الـجـمـهـوريـة الـجـزائـريـة الـديـمـقـراطـيـة الـشـعـبـيـة

People's Democratic Republic of Algeria

وزارة الـــتـــعـــلـــيـــم الـــعـــالـــي و الـــبـــحـــث الـــعـــلـــمـــي

Ministry of Higher Education and Scientific Research

UNIVERSITY OF MUSTAFA STAMBOULI

MASCARA

FACULTY OF EXACT SCIENCES

DEPARTEMENT OF MATHEMATICS

جامعة مصطفى اسطمبولي

مـعـسـكـر

كـلـيـة الـعـلـوم الـدقـيـقـة

قسـم الـرياضيات

PhD Thesis

Speciality : *Mathematics*

Option : *Differential geometry*

Entitled :

---

f-harmonic maps

Presented by: Remli Embarka

PhD Thesis submitted: 16/02/2021

The Jury:

President:	**Mr.** Khaled BENMERIEM	PR	**Mascara University**
Supervesor:	**Mr.** Ahmed MOHAMMED CHERIF	MCA	**Mascara University**
Examinar:	**Mr.** Kadour ZAGGA	MCA	**Mascara University**
Examinar:	**Mr.** Hichem ELHENDI	MCA	**Bechar University**
Examinar:	**Mr.** Mohammed DIDA HAMOU	MCA	**Saida University**
Examinar:	**Mr.** Lakehal BELARBI	MCA	**Mostaganem University**

University year: 2020-2021

الجمهورية الجزائرية الديمقراطية الشعبية

REPUBLIQUE ALGERIENNE DEMOCRATIQUE ET POPULAIRE

وزارة التعليم العالي و البحث العلمي

MINISTERE DE L'ENSEIGNEMENT SUPERIEUR ET DE LA RECHERCHE SCIENTIFIQUE

UNIVERSITÉ DE MUSTAFA STAMBOULI

MASCARA

FACULTÉ DES SCIENCES EXACTES

DEPARTEMENT DE MATHEMATIQUES

جامعة مصطفى اسطمبولي

معسكر

كلية العلوم الدقيقة

قسم الرياضيات

Thèse de doctorat

Spécialité : *Mathématiques*

Option : *Géométrie différentielle*

Sujet de la thèse :

Les applications *f*-harmoniques

Présentée par: Remli Embarka

La soutenance sera prévue le: 16/02/2021

Devant le Jury :

Président:	**Mr.** Khaled BENMERIEM	PR	**Université de Mascara**
Encadreur:	**Mr.** Ahmed MOHAMMED CHERIF	MCA	**Université de Mascara**
Examinateur:	**Mr.** Kadour ZAGGA	MCA	**Université de Mascara**
Examinateur:	**Mr.** Hichem ELHENDI	MCA	**Université de Bechar**
Examinateur:	**Mr.** Mohammed DIDA HAMOU	MCA	**Université de Saida**
Examinateur:	**Mr.** Lakehal BELARBI	MCA	**Université de Mostaganem**

L'Année Universitaire: 2020-2021

Contents

Acknowledgements **6**
0.1 Acknowledgements . . . 6

Publications **7**
0.2 Publications . . . 7

Abstract **8**
0.3 Abstract . . . 8
0.4 Résumé . . . 8

Introduction **10**
0.5 Introduction . . . 10

1 Introduction to differential and Riemannian geometry **13**
1.1 Recall of differential geometry . . . 13
1.1.1 Differentiable manifold . . . 13
1.1.2 Differentiable mapping . . . 15
1.1.3 Tangent space . . . 15
1.1.4 Tangent bundle . . . 16
1.1.5 Cotangent space . . . 16
1.1.6 Cotangent bundle . . . 16
1.1.7 Vectors field . . . 16
1.2 Orientation and manifold with boundary . . . 18
1.2.1 Orientable manifold . . . 18
1.2.2 Half-space . . . 18
1.2.3 Manifolds with boundary . . . 19
1.3 Recall of Riemannian geometry . . . 20
1.3.1 Riemannians metrics . . . 20
1.3.2 Reverse image of a metric tensor . . . 22
1.3.3 Metric induced on the inverse tangent bundle . . . 23
1.4 Linear connection . . . 23
1.4.1 Torsion tensor . . . 24

1.4.2 Levi-Civita connection . . . 24
1.5 Induced connection on the tangent bundle . . . 25
1.6 Second fundamental form . . . 27
1.7 Sub-manifolds . . . 28
1.8 Curvatures . . . 30
1.8.1 Curvature tensor . . . 30
1.8.2 Sectional curvature . . . 31
1.8.3 Ricci curvature . . . 32
1.9 Operators on Riemannian manifold . . . 33
1.9.1 Gradient operator . . . 33
1.9.2 Hessian operator . . . 34
1.9.3 Divergence operator . . . 34
1.9.4 Laplacian operator . . . 35
1.9.5 Divergence Theorem . . . 35

2 Harmonic maps 36
2.1 Geodesics . . . 36
2.2 Harmonic maps . . . 38
2.2.1 First variation of energy . . . 38
2.2.2 Second variation of energy . . . 43
2.3 Biharmonic maps . . . 45
2.3.1 First variation of bi-energy . . . 45
2.4 Somes result on stable harmonic maps . . . 48
2.5 Homothetic vector fields and harmonic maps . . . 56
2.5.1 Homothetic vector fields and harmonic maps . . . 56
2.5.2 Homothetic vector fields and biharmonic maps . . . 59

3 Generalized f-harmonic maps 61
3.1 f-harmonic maps . . . 61
3.1.1 The first variation of the f-energy . . . 62
3.1.2 The second variation of the f-energy . . . 63
3.2 f-biharmonic maps . . . 67
3.2.1 First variation of the f-bi-energy . . . 68
3.3 Main results . . . 71
3.3.1 Some results on stable f-harmonic maps . . . 71
3.3.2 Homothetic vector fields and f-harmonic maps . . . 77
3.3.3 f-biharmonic maps and submanifolds . . . 79

4 L-harmonic maps 84
4.1 The Euler-Lagrange equations . . . 84
4.2 L-harmonic maps . . . 86
4.2.1 The first variation of L-energy . . . 87
4.2.2 The second variation of L-energy . . . 88

4.3 L-biharmonic maps 90
4.3.1 First variation of the L-bienergy 91
4.4 Main results 92
4.4.1 Semi-conformal L-harmonic maps 92
4.4.2 A Liouville type theorem for L-harmonic maps 96

5 (p, f)-harmonic maps 101
5.1 Main results 101
5.1.1 The first variation of the (p, f)-energy 101
5.1.2 A Liouville type Theorem for (p, f)-harmonic maps 103
5.1.3 Stress (p, f)-energy tensor 106
5.1.4 Homothetic vector fields and (p, f)-harmonic maps 108

Bibliography 110

0.1 Acknowledgements

First of all, I thank ALLAH for giving me the courage and the will to do this work.
I express my sincere gratitude to supervisor AHMED MOHAMMED CHERIF who encouraged me to publish this work, his availability, his generosity,his moral help and his very precious advice which helped me to determine this work .

I thank the members of the jury for having honored me with their presence. I am also grateful to Prof. khaled Benmeriem, who accepted to chair this thesis. I am also thankful to Dr. Kadour Zagga, Dr. Hichem Elhendi, Dr. Mohammed Dida Hamou, and Dr. Lakhal Belarbi, who accepted to be members of this thesis.

My thanks to all those who helped me from near and far to develop this work.

Thank you also to all my colleagues from the University of Mascara.

I dedicate this work
AT
My very dear parents.
My husband.
My children AHMED FODIL and MOHAMED NABIL.
My family especially my brothers and my sisters

0.2 Publications

1. E.Remli and A. M. Cherif, *Some Result on stable f-harmonic maps,* Commun. Korean Math. Soc. 33 (2018), No. 3, pp. 935-942

2. E.Remli and A. M. Cherif, *SOME RESULTS ON f-HARMONIC MAPS AND f-BIHARMONIC SUBMANIFOLDS,* Acta Math. Univ. Comenianae Vol. LXXXIX, 2(2020), pp. 299-307.

3. E.Remli and A. M. Cherif, *Semi-conformal L-harmonic maps and Liouville type theorem* (submitted)

4. E.Remli and A. M. Cherif, *On the generalized of p-harmonic and f-harmonic maps,* Preprint in Kyungpook Mathematical Journal.

0.3 Abstract

The purpose of this doctoral thesis is to study some geometric properties of f-harmonic maps(resp. L-harmonic maps) with $f \in C^{\infty}(M \times N)$(resp.$L \in C^{\infty}(M \times N \times \mathbb{R})$). This goal also includes the variationnels problems, where we introduce the notion of (p, f)-harmonic maps with $p \geq 2$ et $f \in C^{\infty}(M)$, establishing the first variation of the functional (p, f)-energy. Then we will define Liouville's theorem relating to (p, f)-harmonic maps, the stress (p, f)-energy tensor.Finally we give a result of homothetic vector fields and (p, f)-harmonicity.
Keywords:f**-harmonic maps (resp.**f**-biharmonic maps),** L**-harmonic maps (resp.**L**-biharmonic maps),** (p, f)**-harmonic maps.**

0.4 Résumé

Le but de cette thèse de doctorat est d'étudier de certaines propriétés géométriques des applications f-harmoniques (resp. L-harmoniques) avec $f \in C^{\infty}(M \times N)$(resp.$L \in C^{\infty}(M \times N \times \mathbb{R})$). Ce but inclut aussi les problèmes variationnels, dont nous introduisons la notion des applications (p, f)-harmoniques avec $p \geq 2$ et $f \in C^{\infty}(M)$, en établissant la première variation de la fonctionnelle (p, f)-énergie. Ensuite, nous caractérisons le théorème de Liouville relatif aux applications (p, f)-harmoniques, le tenseur (p, f)-énergie impulsion. Enfin on donne un résultats de (p, f)-harmonicité et le champs de vecteur homothétique.
Mots-clés :les applications f**-harmoniques (resp.**f**-biharmonique), les applications** L**-harmoniques (resp.**L**-biharmonique), les applications** (p, f)**-harmoniques.**

ملخص

الهدف من هذه الاطروحة هو دراسة بعض الخصائص الهندسية لتطبيقات ƒ- التوافقية و L- التوافقية حيث $ƒ\epsilon C^{\infty}(M\times N)$ و $L\epsilon C^{\infty}(M\times N\times R)$. يتضمن ايضا المشاكل التغييرية حيث سيتم تقديم مفهوم جديد لتطبيقات (ƒ¸p)-التوافقية هذا المفهوم هو عبارة عن تعميم لتطبيقات p-التوافقية و ƒ-التوافقية حيث نقوم بانشاء العبارة الاولى التغييرية التي من خلالها يمكن كتابة معادلة Euler¸-Lagrange ¸ الحصول على نظرية ليوفيل (LiouviLLe)¸ الحصول ايضا على مؤثر (ƒ¸p)-الطاقة و في النهاية نقوم بدراسة التطبيقات (ƒ¸p)-التوافقية وحقل الاشعة المتماثل.

الكلمات المفتاحية التطبيقات ƒ- التوافقية (التطبيقات ƒ- التوافقية الثنائية) ¸ التطبيقات L- التوافقية(التطبيقات L- التوافقية الثنائية)¸ التطبيقات (ƒ¸p)-التوافقية.

0.5 Introduction

Harmonic maps are solutions to a natural geometrical variational problem. This notion appear in various contexts: geodesic, harmonic functions, minimal surfaces...ect. A map $\varphi \in C^\infty(M,N)$ is called harmonic if it is a critical point of the energy functional

$$E(\varphi, D) = \frac{1}{2}\int_D |d\varphi|^2 v^M.$$

Equivalently, φ is harmonic if it satisfies the associated Euler- Lagrange equation $\tau(\varphi) = 0$, where $\tau(\varphi)$ is the tension field of φ defined by $\tau(\varphi) = \operatorname{trace}\nabla d\varphi$.
Biharmonic maps generalize the notion of harmonic map and are defined as critical points of the bienergy functional.

$$E_2(\varphi, D) = \frac{1}{2}\int_D |\tau(\varphi)|^2 v^M,$$

where φ is a smooth map between two Riemannian manifolds M and N, $|d\varphi|$ is the Hilbert Schmidt norm of the differential $d\varphi$. Several authors interested in this type of harmonic maps, we can cite, J.Eells, J.H. Sampson, L.Lemaire [13], [14] and A.Lichnerowiez [24].
Other authors introduced the f-harmonic maps (resp. f-biharmonic maps), (we can cite for example M.Djaa, A.M.Cherif, K. Zagga, S. Ouakkas [11]).
Corresponding to the critical points of the functional f-energy (resp. f-bi-energy) given by

$$E_f(\varphi, D) = \frac{1}{2}\int_D f(x,\varphi(x))|d\varphi|^2 v^M, \tag{1}$$

resp.

$$E_{2,f}(\varphi, D) = \frac{1}{2}\int_D |\tau_f(\varphi)|^2 v^M, \tag{2}$$

where $f : M \times N \to (0,\infty)$ is a smooth positive function, called torsion weight function and $\tau_f(\varphi)$ is the f-tention field of φ defined by

$$\tau_f(\varphi) = f_\varphi \tau(\varphi) + d\varphi(\operatorname{grad}^M f_\varphi) - e(\varphi)(\operatorname{grad}^N f) o\varphi.$$

In their article in 2015. A. M. Cherif and M. Djaa [29] established the L-harmonic maps (resp. L-biharmonic maps), considering the critical points of the functionality L-energy (resp. L-bienergy).

$$E_L(\varphi, D) = \int_D L(x,\varphi(x), e(\varphi)(x)) v^M,$$

resp.

$$E_{2,L}(\varphi, D) = \frac{1}{2}\int_D |\tau_L(\varphi)|^2 v^M,$$

where

$$\begin{array}{rcl} L : M \times N \times \mathbb{R} & \to & (0,\infty) \\ (x,y,r) & \mapsto & L(x,y,r) \end{array}$$

is a smooth positive function, $e(\varphi)$ is called energy density of φ defined by

$$e(\varphi) = \frac{1}{2}|d\varphi|^2,$$

$\tau_L(\varphi)$ is the L-tension field of φ given by

$$\tau_L(\varphi) = L'_\varphi\,\tau(\varphi) + d\varphi\big(\operatorname{grad}^M L'_\varphi\big) - (\operatorname{grad}^N L)\circ\varphi.$$

The principal objective of this work is to study the geometric properties of f-harmonic maps and f-biharmonic maps with f is a smooth positive function corresponding to the critical points of f-energy and f-bienergy associated to φ (see the equations (1) and (2)).
Next we mention some theorems relating to L-harmonic maps, for example we give the proof of Liouville type theorem for L-harmonic maps from complete noncompact Riemannian manifold (M^m, g) with positive Ricci curvature into a Riemannian manifold (N^n, h) with non-positive sectional curvature, where $L \in C^\infty(M^m \times N^n \times \mathbb{R}_+)$ is a smooth positive function which satisfies some suitable conditions, where the Liouville type theorems for harmonic maps between complete smooth Riemannian manifolds have been done by many authors: Eells-Sampson [14], Schoen-Yau [23], Cheng [15].

In a different context, we introduced a new notion of harmonicity, it is the (p, f)-harmonic maps with $p \geq 2$ and $f \in C^\infty(M)$ between riemannian manifolds. This notion is a natural generalization of p-harmonic maps and f-harmonic maps. Our purpose in this party is to study the geometric properties of (p, f)-harmonic maps corresponding to the critical points of functional associated to φ

$$E_{p,f}(\varphi, D) = \frac{1}{p}\int_D f(x)|d\varphi|^p v^M.$$

In the first chapter, we shall give several notations and definitions of differential geometry (resp. Riemannian geometry): differentiable manifolds, tangent space, tangent bundle, etc .... (resp. Riemannian manifolds, Riemannian metric, etc ....).

The second chapter, we shall introduce the theory of harmonic mappings, (resp. biharmonic maps), we shall present some results on the stability of harmonic maps introduced by Y.L. Xin [45].

In the third chapter, we discuss the stabilities of f-harmonic maps on sphere S^n with $n > 2$, we also prove that any f-harmonic map from a complete Riemannian manifold

(M, g) to Riemannian manifold (N, h) is necessarily constant, with (N, h) admitting a proper homothetic vector field satisfying some conditions. Also we present some properties for the f-biharmonicity of submanifolds of $\mathbb{R}^n$, where f is a smooth positive function on $\mathbb{R}^n$.
The results obtained in this chapter are published in the article [39] and [40] .

The fourth chapter is devoted to the study of L-harmonic maps and some geometric properties, we prove that every semi-conformal harmonic map between Riemannian manifolds is L-harmonic map. We also prove a Liouville type theorem for L-harmonic maps.
The results obtained in this chapter are submitted for publication.

In the chapter five we extend the definition of p-harmonic maps between two Riemannian manifolds called the (p, f)-harmonic maps which include the first variation of (p, f)-energy functional , we prove a Liouville type theorem for generalized p-harmonic. We present some new properties for the generalized stress p-energy tensor. We also prove that every generalized p-harmonic map from a complete Riemannian manifold into a Riemannian manifold admitting a homothetic vector field satisfying some condition is constant.
The results obtained in this chapter are published in the article [41].

Chapter 1

Introduction to differential and Riemannian geometry

In this chapter, we present the basic concepts: Differential geometry, manifold on-board, Riemannian geometry, linear connection, induced connection on the inverse tangent bundle, second fundamental form, submanifold, curvatures and operators on a Riemannian manifold ([4], [17], [25], [35], [36], [37]).

1.1 Recall of differential geometry

1.1.1 Differentiable manifold

Let M be a topological space, we assume that M satisfies the Hausdorff separation axiom which states that any two different points in M can be separated by disjoint open sets. M is called a topological manifold if there exists an $m \in \mathbb{N}$ and for every point $x \in M$ an open neighborhood U_x of x, such that U_x is homeomorphic to some open subset V of $\mathbb{R}^m$. The naturel number m is called the dimension of M. So a topological manifold M is locally homeomorphic to the standard m-dimensional vector space $\mathbb{R}^m$. An open chart on M is a pair (U, φ) where U is an open subset of M and φ is a homeomorphism of U onto an open subset of $\mathbb{R}^m$.

Definition 1.1.1. *Let M be a Hausdorff space. A differentiable structure on M of dimension n (atlas $\mathcal{A}$) is a collection of open charts $(U_i, \varphi_i)_{i\in I}$ on M where $\varphi_i(U_i)$ is an open subset of $\mathbb{R}^n$ such that the following conditions are satisfied:*

1. $M = \bigcup_{i\in I} U_i$,
2. *For each pair $i, j \in I$ the mapping $\phi_j \circ \phi_i^{-1}$ is a C^∞-diffeomorphism mapping of $\varphi_i(U_i \cap U_j)$ onto $\varphi_j(U_i \cap U_j)$, with $U_i \cap U_j \neq \emptyset$*

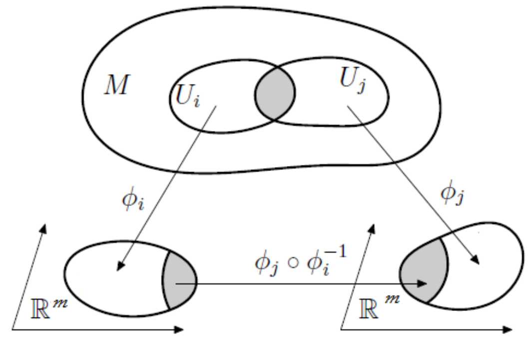

Figure 1.1: Charts change mapping.

Definition 1.1.2. *A differentiable manifold of dimension n is a Hausdorff space has a differentiable structure of dimension n.*

Example 1.1.1. The space $\mathbb{R}^n$ is a differentiable manifold with $\mathcal{A} = (\mathbb{R}^n, Id_{\mathbb{R}^n})$.

Example 1.1.2. Let $\mathbb{S}^n$ denote the unit sphere in $\mathbb{R}^{n+1}$ i.e.

$$\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \ / \ \sum_{i=1}^{n+1} x_i^2 = 1\},$$

equipped with the subset topology $\mathcal{T}_{\mathbb{S}^n}$ induced by $\mathcal{T}$ on $\mathbb{R}^{n+1}$. Let N be the north pole $N = (1, 0) \in \mathbb{R} \times \mathbb{R}^n$ and S be the south pole $S = (-1, 0)$ on $\mathbb{S}^n$, respectively. Put $U_N = \mathbb{S}^n - \{N\}$, $U_S = \mathbb{S}^n - \{S\}$ and define $\varphi_N : U_N \to \mathbb{R}^n$, $\varphi_S : U_S \to \mathbb{R}^n$, by $\varphi_N : (x_1, ..., x_{n+1}) \mapsto \frac{1}{1-x_1}(x_2, ..., x_{n+1})$, $\varphi_S : (x_1, ..., x_{n+1}) \mapsto \frac{1}{1+x_1}(x_2, ..., x_{n+1})$. Then the transition maps $\varphi_S \circ \varphi_N^{-1}$, $\varphi_N \circ \varphi_S^{-1} : \mathbb{R}^n - \{0\} \to \mathbb{R}^n - \{0\}$ are given by $x \mapsto \frac{x}{|x|^2}$. So $\mathcal{A} = \{(U_N, \varphi_N), (U_S, \varphi_S)\}$ is a C^∞-atlas on $\mathbb{S}^n$.

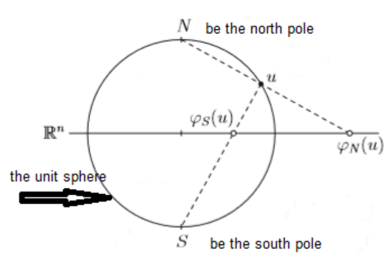

Figure 1.2: Stereographic projection.

1.1.2 Differentiable mapping

Definition 1.1.3. *1. Let M be a differentiable manifold, $f : M \longrightarrow \mathbb{R}$ is called to be differentiable function at point $p \in M$, if there is a chart (U, ϕ) of M with $p \in U$ such as $f \circ \phi^{-1} : \phi(U) \longrightarrow \mathbb{R}$ is differentiable. The function f is differentiable in M if it is differentiable en p for all $p \in M$.*

2. Given two differentiable manifolds M and N, a mapping $f : M \to N$ is said to be differentiable (or C^∞-differentiable), if for every chart (U_i, φ_i) of M and every chart (V_j, ψ_j) of N such that $f(U_i) \subset V_j$, the mapping $\psi_j \circ f \circ \varphi_i^{-1}$ of $\varphi_i(U_i)$ into $\psi_i(V_j)$ is differentiable.

1.1.3 Tangent space

Notation 1.1.1. We note by:
$C^\infty(M)$ is the set of differentiable functions in M.
$C^\infty(M, N)$ is the set of differentiable mapping of M in N.

Definition 1.1.4 (Tangent vector). *Let M be a differentiable manifold and $p \in M$, then a tangent vector X_p at p is a map*

$$\begin{aligned} X_p : C^\infty(M) &\longrightarrow \mathbb{R} \\ f &\longmapsto X_p(f), \end{aligned}$$

such that

C1. $X_p(\lambda f + \mu g) = \lambda X_p(f) + \mu X_p(g)$,

C2. $X_p(fg) = X_p(f)g(p) + f(p)X_p(g)$,

C3. *If f is constant in the neighborhood of p then $X_p(f) = 0$,*

for all λ, $\mu \in \mathbb{R}$ and f, $g \in C^\infty(M)$.

Definition 1.1.5 (Tangent space). *The tangent space T_pM of M at p is the set of all tangent vectors at p, this set has a natural structure of a real vector space given by the following operations $(+)$ and $(.)$*

i. $(X_p + Y_p)(f) = X_p(f) + Y_p(f)$;

ii. $(\lambda.X_p)(f) = \lambda.X_p(f)$;

for X_p, $Y_p \in T_pM$, $f \in C^\infty(M)$ and $\lambda \in \mathbb{R}$.

Remark 1.1.1. $X_p(f)$ is also called the derivative of f by X_p.

1.1.4 Tangent bundle

Definition 1.1.6. *The set of tangent vectors of M, denoted by $TM = \bigcup_{p\in M} T_pM$, is called the tangent bundle of M.*
Then, $A \in TM$ if and only if there is a point $p \in M$ such as $A \in T_pM$. This point is only determined by A and noted by $\pi(A)$, the mapping:

$$\begin{array}{rcl} \pi : TM & \longrightarrow & M \\ A & \longmapsto & \pi(A) = p \end{array}$$

is the canonical projection

1.1.5 Cotangent space

Definition 1.1.7. *Let T_x^*M be the dual space of the tangent space T_xM of M at x. An element of T_x^*M is called a covector at x. An assignment of a covector at each point x is called an 1-form (differential form of degree 1).*

Remark 1.1.2. T_x^*M is the set of linear form on T_xM

$$T_x^*M \ni w_x : \begin{array}{rcl} T_xM & \longrightarrow & \mathbb{R} \\ X_x & \longmapsto & w_x(X_x) \end{array}$$

1.1.6 Cotangent bundle

Definition 1.1.8. *We call cotangent bundle of M the fibre bundle which has as total space*

$$T^*M = \bigcup_{x\in M} T_x^*M.$$

1.1.7 Vectors field

Definition 1.1.9. *A vector field X on a differentiable manifold M is an assignment of a vector X_p to each point p of M. In other words, a vector field X on a manifold M is a mapping,*

$$\begin{array}{rcl} X : M & \longrightarrow & TM \\ p & \longmapsto & X_p \end{array}$$

such that $\pi(X_p) = p$, for all $p \in M$.

Remark 1.1.3.

- We denote by $\Gamma(TM)$ the set of all differentiable vector fields on M.

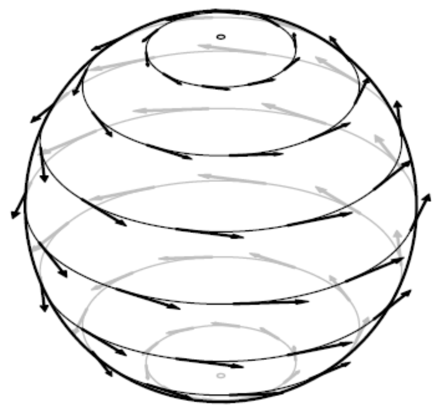

Figure 1.3: Vector field of S^n.

- If f is a differentiable function on M, then $X(f)$ is a differentiable function on M defined by $(X(f))(p) = X(f)$,for all $X \in \Gamma(TM)$ and $p \in M$.

Definition 1.1.10. *Let M be an m-dimensional differentiable manifold, (U, ϕ) be a chart of M and $p \in U$, for $i = 1, ..., m$, we define the map $\frac{\partial}{\partial x_i}\Big|_p : C^\infty(M) \longrightarrow \mathbb{R}$, by:*

$$\frac{\partial}{\partial x_i}\Big|_p (f) = \frac{\partial (f \circ \phi^{-1})}{\partial x_i}\Big|_{\phi(p)}.$$

$\frac{\partial}{\partial x_i}\Big|_p$ is said derivative associated to the chart (U, ϕ).

Remark 1.1.4. **i.** $\{\frac{\partial}{\partial x_i}|_p, \quad i = 1...m\}$ be a frame for the tangent space T_pM, for all $p \in U$.

ii. $\{dx_i|_p, \quad i = 1...m\}$ be a form basis for the cotangent space T_p^*M(the dual basis of the basis $\{\frac{\partial}{\partial x_i}|_p, \quad i = 1...m\}$ for T_pM).

Definition 1.1.11. *Let $T_x^{(r,s)}M = \underbrace{T_xM \otimes \ldots \otimes T_xM}_{r-once} \otimes \underbrace{T_x^*M \otimes \ldots \otimes T_x^*M}_{s-once}$ be the vectorial space, where $x \in M$ and let $T^{(r,s)}M = \bigcup_{x \in M} T_x^{(r,s)}M$. A element $T \in T_x^{(r,s)}M$ is a tensor of type (r, s) above x. A tensor field of type (r, s) on a manifold M is an assignment section of $T^{(r,s)}M$ (i.e. a tensor is a map $T : M \longrightarrow T^{(r,s)}M$, $x \mapsto T(x) \in T_x^{(r,s)}M$).*

Example 1.1.3. **I)** A function on a manifold M is a tensor of type $(0, 0)$.

II) A vector field X is a tensor of type $(1, 0)$.

III) A differential 1-form ω on a manifold M is a tensor of type $(0,1)$.

Definition 1.1.12 (Tangent mapping). *Let $f : M \longrightarrow N$ be a differentiable mapping between differentiable manifold, we define tangent mapping*

$$d_p f : T_p M \longrightarrow T_{f(p)} N,$$

by:

$$(d_p f(v))(g) = v(g \circ f), \quad \forall g \in C^\infty(N) \quad \forall p \in M.$$

Definition 1.1.13. *If X and Y are vector fields, the Lie bracket $[X,Y]$ is given by*

$$[X,Y] = XY - YX.$$

1.2 Orientation and manifold with boundary

1.2.1 Orientable manifold

Definition 1.2.1. *We call an atlas of orientation of manifold M all $\mathcal{A} = \{(U_i, \varphi_i)\}_{i \in I}$ such that the charts changes mapping $\psi_{ij} = \varphi_i \circ \varphi_j^{-1}$ has a positive Jacobian, i.e.*

$$J(\psi_{ij})_x = \det(d_{\varphi_j(x)} \psi_{ij}) > 0.$$

Definition 1.2.2. *An orientable manifold is a manifold for which there are orientation atlases.*

Remark 1.2.1. If φ be a diffeomorphism of $\mathbb{R}^n$, its Jacobian is defined by:

$$J(\varphi)_x = det(d_x \varphi).$$

Example 1.2.1.

- $\mathbb{R}^n$ is an orientable manifold.
- The tangent bundle TM to a manifold M is an oriented manifold even if M is not.
- The real projective plane, the Möbius band and the Klein bottle are non-orientable manifolds.

1.2.2 Half-space

Definition 1.2.3. *The half-space noted $\mathbb{H}^m$ is defined by:*

$$\mathbb{H}^m = \{(x_1, \ldots, x_m) \in \mathbb{R}^m \; / \; x_1 \leq 0\}.$$

The boundary of the half-space noted $\partial \mathbb{H}^m$ is given by:

$$\partial \mathbb{H}^m = \{(x_1, \ldots, x_m) \in \mathbb{R}^m \; / \; x_1 = 0\}.$$

The opens subset of the half-space $\mathbb{H}^m$ are defined by:
$U_i = V_i \cap \mathbb{H}^m$ where V_i is an open subset of $\mathbb{R}^m$, $i = 1...m$.

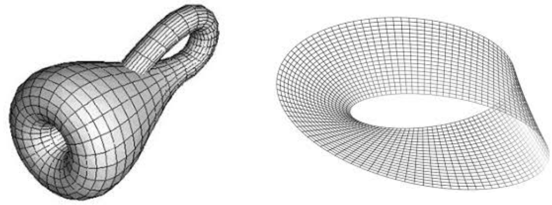

Figure 1.4: The Möbius band and the Klein bottle

Definition 1.2.4. *Let U be an open subset of $\mathbb{H}^m$, the boundary of U noted ∂U is given by:*

$$\partial U = U \cap \partial \mathbb{H}^m.$$

The interior of an open subset U denoted by $Int(U)$ is the open subset of $\mathbb{R}^m$ defined by:

$$Int(U) = U \backslash \partial U.$$

1.2.3 Manifolds with boundary

Definition 1.2.5. *Let M be a separate topological space, we say that M is an m-dimensional smooth manifold with boundary, if there is an atlas $\mathcal{A} = \{(W_i, \phi_i)\}_{i\in I}$ such that ϕ_i is a homeomorphism of an open subset W_i of M on an open subset U_i of the half space $\mathbb{H}^m$, and the charts change mapping $\phi_i \circ \phi_j^{-1}$ is of class C^∞.*

Definition 1.2.6. *The boundary ∂M of a manifold with boundary is the set of points x of M which has a chart (W, φ) such that $\varphi(x)$ into the boundary of $\varphi(W)$.*

Remark 1.2.2. **i.** ∂M is an $(m-1)$-dimensional smooth manifold without boundary ($i.e.$ $\partial(\partial M) = \emptyset$).

ii. If M be an orientable manifold with boundary, then ∂M is an orientable manifold.

Proposition 1.2.1. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a smooth submersion on $f^{-1}(\{0\})$, then $M = \{x \in \mathbb{R}^n \ / \ f(x) \leq 0\}$ be an n-dimensional smooth manifold with boundary, and $\partial M = \{x \in \mathbb{R}^n \ / \ f(x) = 0\}$.*

Example 1.2.2. Let $B^3 = \{(x, y, z) \in \mathbb{R}^3 \ / \ x^2+y^2+z^2 \leq 1\}$, then B^3 is a 3-dimensional smooth manifold with boundary.

Indeed. The mapping:

$$\begin{array}{rcl} f : \mathbb{R}^3 & \longrightarrow & \mathbb{R} \\ (x, y, z) & \longmapsto & x^2 + y^2 + z^2 - 1 \end{array}$$

is a submersion on $f^{-1}(\{0\})$, because $Jac_f = (2x \ 2y \ 2z)$ is of rank 1 on $f^{-1}(\{0\})$. Another, $\partial B^3 = \{x, y, z) \in \mathbb{R}^3 \ / \ x^2 + y^2 + z^2 = 1\} = \mathbb{S}^2$.

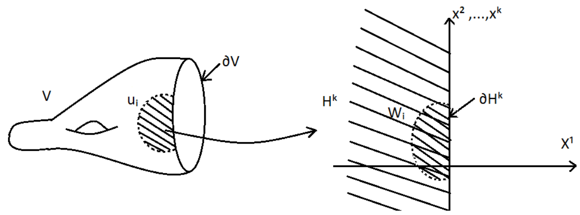

Figure 1.5: Manifold with boundary.

1.3 Recall of Riemannian geometry

1.3.1 Riemannians metrics

Definition 1.3.1. *Let M be an m-dimensional smooth manifold. A Riemannian metric on M is a tensor field,*

$$g : \Gamma(TM) \times \Gamma(TM) \to C^\infty(M)$$

sach that for each $p \in M$ the restriction $g_p = g/_{T_pM \otimes T_pM} : T_pM \otimes T_pM \to \mathbb{R}$ with

$$g_p : (X_p, Y_p) \mapsto g(X, Y)(p)$$

is inner product (that is a symmetric, bilinear, positive-definite form) on the vector space T_pM.
The pair (M^m, g) is called a Riemannian manifold of dimension m.

