$, using (5.3.2)-(5.3.14), yields that

$$\mathcal{I}^m(t) \leq c + c\int_0^t\int_0^1 |\phi_{tt}^m|^2dxdt.$$

Employing Gronwall's inequality we immediately get

$$\mathcal{I}^m(t) \leq c. \tag{5.3.17}$$

Therefore, it is deduced that

$$\begin{aligned} z_t^m \quad & \text{is uniformly bounded in} \quad L^2\Big(0,T;L_z^2(0,1)\Big),\\ u_{tt}^m,\phi_{tt}^m \quad & \text{are uniformly bounded in} \quad L_{\mathrm{loc}}^\infty\Big(0,\infty;L^2(0,1)\Big). \end{aligned} \tag{5.3.18}$$

iii. ***Passage to the limit.*** It follows from the estimates (5.3.7), (5.3.10) and (5.3.17) that it exist subsequences $\{u^n\}_{n=1}^\infty \subset \{u^m\}_{m=1}^\infty$, $\{\phi^n\}_{n=1}^\infty \subset \{\phi^m\}_{m=1}^\infty$ and $\{z^n\}_{n=1}^\infty \subset \{z^m\}_{m=1}^\infty$ verifying, for all $T \geq 0$, the following convergences

$$\begin{array}{lllllll} g_1(\phi_t^n) \longrightarrow f & \text{and} & g_2(z^n) \longrightarrow h & \text{weakly-star in} & L^2\Big((0,1)\times(0,T)\Big),\\ u^n \longrightarrow u & \text{and} & \phi^n \longrightarrow \phi & \text{weakly-star in} & L_{\mathrm{loc}}^\infty\Big(0,\infty;H^2\cap H_0^1\Big),\\ u_t^n \longrightarrow u_t & \text{and} & \phi_t^n \longrightarrow \phi_t & \text{weakly-star in} & L_{\mathrm{loc}}^\infty\Big(0,\infty;H_0^1\Big),\\ u_{tt}^n \longrightarrow u_{tt} & \text{and} & \phi_{tt}^n \longrightarrow \phi_{tt} & \text{weakly-star in} & L_{\mathrm{loc}}^\infty\Big(0,\infty;L^2\Big),\\ z^n \longrightarrow z & \text{and} & z_t^n \longrightarrow z_t & \text{weakly-star in} & L_{\mathrm{loc}}^\infty\Big(0,\infty;L_z^2\Big), \end{array} \tag{5.3.19}$$

We will show that U is a weak solution of system (5.2.1). Firstly, we will prove that $f = g_1(\phi_t)$ and $h = g_2(z(x,1))$.

Lemma 5.3.2. *For each $T > 0$, we have*

$$\begin{aligned} g_1(\phi_t^n) \longrightarrow g_1(\phi_t) \quad & \textit{weakly-star in} \quad L^2\Big((0,1)\times(0,T)\Big),\\ g_2(z^n(x,1)) \longrightarrow g_2(z(x,1)) \quad & \textit{weakly-star in} \quad L^2\Big((0,1)\times(0,T)\Big). \end{aligned} \tag{5.3.20}$$

Proof. From (5.3.10), we have ϕ_t^n is bounded in $L^\infty(0,T;H_0^1)$ and ϕ_{tt}^n is bounded in $L^\infty(0,T;L^2)$. Then, the injection by continuity in L^p gives us the boundedness of ϕ_t^n in $L^2(0,T;H_0^1)$ and ϕ_{tt}^n in $L^2(0,T;L^2)$. Hence, ϕ_t^n is bounded in $H^1(Q)$, where $Q = (0,1)\times(0,T)$. It is known that the embedding $H^1(Q) \hookrightarrow L^2(Q)$ is compact. This enables us to extract a subsequence of ϕ^n, represented again by ϕ^n, such that

$$\phi^n_t \longrightarrow \phi_t \quad \text{strongly in} \quad L^2\big(0,T;L^2(0,1)\big),$$

and so

$$\phi^n_t \qquad \longrightarrow \qquad \phi_t \qquad\qquad \text{a.e. on } Q.$$

By the continuity of g_1, we have

$$g_1(\phi^n_t) \quad \longrightarrow \quad g_1(\phi_t) \qquad \text{a.e. on } Q. \tag{5.3.21}$$

And similarly,

$$g_2(z^n(x,1)) \longrightarrow g_2(z(x,1)) \quad \text{a.e. on } Q. \tag{5.3.22}$$

On the other hand, appealing to the inequalities (5.3.5) and (5.3.6), we get

$$\int_0^1 \Big[g_1^2(\phi^n_t) + g_2^2(z^n(x,1))\Big]dx \le c + \int_0^1 \Big[\phi^n_t g_1(\phi^n_t) + z^n(x,1)g_2(z^n(x,1))\Big]dx.$$

It then follows from (5.3.7) that

$$\int_0^t \int_0^1 \Big[g_1^2(\phi^n_t) + g_2^2(z^n(x,1))\Big]dxdt \le c,$$

which directly gives $g_1(\phi^n_t), g_2(z^n(x,1)) \in L^2(Q)$. Combining these with (5.3.21) − (5.3.22) and using Lemma 1.4.3, we obtain (5.3.20). □

To prove that U is a weak solution of (5.2.1) we discuss as in [**56**] (see also [**100**]). For, we consider functions $v, \omega \in C\big(0,T;L^2(0,1)\big)$ and $y \in C\big(0,T;L^2_z(0,1)\big)$ having the forms

$$\big(v(x,t), \omega(x,t)\big) = \Big(\sum_{i=1}^N \widetilde{c}^{in}(t), \sum_{i=1}^N \widetilde{d}^{in}(t)\Big)\Phi^i(x), \tag{5.3.23}$$

$$y(x,\rho,t) = \sum_{i=1}^N \widetilde{e}^{in}(t)\Psi^i(x,\rho), \tag{5.3.24}$$

where $N \ge n$ is a fixed integer.
Then, by multiplying $(5.3.1)_1$, $(5.3.1)_2$ and $(5.3.1)_3$ by $\widetilde{c}^{in}(t)$, $\widetilde{d}^{in}$ and $\widetilde{e}^{in}$, respectively, and summing the resultants over i from 1 to N, we find that

$$\begin{cases} \displaystyle\int_0^T \int_0^1 \Big[\rho_1 u^n_{tt} - \kappa u^n_{xx} - b\phi^n_x\Big] v dxdt &= 0, \\ \displaystyle\int_0^T \int_0^1 \Big[\rho_2 \phi^n_{tt} - \delta\phi^n_{xx} + bu^n_x + \xi\phi^n_x + \mu_1 g_1(\phi^n_t) + \mu_2 g_2(z^n(x,1))\Big]\omega dxdt &= 0, \\ \displaystyle\int_0^T \int_0^1 \int_0^1 \Big[\tau z^n_t(x,\rho) + z^n_\rho(x,\rho)\Big] y(x,\rho) d\rho dxdt &= 0. \end{cases} \tag{5.3.25}$$

After passing to the limit in (5.3.25) as $n \longrightarrow +\infty$ and using (5.3.19), we arrive at

$$\begin{cases} \int_0^T \int_0^1 \Big[\rho_1 u_{tt} - \kappa u_{xx} - b\phi_x\Big] v dx dt & = 0, \\ \int_0^T \int_0^1 \Big[\rho_2 \phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \mu_1 g_1(\phi_t) + \mu_2 g_2(z(x,1))\Big]\omega dx dt & = 0, \\ \int_0^T \int_0^1 \int_0^1 \Big[\tau z_t(x,\rho) + z_\rho(x,\rho)\Big] y(x,\rho) d\rho dx dt & = 0. \end{cases} \tag{5.3.26}$$

Eqs. (5.3.26) hold for all $(v,\omega,y) \in \big(L^2\big(0,T;L^2\big)\big)^2 \times L^2\big(0,T;L_z^2\big)$ since the functions of the forms (5.3.23) and (5.3.24) are dense, respectively, in $L^2\big(0,T;L^2\big)$ and $L^2\big(0,T;L_z^2\big)$. Next, we must verify that the limit functions u, ϕ, z fulfill the initial conditions

$$u(.,0) = u_0, \quad u_t(.,0) = u_1, \quad \phi(.,0) = \phi_0, \quad \phi_t(.,0) = \phi_1 \tag{5.3.27}$$

and the history value

$$z(.,.,0) = f_0. \tag{5.3.28}$$

For, we take any $v,\omega \in C^2\big(0,T;L^2\big)$ and $y \in C^1(0,T,L_z^2)$ satisfying

$$u(.,T) = u_t(.,T) = \phi(.,T) = \phi_t(.,T) = y(.,\rho,T) = 0.$$

Then integrating with respect to t in (5.3.26), we have

$$\begin{cases} \int_0^T \int_0^1 \Big[\rho_1 u v_{tt} - \big(\kappa u_{xx} + b\phi_x\big) v\Big] dx dt + \rho_1 \int_0^1 \Big[u(0)v_t(0) - u_t(0)v(0)\Big] dx = 0, \\ \int_0^T \int_0^1 \Big[\rho_2 \phi\omega_{tt} - \big(\delta\phi_{xx} - bu_x - \xi\phi_x - \mu_1 g_1(\phi_t) - \mu_2 g_2(z(x,1))\big)\omega\Big] dx dt \\ +\rho_2 \int_0^1 \Big[\phi(0)\omega_t(0) - \phi_t(0)\omega(0)\Big] dx = 0, \\ \int_0^T \int_0^1 \int_0^1 \Big[-\tau z(x,\rho,t) y_t(x,\rho,t) + z_\rho(x,\rho,t) y(x,\rho,t)\Big] d\rho dx dt \\ -\tau \int_0^1 \int_0^1 z(x,\rho,0) y(x,\rho,0) d\rho dx = 0. \end{cases} \tag{5.3.29}$$

On the other hand, proceeding in the same way, we obtain from (5.3.25) that

$$\begin{cases} \int_0^T \int_0^1 \Big[\rho_1 u^n v_{tt} - \big(\kappa u_{xx}^n + b\phi_x^n\big) v\Big] dx dt + \rho_1 \int_0^1 \Big[u^n(0)v_t(0) - u_t^n(0)v(0)\Big] dx = 0, \\ \int_0^T \int_0^1 \Big[\rho_2 \phi^n\omega_{tt} - \big(\delta\phi_{xx}^n - bu_x^n - \xi\phi_x^n - \mu_1 g_1(\phi_t^n) - \mu_2 g_2(z^n(x,1))\big)\omega\Big] dx dt \\ +\rho_2 \int_0^1 \Big[\phi^n(0)\omega_t(0) - \phi_t^n(0)\omega(0)\Big] dx = 0, \\ \int_0^T \int_0^1 \int_0^1 \Big[\tau z^n(x,\rho,t) y_t(x,\rho,t) + z_\rho^n(x,\rho,t) y(x,\rho,t)\Big] d\rho dx dt \\ -\tau \int_0^1 \int_0^1 z^n(x,\rho,0) y(x,\rho,0) d\rho dx = 0. \end{cases}$$

Recalling (5.3.19) and (5.3.2), we get

$$\begin{cases} \int_0^T \int_0^1 \Big[\rho_1 u v_{tt} - \big(\kappa u_{xx} + b\phi_x\big)v\Big]dxdt + \rho_1 \int_0^1 \Big[u_0 v_t(0) - u_1 v(0)\Big]dx = 0, \\ \int_0^T \int_0^1 \Big[\rho_2 \phi \omega_{tt} - \big(\delta\phi_{xx} - bu_x - \xi\phi_x - \mu_1 g_1(\phi_t) - \mu_2 g_2(z(x,1))\big)\omega\Big]dxdt \\ +\rho_2 \int_0^1 \Big[\phi_0 \omega_t(0) - \phi_1 \omega(0)\Big]dx = 0, \\ \int_0^T \int_0^1 \int_0^1 \Big[-\tau z(x,\rho,t)y_t(x,\rho,t) + z_\rho(x,\rho,t)y(x,\rho,t)\Big]d\rho dxdt \\ -\tau \int_0^1 \int_0^1 f_0 y(x,\rho,0)d\rho dx = 0. \end{cases} \tag{5.3.30}$$

As $v(x,0), v_t(x,0), \omega(x,0), \omega_t(x,0), y(x,\rho,0)$ are arbitrary, comparing identities (5.3.29) and (5.3.30), we deduce (5.3.27) and (5.3.28).
Consequently, (5.2.1) admits at least one global weak solution U.

For the uniqueness, we assume that $(\widetilde{u}, \widetilde{\phi}, \widetilde{z})$ and $(\widetilde{\widetilde{u}}, \widetilde{\widetilde{\phi}}, \widetilde{\widetilde{z}})$ are two weak solutions of (5.2.1), then $(u,\phi,z) = (\widetilde{u}, \widetilde{\phi}, \widetilde{z}) - (\widetilde{\widetilde{u}}, \widetilde{\widetilde{\phi}}, \widetilde{\widetilde{z}})$ satisfies the following system

$$\begin{cases} \rho_1 u_{tt} - \kappa u_{xx} - b\phi_x = 0, \\ \rho_2 \phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \mu_1\big(g_1(\widetilde{\phi}_t) - g_1(\widetilde{\widetilde{\phi}}_t)\big) + \mu_2\big(g_2(\widetilde{z}(x,1)) - g_2(\widetilde{\widetilde{z}}(x,1))\big) = 0, \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \\ u(0,t) = u(1,t) = \phi(0,t) = \phi(0,t) = 0, \\ u(x,0) = u_t(x,0) = \phi(x,0) = \phi_t(x,0) = z(x,\rho,0) = 0. \end{cases} \tag{5.3.31}$$

To get the uniqueness result, it suffices to verify that $(u,\phi,z) = (0,0,0)$ is the only stronger weak solution of (5.3.31). For that, multiplying $(5.3.31)_1$ by $2u_t$ and $(5.3.31)_2$ by $2\phi_t$, we yield

$$\begin{aligned} \frac{d}{dt}\int_0^1 \Big[\rho_1 u_t^2 + \rho_2 \phi_t^2 + \kappa u_x^2 + \delta\phi_x^2 + \xi\phi^2 + 2bu_x\phi\Big]dx + 2\mu_1 \int_0^1 \phi_t\big(g_1(\widetilde{\phi}_t) - g_1(\widetilde{\widetilde{\phi}}_t)\big)dx \\ + 2\mu_2 \int_0^1 \phi_t\big(g_2(\widetilde{z}(x,1)) - g_2(\widetilde{\widetilde{z}}(x,1))\big)dx = 0. \end{aligned} \tag{5.3.32}$$

And, we multiply $(5.3.31)_3$ by $2z(x,\rho)$ to get

$$\tau \frac{d}{dt}\int_0^1 \int_0^1 z^2(x,\rho)d\rho dx + \int_0^1 z^2(x,1)dx - \int_0^1 \phi_t^2 dx = 0. \tag{5.3.33}$$

Moreover, setting

$$\Lambda(t) = \int_0^1 \Big[\rho_1 u_t^2 + \rho_2 \phi_t^2 + \kappa u_x^2 + \delta\phi_x^2 + \xi\phi^2 + 2bu_x\phi + \tau \int_0^1 z^2(x,\rho)d\rho\Big]dx$$

and adding the estimates (5.3.32)-(5.3.33), we obtain

$$\begin{aligned}\Lambda'(t) =& -2\mu_1 \int_0^1 \phi_t\big(g_1(\widetilde{\phi}_t) - g_1(\widetilde{\widetilde{\phi}}_t)\big)dx + \int_0^1 \phi_t^2 dx - \int_0^1 z^2(x,1)dx \\ & - 2\mu_2 \int_0^1 \phi_t\big(g_2(\widetilde{z}(x,1)) - g_2(\widetilde{\widetilde{z}}(x,1))\big)dx.\end{aligned} \tag{5.3.34}$$

As g_1 is an increasing function, we can easily see that

$$\big(s_0 - s_1\big)\big(g_1(s_0) - g_1(s_1)\big) > 0 \quad \forall s_0, s_1 \in \mathbb{R}.$$

Thus, (5.3.34) becomes

$$\Lambda'(t) \leq \int_0^1 \phi_t^2 dx - \int_0^1 z^2(x,1)dx - 2\mu_2 \int_0^1 \phi_t\big(g_2(\widetilde{z}(x,1)) - g_2(\widetilde{\widetilde{z}}(x,1))\big)dx.$$

By the Young's inequality, we get

$$\Lambda'(t) \leq c\int_0^1 \phi_t^2 dx - \int_0^1 z^2(x,1)dx + \varepsilon_3 \int_0^1 \big(g_2(\widetilde{z}(x,1)) - g_2(\widetilde{\widetilde{z}}(x,1))\big)^2 dx.$$

Since g_2 is a continous function, it results from (5.2.3) that

$$\big|g_2(s_0) - g_2(s_1)\big| \leq \lambda_2\big|s_0 - s_1\big| \quad \forall s_0, s_1 \in \mathbb{R}.$$

This leads us to

$$\Lambda'(t) \leq c\int_0^1 \phi_t^2 dx - \big(1 - \lambda_2\varepsilon_3\big)\int_0^1 z^2(x,1)dx.$$

Hence, for a suitable ε_3, we have

$$\Lambda'(t) \leq c\int_0^1 \phi_t^2 dx.$$

As $\Lambda(t)$ is positive (for the same raison given in Remark 5.2.1) and $\Lambda(0) = 0$, Gronwall's inequality forces that $\Lambda(t) = 0$ $(0 \leq t \leq T)$, which means that $u = \phi = z = 0$.
Consequently, (5.2.1) possesses only one weak stronger weak solution.

□

5.4 Asymptotic behavior

This last section, which will be divided into three subsections, studies the stability of system (5.2.1). In fact, using the Lyapunov method, we will prove that, under equal and non-equal wave speeds cases, the solution of (5.2.1) converges to zero as t tends to infinity.
At the first, we consider the following additional assumption:
(A_3) With respect to the weights of feedbacks $\mu_i (i = 1, 2)$, we assume that

$$|\mu_2| < \frac{\alpha_1}{\alpha_2}\mu_1.$$

Now, we can suppose that the positive constant γ satisfies the following inequality

$$\frac{(1-\alpha_1)|\mu_2|}{\alpha_1} < \gamma < \frac{\mu_1 - \alpha_2|\mu_2|}{\alpha_2}.$$

Then, by setting $a = a_1 = 0$ in (5.2.8), it results from (5.2.9) that

$$E'(t) \le -\beta_1 \int_0^1 \phi_t g_1(\phi_t)dx - \beta_2 \int_0^1 z(x,1)g_2(z(x,1))dx, \tag{5.4.1}$$

where $\beta_1 = \mu_1 - \gamma\alpha_2 - \alpha_2|\mu_2| > 0$ and $\beta_2 = \gamma\alpha_1 - (1-\alpha_1)|\mu_2| > 0$.

5.4.1 Technical lemmas

In this subsection, we state and prove various Lemmas given for any regular solution of (5.2.1). It would help us to estimate the derivative of the Lyapunov functional.

Lemma 5.4.1. *The functional*

$$F_1(t) = -\rho_1 \int_0^1 u_t u dx$$

satisfies, along the solution of system (5.2.1)*, the estimate*

$$F_1'(t) \le -\rho_1 \int_0^1 u_t^2 dx + \frac{3\kappa}{2} \int_0^1 u_x^2 dx + c\int_0^1 \phi_x^2 dx. \tag{5.4.2}$$

Proof. A simple differentiation with respect to t, using $(5.2.1)_1$, yields

$$F_1'(t) = -\rho_1 \int_0^1 u_t^2 dx + \kappa \int_0^1 u_x^2 dx + b\int_0^1 u_x \phi dx.$$

The Young's and Poincaré's inequalities lead to (5.4.2). □

Lemma 5.4.2. *The functional defined by*

$$F_2(t) = \rho_2 \int_0^1 \phi_t u_x dx + \frac{\delta\rho_1}{\kappa} \int_0^1 u_t \phi_x dx$$

fulfills for any $\eta > 0$*,*

$$\begin{aligned} F_2'(t) \le &-\frac{b}{2}\int_0^1 u_x^2 dx + \eta\big(u_x^2(1,t) + u_x^2(0,t)\big) + \frac{\delta^2}{4\eta}\big(\phi_x^2(1,t) + \phi_x^2(0,t)\big) + c\int_0^1 \phi_x^2 dx \\ &+ c\int_0^1 g_1^2(\phi_t)dx + c\int_0^1 g_2^2(z(x,1))dx + \left(\frac{\delta\rho_1}{\kappa} - \rho_2\right)\int_0^1 \phi_{xt} u_t dx. \end{aligned} \tag{5.4.3}$$

Proof. Direct computations, using $(5.2.1)_1$-$(5.2.1)_2$, lead to

$$\begin{aligned} F_2'(t) = &\int_0^1 u_x \Big[\delta\phi_{xx} - bu_x - \xi\phi - \mu_1 g_1(\phi_t) - \mu_2 g_2(z(x,1))\Big]dx \\ &+ \frac{\delta}{\kappa}\int_0^1 \phi_x \Big[\kappa u_{xx} + b\phi_x\Big]dx + \left(\frac{\delta\rho_1}{\kappa} - \rho_2\right)\int_0^1 \phi_{xt} u_t dx. \end{aligned}$$

An integration by parts gives

$$F_2'(t) = \Big[\delta u_x\phi_x\Big]_{x=0}^{x=1} - b\int_0^1 u_x^2dx + \frac{b\delta}{\kappa}\int_0^1 \phi_x^2dx - \xi\int_0^1 u_x\phi dx - \mu_1\int_0^1 g_1(\phi_t)u_xdx - \mu_2\int_0^1 g_2(z(x,1))u_xdx + \left(\frac{\delta\rho_1}{\kappa} - \rho_2\right)\int_0^1 \phi_{xt}u_tdx.$$

Using Young's and Poincaré's inequalities, (5.4.3) is established. □

Lemma 5.4.3. *Let χ be a solution of*

$$\begin{cases} \chi_{xx} = -\phi_x, \\ \chi(0) = \chi(1) = 0. \end{cases}$$

Then, the functional

$$F_3(t) = \int_0^1 \left(\rho_2\phi_t\phi + \frac{b\rho_1}{\kappa}u_t\chi\right)dx$$

satisfies, for any $\eta_0 > 0$, the following estimate

$$F_3'(t) \leq -\delta\int_0^1 \phi_x^2dx - \frac{1}{2}\left(\xi - \frac{b^2}{\kappa}\right)\int_0^1 \phi^2dx + \eta_0\int_0^1 u_t^2dx + c\int_0^1 \phi_t^2dx + c\int_0^1 g_1^2(\phi_t)dx + c\int_0^1 g_2^2(z(x,1))dx. \tag{5.4.4}$$

Proof. Differentiating F_3 and using $(5.2.1)_1$-$(5.2.1)_2$, we get

$$F_3'(t) = -\xi\int_0^1 \phi^2dx + \frac{b^2}{\kappa}\int_0^1 \chi_x^2dx - \delta\int_0^1 \phi_x^2dx + \rho_2\int_0^1 \phi_t^2dx + \frac{b\rho_1}{\kappa}\int_0^1 u_t\chi_tdx - \mu_1\int_0^1 \phi g_1(\phi_t)dx - \mu_2\int_0^1 \phi g_2(z(x,1))dx. \tag{5.4.5}$$

By Young's inequality, we have

$$\frac{b\rho_1}{\kappa}\int_0^1 u_t\chi_tdx \leq \eta_0\int_0^1 u_t^2dx + c\int_0^1 \chi_t^2dx, \tag{5.4.6}$$

$$\mu_1\int_0^1 \phi g_1(\phi_t)dx \leq \frac{1}{4}\left(\xi - \frac{b^2}{\kappa}\right)\int_0^1 \phi^2dx + c\int_0^1 g_1^2(\phi_t)dx, \tag{5.4.7}$$

$$\mu_2\int_0^1 \phi g_2(z(x,1))dx \leq \frac{1}{4}\left(\xi - \frac{b^2}{\kappa}\right)\int_0^1 \phi^2dx + c\int_0^1 g_2^2(z(x,1))dx. \tag{5.4.8}$$

Inserting (5.4.6)-(5.4.8) into (5.4.5) and using the fact that

$$\int_0^1 \chi_x^2dx \leq \int_0^1 \phi^2dx,$$

$$\int_0^1 \chi_t^2dx \leq \int_0^1 \chi_{tx}^2dx \leq \int_0^1 \phi_t^2dx,$$

we obtain (5.4.4). □

Next, in order to eliminate the boundary terms, appearing in (5.4.3), we introduce the following function

$$m(x) = -4x + 2, \quad x \in [0,1]. \tag{5.4.9}$$

Then, we have the following result.

Lemma 5.4.4. *For any $\eta > 0$, the functional F_4 defined by*

$$F_4(t) = \frac{\eta}{\kappa}\int_0^1 \rho_1 m(x) u_t u_x dx + \frac{\delta}{4\eta}\int_0^1 \rho_2 m(x)\phi_t\phi_x dx$$

satisfies

$$\begin{aligned} F_4'(t) \le & -\eta\Big(u_x^2(1,t) + u_x^2(0,t)\Big) - \frac{\delta^2}{4\eta}\big(\phi_x^2(1,t)+\phi_x^2(0,t)\big) + c\eta\rho_1\int_0^1 u_t^2 dx + c\int_0^1 \phi_t^2 dx \\ & + \Big(\Big(\frac{1}{4}+\frac{\eta}{4}\Big)b + 2\eta\Big)\int_0^1 u_x^2 dx + c\int_0^1 \phi_x^2 dx + c\int_0^1 g_1^2(\phi_t)dx + c\int_0^1 g_2^2(z(x,1))dx. \end{aligned} \tag{5.4.10}$$

Proof. By using $(5.2.1)_1$, $(5.2.1)_2$ and (5.4.9), it holds that

$$\begin{aligned} F_4'(t) = & \frac{\eta}{\kappa}\Big[-\kappa\big(u_x^2(1,t)+u_x^2(0,t)\big) + 2\rho_1\int_0^1 u_t^2 dx + b\int_0^1 m(x)u_x\phi_x dx + 2\kappa\int_0^1 u_x^2 dx\Big] \\ & + \frac{\delta}{4\eta}\Big[-\delta\big(\phi_x^2(1,t)+\phi_x^2(0,t)\big) + 2\rho_2\int_0^1 \phi_t^2 dx + 2\delta\int_0^1 \phi_x^2 dx - b\int_0^1 m(x)\phi_x u_x dx \\ & \quad - \mu_1\int_0^1 m(x)\phi_x g_1(\phi_t)dx - \mu_2\int_0^1 m(x)\phi_x g_2(z(x,1))dx - 2\xi\int_0^1 \phi^2 dx\Big]. \end{aligned}$$

Estimate (5.4.10) follows by exploiting Young's and Poincaré's inequalities. □

Lemma 5.4.5. *The functional*

$$F_5(t) = \tau\int_0^1\int_0^1 e^{-\tau\rho} G(z(x,\rho,t))d\rho dx$$

satisfies an estimate of the form

$$\begin{aligned} F_5'(t) \le & -\tau e^{-\tau}\int_0^1\int_0^1 G(z(x,\rho,t))d\rho dx - \alpha_1 e^{-\tau}\int_0^1 z(x,1)g_2(z(x,1))dx \\ & + c\int_0^1 \phi_t^2 dx + c\int_0^1 g_1^2(\phi_t)dx. \end{aligned} \tag{5.4.11}$$

Proof. Taking the derivative of F_5 and using $(5.2.1)_3$, we have

$$F_5'(t) = \int_0^1\int_0^1 e^{-\tau\rho} z_\rho(x,\rho,t) g_2(z(x,\rho,t))d\rho dx,$$

that is,

$$
\begin{aligned}
F_5'(t) &= -\int_0^1\int_0^1 \frac{d}{d\rho}\Big[e^{-\tau\rho}G(z(x,\rho,t))\Big]d\rho dx - \tau\int_0^1\int_0^1 e^{-\tau\rho}G(z(x,\rho,t))d\rho dx \\
&= -\int_0^1 \Big[e^{-\tau}G(z(x,1,t)) - G(z(x,0,t))\Big]dx - \tau\int_0^1\int_0^1 e^{-\tau\rho}G(z(x,\rho,t))d\rho dx.
\end{aligned}
$$

Using (5.2.4), we can obtain

$$
F_5'(t) \le -\tau\int_0^1\int_0^1 e^{-\tau\rho}G(z(x,\rho,t))d\rho dx - \alpha_1 e^{-\tau}\int_0^1 z(x,1)g_2(z(x,1))dx + \alpha_2\int_0^1 \phi_t g_1(\phi_t)dx.
$$

Estimate (5.4.11) follows by using Young's inequality with the fact that $e^{-\tau} \le e^{-\tau\rho}, \ \forall\, \rho \in [0,1]$.