Definition 1.3.2. *Let $\gamma : I \to M$ be a C^1-curve in M.Then the length $L(\gamma)$ of γ is defined by $L(\gamma) = \int_I \sqrt{g(\dot\gamma(t), \dot\gamma(t))}dt$, where $\dot\gamma(t) = d\gamma(\frac{d}{dt})\Big|_t$.*

Definition 1.3.3. *The standard inner product on the vector space $\mathbb{R}^n$ is given by*

$$g_0(u, v) = \langle u, v\rangle_{\mathbb{R}^n} = \sum_{i=1}^{n} u_i v_i,$$

defines a Riemannian metric on $\mathbb{R}^n$.The Riemannian manifold $E^n = (\mathbb{R}^n, \langle, \ \rangle_{\mathbb{R}^n})$ is called the Euclidean space.

Remark 1.3.1. By multiply the Euclidean metric by a conformal factor we obtain other important examples of Riemannian manifolds

Example 1.3.1. By the hyperbolic space we mean the Riemannian manifold

$$H^n = \Big(B_1^n(0),\ \frac{4}{\left(1-\|x\|^2\right)^2}\, g_0\Big),$$

where $B_1^n(0)$ is the n-dimensional open unit ball

$$B_1^n(0) = \{x \in \mathbb{R}^n\ /\ \|x\| < 1\}.$$

Let $\gamma : (0,\, 1) \to H^n$ be the curve with $\gamma : t \mapsto (t,\, 0,\, ...0)$. Then

$$\begin{aligned} L(\gamma) &= 2\int_0^1 \frac{\sqrt{\langle \dot\gamma,\, \dot\gamma\rangle}}{1-||\gamma||^2}dt = 2\int_0^1 \frac{dt}{1-t^2} = \Big[\ln\Big(\frac{1+t}{1-t}\Big)\Big]_0^1 \\ &= \infty \end{aligned}$$

Example 1.3.2. By the punctured round sphere we mean the Riemannian manifold

$$\Sigma^n = \Big(\mathbb{R}^n,\ \frac{1}{(1+||x||^2_{\mathbb{R}^n})^2}g_0\Big).$$

Let $\gamma : \mathbb{R}^+ \to \Sigma^n$ be the curve with $\gamma : t \mapsto (t,\, 0,\, ...0)$. Then

$$\begin{aligned} L(\gamma) &= 2\int_0^\infty \frac{\sqrt{\langle \dot\gamma,\, \dot\gamma\rangle}}{1+||\gamma||^2} = 2\int_0^\infty \frac{dt}{1+t^2} = 2\Big[arctan(t)\Big]_0^\infty \\ &= \pi. \end{aligned}$$

Definition 1.3.4. *Let (U, ϕ) be a chart of (M^m, g), with the basic fields $\{\partial_1, \dots, \partial_m\}$. The functions g_{ij}, such that $g_{ij} = g(\partial_i, \partial_j)$ for all $i, j = 1, \dots, m$ is called components of the Riemannian metric g.*

Locally: If M has a local coordinate system (x_i), then:

$$g = \sum_{i,j=1}^{m} g_{ij}\, dx_i \otimes dx_j.$$

Example 1.3.3. In the standard chart (D, Id_D), the hyperbolic metric g_H on D has the components:

$$g_{ij}(x) = \frac{4\delta_{ij}}{\left(1-\|x\|^2\right)^2}.$$

Definition 1.3.5. *We define the length of a vector field X of (M^m, g), by:*

$$|X| = \sqrt{g(X, X)}.$$

Definition 1.3.6. *Let (M^m, g) be a Riemannian manifold, we call Riemannian volume measure, noted v^M or v_g, the measure defined locally in a frame by:*

$$v^M = \sqrt{\det(g_{ij})} dx^1 \wedge \ldots \wedge dx^m.$$

Example 1.3.4. We consider the torus of revolution T^2 of $\mathbb{R}^3$ with the Riemannian metric,

$$g = (b + a\cos\alpha)^2 d\theta^2 + a^2 d\alpha^2,$$

where $b > a > 0$. Then:

$$v_g = \sqrt{\det(g_{ij})} d\theta \wedge d\alpha = a(b + a\cos\alpha) d\theta \wedge d\alpha.$$

Example 1.3.5. We consider the manifold $\mathbb{R}^2$ with Riemannian metric,

$$g_0 = dx^2 + dy^2.$$

Then :

$$v_{g_0} = dx \wedge dy.$$

Example 1.3.6. We consider the manifold $S^2 = \{(x, y, z) \in \mathbb{R}^3 \ / \ x^2 + y^2 + z^2 = 1\}$ with Riemannian metric,

$$g = d\theta^2 + sin^2\theta d\varphi^2.$$

Then:

$$v^g = \sqrt{det(g_{ij})} d\theta \wedge d\varphi = |\sin\theta| d\theta \wedge d\varphi.$$

1.3.2 Reverse image of a metric tensor

Definition 1.3.7. *Let (N^n, h) be a Riemannian manifold, and M be a differentiable manifold and let $f : M \longrightarrow N$ be a smooth map, if f is an immersion at every point of M, then f^*h is a metric tensor on M, called inverse image of h by f, where:*

$$(f^*h)(X, Y) = h(df(X), df(Y)), \quad X, Y \in \Gamma(TM).$$

Locally:
Let (U, φ) be a chart on M of associated basis $(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_m})$ and let (V, ψ) be a chart on N of associated basis $(\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n})$, then

$$\begin{aligned} (f^*h)_{ij} &= (f^*h)(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) \\ &= h(df(\frac{\partial}{\partial x_i}), df(\frac{\partial}{\partial x_j})) \\ &= \sum_{\alpha,\beta=1}^{n} \frac{\partial f_\alpha}{\partial x_i} \frac{\partial f_\beta}{\partial x_j} h(\frac{\partial}{\partial y_\alpha}, \frac{\partial}{\partial y_\beta}) \circ f \end{aligned}$$

$$= \sum_{\alpha,\beta=1}^{n} \frac{\partial f_\alpha}{\partial x_i}\frac{\partial f_\beta}{\partial x_j}(h_{\alpha\beta}\circ f)$$

where $f_\alpha = y_\alpha \circ f$ for all $\alpha = 1, \ldots, n$.

1.3.3 Metric induced on the inverse tangent bundle

Definition 1.3.8. *Let $\varphi : (M, g) \longrightarrow (N, h)$ be a smooth map between two Riemannians manifolds, the inverse tangent bundle is defined by:*

$$\varphi^{-1}TN = \{(x, v)/x \in M,\ v \in T_{\varphi(x)}N\}.$$

A smooth map $v : M \longrightarrow TN$ is called section on $\varphi^{-1}TN$, such that

$$v(x) \in T_{\varphi(x)}N \quad \forall x \in M.$$

The set of sections on $\varphi^{-1}TN$ will be denoted by $\Gamma(\varphi^{-1}TN)$.

Definition 1.3.9. *Let $\varphi : M \longrightarrow N$ be a smooth map between two differentiable manifolds and let h be a Riemannian metric on N, then h induces a Riemannian metric on $\Gamma(\varphi^{-1}TN)$ by*

$$h(u, v)(x) = h_{\varphi(x)}(u_x, v_x),\ \forall\ x \in M \text{ and } u,\ v \in \Gamma(\varphi^{-1}TN).$$

1.4 Linear connection

Definition 1.4.1. *A linear connection on a smooth Riemannian manifold M is a map:*

$$\begin{aligned} \nabla : \Gamma(TM)\times\Gamma(TM) &\rightarrow \Gamma(TM), \\ (X, Y) &\mapsto \nabla_X Y \end{aligned}$$

such that:

1. $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z;$

2. $\nabla_X(fY) = f\nabla_X Y + X(f)Y;$

3. $\nabla_{X+fY}(Z) = \nabla_X Z + f\nabla_Y Z,$

for all $X, Y, Z \in \Gamma(TM)$ and $f \in C^\infty(M)$. We say that $\nabla_X Y$ is the covariant derivative of Y with the direction of X.

Definition 1.4.2. *A section $Y \in \Gamma(TM)$ is said to be parallel with respect to the connection ∇ if*

$$\nabla_X Y = 0, \ \forall \ X \in \Gamma(TM).$$

Definition 1.4.3. *If g is a Riemannian metric on M then a connection ∇ is said to be metric or compatible with g if,*

$$\nabla g = 0 \ \ i.e \ \ (\nabla_X g)(Y, Z) = 0,$$

that is:

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \ \ \forall \ X, Y, Z \in \Gamma(TM).$$

1.4.1 Torsion tensor

Definition 1.4.4. *Let M be a smooth manifold, and ∇ be a connection on the tangent bundle TM, then the torsion of ∇ is a tensor field of type $(1, 2)$ defined by:*

$$\begin{aligned} T : \Gamma(TM) \times \Gamma(TM) &\longrightarrow \Gamma(TM) \\ (X, \ Y) &\longmapsto \nabla_X Y - \nabla_Y X - [X, Y], \end{aligned}$$

where $[\ ,\ ]$ is the Lie bracket on $\Gamma(TM)$. The connection ∇ on the tangent bundle TM is said to be torsion-free if the corresponding torsion T vanishes i.e.

$$[X, Y] = \nabla_X Y - \nabla_Y X \quad \forall \ \ X, Y \in \Gamma(TM).$$

Remark 1.4.1. $T(X, Y) = -T(Y, X)$, foll all $X, Y \in \Gamma(TM)$ (T is an antisymmetric).

1.4.2 Levi-Civita connection

Definition 1.4.5. *Let (M, g) be a Riemannian manifold then the map*

$$\nabla : \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM)$$

defined by the Koszul formula:

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\ &\quad + g(Z, [X, Y]) + g(Y, [Z, X]) - g(X, [Y, Z]), \end{aligned} \tag{1.1}$$

is called the Levi-Civita connection of (M, g).

Theorem 1.4.1. *Let (M, g) be a Riemannian manifold. Then the Levi-Civita connection is an unique linear connection compatible with g and torsion free.*

Proposition 1.4.1. *Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇. Further let (U, φ) be a local coordinate on M and put $\partial_i = \frac{\partial}{\partial x_i} \in \Gamma(TU)$. Then $\{\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_m}\}$ is a local frame of TM on U. We define the Christoffel symbols $\Gamma_{ij}^k : U \longrightarrow \mathbb{R}$ of the connection ∇ with respect to (U, φ) by*

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{i=1}^{m} g^{kl} \{ \frac{\partial g_{jl}}{\partial x_i} + \frac{\partial g_{il}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l} \},$$

where $g_{ij} = g(e_i, e_j) = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ are the components of g, and $(g^{ij}) = (g_{ij})^{-1}$ is the inverse matrix.

1.5 Induced connection on the tangent bundle

Definition 1.5.1. *Let $\varphi : M \longrightarrow N$ be a smooth map between two differentiable manifolds M and N and let ∇^N be a linear connection on N, then the Pull-back connection on the tangent bundle $\varphi^{-1}TN$ is defined by:*

$$\begin{aligned} \nabla^{\varphi} : \Gamma(TM) \times \Gamma(\varphi^{-1}TN) &\longrightarrow \Gamma(\varphi^{-1}TN), \\ (X, V) &\longrightarrow \nabla_X^{\varphi} V = \nabla_{d\varphi(X)}^N \widetilde{V} \end{aligned} \tag{1.2}$$

where $\widetilde{V} \in \Gamma(TN)$ such that $\widetilde{V} \circ \varphi = V$.

Locally:

$$\begin{aligned} \nabla_X^{\varphi} V &= \nabla^{\varphi}_{X^i \frac{\partial}{\partial x_i}} V^{\alpha} (\frac{\partial}{\partial y_\alpha} \circ \varphi) \\ &= X^i \left\{ \frac{\partial V^{\alpha}}{\partial x_i} (\frac{\partial}{\partial y_\alpha} \circ \varphi) + V^{\alpha} \nabla^{\varphi}_{\frac{\partial}{\partial x_i}} (\frac{\partial}{\partial y_\alpha} \circ \varphi) \right\} \end{aligned}$$

Note that :

$$\begin{aligned} \nabla^{\varphi}_{\frac{\partial}{\partial x_i}} (\frac{\partial}{\partial y_\alpha} \circ \varphi) &= \nabla^N_{d\varphi(\frac{\partial}{\partial x_i})} \frac{\partial}{\partial y_\alpha} \\ &= \frac{\partial \varphi_\beta}{\partial x_i} \left(\nabla^N_{\frac{\partial}{\partial y_\beta}} \frac{\partial}{\partial y_\alpha} \right) \circ \varphi \\ &= \frac{\partial \varphi_\beta}{\partial x_i} \left(\Gamma^{\gamma}_{\alpha\beta} \frac{\partial}{\partial y_\gamma} \right) \circ \varphi \end{aligned}$$

So that

$$\nabla_X^{\varphi} V = X^i \left\{ \frac{\partial V^{\gamma}}{\partial x_i} + V^{\alpha} \frac{\partial \varphi_\beta}{\partial x_i} \left(\Gamma^{\gamma}_{\alpha\beta} \circ \varphi \right) \right\} \left(\frac{\partial}{\partial y_\gamma} \circ \varphi \right)$$

Then the relation (1.2) is independent of the choice of $\widetilde{V}$ i.e. this connection is well defined.

Definition 1.5.2. *If $\varphi : M \longrightarrow N$ is a map between differeniable manifolds, then two vector fields $X \in \Gamma(TM)$, $\widetilde{X} \in \Gamma(TN)$ are said to be φ-related if*

$$d\varphi_p(X) = \widetilde{X}_{\varphi(p)} \ \ \forall \ p \in M.$$

In that case we write $\widetilde{X} = d\varphi(X)$.

Proposition 1.5.1. *Let $\varphi : M \longrightarrow N$ be a smooth map and let ∇^N be a linear connection compatible with the Riemaniann metric h on N, then the linear connection ∇^φ is compatible with the induce Riemannian metric on $\varphi^{-1}TN$, that is*

$$X(h(V,W)) = h(\nabla^\varphi_X V, W) + h(V, \nabla^\varphi_X W),$$

for all $X \in \Gamma(TM)$ and V, $W \in \Gamma(\varphi^{-1}TN)$.

Proof. Let $X \in \Gamma(TM), V, W \in \Gamma(\varphi^{-1}TN)$ and $\widetilde{X}, \widetilde{V}, \widetilde{W} \in \Gamma(TN)$, such that

$$d\varphi(X) = \widetilde{X} \circ \varphi, \widetilde{V} \circ \varphi = V \text{ and } \widetilde{W} \circ \varphi = W$$

Then:

$$\begin{aligned}
X(h(V,W)) &= X(h(\widetilde{V} \circ \varphi, \widetilde{W} \circ \varphi)) \\
&= X(h(\widetilde{V}, \widetilde{W}) \circ \varphi) \\
&= d(h(\widetilde{V}, \widetilde{W}) \circ \varphi)(X) \\
&= dh(\widetilde{V}, \widetilde{W})(d\varphi(X)) \\
&= d\varphi(X)(h(\widetilde{V}, \widetilde{W})) \\
&= \widetilde{X}(h(\widetilde{V}, \widetilde{W})) \circ \varphi \\
&= h(\nabla^N_{\widetilde{X}} \widetilde{V}, \widetilde{W}) \circ \varphi + h(\widetilde{V}, \nabla^N_{\widetilde{X}} \widetilde{W}) \circ \varphi \\
&= h(\nabla^N_{\widetilde{X} \circ \varphi} \widetilde{V}, \widetilde{W} \circ \varphi) + h(\widetilde{V} \circ \varphi, \nabla^N_{\widetilde{X} \circ \varphi} \widetilde{W}) \\
&= h(\nabla^\varphi_X V, W) + h(V, \nabla^\varphi_X W).
\end{aligned}$$

□

Proposition 1.5.2. *Let ∇^N be a torsion free connection on N, then*

$$\nabla^\varphi_X d\varphi(Y) = \nabla^\varphi_Y d\varphi(X) + d\varphi([X,Y]),$$

For all $X, Y \in \Gamma(TM)$.

Proof. Let $V, W \in \Gamma(TN)$ be a φ-related with X and Y respectively, then:

$$\begin{aligned}
[V,W] \circ \varphi &= d\varphi \circ [X,Y] \\
\nabla^N_V W &= \nabla^N_W V + [V,W].
\end{aligned}$$

From where:

$$\begin{aligned}
\nabla^{\varphi}_X d\varphi(Y) &= \nabla^{\varphi}_X W \circ \varphi \\
&= \nabla^N_{d\varphi(X)} W \\
&= \big(\nabla^N_V W\big) \circ \varphi \\
&= \big(\nabla^N_W V + [V,W]\big) \circ \varphi \\
&= \nabla^{\varphi}_Y d\varphi(X) + d\varphi([X,Y]).
\end{aligned}$$

□

1.6 Second fundamental form

Definition 1.6.1. *Let $\varphi : (M,g) \to (N,h)$ be a smooth map between two Riemannian manifolds. The second fundamental form of φ is the covariant derivative of vectorial 1-form $d\varphi$, defined by:*

$$\nabla d\varphi(X,Y) = \nabla^{\varphi}_X d\varphi(Y) - d\varphi(\nabla^M_X Y)$$

For all $X, Y \in \Gamma(TM)$.

Definition 1.6.2. *A map $\varphi : (M,g) \longrightarrow (N,h)$ is said to be totally geodesic if its second fundamental form vanishes*

Property 1.6.1. *Let $\varphi : (M,g) \longrightarrow (N,h)$ be a smooth map between two Riemannian manifolds, the second fundamental form of φ is a vectorial 1-form $C^\infty(M)$-bilinear symmetric. i.e.*

$$\nabla d\varphi(f_1.X, f_2.Y) = f_1 f_2 \nabla d\varphi(Y,X),$$

for all X, $Y \in \Gamma(TM)$, and $f_1, f_2 \in C^\infty(M)$.

Proposition 1.6.1. *Let $\varphi : M \longrightarrow N$ and $\psi : N \longrightarrow P$ be a two smooth maps , then*

$$\nabla d(\psi \circ \varphi) = d\psi(\nabla d\varphi) + \nabla d\psi(d\varphi, d\varphi).$$

Proof. Let $X, Y \in \Gamma(TM)$, then

$$\begin{aligned}
\nabla d(\psi\circ\varphi)(X,Y) &= \nabla^{\psi\circ\varphi}_X d(\psi\circ\varphi)(Y) - d(\psi\circ\varphi)(\nabla^M_X Y) \\
&= \nabla^{\psi\circ\varphi}_X d\psi(d\varphi(Y)) - d\psi(d\varphi(\nabla^M_X Y)) \\
&= \nabla^P_{d\psi(d\varphi(X))} d\psi(d\varphi(Y)) - d\psi(d\varphi(\nabla^M_X Y)) \\
&= \nabla^{\psi}_{d\varphi(X)} d\psi(d\varphi(Y)) - d\psi(d\varphi(\nabla^M_X Y)) \\
&= \nabla d\psi(d\varphi(X), d\varphi(Y)) + d\psi(\nabla^N_{d\varphi(X)} d\varphi(Y)) - d\psi(d\varphi(\nabla^M_X Y))
\end{aligned}$$

$$= \nabla d\psi(d\varphi(X), d\varphi(Y)) + d\psi(\nabla d\varphi(X,Y)).$$

□

Definition 1.6.3. *Let (M, g) be an m-dimensional Riemannian manifold, the frame $\{e_i\}_{i=1}^m$ is said geodesic frame at $x \in M$, if it is orthonormal that is $g(e_i, e_j) = \delta_{ij}$ on $U \subset M$, and $(\nabla_{e_i} e_j)|_x = 0, \quad \forall i, j = 1...m$.*

1.7 Sub-manifolds

Definition 1.7.1. *Let M^m and N^n be a two differential manifolds such that $M \subset N$ and $\dim M \leq \dim N$. M is said a sub-manifold of N if the inclusion*

$$\begin{aligned} i : M &\hookrightarrow N \\ x &\longmapsto x \end{aligned}$$

is a plongement (i is an immersion and homeomorphism of M on $i(M)$ for induce topology). If (N^n, h) be a Riemannian manifold and M be a sub-manifold of N, then $g : \Gamma(TM) \times \Gamma(TM) \longrightarrow C^\infty(M)$ is the tensor field on M defined by

$$g(X,Y)_p = h_p(X_p, Y_p), \quad \textit{for all } X,\, Y \in \Gamma(TM) \textit{ and } p \in M.$$

It's called the induce metric on M by h.

Definition 1.7.2. *Let (N^n, h) be a Riemannian manifold and (M^m, g) be a Riemannian sub-manifold of (N^n, h). For a point $p \in M$ we define the normal space $T_pM^\perp$ of M at p by*

$$T_pM^\perp = \Big\{ v \in T_pN \ / \ h_p(v, w) = 0, \ \forall\, w \in T_pM \Big\}.$$

For all $p \in M$ we have the orthogonal decomposition

$$T_pN = T_pM \oplus T_pM^\perp.$$

The normal bundle of M in N is defined by $TM^\perp = \{(p, v) \ / \ p \in M, \ v \in T_pM^\perp\}$.

Proposition 1.7.1. *For all $v \in T_pN$, $\exists! v^\top \in T_pM$, $\exists! v^\perp \in T_pM^\perp$ such that $v = v^\top + v^\perp$. The maps $\top : T_pN \to T_pM$, $v \mapsto v^\top$ and $\perp : T_xN \to T_pM^\perp$, $v \mapsto v^\perp$ are $\mathbb{R}$-linear.*

Remark 1.7.1.

- A vector field X of N is said to be normal, if $X_x \in T_xM^\perp$ for all $x \in M$
- $\Gamma(TM)^\perp$ is the set of normal vector fields.

Definition 1.7.3. *Let (N,h) be a Riemannian manifold and M be a sub-manifold of N with the induced metric g. Then we define*

$$\nabla^M : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$$

by

$$\nabla^M_X Y = (\nabla^N_X Y)^\top,$$

∇^M is the Levi-Civita connection of the sub-manifold (M,g). Furthermore let

$$B : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)^\perp$$

be given by

$$B(X,Y) = (\nabla^N_X Y)^\perp,$$

the operator B is called the second fundamental form of M in (N,h).

Definition 1.7.4. *Let (N,h) be a Riemannian manifold and M be a sub-manifold of N with the induced metric g. Then the smooth section $H = \frac{1}{m}\mathrm{trace}B$ of the normal bundle $TM^\perp$ is called the mean curvature of M in N where $m = \dim M$ and $\mathrm{trace}B = \sum_{i=1}^m B(e_i,e_i)$, with $\{e_i\}_{i=1}^m$ is an orthonormal frame on (M,g).*

Remark 1.7.2.

- A sub-manifold M with mean curvature identically equal to zero is called minimal.
- A sub-manifold M is said to be totally geodesic if its second fundamental form vanishes.

Definition 1.7.5. *Let (M,g) be a Riemannian sub-manifold of (N,h) and let $x \in M$, x is said to be umbilical if there is a normal vector $z \in T_xM^\perp$ such that*

$$B(v,w) = g(v,w)z \ \ for\, all\ v,\, w \in T_xM.$$

M is said to be totally umbilical if there is a vector field $Z \in \Gamma(TM)^\perp$ such that $B(X,Y) = g(X,Y)Z$ for all $X,Y \in \Gamma(TM)$.

Remark 1.7.3.

- Z is called the normal vector field of curvature of M.
- We notice that any sub-manifold M which is minimal and totally umbilical is totally geodesic.

Definition 1.7.6. *Let (N,h) an n-dimensional Riemannian manifold. A Riemannian hypersurface of (N,h) is an m-dimensional Riemannian sub-manifold (M,g) of (N,h), where $m = n-1$.*

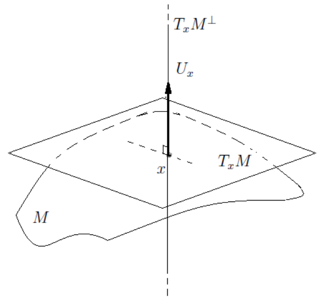

Figure 1.6: The unit vector field normal to M

Definition 1.7.7. *Let (M, g) be a Riemannian hypersurface of (N, h), and let U the unit vector field normal to M. The operator*

$$\begin{aligned} A : \Gamma(TM) &\rightarrow \Gamma(TM) \\ X &\mapsto AX = -\nabla^N_X U \end{aligned}$$

is called a shape operator.

Remark 1.7.4. $\forall X, Y \in \Gamma(TM)$, we have $g(AX, Y) = h(B(X, Y), U)$

Definition 1.7.8. *Let (M, g) be a Riemannian sub-manifold of (N, h)*

$$\begin{aligned} \nabla^{\perp} : \Gamma(TM) \times \Gamma(TM)^{\perp} &\rightarrow \Gamma(TM) \\ (X, Y) &\mapsto \nabla^{\perp}_X Y = (\nabla^N_X Y)^{\perp} \end{aligned}$$

is called the normal connection of M.

1.8 Curvatures

1.8.1 Curvature tensor

Definition 1.8.1. *The curvature tensor R is a tensor field of type $(1, 3)$ defined by*

$$\begin{aligned} R(X, Y)Z &:= [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z \\ &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z, \qquad \forall X, Y, Z \in \Gamma(TM). \end{aligned}$$

The curvature tensor of type $(0, 4)$ is given by

$$R(X, Y, Z, W) = g(R(X, Y)Z, W).$$

Proposition 1.8.1. *Let (M, g) be a smooth Riemannian manifold. For vector fields X, Y, Z, W on M we have*

1. $R(X,Y)Z = -R(Y,X)Z$ *(antisymmetric).*
2. $g(R(X,Y)Z,W) = g(R(Z,W)X,Y)$.
3. $g(R(X,Y)Z,Z) = 0$.
4. *R verified Bianchi's identity algebraic:*
$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0.$$
5. *R verified Bianchi's identity differential:*
$$(\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) + (\nabla_Z R)(X,Y) = 0.$$

1.8.2 Sectional curvature

Definition 1.8.2. *For a point $p \in M$ the function*

$$\begin{array}{rcl} K_p : \Gamma(TM) \times \Gamma(TM) & \to & \mathbb{R} \\ (X,Y) & \mapsto & \dfrac{g(R(X,Y)Y,X)}{g(X,X)g(Y,Y) - g(X,Y)^2}. \end{array}$$

is called the sectional curvature at p.
The Riemannian manifold M is said to be of constant curvature if there exists $k \in \mathbb{R}$ such that $K(X,Y) = k$.

Definition 1.8.3. *Let (M, g) be a smooth Riemannian manifold. We define the smooth tensor field $R_1 : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ of type $(3, 1)$ by*

$$R_1(X,Y)Z = g(Y,Z)X - g(X,Z)Y.$$

for all $X, Y, Z \in \Gamma(TM)$.

Corollary 1.8.1. *Let $(M^m, g)(m \geq 2)$ be a Riemannian manifold of constant curvature k. Then the curvature tensor R is given by*

$$R(X,Y)Z = k[R_1(X,Y)Z].$$

for all $X, Y, Z \in \Gamma(TM)$.

1.8.3 Ricci curvature

Definition 1.8.4. *Let (M^m, g) be a Riemannian manifold, $p \in M$ and $\{e_1, ...e_m\}$ be an orthonormal frame of T_pM. Then*

1. *the Ricci tensor at p is defined by*

$$\text{Ricci}(X) = \sum_{i=1}^{m} R(X, e_i)e_i, \qquad \forall\ X \in T_pM.$$

2. *the Ricci curvature at p is defined by*

$$\text{Ric}(X, Y) = \sum_{i=1}^{m} g(R(X, e_i)e_i, Y), \qquad \forall\, X,\, Y \in T_pM.$$

3. *the scalar curvature S is defined by*

$$\begin{aligned} S &= \text{trace}_g \text{Ric} \\ &= \sum_{i,j=1}^{m} g(R(e_i, e_j)e_j, e_i) \end{aligned}$$

Remark 1.8.1. For all $X, Y \in \Gamma(TM)$ we have:

$$\text{Ric}(X, Y) = g(\text{Ricci}(X), Y)$$

Corollary 1.8.2. *Let (M^m, g) be a Riemannian manifold of constant curvature k, then:*

1. $\text{Ricci}(X) = (m-1)kX$.
2. $\text{Ric}(X, Y) = (m-1)kg(X, Y)$.
3. $S = m(m-1)k$.

Example 1.8.1.

1. The sphere $\mathbb{S}^n$ has constant sectional curvature $+1$.
2. The space $\mathbb{R}^n$ has curvature 0 .
3. $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2\ y > 0\}$ The hyperbolic space with the metric $g = \frac{dx^2+dy^2}{y^2}$, has constant sectional curvature -1.

1.9 Operators on Riemannian manifold

1.9.1 Gradient operator

Let (M,g) be a Riemannian manifold,

$$\begin{array}{rcl} \sharp : \Gamma(T^*M) & \to & \Gamma(TM) \\ \omega & \mapsto & \omega^\sharp \end{array}$$

be a isomorphism map between the cotangent bundle and the tangent bundle given by

$$\forall X \in \Gamma(TM),\ g(\omega^\sharp, X) = \omega(X).$$

Definition 1.9.1. *Let (M,g) be a Riemannian manifold, the gradient operator is given by*

$$\begin{array}{rcl} \operatorname{grad} : C^\infty(M) & \longrightarrow & \Gamma(TM). \\ f & \mapsto & \operatorname{grad} f = (df)^\sharp \end{array}$$

So that for all $X \in \Gamma(TM)$ we have

$$g(\operatorname{grad} f, X) = X(f) = df(X).$$

Locally:

$$\operatorname{grad} f = \sum_{i=1}^{m} g^{ij} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j},$$

where $(\frac{\partial}{\partial x_i})_{i=1,\dots m}$ is a local coordinate. Let $\{e_i\}_{i=1,\dots m}$ be an orthonormal frame on (M,g). Then

$$\operatorname{grad} f = \sum_{i=1}^{m} e_i(f) e_i.$$

Proposition 1.9.1. *Let (M,g) be a Riemannian manifold, then*

1. $\operatorname{grad}(f+h) = \operatorname{grad} f + \operatorname{grad} h$;
2. $\operatorname{grad}(fh) = h \operatorname{grad} f + f \operatorname{grad} h$;
3. $(\operatorname{grad} f)(h) = (\operatorname{grad} h)(f)$.
4. $g(\nabla_X \operatorname{grad} f, Y) = g(\nabla_Y \operatorname{grad} f, X)$,

where $f,\ h \in \mathbf{C}^\infty(M)$ and $X, Y \in \Gamma(TM)$.

1.9.2 Hessian operator

Definition 1.9.2. *Let f be a differentiable function on (M^m, g), then*

$$\begin{array}{rcl} \text{Hess}\, f : \Gamma(TM) \times \Gamma(TM) & \longrightarrow & C^\infty(M) \\ (X, \quad Y) & \mapsto & (\text{Hess}\, f)(X,Y) = g(\nabla_X \text{grad}\, f, Y) \end{array}$$

we have

1. Hess f *be a tensor of type* $(0,2)$.
2. Hess f *is symmetric.*

Locally:

$$\text{Hess}\, f = \sum_{i,j=1}^{m} (\text{Hess}\, f)_{ij}\, dx_i \otimes dx_j,$$

where

$$\begin{aligned} (\text{Hess}\, f)_{ij} &= g(\nabla_{\partial_i} \text{grad}\, f, \partial_j) \\ &= \frac{\partial^2 f}{\partial x_i \partial x_j} - \sum_{k=1}^{m} \Gamma_{ij}^k \frac{\partial f}{\partial x_k}. \end{aligned}$$

1.9.3 Divergence operator

Let X be a vector field on (M, g), then

$$\begin{array}{rcl} \nabla X : \Gamma(TM) & \longrightarrow & \Gamma(TM) \\ Z & \mapsto & \nabla_Z X \end{array}$$

is a smooth linear mapping.

Definition 1.9.3. *The divergence of the vector field $X \in \Gamma(TM)$, denoted* $\text{div}\, X$ *is defined by*

$$\text{div}\, X = \text{trace} \nabla X.$$

Let $\{e_i\}_{i=1,\dots m}$ be an orthonormal frame on M, then

$$\text{div}\, X = \sum_{i=1}^{m} g(\nabla_{e_i} X, e_i).$$

Propertys 1.9.1. *Let (M, g) be a Riemannian manifold, then*

1. $\text{div}\,(X+Y) = \text{div}\, X + \text{div}\, Y$;

2. $\operatorname{div}(fX) = f\operatorname{div} X + X(f)$,

for all $X, Y \in \Gamma(TM)$ *and* $f \in C^\infty(M)$.

Definition 1.9.4. *The divergence of 1-form* $\omega \in \Gamma(T^*M)$ *is defined by*

$$\operatorname{div}^M \omega = \sum_{i=1}^{m} (e_i(\omega(e_i)) - \omega(\nabla^M_{e_i} e_i))$$

Proposition 1.9.2. *Let* $\omega, \eta \in \Gamma(T^*M)$ *and* $f \in \mathbf{C}^\infty(M)$, *then*

1. $div(\omega + \eta) = div\, \omega + div\, \eta$.
2. $div(f\,\omega) = f div\, \omega + \omega(\text{grad } f)$.

1.9.4 Laplacian operator

Definition 1.9.5. *Let* (M^m, g) *be a Riemannian manifold, the Laplacian operator noted* $\triangle$, *on* M *is defined by*

$$\begin{array}{rcl} \triangle : C^\infty(M) & \longrightarrow & C^\infty(M) \\ f & \mapsto & \triangle(f) = \operatorname{div}(\operatorname{grad} f) \end{array}$$

Propertys 1.9.2. *Let* (M^m, g) *be a Riemannian manifold, then*

1. $\triangle(f + h) = \triangle(f) + \triangle(h)$;
2. $\triangle(fh) = h \triangle (f) + f \triangle (h) + 2g(\operatorname{grad} f, \operatorname{grad} h)$,

for all $f, h \in C^\infty(M)$.

1.9.5 Divergence Theorem

Proposition 1.9.3. *Let* (M^m, g) *be a Riemannian manifold, and let* D *be a compact domain with boundary on* M. *Let* ω *be an* 1*-form and* X *a vector field defined on a neighborhood in* D, *then*

$$\int_D (\operatorname{div} X) v_g = \int_{\partial D} g(X, \mathfrak{n}) v^{\partial D} \qquad \textit{and} \qquad \int_D (\operatorname{div} \omega) v_g = \int_{\partial D} \omega(\mathfrak{n}) v^{\partial D}$$

where ∂D *is the boundary of* D *and* $\mathfrak{n} \approx \mathfrak{n}(x)$ *is the unit normal at a point* $x \in \partial D$.

Corollary 1.9.1. *Let* X *be a vector field (resp.* Ω *an* 1*-form) with compact supports in a domain* D, *then:*

$$\int_D (\operatorname{div} X) v_g = 0 \qquad \textit{and} \qquad \int_D (\operatorname{div} \omega) v_g = 0.$$

Chapter 2

Harmonic maps

In this chapter we define the harmonic and bi-harmonic maps, we give some results on stable harmonic maps [22] [45], some properties of harmonicity and homothetic vector field, established by Ahmed Mohammed Cherif [27].
We start this chapter by studying the geodesics which will play very important roles in the following works.

2.1 Geodesics

Let (M, g) be a Riemannian manifold, and let $\gamma : \mathbb{R} \supset I \to M$ be a C^∞-curve on M. The set of the vector fields along γ, is defined by

$$\Gamma(\gamma^{-1}TM) = \Big\{ Y : I \to TM \ \ /Y(t) \in T_{\gamma(t)}M, \ \forall t \in I \Big\}.$$

Definition 2.1.1. *Let (M^m, g) be an m-dimensional Riemannian manifold. A curve γ on (M^m, g) is called geodesic if $\nabla^\gamma_{\frac{d}{dt}} \gamma(\frac{d}{dt}) = 0$. i.e.*

$$\frac{d^2\gamma_k}{dt^2} + \sum_{i,j=1}^{m} \frac{d\gamma_i}{dt}\frac{d\gamma_j}{dt}(\Gamma^k_{ij} o \gamma) = 0, \quad \forall \ k = 1...m.$$

Example 2.1.1. Let $(\mathbb{R},\, g_0 = dx^2)$ be a Riemannian manifold, then a curve $\gamma : I \to \mathbb{R}$ is geodesic if and only if $\frac{d^2\gamma_k}{dt^2} = 0$, because $\Gamma^1_{11} = 0$, which implies $\gamma(t) = at + b,$ where $a,\, b \in \mathbb{R}$

Definition 2.1.2. *Let (M, g) be a Riemannian manifold and $\gamma : I \to M$ be a C^r-curve on M. A variation of γ is a C^r-map $\varphi : (-\epsilon, \epsilon) \times I \to M$ such that for all $s \in I$, $\varphi_0(s) = \varphi(0, s) = \gamma(s)$. If the interval is compact i.e. of the form $I = [a, b]$, then the variation φ is called proper if for all $t \in (-\epsilon, \epsilon)$, $\varphi_t(a) = \gamma(a)$ and $\varphi_t(b) = \gamma(b)$.*

Definition 2.1.3. *Let (M, g) be a Riemannian manifold and $\gamma : I \to M$ be a C^2-curve on M. For every compact interval $[a, b] \subset I$ we define the energy functional $E_{[a,b]}$ by*

$$E_{[a,b]}(\gamma) = \frac{1}{2}\int_a^b g(\dot\gamma(t), \dot\gamma(t))dt.$$

A C^2-curve γ is called a critical point for the energy functional if every proper variation φ of $\gamma/_{[a,b]}$ satisfies

$$\frac{d}{dt}(E_{[a,b]}(\varphi_t))\Big|_{t=0} = 0.$$

Theorem 2.1.1. *[17]. A C^2-curve γ is a critical point for the energy functional if and only if it is a geodesic.*

Proof. For a C^2-map

$$\begin{aligned} \varphi : (-\epsilon, \epsilon) \times I &\to M \\ (t, s) &\mapsto \varphi(t, s) \end{aligned}$$

We define the vector field $X = d\varphi(\frac{\partial}{\partial s})$ and $Y = d\varphi(\frac{\partial}{\partial t})$ along φ. The following shows that the vector fields X and Y commute:

$$\nabla_X Y - \nabla_Y X = [X, Y] = \Big[d\varphi(\frac{\partial}{\partial s}),\, d\varphi(\frac{\partial}{\partial t})\Big] = d\varphi(\Big[\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\Big]) = 0,$$

since $\left[\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right] = 0$. We now assume that φ is a proper variation of $\gamma/_{[a,b]}$. Then

$$\begin{aligned} \frac{d}{dt}\Big(E_{[a,b]}(\varphi_t)\Big) &= \frac{1}{2}\frac{d}{dt}\Big(\int_a^b g(X, X)ds\Big) \\ &= \frac{1}{2}\int_a^b \frac{d}{dt}\Big(g(X, X)\Big)ds \\ &= \int_a^b g(\nabla_Y X, X)ds \\ &= \int_a^b g(\nabla_X Y, X)ds \\ &= \int_a^b \Big(\frac{d}{ds}\Big(g(Y, X)\Big) - g(Y, \nabla_X X)\Big)ds \\ &= \Big[g(Y, X)\Big]_a^b - \int_a^b g(Y, \nabla_X X)ds \end{aligned}$$

The variation is proper, so $Y(a) = Y(b) = 0$. Furthermore

$$X(0, s) = \frac{\partial\varphi}{\partial s}(0, s) = \dot\gamma(s).$$

So

$$\frac{d}{dt}(E_{[a,b]}(\varphi_t))\Big|_{t=0} = -\int_a^b g(Y(0,s),(\nabla_{\dot\gamma}\dot\gamma)(s))ds.$$

The last integral vanishes for every proper variation φ of γ if and only if $\nabla_{\dot\gamma}\dot\gamma = 0$.
A geodesic $\gamma : I \to (M,g)$ is special case of what is called a harmonic map $\varphi : (M,g) \to (N,h)$ between Riemannian manifolds. □

2.2 Harmonic maps

Definition 2.2.1. *Consider a smooth map $\varphi : (M^m, g) \to (N^n, h)$ between Riemannian manifolds, for any compact domain D of M the energy functional of φ is defined by*

$$E(\varphi; D) = \frac{1}{2}\int_D |d\varphi|^2 \, v_g, \tag{2.1}$$

where $|d\varphi|$ is the Hilbert Schmidt norm of differential of φ given by

$$|d\varphi|^2 = \sum_{i=1}^m h(d\varphi(e_i), d\varphi(e_i))$$

$\{e_1, \dots, e_m\}$ *be an orthonormal frame on M*

Definition 2.2.2. *A variation of φ to support in a compact domain $D \subset M$, is a smooth family maps $(\varphi_t)_{t\in(-\epsilon,\epsilon)} : M \to N$, such that $\varphi_0 = \varphi$ and $\varphi_t = \varphi$ on $M \setminus \mathrm{int}(D)$.*

Definition 2.2.3. *A map is called harmonic if it is a critical point of the energy functional over any compact subset D of M. i.e*

$$\frac{d}{dt}E(\varphi_t; D)\Big|_{t=0} = 0.$$

2.2.1 First variation of energy

Theorem 2.2.1. *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map and let $(\varphi_t)_{t\in(-\epsilon,\epsilon)}$ be a smooth variation of φ supported in D. Then*

$$\frac{d}{dt}E(\varphi_t; D)\Big|_{t=0} = -\int_D h(v, \tau(\varphi))\, v_g,$$

where $v = \dfrac{d\varphi_t}{dt}\Big|_{t=0}$ *denotes the variation vector field of $\{\varphi_t\}$,*

$$\tau(\varphi) = \mathrm{trace}_g \nabla d\varphi = \sum_{i=1}^m \left\{\nabla^\varphi_{e_i} d\varphi(e_i) - d\varphi(\nabla^M_{e_i} e_i)\right\} \tag{2.2}$$

is called tension field of φ where $\{e_1, ..., e_m\}$ is an orthonormal frame on (M^m, g).

Proof. Defined $\phi : M \times (-\epsilon, \epsilon) \to N$ by $\phi(x,t) = \varphi_t(x)$, let ∇^ϕ denote the pull-back connection on $\phi^{-1}TN$. Note that, for any vector field X on M considered as a vector field on $M \times (-\epsilon, \epsilon)$, we have $[\partial_t, X] = 0$. Using (2.1) we obtain

$$\begin{aligned}
\frac{d}{dt}E(\varphi_t; D)\Big|_{t=0} &= \frac{1}{2}\frac{d}{dt}\int_D \sum_{i=1}^m h(d\varphi_t(e_i), d\varphi_t(e_i))\, v_g\Big|_{t=0} \\
&= \frac{1}{2}\frac{d}{dt}\int_D \sum_{i=1}^m h(d\phi(e_i,0), d\phi(e_i,0))\, v_g\Big|_{t=0} \\
&= \frac{1}{2}\int_D \frac{\partial}{\partial t}\sum_{i=1}^m h(d\phi(e_i,0), d\phi(e_i,0))\, v_g\Big|_{t=0} \\
&= \int_D \sum_{i=1}^m h(\nabla^\phi_{(0,\frac{d}{dt})} d\phi(e_i,0), d\phi(e_i,0))\, v_g\Big|_{t=0} \\
&= \int_D \sum_{i=1}^m h(\nabla^\phi_{(e_i,0)} d\phi(0,\frac{d}{dt}), d\phi(e_i,0))\, v_g\Big|_{t=0} \\
&= \int_D \sum_{i=1}^m h(\nabla^N_{d\varphi(e_i)} v, d\varphi(e_i))\, v_g \\
&= \int_D \sum_{i=1}^m h(\nabla^\varphi_{e_i} v, d\varphi(e_i))\, v_g. \qquad (2.3)
\end{aligned}$$

Define an 1-form on M by

$$\omega(X) = h(v, d\varphi(X)), \quad X \in \Gamma(TM).$$

We have

$$\begin{aligned}
\mathrm{div}^M \omega &= (\nabla_{e_i}\omega)(e_i) \\
&= \sum_{i=1}^m \left\{ e_i(\omega(e_i)) - \omega(\nabla^M_{e_i} e_i) \right\} \\
&= \sum_{i=1}^m \left\{ h(\nabla^\varphi_{e_i} v, d\varphi(e_i)) + h(v, \nabla^\varphi_{e_i} d\varphi(e_i)) - h(v, d\varphi(\nabla^M_{e_i} e_i)) \right\} \\
&= \sum_{i=1}^m h(\nabla^\varphi_{e_i} v, d\varphi(e_i)) + h(v, \tau(\varphi)), \qquad (2.4)
\end{aligned}$$

according to formulas (2.3), (2.4), and divergence Theorem, we obtain

$$\frac{d}{dt}E(\varphi_t; D)\Big|_{t=0} = -\int_D h(v, \tau(\varphi)) v_g.$$

□

Theorem 2.2.2. *A smooth map $\varphi : (M^m, g) \to (N^n, h)$ between Riemannian manifolds is harmonic if and only if*

$$\tau(\varphi) = \mathrm{trace}\nabla d\varphi = 0.$$

Remark 2.2.1. **Locally:**

$$\tau(\varphi) = g^{ij}\left(\frac{\partial^2\varphi_\gamma}{\partial x_i \partial x_j} + \frac{\partial\varphi_\alpha}{\partial x_i}\frac{\partial\varphi_\beta}{\partial x_j}\ {}^N\Gamma^\gamma_{\alpha\beta}\circ\varphi - \frac{\partial\varphi_\gamma}{\partial x_k}\ {}^M\Gamma^k_{ij}\right)\frac{\partial}{\partial y_\gamma}\circ\varphi. \tag{2.5}$$

$(\frac{\partial}{\partial x_i})$ (resp. $(\frac{\partial}{\partial y_\alpha})$) is a local frame of vector fields on M (resp. on N).

Example 2.2.1. Any constant map $\varphi : (M, g) \longrightarrow (N, h)$ is a harmonic map (because $d\varphi = 0$).

Example 2.2.2. The second fundamental form of the identity mapping $\mathrm{Id_M} : (M, g) \longrightarrow (M, g)$ is zero, i.e. $\mathrm{Id_M}$ is totally geodesic, therefore $\mathrm{Id_M}$ is harmonic.

Remark 2.2.2. Any totally geodesic map is a harmonic map, the reverse is not always true.

Example 2.2.3. Let φ be a map defined by

$$\begin{array}{rcl} \varphi : \mathbb{R}^2 & \to & \mathbb{R} \\ (x, y) & \mapsto & \varphi(x, y) = x^2 - y^2 \end{array}$$

We have $\Delta\varphi = 0$ then, φ is harmonic, and other hand we have

$$\begin{array}{rcl} (\nabla d\varphi)(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) & = & \nabla^\varphi_{\frac{\partial}{\partial x}} d\varphi(\frac{\partial}{\partial x}) - d\varphi(\nabla^{\mathbb{R}^2}_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x}) \\ & = & \nabla^\varphi_{\frac{\partial}{\partial x}} d\varphi(\frac{\partial}{\partial x}) \\ & = & \dfrac{\partial^2\varphi}{\partial^2 x} = 2. \end{array}$$

Then φ is not totaly geodesic.

Example 2.2.4. Let (M, g) be a Riemannian manifold and let $f : M \longrightarrow \mathbb{R}$ be a smooth function, then

$$\begin{array}{rcl} \tau(f) & = & \mathrm{trace}\nabla df \\ & = & \nabla df(e_i, e_i) \\ & = & \nabla^f_{e_i} df(e_i) - df(\nabla^M_{e_i} e_i) \\ & = & e_i(e_i(f)) - (\nabla^M_{e_i} e_i)(f) \\ & = & g(\nabla_{e_i}\mathrm{grad}\, f, e_i) \\ & = & \mathrm{div}\,\mathrm{grad} f \\ & = & \triangle(f), \end{array}$$

where $\{e_i\}$ is an orthonormal frame on M.

Example 2.2.5. Let $\mathbb{R}^n$ be provided with the canonical metric g_0 and let

$$\varphi : (M, g) \longrightarrow (\mathbb{R}^n, g_0),\ \varphi(x) = (\varphi_1(x), \dots, \varphi_n(x)),$$

be a differentiable map. According to the formula (2.5) with ${}^{\mathbb{R}^n}\Gamma^{\gamma}_{\alpha\beta} = 0$, we have

$$\tau(\varphi) = g^{ij}\left(\frac{\partial^2 \varphi_\gamma}{\partial x_i \partial x_j} - \frac{\partial \varphi_\gamma}{\partial x_k}\ {}^{M}\Gamma^k_{ij}\right)\frac{\partial}{\partial y_\gamma} \circ \varphi$$

that is

$$\tau(\varphi) = (\triangle(\varphi_1), \dots, \triangle(\varphi_n)),$$

hence the map φ is harmonic if and only if $\triangle(\varphi_\alpha) = 0$, $\forall \alpha = 1, \dots, n$, i.e. φ_α are harmonic functions.

Example 2.2.6. If $M =]a, b[$ be an interval of $\mathbb{R}$, then a curve $\gamma : (a, b) \longrightarrow (N^n, h)$ is harmonic if

$$\frac{d^2\gamma^\alpha}{dt^2} +^{N} \Gamma^{\alpha}_{\beta\delta}\frac{d\gamma^\beta}{dt}\frac{d\gamma^\delta}{dt} = 0,$$

therefore, γ is harmonic if and only if it is a geodesic.

Remark 2.2.3. If φ is a harmonic map and if ψ is a totally geodesic map, then $\psi \circ \varphi$ is a harmonic map.

Remark 2.2.4. The compound of two harmonic maps is not generally a harmonic map.

Example 2.2.7. We define the maps φ and ψ by

$$\begin{array}{rcl} \varphi : \mathbb{R}^2 & \to & \mathbb{R} \\ (x, y) & \mapsto & x^2 - y^2 \end{array}$$

and

$$\begin{array}{rcl} \psi : \mathbb{R} & \to & \mathbb{R}^2 \\ z & \mapsto & (z, 0) \end{array}$$

We have

$$\begin{aligned} \Delta\varphi &= \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} \\ &= 2 - 2 \\ &= 0, \end{aligned}$$

then, φ is harmonic, and

$$\begin{aligned} \Delta\psi &= (\Delta\psi^1, \Delta\psi^2) \\ &= (0, 0) \\ &= 0, \end{aligned}$$

then, ψ is harmonic.

$$\begin{aligned}\varphi\circ\psi:\mathbb{R}&\to\mathbb{R}^2\to\mathbb{R}\\ z&\mapsto(z,0)\mapsto z^2\end{aligned}$$

$$\Delta(\varphi\circ\psi)=2\neq 0.$$

Then, $\varphi o\psi$ is not harmonic.

Example 2.2.8. Let S be a surface in Euclidean space $\mathbb{R}^3$, and let

$$\varphi:(\Omega,<>_{\mathbb{R}^2})\longrightarrow(\mathbb{R}^3,<>_{\mathbb{R}^3})$$

be a local parametrization of S, where Ω is an open subset of $\mathbb{R}^2$, such that:

$$\left|\frac{\partial\varphi}{\partial x}\right|^2=\left|\frac{\partial\varphi}{\partial y}\right|^2,\ \left\langle\frac{\partial\varphi}{\partial x},\frac{\partial\varphi}{\partial y}\right\rangle_{\mathbb{R}^3}=0.$$

Let

$$N=\frac{\frac{\partial\varphi}{\partial x}\wedge\frac{\partial\varphi}{\partial y}}{|\frac{\partial\varphi}{\partial x}\wedge\frac{\partial\varphi}{\partial y}|}$$

The normal unit vector,

$$E=\left|\frac{\partial\varphi}{\partial x}\right|^2,\quad F=\left\langle\frac{\partial\varphi}{\partial x},\frac{\partial\varphi}{\partial y}\right\rangle_{\mathbb{R}^3},\quad G=\left|\frac{\partial\varphi}{\partial y}\right|^2$$

$$e=\left\langle N,\frac{\partial^2\varphi}{\partial x^2}\right\rangle_{\mathbb{R}^3},\quad f=\left\langle N,\frac{\partial^2\varphi}{\partial x\partial y}\right\rangle_{\mathbb{R}^3},\quad g=\left\langle N,\frac{\partial^2\varphi}{\partial y^2}\right\rangle_{\mathbb{R}^3},$$

$$H=\frac{1}{2}\frac{eG+gE-2fF}{EG-F^2}$$

The mean curvature of S. We have

$$\begin{aligned}\left\langle\frac{\partial\varphi}{\partial x},\tau(\varphi)\right\rangle_{\mathbb{R}^3}&=\left\langle\frac{\partial\varphi}{\partial x},\frac{\partial^2\varphi}{\partial x^2}\right\rangle_{\mathbb{R}^3}+\left\langle\frac{\partial\varphi}{\partial x},\frac{\partial^2\varphi}{\partial y^2}\right\rangle_{\mathbb{R}^3}\\&=\left\langle\frac{\partial\varphi}{\partial x},\frac{\partial^2\varphi}{\partial x^2}\right\rangle_{\mathbb{R}^3}-\left\langle\frac{\partial^2\varphi}{\partial x\partial y},\frac{\partial\varphi}{\partial y}\right\rangle_{\mathbb{R}^3}\\&=\frac{1}{2}\frac{\partial}{\partial x}\left|\frac{\partial\varphi}{\partial x}\right|^2-\frac{1}{2}\frac{\partial}{\partial x}\left|\frac{\partial\varphi}{\partial y}\right|^2\\&=0.\end{aligned}$$

In the same way, $\left\langle\frac{\partial\varphi}{\partial y},\tau(\varphi)\right\rangle_{\mathbb{R}^3}=0$. Therefore $\tau(\varphi)$ is normal on the surface S, and we have

$$H=\frac{e+g}{2E}=\frac{\langle N,\tau(\varphi)\rangle_{\mathbb{R}^3}}{2E}.$$

Then, S is minimal if and only if φ is harmonic.

2.2.2 Second variation of energy

Theorem 2.2.3. *Let $\varphi : (M, g) \longrightarrow (N, h)$ be a harmonic map between Riemannian manifolds, and $\{\varphi_{t,s}\}$ be a two-parameter variation with compact support in D. We set:*

$$v = \frac{\partial \varphi_{t,s}}{\partial t}\bigg|_{(t,s)=(0,0)} \quad \text{and} \quad w = \frac{\partial \varphi_{t,s}}{\partial s}\bigg|_{(t,s)=(0,0)}$$

denotes the variation vector fields of φ.
Under the notation above we have the following

$$\frac{\partial^2}{\partial t \partial s} E(\varphi_{t,s}; D)\bigg|_{(t,s)=(0,0)} = \int_D h(J_\varphi(v), w)\, v_g,$$

where $J_\varphi(v) \in \Gamma(\varphi^{-1}TN)$ given by

$$J_\varphi(v) = -\operatorname{trace} R^N(v, d\varphi)d\varphi - \operatorname{trace}(\nabla^\varphi)^2 v.$$

R^N is the curvature tensor on (N, h), and

$$\operatorname{trace}(\nabla^\varphi)^2 v = \sum_{i=1}^m \left[\nabla^\varphi_{e_i} \nabla^\varphi_{e_i} v - \nabla^\varphi_{\nabla^M_{e_i} e_i} v\right]$$

Proof. Define $\phi : M \times (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \longrightarrow N$ by $\phi(x, t, s) = \varphi_{t,s}(x)$. Let ∇^ϕ denote the pull-back connection on $\phi^{-1}TN$. Note that, for any vector field X on M considered as a vector field on $M \times (-\epsilon, \epsilon) \times (-\epsilon, \epsilon)$, we have

$$[\partial_t, X] = 0, \;\; [\partial_s, X] = 0, \;\; [\partial_t, \partial_s] = 0.$$

We put $E_i = (e_i, 0, 0)$, $\frac{\partial}{\partial t} = (0, \frac{d}{dt}, 0)$ and $\frac{\partial}{\partial s} = (0, 0, \frac{d}{ds})$. Then, by (2.1) we obtain

$$\frac{\partial^2}{\partial t \partial s} E(\varphi_{t,s}; D)\bigg|_{(t,s)=(0,0)} = \frac{1}{2} \int_D \sum_{i=1}^m \frac{\partial^2}{\partial t \partial s} h(d\phi(E_i), d\phi(E_i))\, v_g \bigg|_{(t,s)=(0,0)}, \quad (2.6)$$

first, note that

$$\begin{aligned} \frac{1}{2}\frac{\partial^2}{\partial t \partial s} h(d\phi(E_i), d\phi(E_i)) &= \frac{\partial}{\partial t} h(\nabla^\phi_{\frac{\partial}{\partial s}} d\phi(E_i), d\phi(E_i)) \\ &= h(\nabla^\phi_{\frac{\partial}{\partial t}} \nabla^\phi_{\frac{\partial}{\partial s}} d\phi(E_i), d\phi(E_i)) \\ &\quad + h(\nabla^\phi_{\frac{\partial}{\partial s}} d\phi(E_i), \nabla^\phi_{\frac{\partial}{\partial t}} d\phi(E_i)), \end{aligned} \quad (2.7)$$

the first term on the left-hand side of (2.7) is

$$h(\nabla^\phi_{\frac{\partial}{\partial t}} \nabla^\phi_{\frac{\partial}{\partial s}} d\phi(E_i), d\phi(E_i)) = h(\nabla^\phi_{\frac{\partial}{\partial t}} \nabla^\phi_{E_i} d\phi(\frac{\partial}{\partial s}), d\phi(E_i))$$

$$\begin{aligned} &= h(R^N(d\phi(\frac{\partial}{\partial t}), d\phi(E_i))d\phi(\frac{\partial}{\partial s}), d\phi(E_i)) \\ &+h(\nabla^\phi_{E_i}\nabla^\phi_{\frac{\partial}{\partial t}} d\phi(\frac{\partial}{\partial s}), d\phi(E_i)) \\ &+h(\nabla^\phi_{\left[\frac{\partial}{\partial t}, E_i\right]} d\phi(\frac{\partial}{\partial s}), d\phi(E_i)). \end{aligned} \tag{2.8}$$

Define an 1-form on M by

$$\omega(X) = h(\nabla^\phi_{\frac{\partial}{\partial t}} d\phi(\frac{\partial}{\partial s})\Big|_{(t,s)=(0,0)}, d\varphi(X)), \quad X \in \Gamma(TM).$$

We calculate the divergence of ω.

$$\begin{aligned} \mathrm{div}^M\omega &= \sum_{i=1}^m \{e_i(\omega(e_i)) - \omega(\nabla^M_{e_i} e_i)\} \\ &= \sum_{i=1}^m \{e_i(h(\nabla^\phi_{\frac{\partial}{\partial t}} d\phi(\frac{\partial}{\partial s})\Big|_{(t,s)=(0,0)}, d\varphi(e_i))) - h(\nabla^\phi_{\frac{\partial}{\partial t}} d\phi(\frac{\partial}{\partial s})\Big|_{(t,s)=(0,0)}, d\varphi(\nabla^M_{e_i} e_i))\} \\ &= \sum_{i=1}^m \{h(\nabla^\phi_{E_i}\nabla^\phi_{\frac{\partial}{\partial t}} d\phi(\frac{\partial}{\partial s})\Big|_{(t,s)=(0,0)}, d\varphi(e_i)) \\ &\quad +h(\nabla^\phi_{\frac{\partial}{\partial t}} d\phi(\frac{\partial}{\partial s})\Big|_{(t,s)=(0,0)}, \nabla^\varphi_{e_i} d\varphi(e_i)) - h(\nabla^\phi_{\frac{\partial}{\partial t}} d\phi(\frac{\partial}{\partial s})\Big|_{(t,s)=(0,0)}, d\varphi(\nabla^M_{e_i} e_i))\} \\ &= \sum_{i=1}^m \{h(\nabla^\phi_{E_i}\nabla^\phi_{\frac{\partial}{\partial t}} d\phi(\frac{\partial}{\partial s})\Big|_{(t,s)=(0,0)}, d\varphi(e_i)) + h(\nabla^\phi_{\frac{\partial}{\partial t}} d\phi(\frac{\partial}{\partial s})\Big|_{(t,s)=(0,0)}, \tau(\varphi)). \end{aligned}$$

According to the harmonicity of φ we obtain

$$\mathrm{div}^M\omega = \sum_{i=1}^m \{h(\nabla^\phi_{E_i}\nabla^\phi_{\frac{\partial}{\partial t}} d\phi(\frac{\partial}{\partial s})\Big|_{(t,s)=(0,0)}, d\varphi(e_i)). \tag{2.9}$$

From the formulas (2.8) and (2.9), with $\left[\frac{\partial}{\partial t}, E_i\right] = 0$, we get

$$\begin{aligned} h(\nabla^\phi_{\frac{\partial}{\partial t}}\nabla^\phi_{\frac{\partial}{\partial s}} d\phi(E_i), d\phi(E_i))\Big|_{(t,s)=(0,0)} &= \sum_{i=1}^m h(R^N(v, d\varphi(e_i))w, d\varphi(e_i)) \\ &\quad +\mathrm{div}^M\omega. \end{aligned} \tag{2.10}$$

The second term on the left-hand side of (2.7) is

$$\begin{aligned} h(\nabla^\phi_{\frac{\partial}{\partial s}} d\phi(E_i), \nabla^\phi_{\frac{\partial}{\partial t}} d\phi(E_i)) &= h(\nabla^\phi_{E_i} d\phi(\frac{\partial}{\partial s}), \nabla^\phi_{E_i} d\phi(\frac{\partial}{\partial t})) \\ &= E_i\left(h(d\phi(\frac{\partial}{\partial s}), \nabla^\phi_{E_i} d\phi(\frac{\partial}{\partial t}))\right) \end{aligned}$$

$$-h(d\phi(\frac{\partial}{\partial s}), \nabla^{\phi}_{E_i}\nabla^{\phi}_{E_i} d\phi(\frac{\partial}{\partial t})). \tag{2.11}$$

Define an 1-form on M by

$$\eta(X) = h(w, \nabla^{\varphi}_{X} v), \quad X \in \Gamma(TM).$$

Then

$$\begin{aligned} \operatorname{div}^M \eta &= \sum_{i=1}^{m} \{e_i(\eta(e_i)) - \eta(\nabla^M_{e_i} e_i)\} \\ &= \sum_{i=1}^{m} \{e_i(h(w, \nabla^{\varphi}_{e_i} v)) - h(w, \nabla^{\varphi}_{\nabla^M_{e_i} e_i} v)\}. \end{aligned} \tag{2.12}$$

According to formulas (2.11) and (2.12), we obtain

$$\begin{aligned} \sum_{i=1}^{m} h(\nabla^{\phi}_{\frac{\partial}{\partial t}} d\phi(E_i), \nabla^{\phi}_{\frac{\partial}{\partial s}} d\phi(E_i))\Big|_{(t,s)=(0,0)} &= \operatorname{div}^M \eta + \sum_{i=1}^{m} h(w, \nabla^{\varphi}_{\nabla^M_{e_i} e_i} v) \\ &\quad - \sum_{i=1}^{m} h(w, \nabla^{\varphi}_{e_i} \nabla^{\varphi}_{e_i} v). \end{aligned} \tag{2.13}$$

From the formulas (2.6), (2.7), (2.10), (2.13) and the divergence Theorem, the Theorem 2.2.3 follows. □

2.3 Biharmonic maps

The bienergy functional of a smooth map $\varphi : (M^m, g) \longrightarrow (N^n, h)$ is defined by

$$E_2(\varphi, D) = \frac{1}{2} \int_D |\tau(\varphi)|^2 v_g. \tag{2.14}$$

Definition 2.3.1. *A map is called biharmonic if it is a critical point of the bienergy functional over any compact subset D of M.*

2.3.1 First variation of bi-energy

Theorem 2.3.1. *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map between Riemannian manifolds, D a compact subset of M and let $\{\varphi_t\}_{t\in(-\epsilon,\epsilon)}$ be a smooth variation with compact support in D. Then*

$$\frac{d}{dt} E_2(\varphi_t; D)|_{t=0} = -\int_D h(v, \tau_2(\varphi)) v_g,$$

where $v = \frac{d\varphi_t}{dt}|_{t=0}$ *denotes the variation vector field of* φ *and in locale frame at* $x \in M$*, we have*

$$\begin{aligned}\tau_2(\varphi) &= -\mathrm{trace}_g R^N(\tau(\varphi), d\varphi)d\varphi - \mathrm{trace}_g(\nabla^\varphi)^2\tau(\varphi)\\ &= -\sum_{i=1}^m R^N(\tau(\varphi), d\varphi(e_i))d\varphi(e_i) - \sum_{i=1}^m \{\nabla^\varphi_{e_i}\nabla^\varphi_{e_i}\tau(\varphi)\\ &\quad -\nabla^\varphi_{\nabla^M_{e_i}e_i}\tau(\varphi)\} \qquad (2.15)\end{aligned}$$

$\tau_2(\varphi)$ *is called the bi-tension field of* φ*.*

Proof. Define $\phi : M \times (-\epsilon, \epsilon) \longrightarrow N$ by $\phi(x,t) = \varphi_t(x)$.
First note that

$$\frac{d}{dt}E_2(\varphi_t; D)|_{t=0} = \int_D \sum_{i=1}^m h\left(\nabla^\phi_{(0,\frac{d}{dt})}\nabla d\phi((e_i,0),(e_i,0)), \nabla d\phi((e_i,0),(e_i,0))\right) v_g|_{t=0}. \qquad (2.16)$$

Calculating in a normal frame at $x \in M$ we have

$$\begin{aligned}\nabla^\phi_{(0,\frac{d}{dt})}d\phi(e_i,0) &= \nabla^\phi_{(e_i,0)}d\phi(0,\frac{d}{dt}) + d\phi([(0,\frac{d}{dt}),(e_i,0)])\\ &= \nabla^\phi_{(e_i,0)}d\phi(0,\frac{d}{dt}). \qquad (2.17)\\ \nabla^\phi_{(0,\frac{d}{dt})}d\phi(\nabla^M_{e_i}e_i,0) &= \nabla^\phi_{(\nabla^M_{e_i}e_i,0)}d\phi(0,\frac{d}{dt}). \qquad (2.18)\\ \nabla^\phi_{(0,\frac{d}{dt})}\nabla d\phi((e_i,0),(e_i,0)) &= \nabla^\phi_{(0,\frac{d}{dt})}\nabla^\phi_{(e_i,0)}d\phi(e_i,0) - \nabla^\phi_{(0,\frac{d}{dt})}d\phi\left(\nabla^{M\times(-\epsilon,\epsilon)}_{(e_i,0)}(e_i,0)\right)\\ &= R^N(d\phi(0,\frac{d}{dt}), d\phi(e_i,0))d\phi(e_i,0) + \nabla^\phi_{(e_i,0)}\nabla^\phi_{(0,\frac{d}{dt})}d\phi(e_i,0)\\ &\quad +\nabla^\phi_{[(0,\frac{d}{dt}),(e_i,0)]}d\phi(e_i,0) - \nabla^\phi_{(0,\frac{d}{dt})}d\phi(\nabla^M_{e_i}e_i,0).\\ &= R^N(d\phi(0,\frac{d}{dt}), d\phi(e_i,0))d\phi(e_i,0) + \nabla^\phi_{(e_i,0)}\nabla^\phi_{(e_i,0)}d\phi(0,\frac{d}{dt})\\ &\quad -\nabla^\phi_{(\nabla^M_{e_i}e_i,0)}d\phi(0,\frac{d}{dt}). \qquad (2.19)\end{aligned}$$

From where

$$\begin{aligned}&h(\nabla^\phi_{(0,\frac{d}{dt})}\nabla d\phi((e_i,0),(e_i,0)),\ \ \nabla d\phi((e_i,0),(e_i,0)))|_{t=0} =\\ &h(R^N(v, d\varphi(e_i))d\varphi(e_i), \tau(\varphi)) + h(\nabla^\varphi_{e_i}\nabla^\varphi_{e_i}v, \tau(\varphi)) - h(\nabla^\varphi_{\nabla^M_{e_i}e_i}v, \tau(\varphi)). \qquad (2.20)\end{aligned}$$

Let $\omega \in \Gamma(T^*M)$, be a 1-form to support in D, defined by:

$$\omega(X) = h(\nabla^\varphi_X v, \tau(\varphi)), \qquad X \in \Gamma(TM).$$

We calculate the divergence of ω

$$\begin{aligned}
\operatorname{div}^M\omega &= \sum_{i=1}^m\{e_i(\omega(e_i))-\omega(\nabla^M_{e_i}e_i)\}\\
&= \sum_{i=1}^m\{e_i(h(\nabla^\varphi_{e_i}v,\tau(\varphi)))-h(\nabla^\varphi_{\nabla^M_{e_i}e_i}v,\tau(\varphi))\}\\
&= \sum_{i=1}^m\{h(\nabla^\varphi_{e_i}\nabla^\varphi_{e_i}v,\tau(\varphi))+h(\nabla^\varphi_{e_i}v,\nabla^\varphi_{e_i}\tau(\varphi))-h(\nabla^\varphi_{\nabla^M_{e_i}e_i}v,\tau(\varphi))\}. \quad (2.21)
\end{aligned}$$

From the formulas (2.20) and (2.21), we obtain:

$$\begin{aligned}
&\sum_{i=1}^m h(\nabla^\phi_{(0,\frac{d}{dt})}\nabla d\phi((e_i,0),(e_i,0)),\ \ \nabla d\phi((e_i,0),(e_i,0)))|_{t=0} =\\
&\sum_{i=1}^m h(R^N(v,d\varphi(e_i))d\varphi(e_i),\tau(\varphi))+\operatorname{div}^M\omega-\sum_{i=1}^m h(\nabla^\varphi_{e_i}v,\nabla^\varphi_{e_i}\tau(\varphi)). \quad (2.22)
\end{aligned}$$

Let $\eta\in\Gamma(T^*M)$, be an 1-form to support in D, given by

$$\eta(X)=h(v,\nabla^\varphi_X\tau(\varphi)),\qquad X\in\Gamma(TM).$$

We calculate the divergence of η

$$\begin{aligned}
\operatorname{div}^M\eta &= \sum_{i=1}^m\{e_i(\eta(e_i))-\eta(\nabla^M_{e_i}e_i)\}\\
&= \sum_{i=1}^m\{e_i(h(v,\nabla^\varphi_{e_i}\tau(\varphi)))-h(v,\nabla^\varphi_{\nabla^M_{e_i}e_i}\tau(\varphi))\}\\
&= \sum_{i=1}^m\{h(\nabla^\varphi_{e_i}v,\nabla^\varphi_{e_i}\tau(\varphi))+h(v,\nabla^\varphi_{e_i}\nabla^\varphi_{e_i}\tau(\varphi))-h(v,\nabla^\varphi_{\nabla^M_{e_i}e_i}\tau(\varphi))\}. \quad (2.23)
\end{aligned}$$

Substituting (2.23) in (2.22), we obtain

$$\begin{aligned}
&\sum_{i=1}^m h(\nabla^\phi_{(0,\frac{d}{dt})}\nabla d\phi((e_i,0),(e_i,0)),\ \ \nabla d\phi((e_i,0),(e_i,0)))|_{t=0} =\\
&\sum_{i=1}^m h(R^N(\tau(\varphi),d\varphi(e_i))d\varphi(e_i),v)+\operatorname{div}^M\omega-\operatorname{div}^M\eta\\
&+\sum_{i=1}^M h(v,\nabla^\varphi_{e_i}\nabla^\varphi_{e_i}\tau(\varphi))-\sum_{i=1}^M h(v,\nabla^\varphi_{\nabla^M_{e_i}e_i}\tau(\varphi)). \quad (2.24)
\end{aligned}$$

From the formulas (2.16), (2.24) and according the divergence Theorem, we obtain

$$\frac{d}{dt}E_2(\varphi_t;D)|_{t=0} = -\int_D \sum_{i=1}^{m} h\left(-R^N(\tau(\varphi), d\varphi(e_i))d\varphi(e_i) - \nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}\tau(\varphi) + \nabla^{\varphi}_{\nabla^M_{e_i}e_i}\tau(\varphi), v\right) v_g.$$

□

Theorem 2.3.2. *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map between two Riemannian manifolds, then φ is said biharmonic if and only if*

$$\tau_2(\varphi) = -\mathrm{trace}_g R^N(\tau(\varphi), d\varphi)d\varphi - \mathrm{trace}_g(\nabla^{\varphi})^2\tau(\varphi) = 0. \tag{2.25}$$

Remark 2.3.1.