□

Lemma 5.4.6. *For a suitable choice of N and N_i, $(i = 1, 2, ..., 5)$, the functional defined by*

$$
\mathcal{L}(t) = NE(t) + \sum_{i=1}^{5} N_i F_i(t) \tag{5.4.12}
$$

satisfies, for a fixed positive constant m_0, the estimate

$$
\mathcal{L}'(t) \le -m_0E(t) + \left(\frac{\delta\rho_1}{\kappa} - \rho_2\right)\int_0^1 \phi_{xt}u_t dx + c\int_0^1 \phi_t^2 dx + c\int_0^1 g_1^2(\phi_t)dx + c\int_0^1 g_2^2(z(x,1))dx. \tag{5.4.13}
$$

Proof. It follows from (5.4.1), (5.4.2), (5.4.3), (5.4.4), (5.4.10) and (5.4.11) that for any $t \ge 0$,

$$
\begin{aligned}
\mathcal{L}'(t) \le & -(N_4 - N_2)\Big[\eta\big(u_x^2(1,t) + u_x^2(0,t)\big) + \frac{\delta^2}{4\eta}\big(\phi_x^2(1,t) + \phi_x^2(0,t)\big)\Big] \\
& - \Big[\rho_1 N_1 - \eta_0 N_3 - \eta c\rho_1 N_4\Big]\int_0^1 u_t^2 dx + \Big[N_3 + N_4 + N_5\Big]c\int_0^1 \phi_t^2 dx \\
& - \Big[\frac{b}{2}N_2 - \frac{3\kappa}{2}N_1 - \Big(\Big(\frac{1}{4} + \frac{\eta}{4}\Big)b + 2\eta\Big)N_4\Big]\int_0^1 u_x^2 dx \\
& - \frac{1}{2}\Big(\xi - \frac{b^2}{\kappa}\Big)N_3\int_0^1 \phi^2 dx - \Big[\delta N_3 - \big(N_1 + N_2 + N_4\big)c\Big]\int_0^1 \phi_x^2 dx \\
& - \tau e^{-\tau}N_5\int_0^1\int_0^1 G(z(x,\rho))d\rho dx + \Big[N_2 + N_3 + N_4 + N_5\Big]c\int_0^1 g_1^2(\phi_t)dx \\
& + \Big[N_2 + N_3 + N_4\Big]c\int_0^1 g_2^2(z(x,1))dx + N_2\left(\frac{\delta\rho_1}{\kappa} - \rho_2\right)\int_0^1 \phi_{xt}u_t dx.
\end{aligned}
$$

Furthermore, we take

$$
N_1 = 3\eta c, \quad N_2 = N_4 = N_5 = 1, \quad \eta_0 = \frac{\eta c\rho_1}{N_3},
$$

to get

$$\begin{aligned}\mathcal{L}'(t) \leq & -\eta c\rho_1 \int_0^1 u_t^2 dx + c\int_0^1 \phi_t^2 dx - \frac{1}{4}\Big(b - \eta\big(18\kappa c + b + 8\big)\Big)\int_0^1 u_x^2 dx \\ & - \Big(\delta N_3 - c\Big)\int_0^1 \phi_x^2 dx - \frac{1}{2}\Big(\xi - \frac{b^2}{\kappa}\Big)N_3 \int_0^1 \phi^2 dx + c\int_0^1 g_2^2(z(x,1))dx \\ & - \tau e^{-\tau}\int_0^1\int_0^1 G(z(x,\rho))d\rho dx + c\int_0^1 g_1^2(\phi_t)dx + \left(\frac{\delta\rho_1}{\kappa} - \rho_2\right)\int_0^1 \phi_{xt}u_t dx.\end{aligned} \tag{5.4.14}$$

Now, we select $\eta < \frac{b}{18\kappa c + b + 8}$ and then we choose N_3 large enough such that

$$\delta N_3 - c > 0.$$

Thus, due to $\kappa\xi > b^2$ and (5.4.14), we end up with

$$\begin{aligned}\mathcal{L}'(t) \leq & -c\int_0^1 \left[u_t^2 + \phi_t^2 + u_x^2 + \phi_x^2 + \phi^2 + \int_0^1 G(z(x,\rho))d\rho\right]dx + c\int_0^1 \phi_t^2 dx \\ & + c\int_0^1 g_1^2(\phi_t)dx + c\int_0^1 g_2^2(z(x,1))dx + \left(\frac{\delta\rho_1}{\kappa} - \rho_2\right)\int_0^1 \phi_{xt}u_t dx.\end{aligned} \tag{5.4.15}$$

In the other hand, from (5.2.5), we obtain by using Young's inequality that

$$E(t) \leq \int_0^1 \left[\rho_1 u_t^2 + \rho_2\phi_t^2 + \big(\kappa + b\big)u_x^2 + \delta\phi_x^2 + \big(\xi + b\big)\phi^2 + 2\tau\gamma\int_0^1 G(z(x,\rho))d\rho\right]dx.$$

This relation, together with (5.4.15), gives the desired estimate (5.4.13). □

5.4.2 General decay rates for equal speeds of wave propagation.

In this subsection, we study the decay of solution of our problem (5.2.1) in the case $\frac{\rho_1}{\kappa} = \frac{\rho_2}{\delta}$.

Theorem 5.4.7. *Let $U_0 \in \mathcal{H}$. Assuming that (A_1)-(A_3) are fulfilled, $\kappa\xi > b^2$ and that*

$$\frac{\rho_1}{\kappa} = \frac{\rho_2}{\delta}.$$

Then, there exist some positive constants ς, ς_1, ς_2 and ϵ_0 such that the solution of (2.2.1) *satisfies*

$$E(t) \leq \varsigma K_1^{-1}\big(\varsigma_1 t + \varsigma_2\big) \qquad \forall t > 0, \tag{5.4.16}$$

where

$$K_1(t) = \int_t^1 \frac{1}{K(s)}ds \qquad \text{and} \qquad K(t) = tH'(\epsilon_0 t). \tag{5.4.17}$$

Proof. Since $\frac{\rho_1}{\kappa} = \frac{\rho_2}{\delta}$, then we can easily show, for N sufficiently large, that the functional $\mathcal{L}$ given by (5.4.12) is equivalent to E, i.e.,

$$\mathcal{L} \sim E.$$

Consider the following two sets:

$$\mathcal{D}_1 = \Big\{ x \in [0,1] : \quad |\phi_t| \leq \epsilon \ \Big\}, \qquad \mathcal{D}_2 = \Big\{ x \in [0,1] : \quad |\phi_t| > \epsilon \ \Big\}.$$

Then, by recalling (5.2.2), (A_2) and (5.4.1), we obtain that

$$\mathcal{L}'(t) \leq -\, m_0 E(t) - cE'(t) + c\int_{\mathcal{D}_1} H^{-1}\big(\phi_t g_1(\phi_t)\big)dx.$$

Hence, the function $\mathcal{L}_0 = \mathcal{L} + cE$ satisfies

$$\mathcal{L}_0 \sim E$$

and

$$\mathcal{L}_0'(t) \leq -\, m_0 E(t) + c\int_{\mathcal{D}_1} H^{-1}\big(\phi_t g_1(\phi_t)\big)dx. \tag{5.4.18}$$

Now, we discuss the following two cases:

1. H is linear on $[\mathbf{0}, \epsilon]$: In this case, one has for some positive constant c',

$$\mathcal{L}_0'(t) \leq -m_0 E(t) - c'E'(t).$$

Then, $\mathcal{L}_1 = \mathcal{L}_0 + c'E \sim E$ satisfies

$$\mathcal{L}_1(t) \leq \mathcal{L}_1(0)e^{-ct},$$

which implies that

$$E(t) \leq C\big(E(0)\big)e^{-ct}.$$

2. H is non-linear on $[\mathbf{0}, \epsilon]$: We note that, by using Jensen's inequality and the concavity of H^{-1}, the following inequality holds

$$\int_{\mathcal{D}_1} H^{-1}\big(\phi_t g_1(\phi_t)\big)dx \leq cH^{-1}\left(\int_{\mathcal{D}_1} \phi_t g_1(\phi_t)dx\right).$$

Substituting this latter estimate in (5.4.18), we get

$$\mathcal{L}_0'(t) \leq -\, m_0 E(t) + cH^{-1}\left(\int_{\mathcal{D}_1} \phi_t g_1(\phi_t)dx\right). \tag{5.4.19}$$

Let us define for $\epsilon_0 < \epsilon$ and $m_1 > 0$

$$L(t) = H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right)\mathcal{L}_0(t) + m_1 E(t).$$

Then, one can easily see that, for some fixed positive constants v_0 and v_1,

$$v_0 L(t) \leq E(t) \leq v_1 L(t) \tag{5.4.20}$$

and

$$L'(t) = \epsilon_0 \frac{E'(t)}{E(0)} H''\left(\epsilon_0 \frac{E(t)}{E(0)}\right)\mathcal{L}_0(t) + H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right)\mathcal{L}_0'(t) + m_1 E'(t).$$

As H is an increasing-convex function, we have that $H' > 0$ and $H'' > 0$. Using these facts with (5.4.19) and $E' \leq 0$, it results

$$L'(t) \leq -\, m_0 E(t) H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) + cH'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) H^{-1}\left(\int_{\mathcal{D}_1} \phi_t g_1(\phi_t)dx\right) + m_1 E'(t). \tag{5.4.21}$$

Let H^* be the convex conjugate of H, then testing (5.2.7) with

$$A = H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) \quad \text{and} \quad B = H^{-1}\left(\int_{\mathcal{D}_1} \phi_t g_1(\phi_t)dx\right),$$

we get

$$H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) H^{-1}\left(\int_{\mathcal{D}_1} \phi_t g_1(\phi_t)dx\right) \leq H^*\left(H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right)\right) + \int_{\mathcal{D}_1} \phi_t g_1(\phi_t)dx.$$

Making use of $H^*(s) \leq s(H')^{-1}(s)$ and (5.4.1), we have

$$H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) H_1^{-1}\left(\int_{\mathcal{D}_1} \phi_t g_1(\phi_t)dx\right) \leq \epsilon_0 \frac{E(t)}{E(0)} H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) - cE'(t). \tag{5.4.22}$$

A simple substitution of (5.4.22) into (5.4.21) gives us

$$L'(t) \leq -\big(m_0 E(0) - c\epsilon_0\big) \frac{E(t)}{E(0)} H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) + \big(m_1 - c\big) E'(t).$$

Fexing ϵ_0 sufficiently small, so that $m_0 E(0) - c\epsilon_0 > 0$, then for $m_1 > c$, we can find a fixed positive constant ς_0 such that

$$L'(t) \leq -\varsigma_0 \frac{E(t)}{E(0)} H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) = -\varsigma_0 K\left(\frac{E(t)}{E(0)}\right), \tag{5.4.23}$$

where $K(t) = tH'(\epsilon_0 t)$. Moreover, with $L_1(t) = \frac{v_0 L(t)}{E(0)}$ it obvious that $L_1(t) \leq \frac{E(t)}{E(0)} \leq 1$ and $L_1 \sim E$. Thus, inequality (5.4.23) may be transformed into

$$L_1'(t) \leq -\varsigma_1 K\big(L_1(t)\big). \tag{5.4.24}$$

By the definition of K_1, we know that

$$K_1'(t) = -\frac{1}{K(t)} < 0, \quad \forall t \geq 0,$$

which, combined with (5.4.24), implies

$$L_1'(t) \leq \frac{\varsigma_1}{K_1'\big(L_1(t)\big)},$$

that is,

$$\big[K_1(L_1(t))\big]' \geq \varsigma_1,$$

by integrating over $[0, t]$, we yield that

$$K_1\big(L_1(t)\big) \geq \varsigma_1 t + K_1\big(L_1(0)\big),$$

using then the non-increasing property of K_1^{-1}, we obtain

$$L_1(t) \leq K_1^{-1}\big(\varsigma_1 t + \varsigma_2\big).$$

This, together with $L_1 \sim E$, gives us the desired result in Theorem 5.4.7. □

5.4.3 General decay rates for non-equal speeds of wave propagation

In this subsection, we investigate the situation when $\frac{\rho_1}{\kappa} \neq \frac{\rho_2}{\delta}$, which is more realistic in the view of physics. For that purpose, we consider the following hypotheses:
(A_4) $g_1 : \mathbb{R} \longrightarrow \mathbb{R}$ is a strictly increasing function of class C^1 such that it exist $\epsilon < 1$, c_3 and a C^1-function $H : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ which is linear on $[0, \epsilon]$ or non-decreasing and convex function of class C^2 with $H(0) = H'(0) = 0$ such that

$$\begin{cases} |g_1(s)| \leq c_3|s| & \text{if } |s| > \epsilon, \\ g_1^2(s) \leq H^{-1}\big(sg_1(s)\big) & \text{if } |s| \leq \epsilon. \end{cases} \tag{5.4.25}$$

(A_5) Also, we assume that

$$|\mu_2| < \min\Big(\frac{\alpha_1}{\alpha_2}, \frac{\lambda_1}{\lambda_2}\Big)\mu_1.$$

We now present the general decay result in the non-equal speeds case.

Theorem 5.4.8. *Let $U_0 \in \mathcal{H}_0$. Assuming that (A_2), (A_4), (A_5) hold, $\kappa\xi > b^2$ and that*

$$\frac{\rho_1}{\kappa} \neq \frac{\rho_2}{\delta}.$$

Then, it exist some positive constants w and w_1 such that for any $t > 0$,

$$E(t) \leq wK^{-1}\Big(\frac{w_1}{t}\Big). \tag{5.4.26}$$

Proof. In view of Lemma 5.2.5 and (A_5), we obtain that

$$\mathcal{F}'(t) \leq -\beta_3 \int_0^1 \phi_{xt}^2 dx \qquad \forall t \geq 0, \tag{5.4.27}$$

where $\beta_3 = \lambda_1\mu_1 - \lambda_2|\mu_2| > 0$.

In the sequel, we introduce the following Lyapunov functional

$$\mathcal{G}_0(t) = M\mathcal{F}(t) + \mathcal{L}(t), \tag{5.4.28}$$

where $\mathcal{L}$ is defined in Lemma 5.4.6 and M is a fixed positive constant to be determined posteriori. Before go further, it should be mentioned that $\mathcal{G}_0$ is not equivalent to E. Then, by combining (5.4.13) and (5.4.27), we find that for any $t \geq 0$,

$$\begin{aligned} \mathcal{G}_0'(t) \leq & - m_0E(t) - \beta_3 M \int_0^1 \phi_{xt}^2 dx + \Big(\frac{\delta\rho_1}{\kappa} - \rho_2\Big) \int_0^1 \phi_{xt}u_t dx \\ & + c\int_0^1 \phi_t^2 dx + c\int_0^1 g_1^2(\phi_t)dx + c\int_0^1 g_2^2(z(x,1))dx. \end{aligned}$$

Utilizing Young's and Poincaré's inequalities and (5.2.5), it follows that

$$\mathcal{G}_0'(t) \leq -\big(m_0 - \eta_1\big)E(t) - \big(\beta_3 M - c_{\eta_1} - c\big) \int_0^1 \phi_{xt}^2 dx + c\int_0^1 g_1^2(\phi_t)dx + c\int_0^1 g_2^2(z(x,1))dx.$$

Fixing $\eta_1 < m_0$ and then taking M sufficiently large, so that $\beta_3 M - c_{\eta_1} - c \geq 0$, we obtain for $d_0 > 0$,

$$\mathcal{G}_0'(t) \leq -d_0 E(t) + c\int_0^1 g_1^2(\phi_t)dx + c\int_0^1 g_2^2(z(x,1))dx.$$

By exploiting (A$_2$), (5.4.25) and (5.4.1), it holds that

$$\mathcal{G}_0'(t) \leq -d_0 E(t) - cE'(t) + \int_{\mathcal{D}_1} H^{-1}\big(\phi_t g_1(\phi_t)\big)dx.$$

In summary, the functional $\mathcal{G}_1(t) = \mathcal{G}_0(t) + cE(t)$ fulfills

$$\mathcal{G}_1'(t) \leq -d_0 E(t) + \int_{\mathcal{D}_1} H^{-1}\big(\phi_t g_1(\phi_t)\big)dx. \tag{5.4.29}$$

As in the proof of Theorem 5.4.7, we distinguish the following two cases:

1. H is linear on $[\mathbf{0}, \epsilon]$: By (5.4.1), one obtains for a fixed positive constant c',

$$\mathcal{G}_1'(t) \leq -d_0 E(t) - c'E'(t).$$

Then, the functional $\mathcal{G}_2(t) = \mathcal{G}_1(t) + c'E(t)$, satisfies

$$\mathcal{G}_2'(t) \leq -d_0 E(t).$$

Integrating the above inequality on $[0,t]$ and using the non-increasing property of E, we yield that

$$tE(t) \leq \int_0^t E(s)ds \leq \frac{1}{d_0}\mathcal{G}_2(0).$$

Hence, for $d > 0$ we have

$$E(t) \leq \frac{d}{t} \quad \forall t > 0.$$

2. H is non-linear on $[\mathbf{0}, \epsilon]$: Analogously to the second part of the proof of Theorem 5.4.7, we find that the functional

$$\mathcal{G}_3(t) = H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right)\mathcal{G}_1(t) + d_1 E(t)$$

satisfies, for a fixed positive constant w_0, the following property

$$\mathcal{G}_3'(t) \leq -w_0 K\left(\epsilon_0 \frac{E(t)}{E(0)}\right).$$

An integration over $[0,t]$ yields

$$\int_0^t K\left(\epsilon_0 \frac{E(s)}{E(0)}\right)ds \leq \frac{1}{w_0}\mathcal{G}_3(0).$$

Since $E' \leq 0$ and $K' > 0$, then the map $t \longmapsto K\big(\epsilon_0 \frac{E(t)}{E(0)}\big)$ is non-increasing. This gives that

$$tK\left(\epsilon_0 \frac{E(t)}{E(0)}\right) \leq \int_0^t K\left(\epsilon_0 \frac{E(s)}{E(0)}\right)ds,$$

and so

$$tK\left(\epsilon_0 \frac{E(t)}{E(0)}\right) \leq \frac{1}{w_0}\mathcal{G}_3(0).$$

Consequently, for $w, w_1 > 0$ we have

$$E(t) \leq wK^{-1}\left(\frac{w_1}{t}\right) \qquad \forall t > 0.$$

This ends the proof of Theorem 5.4.8. □

Chapter 6

Stability result of the Bresse system with delay and boundary feedback

6.1 Introduction

Let $0 < T \leq \infty$, $L > 0$. We denote by $\varphi = \varphi(x,t) : [0,L] \times [0,T] \longrightarrow \mathbb{R}$, $\psi = \psi(x,t) : [0,L] \times [0,T] \longrightarrow \mathbb{R}$ and $\omega = \omega(x,t) : [0,L] \times [0,T] \longrightarrow \mathbb{R}$, the longitudinal, vertical and shear angle displacements of the cross section at $x \in (0,L)$ and at time $t \in (0,t)$, respectively. The original Bresse system is given by the following equations (see [**70**]):

$$\begin{cases} \rho_1\varphi_{tt} &= Q_x + lN + F_1, \\ \rho_2\psi_{tt} &= M_x - Q + F_2, \\ \rho_1\omega_{tt} &= N_x - lQ + F_3, \end{cases} \tag{6.1.1}$$

where we use N, Q and M to denote the axial force, the shear force and the bending moment respectively. These forces are stress-strain relations for elastic behavior and given by

$$N = K_0(\omega_x - l\varphi), \quad Q = K(\varphi_x + \psi + l\omega) \quad \text{and } M = b\psi_x,$$

where K, K_0 and b are positive constants. Here $\rho_1 = \rho A$, $\rho_2 = \rho I$, $K_0 = EA$, $K = K'GA$, $b = EI$ and $l = R^{-1}$. Coefficients aforementioned, all assumed positive, represent:

- ρ the density,
- E the modulus of elasticity,
- G the shear modulus,
- K' the shear factor,
- A the cross-sectional area,
- I the second moment of area of the cross section,
- R the radius of curvature,
- l the curvature $l = 1/R$.

Finally, by the terms F_i we are denoting external forces. Therefore, the evolutive problem can be written as

$$\begin{cases} \rho_1\varphi_{tt} - K(\varphi_x + \psi + l\omega)_x - K_0 l(\omega_x - l\varphi) = 0 & \text{in } [0,L] \times [0,T], \\ \rho_2\psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi + l\omega) = 0 & \text{in } [0,L] \times [0,T], \\ \rho_1\omega_{tt} - K_0(\omega_x - l\varphi)_x + Kl(\varphi_x + \psi + l\omega) = 0 & \text{in } [0,L] \times [0,T] \end{cases} \tag{6.1.2}$$

when the external forces are null.

It is known that the system (6.1.2) for $l = 0$ reduces to the standard Timoshenko system when $\omega = 0$. Many authors have established several results dealing with global existence and the stability behavior of the two systems using different kinds of dampings. It has been shown that the stability depends on the nature and position of the controls and some relations between the coefficients.

A few works adressed the issue of stability of the Bresse system with delays (see **[117-119]**.) The authors of **[117]** have treated (6.1.1) when

$$\begin{aligned} F_1 &= -\mu_1\varphi_t - \mu_2\varphi_t(x, t-\tau_1), \\ F_2 &= -\widetilde{\mu_1}\psi_t - \widetilde{\mu_2}\psi_t(x, t-\tau_2), \\ F_3 &= -\widetilde{\widetilde{\mu_1}}\omega_t - \widetilde{\widetilde{\mu_2}}\omega_t(x, t-\tau_3), \end{aligned} \tag{6.1.3}$$

with homogeneous Dirichlet boundary conditions. Under suitable assumptions on the weight of the delayed feedbacks and the weight of the non-delayed ones, they obtained an exponential rate of decay of solutions by making use of a multiplier method. This work was extended by the same authors in **[118]** to the nonlinear case.

In this paper we investigate the global well-posedness and the boundary stabilization of the linear Bresse system in bounded interval $[0, L]$.

$$\begin{cases} \rho_1\varphi_{tt} - \kappa(\varphi_x + \psi + l\omega)_x - \kappa_0 l(\omega_x - l\varphi) + a_1\varphi_t(x, t-\tau) = 0 & \text{in } [0, L] \times [0, +\infty[, \\ \rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi + l\omega) + a_2\psi_t(x, t-\tau) = 0 & \text{in } [0, L] \times [0, +\infty[, \\ \rho_1\omega_{tt} - \kappa_0(\omega_x - l\varphi)_x + \kappa l(\varphi_x + \psi + l\omega) + a_3\omega_t(x, t-\tau) = 0 & \text{in } [0, L] \times [0, +\infty[. \end{cases} \tag{6.1.4}$$

System (6.1.4) is subjected to the following boundary conditions:

$$\begin{cases} \kappa(\varphi_x + \psi + l\omega)(L, t) = -\varphi_t(L, t) & \text{in } [0, +\infty[, \\ b\psi_x(L, t) = -\psi_t(L, t) & \text{in } [0, +\infty[, \\ \kappa_0(\omega_x - l\varphi)(L, t) = -\omega_t(L, t) & \text{in } [0, +\infty[, \\ \varphi(0, t) = \psi(0, t) = \omega(0, t) = 0 & \text{in } [0, +\infty[, \end{cases} \tag{6.1.5}$$

where $(x, t) \in (0, L) \times (0, +\infty)$ and $L > 0$ and the parameters $a_1, a_2, a_3, \alpha, \mu$ and γ are positive constants. The system is completed with the following initial conditions:

$$\begin{cases} \varphi(x, 0) = \varphi_0(x), \quad \varphi_t(x, 0) = \varphi_1(x), & \text{in} \quad [0, L], \\ \psi(x, 0) = \psi_0(x), \quad \psi_t(x, 0) = \psi_1(x), & \text{in} \quad [0, L] \\ \omega(x, 0) = \omega_0(x), \quad \omega_t(x, 0 = \omega_1(x) & \text{in} \quad [0, L] \\ \varphi_t(x, t-\tau) = f_1(x, t-\tau) & \text{in } [0, L] \times [0, \tau], \\ \psi_t(x, t-\tau) = f_2(x, t-\tau) & \text{in } [0, L] \times [0, \tau], \\ \omega_t(x, t-\tau) = f_3(x, t-\tau) & \text{in } [0, L] \times [0, \tau], \end{cases} \tag{6.1.6}$$

where $\tau > 0$ is the time delay. The initial data $(\varphi_0, \varphi_1, \psi_0, \psi_1, \omega_0, \omega_1, f_1, f_2, f_3)$ belong to a suitable Sobolev space. By ω, ψ and φ we are denoting the longitudinal, vertical and shear angle displacements.

Concerning the boundary stabilization of the Timoshenko system with delays, we would like to mention the contribution of Said-Houari and Soufyane, (see [**75**]) in which the authors proved the global well-posedness and exponential decay of energy by assuming the weights of the delay are small enough. For more results concerning Timoshenko system with delay, one can refer to the previous studies [**68**]-[**74**] and so on.

Comparing our result with the work of Feng, (see [**77**]) he studied for laminated Timoshenko beams with time delays and boundary feedbacks, he has proved the global well-posedness and exponential decay of energy by assuming the weights of the delay are small enough.

The main objectives of the present chapter are to establish the global well-posedness and exponential stability of the problem $(6.1.4)-(6.1.6)$.

Our purpose in this paper is to give a global solvability in Sobolev spaces and energy decay estimates of the solutions to the problem $(6.1.4)-(6.1.6)$ for linear damping and delay terms. To obtain global solutions to the problem $(6.1.4)-(6.1.6)$, we use the argument combining the semigroup theory (see [**36**] and [**72**]) with the energy estimate method. To prove decay estimates, we use a multiplier method.

6.2 Well-posedness of the problem

In this section, we prove the global existence and the uniqueness of the solution of system (6.1.4)-(6.1.6). For this purpose, we adopt the technique of [**36**] (see also [**76**]) to prove that the operator $\mathcal{A}$ defined in (6.2) generates a contraction semigroup on the Hilbert space $\mathcal{H}$ given by (6.2).