1. The equation (2.25) is called the Euler-Lagrange equation.

2. Let M and N be two Riemanian manifolds with the coordinates (x^i) and (y^α) respectively, then, in the neighborhood of the points $x \in M$ and $\varphi(x) \in N$ we have

$$\begin{aligned}\tau_2(\varphi) \;=\; & g^{ij}\{\frac{\partial^2\tau^\sigma}{\partial x^i\partial x^j} + 2\frac{\partial\tau^\sigma\partial\tau^\beta}{\partial x^j\partial x^j}{}^N\Gamma^\sigma_{\alpha\beta} + \tau^\alpha\frac{\partial^2\varphi^\beta}{\partial x^i\partial x^j}{}^N\Gamma^\sigma_{\alpha\beta} \\ & + \tau^\alpha\frac{\partial\varphi^\beta}{\partial x^i}\frac{\partial^N\Gamma^\sigma_{\alpha\beta}}{\partial x^j} + \tau^\alpha\frac{\partial\varphi^\beta}{\partial x^i}\frac{\partial\varphi^\rho}{\partial x^j}{}^N\Gamma^\upsilon_{\alpha\beta}{}^N\Gamma^\sigma_{\upsilon\rho} \\ & - {}^M\Gamma^k_{ij}(\frac{\partial\tau^\sigma}{\partial x^k} + \tau^\alpha\frac{\partial\varphi^\beta}{\partial x^k}{}^N\Gamma^\sigma_{\alpha\beta}) - \tau^\upsilon\frac{\partial\varphi^\alpha}{\partial x^i}\frac{\partial\varphi^\beta}{\partial x^j}{}^N R^\sigma_{\beta\alpha\upsilon}\}\frac{\partial}{\partial y^\sigma}\circ\varphi,\end{aligned}$$

   where $\tau^\gamma = g^{ij}\left(\frac{\partial^2\varphi^\gamma}{\partial x^i\partial x^j} + \frac{\partial\varphi^\alpha}{\partial x^i}\frac{\partial\varphi^\beta}{\partial x^j}{}^N\Gamma^\gamma_{\alpha\beta}\circ\varphi - \frac{\partial\varphi^\gamma}{\partial x^k}{}^M\Gamma^k_{ij}\right)$ and ${}^N R^\sigma_{\beta\alpha\upsilon}$ designate the components of the curvature tensor of (N^n, h).

3. Any harmonic map is a biharmonic.

4. Biharomnic maps are not generally harmonic maps.

Example 2.3.1.

1. The polynomials of degrees 3 on $\mathbb{R}$ are biharmonic non-harmonic maps.

2. The identity map $Id : (M^m, g) \longrightarrow (M^m, g)$ is biharmonic.

3. A smooth map $\varphi : (M^m, g) \longrightarrow (\mathbb{R}^n, <,>_{\mathbb{R}^n})$, is biharmonic if and only if $\Delta^M(\Delta^M\varphi^\sigma) = 0$, for all $\sigma = 1, \ldots, n$.

2.4 Somes result on stable harmonic maps

Definition 2.4.1. *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a harmonic map between two Riemannian manifolds, the Hessian of φ (of the energy E) is defined by*

$$H(E)_\varphi(v, w) = \int_M h(J_\varphi(v), w)v_g, \quad \forall v, w \in \Gamma(\varphi^{-1}TN).$$

Remark 2.4.1.

$$\begin{aligned} J_\varphi(v) &= -\operatorname{trace} R^N(v, d\varphi)d\varphi - \operatorname{trace}(\nabla^\varphi)^2 v \\ &= -\sum_{i=1}^m R^N(v, d\varphi(e_i))d\varphi(e_i) - \sum_{i=1}^m \Big[\nabla^\varphi_{e_i}\nabla^\varphi_{e_i} v - \nabla^\varphi_{\nabla^M_{e_i} e_i} v\Big]. \end{aligned} \tag{2.26}$$

Definition 2.4.2. *A harmonic map $\varphi : (M, g) \longrightarrow (N, h)$ between two Riemannian manifolds is called stable if $H(E)_\varphi(v, v) \geq 0$, for all $v \in \Gamma(\varphi^{-1}TN)$.*

Proposition 2.4.1. *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a harmonic map between two Riemannian manifolds, where M is compact manifold without boundary, if N has a sectional curvature ≤ 0, then φ is stable map.*

Proof. Let $\{e_i\}_{i=1}^m$ be an orthonormal frame on M, such that

$$\left(\nabla^M_{e_i} e_j\right)_x = 0, \; \forall i, j = 1, \ldots, m,$$

where $x \in M$, then $\forall v \in \Gamma(\varphi^{-1}TN)$ we have (at point x):

$$\begin{aligned} < J_\varphi v, v > &= -\sum_{i=1}^m < \nabla^\varphi_{e_i}\nabla^\varphi_{e_i} v, v > - \sum_{i=1}^m < R^N(v, d\varphi(e_i))d\varphi(e_i), v > \\ &= -\sum_{i=1}^m e_i < \nabla^\varphi_{e_i} v, v > + \sum_{i=1}^m < \nabla^\varphi_{e_i} v, \nabla^\varphi_{e_i} v > \\ &\quad -\sum_{i=1}^m < R^N(v, d\varphi(e_i))d\varphi(e_i), v > \\ &= -\operatorname{div} < \nabla^\varphi v, v > + |\nabla^\varphi v|^2 - \sum_{i=1}^m < R^N(v, d\varphi(e_i))d\varphi(e_i), v > . \end{aligned}$$

Since sectional curvature ≤ 0 and according to the divergence Theorem, we get:

$$H(E)_\varphi(v, v) = \int_M |\nabla^\varphi v|^2 \, v_g - \int_M \sum_{i=1}^m < R^N(v, d\varphi(e_i))d\varphi(e_i), v > v_g \geq 0.$$

□

Proposition 2.4.2. *Let (M, g) be a compat Riemannian manifold, without boundary, of dimension $m \leq 2$, then the identity mapping $I : (M, g) \to (M, g)$ is stable harmonic map.*

For the proof of the proposition 2.4.2 we need the following lemma:

Lemma 2.4.1 (Yano's formula). *Let $(L_v g)$ an operator of the derivative of lie on the metric g then:*

$$\frac{1}{2}\int_M |L_v g|^2 v_g = \int_M \Big[|\nabla v|^2 - Ricci(v,v) + (\operatorname{div} v)^2\Big] v_g$$

Proof. Let $\{e_i\}_{i=1}^m$ be a orthonormal frame on M with $(\nabla_{e_i} e_j)_x = 0$, $\forall i, j = 1, ...m$. We compute

$$\begin{aligned}
|L_v g|^2 &= L_v g(e_i,e_j) L_v g(e_i,e_j) \\
&= \Big[g(\nabla_{e_i} v, e_j) + g(\nabla_{e_j} v, e_i)\Big]\Big[g(\nabla_{e_i} v, e_j) + g(\nabla_{e_j} v, e_i)\Big] \\
&= g(\nabla_{e_i} v, e_j) g(\nabla_{e_i} v, e_j) + 2g(\nabla_{e_i} v, e_j) g(\nabla_{e_j} v, e_i) + g(\nabla_{e_j} v, e_i) g(\nabla_{e_j} v, e_i) \\
&= g(\nabla_{e_i} v, g(\nabla_{e_i} v, e_j) e_j) + 2e_i\Big[g(v,e_j) g(\nabla_{e_j} v, e_i)\Big] \\
&\quad -2g(v,e_j) g(\nabla_{e_i}\nabla_{e_j} v, e_i) + g(\nabla_{e_j} v, g(\nabla_{e_j} v, e_i) e_i).
\end{aligned} \tag{2.27}$$

We put $\theta(X) = g(v,e_j) g(\nabla_{e_j} v, X)$ then

$$\begin{aligned}
|L_v g|^2 &= 2|\nabla v|^2 + 2\operatorname{div}\theta + 2g(v,e_j)\Big[-g(\nabla_{e_j}\nabla_{e_i} v, e_i) - g(R(e_i,e_j)v, e_i)\Big] \\
&= 2|\nabla v|^2 + 2\operatorname{div}\theta + 2\Big[-g(\nabla_{g(v,e_j)e_j}\nabla_{e_i} v, e_i) - g(R(e_i, g(v,e_j)e_j)v, e_i)\Big] \\
&= 2|\nabla v|^2 + 2\operatorname{div}\theta - 2g(\nabla_v \nabla_{e_i} v, e_i) - 2g(R(e_i,v)v, e_i) \\
&= 2|\nabla v|^2 + 2\operatorname{div}\theta - 2g(\nabla_v \nabla_{e_i} v, e_i) - 2Ricci(v,v).
\end{aligned} \tag{2.28}$$

We know that

$$\begin{aligned}
(\operatorname{div} v)^2 &= \Big[g(\nabla_{e_i} v, e_i)\Big]\Big[g(\nabla_{e_j} v, e_j)\Big] \\
&= e_i\Big[g(v,e_i) g(\nabla_{e_j} v, e_j)\Big] - \Big[g(v,e_i) g(\nabla_{e_i}\nabla_{e_j} v, e_j)\Big] \\
&= \operatorname{div}\vartheta - g(\nabla_{g(v,e_i)e_i}\nabla_{e_j} v, e_j) \\
&= \operatorname{div}\vartheta - g(\nabla_v \nabla_{e_j} v, e_j)
\end{aligned}$$

Then

$$g(\nabla_v \nabla_{e_j} v, e_j) = \operatorname{div}\vartheta - (\operatorname{div} v)^2, \tag{2.29}$$

where

$$\vartheta(X) = g(v,X) g(\nabla_{e_j} v, e_j)$$

Substituting (2.29) in (2.28) we get

$$\frac{1}{2}|L_v g|^2 = |\nabla v|^2 + \operatorname{div}\theta - \operatorname{div}\vartheta + (\operatorname{div} v)^2 - Ricci(v,v)$$

According to the divergence theorem we obtain

$$\frac{1}{2}\int_M |L_v g|^2 v_g \;=\; \int_M \Big[|\nabla v|^2 + (\operatorname{div} v)^2 - Ricci(v,v)\Big] v_g.$$

□

Proof. (of proposition 2.4.2) Let $\{e_i\}_{i=1}^m$ be an orthonormal frame on M such that $(\nabla_{e_j} e_i)_x = 0$ where $x \in M$, and let $v \in \Gamma(TM)$, then at point x we have

$$\begin{aligned} < J_I v, v > &= -\sum_{i=1}^m < \nabla_{e_i}\nabla_{e_i} v, v > - \sum_{i=1}^m < R(v,e_i)e_i, v > \\ &= -\sum_{i=1}^m e_i < \nabla_{e_i} v, v > + \sum_{i=1}^m < \nabla_{e_i} v, \nabla_{e_i} v > - \sum_{i=1}^m < R(v,e_i)e_i, v > \\ &= -\operatorname{div}\varpi + |\nabla v|^2 - < Ricci v, v > \end{aligned}$$

Such that, $g =<,>$ and $\varpi(X) =< \nabla_X v, v > \forall X \in \Gamma(TM)$ from where

$$< J_I v, v > \;=\; -\operatorname{div}\varpi + |\nabla v|^2 - Ric(v,v)$$

As M is compact manifold without boundary, according to the Green Theorem we obtain

$$\int_M < J_I v, v > v_g \;=\; \int_M \Big[|\nabla v|^2 - Ric(v,v)\Big] v_g$$

By the lemma 2.4.1 we have

$$\int_M \Big[|\nabla v|^2 - Ric(v,v)\Big] v_g \;=\; \int_M \Big[\frac{1}{2}|L_v g|^2 - (\operatorname{div} v)^2\Big] v_g. \tag{2.30}$$

Since

$$\begin{aligned} |L_v g|^2 &\geq (L_v g(e_i,e_i))^2 = 4\sum_{i=1}^m (g(\nabla_{e_i} v, e_i)^2 \\ &\geq \frac{4}{m}(\sum_{i=1}^m g(\nabla_{e_i} v, e_i))^2 \\ &\geq \frac{4}{m}(\operatorname{div} v)^2. \end{aligned} \tag{2.31}$$

Substituting (2.31) in (2.30) we obtain:

$$\begin{aligned} \int_M < J_I v, v > v_g &\geq \int_M \Big[\frac{2}{m}(\operatorname{div} v)^2 - (\operatorname{div} v)^2\Big] v_g \\ &\geq \int_M \frac{2-m}{m}(\operatorname{div} v)^2 v_g. \end{aligned}$$

By the condition $m \leq 2$ we have $Hess_I(v,v) \geq 0$, then I is stable harmonic map. □

Example 2.4.1. Any closed geodesic $\gamma : \mathbb{S}^1 \longrightarrow \mathbb{H}^n$ is stable, where

$$\mathbb{H}^n = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n \,|\, x_n > 0\},$$

is the hyperbolic space with the metric $g_{ij} = \frac{1}{x_n^2}\delta_{ij}$ (because the sectional curvature of $\mathbb{H}^n$ is equal to -1).

Theorem 2.4.1 (Y. L. Xin 1980,[45])**.** *Any stable harmonic map φ from sphere $(\mathbb{S}^n, g)$ $(n > 2)$ to Riemannian manifold (N, h) is constant.*

The proof of Theorem 2.4.1 is based on the following lemma:

Lemma 2.4.2. *Let $\mathbb{S}^n$ be an unit sphere of $\mathbb{R}^{n+1}$, $\lambda(x) =< \alpha, x >_{\mathbb{R}^{n+1}}$ be a function defined on $\mathbb{S}^n$, where $\alpha \in \mathbb{R}^{n+1}$ fixed, and $v = \operatorname{grad}\lambda$, then:*

- *In the orthonormal frame $\{e_i\}$ on $\mathbb{S}^n$, we have*

$$v = \sum_{i=1}^{m} < \alpha, e_i >_{\mathbb{R}^{n+1}} e_i,$$

- *Let ∇ the Levi-Civita connection of the metric induced on $\mathbb{S}^n$, we have:*

  1. $\nabla_X v = -\lambda X, \ \forall X \in \Gamma(T\mathbb{S}^n)$;
  2. $\operatorname{trace}\nabla^2 v = -v$.

Proof. • In the canonical frame $\{\partial_i\}_{i=1,\dots,n+1}$ of $\mathbb{R}^{n+1}$, if

$x = \sum_{i=1}^{n+1} x^i\partial_i = x^i\partial_i$ and $\alpha = \sum_{j=1}^{n+1} \alpha^j\partial_j = \alpha^j\partial_j$, then

$$\lambda(x) = \sum_{i=1}^{n+1} x^i\alpha^i. \tag{2.32}$$

Let $\{e_i\}_{i=1}^n$ be an orthonormal frame on $\mathbb{S}^n$, we have

$$\operatorname{grad}\lambda = e_i(\lambda)e_i,$$

and $\forall\, i = 1, \dots, n$, we have $e_i = e_i^m\partial_m$.
Then by (2.32) we get

$$\begin{aligned} v = \operatorname{grad}\lambda &= e_i^m\partial_m(\lambda)e_i \\ &= e_i^m\alpha^m e_i \\ &= < e_i, \alpha > e_i. \end{aligned}$$

- Let $\{e_i\}_{i=1}^n$ be an orthonormal frame on $\mathbb{S}^n$, such that $(\nabla_{e_i}e_j)_x = 0$, $\forall i,j = 1,\dots,n$, where $x \in \mathbb{S}^n$. Then at point x we have:

$$\begin{aligned}\nabla_X v &= \nabla_X <\alpha, e_i> e_i \\ &= X(<\alpha, e_i>)e_i + <\alpha, e_i> \nabla_X e_i \\ &= X(<\alpha, e_i>)e_i,\end{aligned}$$

with $\nabla_X e_i = \nabla_{X_j e_j} e_i = X_j \nabla_{e_j} e_i = 0$.
So that

$$\begin{aligned}\nabla_X v &= \Big[<\overline{\nabla}_X \alpha, e_i> + <\alpha, \overline{\nabla}_X e_i> \Big] e_i \\ &= <\alpha, \overline{\nabla}_X e_i> e_i \\ &= <\alpha, (\overline{\nabla}_X e_i)^\top> e_i + <\alpha, (\overline{\nabla}_X e_i)^\perp> e_i \\ &= <\alpha, \nabla_X e_i> e_i + <\alpha, B(X, e_i)> e_i \\ &= <\alpha, B(X, e_i)> e_i, \qquad (2.33)\end{aligned}$$

where $\overline{\nabla}$ is the Levi-Civita connection of $\mathbb{R}^{n+1}$, $\overline{\nabla}_X \alpha = 0$, and B is the second fundamental form of $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$.
We have:

$$B(X, e_i) = Ax, \qquad (2.34)$$

because $B(X, e_i)$ is a normal, and $x \in (T_x\mathbb{S}^n)^\perp$, where $A \in C^\infty(\mathbb{S}^n)$.

$$h(B(X, e_i), x) = A, \qquad (2.35)$$

because $h(x, x) = 1$ with $h = <,>_{\mathbb{R}^{n+1}}$, that is

$$A = h((\overline{\nabla}_X e_i)^\perp, x) = h(\overline{\nabla}_X e_i, x).$$

Then

$$A = X(h(e_i, x)) - h(e_i, \overline{\nabla}_X x) = -h(e_i, \overline{\nabla}_X x). \qquad (2.36)$$

We have

$$\begin{aligned}\overline{\nabla}_X x &= \overline{\nabla}_{X_i \partial_i} x_j \partial_j \\ &= X_i \overline{\nabla}_{\partial_i} x_j \partial_j \\ &= X_i \{\partial_i(x_j)\partial_j + x_j \overline{\nabla}_{\partial_i}\partial_j\} \\ &= X_i \delta_{ij}\, \partial_j \\ &= X_j \partial_j \\ &= X, \qquad (2.37)\end{aligned}$$

substituting the equation (2.37) in (2.36) we get

$$A = -h(e_i, X). \qquad (2.38)$$

From the equations (2.34) and (2.38) we obtain:

$$\begin{aligned}\nabla_X v &= <\alpha, -h(e_i, X)x> e_i \\ &= -h(e_i, X) <\alpha, x> e_i \\ &= -\lambda X, \end{aligned} \tag{2.39}$$

and by the equation (2.39) we get:

$$\begin{aligned}\nabla^2_{e_i} v &= \nabla_{e_i}\nabla_{e_i} v \\ &= \nabla_{e_i}(-\lambda e_i) \\ &= -e_i(\lambda)e_i - \lambda \nabla_{e_i} e_i \\ &= -\operatorname{grad}\lambda \\ &= -v.\end{aligned}$$

□

Proof. (of Theorem 2.4.1)
Choose a normal orthonormal frame $\{e_i\}$ at point x_0 in $\mathbb{S}^n$, from the lemma 2.4.2 we have

$$\begin{aligned}\nabla^\varphi_{e_i}\nabla^\varphi_{e_i} d\varphi(v) &= \nabla^\varphi_{e_i}\nabla^\varphi_v d\varphi(e_i) + \nabla^\varphi_{e_i} d\varphi([e_i, v]) \\ &= R^N(d\varphi(e_i), d\varphi(v))d\varphi(e_i) + \nabla^\varphi_v \nabla^\varphi_{e_i} d\varphi(e_i) \\ &\quad + d\varphi([e_i,[e_i,v]]) + 2f_\varphi \nabla^\varphi_{[e_i,v]} d\varphi(e_i),\end{aligned} \tag{2.40}$$

from the definition of tension field, we get

$$\begin{aligned}\nabla^\varphi_{e_i}\nabla^\varphi_{e_i} d\varphi(v) &= -R^N(d\varphi(v), d\varphi(e_i))d\varphi(e_i) + \nabla^\varphi_v \tau(\varphi) \\ &\quad + \nabla^\varphi_v d\varphi(\nabla^{\mathbb{S}^n}_{e_i} e_i) + d\varphi(\nabla^{\mathbb{S}^n}_{e_i}\nabla^{\mathbb{S}^n}_{e_i} v) \\ &\quad - d\varphi(\nabla^{\mathbb{S}^n}_{e_i}\nabla^{\mathbb{S}^n}_{v} e_i) + 2\nabla^\varphi_{[e_i,v]} d\varphi(e_i)\end{aligned} \tag{2.41}$$

by equations (2.40), (2.41), and the harmonicity condition of φ, we have

$$\begin{aligned}\nabla^\varphi_{e_i}\nabla^\varphi_{e_i} d\varphi(v) &= R^N(d\varphi(v), d\varphi(e_i))d\varphi(e_i) \\ &\quad + d\varphi(\nabla^{\mathbb{S}^n}_{v}\nabla^{\mathbb{S}^n}_{e_i} e_i) + d\varphi(\nabla^{\mathbb{S}^n}_{e_i}\nabla^{\mathbb{S}^n}_{e_i} v) \\ &\quad - d\varphi(\nabla^{\mathbb{S}^n}_{e_i}\nabla^{\mathbb{S}^n}_{v} e_i) + 2\nabla^\varphi_{\nabla^{\mathbb{S}^n}_{e_i} v} d\varphi(e_i),\end{aligned} \tag{2.42}$$

by the definition of Ricci tensor, we get

$$\begin{aligned}\nabla^\varphi_{e_i}\nabla^\varphi_{e_i} d\varphi(v) &= -R^N(d\varphi(v), d\varphi(e_i))d\varphi(e_i) \\ &\quad + d\varphi(\mathrm{Ricci}^{\mathbb{S}^n} v) + d\varphi(\mathrm{trace}(\nabla^{\mathbb{S}^n})^2 v) \\ &\quad + 2\nabla^\varphi_{\nabla^{\mathbb{S}^n}_{e_i} v} d\varphi(e_i),\end{aligned} \tag{2.43}$$

from the property $\nabla^{\mathbb{S}^n}_X v = -\lambda X$, we obtain

$$\begin{aligned} \nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i} d\varphi(v) &= -R^N(d\varphi(v), d\varphi(e_i))d\varphi(e_i) \qquad (2.44)\\ &+d\varphi(\mathrm{Ricci}^{\mathbb{S}^n}\, v)\\ &+d\varphi(\mathrm{trace}(\nabla^{\mathbb{S}^n})^2 v) - 2\lambda\tau(\varphi). \qquad (2.45)\end{aligned}$$

From the definition of Jacobi operator (2.26), the harmonicity condition of φ and equation (2.44) we have

$$J_{\varphi}(d\varphi(v)) = -d\varphi(\mathrm{Ricci}^{\mathbb{S}^n}\, v) - d\varphi(\mathrm{trace}(\nabla^{\mathbb{S}^n})^2 v) \qquad (2.46)$$

since $\mathrm{trace}_g(\nabla^{\mathbb{S}^n})^2 v = -v$, and $\mathrm{Ricci}^{\mathbb{S}^n}\, v = (n-1)v$ (see [1, 44]), we conclude

$$h(J_{\varphi}(d\varphi(v)), d\varphi(v)) = -(n-2)h(d\varphi(v), d\varphi(v)) \qquad (2.47)$$

by (2.47), it follows that

$$\mathrm{trace}_{\alpha}\, h(J_{\varphi}(d\varphi(v)), d\varphi(v)) = -(n-2)|d\varphi|^2, \qquad (2.48)$$

from the stable harmonic condition, and equation (2.48), we get

$$0 \le \mathrm{trace}_{\alpha}\, I^{\varphi}(d\varphi(v), d\varphi(v)) = -(n-2)\int_{\mathbb{S}^n} |d\varphi|^2 v^{\mathbb{S}^n} \le 0.$$

Consequently, $|d\varphi| = 0$, that is φ is constant, because $n > 2$. □

Using the similar technique we have

Theorem 2.4.2 (P. F. Leung. 1982 [22])**.** *Let (Mg) be a compact Riemannian manifold. When $n > 2$, any stable harmonic map $\varphi : M \to S^n$ must be constant.*

Proof. Choose a normal orthonormal frame $\{e_i\}$ at point x_0 in M. When the same data of previous proof, we have

$$\begin{aligned} \nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}(v\circ\varphi) &= -\nabla^{\varphi}_{e_i}(\lambda\circ\varphi)d\varphi(e_i)\\ &= -d\varphi(\mathrm{grad}^M(\lambda\circ\varphi)) - (\lambda\circ\varphi)\tau(\varphi), \qquad (2.49)\end{aligned}$$

by the definition of gradient operator, we get

$$-d\varphi(\mathrm{grad}^M(\lambda\circ\varphi)) = - < d\varphi(e_i), v\circ\varphi > d\varphi(e_i), \qquad (2.50)$$

substituting the formula (2.50) into (2.49) gives

$$\begin{aligned} \nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}(v\circ\varphi) &= - < d\varphi(e_i), v\circ\varphi > d\varphi(e_i)\\ &-(\lambda\circ\varphi)\tau(\varphi), \qquad (2.51)\end{aligned}$$

from the harmonicity condition of φ, and equation (2.51), we have

$$< \nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}(v\circ\varphi), v\circ\varphi > \;=\; -< d\varphi(e_i), v\circ\varphi >< d\varphi(e_i), v\circ\varphi > \tag{2.52}$$

since the sphere $\mathbb{S}^n$ has constant curvature, we obtain

$$\begin{aligned}< R^{\mathbb{S}^n}(v\circ\varphi, d\varphi(e_i))d\varphi(e_i), v\circ\varphi >= |d\varphi|^2 < v\circ\varphi, v\circ\varphi > \\ -< d\varphi(e_i), v\circ\varphi >< d\varphi(e_i), v\circ\varphi >,\end{aligned} \tag{2.53}$$

by the definition of Jacobi operator and equations (2.52), (2.53), we get

$$\begin{aligned}< J_{\varphi}(v\circ\varphi), v\circ\varphi > \;&=\; 2 < d\varphi(e_i), v\circ\varphi >< d\varphi(e_i), v\circ\varphi > \\ &\quad -|d\varphi|^2 < v\circ\varphi, v\circ\varphi > .\end{aligned} \tag{2.54}$$

So

$$\operatorname{trace}_{\alpha} < J_{\varphi}(v\circ\varphi), v\circ\varphi > \;=\; (2-n)|d\varphi|^2. \tag{2.55}$$

So that

$$\operatorname{trace}_{\alpha} I^{\varphi}(v\circ\varphi, v\circ\varphi) \;=\; (2-n)\int_M |d\varphi|^2 v^M \tag{2.56}$$

Hence Theorem 2.4.2 follows from (2.56) and the stable harmonicity condition of φ with $n > 2$. □

2.5 Homothetic vector fields and harmonic maps

2.5.1 Homothetic vector fields and harmonic maps

A vector fields ξ on a Riemannian manifold (M, g) is called a homothetic if $\mathcal{L}_{\xi}g = 2kg$, for some constant $k \in \mathbb{R}$, where $\mathcal{L}_{\xi}g$ is the Lie derivative of the metric g with respect to ξ, that is

$$g(\nabla_X\xi, Y) + g(\nabla_Y\xi, X) = 2kg(X, Y), \quad \forall X, Y \in \Gamma(TM). \tag{2.57}$$

The constant k is then called the homothetic constant. If ξ is homothetic and $k \neq 0$, then it is called proper homothetic while $k = 0$ it is Killing (see [1], [20], [43]). It follows that grad$f$ is homothetic if and only if the Hessian satisfies the equation: $(\mathrm{Hess}f)(X, Y) = kg(X, Y), \quad \forall \quad X, Y \in \Gamma(TM), \quad k \in \mathbb{R}$. If (M, g) is a complete n-dimensional Riemannian manifold, and suppose that there exists a non-constant smooth function f in M satisfying Hess$f = kg$, for some constant $k \neq 0$, then M is isometric to $\mathbb{R}^n$ (see [34]). Note that, if a complete Riemannian manifold of dimension ≥ 2 admits a proper homothetic vector field then the manifold is isometric to the Euclidean space (see [19], [43]).

Example 2.5.1. The position vector fields $P = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}$, in $\mathbb{R}^n$ is an homothetic vector fields.
Indeed, we have

$$\nabla_X^{\mathbb{R}^n} p = X, \qquad \forall X \in \Gamma(T\mathbb{R}^n),$$

then, $\forall X, Y \in \Gamma(T\mathbb{R}^n)$:

$$\begin{aligned}(\mathcal{L}_p g)(X,Y) &= g(\nabla_X P, Y) + g(\nabla_Y P, X) \\ &= g(X,Y) + g(Y,X) \\ &= 2g(X,Y).\end{aligned}$$

So, P is an homothetic vector fields with the homothetic constant $k = 1$.

Theorem 2.5.1 (A.M.Cherif.2017.[27]). *Let (M, g) be a compact orientable Riemannian manifold without boundary, and (N, h) be a Riemannian manifold admitting a proper homothetic vector field ξ with homothetic constant $k \neq 0$. Then, any harmonic map φ from (M, g) to (N, h) is constant.*

Proof. Let $X \in \Gamma(TM)$, we set

$$\omega(X) = h(\xi \circ \varphi, d\varphi(X)). \tag{2.58}$$

Let $\{e_i\}$ be a normal orthonormal frame at $x \in M$, we have

$$\operatorname{div}^M \omega = e_i[h(\xi \circ \varphi, d\varphi(e_i))], \tag{2.59}$$

by equation (2.59), and the harmonicity condition of φ, we get

$$\operatorname{div}^M \omega = h(\nabla^{\varphi}_{e_i}(\xi \circ \varphi), d\varphi(e_i)), \tag{2.60}$$

since ξ is a homothetic vector field, we find that

$$\operatorname{div}^M \omega = kh(d\varphi(e_i), d\varphi(e_i)) = k|d\varphi|^2. \tag{2.61}$$

Theorem 2.5.1 follows from equation (2.61), and the divergence theorem, with $k \neq 0$. □

In the case of non-compact Riemannian manifold, we obtain the following result.

Theorem 2.5.2 (A.M.Cherif.2017.[27]). *Let (M, g) be a complete non-compact Riemannian manifold, and (N, h) be a Riemannian manifold admitting a proper homothetic vector field ξ with homothetic constant $k \neq 0$. If $\varphi : (M, g) \longrightarrow (N, h)$ is harmonic map, satisfying:*

$$\int_M |\xi \circ \varphi|^2 v^g < \infty, \tag{2.62}$$

then φ is constant.

To prove the Theorem 2.5.2, we need the following lemma:

Lemma 2.5.1 (Young's inequality). *Let (M, g) be a Riemannian manifold and $X, Y \in \Gamma(TM)$, then $\forall \epsilon > 0$ we have $-2g(X, Y) \leq \epsilon |X|^2 + \frac{1}{\epsilon}|Y|^2$.*

Proof. Let $\epsilon > 0$, we have

$$\begin{aligned} |\sqrt{\epsilon}X + \frac{1}{\sqrt{\epsilon}}Y|^2 &= g(\sqrt{\epsilon}X + \frac{1}{\sqrt{\epsilon}}Y, \sqrt{\epsilon}X + \frac{1}{\sqrt{\epsilon}}Y) \\ &= \epsilon g(X, X) + 2g(X, Y) + \frac{1}{\epsilon}g(Y, Y) \geq 0, \end{aligned}$$

$$\Rightarrow -2g(X, Y) \leq \epsilon |X|^2 + \frac{1}{\epsilon}|Y|^2.$$

□

Proof of Theorem 2.5.2. Let ρ be a smooth function with compact support on M, we set

$$\omega(X) = h(\xi \circ \varphi, \rho^2 d\varphi(X)), \quad X \in \Gamma(TM). \tag{2.63}$$

Let $\{e_i\}$ be a normal orthonormal frame at $x_0 \in M$, we have

$$\operatorname{div}^M \omega = e_i[h(\xi \circ \varphi, \rho^2 d\varphi(e_i))], \tag{2.64}$$

by the equation (2.64), and the harmonicity condition of φ, we get:

$$\begin{aligned} \operatorname{div}^M \omega &= h(\nabla^\varphi_{e_i}(\xi \circ \varphi), \rho^2 d\varphi(e_i)) + h(\xi \circ \varphi, \nabla^\varphi_{e_i}\rho^2 d\varphi(e_i)) \\ &= \rho^2 h(\nabla^\varphi_{e_i}(\xi \circ \varphi), d\varphi(e_i)) + 2\rho e_i(\rho) h(\xi \circ \varphi, d\varphi(e_i)), \end{aligned} \tag{2.65}$$

since ξ is a homothetic vector field with homothetic constant k, we find that:

$$\rho^2 h(\nabla^\varphi_{e_i}(\xi \circ \varphi), d\varphi(e_i)) = k\rho^2 h(d\varphi(e_i)), d\varphi(e_i)), \tag{2.66}$$

by the Young's inequality, we have:

$$-2\rho e_i(\rho) h(\xi \circ \varphi, d\varphi(e_i)) \leq \epsilon \rho^2 |d\varphi|^2 + \frac{1}{\epsilon} e_i(\rho)^2 |\xi \circ \varphi|^2, \tag{2.67}$$

$\forall \epsilon > 0$. From (2.65)-(2.67), we deduce the inequality:

$$k\rho^2 |d\varphi|^2 - \operatorname{div}^M \omega \leq \varepsilon \rho^2 |d\varphi|^2 + \frac{1}{\varepsilon} e_i(\rho)^2 |\xi \circ \varphi|^2. \tag{2.68}$$

We suppose that $k > 0$, and let $\epsilon = \frac{k}{2}$ by (2.68), we have

$$\frac{k}{2}\rho^2 |d\varphi|^2 - \operatorname{div}^M \omega \leq \frac{2}{k} e_i(\rho)^2 |\xi \circ \varphi|^2, \tag{2.69}$$

by the divergence theorem, and (2.69), we have

$$\frac{k}{2}\int_M \rho^2 |d\varphi|^2 v^g \leq \frac{2}{k}\int_M e_i(\rho)^2 |\xi\circ\varphi|^2 v^g. \tag{2.70}$$

Consider the smooth function $\rho = \rho_R$ such that,$\rho \leq 1$ on$M$, $\rho = 1$ on the ball $B(x_0, R)$, $\rho = 0$ on $M\backslash B(x_0, 2R)$ and $|\operatorname{grad}^M \rho| \leq \frac{2}{R}$ (see [42]). Then, from (2.70), we get:

$$\frac{k}{2}\int_M \rho^2 |d\varphi|^2 v^g \leq \frac{8}{kR^2}\int_M |\xi\circ\varphi|^2 v^g, \tag{2.71}$$

since $\int_M |\xi\circ\varphi|^2 v^g < \infty$, when $R \to \infty$, we obtain:

$$\frac{k}{2}\int_M |d\varphi|^2 v^g = 0, \tag{2.72}$$

Consequently, $|d\varphi| = 0$, that is φ is constant.(If $k < 0$,consider the proper homothetic vector field $\bar{\xi} = -\xi$). □

If $(M, g) = (N, h)$ and $\varphi = Id_M$, from Theorem 2.5.2, we get the following.