So, let us introduce the following new variables:

$$\begin{array}{rcll} z_1(x,\rho,t) &=& \varphi_t(x,t-\tau\rho), & x\in[0,L],\ \rho\in[0,1], \quad t>0,\\ z_2(x,\rho,t) &=& \psi_t(x,t-\tau\rho), & x\in[0,L],\ \rho\in[0,1], \quad t>0,\\ z_3(x,\rho,t) &=& \omega_t(x,t-\tau\rho), & x\in[0,L],\ \rho\in[0,1], \quad t>0. \end{array}$$

Then, it is easy to check that

$$\tau z_{it}(x,\rho,t)+z_{i\rho}(x,\rho,t)=0, \quad \text{in } [0,L]\times[0,1]\times[0,+\infty] \quad \text{for } i=1,2,3.$$

Therefore, our problem $(6.1.4)-(6.1.6)$ is equivalent to:

$$\begin{cases} \rho_1\varphi_{tt}(x,t)-K(\varphi_x+\psi+l\omega)_x(x,t)-K_0 l(\omega_x-l\varphi)(x,t)+a_1z_1(x,1,t)=0,\\ \tau z_{1t}(x,\rho,t)+z_{1\rho}(x,\rho,t)=0,\\ \rho_2\psi_{tt}(x,t)-b\psi_{xx}(x,t)+K(\varphi_x+\psi+l\omega)(x,t)+a_2z_2(x,1,t)=0,\\ \tau z_{2t}(x,\rho,t)+z_{2\rho}(x,\rho,t)=0,\\ \rho_1\omega_{tt}(x,t)-K_0(\omega_x-l\varphi)_x(x,t)+Kl(\varphi_x+\psi+l\omega)(x,t)+a_3z_3(x,1,t)=0,\\ \tau z_{3t}(x,\rho,t)+z_{3\rho}(x,\rho,t)=0. \end{cases} \tag{6.2.1}$$

Now, we present a short discussion of the well-posedness, and semigroup formulation of the initial boundary value problem (6.2.1), (6.1.5) and (6.1.6). For this purpose, let $U=$

$(\varphi, \varphi_t, z_1, \psi, \psi_t, z_2, \omega, \omega_t, z_3)^T$, then U satisfies the problem

$$\begin{cases} U' = \mathcal{A}U, \\ U(0) = \big(\varphi_0, \varphi_1, f_1(., -.\tau), \psi_0, \psi_1, f_2(., -.\tau), \omega_0, \omega_1, f_3(., -.\tau)\big)^T, \end{cases}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\begin{pmatrix} \varphi \\ u \\ z_1 \\ \psi \\ v \\ z_2 \\ \omega \\ \tilde{\omega} \\ z_3 \end{pmatrix} = \begin{pmatrix} u \\ \dfrac{K}{\rho_1}(\varphi_x + \psi + l\omega)_x + \dfrac{lK_0}{\rho_1}(\omega_x - l\varphi) - \dfrac{a_2}{\rho_1} z_1(.,1) \\ -\tau^{-1} z_{1\rho} \\ v \\ \dfrac{b}{\rho_2}\psi_{xx} - \dfrac{K}{\rho_2}(\varphi_x + \psi + l\omega) - \dfrac{a_2}{\rho_2} z_2(.,1) \\ -\tau^{-1} z_{2\rho} \\ \tilde{\omega} \\ \dfrac{K_0}{\rho_1}(\omega_x - l\varphi)_x - \dfrac{lK}{\rho_1}(\varphi_x + \psi + l\omega) - \dfrac{a_3}{\rho_1} z_3(.,1) \\ -\tau^{-1} z_{3\rho} \end{pmatrix}$$

with domain

$$D(\mathcal{A}) = \left\{ \begin{array}{c} (\varphi, u, z_1, \psi, v, z_2, \omega, \tilde{\omega}, z_3)^T \ \in H, \\ u = z_1(.,0), v = z_2(.,0), \tilde{\omega} = z_3(.,0), \text{ in } (0,L), \\ K(\varphi_x + \psi + l\omega)(L) = -\alpha u(L), \quad b\psi_x(L) = -\mu v(L), \\ K_0(\omega_x - l\varphi)(L) = -\gamma \tilde{\omega}(L) \end{array} \right\},$$

where

$$H = (H^2(0,L) \cap H^1_*(0,L) \times H^1_*(0,L) \times (H^2(0,L) \cap H^1_*(0,L) \times H^1_*(0,L) \\ \times L^2(0,1,H^1(0,L)) \times L^2(0,1,H^1(0,L)),$$

and

$$H^1_*(0,L) = \Big\{ f \in H^1(0,L) : f(0) = 0 \Big\}.$$

Now, the energy space $\mathcal{H}$ is defined as follows:

$$\mathcal{H} := H^1_*(0,L) \times L^2(0,L) \times H^1_*(0,L) \times L^2(0,L) \times L^2((0,L) \times (0,1)) \times L^2((0,L) \times (0,1)).$$

For $U = (\varphi, u, z_1, \psi, v, z_2, \omega, \tilde{\omega}, z_3)^T, \overline{U} = (\overline{\varphi}, \overline{u}, \overline{z}_1, \overline{\psi}, \overline{v}, \overline{z}_2, \overline{\omega}, \overline{\tilde{\omega}}, \overline{z}_3)^T$ and for ξ_i positive constants, we define the inner product in $\mathcal{H}$ as follows:

$$\langle U, \overline{U} \rangle_{\mathcal{H}} = \int_0^L \Big[\rho_1 u\overline{u} + \rho_2 v\overline{v} + \rho_1 \tilde{\omega}\overline{\tilde{\omega}} + b\psi_x \overline{\psi}_x + K(\varphi_x + \psi + l\omega)(\overline{\varphi}_x + \overline{\psi} + l\overline{\omega}) \\ + K_0(\omega_x - l\varphi)(\overline{\omega}_x - l\overline{\varphi}) + \sum_{i=1}^{3} \xi_i \int_0^1 z_i(x,\rho)\overline{z}_i(x,\rho)\, d\rho \Big] dx.$$

The existence and uniqueness results read as follows.

Theorem 6.2.1. *For any $U_0 \in \mathcal{H}$, there exists a unique solution $U \in C([0,+\infty),\mathcal{H})$ of problem (6.2). Moreover, if $U_0 \in D(\mathcal{A})$, then*

$$U \in C([0,+\infty); D(\mathcal{A})) \cap C^1([0,+\infty); \mathcal{H}).$$

Proof. In order to prove the result stated in Theorem 6.2.1, we will use the semigroup approach. That is, we will show that the operator $\mathcal{A}$ generates a C_0-semigroup in $\mathcal{H}$. In this step, we concern ourselves to prove that the operator $\mathcal{A}$ is dissipative. Indeed, for $U = (\varphi, u, z_1, \psi, v, z_2, \omega, \tilde{\omega}, z_3)^T$, we have

$$\begin{aligned}\langle \mathcal{A}U, U\rangle_{\mathcal{H}} =& -\alpha u^2(L) - \mu v^2(L) - \gamma\tilde{\omega}^2(L) - a_1\int_0^L z_1(x,1)u\,dx - a_2\int_0^L z_2(x,1)v\,dx \\ & - a_3\int_0^L z_3(x,1)\tilde{\omega}dx - \sum_{i=1}^3 \frac{\xi_i}{\tau}\int_0^L\int_0^1 z_i(x,\rho)z_{i\rho}(x,\rho)\,d\rho\,dx.\end{aligned} \tag{6.2.2}$$

Looking now at the last two terms of the right-hand side of (6.2.2), we have

$$\begin{aligned}\sum_{i=1}^3 \xi_i\int_0^L\int_0^1 z_i(x,\rho)z_{i\rho}(x,\rho)\,d\rho\,dx &= \sum_{i=1}^3 \xi_i\int_0^L\int_0^1 \frac{1}{2}\frac{\partial}{\partial\rho}z_i^2(x,\rho)\,d\rho\,dx \\ &= \sum_{i=1}^3 \frac{\xi_i}{2}\int_0^L \Big[z_i^2(x,1) - z_i^2(x,0)\Big]\,dx.\end{aligned} \tag{6.2.3}$$

Consequently, (6.2.3) becomes

$$\begin{aligned}\langle \mathcal{A}U, U\rangle_{\mathcal{H}} =& -\alpha u^2(L) - \mu v^2(L) - \gamma\tilde{\omega}^2(L) - a_1\int_0^L z_1(x,1)u\,dx - a_2\int_0^L z_2(x,1)v\,dx \\ & - a_3\int_0^L z_3(x,1)\tilde{\omega}\,dx - \sum_{i=1}^3 \frac{\xi_i}{2\tau}\int_0^L \Big[z_i^2(x,1) - z_i^2(x,0)\Big]\,dx.\end{aligned} \tag{6.2.4}$$

By using Young's inequality, we obtain from (6.2.4) that

$$\begin{aligned}\langle \mathcal{A}U, U\rangle_{\mathcal{H}} \leq& -\alpha u^2(L) - \mu v^2(L) - \gamma\tilde{\omega}^2(L) - \frac{\xi_1}{4\tau}\int_0^L z_1^2(x,1)\,dx \\ & + \Big(\frac{a_1^2\tau}{\xi_1} + \frac{\xi_1}{2\tau}\Big)\int_0^L u^2\,dx - \frac{\xi_2}{4\tau}\int_0^L z_2^2(x,1)\,dx + \Big(\frac{a_2^2\tau}{\xi_2} + \frac{\xi_2}{2\tau}\Big)\int_0^L v^2\,dx \\ & - \frac{\xi_3}{4\tau}\int_0^L z_3^2(x,1)\,dx + \Big(\frac{a_3^2\tau}{\xi_3} + \frac{\xi_3}{2\tau}\Big)\int_0^L \tilde{\omega}^2\,dx \\ \leq& \max\left(\frac{1}{\rho_1}\Big(\frac{a_1^2\tau}{\xi_1} + \frac{\xi_1}{2\tau}\Big), \frac{1}{\rho_2}\Big(\frac{a_2^2\tau}{\xi_2} + \frac{\xi_2}{2\tau}\Big), \frac{1}{\rho_1}\Big(\frac{a_3^2\tau}{\xi_3} + \frac{\xi_3}{2\tau}\Big)\right)\langle U, U\rangle_{\mathcal{H}} \\ =& c_1\langle U, U\rangle_{\mathcal{H}}.\end{aligned}$$

Consequently, the operator $\mathcal{A} - c_1 I$ is dissipative. To show that $\mathcal{A}$ is maximal monotone, it is sufficient to show that the operator $\lambda I - \mathcal{A}$ is surjective for fixed $\lambda > 0$. Indeed, given $(h_1,h_2,h_3,h_4,h_5,h_6,h_7,h_8,h_9)^T \in \mathcal{H}$, we seek $U=(\varphi,u,z_1,\psi,v,z_2,\omega,\tilde{\omega},z_3)^T \in D(\mathcal{A})$ solution of the following system of equations

$$\begin{cases} \lambda\varphi - u &= h_1, \\ \lambda u - \dfrac{K}{\rho_1}(\varphi_x+\psi+l\omega)_x - \dfrac{lK_0}{\rho_1}(\omega_x - l\varphi) + \dfrac{a_1}{\rho_1} z_1(.,1) &= h_2, \\ \lambda z_1 + \dfrac{1}{\tau} z_{1\rho} &= h_3, \\ \lambda\psi - v &= h_4, \\ \lambda v - \dfrac{b}{\rho_2}\psi_{xx} + \dfrac{K}{\rho_2}(\varphi_x+\psi+l\omega) + \dfrac{a_2}{\rho_2} z_2(.,1) &= h_5, \\ \lambda z_2 + \dfrac{1}{\tau} z_{2\rho} &= h_6, \\ \lambda\omega - \tilde{\omega} &= h_7, \\ \lambda\tilde{\omega} - \dfrac{K_0}{\rho_1}(\omega_x - l\varphi)_x + \dfrac{lK}{\rho_1}(\varphi_x+\psi+l\omega) + \dfrac{a_3}{\rho_1} z_3(.,1) &= h_8, \\ \lambda z_3 + \dfrac{1}{\tau} z_{3\rho} &= h_9. \end{cases} \tag{6.2.5}$$

Suppose that we have found (φ,ψ,ω) with the appropriate regularity, then

$$\begin{cases} u = \lambda\varphi - h_1, \\ v = \lambda\psi - h_4, \\ \tilde{\omega} = \lambda\omega - h_7. \end{cases} \tag{6.2.6}$$

It is clear that $u \in H^1_*(0,L), v \in H^1_*(0,L)$ and $\omega \in H^1_*(0,L)$. Furthermore, by (6.2.5), we can find $z_i (i=1,2,3)$ as

$$z_1(x,0) = u(x), \quad z_2(x,0) = v(x), \quad z_3(x,0) = \tilde{\omega}(x), \qquad \text{for } x \in (0,L). \tag{6.2.7}$$

Following the same approach as in [**36**], we obtain, by using equations for z_i in (6.2.5),

$$\begin{aligned} z_1(x,\rho) &= u(x)e^{-\lambda\tau\rho} + \tau_1 e^{-\lambda\tau\rho}\int_0^\rho h_3(x,s)e^{\lambda\tau s}\,ds, \\ z_2(x,\rho) &= v(x)e^{-\lambda\tau\rho} + \tau e^{-\lambda\tau\rho}\int_0^\rho h_6(x,s)e^{\lambda\tau s}\,ds, \\ z_3(x,\rho) &= \tilde{\omega}(x)e^{-\lambda\tau\rho} + \tau e^{-\lambda\tau\rho}\int_0^\rho h_9(x,s)e^{\lambda\tau s}\,ds. \end{aligned}$$

From (6.2.6), we obtain

$$\begin{cases} z_1(x,\rho) = \lambda\varphi(x)e^{-\lambda\tau\rho} - h_1 e^{-\lambda\tau\rho} + \tau e^{-\lambda\tau\rho}\displaystyle\int_0^\rho h_3(x,s)e^{\lambda\tau s}\,ds, \\ z_2(x,\rho) = \lambda\psi(x)e^{-\lambda\tau\rho} - h_4 e^{-\lambda\tau\rho} + \tau_2 e^{-\lambda\tau\rho}\displaystyle\int_0^\rho h_6(x,s)e^{\lambda\tau s}\,ds, \\ z_3(x,\rho) = \lambda\omega(x)e^{-\lambda\tau\rho} - h_7 e^{-\lambda\tau\rho} + \tau e^{-\lambda\tau\rho}\displaystyle\int_0^\rho h_9(x,s)e^{\lambda\tau s}\,ds. \end{cases} \tag{6.2.8}$$