Corollary 2.5.1. *Let (M, g) be a complete non-compact Riemannian manifold and let ξ be a proper homothetic vector field on (M, g). Then,*

$$\int_M |\xi|^2 v^g = \infty.$$

.

2.5.2 Homothetic vector fields and biharmonic maps

A vector field ξ on a Riemannian manifold (M, g) is said to be a Jacobi-type vector field if it satisfies:

$$\nabla_X \nabla_X \xi - \nabla_{\nabla_X X}\xi + R(\xi, X)X = 0, \quad \forall\, X \in \Gamma(TM).$$

Theorem 2.5.3 (A.M.Cherif.2017.[27]). *Let (M, g) be a compact orientable Riemannian manifold without boundary and (N, h) be a Riemannian manifold admitting a proper homothetic vector field ξ with homothetic constant $k \neq 0$. Then, any biharmonic map φ from (M, g) to (N, h) is constant.*

For the proof of Theorem 2.5.3, we need the following lemma.

Lemma 2.5.2. *A homothetic vector field on a Riemannian manifold is a Jacobi-type vector field.*

Proof. (see [27].) □

Proof. (of theorem 2.5.3) We set

$$\eta(X) \quad = \quad h(\xi\circ\varphi, \nabla^{\varphi}_{X}\tau(\varphi)), \quad X\in\Gamma(TM),$$

calculating in a normal frame at $x \in M$, we have

$$\begin{aligned} \operatorname{div}^{M}\eta &= e_i[h(\xi\circ\varphi, \nabla^{\varphi}_{e_i}\tau(\varphi))] \\ &= h(\nabla^{\varphi}_{e_i}(\xi\circ\varphi), \nabla^{\varphi}_{e_i}\tau(\varphi)) + h(\xi\circ\varphi, \nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}\tau(\varphi)), \end{aligned} \tag{2.73}$$

from the equation (2.73), and the biharmonicity condition of φ, we get:

$$\operatorname{div}^{M}\eta \quad = \quad h(\nabla^{\varphi}_{e_i}(\xi\circ\varphi), \nabla^{\varphi}_{e_i}\tau(\varphi)) - h(R^{N}(\tau(\varphi), d\varphi(e_i))d\varphi(e_i), \xi\circ\varphi), \tag{2.74}$$

the first term on the left-hand side of (2.74) is

$$h(\nabla^{\varphi}_{e_i}(\xi\circ\varphi), \nabla^{\varphi}_{e_i}\tau(\varphi)) = e_i[h(\nabla^{\varphi}_{e_i}(\xi\circ\varphi), \tau(\varphi))] - h(\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}(\xi\circ\varphi), \tau(\varphi)), \tag{2.75}$$

by the equations (2.74), (2.75 and the property:

$$h(R^{N}(X,Y)Z,W) = h(R^{N}(W,Z)Y,X),$$

where $X, Y, Z, X \in \Gamma(TM)$, we conclude that

$$\begin{aligned} \operatorname{div}^{M}\eta &= \operatorname{div}^{M} h(\nabla^{\varphi}_{\cdot}(\xi\circ\varphi), \tau(\varphi)) - h(\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}(\xi\circ\varphi), \tau(\varphi)) \\ &\quad -h(R^{N}(\xi\circ\varphi, d\varphi(e_i))d\varphi(e_i), \tau(\varphi)), \end{aligned} \tag{2.76}$$

from Lemma 2.5.2, we have

$$\operatorname{div}^{M}\eta = \operatorname{div}^{M} h(\nabla^{\varphi}_{\cdot}(\xi\circ\varphi), \tau(\varphi)) - h(\nabla^{N}_{\tau(\varphi)}\xi, \tau(\varphi)), \tag{2.77}$$

since ξ is a homothetic vector field with homothetic constant k, we get:

$$\operatorname{div}^{M}\eta = \operatorname{div}^{M} h(\nabla^{\varphi}_{\cdot}(\xi\circ\varphi), \tau(\varphi)) - k|\tau(\varphi)|^{2},$$

from the equation (2.78), and the divergence theorem, with $k \neq 0$, we get $\tau(\varphi) = 0$, that is φ is harmonic map, so by Theorem 2.5.1, φ is constant. □

If $\xi = \operatorname{grad}^{N} f$, from Theorem 2.5.2, we deduce.

Corollary 2.5.2. *Let (M, g) be a compact orientable Riemannian manifold without boundary and (N, h) be a Riemannian manifold admitting a smooth function f satisfying* $\operatorname{hess} f = kh$*, for some constant $k \neq 0$. Then, any bi-harmonic map φ from (M, g) to (N, h) is constant.*

Chapter 3

Generalized f-harmonic maps

In this chapter, we define f-harmonic maps and f-biharmonic maps between two Riemannian manifolds M and N, where f is a positive function in $C^{\infty}(M \times N)$, and we present some properties for f-harmonic maps and f-biharmonic maps. The case where $f = 1$ we find the results of chapter 2.

3.1 f-harmonic maps

Definition 3.1.1. *Consider a smooth map $\varphi : M^m \longrightarrow N^n$ between Riemannian manifolds, and*

$$\begin{array}{rcl} f : M \times N & \to & (0, \infty) \\ (x, y) & \mapsto & f(x, y) \end{array}$$

be a smooth positive function. The f-energy functional of φ is defined by

$$E_f(\varphi; D) = \frac{1}{2} \int_D f(x, \varphi(x)) |d\varphi|^2 v_g. \tag{3.1}$$

Definition 3.1.2. *A map is called f-harmonic if it is a critical point of the f-energy functional over any compact subset of M, that is*

$$\frac{d}{dt} E_f(\varphi_t; D)|_{t=0} = 0, \tag{3.2}$$

where $\{\varphi_t\}$ is a smooth variation of φ with compact support in D.

Definition 3.1.3. *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map. The f-tension field of φ it is a section $\tau_f(\varphi) \in \Gamma(\varphi^{-1}TN)$ defined by*

$$\begin{array}{rcl} \tau_f(\varphi) & = & \operatorname{trace}_g \nabla f_\varphi d\varphi - e(\varphi)(\operatorname{grad}^N f) o\varphi. \\ & = & f_\varphi \tau(\varphi) + d\varphi(\operatorname{grad}^M f_\varphi) - e(\varphi)(\operatorname{grad}^N f) o\varphi. \end{array}$$

3.1.1 The first variation of the f-energy

Theorem 3.1.1. *[11]. Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map and $\{\varphi_t\}_{t\in I}$,$(I = (-\epsilon, \epsilon) \subset \mathbb{R})$ a smooth variation of φ to support in D then*

$$\frac{d}{dt}E_f(\varphi_t; D)|_{t=0} = -\int_D h(v, \tau_f(\varphi))v_g,$$

where $v = \frac{d\varphi_t}{dt}\Big|_{t=0}$ denotes the variation vector field of variation $\{\varphi_t\}_{t\in I}$ and f_φ is a smooth function defined by

$$\begin{aligned} f_\varphi : \quad M \to & \quad (0, \infty) \\ x \mapsto & \quad f_\varphi(x) = f(x, \varphi(x)). \end{aligned}$$

Proof. Let $\{e_1, \ldots, e_m\}$ be an orthonormal frame and $v \in \Gamma(\varphi^{-1}TN)$ the variation vector field associated to the variation $\{\varphi_t\}_{t\in I}$ given by

$$v = \frac{d\varphi_t}{dt}\Big|_{t=0}.$$

Let $\phi : M \times (-\epsilon, \epsilon) \longrightarrow N$ defined by

$$\phi(x, t) = \varphi_t(x).$$

We have

$$\begin{aligned} \frac{d}{dt}E_f(\varphi_t; D)|_{t=0} &= \int_D \sum_{i=1}^m f_{\varphi_t} h(\nabla^\phi_{(0,\frac{d}{dt})} d\phi(e_i, 0), d\phi(e_i, 0))v_g|_{t=0} \\ &\quad + \frac{1}{2}\int_D \sum_{i=1}^m \frac{\partial f_{\varphi_t}}{\partial t} h(d\phi(e_i, 0), d\phi(e_i, 0))v_g|_{t=0} \\ &= \int_D \sum_{i=1}^m f_{\varphi_t} h(\nabla^\phi_{(e_i,0)} d\phi(0, \frac{d}{dt}), d\phi(e_i, 0))v_g|_{t=0} \\ &\quad + \frac{1}{2}\int_D \sum_{i=1}^m \frac{\partial f_{\varphi_t}}{\partial t} h(d\phi(e_i, 0), d\phi(e_i, 0))v_g|_{t=0} \\ &= \int_D \sum_{i=1}^m f\varphi h(\nabla^\varphi_{e_i} v, d\varphi(e_i))v_g + \int_D v(f)e(\varphi)v_g, \end{aligned} \tag{3.3}$$

where

$$e(\varphi) = \frac{1}{2}\sum_{i=1}^m h(d\phi(e_i, 0), d\phi(e_i, 0))\Big|_{t=0} \ et \ \frac{\partial f_{\varphi_t}}{\partial t}\Big|_{t=0} = v(f) = h(v, (\mathrm{grad}^N f)o\varphi).$$

Let $\omega \in (T^*M)$ defined by

$$\omega(X) = h(v, f_\varphi d\varphi(X)), \quad X \in \Gamma(TM).$$

Then

$$\begin{aligned}
\operatorname{div}^M \omega &= \sum_{i=0}^{m} \Big\{ e_i(\omega(e_i)) - \omega(\nabla^M_{e_i} e_i) \Big\} \\
&= \sum_{i=0}^{m} \Big\{ h(\nabla^\varphi_{e_i} v, f_\varphi d\varphi(e_i)) + h(v, \nabla^\varphi_{e_i} f_\varphi d\varphi(e_i)) - h(v, f_\varphi d\varphi(\nabla^M_{e_i} e_i)) \Big\}. \quad (3.4)
\end{aligned}$$

By using the formulas (3.3), (3.4) and the divergence Theorem, we get

$$\frac{d}{dt} E_f(\varphi_t; D)|_{t=0} = -\int_D \sum_{i=1}^{m} \Big\{ h(v, \nabla^\varphi_{e_i} f_\varphi d\varphi(e_i) - f_\varphi d\varphi(\nabla^M_{e_i} e_i) - e(\varphi)(\operatorname{grad}^N f) o\varphi) \Big\} v_g.$$

□

Theorem 3.1.2. *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map, then φ is f- harmonic, if and only if*

$$\tau_f(\varphi) = f_\varphi \tau(\varphi) + d\varphi(\operatorname{grad}^M f_\varphi) - e(\varphi)(\operatorname{grad}^N f) o\varphi = 0. \quad (3.5)$$

Remark 3.1.1. 1. If $f = 1$ on M then $\tau_f(\varphi) = \tau(\varphi)$ is the natural tension field of φ.

2. The equation (3.5) is called Euler-Lagrange equation associated to the f-energy functional.

3.1.2 The second variation of the f-energy

Notation 3.1.1. Let $v \in \Gamma(\varphi^{-1}TN)$,

$$\begin{aligned}
J^\varphi_f(v) &= -f_\varphi \operatorname{trace}_g R^N(v, d\varphi)d\varphi - \operatorname{trace}_g \nabla^\varphi f_\varphi \nabla^\varphi v \\
&\quad + e(\varphi)(\nabla^N_v \operatorname{grad}^N f) \circ \varphi - d\varphi(\operatorname{grad}^M v(f)) \\
&\quad - v(f)\tau(\varphi) + < \nabla^\varphi v, d\varphi > (\operatorname{grad}^N f) \circ \varphi, \quad (3.6)
\end{aligned}$$

J^φ_f is called the f-Jacobi operator corresponding to φ.

Theorem 3.1.3. *[11]. Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be an f-harmonic map between Riemannian manifolds and $\varphi_{t,s} : M \to N(-\epsilon < t, s < \epsilon))$ be a two-parameter variation with compact support, such that $\varphi_{(0,0)} = \varphi$, then*

$$\frac{\partial^2}{\partial t \partial s} E_f(\varphi_{t,s}; D)\Big|_{(t,s)=(0,0)} = \int_D h(J^\varphi_f(v), w) v_g,$$

where $v = \frac{\partial \varphi_{t,s}}{\partial t}|_{(t,s)=(0,0)}$ and $w = \frac{\partial \varphi_{t,s}}{\partial S}|_{(t,s)=(0,0)}$ denote the vectors filed of variations.

Proof. Let $\{e_1, \ldots, e_m\}$ be an orthonormal frame on M, such that $\nabla_{e_i} e_j = 0$, at fixed point $x \in M$ for all $i, j = 1, ..., m$. We put

$$E_i = (e_i, 0, 0), \quad \frac{\partial}{\partial t} = (0, \frac{d}{dt}, 0) \;\; and \;\; \frac{\partial}{\partial s} = (0, 0, \frac{d}{ds}).$$

Let $\phi : M \times (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \longrightarrow N$ defined by $\phi(x, t, s) = \varphi_{t,s}(x)$.We compute

$$\frac{\partial^2}{\partial t \partial s} E_f(\varphi_{t,s}) = \frac{1}{2} \int_M \sum_{i=1}^{m} \frac{\partial^2}{\partial t \partial s} \Big[f(x, \varphi_{t,s}(x)) h(d\varphi_{t,s}(e_i), d\varphi_{t,s}(e_i)) \Big] v_g,$$

we have

$$\begin{aligned} \frac{1}{2} \frac{\partial}{\partial t} \Big[f(x, \varphi_{t,s}(x)) h(d\varphi_{t,s}(e_i) \quad , \quad & d\varphi_{t,s}(e_i)) \Big] = \\ & \frac{1}{2} \frac{\partial}{\partial t} f(x, \varphi_{t,s}(x)).h(d\varphi_{t,s}(e_i), d\varphi_{t,s}(e_i)) \\ & + f(x, \varphi_{t,s}(x)).h(\nabla^{\phi}_{\frac{\partial}{\partial t}} d\varphi_{t,s}(e_i), d\varphi_{t,s}(e_i)). \end{aligned}$$

Hence

$$\begin{aligned} \frac{1}{2} \frac{\partial^2}{\partial t \partial s} \Big[f(x, \varphi_{t,s}(x)) h(d\varphi_{t,s}(e_i) \quad , \quad & d\varphi_{t,s}(e_i)) \Big] = \\ & \frac{1}{2} \frac{\partial^2}{\partial t \partial s} f(x, \varphi_{t,s}(x)).h(d\varphi_{t,s}(e_i), d\varphi_{t,s}(e_i)) \\ & + \frac{\partial}{\partial t} f(x, \varphi_{t,s}(x)).h(\nabla^{\phi}_{\frac{\partial}{\partial s}} d\varphi_{t,s}(e_i), d\varphi_{t,s}(e_i)) \\ & + \frac{\partial}{\partial s} f(x, \varphi_{t,s}(x)).h(\nabla^{\phi}_{\frac{\partial}{\partial t}} d\varphi_{t,s}(e_i), d\varphi_{t,s}(e_i)) \\ & + f(x, \varphi_{t,s}(x)).h(\nabla^{\phi}_{\frac{\partial}{\partial s}} \nabla^{\phi}_{\frac{\partial}{\partial t}} d\varphi_{t,s}(e_i), d\varphi_{t,s}(e_i)) \\ & + f(x, \varphi_{t,s}(x)).h(\nabla^{\phi}_{\frac{\partial}{\partial t}} d\varphi_{t,s}(e_i), \nabla^{\phi}_{\frac{\partial}{\partial s}} d\varphi_{t,s}(e_i)). \end{aligned} \tag{3.7}$$

On the other hand

$$\begin{aligned} \frac{\partial}{\partial t} f(x, \varphi_{t,s}(x)) &= df(0, \frac{\partial \varphi_{t,s}(x)}{\partial t}) \\ &= df(0, d\phi(\frac{\partial}{\partial t})) \\ &= h((\mathrm{grad}^N f) o\phi, d\phi(\frac{\partial}{\partial t})); \end{aligned}$$

$$\frac{\partial}{\partial s} f(x, \varphi_{t,s}(x)) = h((\mathrm{grad}^N f) o\phi, d\phi(\frac{\partial}{\partial s}));$$

and

$$\begin{aligned}\frac{\partial^2}{\partial t\partial s}f(x,\varphi_{t,s}(x)) &= h(\nabla^{\phi}_{\frac{\partial}{\partial s}}(\operatorname{grad}^N f)o\phi, d\phi(\frac{\partial}{\partial t}))\\ &\quad +h((\operatorname{grad}^N f)o\phi, \nabla^{\phi}_{\frac{\partial}{\partial s}}d\phi(\frac{\partial}{\partial t})).\end{aligned}$$

From where

$$\begin{aligned}\frac{1}{2}\sum_{i=1}^{m}\frac{\partial^2}{\partial t\partial s}f(x,\varphi_{t,s}(x)).h\big(d\varphi_{t,s}(e_i) \quad &, \quad d\varphi_{t,s}(e_i)\big)\Big|_{(t,s)=(0,0)} =\\ & h(\nabla^N_w(\operatorname{grad}^N f)o\varphi, v)e(\varphi) \qquad (3.8)\\ & +h\big((\operatorname{grad}^N f)o\varphi, \nabla^{\phi}_{\frac{\partial}{\partial s}}d\phi(\frac{\partial}{\partial t})\Big|_{(t,s)=(0,0)}\big)e(\varphi).\end{aligned}$$

We have

$$\begin{aligned}\frac{\partial}{\partial t}f(x,\varphi_{t,s}(x)).h\big(\nabla^{\phi}_{\frac{\partial}{\partial s}}d\varphi_{t,s}(e_i) \quad &, \quad d\varphi_{t,s}(e_i)\big)\Big|_{(t,s)=(0,0)}\\ &= +h\big((\operatorname{grad}^N f)o\varphi, v\big)h\big(\nabla^{\phi}_{\frac{\partial}{\partial s}}d\phi(E_i), d\phi(E_i)\big)\Big|_{(t,s)=(0,0)}\\ &= v(f))h\big(\nabla^{\phi}_{E_i}d\phi(\frac{\partial}{\partial s}), d\phi(E_i)\big)\Big|_{(t,s)=(0,0)},\\ &= v(f)\Big[e_i(h(w,d\varphi(e_i))) - h(w,\tau(\varphi))\Big].\end{aligned}$$

If w_1 denotes the differential 1-form with support in D, defined on M by

$$w_1(Y) = h(w, d\varphi(Y)), \quad Y \in \Gamma(TM).$$

Then

$$\begin{aligned}\frac{\partial}{\partial t}f(x,\varphi_{t,s}(x)).h\big(\nabla^{\phi}_{\frac{\partial}{\partial s}}d\varphi_{t,s}(e_i) \quad &, \quad d\varphi_{t,s}(e_i)\big)\Big|_{(t,s)=(0,0)}\\ &= v(f)\operatorname{div}^M w_1 - h(w, v(f)\tau(\varphi))\\ &= \operatorname{div}^M(v(f)w_1) - h(w, d\varphi(\operatorname{grad}^M v(f)))\\ &\quad -h(w, v(f)\tau(\varphi)); \qquad (3.9)\end{aligned}$$

$$\begin{aligned}\sum_{i=1}^{m}\frac{\partial}{\partial s}f(x,\varphi_{t,s}(x)).h\big(\nabla^{\phi}_{\frac{\partial}{\partial t}}d\varphi_{t,s}(e_i) \quad &, \quad d\varphi_{t,s}(e_i)\big)\Big|_{(t,s)=(0,0)}\\ &= h((\operatorname{grad}^N f)o\varphi, w). < \nabla^{\varphi}v, d\varphi >\\ &= h(< \nabla^{\varphi}v, d\varphi > (\operatorname{grad}^N f)o\varphi, w); \qquad (3.10)\end{aligned}$$

$$\begin{aligned}
\sum_{i=1}^{m} f(x,\varphi_{t,s}(x)).h\big(\nabla^{\phi}_{\frac{\partial}{\partial s}}\nabla^{\varphi}_{\frac{\partial}{\partial t}}d\varphi_{t,s}(e_i) \quad &, \quad d\varphi_{t,s}(e_i)\big)\Big|_{(t,s)=(0,0)} \\
&= \sum_{i=1}^{m} f_\varphi h\big(\nabla^{\phi}_{\frac{\partial}{\partial s}}\nabla^{\phi}_{E_i}d\phi(\frac{\partial}{\partial t}),d\varphi(e_i)\big)|_{(t,s)=(0,0)} \\
&= \sum_{i=1}^{m} f_\varphi h(R^N(w,d\varphi(e_i))v,d\varphi(e_i)) \\
&\quad +\sum_{i=1}^{m} f_\varphi h\big(\nabla^{\phi}_{E_i}\nabla^{\phi}_{\frac{\partial}{\partial s}}d\phi(\frac{\partial}{\partial t}),d\varphi(e_i)\big)\Big|_{(t,s)=(0,0)} \\
&= -\sum_{i=1}^{m} f_\varphi h(R^N(v,d\varphi(e_i))d\varphi(e_i),w) \\
&\quad +\sum_{i=1}^{m} f_\varphi e_i\big(h\big(\nabla^{\phi}_{\frac{\partial}{\partial s}}d\phi(\frac{\partial}{\partial t}),d\varphi(e_i)\big)\big)\Big|_{(t,s)=(0,0)} \\
&\quad -f_\varphi h\big(\nabla^{\phi}_{\frac{\partial}{\partial s}}d\phi(\frac{\partial}{\partial t}),\tau(\varphi)\big)\Big|_{(t,s)=(0,0)}.
\end{aligned}$$

If w_2 denotes the differential 1-form with support in D, defined on M by

$$w_2(Y)=h(\nabla^{\phi}_{\frac{\partial}{\partial s}}d\phi(\frac{\partial}{\partial t}),d\varphi(Y))\Big|_{(t,s)=(0,0)}, \quad Y\in\Gamma(TM).$$

Then

$$\begin{aligned}
\sum_{i=1}^{m} f(x,\varphi_{t,s}(x)).h\big(\nabla^{\phi}_{\frac{\partial}{\partial s}}\nabla^{\varphi}_{\frac{\partial}{\partial t}}d\varphi_{t,s}(e_i) \quad &, \quad d\varphi_{t,s}(e_i)\big)\Big|_{(t,s)=(0,0)} \\
&= -f_\varphi h(\operatorname{trace}_g R^N(v,d\varphi)d\varphi,w) \\
&\quad +f_\varphi \operatorname{div}^M w_2 - f_\varphi h\big(\nabla^{\phi}_{\frac{\partial}{\partial s}}d\phi(\frac{\partial}{\partial t}),\tau(\varphi)\big)\big)\Big|_{(t,s)=(0,0)} \\
&= -f_\varphi h(\operatorname{trace}_g R^N(v,d\varphi)d\varphi,w)+\operatorname{div}^M(f_\varphi w_2) \\
&\quad -h\big(\nabla^{\phi}_{\frac{\partial}{\partial s}}d\phi(\frac{\partial}{\partial t}),d\varphi(\operatorname{grad}^M f_\varphi)\big)\Big|_{(t,s)=(0,0)} \\
&\quad -f_\varphi h\big(\nabla^{\phi}_{\frac{\partial}{\partial s}}d\phi(\frac{\partial}{\partial t}),\tau(\varphi)\big)\big)\Big|_{(t,s)=(0,0)} \qquad (3.11)
\end{aligned}$$

So that

$$\begin{aligned}
f(x,\varphi_{t,s}(x)).h(\nabla^{\phi}_{\frac{\partial}{\partial t}}d\varphi_{t,s}(e_i),\nabla^{\phi}_{\frac{\partial}{\partial s}}d\varphi_{t,s}(e_i)) &= f_\varphi h(\nabla^{\phi}_{E_i}d\phi(\frac{\partial}{\partial t}),\nabla^{\phi}_{E_i}d\phi(\frac{\partial}{\partial s}))\Big|_{(t,s)=(0,0)} \\
&= f_\varphi\Big[e_i(h(\nabla^{\varphi}v,w))-h(\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}v,w)\Big];
\end{aligned}$$

$$\begin{aligned}\sum_{i=1}^{m} f(x,\varphi_{t,s}(x)).h(\nabla^{\phi}_{\frac{\partial}{\partial t}} d\varphi_{t,s}(e_i) \quad &, \quad \nabla^{\phi}_{\frac{\partial}{\partial s}} d\varphi_{t,s}(e_i))\Big|_{(t,s)=(0,0)} \\ &= f_\varphi\Big[\operatorname{div}^M w_3 - h(\operatorname{trace}_g(\nabla^\varphi)^2 v, w)\Big] \\ &= \operatorname{div}^M(f_\varphi w_3) - h(\nabla^\varphi_{\operatorname{grad}^M f_\varphi} v, w) \\ &\quad -h(f_\varphi \operatorname{trace}_g(\nabla^\varphi)^2 v, w) \\ &= \operatorname{div}^M(f_\varphi w_3) - h(\operatorname{trace}_g \nabla^\varphi f_\varphi \nabla^\varphi v, w), \quad (3.12)\end{aligned}$$

where w_3 denotes the differential 1-form with support in D, defined on M by

$$w_3(Y) = h(\nabla^\varphi_Y v, w), \quad Y \in \Gamma(TM).$$

By the formulas (3.7),(3.8),(3.9),(3.10), (3.11), (3.12) and the divergence Theorem, the Theorem 3.1.3 follows. □

3.2 f-biharmonic maps

A natural generalization of f-biharmonic maps is given by integrating the square of the norm of the f-tension field. More precisely, the f-bienergy functional of a smooth map $\varphi : (M^m, g) \to (N^n, h)$ is defined by

$$E_{2,f}(\varphi, D) = \frac{1}{2}\int_D |\tau_f(\varphi)|^2 v_g, \tag{3.13}$$

where D is a compact subset M.
A map φ is called f-biharmonic if it is a critical point of the f-bi-energy functional over any compact subset of M that is,

$$\frac{d}{dt}E_{2,f}(\varphi_t, D)|_{t=0} = 0, \tag{3.14}$$

where $\{\varphi_t\}$ is a smooth variation of φ with support in D.

Definition 3.2.1. *Let $\varphi : M^m \longrightarrow N^n$ be a smooth map between two Riemannian manifolds. $\tau_{2,f}(\varphi) \in \Gamma(\varphi^{-1}(TN))$ defined by*

$$\begin{aligned}\tau_{2,f}(\varphi) \;=\; & -f_\varphi \operatorname{trace}_g R^N(\tau_f(\varphi), d\varphi)d\varphi - \operatorname{trace}_g \nabla^\varphi f_\varphi \nabla^\varphi \tau_f(\varphi) \\ & + e(\varphi)(\nabla^N_{\tau_f(\varphi)} \operatorname{grad}^N f) o\varphi - d\varphi(\operatorname{grad}^M \tau_f(\varphi)(f)) \\ & - \tau_f(\varphi)(f)\tau(\varphi) + < \nabla^\varphi \tau_f(\varphi), d\varphi > (\operatorname{grad}^N f) o\varphi\end{aligned}$$

is called the f-bi-tension field of φ, where

$$\operatorname{trace}_g R^N(\tau_f(\varphi), d\varphi)d\varphi \;=\; \sum_{i=1}^{m} R^N(\tau_f(\varphi), d\varphi(e_i))d\varphi(e_i);$$

$$\begin{aligned}\mathrm{trace}_g(\nabla^\varphi f_\varphi \nabla^\varphi \tau_f(\varphi) &= \sum_{i=1}^m \big(\nabla^\varphi_{e_i} f_\varphi \nabla^\varphi_{e_i}\tau_f(\varphi) - f_\varphi \nabla^\varphi_{\nabla^M_{e_i}e_i}\tau_f(\varphi)\big);\\ <\nabla^\varphi\tau_f(\varphi), d\varphi> &= \sum_{i=1}^m h(\nabla^\varphi_{e_i}\tau_f(\varphi), d\varphi(e_i)),\end{aligned}$$

where $\{e_1, \ldots, e_m\}$ *is an orthonormal frame on* M.

3.2.1 First variation of the f-bi-energy

Theorem 3.2.1. *[11]. Let* $\varphi : (M^m, g) \longrightarrow (N^n, h)$ *be a smooth map and let* $\{\varphi_t\}$ *be a smooth variation of* φ *with support in* D*. Then*

$$\frac{d}{dt}E_{2,f}(\varphi_t; D)|_{t=0} = -\int_D h(v, \tau_{2,f}(\varphi))v_g,$$

where $v = \frac{d\varphi_t}{dt}|_{t=0}$ *denotes the variation vector field of* $\{\varphi_t\}$.

Proof. Let $\{\varphi_t\}$ be a smooth variation of φ with support in compact subset D of M and $v \in \Gamma(\varphi^{-1}TN)$ is a variation vector field of $\{\varphi_t\}$.
Let $\phi : M \times (-\epsilon, \epsilon) \to N$ defined by $\phi(x,t) = \varphi_t(x)$, then

$$\frac{d}{dt}E_{2,f}(\varphi_t, D) = \int_D h(\nabla^\phi_{\frac{\partial}{\partial t}}\tau_f(\varphi_t), \tau_f(\varphi_t))v_g. \tag{3.15}$$

Let $\{e_1, \ldots, e_m\}$ be an orthonormal frame on M such that $(\nabla^M_{e_i}e_j) = 0\ \forall i, j = 1, \ldots, m$ at a fixed point $x \in M$, we have

$$\nabla^\phi_{\frac{\partial}{\partial t}}\tau_f(\varphi_t) = \sum_{i=1}^m \nabla^\phi_{\frac{\partial}{\partial t}}\nabla^\phi_{(e_i,0)} f_{\varphi_t} d\varphi_t(e_i) - \nabla^\phi_{\frac{\partial}{\partial t}} e(\varphi_t)(\mathrm{grad}^N f) o\varphi_t. \tag{3.16}$$

The first term of (3.16) is given by

$$\begin{aligned}\sum_{i=1}^m \nabla^\phi_{\frac{\partial}{\partial t}}\nabla^\phi_{(e_i,0)} f_{\varphi_t} d\varphi_t(e_i) &= \sum_{i=1}^m R^N(d\phi\frac{\partial}{\partial t}), d\phi(e_i,0)) f_{\varphi_t} d\varphi_t(e_i)\\ &\quad + \sum_{i=1}^m \nabla^\phi_{(e_i,0)}\nabla^\phi_{\frac{\partial}{\partial t}} f_{\varphi_t} d\varphi_t(e_i).\end{aligned} \tag{3.17}$$

We have

$$\begin{aligned}\sum_{i=1}^m h(\nabla^\phi_{(e_i,0)}\nabla^\phi_{\frac{\partial}{\partial t}} f_{\varphi_t} d\varphi_t(e_i), \tau_f(\varphi_t)) &= \sum_{i=1}^m e_i h(\nabla^\phi_{\frac{\partial}{\partial t}} f_{\varphi_t} d\varphi_t(e_i), \tau_f(\varphi_t))\\ &\quad - \sum_{i=1}^m h(\nabla^\phi_{\frac{\partial}{\partial t}} f_{\varphi_t} d\varphi_t(e_i), \nabla^\phi_{(e_i,0)}\tau_f(\varphi_t))\end{aligned}$$

$$\begin{aligned} &= \sum_{i=1}^m e_i h(\nabla^\phi_{\frac{\partial}{\partial t}} f_{\varphi_t} d\varphi_t(e_i), \tau_f(\varphi_t)) \\ &\quad - \sum_{i=1}^m \frac{\partial f_{\varphi_t}}{\partial t}.h(d\varphi_t(e_i), \nabla^\phi_{(e_i,0)}\tau_f(\varphi_t)) \\ &\quad - \sum_{i=1}^m f_{\varphi_t} h(\nabla^\phi_{\frac{\partial}{\partial t}} d\varphi_t(e_i), \nabla^\phi_{(e_i,0)}\tau_f(\varphi_t)). \end{aligned} \tag{3.18}$$

Let w_1 be a differential 1-form with support in D, defined on M by

$$w_1(Y) = h(\nabla^\phi_{\frac{\partial}{\partial t}} f_{\varphi_t} d\varphi_t(Y), \tau_f(\varphi_t))|_{t=0}, \quad Y \in \Gamma(TM).$$

By (3.18) we obtain

$$\begin{aligned} \sum_{i=1}^m h(\nabla^\phi_{(e_i,0)}\nabla^\phi_{\frac{\partial}{\partial t}} f_{\varphi_t} d\varphi_t(e_i), \tau_f(\varphi_t))\Big|_{t=0} &= \operatorname{div}^M w_1 - h((\operatorname{grad}^N f)o\varphi, v). < d\varphi, \nabla^\varphi \tau_f(\varphi) > \\ &\quad - \sum_{i=1}^m f_{\varphi_t} h(\nabla^\phi_{\frac{\partial}{\partial t}} d\varphi_t(e_i), \nabla^\phi_{(e_i,0)}\tau_f(\varphi_t))|_{t=0}; \end{aligned} \tag{3.19}$$

$$\begin{aligned} \sum_{i=1}^m f_{\varphi_t} h(\nabla^\phi_{\frac{\partial}{\partial t}} d\varphi_t(e_i), \nabla^\phi_{(e_i,0)}\tau_f(\varphi_t))\Big|_{t=0} &= \sum_{i=1}^m h(\nabla^\phi_{(e_i,0)} d\phi_t(\frac{\partial}{\partial t}), f_{\varphi_t}\nabla^\phi_{(e_i,0)}\tau_f(\varphi_t))|_{t=0} \\ &= \sum_{i=1}^m \big[e_i(h(v, f_\varphi \nabla^\varphi_{e_i}\tau_f(\varphi_t))) \\ &\quad - h(v, \nabla^\varphi_{e_i} f_\varphi \nabla^\varphi_{e_i}\tau_f(\varphi_t))\big]. \end{aligned} \tag{3.20}$$

If ω_2 denotes the differential 1-form with support in D, defined on M by

$$\omega_2(X) = h(v, f_\varphi \nabla^\varphi_Y \tau_f(\varphi)), \qquad Y \in \Gamma(TM).$$

Then

$$\sum_{i=1}^m f_{\varphi_t} h(\nabla^\phi_{\frac{\partial}{\partial t}} d\varphi_t(e_i), \nabla^\phi_{(e_i,0)}\tau_f(\varphi_t))|_{t=0} = \operatorname{div}^M \omega_2 - h(v, \operatorname{trace}_g \nabla^\varphi f_\varphi \nabla^\varphi \tau_f(\varphi_t)). \tag{3.21}$$

By the formulas (3.17) (3.19) and (3.21), we obtain

$$\begin{aligned} \sum_{i=1}^m h(\nabla^\phi_{\frac{\partial}{\partial t}}\nabla^\phi_{(e_i,0)} f_{\varphi_t} d\varphi_t(e_i), \tau_f(\varphi_t))|_{t=0} &= h(f_\varphi \operatorname{trace}_g R^N(\tau_f(\varphi), d\varphi)d\varphi, v) \\ + \operatorname{div}^M \omega_1 - h(< d\varphi, \nabla^\varphi \tau_f(\varphi) > (\operatorname{grad}^N f)o\varphi, v) \end{aligned}$$

$$-\mathrm{div}^M\omega_2 + h(\mathrm{trace}_g\,\nabla^\varphi f_\varphi \nabla^\varphi \tau_f(\varphi), v). \tag{3.22}$$

The second term of (3.16) is given by

$$\nabla^\phi_{\frac{\partial}{\partial t}} e(\varphi_t)(\mathrm{grad}^N f)o\varphi_t = \frac{\partial e(\varphi_t)}{\partial t}(\mathrm{grad}^N f)o\varphi_t + e(\varphi_t)\nabla^\phi_{\frac{\partial}{\partial t}}(\mathrm{grad}^N f)o\varphi_t. \tag{3.23}$$

Calculate

$$\begin{aligned}
\frac{\partial e(\varphi_t)}{\partial t}|_{t=0} &= \frac{1}{2}\sum_{i=1}^m \frac{\partial}{\partial t} h(d\phi(e_i,0), d\phi(e_i,0))|_{t=0} \\
&= \sum_{i=1}^m h(\nabla^\phi_{\frac{\partial}{\partial t}} d\phi(e_i,0), d\phi(e_i,0))|_{t=0} \\
&= \sum_{i=1}^m h(\nabla^\phi_{(e_i,0)} d\phi(\frac{\partial}{\partial t}), d\phi(e_i,0))|_{t=0} \\
&= \sum_{i=1}^m e_i(h(v, d\varphi(e_i))) - h(v,\tau(\varphi)) \\
&= \mathrm{div}^M\,\omega_3 - h(v,\tau(\varphi)), \qquad (3.24)
\end{aligned}$$

where ω_3 is a differential 1-form with support in D, defined on M by

$$\omega_3(Y) = h(v, d\varphi(Y)), \quad Y \in \Gamma(TM)$$

By the formulas (3.23) and (3.24),we get

$$\begin{aligned}
h(\nabla^\phi_{\frac{\partial}{\partial t}} e(\varphi_t)(\mathrm{grad}^N f)o\varphi_t, \tau_f(\varphi_t))|_{t=0} &= \tau_f(\varphi)(f)\,\mathrm{div}^M\,\omega_3 - \tau_f(\varphi)(f)h(v,\tau(\varphi)) \\
&\quad + e(\varphi)h(\nabla^\phi_{\frac{\partial}{\partial t}}(\mathrm{grad}^N f)o\varphi_t, \tau_f(\varphi_t))|_{t=0} \\
&= \mathrm{div}^M(\tau_f(\varphi)(f)\omega_3) - h(v, d\varphi(\mathrm{grad}^M \tau_f(\varphi)(f))) \\
&\quad - \tau_f(\varphi)(f)h(v,\tau(\varphi)) \\
&\quad + e(\varphi)h(\nabla^N_{\tau_f(\varphi)}\mathrm{grad}^N f, v). \qquad (3.25)
\end{aligned}$$

By the equations (3.15),(3.16), (3.22), (3.25) and the divergence Theorem, we obtain

$$\begin{aligned}
\frac{d}{dt}E_{2,f}(\varphi_t; D)|_{t=0} &= -\int_D h(-f_\varphi \mathrm{trace}_g R^N(\tau_f(\varphi), d\varphi)d\varphi - \mathrm{trace}_g \nabla^\varphi f_\varphi \nabla^\varphi \tau_f(\varphi) \\
&\quad + e(\varphi)(\nabla^N_{\tau_f(\varphi)}\mathrm{grad}^N f)o\varphi - d\varphi(\mathrm{grad}^M \tau_f(\varphi)(f)) \\
&\quad - \tau_f(\varphi)(f)\tau(\varphi) + < \nabla^\varphi \tau_f(\varphi), d\varphi > (\mathrm{grad}^N f)o\varphi, v)v_g.
\end{aligned}$$

□

Theorem 3.2.2. *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map . Then φ is f-biharmonic if and only if*

$$\begin{aligned}
\tau_{2,f}(\varphi) &= -f_\varphi \mathrm{trace}_g R^N(\tau_f(\varphi), d\varphi)d\varphi - \mathrm{trace}_g \nabla^\varphi f_\varphi \nabla^\varphi \tau_f(\varphi) \\
&\quad + e(\varphi)(\nabla^N_{\tau_f(\varphi)} \mathrm{grad}^N f) o\varphi - d\varphi(\mathrm{grad}^M \tau_f(\varphi)(f)) \\
&\quad - \tau_f(\varphi)(f)\tau(\varphi) + < \nabla^\varphi \tau_f(\varphi), d\varphi > (\mathrm{grad}^N f) o\varphi \\
&= 0. \qquad (3.26)
\end{aligned}$$

Remark 3.2.1.
- (1) If $f = 1$, then $\tau_{2,f}(\varphi) = \tau_2(\varphi)$, is the natural bi-tension field of φ.

- The equation (3.26) is called the Euler-Lagrange equation associated to f-bi-energy functional .

3.3 Main results

3.3.1 Some results on stable f-harmonic maps

Theorem 3.3.1. *[39]. Any stable f-harmonic map φ from sphere $(\mathbb{S}^n, g)$ $(n > 2)$ to Riemannian manifold (N, h) is constant, where f is a smooth positive function on $\mathbb{S}^n \times N$ satisfying* $\mathrm{trace}_g\, h((\nabla d\varphi)(\cdot, \mathrm{grad}^{\mathbb{S}^n} f), d\varphi(\cdot)) \geq 0$.

Proof. Choose a normal orthonormal frame $\{e_i\}$ at point x_0 in $\mathbb{S}^n$. Set

$$\lambda(x) = < \alpha, x >_{\mathbb{R}^{n+1}},$$

for all $x \in \mathbb{S}^n$, where $\alpha \in \mathbb{R}^{n+1}$ and let $v = \mathrm{grad}^{\mathbb{S}^n} \lambda$. Note that

$$v = < \alpha, e_i > e_i,\ \nabla^{\mathbb{S}^n}_X v = -\lambda X, \text{ for all } X \in \Gamma(T\mathbb{S}^n),$$
$$\mathrm{trace}_g (\nabla^{\mathbb{S}^n})^2 v = \nabla^{\mathbb{S}^n}_{e_i} \nabla^{\mathbb{S}^n}_{e_i} v - \nabla^{\mathbb{S}^n}_{\nabla^{\mathbb{S}^n}_{e_i} e_i} v = -v,$$

where $\nabla^{\mathbb{S}^n}$ is the Levi-Civita connection on $\mathbb{S}^n$ with respect to the standard metric g of the sphere (see [44]). At point x_0, we have

$$\nabla^\varphi_{e_i} f_\varphi \nabla^\varphi_{e_i} d\varphi(v) = \nabla^\varphi_{\mathrm{grad}^{\mathbb{S}^n} f_\varphi} d\varphi(v) + f_\varphi \nabla^\varphi_{e_i} \nabla^\varphi_{e_i} d\varphi(v), \qquad (3.27)$$

the first term of (3.27) is given by

$$\begin{aligned}
\nabla^\varphi_{\mathrm{grad}\, f_\varphi} d\varphi(v) &= \nabla^\varphi_v d\varphi(\mathrm{grad}^{\mathbb{S}^n} f_\varphi) + d\varphi([\mathrm{grad}^{\mathbb{S}^n} f_\varphi, v]) \\
&= \nabla^\varphi_v d\varphi(\mathrm{grad}^{\mathbb{S}^n} f_\varphi) + d\varphi(\nabla^{\mathbb{S}^n}_{\mathrm{grad}^{\mathbb{S}^n} f_\varphi} v) \\
&\quad - d\varphi(\nabla^{\mathbb{S}^n}_v \mathrm{grad}^{\mathbb{S}^n} f_\varphi), \qquad (3.28)
\end{aligned}$$

the seconde term of (3.27) is given by

$$\begin{aligned} f_\varphi \nabla^\varphi_{e_i}\nabla^\varphi_{e_i} d\varphi(v) &= f_\varphi \nabla^\varphi_{e_i}\nabla^\varphi_{v} d\varphi(e_i) + f_\varphi \nabla^\varphi_{e_i} d\varphi([e_i,v]) \\ &= f_\varphi R^N(d\varphi(e_i), d\varphi(v))d\varphi(e_i) + f_\varphi \nabla^\varphi_v \nabla^\varphi_{e_i} d\varphi(e_i) \\ &\quad + f_\varphi d\varphi([e_i,[e_i,v]]) + 2 f_\varphi \nabla^\varphi_{[e_i,v]} d\varphi(e_i), \end{aligned} \tag{3.29}$$

from the definition of tension field, we get

$$\begin{aligned} f_\varphi \nabla^\varphi_{e_i}\nabla^\varphi_{e_i} d\varphi(v) &= -f_\varphi R^N(d\varphi(v), d\varphi(e_i))d\varphi(e_i) + f_\varphi \nabla^\varphi_v \tau(\varphi) \\ &\quad + f_\varphi \nabla^\varphi_v d\varphi(\nabla^{\mathbb{S}^n}_{e_i} e_i) + f_\varphi d\varphi(\nabla^{\mathbb{S}^n}_{e_i}\nabla^{\mathbb{S}^n}_{e_i} v) \\ &\quad - f_\varphi d\varphi(\nabla^{\mathbb{S}^n}_{e_i}\nabla^{\mathbb{S}^n}_{v} e_i) + 2 f_\varphi \nabla^\varphi_{[e_i,v]} d\varphi(e_i) \\ &= -f_\varphi R^N(d\varphi(v), d\varphi(e_i))d\varphi(e_i) + \nabla^\varphi_v f_\varphi \tau(\varphi) - v(f_\varphi)\tau(\varphi) \\ &\quad + f_\varphi \nabla^\varphi_v d\varphi(\nabla^{\mathbb{S}^n}_{e_i} e_i) + f_\varphi d\varphi(\nabla^{\mathbb{S}^n}_{e_i}\nabla^{\mathbb{S}^n}_{e_i} v) \\ &\quad - f_\varphi d\varphi(\nabla^{\mathbb{S}^n}_{e_i}\nabla^{\mathbb{S}^n}_{v} e_i) + 2 f_\varphi \nabla^\varphi_{[e_i,v]} d\varphi(e_i), \end{aligned} \tag{3.30}$$

by equations (3.27), (3.28), (3.30), and the f-harmonicity condition of φ, we have

$$\begin{aligned} \nabla^\varphi_{e_i} f_\varphi \nabla^\varphi_{e_i} d\varphi(v) &= d\varphi(\nabla^{\mathbb{S}^n}_{\mathrm{grad}^{\mathbb{S}^n} f_\varphi} v) - d\varphi(\nabla^{\mathbb{S}^n}_v \mathrm{grad}^{\mathbb{S}^n} f_\varphi) \\ &\quad - f_\varphi R^N(d\varphi(v), d\varphi(e_i))d\varphi(e_i) \\ &\quad + \nabla^\varphi_v e(\varphi)(\mathrm{grad}^N f)\circ\varphi - v(f_\varphi)\tau(\varphi) \\ &\quad + f_\varphi d\varphi(\nabla^{\mathbb{S}^n}_{v}\nabla^{\mathbb{S}^n}_{e_i} e_i) + f_\varphi d\varphi(\nabla^{\mathbb{S}^n}_{e_i}\nabla^{\mathbb{S}^n}_{e_i} v) \\ &\quad - f_\varphi d\varphi(\nabla^{\mathbb{S}^n}_{e_i}\nabla^{\mathbb{S}^n}_{v} e_i) + 2 f_\varphi \nabla^\varphi_{\nabla^{\mathbb{S}^n}_{e_i} v} d\varphi(e_i), \end{aligned} \tag{3.31}$$

by the definition of Ricci tensor, we get

$$\begin{aligned} \nabla^\varphi_{e_i} f_\varphi \nabla^\varphi_{e_i} d\varphi(v) &= d\varphi(\nabla^{\mathbb{S}^n}_{\mathrm{grad}^{\mathbb{S}^n} f_\varphi} v) - d\varphi(\nabla^{\mathbb{S}^n}_v \mathrm{grad}^{\mathbb{S}^n} f_\varphi) \\ &\quad - f_\varphi R^N(d\varphi(v), d\varphi(e_i))d\varphi(e_i) + \nabla^\varphi_v e(\varphi)(\mathrm{grad}^N f)\circ\varphi \\ &\quad - v(f_\varphi)\tau(\varphi) + f_\varphi d\varphi(\mathrm{Ricci}^{\mathbb{S}^n} v) + f_\varphi d\varphi(\mathrm{trace}(\nabla^{\mathbb{S}^n})^2 v) \\ &\quad + 2 f_\varphi \nabla^\varphi_{\nabla^{\mathbb{S}^n}_{e_i} v} d\varphi(e_i), \end{aligned} \tag{3.32}$$

from the property $\nabla^{\mathbb{S}^n}_X v = -\lambda X$, we obtain

$$\begin{aligned} \nabla^\varphi_{e_i} f_\varphi \nabla^\varphi_{e_i} d\varphi(v) &= -\lambda d\varphi(\mathrm{grad}^{\mathbb{S}^n} f_\varphi) - d\varphi(\nabla^{\mathbb{S}^n}_v \mathrm{grad}^{\mathbb{S}^n} f_\varphi) \\ &\quad - f_\varphi R^N(d\varphi(v), d\varphi(e_i))d\varphi(e_i) \\ &\quad + < \nabla^\varphi_{e_i} d\varphi(v), d\varphi(e_i) > (\mathrm{grad}^N f)\circ\varphi \\ &\quad - h(d\varphi(\nabla^{\mathbb{S}^n}_{e_i} v), d\varphi(e_i))(\mathrm{grad}^N f)\circ\varphi \\ &\quad + e(\varphi)\nabla^\varphi_v (\mathrm{grad}^N f)\circ\varphi \\ &\quad - v(f)\tau(\varphi) - d\varphi(v)(f)\tau(\varphi) + f_\varphi d\varphi(\mathrm{Ricci}^{\mathbb{S}^n} v) \end{aligned}$$

$$+f_\varphi d\varphi(\operatorname{trace}(\nabla^{\mathbb{S}^n})^2 v) - 2\lambda f_\varphi \tau(\varphi). \tag{3.33}$$

From the definition of Jacobi operator (3.6) and equation (3.33) we have

$$\begin{aligned} J_f^\varphi(d\varphi(v)) &= \lambda d\varphi(\operatorname{grad}^{\mathbb{S}^n} f_\varphi) + d\varphi(\nabla_v^{\mathbb{S}^n} \operatorname{grad}^{\mathbb{S}^n} f_\varphi) \\ &\quad -\lambda h(d\varphi(e_i), d\varphi(e_i))(\operatorname{grad}^N f)\circ\varphi + v(f)\tau(\varphi) \\ &\quad -f_\varphi d\varphi(\operatorname{Ricci}^{\mathbb{S}^n} v) - f_\varphi d\varphi(\operatorname{trace}(\nabla^{\mathbb{S}^n})^2 v) \\ &\quad -d\varphi(\operatorname{grad}^{\mathbb{S}^n} d\varphi(v)(f)) + 2\lambda f_\varphi \tau(\varphi), \end{aligned} \tag{3.34}$$

since $\operatorname{trace}_g(\nabla^{\mathbb{S}^n})^2 v = -v$ and $\operatorname{Ricci}^{\mathbb{S}^n} v = (n-1)v$ (see [1, 44]), we conclude

$$\begin{aligned} h(J_f^\varphi(d\varphi(v)), d\varphi(v)) &= \lambda h(d\varphi(\operatorname{grad}^{\mathbb{S}^n} f_\varphi), d\varphi(v)) \\ &\quad +h(d\varphi(\nabla_v^{\mathbb{S}^n} \operatorname{grad}^{\mathbb{S}^n} f_\varphi), d\varphi(v)) \\ &\quad -2\lambda h(e(\varphi)(\operatorname{grad}^N f)\circ\varphi, d\varphi(v)) \\ &\quad +v(f)h(\tau(\varphi), d\varphi(v)) \\ &\quad -(n-2)f_\varphi h(d\varphi(v), d\varphi(v)) \\ &\quad -h(d\varphi(\operatorname{grad}^{\mathbb{S}^n} d\varphi(v)(f)), d\varphi(v)) \\ &\quad +2\lambda f_\varphi h(\tau(\varphi), d\varphi(v)), \end{aligned} \tag{3.35}$$

by (3.35) and the f-harmonicity condition of φ, it follows that

$$\begin{aligned} \operatorname{trace}_\alpha h(J_f^\varphi(d\varphi(v)), d\varphi(v)) &= h(d\varphi(\nabla_{e_j}^{\mathbb{S}^n} \operatorname{grad}^{\mathbb{S}^n} f_\varphi), d\varphi(e_j)) \\ &\quad +h(\tau(\varphi), d\varphi(\operatorname{grad}^{\mathbb{S}^n} f)) - (n-2)f_\varphi |d\varphi|^2 \\ &\quad -h(d\varphi(\operatorname{grad}^{\mathbb{S}^n} d\varphi(e_j)(f)), d\varphi(e_j)), \end{aligned} \tag{3.36}$$

by the following formulas

$$\begin{aligned} h(d\varphi(\nabla_{e_j}^{\mathbb{S}^n} \operatorname{grad}^{\mathbb{S}^n} f_\varphi), d\varphi(e_j)) &= h(d\varphi(\nabla_{e_j}^{\mathbb{S}^n} \operatorname{grad}^{\mathbb{S}^n} f), d\varphi(e_j)) \\ &\quad + h(\nabla_{e_j}^\varphi d\varphi(e_i), \operatorname{grad}^N f) h(d\varphi(e_i), d\varphi(e_j)) \\ &\quad + h(d\varphi(e_i), \nabla_{e_j}^\varphi \operatorname{grad}^N f) h(d\varphi(e_i), d\varphi(e_j)), \\ d\varphi(\operatorname{grad}^{\mathbb{S}^n} d\varphi(e_j)(f)) &= e_i[h(d\varphi(e_j), \operatorname{grad}^N f)] d\varphi(e_i) \\ &= h(\nabla_{e_i}^\varphi d\varphi(e_j), \operatorname{grad}^N f) d\varphi(e_i) \\ &\quad + h(d\varphi(e_j), \nabla_{e_i}^\varphi \operatorname{grad}^N f) d\varphi(e_i); \end{aligned}$$

$$\begin{aligned} -h(d\varphi(\operatorname{grad}^{\mathbb{S}^n} d\varphi(e_j)(f)), d\varphi(e_j)) &= -h(\nabla_{e_i}^\varphi d\varphi(e_j), \operatorname{grad}^N f) h(d\varphi(e_i), d\varphi(e_j)) \\ &\quad -h(d\varphi(e_j), \nabla_{e_i}^\varphi \operatorname{grad}^N f) h(d\varphi(e_i), d\varphi(e_j)), \end{aligned}$$

and equation (3.36), it follows that

$$\operatorname{trace}_\alpha h(J_f^\varphi(d\varphi(v)), d\varphi(v)) = h(d\varphi(\nabla_{e_j}^{\mathbb{S}^n} \operatorname{grad}^{\mathbb{S}^n} f), d\varphi(e_j))$$

$$+h(\tau(\varphi), d\varphi(\operatorname{grad}^{\mathbb{S}^n} f)) - (n-2) f_\varphi |d\varphi|^2,$$

note that

$$\begin{aligned} h(\tau(\varphi), d\varphi(\operatorname{grad}^{\mathbb{S}^n} f)) &= h(\nabla^\varphi_{e_i} d\varphi(e_i), d\varphi(\operatorname{grad}^{\mathbb{S}^n} f)) \\ &= \operatorname{div}^{\mathbb{S}^n} \eta - h(d\varphi(e_i), \nabla^\varphi_{e_i} d\varphi(\operatorname{grad}^{\mathbb{S}^n} f)), \end{aligned}$$

with $\eta(X) = h(d\varphi(X), d\varphi(\operatorname{grad}^{\mathbb{S}^n} f)$, $\forall X \in \Gamma(T\mathbb{S}^n)$. We obtain

$$\begin{aligned} \operatorname{trace}_\alpha h(J^\varphi_f(d\varphi(v)), d\varphi(v)) &= -h((\nabla d\varphi)(e_j, \operatorname{grad}^{\mathbb{S}^n} f), d\varphi(e_j)) \\ &\quad + \operatorname{div}^{\mathbb{S}^n} \eta - (n-2) f_\varphi |d\varphi|^2, \end{aligned} \tag{3.37}$$

since $h((\nabla d\varphi)(e_j, \operatorname{grad}^{\mathbb{S}^n} f), d\varphi(e_j)) \geq 0$, from the stable f-harmonic condition, and equation (3.37), we get

$$\begin{aligned} 0 \leq \operatorname{trace}_\alpha I^\varphi_f(d\varphi(v), d\varphi(v)) &+ \int_{\mathbb{S}^n} h((\nabla d\varphi)(e_j, \operatorname{grad}^{\mathbb{S}^n} f), d\varphi(e_j)) v^{\mathbb{S}^n} \\ &= -(n-2) \int_{\mathbb{S}^n} f_\varphi |d\varphi|^2 v^{\mathbb{S}^n} \leq 0. \end{aligned}$$

Consequently, $|d\varphi| = 0$, that is φ is constant, because $n > 2$. □

Using the similar technique we have

Theorem 3.3.2. *[39]. Let (M, g) be a compact Riemannian manifold. When $n > 2$, any stable f-harmonic map $\varphi : M \to \mathbb{S}^n$ must be constant, where f is a smooth positive function on $M \times \mathbb{S}^n$, with $\Delta^{\mathbb{S}^n}(f) \circ \varphi \leq 0$.*

Proof. Choose a normal orthonormal frame $\{e_i\}$ at point x_0 in M. When the same data of previous proof, we have

$$\nabla^\varphi_{e_i} f_\varphi \nabla^\varphi_{e_i}(v \circ \varphi) = \nabla^\varphi_{\operatorname{grad}^M f_\varphi}(v \circ \varphi) + f_\varphi \nabla^\varphi_{e_i} \nabla^\varphi_{e_i}(v \circ \varphi), \tag{3.38}$$

the first term of (3.38) is given by

$$\nabla^\varphi_{\operatorname{grad}^M f_\varphi}(v \circ \varphi) = -(\lambda \circ \varphi) d\varphi(\operatorname{grad}^M f_\varphi); \tag{3.39}$$

the seconde term of (3.38) is given by

$$\begin{aligned} f_\varphi \nabla^\varphi_{e_i} \nabla^\varphi_{e_i}(v \circ \varphi) &= -f_\varphi \nabla^\varphi_{e_i}(\lambda \circ \varphi) d\varphi(e_i) \\ &= -f_\varphi d\varphi(\operatorname{grad}^M(\lambda \circ \varphi)) - (\lambda \circ \varphi) f_\varphi \tau(\varphi), \end{aligned} \tag{3.40}$$

by the definition of gradient operator, we get

$$-f_\varphi d\varphi(\operatorname{grad}^M(\lambda \circ \varphi)) = -f_\varphi < d\varphi(e_i), v \circ \varphi > d\varphi(e_i), \tag{3.41}$$

substituting the formulas (3.39), (3.40), (3.41) into (3.38), we get

$$\begin{aligned}\nabla^{\varphi}_{e_i} f_\varphi \nabla^{\varphi}_{e_i}(v\circ\varphi) &= -(\lambda\circ\varphi)d\varphi(\operatorname{grad}^M f_\varphi) - f_\varphi < d\varphi(e_i), v\circ\varphi > d\varphi(e_i)\\ &\quad -(\lambda\circ\varphi)f_\varphi\tau(\varphi), \end{aligned}\tag{3.42}$$

from the f-harmonicity condition of φ, and equation (3.42), we have

$$\begin{aligned}< \nabla^{\varphi}_{e_i} f_\varphi \nabla^{\varphi}_{e_i}(v\circ\varphi), v\circ\varphi > &= -f_\varphi < d\varphi(e_i), v\circ\varphi >< d\varphi(e_i), v\circ\varphi >\\ &\quad -(\lambda\circ\varphi)e(\varphi) < (\operatorname{grad}^{\mathbb{S}^n} f)\circ\varphi, v\circ\varphi >, \end{aligned}\tag{3.43}$$

since the sphere $\mathbb{S}^n$ has constant curvature, we obtain

$$\begin{aligned}< f_\varphi R^{\mathbb{S}^n}(v\circ\varphi, d\varphi(e_i))d\varphi(e_i), v\circ\varphi >&= f_\varphi|d\varphi|^2 < v\circ\varphi, v\circ\varphi >\\ &\quad -f_\varphi < d\varphi(e_i), v\circ\varphi >< d\varphi(e_i), v\circ\varphi >, \end{aligned}\tag{3.44}$$

by the definition of Jacobi operator and equations (3.43), (3.44), we get

$$\begin{aligned}< J^{\varphi}_f(v\circ\varphi), v\circ\varphi > &= 2f_\varphi < d\varphi(e_i), v\circ\varphi >< d\varphi(e_i), v\circ\varphi >\\ &\quad -f_\varphi|d\varphi|^2 < v\circ\varphi, v\circ\varphi >\\ &\quad +(\lambda\circ\varphi)e(\varphi) < (\operatorname{grad}^{\mathbb{S}^n} f)\circ\varphi, v\circ\varphi >\\ &\quad +e(\varphi) < (\nabla^{\mathbb{S}^n}_{v\circ\varphi}\operatorname{grad}^{\mathbb{S}^n} f)\circ\varphi, v\circ\varphi >\\ &\quad - < d\varphi(\operatorname{grad}^M(v\circ\varphi)(f)), v\circ\varphi >\\ &\quad - < (v\circ\varphi)(f)\tau(\varphi), v\circ\varphi >\\ &\quad + < \nabla^{\varphi} v\circ\varphi, d\varphi >< (\operatorname{grad}^{\mathbb{S}^n} f)\circ\varphi, v\circ\varphi >, \end{aligned}\tag{3.45}$$

so that

$$\begin{aligned}\operatorname{trace}_\alpha < J^{\varphi}_f(v\circ\varphi), v\circ\varphi > &= (2-n)f_\varphi|d\varphi|^2\\ &\quad +e(\varphi)\operatorname{trace}_\alpha(\operatorname{Hess}^{\mathbb{S}^n} f)(v\circ\varphi, v\circ\varphi)\\ &\quad -\operatorname{trace}_\alpha < d\varphi(\operatorname{grad}^M(v\circ\varphi)(f)), v\circ\varphi >\\ &\quad -\operatorname{trace}_\alpha < \tau(\varphi), v\circ\varphi > (v\circ\varphi)(f)\\ &\quad +\operatorname{trace}_\alpha < \nabla^{\varphi} v\circ\varphi, d\varphi >\\ &\quad < (\operatorname{grad}^{\mathbb{S}^n} f)\circ\varphi, v\circ\varphi >, \end{aligned}\tag{3.46}$$

where $\operatorname{Hess}^{\mathbb{S}^n} f$ is the hessian of the function f on $\mathbb{S}^n$, by the following formulas

$$\begin{aligned}d\varphi(\operatorname{grad}^M(v\circ\varphi)(f)) &= e_i < v\circ\varphi, \operatorname{grad}^{\mathbb{S}^n} f > d\varphi(e_i)\\ &= < \nabla^{\varphi}_{e_i}(v\circ\varphi), \operatorname{grad}^{\mathbb{S}^n} f > d\varphi(e_i)\\ &\quad + < v\circ\varphi, \nabla^{\varphi}_{e_i}\operatorname{grad}^{\mathbb{S}^n} f > d\varphi(e_i)\end{aligned}$$

$$\begin{aligned} &= -(\lambda\circ\varphi) < d\varphi(e_i), \operatorname{grad}^{\mathbb{S}^n} f > d\varphi(e_i) \\ &\quad + < v\circ\varphi, \nabla^{\mathbb{S}^n}_{d\varphi(e_i)} \operatorname{grad}^{\mathbb{S}^n} f > d\varphi(e_i); \end{aligned}$$

$$-\operatorname{trace}_\alpha < d\varphi(\operatorname{grad}^M(v\circ\varphi)(f)), v\circ\varphi > \;=\; -(\operatorname{Hess}^{\mathbb{S}^n} f)(d\varphi(e_i), d\varphi(e_i));$$

$$\begin{aligned} < \nabla^\varphi v\circ\varphi, d\varphi > &= < \nabla^\varphi_{e_i} v\circ\varphi, d\varphi(e_i) > \\ &= -(\lambda\circ\varphi) < d\varphi(e_i), d\varphi(e_i) > \\ &= -(\lambda\circ\varphi)|d\varphi|^2; \end{aligned}$$

$$\operatorname{trace}_\alpha < \nabla^\varphi v\circ\varphi, d\varphi >< (\operatorname{grad}^{\mathbb{S}^n} f)\circ\varphi, v\circ\varphi > \;=\; 0;$$

$$\begin{aligned} -\operatorname{trace}_\alpha < \tau(\varphi), v\circ\varphi > (v\circ\varphi)(f) &= - < \tau(\varphi), \operatorname{grad}^{\mathbb{S}^n} f > \\ &= - < \nabla^\varphi_{e_i} d\varphi(e_i), \operatorname{grad}^{\mathbb{S}^n} f > \\ &= -\operatorname{div} w + < d\varphi(e_i), \nabla^{\mathbb{S}^n}_{d\varphi(e_i)} \operatorname{grad}^{\mathbb{S}^n} f >, \end{aligned}$$

where $w(X) =< d\varphi(X), \operatorname{grad}^{\mathbb{S}^n} f >, \forall X \in \Gamma(TM)$, and (3.46), we have

$$\begin{aligned} \operatorname{trace}_\alpha < J^\varphi_f(v\circ\varphi), v\circ\varphi > &= (2-n) f_\varphi |d\varphi|^2 \\ &\quad + e(\varphi)\Delta^{\mathbb{S}^n}(f)\circ\varphi \\ &\quad -(\operatorname{Hess}^{\mathbb{S}^n} f)(d\varphi(e_i), d\varphi(e_i)) \\ &\quad -\operatorname{div} w + (\operatorname{Hess}^{\mathbb{S}^n} f)(d\varphi(e_i), d\varphi(e_i)), \end{aligned}$$

where $\Delta^{\mathbb{S}^n}(f)\circ\varphi = \operatorname{trace}_\alpha(\operatorname{Hess}^{\mathbb{S}^n} f)(v\circ\varphi, v\circ\varphi)$, so that

$$\begin{aligned} \operatorname{trace}_\alpha I^\varphi_f(v\circ\varphi, v\circ\varphi) &= (2-n)\int_M f_\varphi |d\varphi|^2 v^M \\ &\quad + \int_M e(\varphi)[\Delta^{\mathbb{S}^n}(f)\circ\varphi] v^M. \end{aligned} \tag{3.47}$$

Hence Theorem 3.3.2 follows from (3.47) and the stable f-harmonicity condition of φ with $n > 2$ and $\Delta^{\mathbb{S}^n}(f)\circ\varphi \leq 0$. □

If $f(x,y) = f_1(x)\ \forall\ (x,y) \in M \times N$, such that f_1 be a smooth positive function positive on M, we get the following result:

Corollary 3.3.1. *[26]. any stable f-harmonic map $\varphi : \mathbb{S}^n \to N$ must be constant, where f is a smooth positive function on $\mathbb{S}^n$. with $h((\nabla d\varphi)(., \operatorname{grad} f), d\varphi(.)) \geq 0$.*

Corollary 3.3.2. *[26]. Let (M, g) be a compact Riemannian manifold. When $n > 2$, any stable f-harmonic map $\varphi : M \to \mathbb{S}^n$ must be constant, where f is a smooth positive function on M.*

3.3.2 Homothetic vector fields and f-harmonic maps

Theorem 3.3.3. *.[40] Let (M, g) be a compact orientable Riemannian manifold without boundary, (N, h) a Riemannian manifold admitting a homothetic vector field ξ with homothetic constant k, and let f be a smooth positive function on $M \times N$ such that $2kf + \xi(f) \neq 0$ at any point. Then, any f-harmonic map φ from (M, g) to (N, h) is constant.*

Proof. We set

$$\omega(X) = h(\xi \circ \varphi, f_\varphi d\varphi(X)), \quad \forall X \in \Gamma(TM), \tag{3.48}$$

let $\{e_i\}$ be a normal orthonormal frame at $x \in M$, we have

$$\begin{aligned} \operatorname{div}^M \omega &= e_i\big[h(\xi \circ \varphi, f_\varphi d\varphi(e_i))\big] \\ &= h(\nabla^\varphi_{e_i}(\xi \circ \varphi), f_\varphi d\varphi(e_i)) + h(\xi \circ \varphi, \nabla^\varphi_{e_i} f_\varphi d\varphi(e_i)) \\ &= h(\nabla^\varphi_{e_i}(\xi \circ \varphi), f_\varphi d\varphi(e_i)) + h(\xi \circ \varphi, f_\varphi \tau(\varphi) + d\varphi(\operatorname{grad}^M f_\varphi)) \end{aligned} \tag{3.49}$$

by equation (3.49) and the f-harmonicity of φ, we get:

$$\begin{aligned} \operatorname{div}^M \omega &= h(\nabla^\varphi_{e_i}(\xi \circ \varphi), f_\varphi d\varphi(e_i)) + h(\xi \circ \varphi, e(\varphi)(\operatorname{grad}^N f) \circ \varphi) \\ &= f_\varphi h(\nabla^N_{d\varphi(e_i)}\xi, d\varphi(e_i)) + h(\xi \circ \varphi, e(\varphi)(\operatorname{grad}^N f) \circ \varphi), \end{aligned}$$

since ξ is a homothetic vector field with homothetic constant k, we find that

$$\begin{aligned} \operatorname{div}^M \omega &= f_\varphi k h(d\varphi(e_i), d\varphi(e_i)) + e(\varphi) h(\xi \circ \varphi, (\operatorname{grad}^N f) \circ \varphi) \\ &= k f_\varphi |d\varphi|^2 + \frac{1}{2}|d\varphi|^2 h(\xi \circ \varphi, (\operatorname{grad}^N f) \circ \varphi) \\ &= \frac{|d\varphi|^2}{2}\big[2k f_\varphi + h(\xi \circ \varphi, (\operatorname{grad}^N f) \circ \varphi)\big] \\ &= \frac{|d\varphi|^2}{2}\big[2k f_\varphi + \xi(f) \circ \varphi\big]. \end{aligned}$$

Theorem 3.3.3 follows from the last equation, and the Green Theorem [1], with $2kf + \xi(f) \neq 0$. □

Remark 3.3.1. If $f = 1$ on $M \times N$ we obtain the following result of chapter 2 [27].