By using (6.2.5) and (6.2.6) the functions φ, ψ and ω satisfy the following system

$$\begin{cases} \lambda^2\varphi - \dfrac{K}{\rho_1}(\varphi_x + \psi + l\omega)_x - \dfrac{lK_0}{\rho_1}(\omega_x - l\varphi) + \dfrac{a_1}{\rho_1}z_1(.,1) = h_2 + \lambda h_1, \\ \lambda^2\psi - \dfrac{b}{\rho_2}\psi_{xx} + \dfrac{K}{\rho_2}(\varphi_x + \psi + l\omega) + \dfrac{a_2}{\rho_2}z_2(.,1) \qquad\qquad = h_5 + \lambda h_4, \\ \lambda^2\omega - \dfrac{K_0}{\rho_1}(\omega_x - l\varphi)_x + \dfrac{lK}{\rho_1}(\varphi_x + \psi + l\omega) + \dfrac{a_3}{\rho_1}z_3(.,1) = h_8 + \lambda h_7. \end{cases} \tag{6.2.9}$$

Using the following

$$\begin{cases} z_1(x,1) &= u(x)e^{-\lambda\tau} + \tau e^{-\lambda\tau}\displaystyle\int_0^1 h_3(x,s)e^{\lambda\tau s}\,ds = \lambda\varphi e^{-\lambda\tau} + z_1^0(x), \\ z_2(x,1) &= v(x)\tau e^{-\lambda\tau} + \tau e^{-\lambda\tau}\displaystyle\int_0^1 h_6(x,s)e^{\lambda\tau s}\,ds = \lambda\psi e^{-\lambda\tau} + z_2^0(x), \\ z_3(x,1) &= \tilde{\omega}(x)e^{-\lambda\tau} + \tau e^{-\lambda\tau}\displaystyle\int_0^1 h_9(x,s)e^{\lambda\tau s}\,ds = \lambda\omega e^{-\lambda\tau} + z_3^0(x), \end{cases}$$

where for $x \in (0,L)$,

$$\begin{cases} z_1^0(x) &= -h_1(x)e^{-\lambda\tau} + \tau e^{-\lambda\tau}\displaystyle\int_0^1 h_3(x,s)e^{\lambda\tau_1 s}\,ds, \\ z_2^0(x) &= -h_4(x)e^{-\lambda\tau} + \tau e^{-\lambda\tau}\displaystyle\int_0^1 h_6(x,s)e^{\lambda\tau s}\,ds, \\ z_3^0(x) &= -h_7(x)e^{-\lambda\tau} + \tau e^{-\lambda\tau}\displaystyle\int_0^1 h_9(x,s)e^{\lambda\tau s}\,ds. \end{cases}$$

The problem (6.2.9) can be reformulated as

$$\begin{cases} \displaystyle\int_0^L \left[\lambda^2\varphi - \frac{K}{\rho_1}(\varphi_x + \psi + l\omega)_x - \frac{lK_0}{\rho_1}(\omega_x - l\varphi) + \frac{a_1}{\rho_1}\lambda\varphi e^{-\lambda\tau}\right]\omega_1\,dx \\ = \displaystyle\int_0^L \left[h_2 + \lambda h_1 - \frac{a_1}{\rho_1}z_1^0(x)\right]\omega_1\,dx, \qquad \forall\omega_1 \in H_*^1(0,L), \\ \displaystyle\int_0^L \left[\lambda^2\psi - \frac{b}{\rho_2}\psi_{xx} + \frac{K}{\rho_2}(\varphi_x + \psi + l\omega) + \frac{a_2}{\rho_2}\lambda\psi e^{-\lambda\tau}\right]\omega_2\,dx \\ = \displaystyle\int_0^L \left[h_5 + \lambda h_4 - \frac{a_2}{\rho_2}z_2^0(x)\right]\omega_2\,dx, \qquad \forall\omega_2 \in H_*^1(0,L), \\ \displaystyle\int_0^L \left[\lambda^2\omega - \frac{K_0}{\rho_1}(\omega_x - l\varphi)_x + \frac{lK}{\rho_1}(\varphi_x + \psi + l\omega) + \frac{a_3}{\rho_1}\lambda\omega e^{-\lambda\tau}\right]\omega_3\,dx \\ = \displaystyle\int_0^L \left[h_8 + \lambda h_7 - \frac{a_3}{\rho_1}z_3^0(x)\right]\omega_3\,dx, \qquad \forall\omega_3 \in H_*^1(0,L). \end{cases} \tag{6.2.10}$$

Integrating Eqs. $(6.2.10)_1$-$(6.2.10)_3$ by parts and then summing the resultants, we obtain the following problem which is equivalent to (6.2.10)

$$\phi((\varphi,\psi,\omega),(\omega_1,\omega_2,\omega_3)) = \mathcal{I}(\omega_1,\omega_2,\omega_3)), \tag{6.2.11}$$

where the bilinear form $\phi : [H^1_*(0,L)\times H^1_*(0,L)\times H^1_*(0,L)]^2 \to \mathbb{R}$ and the linear form $\mathcal{I} : H^1_*(0,L)\times H^1_*(0,L)\times H^1_*(0,L) \to \mathbb{R}$ are defined by

$$\begin{aligned}
\phi\big((\varphi,\psi,\omega),(\omega_1,\omega_2,\omega_3)\big) = &\int_0^L \left(\lambda^2+\frac{a_1}{\rho_1}\lambda e^{-\lambda\tau}\right)\varphi\omega_1\,dx + \int_0^L \frac{K}{\rho_1}(\varphi_x+\psi+l\omega)(\omega_1)_x dx\\
&-\frac{lK_0}{\rho_1}\int_0^L(\omega_x-l\varphi)\omega_1 dx + \frac{\alpha}{\rho_1}\lambda\varphi(L)\omega_1(L)\\
&+\int_0^L\left(\lambda^2+\frac{a_2}{\rho_2}\lambda e^{-\lambda\tau}\right)\psi\omega_2\,dx + \frac{b}{\rho_2}\int_0^L \psi_x(\omega_2)_x\,dx\\
&+\frac{K}{\rho_2}\int_0^L(\varphi_x+\psi+l\omega)\omega_2\,dx + \frac{\mu}{\rho_2}\lambda\psi(L)\omega_2(L)\\
&+\int_0^L\left(\lambda^2+\frac{a_3}{\rho_1}\lambda e^{-\lambda\tau}\right)\omega\omega_3\,dx + \int_0^L\frac{K_0}{\rho_1}(\omega_x-l\varphi)(\omega_3)_x\,dx\\
&+\frac{lK}{\rho_1}\int_0^L(\varphi_x+\psi+l\omega)\omega_3\,dx + \frac{\gamma}{\rho_1}\lambda\omega(L)\omega_3(L),
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{I}\big(\omega_1,\omega_2,\omega_3\big) = &\int_0^L\Big(h_2+\lambda h_1-\frac{a_1}{\rho_1}z_1^0(x)\Big).\omega_1\,dx + \int_0^L\Big(h_5+\lambda h_4-\frac{a_2}{\rho_2}z_2^0(x)\Big).\omega_2\,dx\\
&+\int_0^L\Big(h_8+\lambda h_7-\frac{a_3}{\rho_1}z_3^0(x)\Big)\omega_3\,dx + \frac{\alpha}{\rho_1}h_1(L)\omega_1(L)\\
&+\frac{\mu}{\rho_2}h_4(L)\omega_2(L) + \frac{\gamma}{\rho_1}h_7(L)\omega_3(L).
\end{aligned}$$

It is easy to verify that ϕ is continuous and coercive, and $\mathcal{I}$ is continuous. So applying the Lax-Milgram theorem, we deduce that for all $(\omega_1,\omega_2,\omega_3)\in H^1_*(0,L)\times H^1_*(0,L)\times H^1_*(0,L)$, problem (6.2.11) admits a unique solution $(\varphi,\psi,\omega)\in H^1_*(0,L)\times H^1_*(0,L)\times H^1_*(0,L)$. By the classical elliptic regularity, we deduce that $(\varphi,\psi,\omega)\in (H^2(0,L)\cap H^1_*(0,L))\times(H^2(0,L)\cap H^1_*(0,L))\times(H^2(0,L)\cap H^1_*(0,L))$. Therefore, the operator $\lambda I-\mathcal{A}$ is surjective for any $\lambda>0$. Hence, $-\mathcal{A}$ is maximal monotone operator. Thanks to Lummer-Phillips theorem, we conclude that the operator $\mathcal{A}$ generates a linear C_0-semigroup in $\mathcal{H}$ and so (4.2.1) is well-posed (see Pazy [**94**]). □

6.3 Asymptotic stability

In this section, we study the asymptotic behaviour of system (6.1.4)-(6.1.6). For any regular solution of (6.1.4)-(6.1.6), we define the energy by the following formula

$$\begin{aligned}
\mathcal{E}(t) =&\frac{1}{2}\int_0^L\Big[\rho_1|\varphi_t|^2+\rho_2|\psi_t|^2+\rho_1|\omega_t|^2+b|\psi_x|^2+K|\varphi_x+\psi+l\omega|^2+K_0|\omega_x-l\varphi|^2\Big]\,dx\\
&+\frac{\xi_1}{2}\int_{t-\tau}^t\int_0^L\varphi_t^2(x,s)\,dx\,ds+\frac{\xi_2}{2}\int_{t-\tau}^t\int_0^L\psi_t^2(x,s)\,dx\,ds+\frac{\xi_3}{2}\int_{t-\tau}^t\int_0^L\omega_t^2(x,s)\,dx\,ds,
\end{aligned} \tag{6.3.1}$$

where ξ_1, ξ_2 and ξ_3 are strictly positive numbers that will be chosen later. The main result of this section is:

Theorem 6.3.1. *Let (φ, ψ, ω) be a regular solution of (6.1.4)-(6.1.6). Assume that*

$$\frac{6KL^2}{\pi^2} \leq b, \quad l \leq \frac{1}{4L} \min\left(\sqrt{\frac{K}{K_0}}, \sqrt{\frac{K_0}{K}}\right),$$

and that it exist small enough positive constants a_i^0 satisfying $0 \leq a_i < a_i^0$, $i = 1, 2, 3$. Then,

$$\mathcal{E}(t) \leq C_1 e^{-C_2 t}, \qquad \forall\, t \geq 0, \tag{6.3.2}$$

while C_1 and C_2 are two fixed positive constants.

The proof of Theorem 6.3.1 will be done through some Lemmas.

Lemma 6.3.2. *For any regular solution of (6.1.4)-(6.1.6) the following estimate holds:*

$$\begin{aligned}
\mathcal{E}'(t) \leq & -\alpha\varphi_t^2(L,t) + \left(\frac{a_1+\xi_1}{2}\right)\int_0^L \varphi_t^2(x,t)\,dx + \left(\frac{a_1-\xi_1}{2}\right)\int_0^L \varphi_t^2(x,t-\tau)\,dx \\
& -\mu\psi_t^2(L,t) + \left(\frac{a_2+\xi_2}{2}\right)\int_0^L \psi_t^2(x,t)\,dx + \left(\frac{a_2-\xi_2}{2}\right)\int_0^L \psi_t^2(x,t-\tau)\,dx \\
& -\gamma\omega_t^2(L,t) + \left(\frac{a_3+\xi_3}{2}\right)\int_0^L \omega_t^2(x,t)\,dx + \left(\frac{a_3-\xi_3}{2}\right)\int_0^L \omega_t^2(x,t-\tau)\,dx.
\end{aligned} \tag{6.3.3}$$

Proof. Differentiating (6.3.1), we get

$$\begin{aligned}
\mathcal{E}'(t) = & \int_0^L \rho_1\varphi_t\varphi_{tt}dx + \int_0^L \rho_2\psi_t\psi_{tt}dx + \int_0^L \rho_1\omega_t\omega_{tt}dx + \int_0^L b\psi_x\psi_{xt}dx \\
& + \int_0^L K(\varphi_x+\psi+l\omega)(\varphi_x+\psi+l\omega)_t dx + \int_0^L K_0(\omega_x - l\varphi)(\omega_x - l\varphi)_t dx \\
& + \frac{\xi_1}{2}\int_0^L \varphi_t^2(x,t)\,dx - \frac{\xi_1}{2}\int_0^L \varphi_t^2(x,t-\tau)\,dx + \frac{\xi_2}{2}\int_0^L \psi_t^2(x,t)\,dx \\
& - \frac{\xi_2}{2}\int_0^L \psi_t^2(x,t-\tau)\,dx + \frac{\xi_3}{2}\int_0^L \omega_t^2(x,t)\,dx - \frac{\xi_3}{2}\int_0^L \omega_t^2(x,t-\tau)\,dx.
\end{aligned}$$

Now, using the equations in (6.1.4) and exploiting the boundary conditions in (6.1.5), we obtain

$$\begin{aligned}
\mathcal{E}'(t) \leq & -\alpha\varphi_t^2(L,t) + \frac{\xi_1}{2}\int_0^L \varphi_t^2(x,t)\,dx - a_1\int_0^L \varphi_t(x,t-\tau)\varphi_t(x,t)\,dx \\
& -\mu\psi_t^2(L,t) + \frac{\xi_2}{2}\int_0^L \psi_t^2(x,t)\,dx - a_2\int_0^L \psi_t(x,t-\tau)\psi_t(x,t)\,dx \\
& -\gamma\omega_t^2(L,t) + \frac{\xi_3}{2}\int_0^L \omega_t^2(x,t)\,dx - a_3\int_0^L \varphi_t(x,t-\tau)\varphi_t(x,t)\,dx \\
& - \frac{\xi_1}{2}\int_0^L \varphi_t^2(x,t-\tau)\,dx - \frac{\xi_2}{2}\int_0^L \psi_t^2(x,t-\tau)\,dx - \frac{\xi_3}{2}\int_0^L \omega_t^2(x,t-\tau)\,dx.
\end{aligned} \tag{6.3.4}$$

Applying Young's inequality to the first three terms in (6.3.4), then (6.3.3) holds true. This completes the proof of Lemma 6.3.2. □

Next, we define the following functional:

$$\mathcal{F}(t)=\int_0^L\Big(\rho_1 x\varphi_t(\varphi_x+l\omega)+\rho_2 x\psi_t\psi_x+\rho_1 x\omega_t(\omega_x-l\varphi)\Big)dx. \tag{6.3.5}$$

Then we have the following estimate:

Lemma 6.3.3. *Let (φ,ψ,ω) be the solution of (6.1.4)-(6.1.6). Then we have, for any ϵ_1,ϵ_2, ϵ_3, δ_1, β_1, β_2, $\beta_3>0$,*

$$\begin{aligned}\mathcal{F}'(t)\le&-\frac{\rho_1}{2}\int_0^L\varphi_t^2dx-\frac{\rho_2}{2}\int_0^L\psi_t^2dx-\frac{\rho_1}{2}\int_0^L\omega_t^2\,dx\\&+\left(a_1L\epsilon_1+\frac{L^2}{4\beta_1}\right)\int_0^L(\varphi_x+l\omega)^2\,dx\\&+\left(a_2L\epsilon_2+\frac{c^*}{4\beta_2}+\delta_1c_2-\frac{b}{2}\right)\int_0^L\psi_x^2\,dx\\&+\left(\beta_2K^2+\beta_3K^2l^2-\frac{K}{2}\right)\int_0^L(\varphi_x+\psi+l\omega)^2\,dx\\&+\left(a_3L\epsilon_3+\frac{L^2}{4\beta_3}+\beta_1K_0^2l^2-\frac{K_0}{2}\right)\int_0^L(\omega_x-l\varphi)^2\,dx\\&+\left(\frac{\rho_2L}{2}+\frac{\mu^2L}{2b}\right)\psi_t^2(L,t)+\left(\frac{\rho_2L}{2}+\frac{\alpha^2L}{2K}+\frac{\alpha^2L^2}{4\delta_1}\right)\varphi_t^2(L,t)\\&+\left(\frac{\rho_1L}{2}+\frac{\gamma^2L}{2K_0}\right)\omega_t^2(L,t)+\frac{a_1L}{4\epsilon_1}\int_0^L\varphi_t^2(x,t-\tau)\,dx\\&+\frac{a_2L}{4\epsilon_2}\int_0^L\psi_t^2(x,t-\tau)\,dx+\frac{a_3L}{4\epsilon_3}\int_0^L\omega_t^2(x,t-\tau)\,dx,\end{aligned}\tag{6.3.6}$$

where $c^*=L^2/\pi^2$ is the Poincaré constant.

Proof. Differentiating the functional $\mathcal{F}$ with respect to t and using (6.1.4), we find

$$\begin{aligned}\mathcal{F}'(t)=&\int_0^LKx(\varphi_x+\psi+l\omega)_x(\varphi_x+l\omega)dx+\int_0^LK_0lx(\omega_x-l\varphi)(\varphi_x+l\omega)dx\\&+\int_0^Lbx\psi_{xx}\psi_xdx-\int_0^LKx(\varphi_x+\psi+l\omega)\psi_xdx+\int_0^LK_0x(\omega_x-l\varphi)_x(\omega_x-l\varphi)dx\\&-\int_0^LKlx(\varphi_x+\psi+l\omega)(\omega_x-l\varphi)dx+\int_0^L\rho_1\frac{x}{2}\frac{d\varphi_t^2}{dx}dx+\int_0^L\rho_2\frac{x}{2}\frac{d\psi_t^2}{dx}dx\\&+\int_0^L\rho_1\frac{x}{2}\frac{d\omega_t^2}{dx}dx-a_1\int_0^Lx(\varphi_x+l\omega)\varphi_t(x,t-\tau)\,dx\\&-a_2\int_0^Lx\psi_x\psi_t(x,t-\tau)\,dx-a_3\int_0^Lx(\omega_x-l\varphi)\omega_t(x,t-\tau)\,dx.\end{aligned}\tag{6.3.7}$$

Remark that

$$
\begin{aligned}
-\int_0^L Kx(\varphi_x+\psi+l\omega)\psi_x\,dx = &-KL(\varphi_x+\psi+l\omega)(L,t)\psi(L,t)\\
&+\int_0^L K(\varphi_x+\psi+l\omega)\psi\,dx\\
&+\int_0^L Kx(\varphi_x+\psi+l\omega)_x\psi\,dx.
\end{aligned}
\tag{6.3.8}
$$

Substituting (6.3.8) into (6.3.7) and using integration by parts, we obtain that

$$
\begin{aligned}
\mathcal{F}'(t) = &-\int_0^L \frac{\rho_1}{2}\varphi_t^2dx-\int_0^L \frac{\rho_2}{2}\psi_t^2dx-\int_0^L \frac{\rho_1}{2}\omega_t^2dx-\int_0^L \frac{K}{2}(\varphi_x+\psi+l\omega)^2dx\\
&-\int_0^L \frac{b}{2}\psi_x^2dx-\int_0^L \frac{K_0}{2}(\omega_x-l\varphi)^2dx+\frac{bL}{2}\psi_x^2(L,t)+\frac{KL}{2}(\varphi_x+\psi+l\omega)^2(L,t)\\
&+\frac{K_0L}{2}(\omega_x-l\varphi)^2(L,t)+\frac{\rho_1L}{2}\varphi_t^2(L,t)+\frac{\rho_2L}{2}\psi_t^2(L,t)+\frac{\rho_1L}{2}\omega_t^2(L,t)\\
&-KL(\varphi_x+\psi+l\omega)(L,t)\psi(L,t)+\int_0^L K(\varphi_x+\psi+l\omega)\psi\,dx\\
&+K_0l\int_0^L x(\omega_x-l\varphi)(\varphi_x+l\omega)dx-Kl\int_0^L x(\varphi_x+\psi+l\omega)(\omega_x-l\varphi)dx\\
&-a_1\int_0^L x(\varphi_x+l\omega)\varphi_t(x,t-\tau)\,dx-a_2\int_0^L x\psi_x(x,t)\psi_t(x,t-\tau)\,dx\\
&-a_3\int_0^L x(\omega_x-l\varphi)\omega_t(x,t-\tau)\,dx.
\end{aligned}
\tag{6.3.9}
$$

Using then the boundary conditions (6.1.5), we write

$$
\frac{bL}{2}\psi_x^2(L,t)=\frac{\mu^2L}{2b}\psi_t^2(L,t). \tag{6.3.10}
$$

Similarly, we get

$$
\frac{KL}{2}(\varphi_x+\psi+l\omega)^2(L,t)=\frac{\alpha^2L}{2K}\varphi_t^2(L,t), \tag{6.3.11}
$$

$$
\frac{K_0L}{2}(\omega_x-l\varphi)^2(L,t)=\frac{\gamma^2L}{2K_0}\omega_t^2(L,t). \tag{6.3.12}
$$

By the imbedding of $W^{1,1}(0,L)$ in $L^\infty(0,L)$, one has

$$
|\psi(L,t)|^2\le c_1\int_0^L(\psi^2+\psi_x^2)\,dx,
$$

which implies by Poincaré's inequality

$$
|\psi(L,t)|^2\le c_2\int_0^L \psi_x^2\,dx, \tag{6.3.13}
$$

where c_1 and c_2 are two fixed positive constants.

Making use of (6.1.5), Young's inequality and (6.3.13) we obtain for all $\delta_1 > 0$ that

$$\begin{aligned} -KL(\varphi_x+\psi+l\omega)(L,t)\psi(L,t) &= \alpha L\varphi_t(L,t)\psi(L,t) \\ &\leq \delta_1 c_2 \int_0^L \psi_x^2\,dx + \frac{\alpha^2L^2}{4\delta_1}\varphi_t^2(L,t). \end{aligned} \tag{6.3.14}$$

Once again, by Young's and Poincaré's inequalities we can get, for any $\beta_1, \beta_2, \beta_3 > 0$

$$K_0 l \int_0^L x(\omega_x - l\varphi)(\varphi_x + l\omega)dx \leq \beta_1 K_0^2 l^2 \int_0^L (\omega_x - l\varphi)^2\,dx + \frac{L^2}{4\beta_1}\int_0^L (\varphi_x + l\omega)^2\,dx, \tag{6.3.15}$$

$$K\int_0^L (\varphi_x+\psi+l\omega)\psi\,dx \leq \beta_2 K^2 \int_0^L (\varphi_x+\psi+l\omega)^2\,dx + \frac{c^*}{4\beta_2}\int_0^L \psi_x^2\,dx, \tag{6.3.16}$$

$$-Kl\int_0^L x(\varphi_x+\psi+l\omega)(\omega_x-l\varphi)\,dx \leq \beta_3 K^2 l^2 \int_0^L (\varphi_x+\psi+l\omega)^2\,dx + \frac{L^2}{4\beta_3}\int_0^L (\omega_x - l\varphi)^2\,dx, \tag{6.3.17}$$

where $c^* = L^2/\pi^2$ is the Poincaré constant.

On the other hand, for all $\epsilon_1, \epsilon_2, \epsilon_3 > 0$, using Young's inequality then the last two terms in the right-hand side of (6.3.9) can be estimated as follows:

$$\left| a_1 \int_0^L x(\varphi_x + l\omega)\varphi_t(x,t-\tau)\,dx \right| \leq a_1 L\epsilon_1 \int_0^L (\varphi_x + l\omega)^2\,dx + \frac{a_1 L}{4\epsilon_1}\int_0^L \varphi_t^2(x,t-\tau)\,dx, \tag{6.3.18}$$

$$\left| a_2 \int_0^L x\psi_x(x,t)\psi_t(x,t-\tau)\,dx \right| \leq a_2 L\epsilon_2 \int_0^L \psi_x^2(x,t)\,dx + \frac{a_2 L}{4\epsilon_2}\int_0^L \psi_t^2(x,t-\tau)\,dx \tag{6.3.19}$$

and

$$\left| a_3 \int_0^L x(\omega_x - l\varphi)\omega_t(x,t-\tau)\,dx \right| \leq a_3 L\epsilon_3 \int_0^L (\omega_x - l\varphi)^2\,dx + \frac{a_3 L}{4\epsilon_3}\int_0^L \omega_t^2(x,t-\tau)\,dx \tag{6.3.20}$$

Inserting (6.3.10)-(6.3.20) into (6.3.9), we get (6.3.6). Thus, the proof of Lemma 6.3.3 is completed. □

Next, let us introduce

$$\mathcal{F}_1(t) =: \int_0^L \int_{t-\tau}^t e^{s-t}\varphi_t^2(x,s)\,ds\,dx, \tag{6.3.21}$$

$$\mathcal{F}_2(t) =: \int_0^L \int_{t-\tau}^t e^{s-t}\psi_t^2(x,s)\,ds\,dx \tag{6.3.22}$$

and

$$\mathcal{F}_3(t) =: \int_0^L \int_{t-\tau}^t e^{s-t}\omega_t^2(x,s)\,ds\,dx. \tag{6.3.23}$$

Then, the following estimates hold.

Lemma 6.3.4. *Let (φ, ψ, ω) be the solution of (6.1.4)-(6.1.6), Then we have*

$$\mathcal{F}_1'(t) \le \int_0^L \varphi_t^2\, dx - e^{-\tau} \int_0^L \varphi_t^2(x, t-\tau)\, dx - e^{-\tau} \int_0^L \int_{t-\tau}^t \varphi_t^2(x,s)\, ds\, dx, \tag{6.3.24}$$

$$\mathcal{F}_2'(t) \le \int_0^L \psi_t^2\, dx - e^{-\tau} \int_0^L \psi_t^2(x, t-\tau)\, dx - e^{-\tau} \int_0^L \int_{t-\tau}^t \psi_t^2(x,s)\, ds\, dx \tag{6.3.25}$$

and

$$\mathcal{F}_3'(t) \le \int_0^L \omega_t^2\, dx - e^{-\tau} \int_0^L \omega_t^2(x, t-\tau)\, dx - e^{-\tau} \int_0^L \int_{t-\tau}^t \omega_t^2(x,s)\, ds\, dx. \tag{6.3.26}$$

Proof. Taking the derivative of $\mathcal{F}_1$ with respect to t, we have

$$\mathcal{F}_1'(t) = \int_0^L \varphi_t^2(x,t)\, dx - e^{-\tau} \int_0^L \varphi_t^2(x, t-\tau)\, dx - \int_0^L \int_{t-\tau}^t e^{s-t} \varphi_t^2(x,s)\, ds\, dx. \tag{6.3.27}$$