If $f(x, y) = f_1(x)$, for all $(x, y) \in M \times N$, where f_1 is a smooth positive function on M, we have the following.

Corollary 3.3.3. *Let (M, g) be a compact orientable Riemannian manifold without boundary, (N, h) a Riemannian manifold admitting a proper homothetic vector field, and let f_1 be a smooth positive function on M. Then, any f_1-harmonic map φ from (M, g) to (N, h) is constant.*

In the case of non-compact Riemannian manifold, we obtain the following result.

Theorem 3.3.4. *.[40] Let (M, g) be a complete non-compact orientable Riemannian manifold, (N, h) a Riemannian manifold admitting a homothetic vector field ξ with homothetic constant k, and let f be a smooth positive function on $M \times N$ such that $2(k-\mu)f + \xi(f) \neq 0$ (at any point) for some constant $\mu > 0$. If $\varphi : (M, g) \to (N, h)$ is f-harmonic map satisfying*

$$\int_M f_\varphi |\xi \circ \varphi|^2 v^g < \infty,$$

then φ is constant.

Proof. Let ρ be a smooth function with compact support on M, we set

$$\omega(X) = h(\xi \circ \varphi, \rho^2 f_\varphi d\varphi(X)), \quad \forall X \in \Gamma(TM),$$

and let $\{e_i\}$ be a normal orthonormal frame at $x \in M$, we have

$$\begin{aligned}
\operatorname{div}^M \omega &= e_i[h(\xi \circ \varphi, \rho^2 f_\varphi d\varphi(e_i))] \\
&= h(\nabla^\varphi_{e_i}(\xi \circ \varphi), \rho^2 f_\varphi d\varphi(e_i)) + h(\xi \circ \varphi, \nabla^\varphi_{e_i} \rho^2 (f_\varphi d\varphi(e_i))) \\
&= h(\nabla^\varphi_{e_i}(\xi \circ \varphi), \rho^2 f_\varphi d\varphi(e_i)) + h(\xi \circ \varphi, e_i(\rho^2) f_\varphi d\varphi(e_i)) \\
&\quad + h(\xi \circ \varphi, \rho^2 \nabla^\varphi_{e_i} f_\varphi d\varphi(e_i)),
\end{aligned}$$

so that

$$\begin{aligned}
\operatorname{div}^M \omega &= h(\nabla^\varphi_{e_i}(\xi \circ \varphi), \rho^2 f_\varphi d\varphi(e_i)) + h(\xi \circ \varphi, 2\rho e_i(\rho) f_\varphi d\varphi(e_i)) \\
&\quad + h(\xi \circ \varphi, \rho^2 [f_\varphi \tau(\varphi) + d\varphi(\operatorname{grad}^M f_\varphi)])
\end{aligned} \tag{3.50}$$

by equation (5.26), and f-harmonicity condition of φ, we get

$$\begin{aligned}
\operatorname{div}^M \omega &= \rho^2 f_\varphi h(\nabla^N_{d\varphi(e_i)} \xi, d\varphi(e_i)) + 2\rho e_i(\rho) f_\varphi h(\xi \circ \varphi, d\varphi(e_i)) \\
&\quad + \rho^2 h(\xi \circ \varphi, e(\varphi)(\operatorname{grad}^N f) \circ \varphi)
\end{aligned}$$

since ξ is a homothetic vector field with homothetic constant k, we find that

$$\begin{aligned}
\operatorname{div}^M \omega &= k\rho^2 f_\varphi h(d\varphi(e_i), d\varphi(e_i)) + 2\rho e_i(\rho) f_\varphi h(\xi \circ \varphi, d\varphi(e_i)) \\
&\quad + \frac{1}{2} |d\varphi|^2 \rho^2 \xi(f) \circ \varphi,
\end{aligned}$$

that is,

$$\begin{aligned}
\operatorname{div}^M \omega &= k\rho^2 f_\varphi |d\varphi|^2 + 2\rho e_i(\rho) f_\varphi h(\xi \circ \varphi, d\varphi(e_i)) \\
&\quad + \frac{1}{2} |d\varphi|^2 \rho^2 \xi(f) \circ \varphi,
\end{aligned} \tag{3.51}$$

by the Young's inequality, we have

$$-2\rho e_i(\rho) h(\xi \circ \varphi, d\varphi(e_i)) \leq \epsilon \rho^2 |d\varphi|^2 + \frac{1}{\epsilon} e_i(\rho)^2 |\xi \circ \varphi|^2,$$

for all $\epsilon > 0$, multiplying the last inequality by f_φ, we find that

$$-2f_\varphi \rho e_i(\rho) h(\xi \circ \varphi, d\varphi(e_i)) \leq \epsilon f_\varphi \rho^2 |d\varphi|^2 + \frac{1}{\epsilon} f_\varphi e_i(\rho)^2 |\xi \circ \varphi|^2, \tag{3.52}$$

from (5.27), (5.28), we deduce the inequality

$$\begin{aligned} k\rho^2 f_\varphi |d\varphi|^2 - \operatorname{div}^M \omega &+ \frac{1}{2}|d\varphi|^2 \rho^2 \xi(f) \circ \varphi \\ &\leq \epsilon f_\varphi \rho^2 |d\varphi|^2 + \frac{1}{\epsilon} f_\varphi e_i(\rho)^2 |\xi \circ \varphi|^2, \end{aligned} \tag{3.53}$$

we set $\epsilon = \mu$, by (5.29), we have

$$\begin{aligned} (k-\mu)\rho^2 f_\varphi |d\varphi|^2 - \operatorname{div}^M \omega &+ \frac{1}{2}|d\varphi|^2 \rho^2 \xi(f) \circ \varphi \\ &\leq \frac{1}{\mu} f_\varphi e_i(\rho)^2 |\xi \circ \varphi|^2, \end{aligned} \tag{3.54}$$

by the divergence Theorem, and (5.30), we have

$$\frac{1}{2}\int_M \rho^2 |d\varphi|^2 \big[2(k-\mu) f_\varphi + \xi(f) \circ \varphi\big] v^g \leq \frac{1}{\mu}\int_M f_\varphi e_i(\rho)^2 |\xi \circ \varphi|^2 v^g. \tag{3.55}$$

Now, consider the cut-off smooth function $\rho = \rho_R$ such that, $\rho \leq 1$ on M, $\rho = 1$ on the ball $B(\rho, R)$, $\rho = 0$ on $M \setminus B(\rho, 2R)$ and $|\operatorname{grad}^M \rho| \leq \frac{2}{R}$ (see [42]), from (5.31) we get:

$$\frac{1}{2}\int_M \rho^2 |d\varphi|^2 \big[2(k-\mu) f_\varphi + \xi(f) \circ \varphi\big] v^g \leq \frac{4}{\mu R^2}\int_M f_\varphi |\xi \circ \varphi|^2 v^g, \tag{3.56}$$

since $\int_M f_\varphi |\xi \circ \varphi|^2 v^g < \infty$, when $R \to \infty$ we obtain:

$$\int_M |d\varphi|^2 \big[2(k-\mu) f_\varphi + \xi(f) \circ \varphi\big] v^g = 0. \tag{3.57}$$

Consequently, $|d\varphi| = 0$ that is φ is constant, because $2(k-\mu)f + \xi(f) \neq 0$ at any point. □

3.3.3 f-biharmonic maps and submanifolds

Let M be a submanifold of $\mathbb{R}^n$ of dimension m, $\mathbf{i} : M \to \mathbb{R}^n$ the canonical inclusion, $f \in C^\infty(\mathbb{R}^n)$ a smooth positive function such that $f \circ \mathbf{i} = 1$, and let $\{e_i\}$ be an orthonormal frame with respect to induced Riemannian metric on M by the inner product $<,>$ on $\mathbb{R}^n$. We denote by ∇ (resp. ∇^M) the Levi-Civita connection of $\mathbb{R}^n$ (resp. of M), by grad (resp. grad^M) the gradient operator in $\mathbb{R}^n$ (resp. in M), by B the second fundamental form of the submanifold M, by A the shape operator, by H the mean curvature vector field of M, and by $\nabla^\perp$ the normal connection of M (see for example [1]). Under the notation above we have the following results.

Theorem 3.3.5. *.[40] The map* $\mathbf{i}$ *is* f*-biharmonic if and only if*

$$\begin{aligned}
&\frac{m}{2}\operatorname{grad}^M|H|^2 - 2A_{\nabla^\perp_{e_i}H}(e_i) - m(\nabla^\perp_{e_i}H)(f)e_i\\
&+A_{\nabla^\perp_{e_i}\operatorname{grad}f}(e_i) + \frac{m-2}{2}\operatorname{grad}^M H(f) - \frac{m-4}{8}\operatorname{grad}^M|\operatorname{grad}f|^2 = 0,
\end{aligned}$$

$$\begin{aligned}
&-B(e_i,A_H(e_i)) - \Delta^\perp H + \frac{1}{2}B(e_i,A_{\operatorname{grad}f}(e_i))\\
&+\frac{1}{2}\Delta^\perp\operatorname{grad}f + \frac{m}{2}(\nabla_H\operatorname{grad}f)^\perp - \frac{m}{4}(\nabla_{\operatorname{grad}f}\operatorname{grad}f)^\perp\\
&-mH(f)H + \frac{m}{2}|\operatorname{grad}f|^2H - m|H|^2\operatorname{grad}f + \frac{m}{2}H(f)\operatorname{grad}f = 0.
\end{aligned}$$

We need the following lemmas to prove Theorem (5.1.1).

Lemma 3.3.1. *[49] Let* $\Delta^\perp$ *the Laplacian in the normal bundle of* M*, then*

$$\begin{aligned}
\operatorname{trace}\nabla^2 H &= -\frac{m}{2}\operatorname{grad}^M(|H|^2) + 2A_{\nabla^\perp_{e_i}H}(e_i)\\
&\quad +B(e_i,A_H(e_i)) + \Delta^\perp H.
\end{aligned}$$

Lemma 3.3.2. *On taking the trace of* $\nabla^2\operatorname{grad}f$*, we obtain*

$$\begin{aligned}
\operatorname{trace}\nabla^2\operatorname{grad}f &= -m(\nabla^\perp_{e_i}H)(f)e_i + 2A_{\nabla^\perp_{e_i}\operatorname{grad}f}(e_i)\\
&\quad +B(e_i,A_{\operatorname{grad}f}(e_i)) + \Delta^\perp\operatorname{grad}f.
\end{aligned}$$

Proof. First, note that $\operatorname{grad}f$ is normal to M because f is constant on M. We suppose that $\nabla^M_{e_i}e_j = 0$ at $x \in M$ for all $i,j = 1,...,m$. Then calculating at x

$$\begin{aligned}
\nabla_{e_i}\nabla_{e_i}\operatorname{grad}f &= \nabla_{e_i}\big(A_{\operatorname{grad}f}(e_i) + (\nabla_{e_i}\operatorname{grad}f)^\perp\big)\\
&= \nabla^M_{e_i}A_{\operatorname{grad}f}(e_i) + B(e_i,A_{\operatorname{grad}f}(e_i))\\
&\quad +A_{(\nabla_{e_i}\operatorname{grad}f)^\perp}(e_i) + \big(\nabla_{e_i}(\nabla_{e_i}\operatorname{grad}f)^\perp\big)^\perp, \qquad (3.58)
\end{aligned}$$

since $< A_{\operatorname{grad}f}(X),Y >= - < B(X,Y),\operatorname{grad}f >$, for all $X,Y \in \Gamma(TM)$, we get the following

$$\begin{aligned}
\nabla^M_{e_i}A_{\operatorname{grad}f}(e_i) &= \big\langle\nabla^M_{e_i}A_{\operatorname{grad}f}(e_i),e_j\big\rangle\, e_j\\
&= e_i\big(\big\langle A_{\operatorname{grad}f}(e_i),e_j\big\rangle\big)\, e_j\\
&= -e_i\big(\big\langle B(e_i,e_j),\operatorname{grad}f\big\rangle\big)\, e_j\\
&= -e_i\big(\big\langle\nabla_{e_j}e_i,\operatorname{grad}f\big\rangle\big)\, e_j,
\end{aligned}$$

and since $\nabla_X\nabla_Y = \nabla_Y\nabla_X + \nabla_{[X,Y]}Z$, for all $X,Y,Z \in \Gamma(TM)$, we have

$$\nabla^M_{e_i}A_{\operatorname{grad}f}(e_i) = -\big\langle\nabla_{e_i}\nabla_{e_j}e_i,\operatorname{grad}f\big\rangle\, e_j - \big\langle\nabla_{e_j}e_i,\nabla_{e_i}\operatorname{grad}f\big\rangle\, e_j$$

$$= -\left\langle \nabla_{e_j}\nabla_{e_i}e_i, \operatorname{grad} f\right\rangle e_j - \left\langle B(e_i,e_j), (\nabla_{e_i}\operatorname{grad} f)^{\perp}\right\rangle e_j,$$

here, the Riemannian curvature tensor of $\mathbb{R}^n$ is null, so that

$$\begin{aligned}
\nabla^M_{e_i} A_{\operatorname{grad} f}(e_i) &= -e_j\big(\left\langle \nabla_{e_i}e_i, \operatorname{grad} f\right\rangle\big)\, e_j + \left\langle \nabla_{e_i}e_i, \nabla_{e_j}\operatorname{grad} f\right\rangle e_j \\
&\quad + \left\langle A_{(\nabla_{e_i}\operatorname{grad} f)^{\perp}}(e_i), e_j\right\rangle e_j \\
&= -m e_j\big(\left\langle H, \operatorname{grad} f\right\rangle\big)\, e_j + m\left\langle H, \nabla_{e_j}\operatorname{grad} f\right\rangle e_j \\
&\quad + A_{(\nabla_{e_i}\operatorname{grad} f)^{\perp}}(e_i) \\
&= -m\left\langle \nabla_{e_j}H, \operatorname{grad} f\right\rangle e_j + A_{(\nabla_{e_i}\operatorname{grad} f)^{\perp}}(e_i). \qquad (3.59)
\end{aligned}$$

By (3.58) and (3.59) the lemma is follows. □

Proof of theorem 5.1.1. Note that the f-tension field of $\mathbf{i}$ is given by

$$\begin{aligned}
\tau_f(\mathbf{i}) &= \tau(\mathbf{i}) - e(\mathbf{i})(\operatorname{grad} f)\circ \mathbf{i} \\
&= mH - \frac{m}{2}\operatorname{grad} f.
\end{aligned}$$

we suppose such that $\nabla^M_{e_i}e_j = 0$ at $x \in M$ for all $i, j = 1, ..., m$. Then calculating at x

$$\nabla^{\mathbf{i}}_{e_i}\nabla^{\mathbf{i}}_{e_i}\tau_f(\mathbf{i}) = m\nabla_{e_i}\nabla_{e_i}H - \frac{m}{2}\nabla_{e_i}\nabla_{e_i}\operatorname{grad} f,$$

so by lemmas 3.3.1 and 3.3.2, we have

$$\begin{aligned}
-\nabla^{\mathbf{i}}_{e_i}\nabla^{\mathbf{i}}_{e_i}\tau_f(\mathbf{i}) &= \frac{m^2}{2}\operatorname{grad}^M(|H|^2) - 2mA_{\nabla^{\perp}_{e_i}H}(e_i) \\
&\quad - mB(e_i, A_H(e_i)) - m\Delta^{\perp}H - \frac{m^2}{2}(\nabla^{\perp}_{e_i}H)(f)e_i \\
&\quad + mA_{\nabla^{\perp}_{e_i}\operatorname{grad} f}(e_i) + \frac{m}{2}B(e_i, A_{\operatorname{grad} f}(e_i)) + \frac{m}{2}\Delta^{\perp}\operatorname{grad} f. \qquad (3.60)
\end{aligned}$$

In the same way, we have the following formulas

$$\begin{aligned}
e(\mathbf{i})(\nabla_{\tau_f(\mathbf{i})}\operatorname{grad} f)\circ \mathbf{i} &= \frac{m^2}{2}\nabla_H \operatorname{grad} f - \frac{m^2}{4}\nabla_{\operatorname{grad} f}\operatorname{grad} f \\
&= \frac{m^2}{2}(\nabla_H \operatorname{grad} f)^{\perp} - \frac{m^2}{4}(\nabla_{\operatorname{grad} f}\operatorname{grad} f)^{\perp} \\
&\quad + \frac{m^2}{2} < \nabla_{e_i}\operatorname{grad} f, H > e_i \\
&\quad - \frac{m^2}{4} < \nabla_{e_i}\operatorname{grad} f, \operatorname{grad} f > e_i \\
&= \frac{m^2}{2}(\nabla_H \operatorname{grad} f)^{\perp} - \frac{m^2}{4}(\nabla_{\operatorname{grad} f}\operatorname{grad} f)^{\perp}
\end{aligned}$$

$$+\frac{m^2}{2}\operatorname{grad}^M H(f) - \frac{m^2}{2}(\nabla^\perp_{e_i} H)(f)e_i$$
$$-\frac{m^2}{8}\operatorname{grad}^M |\operatorname{grad} f|^2, \tag{3.61}$$

$$-d\mathbf{i}(\operatorname{grad}^M \tau_f(\mathbf{i})(f)) = -m\operatorname{grad}^M H(f) + \frac{m}{2}\operatorname{grad}^M |\operatorname{grad} f|^2, \tag{3.62}$$

$$-\tau_f(\mathbf{i})(f)\tau(\mathbf{i}) = -m^2 H(f)H + \frac{m^2}{2}|\operatorname{grad} f|^2 H, \tag{3.63}$$

$$\begin{aligned} < \nabla^{\mathbf{i}}\tau_f(\mathbf{i}), d\mathbf{i} > (\operatorname{grad} f)\circ \mathbf{i} &= \big[m < \nabla_{e_i} H, e_i > \\ &\quad -\frac{m}{2} < \nabla_{e_i}\operatorname{grad} f, e_i > \big]\operatorname{grad} f \\ &= \big[-m < H, B(e_i,e_i) > \\ &\quad +\frac{m}{2} < \operatorname{grad} f, B(e_i,e_i) > \big]\operatorname{grad} f \\ &= \big[-m^2|H|^2 + \frac{m^2}{2}H(f)\big]\operatorname{grad} f, \end{aligned} \tag{3.64}$$

by definition (3.6), and equations (5.24 - 3.64), the Theorem is follows. □

Example 3.3.1. . Let $\epsilon \in \mathbb{R}$, the plane $M = \{(x,y,z) \in \mathbb{R}^3 | z = \epsilon\}$ is proper f-biharmonic, i.e. the canonical inclusion $\mathbf{i} : M \hookrightarrow \mathbb{R}^3$ is f-biharmonic non-f-harmonic map, for $f(x,y,z) = F(z-\epsilon)$, where F is a smooth positive function such that $F(0) = 1$, $F'(0) \neq 0$ and $F''(0) = 0$. For example, we consider the function

$$F(t) = \frac{1}{2} + \frac{1}{2}\left[t^2 - \exp(t)\right]^2.$$

Indeed; the function f satisfies the following formulas

$$\operatorname{grad} f = F'(z-\epsilon)\partial_z, \quad |\operatorname{grad} f|^2 = F'(0)^2 \text{ on } M,$$

$$\nabla_Z \operatorname{grad} f = F''(z-\epsilon) < Z, \partial_z > \partial_z,$$

for all $Z \in \Gamma(T\mathbb{R}^3)$, and for $X \in \Gamma(TM)$ we have

$$\nabla_X \operatorname{grad} f = 0,$$

and note that a unit normal vector field U on M is evidently parallel in $\mathbb{R}^3$ (constant Euclidean coordinates), hence $A_U X = \nabla_X U = 0$, for all tangent vectors X to M. Thus the shape operator is identically zero, so that $B = 0$ and $H = 0$. According to Theorem 5.1.1, the map $\mathbf{i}$ is f-biharmonic if and only if $F''(0)F'(0) = 0$.

Using the similar technique of Example 3.3.1, we have

Example 3.3.2. The sphere $\mathbb{S}^m$ of $\mathbb{R}^{m+1}$ is proper f-biharmonic for

$$f(y) = F\left(\tfrac{|y|^2}{2}\right),\ \forall y \in \mathbb{R}^n, \text{ where } F(t) = \tfrac{1}{5}\exp\left(\tfrac{5}{2} - 5t\right) - \tfrac{2}{5}t + 1.$$

Here, $H = -P$, where P is the position vector field on $\mathbb{R}^{m+1}$,

$$|H| = 1, \quad \nabla^{\perp}_X H = 0, \quad A_H X = -X, \quad B(X,Y) = - < X, Y > P,$$

$$\operatorname{grad} f = F'\big(\tfrac{|y|^2}{2}\big)P, \quad H(f) = -F'\big(\tfrac{1}{2}\big), \quad A_{\operatorname{grad} f}X = F'\big(\tfrac{1}{2}\big)X$$

$$\nabla_Z \operatorname{grad} f =< Z, P > F''\big(\tfrac{|y|^2}{2}\big)P + F'\big(\tfrac{|y|^2}{2}\big)Z,$$

where $X, Y \in \Gamma(T\mathbb{S}^m)$ and $Z \in \Gamma(T\mathbb{R}^{m+1})$. According to Theorem 5.1.1, the map $\mathbf{i}$ is f-biharmonic if and only if

$$\frac{1}{2}F''\left(\frac{1}{2}\right) + 3F'\left(\frac{1}{2}\right) + \frac{5}{4}F'\left(\frac{1}{2}\right)^2 + \frac{1}{4}F'\left(\frac{1}{2}\right)F''\left(\frac{1}{2}\right) + 1 = 0.$$

Chapter 4

L-harmonic maps

In this chapter, we prove that every semi-conformal harmonic map between Riemannian manifolds is L-harmonic map. We also prove a Liouville type theorem for L-harmonic maps.

4.1 The Euler-Lagrange equations

Definition 4.1.1. *The Lagrangian on an open set U of $\mathbb{R}^n$ is a smooth function, defined by*

$$\begin{array}{rcl} L : U \times \mathbb{R}^n \times [t_1, t_2] & \longrightarrow & \mathbb{R} \\ (x, y, t) & \mapsto & L(x, y, t) \end{array}$$

Let $E(\varphi)$ the energy functional defined by

$$E(\varphi) = \int_{t_1}^{t_2} L(\varphi(t), \frac{d\varphi}{dt}(t), t)dt. \tag{4.1}$$

Let $x_1, x_2 \in U$, the associated variational problem consists in looking for the curves $\varphi : [t_1, t_2] \longrightarrow U$ plotted in U, as $\varphi(t_1) = x_1$ and $\varphi(t_2) = x_2$, which minimize the energy functional (4.1). *To characterize the function φ, we consider the variation $\varphi_s(t) = \varphi(t) + sv(t)$ where $v(t)$ is a non-zero function, except at the limits t_1 and t_2, then we have*
$v(t_1) = v(t_2) = 0$, $\varphi_s(t_1) = \varphi(t_1) = x_1$ and $\varphi_s(t_2) = \varphi(t_2) = x_2$, where $s \in (-\varepsilon, \varepsilon)$.

Theorem 4.1.1.

$$\frac{d}{ds} E(\varphi_s)\Big|_{s=0} = \int_{t_1}^{t_2} \Big\langle v(t), \frac{\partial L}{\partial x}\Big(\varphi(t), \frac{d\varphi}{dt}(t), t\Big) - \frac{\partial}{\partial t}\Big(\frac{\partial L}{\partial y}\Big(\varphi(t), \frac{d\varphi}{dt}(t), t\Big)\Big)\Big\rangle_{\mathbb{R}^n} dt$$

where

$$\frac{\partial L}{\partial x} = \Big(\frac{\partial L}{\partial x^1}, ..., \frac{\partial L}{\partial x^n}\Big), \quad \frac{\partial L}{\partial y} = \Big(\frac{\partial L}{\partial y^1}, ..., \frac{\partial L}{\partial y^n}\Big)$$

$\langle , \rangle$ denote the scalar product on $\mathbb{R}^n$.

Proof. Define $\phi : (-\epsilon, \epsilon) \times [t_1, t_2] \to \mathbb{R}^n$ by

$$\phi(s,t) = \varphi_s(t) = \varphi(t) + sv(t) \tag{4.2}$$

By (4.1) and (4.2) we have

$$\frac{d}{ds}E(\varphi_s)\big|_{s=0} = \int_{t_1}^{t_2} \frac{\partial}{\partial s} L(\phi(s,t), \frac{\partial}{\partial t}\varphi(s,t), t)dt. \tag{4.3}$$

Since

$$\begin{aligned}\frac{\partial}{\partial s} L(\phi(s,t), \frac{\partial \varphi}{\partial t}(s,t), t) &= \sum_{i=1}^{n} \frac{\partial \phi^i}{\partial s}(s,t) \frac{\partial L}{\partial x^i}\big(\phi(s,t), \frac{\partial \phi}{\partial t}(s,t), t\big) \\ &+ \sum_{i=1}^{n} \frac{\partial}{\partial s}(\frac{\partial \phi^i}{\partial t})(s,t) \frac{\partial L}{\partial y^i}\big(\phi(s,t), \frac{\partial \phi}{\partial t}(s,t), t\big) \end{aligned} \tag{4.4}$$

Integrating by party we find

$$\begin{aligned} &\int_{t_1}^{t_2} \sum_{i=1}^{n} \frac{\partial}{\partial s}(\frac{\partial \phi^i}{\partial t})(s,t) \frac{\partial L}{\partial y^i}\Big(\phi(s,t), \frac{\partial \phi}{\partial t}(s,t), t\Big)dt \\ =& \sum_{i=1}^{n} \int_{t_1}^{t_2} \frac{\partial}{\partial t}(\frac{\partial \phi^i}{\partial s})(s,t) \frac{\partial L}{\partial y^i}\Big(\phi(s,t), \frac{\partial \phi}{\partial t}(s,t), t\Big)dt \\ =& \sum_{i=1}^{n} \frac{\partial \phi^i}{\partial s}(s,t) \frac{\partial L}{\partial y^i}\Big(\phi(s,t), \frac{\partial \phi}{\partial t}(s,t), t\Big)\Big|_{t_1}^{t_2} \\ & -\sum_{i=1}^{n} \int_{t_1}^{t_2} (\frac{\partial \phi^i}{\partial s})(s,t) \frac{\partial}{\partial t}\Big(\frac{\partial L}{\partial y^i}\Big(\phi(s,t), \frac{\partial \phi}{\partial t}(s,t), t\Big)\Big)dt \end{aligned} \tag{4.5}$$

According to formulas (4.2), (4.3),(4.4), (4.5) we get

$$\begin{aligned}\frac{d}{ds}E(\varphi_s)\big|_{s=0} &= \int_{t_1}^{t_2} \Big\langle v(t), \frac{\partial L}{\partial x}\Big(\varphi(t), \frac{\partial \varphi}{\partial t}(t), t\Big)\Big\rangle_{\mathbb{R}^n} dt + \Big\langle v(t), \frac{\partial L}{\partial y}\Big(\varphi(t), \frac{\partial \varphi}{\partial t}(t), t\Big)\Big\rangle_{\mathbb{R}^n}\Big|_{t_1}^{t_2} \\ & - \int_{t_1}^{t_2} \Big\langle v(t), \frac{\partial}{\partial t}\Big(\frac{\partial L}{\partial y}(\varphi(t), \frac{\partial \varphi}{\partial t}(t), t)\Big)\Big\rangle_{\mathbb{R}^n} dt \end{aligned} \tag{4.6}$$

Since $v(t_1) = v(t_2) = 0$ then

$$\begin{aligned}\frac{d}{ds}E(\varphi_s)\big|_{s=0} &= \int_{t_1}^{t_2} \Big\langle v(t), \frac{\partial L}{\partial x}\Big(\varphi(t), \frac{\partial \varphi}{\partial t}(t), t\Big)\Big\rangle_{\mathbb{R}^n} dt \\ & - \int_{t_1}^{t_2} \Big\langle v(t), \frac{\partial}{\partial t}\Big(\frac{\partial L}{\partial y}(\varphi(t), \frac{\partial \varphi}{\partial t}(t), t)\Big)\Big\rangle_{\mathbb{R}^n} dt \end{aligned} \tag{4.7}$$

□

Theorem 4.1.2. *The curve $\varphi : [t_1, t_2] \to U$ is a critical point of energy functional* (4.1) *if and ally if*

$$\frac{\partial L}{\partial x}\Big(\varphi(t), \frac{d\varphi}{dt}(t), t\Big) - \frac{\partial}{\partial t}\Big(\frac{\partial L}{\partial y}\Big(\varphi(t), \frac{d\varphi}{dt}(t), t\Big)\Big) = 0. \tag{4.8}$$

This system of n second order differential equations is called the Euler-Lagrange equation system.