Then, (6.3.24) easily holds, and similarly (6.3.25) and (6.3.26). □

To prove Theorem 6.3.1, we define the Lyapunov functionnal functional $\mathcal{L}(t)$ as follows:

$$\mathcal{L}(t) := \mathcal{E}(t) + N\mathcal{F}(t) + N_1\mathcal{F}_1(t) + N_2\mathcal{F}_2(t) + N_3\mathcal{F}_3(t) \tag{6.3.28}$$

where N, N_1, N_2 and N_3 are positive real numbers that be chosen later. Now, from (6.3.3), (6.3.6), (6.3.24), (6.3.25) and (6.3.26) and using the trivial inequality

$$\int_0^L (\varphi_x + l\omega)^2\, dx \le 2\int_0^L (\varphi_x + \psi + l\omega)^2\, dx + 2c^* \int_0^L \psi_x^2\, dx,$$

we get

$$\begin{aligned}
\mathcal{L}'(t) \le & - A_1\varphi_t^2(L,t) - A_2\psi_t^2(L,t) - A_3\omega_t^2(L,t) + \left[\left(\frac{a_1 + \xi_1}{2} \right) + N_1 - \frac{N\rho_1}{2} \right] \int_0^L \varphi_t^2\, dx \\
& + \left[\left(\frac{a_2 + \xi_2}{2} \right) + N_2 - \frac{N\rho_2}{2} \right] \int_0^L \psi_t^2\, dx + \left[\left(\frac{a_3 + \xi_3}{2} \right) + N_3 - \frac{N\rho_1}{2} \right] \int_0^L \omega_t^2\, dx \\
& + \left[\left(\frac{a_1 - \xi_1}{2} \right) + N\frac{a_1 L}{4\epsilon_1} - N_1 e^{-\tau} \right] \int_0^L \varphi_t^2(x, t-\tau)\, dx \\
& + \left[\left(\frac{a_2 - \xi_2}{2} \right) + N\frac{a_2 L}{4\epsilon_2} - N_2 e^{-\tau} \right] \int_0^L \psi_t^2(x, t-\tau)\, dx \\
& + \left[\left(\frac{a_3 - \xi_3}{2} \right) + N\frac{a_3 L}{4\epsilon_3} - N_3 e^{-\tau} \right] \int_0^L \omega_t^2(x, t-\tau)\, dx \\
& + N\left(2a_1 L\epsilon_1 + \frac{L^2}{2\beta_1} + \beta_2 K^2 + \beta_3 K^2 l^2 - \frac{K}{2} \right) \int_0^L (\varphi_x + \psi + l\omega)^2\, dx \\
& + N\left(2a_1 L\epsilon_1 c^* + \frac{c^* L^2}{2\beta_1} + a_2 L\epsilon_2 + \frac{c^*}{4\beta_2} + \delta_1 c_2 - \frac{b}{2} \right) \int_0^L \psi_x^2\, dx \\
& + N\left(a_3 L\epsilon_3 + \frac{L^2}{4\beta_3} + \beta_1 K_0^2 l^2 - \frac{K_0}{2} \right) \int_0^L (\omega_x - l\varphi)^2\, dx \\
& - \eta_0 \int_0^L \int_{t-\tau}^t \left[\varphi_t^2(x,s) + \psi_t^2(x,s) + \omega_t^2(x,s) \right] ds\, dx,
\end{aligned} \tag{6.3.29}$$

where $\eta_0 = e^{-\tau}\min\big(N_1, N_2, N_3\big)$ and

$$A_1 = \alpha - N\bigg(\frac{\rho_2 L}{2} + \frac{\alpha^2 L}{2K} + \frac{\alpha^2 L^2}{4\delta_1}\bigg),$$
$$A_2 = \mu - N\bigg(\frac{\rho_2 L}{2} + \frac{\mu^2 L}{2b}\bigg),$$
$$A_3 = \gamma - N\bigg(\frac{\rho_1 L}{2} + \frac{\gamma^2 L}{2K_0}\bigg).$$

At this point, we have to select our constants very carefully in order to get

$$\mathcal{L}'(t) \leq -\eta E(t) \quad \forall t \geq 0,$$

where η is a fixed postive constant. First, it is clear that for any $\alpha > 0$, $\mu > 0$ and $\gamma > 0$, and for N sufficiently small, we get $A_i \geq 0, \quad i = 1, 2, 3$.

Second, we may choose β_1, β_2 and β_3 such that

$$\frac{L^2}{2\beta_1} + \beta_2 K^2 + \beta_3 K^2 l^2 - \frac{K}{2} \leq -\frac{K}{4},$$
$$\frac{c^* L^2}{2\beta_1} + \frac{c^*}{4\beta_2} - \frac{b}{2} \leq -\frac{b}{8},$$
$$\frac{L^2}{4\beta_3} + \beta_1 K_0^2 l^2 - \frac{K_0}{2} \leq -\frac{K_0}{8}.$$

Letting

$$\beta_1 = \frac{1}{8K_0 l^2}, \qquad \beta_2 = \frac{1}{8K} \quad \text{and} \quad \beta_3 = \frac{1}{8Kl^2}.$$

So, we need that

$$L^2 l^2 K_0 \leq \frac{K}{16},$$
$$4c^* K_0 l^2 L^2 + 2c^* K \leq \frac{3b}{8}, \tag{6.3.30}$$
$$L^2 l^2 K \leq \frac{K_0}{16}.$$

Of course, in order to obtain (6.3.30), we have to assume that

$$6Kc^* \leq b,$$
$$l \leq \frac{1}{4L}\min\Big(\sqrt{\frac{K}{K_0}}, \sqrt{\frac{K_0}{K}}\Big).$$

As long as β_i, $i = 1, 2, 3$ are fixed, we pick $\delta_1, \epsilon_1, \epsilon_2 > 0$ and $\epsilon_3 > 0$ so small such that

$$2a_1 L\epsilon_1 \leq \frac{K}{8},$$
$$2a_1 L\epsilon_1 c^* + a_2 L\epsilon_2 + \delta_1 c_2 \leq \frac{b}{16},$$
$$2a_3 L\epsilon_3 \leq \frac{K_0}{16}.$$

After that, we fix N_1, N_2 and N_3 such that $\frac{N\rho_1}{2} - N_1 > 0$, $\frac{N\rho_2}{2} - N_2 > 0$ and $\frac{N\rho_3}{2} - N_3 > 0$. Now the main goal is to choose the sets of pairs (a_1, ξ_1), (a_2, ξ_2) and (a_3, ξ_3) such that

$$\begin{cases} \dfrac{a_1 + \xi_1}{2} < \dfrac{N\rho_1}{2} - N_1, \\ a_1\Big(\dfrac{1}{2} + \dfrac{NL}{4\epsilon_1}\Big) - \dfrac{\xi_1}{2} \le N_1 e^{-\tau}, \end{cases}$$

$$\begin{cases} \dfrac{a_2 + \xi_2}{2} < \dfrac{N\rho_2}{2} - N_2, \\ a_2\Big(\dfrac{1}{2} + \dfrac{NL}{4\epsilon_2}\Big) - \dfrac{\xi_2}{2} \le N_2 e^{-\tau}, \end{cases}$$

and

$$\begin{cases} \dfrac{a_3 + \xi_3}{2} < \dfrac{N\rho_1}{2} - N_3, \\ a_3\Big(\dfrac{1}{2} + \dfrac{NL}{4\epsilon_3}\Big) - \dfrac{\xi_3}{2} \le N_3 e^{-\tau}. \end{cases}$$

Clearly, for $a_i,\ i = 1, 2, 3$, small enough satisfies

$$a_i < a_i^0 = \min\left\{\frac{N_i e^{-\tau} + (N\rho_i/2 - N_i)}{1 + NL/(4\epsilon_i)}, (N\rho_i - 2N_i)\right\}, \quad i = 1, 2, 3$$

it exists $\xi_i,\ i = 1, 2, 3$ such that

$$a_i\left(1 + \frac{NL}{2\epsilon_i}\right) - 2N_i e^{-\tau} \le \xi_i < (N\rho_i - 2N_i) - a_i,\ i = 1, 2, 3.$$

From this we can infer that $A_i \geq 0$ if $(\alpha, \mu, \gamma) \longrightarrow (0, 0, 0)$ or if $(\alpha, \mu, \gamma) \longrightarrow (\infty, \infty, \infty)$, then $(N, N_i) \longrightarrow (0, 0)$ and consequently a_i^0 goes to zero.

Then, from above, we conclude that it exists a positive constant $\eta > 0$ such that (6.3.29) becomes

$$\begin{aligned} \mathcal{L}'(t) \le &- \eta \int_0^L \left[\psi_t^2 + \varphi_t^2 + \omega_t^2 + (\varphi_x + \psi + l\omega)^2 + \psi_x^2 + (\omega_x - l\varphi)^2\right] dx \\ &- \eta_0 \int_0^L \int_{t-\tau}^t \Big(\varphi_t^2(x, s) + \psi_t^2(x, s) + \omega_t^2(x, s)\Big)\, ds dx, \qquad \forall t \geq 0, \end{aligned}$$

which implies by (6.3.1), that it exists also η_1, such that

$$\mathcal{L}'(t) \le -\eta_1 \mathcal{E}(t), \quad \forall t \geq 0. \tag{6.3.31}$$

On the other hand, from (6.3.1), (6.3.5), (6.3.21), (6.3.22), (6.3.23), (6.3.28) and for N sufficiently small, we deduce that there exist two positive constants λ_1 and λ_2 depending on N, N_1, N_2, N_3 and L such that

$$\lambda_1 \mathcal{E}(t) \le \mathcal{L}(t) \le \lambda_2 \mathcal{E}(t), \quad \forall t \geq 0. \tag{6.3.32}$$

Now, combining (6.3.31) and (6.3.32), there exists $\Lambda > 0$, such that

$$\frac{d\mathcal{L}(t)}{dt} \le -\Lambda \mathcal{L}(t), \quad \forall t \geq 0. \tag{6.3.33}$$

Consequently, integrating (6.3.33) and using once again (6.3.32), we obtain (6.3.2). This ends the proof of Theorem 6.3.1.

Conlusion and Prospects

In this PhD thesis, we have studied the effect of delay on the global existence and the stability of the global solutions for some evolution systems. In chapter 2, we have investigated a second-order abstract viscoelastic equation with a weak internal damping, non-constant delay term in an internal feedback and nonlinear weights. We established the well-posedness result without any relation between the non-linear weights μ_1 and μ_2. Furthermore, we realized new optimal explicit and gereral decay results which include the exponential, polynomial and logarithmic decay rates. This ones have been obtained under a very general condition on g. Precisely, we have assumed that

$$g'(t) \leq -\xi(t)H(g(t)) \quad \forall t \in \mathbb{R}_+,$$

and we did not require that the function ξ be non-increasing which improves several results such as [**15, 17, 26, 55**]. The result we obtained in the third chapter is a generalization of the important manuscript [**67**] of Messaoudi and Soufyane where we realized a general energy decay for a non-dissipative problem which, to the best of our knowledge, has never been studied before. In the chapter 4, we investigated a linear Timoshenko system with a strong damping and a strong delay term in the first equation. Unexpectedly, we obtained that the system is lack of exponential stability whether the equal-speeds condition (4.1.3) holds or not. In addition, by introducing a second-order energy, we established the polynomial decay with optimal rate. The result we obtained is different from the work of Raposo et al.[**84**], where they considered a strong delayed thermoviscoelastic Timoshenko system with heat conduction modeled by the Cattaneo law and established exponential decay. In chapter 5, a nonlinear damped Porous system with a nonlinear delay term was considered. We proved the well-posedness of the system without the restrictions of $\mu_i > 0$ and $\mu_2 < \frac{\alpha_1}{\alpha_2}\mu_1$. Also, we established two general decay estimates with rates depending on the speeds of wave propagation and the regularity of initial data. For the equal-speeds case, we got a similar result as in the Timoshenko beam with the same hypotheses imposed on μ_1 and μ_2 (see [**97, 101**]). Otherwise, a slow decay result was given subject to a new relationship between μ_1 and μ_2. The result of the last chapter is an extension of the works [**65, 75, 77**] to Bresse system where the exponential stability was gotten also under the smallness of the weights of the interior delays a_i, $i = 1, 2, 3$.

Many interesting problems in connection with the systems we have considered here are still open. We propose in what follows some of them.

i. It is remarkable that the fact $E' < 0$ was employed in several sities in the proof of our stability results (2.4.4), see for example, Eqs.(2.4.25),(2.4.34). And, in the case of $\mu_1 \equiv 0$ and

$|\mu_2| > 0$, we obtain that the modified energy functional E defined by (2.2.10) satisfies, for any $t \geq 0$

$$E'(t) \leq \frac{1}{2}\Big(|\mu_2(t)| + \eta(t)\Big)\|u_t(t)\|^2 + \frac{1}{2}\Big(|\mu_2(t)| - \eta(t)\Big)\|z(1,t)\|^2 + \frac{1}{2}\big(g' \circ u\big)(t) - \frac{1}{2}g(t)\|\mathcal{A}^{\frac{1}{2}}u(t)\|^2,$$

from which we conclude that the system is not dissipative in general in the sense that E' is not necessarily negative. So, one could adress this model in the case when H is linear by combining the method of the present paper with the one in Feng [**50**]. But, when H is non-linear the problem stills open.

ii. Inspired by the work of Chellaoua et al.[**31**] and Benaissa et al.[**61**], it would be interesting to study problem (2.1.1) with the nonlinear damping $\mu_1(t)F_1(u_t(t))$ and the nonlinear delay term $\mu_2(t)F_2(u_t(t-\tau(t)))$.

iii. Motivated by Guesmia and Tatar [**62**], it is an interesting problem to consider (2.1.1) with the distributed delay $\int_0^\infty \mu_2(t,s)u(t-s)ds$ instead of $\mu_2(t)u_t(t-\tau(t))$.

iv. In [**63**], Messaoudi considered the following weak viscoelastic wave equation

$$u_{tt}(x,t) - \Delta u(t) + \kappa(t)\big(g * \Delta u\big)(x,t) = 0 \qquad \text{in } \Omega\times]0,+\infty[$$

with Dirichlet boundary condition where $\kappa, g : \mathbb{R}_+ \to]0,+\infty[$ are differentiable positive non-increasing functions such that it exists a differentiable decreasing function $\xi : \mathbb{R}_+ \to]0,+\infty[$ satisfying

$$g(t) \leq -\xi(t)g(t) \qquad \forall t \in \mathbb{R}_+,$$
$$\lim_{t\to+\infty} \frac{-\kappa'(t)}{\xi(t)\kappa(t)} = 0.$$

And, he proved that the energy of solutions E has the following general decay property:

$$E(t) \leq c\exp\Big(-c\int_0^t \kappa(s)\xi(s)ds\Big) \qquad \forall t \in \mathbb{R}_+.$$

Motivated by this study, we will consider in a forthcoming work the following more general problem

$$u_{tt}(t) + \mathcal{A}u(t) - \kappa(t)\big(g * \mathcal{A}u\big)(t) + \mu_1(t)u_t(t) + \mu_2(t)u_t(t-\tau(t)) = 0, \quad t > 0.$$

We will adress the well-posedness and moreover we will show whether the conditions imposed in [**63**], give analogous decay results to those we established, when combined with the ones imposed here.

v. As was mentioned above, Mustafa [**24**] examined a wave equation with a viscoelastic boundary damping localized on Γ_1 and established an explicit and general decay rate of its solutions by assuming that $u_0 = 0$ on Γ_1. So, it is very interesting to consider the case when $u_0 \neq 0$.

vi. One could derive the stability result (4.5.1) under the homogeneous Dirichlet-Dirichlet boundary conditions. But showing the non-exponential decay as well as the optimality of the obtained rate of decay in this case is an important open problem.

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