Example 4.1.1. Let U be an open set of $\mathbb{R}^n$ and L is the Lagrangian defined by

$$L(x, y, t) = \frac{y^2}{2},$$

the Lagrangian represents kinetic energy

$$E(\varphi) = \frac{1}{2}\int_{t_1}^{t_2}\Big(\frac{d\varphi}{dt}\Big)^2 dt.$$

the system (4.8) is reduced to the equation

$$\frac{d^2\varphi}{dt^2} = 0.$$

Then the Euler-Lagrange solutions are the affine lines (geodesics)

$$\varphi(t) = at + b, \quad a, b \in \mathbb{R}^n.$$

4.2 L-harmonic maps

Consider $\varphi : (M, g) \to (N, h)$ a smooth map between two Riemannian manifold and

$$\begin{array}{rcl} L : M \times N \times \mathbb{R} & \to & (0, \infty) \\ (x, y, r) & \mapsto & L(x, y, r) \end{array}$$

a positive function, for any compact domain D of M the function L-energy of φ is defined by

$$E_L(\varphi; D) = \int_D L\big(x, \varphi(x), e(\varphi)(x)\big)\, v_g, \tag{4.9}$$

where $e(\varphi)$ is the energy density of φ defined by

$$e(\varphi) = \frac{1}{2}\, h\big(d\varphi(e_i), d\varphi(e_i)\big), \tag{4.10}$$

v_g is the volume element, here $\{e_i\}$ is an orthonormal frame on (M, g). φ is said to be L-harmonic if it is a critical point of the functional L-energy on any compact domain D of M. We note by $\partial_r = \partial/\partial r$, $L' = \partial_r(L)$, $L'' = \partial_r\big(\partial_r(L)\big)$ and let L'_φ, $L''_\varphi \in C^\infty(M)$ defined by

$$L'_\varphi(x) = L'\big(x, \varphi(x), e(\varphi)(x)\big), \quad L''_\varphi(x) = L''\big(x, \varphi(x), e(\varphi)(x)\big). \tag{4.11}$$

4.2.1 The first variation of L-energy

Theorem 4.2.1. *[29]. Let $\varphi : (M, g) \to (N, h)$ be a smooth map and let $\{\varphi_t\}_{t\in(-\epsilon,\epsilon)}$ be a smooth variation of φ supported in D. Then*

$$\frac{d}{dt}E_L(\varphi_t; D)\Big|_{t=0} = -\int_D h(\tau_L(\varphi), v)\, v_g, \tag{4.12}$$

where $v = \frac{\partial \varphi_t}{\partial t}\Big|_{t=0}$ denotes the variation vector field of φ,

$$\tau_L(\varphi) = L'_\varphi\, \tau(\varphi) + d\varphi\big(\operatorname{grad}^M L'_\varphi\big) - (\operatorname{grad}^N L)\circ\varphi, \tag{4.13}$$

and $\tau(\varphi)$ is the tension field of φ given by

$$\tau(\varphi) = \operatorname{trace}\nabla d\varphi. \tag{4.14}$$

Proof. Defined $\phi : M \times (-\epsilon, \epsilon) \to N$ by

$$\phi(x, t) = \varphi_t(x), \quad (x, t) \in M \times (-\epsilon, \epsilon), \tag{4.15}$$

let ∇^ϕ denote the pull-back connection on $\phi^{-1}TN$. Note that, for any vector field X on M considered as a vector field on $M \times (-\epsilon, \epsilon)$, we have

$$[\partial_t, X] = 0$$

Using (5.7) we obtain

$$\frac{d}{dt}E_L(\varphi_t; D)\Big|_{t=0} = \int_D \partial_t\Big(L(x, \varphi(x), e(\varphi)(x))\Big)\Big|_{t=0} v_g, \tag{4.16}$$

First, note that

$$\partial_t\Big(L(x, \varphi(x), e(\varphi)(x))\Big)\Big|_{t=0} = dL(d\phi(\partial_t))\Big|_{t=0} + dL(\partial_t(e(\varphi_t)))\Big|_{t=0}, \tag{4.17}$$

the first term on the left-hand side of (4.17) is

$$dL(d\phi(\partial_t))\Big|_{t=0} = h((\operatorname{grad}^N L) o\varphi, v) \tag{4.18}$$

Calculating in a normal frame at $x \in M$, we have

$$\begin{aligned} \partial_t(e(\varphi_t)) &= h(\nabla^\phi_{\partial_t} d\varphi_t(e_i), d\varphi_t(e_i)) \\ &= h(\nabla^\phi_{e_i} d\phi(\partial_t), d\varphi_t(e_i)) \end{aligned} \tag{4.19}$$

the second term on the left-hand side of (4.17) is

$$dL(\partial_t(e(\varphi_t)))\Big|_{t=0} = L'_\varphi h(\nabla^\varphi_{e_i} v, d\varphi(e_i))$$

$$= \quad e_i(h(v, L'_{\varphi} d\varphi(e_i))) - h(v, \nabla^{\varphi}_{e_i} L'_{\varphi} d\varphi(e_i)), \qquad (4.20)$$

where the last equality holds since $d\phi(\partial_t)\Big|_{t=0} = v$, define a 1-form on M by

$$\omega(X) = h(v, L'_{\varphi} d\varphi(X)), \quad X \in \Gamma(TM) \qquad (4.21)$$

By (4.20) and (4.21) we get

$$\begin{aligned} dL(\partial_t(e(\varphi_t)))\Big|_{t=0} &= \operatorname{div}\omega - h(v, d\varphi(\operatorname{grad}^M L'_{\varphi})) \\ &- h(v, L'_{\varphi}\tau(\varphi)) \end{aligned} \qquad (4.22)$$

Substituting (4.18), (4.17) and (4.21) in (4.16), and consider the divergence Theorem, the Theorem 4.2.1 follows. □

Corollary 4.2.1. *A smooth map $\varphi : (M, g) \to (N, h)$ between Riemannian manifolds, is L-harmonic if and only if $\tau_L(\varphi) = 0$.*

4.2.2 The second variation of L-energy

Theorem 4.2.2. *[29]. Let $\varphi : (M, g) \to (N, h)$ be an f-harmonic map between Riemannian manifolds and $\{\varphi_{t,s}\}_{t,s\in(-\epsilon,\epsilon)}$ be a two-parameter variation with compact support in D. Set*

$$v = \frac{\partial \varphi_{t,s}}{\partial t}\Big|_{t=s=0}, \quad w = \frac{\partial \varphi_{t,s}}{\partial s}\Big|_{t=s=0}. \qquad (4.23)$$

Under the notation above we have the following

$$\frac{\partial^2}{\partial t \partial s} E_L(\varphi_{t,s}; D)\Big|_{t=s=0} = \int_D h(J_{\varphi,L}(v), w)\, v_g, \qquad (4.24)$$

where $J_{\varphi,L}(v) \in \Gamma(\varphi^{-1}TN)$ given by

$$\begin{aligned} J_{\varphi,L}(v) &= -L'_{\varphi} \operatorname{trace} R^N(v, d\varphi)d\varphi - \operatorname{trace} \nabla^{\varphi} L'_{\varphi} \nabla^{\varphi} v \\ &+ \left(\nabla^N_v \operatorname{grad}^N L\right) \circ \varphi + < \nabla^{\varphi} v, d\varphi > (\operatorname{grad}^N L') \circ \varphi \\ &- \operatorname{trace} \nabla^{\varphi} < \nabla^{\varphi} v, d\varphi > L''_{\varphi} d\varphi. \end{aligned} \qquad (4.25)$$

Here $<,>$ denote the inner product on $T^*M \otimes \varphi^{-1}TN$ and R^N is the curvature tensor on (N, h)

Proof. Let

$$\begin{aligned} \phi : M \times (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) &\to N \\ (x, t, s) &\mapsto \phi(x, t, s) = \varphi_{t,s}(x) \end{aligned} \qquad (4.26)$$

Let∇^{ϕ} denote the pull-back connection on $\varphi^{-1}TN$. Note that, for any vector field X on M considered as a vector field on $M \times (-\epsilon, \epsilon) \times (-\epsilon, \epsilon)$, we have

$$[\partial_t, X] = 0, \quad [\partial_s, X] = 0, \quad [\partial_t, \partial_s] = 0 \tag{4.27}$$

Then, by (5.7) we obtain

$$\frac{\partial^2}{\partial_t \partial_s} E_L(\varphi_{t,s}; D)\Big|_{t=s=0} = \int_D \frac{\partial^2}{\partial_t \partial_s} L\big(x, \varphi_{t,s}(x), e(\varphi_{t,s})(x)\big)\Big|_{t=s=0} v_g, \tag{4.28}$$

first, note that

$$\partial_t\Big(L(x, \varphi_{t,s}(x), e(\varphi_{t,s})(x))\Big) = dL(d\phi(\partial_t)) + dL(\partial_t(e(\varphi_{t,s}))), \tag{4.29}$$

$$dL(d\phi(\partial_t)) = h((\operatorname{grad}^N L)o\varphi, v) \tag{4.30}$$

and

$$dL(\partial_t(e(\varphi_{t,s}))) = h(\nabla^{\phi}_{\partial_t} d\phi(e_i), d\phi(e_i))L'_{\varphi_{t,s}} \tag{4.31}$$

when we pass to the seconde derivative, we get

$$\begin{aligned} \frac{\partial^2}{\partial_t \partial_s}\Big(L(x, \varphi_{t,s}(x), e(\varphi_{t,s})(x))\Big) \;&=\; h(\nabla^{\phi}_{\partial_s} d\phi(\partial_t), (\operatorname{grad}^N L)o\varphi) \\ &\quad + h(d\phi(\partial_t), \nabla^{\phi}_{\partial_s}(\operatorname{grad}^N L)o\varphi) \\ &\quad + h(\nabla^{\phi}_{\partial_s}\nabla^{\phi}_{\partial_t} d\phi(e_i), d\phi(e_i))L'_{\varphi_{t,s}} \\ &\quad + h(\nabla^{\phi}_{\partial_t} d\phi(e_i), \nabla^{\phi}_{\partial_s} d\phi(e_i))L'_{\varphi_{t,s}} \\ &\quad + h(\nabla^{\phi}_{\partial_t} d\phi(e_i), d\phi(e_i))\partial_s(L'_{\varphi_{t,s}}). \end{aligned} \tag{4.32}$$

By (4.23) and the property of the gradient operator we have

$$h(d\phi(\partial_t), \nabla^{\phi}_{\partial_s}(\operatorname{grad}^N L)o\varphi)\Big|_{t=s=0} = h(w, (\nabla^N_v \operatorname{grad}^N L)o\varphi) \tag{4.33}$$

By (4.28) and the definition of the curvature tensor of (N, h) we have

$$\begin{aligned} h(\nabla^{\phi}_{\partial_s}\nabla^{\phi}_{\partial_t} d\phi(e_i), d\phi(e_i))L'_{\varphi_{t,s}}\Big|_{t=s=0} \;&=\; L'_{\varphi} h(R^N(w, d\varphi(e_i))v, d\varphi(e_i)) \\ &\quad + L'_{\varphi} h(\nabla^{\phi}_{e_i}\nabla^{\phi}_{\partial_s} d\phi(\partial_t), d\varphi(e_i))\Big|_{t=s=0} \end{aligned} \tag{4.34}$$

By (4.34), the property of the curvature tensor of (N, h) and the compatibility of ∇^{ϕ} with the metric h we have

$$h(\nabla^{\phi}_{\partial_s}\nabla^{\phi}_{\partial_t} d\phi(e_i), d\phi(e_i))L'_{\varphi_{t,s}}\Big|_{t=s=0} \;=\; -L'_{\varphi} h(R^N(v, d\varphi(e_i))d\varphi(e_i), w)$$

$$+e_i(h(\nabla^{\phi}_{\partial_s} d\phi(\partial_t), L'_{\varphi} d\varphi(e_i)))\Big|_{t=s=0}$$
$$-(h(\nabla^{\phi}_{\partial_s} d\phi(\partial_t), \nabla^{\varphi}_{e_i} L'_{\varphi} d\varphi(e_i)))\Big|_{t=s=0} \quad (4.35)$$

$$h(\nabla^{\phi}_{\partial_t} d\phi(e_i), \nabla^{\phi}_{\partial_s} d\phi(e_i))L'_{\varphi_{t,s}} = e_i(h(L'_{\varphi}\nabla^{\varphi}_{e_i} v, w)) - h(\nabla^{\varphi}_{e_i} L'_{\varphi}\nabla^{\varphi}_{e_i} v, w). \quad (4.36)$$

Note that

$$\begin{aligned} \partial_s(L'_{\varphi_{t,s}}) &= \partial_s(L'(x, \varphi_{t,s}(x), e(\varphi_{t,s})(x))) \\ &= dL'(d\phi(\partial_s)) + dL'(\partial_s(e(\varphi_{t,s}))) \end{aligned} \quad (4.37)$$

by a simple calculation we have

$$dL'(d\phi(\partial_s))\Big|_{t=s=0} = h(w, (\mathrm{grad}^N L')o\varphi) \quad (4.38)$$

$$dL'(\partial_s(e(\varphi_{t,s})))\Big|_{t=s=0} = L''_{\varphi} h(\nabla^{\varphi}_{e_i} w, d\varphi(e_i)). \quad (4.39)$$

Then we get

$$\begin{aligned} h(\nabla^{\phi}_{\partial_t} d\phi(e_i), d\phi(e_i))\partial_s(L'_{\varphi_{t,s}})\Big|_{t=s=0} &= <\nabla^{\varphi} v, d\varphi> h(w, (\mathrm{grad}^N L')o\varphi) \\ &\quad + <\nabla^{\varphi} v, d\varphi> L''_{\varphi} h(\nabla^{\varphi}_{e_i} w, d\varphi(e_i)) \\ &= h(w, <\nabla^{\varphi} v, d\varphi> (\mathrm{grad}^N L')o\varphi) \\ &\quad + e_i(h(w, <\nabla^{\varphi} v, d\varphi> L''_{\varphi} d\varphi(e_i))) \\ &\quad - h(w, \nabla^{\varphi}_{e_i} <\nabla^{\varphi} v, d\varphi> L''_{\varphi} d\varphi(e_i)) \end{aligned} \quad (4.40)$$

From the formulas (4.28), (4.32),(4.33),(4.35),(4.36),(4.40), the divergence Theorem and the L-harmonicity of φ, the Theorem 4.2.2 follows. □

4.3 L-biharmonic maps

A natural generalization of L-harmonic maps is given by integrating the square of the norm of the L-tension field. More precisely, the L-bienergy functional of a smooth map $\varphi : (M, g) \to (N, h)$ is defined by

$$E_{2,L}(\varphi, D) = \frac{1}{2}\int_D |\tau_L(\varphi)|^2 v_g. \quad (4.41)$$

Definition 4.3.1. *A map is called L-biharmonic if it is a critical point of the L-bienergy functional over any compact subset D of M.*

4.3.1 First variation of the L-bienergy

Theorem 4.3.1. *[29] Let $\varphi : (M, g) \to (N, h)$ be a smooth map between Riemannian manifolds, D a compact subset of M and let $\{\varphi_t\}_{t\in(-\epsilon,\epsilon)}$ be a smooth variation with compact support in D. Then*

$$\frac{d}{dt}E_{2,L}(\varphi_t, D)\Big|_{t=0} = \int_D h(\tau_{2,L}(\varphi_t), v)v_g \tag{4.42}$$

where in normal frame at $x \in M$, we have

$$\begin{aligned}\tau_{(2,L)}(\varphi) &= -L'_\varphi \text{ trace } R^N(\tau_L(\varphi), d\varphi)d\varphi - \text{trace } \nabla^\varphi L'_\varphi \nabla^\varphi \tau_L(\varphi)\\ &\quad +\big(\nabla^N_{\tau_L(\varphi)} \text{ grad}^N L\big) \circ \varphi + < \nabla^\varphi \tau_L(\varphi), d\varphi > (\text{grad}^N L') \circ \varphi\\ &\quad - \text{trace } \nabla^\varphi < \nabla^\varphi \tau_L(\varphi), d\varphi > L''_\varphi d\varphi.\end{aligned} \tag{4.43}$$

Proof. Define $\phi : M \times (-\epsilon, \epsilon) \to N$ by $\phi(x, t) = \varphi_t(x)$.
First note that

$$\frac{d}{dt}E_{(2,L)}(\varphi_t, D)\Big|_{t=0} = \int_D h(\nabla^\phi_{\partial t}\tau_L(\varphi_t), \tau_L(\varphi_t))v_g \tag{4.44}$$

Calculating in a normal frame at $x \in M$ we have

$$\nabla^\phi_{\partial_t}\tau_L(\varphi_t) = \nabla^\phi_{\partial_t}\nabla^\phi_{e_i}L'_{\varphi_t}d\varphi_t(e_i) - \nabla^\phi_{\partial_t}(\text{grad}^N)o\varphi_t \tag{4.45}$$

by the definition of the curvature tensor of (N, h) we have

$$\nabla^\phi_{\partial_t}\nabla^\phi_{e_i}L'_{\varphi_t}d\varphi_t(e_i) = L'_{\varphi_t}\mathrm{R}^N(d\phi(\partial_t), d\varphi_t(e_i))d\varphi_t(e_i) + \nabla^\phi_{e_i}\nabla^\phi_{\partial_t}L'_{\varphi_t}d\varphi_t(e_i), \tag{4.46}$$

by the compatibility of ∇^ϕ with h we have

$$\begin{aligned}h(\nabla^\phi_{e_i}\nabla^\phi_{\partial_t}L'_{\varphi_t}d\varphi_t(e_i), \tau_L(\varphi_t)) &= e_i(h(\nabla^\phi_{\partial_t}L'_{\varphi_t}d\varphi_t(e_i), \tau_L(\varphi_t)))\\ &\quad -h(\nabla^\phi_{\partial_t}L'_{\varphi_t}d\varphi_t(e_i), \nabla^\phi_{e_i}\tau_L(\varphi_t)))\end{aligned} \tag{4.47}$$

the second term on the left-hand side of (4.47) is

$$\begin{aligned}-h(\nabla^\phi_{\partial_t}L'_{\varphi_t}d\varphi_t(e_i), \nabla^\phi_{e_i}\tau_L(\varphi_t)) &= -\partial_t(L'_{\varphi_t})h(d\varphi_t(e_i), \nabla^\phi_{e_i}\tau_L(\varphi_t))\\ &\quad -L'_{\varphi_t}h(\nabla^\phi_{\partial_t}d\varphi_t(e_i), \nabla^\phi_{e_i}\tau_L(\varphi_t))\end{aligned} \tag{4.48}$$

be a simple calculation we have

$$\partial_t(L'_{\varphi_t}) = d\phi(\partial_t)(L') + L''_{\varphi_t}h(\nabla^\phi_{e_j}d\phi(\partial_t), d\varphi_t(e_j)), \tag{4.49}$$

then the first term on the left-hand side of (4.48) is

$$\begin{aligned}-\partial_t(L'_{\varphi_t})h(d\varphi_t(e_i), \nabla^\phi_{e_i}\tau_L(\varphi_t)) &= -h(d\varphi_t(e_i), \nabla^\phi_{e_i}\tau_L(\varphi_t))h((\text{grad}^N L')o\varphi_t, d\phi(\partial_t))\\ &\quad -e_j(h(d\phi(\partial_t), L''_{\varphi_t}h(d\varphi_t(e_i), \nabla^\phi_{e_i}\tau_L(\varphi_t))d\varphi_t(e_j))\end{aligned}$$

$$+(h(d\phi(\partial_t), \nabla^{\phi}_{e_j} L''_{\varphi_t} h(d\varphi_t(e_i), \nabla^{\phi}_{e_i}\tau_L(\varphi_t))d\varphi_t(e_j)), \tag{4.50}$$

the second term on the left-hand side of (4.48) is

$$\begin{aligned} -L'_{\varphi_t} h(\nabla^{\phi}_{\partial_t} d\varphi_t(e_i), \nabla^{\phi}_{e_i}\tau_L(\varphi_t)) &= -e_i(h(d\phi(\partial_t), L'_{\varphi_t}\nabla^{\phi}_{e_i}\tau_L(\varphi_t))) \\ &\quad + h(d\phi(\partial_t), \nabla^{\phi}_{e_i} L'_{\varphi_t}\nabla^{\phi}_{e_i}\tau_L(\varphi_t)), \end{aligned} \tag{4.51}$$

and notice that

$$h(\nabla^{\phi}_{\partial_t}(\mathrm{grad}^N L) o\varphi_t, \tau_L(\varphi_t)) = h((\nabla^N_{\tau_L(\varphi_t)}\mathrm{grad}^N L) o\varphi_t, d\phi(\partial_t)). \tag{4.52}$$

From (4.45),(4.46), (4.47),(4.48),(4.49),(4.50),(4.51),(4.52), $v = d\phi(\partial_t)$ when $t = 0$ and the divergence Theorem, we deduce the Theorem 4.3.1 □

4.4 Main results

Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map between Riemannian manifolds. Let $x \in M^m$, the tangent space at x splits $T_x M^m = H_x \oplus V_x$ where $V_x = \mathrm{Ker}\, d_x\varphi$ and $H_x = V_x^{\perp}$ is the orthogonal complement of the vertical space V_x. The map φ is called semi-conformal if for each $x \in M^m$ where $d_x\varphi \neq 0$ the restriction $d_x\varphi : H_x \longrightarrow T_{\varphi(x)}N^n$ is conformal and surjective. On setting $\lambda(x) = 0$ at points x where $d_x\varphi = 0$, we obtain a continuous function $\lambda : M^m \longrightarrow \mathbb{R}_+$ such that for any $X, Y \in H_x$

$$h(d_x\varphi(X), d_x\varphi(Y)) = \lambda^2(x) g(X, Y),$$

the function λ is called the dilation of φ. Note that the generalized conformal maps is discussed in [28].

4.4.1 Semi-conformal L-harmonic maps

Let (M^m, g) be a Riemannian manifold and let N^n the Euclidian space $\mathbb{R}^n$ equipped with the Riemannian metric $h = dy_1^2 + ... + dy_n^2$. We have the following results.

Theorem 4.4.1. *Any semi-conformal harmonic map $\varphi : M^m \longrightarrow \mathbb{R}^n$ is L-harmonic with $L(x, y, r) = F\big(2\, y + (n-2)\, \varphi(x)\big) r$, for all $(x, y, r) \in M^m \times \mathbb{R}^n \times \mathbb{R}_+$, where $F \in C^\infty(\mathbb{R}^n)$ be a smooth positive function.*

Proof. A semi-conformal harmonic map φ is L-harmonic if and only if

$$d\varphi\big(\, \mathrm{grad}^{M^m} L'_{\varphi}\big) - (\mathrm{grad}^{\mathbb{R}^n} L) \circ \varphi = 0,$$

where $L'_{\varphi} : M^m \longrightarrow (0, +\infty)$ is a smooth positive function given by

$$L'_{\varphi}(x) = F\big(n\, \varphi(x)\big).$$

Let us choose $\{e_1, ..., e_m\}$ to be an orthonormal frame on a domain of M^m such that the vectors $\{e_1, ..., e_n\}$ are horizontal and the vectors $\{e_{n+1}, ..., e_m\}$ are vertical, so that $d\varphi(e_i) = \lambda\,(\widetilde{e}_i \circ \varphi)$ for $i = 1, ..., n$ where $\{\widetilde{e}_1, ..., \widetilde{e}_n\}$ is an orthonormal frame on a domain of $\mathbb{R}^n$. Then, we get

$$\begin{aligned} d\varphi(\operatorname{grad}^{M^m} L'_\varphi) &= \sum_{i=1}^{m} e_i(L'_\varphi)\, d\varphi(e_i) \\ &= n \sum_{i=1}^{n} d\varphi(e_i)(F)\, d\varphi(e_i) \\ &= n\,\lambda^2 \sum_{i=1}^{n} (\widetilde{e}_i \circ \varphi)(F)\, (\widetilde{e}_i \circ \varphi) \\ &= n\,\lambda^2 (\operatorname{grad}^{\mathbb{R}^n} F) \circ \varphi, \end{aligned} \tag{4.53}$$

and the term $(\operatorname{grad}^{\mathbb{R}^n} L) \circ \varphi$ is given by

$$\begin{aligned} (\operatorname{grad}^{\mathbb{R}^n} L) \circ \varphi &= \sum_{i=1}^{n} \left[\frac{\partial L}{\partial y_i} \frac{\partial}{\partial y_i} \right] \circ \varphi \\ &= 2e(\varphi) \sum_{i=1}^{n} \left[\frac{\partial F}{\partial y_i} \frac{\partial}{\partial y_i} \right] \circ \varphi \\ &= 2e(\varphi)\, (\operatorname{grad}^{\mathbb{R}^n} F) \circ \varphi, \end{aligned} \tag{4.54}$$

since $e(\varphi) = \frac{n}{2}\lambda^2$, we get $(\operatorname{grad}^{\mathbb{R}^n} L) \circ \varphi = n\,\lambda^2 (\operatorname{grad}^{\mathbb{R}^n} F) \circ \varphi$. □

Using Theorem 4.4.1, we can construct many examples for semi conformal L-harmonic maps.

Example 4.4.1. The Hopf construction map $\varphi : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$, defined by

$$\varphi(x) = (x_1^2 + x_2^2 - x_3^2 - x_4^2, 2\,x_1 x_3 + 2\,x_2 x_4, 2\,x_2 x_3 - 2\,x_1 x_4),$$

is a semi conformal harmonic map with dilation

$$\lambda(x) = 2|x|,\ \forall x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4,$$

see [1]. According to Theorem 4.4.1 the map φ is L-harmonic, where L is the form

$$F\left(x_1^2 + x_2^2 - x_3^2 - x_4^2 + 2\,y_1, 2\,x_1 x_3 + 2\,x_2 x_4 + 2\,y_2, 2\,x_2 x_3 - 2\,x_1 x_4 + 2\,y_3\right) r,$$

for all $(x, y, r) \in \mathbb{R}^4 \times \mathbb{R}^3 \times \mathbb{R}_+$, where $F \in C^\infty(\mathbb{R}^3)$ is a smooth positive function.

If $n = 1$, we have the following corollary.

Corollary 4.4.1. *A smooth function $\varphi \in C^\infty(M^m)$ is harmonic if and only if it is L-harmonic for $L(x, y, r) = F\big(2\,y - \varphi(x)\big)r$, for all $(x, y, r) \in M^m \times \mathbb{R} \times \mathbb{R}_+$ where $F \in C^\infty(\mathbb{R})$ is a smooth positive function.*

Proof. First note that, the function φ is L-harmonic if and only if

$$\tau_L(\varphi) = L'_\varphi \tau(\varphi) + d\varphi(\operatorname{grad}^{M^m} L'_\varphi) - (\operatorname{grad}^{\mathbb{R}} L)\circ\varphi = 0, \tag{4.55}$$

where $L'_\varphi(x) = F(\varphi(x))$, for all $x \in M^m$. We compute

$$\begin{aligned} d\varphi(\operatorname{grad}^M L'_\varphi) &= \sum_{i=1}^m e_i(L'_\varphi) d\varphi(e_i) \\ &= \sum_{i=1}^m e_i(F\circ\varphi) e_i(\varphi) \\ &= \sum_{i=1}^m e_i(\varphi)(F'\circ\varphi) e_i(\varphi) \\ &= (F'\circ\varphi)|\operatorname{grad}^{M^m}\varphi|^2, \end{aligned} \tag{4.56}$$

here $\{e_i\}$ is a orthonormal frame in M^m, and $e_i(\varphi) = d\varphi(e_i)$. The term $-(\operatorname{grad}^{\mathbb{R}} L)\circ\varphi$ of (4.55) is given by

$$\begin{aligned} -(\operatorname{grad}^{\mathbb{R}} L)\circ\varphi &= -\frac{1}{2}\sum_{i=1}^m e_i(\varphi)^2[2(F'\circ\varphi)] \\ &= -|\operatorname{grad}^{M^m}\varphi|^2(F'\circ\varphi). \end{aligned} \tag{4.57}$$

The Corollary follows from (4.55), (4.56) and (4.57). □

Example 4.4.2. The harmonic function

$$\varphi : \mathbb{R}^2\backslash\{0\} \longrightarrow \mathbb{R}, \quad (x_1, x_2) \longmapsto \frac{x_1}{x_1^2 + x_2^2},$$

is L-harmonic with

$$L(x_1, x_2, y, r) = F\left(2\,y - \frac{x_1}{x_1^2 + x_2^2}\right) r, \quad \forall (x_1, x_2, y, r) \in \mathbb{R}^2\times\mathbb{R}\times\mathbb{R}_+,$$

where $F \in C^\infty(\mathbb{R})$ is a smooth positive function.

For $n = 2$, we have the following result.

Theorem 4.4.2. *A semi-conformal map $\varphi : M^m \longrightarrow N^2$ from a Riemannian manifold to a Riemannian 2-manifold is L-harmonic with*

$$L(x, y, r) = re^{\alpha(x)\beta(y)}, \quad \forall (x, y, r) \in M^m\times N^2\times\mathbb{R}_+,$$

where $\alpha \in C^\infty(M^m)$ and $\beta \in C^\infty(N^2)$ if and only if

$$\tau(\varphi) + (\beta\circ\varphi)\, d\varphi(\operatorname{grad}^M \alpha) = 0.$$

Proof. First, note that the function L'_φ is given by

$$L'_\varphi(x) = e^{\alpha(x)\,\beta(\varphi(x))}, \quad \forall x \in M^m.$$

Let us choose $\{e_1, ..., e_m\}$ to be an orthonormal frame on a domain of M^m such that the vectors $\{e_1, e_2\}$ are horizontal and the vectors $\{e_3, ..., e_m\}$ are vertical, so that $d\varphi(e_i) = \lambda\,(\widetilde{e}_i \circ \varphi)$ for $i = 1, 2$ where $\{\widetilde{e}_1, \widetilde{e}_2\}$ is an orthonormal frame on a domain of N^2, then we get

$$\begin{aligned} d\varphi(\operatorname{grad}^{M^m} L'_\varphi) &= \sum_{i=1}^m e_i(L'_\varphi)\, d\varphi(e_i) \\ &= \sum_{i=1}^m e^{\alpha(\beta\circ\varphi)}\, e_i\big(\alpha(\beta\circ\varphi)\big)\, d\varphi(e_i) \\ &= e^{\alpha(\beta\circ\varphi)}\big\{(\beta\circ\varphi)\, d\varphi(\operatorname{grad}^{M^m} \alpha) + \alpha\, d\varphi(\operatorname{grad}^{M^m} (\beta\circ\varphi))\big\}, \end{aligned}$$

we compute the term $d\varphi(\operatorname{grad}^M (\beta\circ\varphi))$,

$$\begin{aligned} d\varphi(\operatorname{grad}^{M^m} (\beta\circ\varphi)) &= \sum_{i=1}^m e_i(\beta\circ\varphi)\, d\varphi(e_i) \\ &= \sum_{i=1}^2 d\varphi(e_i)(\beta)\, d\varphi(e_i) \\ &= \sum_{i=1}^2 \lambda^2\,(\widetilde{e}_i\circ\varphi)(\beta)\,(\widetilde{e}_i\circ\varphi) \\ &= \lambda^2\,(\operatorname{grad}^{N^2} \beta)\circ\varphi, \end{aligned}$$

we conclude that

$$d\varphi(\operatorname{grad}^{M^m} L'_\varphi) = e^{\alpha(\beta\circ\varphi)}\big\{(\beta\circ\varphi)\, d\varphi(\operatorname{grad}^{M^m} \alpha) + \alpha\,\lambda^2\,(\operatorname{grad}^{N^2} \beta)\circ\varphi\big\},$$

since $e(\varphi) = \lambda^2$, we get the following

$$\begin{aligned} (\operatorname{grad}^{N^2} L)\circ\varphi &= \sum_{i=1}^2 (\widetilde{e}_i\circ\varphi)(L)(\widetilde{e}_i\circ\varphi) \\ &= \lambda^2 \sum_{i=1}^2 (\widetilde{e}_i\circ\varphi)(\alpha\,\beta)\, e^{\alpha\,(\beta\circ\varphi)}\,(\widetilde{e}_i\circ\varphi) \\ &= \alpha\lambda^2\, e^{\alpha\,(\beta\circ\varphi)}\,(\operatorname{grad}^{N^2} \beta)\circ\varphi, \end{aligned}$$

so that, the L-tension field of φ is given by

$$\tau_L(\varphi) \;=\; e^{\alpha\,(\beta\circ\varphi)}\big[\tau(\varphi) + (\beta\circ\varphi)\, d\varphi(\operatorname{grad}^{M^m} \alpha)\big].$$

This completes the proof of Theorem 4.4.2. □

Example 4.4.3 (The foliation by the circles of Villarceau, [3]). Let M^3 the manifold $\mathbb{R}\times\mathbb{R}^2\backslash\{0\}$ and let $\varphi : M^3 \longrightarrow \mathbb{R}^2$ defined by

$$\varphi(x_1,x_2,x_3)=\left(\frac{\left(1-\frac{|x|^2}{2}\right)x_2+\sqrt{2}x_1x_3}{x_2^2+x_3^2},\frac{\left(1-\frac{|x|^2}{2}\right)x_3-\sqrt{2}x_1x_2}{x_2^2+x_3^2}\right),$$

the map φ is semi-conformal, its dilation is given by the function

$$\lambda(x)=\frac{\left(1-\frac{|x|^2}{2}\right)^2}{(x_2^2+x_3^2)^2},\quad \forall x=(x_1,x_2,x_3)\in M^3.$$

The tension field of φ is

$$\tau(\varphi)(x)=\left(-\frac{x_2}{x_2^2+x_3^2},-\frac{x_3}{x_2^2+x_3^2}\right).$$

According to Theorem 4.4.2, with $\alpha(x)=c_1\ln(2+|x|^2)+c_2$ and $\beta(y)=-\frac{1}{c_1}$, where $c_1\in\mathbb{R}^*$, $c_2\in\mathbb{R}$, the map φ is L-harmonic with

$$L(x,y,r)=\frac{e^{-\frac{c_2}{c_1}}r}{2+|x|^2},\quad \forall(x,y,r)\in M^3\times\mathbb{R}^2\times\mathbb{R}_+.$$

4.4.2 A Liouville type theorem for L-harmonic maps

Liouville type theorems for harmonic maps between complete smooth Riemannian manifolds have been done by many authors. Eells-Sampson [14] proved that any (bounded) harmonic map from a compact Riemannian manifold with positive Ricci curvature into a complete manifold with non-positive curvature is a constant map. Schoen-Yau [23] also proved that any harmonic map with finite energy from a complete smooth Riemannian manifold with non-negative Ricci curvature into a complete manifold with non-positive curvature is a constant map. Cheng [15] showed that any harmonic map with sublinear growth from a complete Riemannian manifold with non-negative Ricci curvature into an Hadamard manifold is a constant map. Bair-Fardoun-Ouakkas [7] proved the Liouville-type theorem for bi-harmonic maps. The purpose of this party is to provide a proof of the Liouville type theorem for L-harmonic maps from complete non-compact Riemannian manifold (M^m,g) with positive Ricci curvature into a Riemannian manifold (N^n,h) with non-positive sectional curvature, where $L\in C^\infty(M^m\times N^n\times\mathbb{R}_+)$ is a smooth positive function which satisfies some suitable conditions.

Theorem 4.4.3. *Let (M^m,g) be a complete noncompact Riemannian manifold with positive Ricci curvature $\mathrm{Ricci}^{M^m}\geq 0$, (N^n,h) a Riemannian manifold with non-positive sectional curvature $\mathrm{Sect}^{N^n}\leq 0$. Consider an L-harmonic map φ from (M^m,g) to (N^n,h), where $L\in C^\infty(M^m\times N^n\times\mathbb{R}_+)$ is a smooth positive function. Suppose that*

$$L'_\varphi>0,\quad \mathrm{Hess}^{M^m}L'_\varphi\leq 0,\quad \mathrm{Hess}^{N^n}L\geq 0,$$

$$d\varphi(\operatorname{grad}^{M^m} L'_\varphi)(L) \leq 0, \quad \int_{M^m} L'_\varphi\, v^g = \infty, \quad \int_{M^m} L'_\varphi |d\varphi|^2\, v^g < \infty.$$

Then φ is constant.

We need the following lemmas to prove Theorem 4.4.3.

Lemma 4.4.1 ([12, 38])**.** *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ a smooth mapping between Riemannian manifolds and let $f \in C^\infty(M^m)$, then*

$$\big\langle d\varphi, \nabla^\varphi d\varphi(\operatorname{grad}^{M^m} f)\big\rangle = \frac{1}{2}\big(\operatorname{grad}^{M^m} f\big)\big(|d\varphi|^2\big) + \big\langle d\varphi, d\varphi(\nabla^{M^m} \operatorname{grad}^{M^m} f)\big\rangle.$$

Here $\langle,\rangle$ denote the inner product on $T^*M^m \otimes T^*M^m$.

Lemma 4.4.2. *Let (M^m, g), (N^n, h) be two Riemannian manifolds, and $L \in C^\infty(M^m \times N^n \times \mathbb{R}_+)$ a smooth positive function. Consider an L-harmonic map $\varphi : (M^m, g) \longrightarrow (N^n, h)$, then we have*

$$\begin{aligned}
\frac{1}{2}\Delta^{M^m}|d\varphi|^2 \;=\;& |\nabla d\varphi|^2 + \frac{1}{{L'_\varphi}^2}|d\varphi(\operatorname{grad}^{M^m} L'_\varphi)|^2 - \frac{1}{2L'_\varphi}\big(\operatorname{grad}^{M^m} L'_\varphi\big)\big(|d\varphi|^2\big) \\
& - \frac{1}{L'_\varphi}\big\langle d\varphi, d\varphi(\nabla^{M^m} \operatorname{grad}^{M^m} L'_\varphi)\big\rangle - \frac{1}{{L'_\varphi}^2} d\varphi(\operatorname{grad}^{M^m} L'_\varphi)(L) \\
& + \frac{1}{L'_\varphi}\big\langle d\varphi, \nabla^\varphi(\operatorname{grad}^{N^n} L)\circ\varphi\big\rangle + \sum_{i=1}^m h\big(d\varphi\big(\operatorname{Ricci}^{M^m} e_i\big), d\varphi(e_i)\big) \\
& - \sum_{i,j=1}^m h\big(R^{N^n}\big(d\varphi(e_i), d\varphi(e_j)\big)d\varphi(e_j), d\varphi(e_i)\big),
\end{aligned}$$

where $\{e_1, ..., e_m\}$ be a orthonormal frame on (M^m, g).

Proof. We start recalling the standard Bochner formula for the smooth map φ. Let $\{e_1, ..., e_m\}$ be a orthonormal frame on (M^m, g), we have

$$\begin{aligned}
\frac{1}{2}\Delta^{M^m}|d\varphi|^2 =& |\nabla d\varphi|^2 + \big\langle d\varphi, \nabla^\varphi \tau(\varphi)\big\rangle + \sum_{i=1}^m h\big(d\varphi\big(\operatorname{Ricci}^{M^m} e_i\big), d\varphi(e_i)\big) \\
& - \sum_{i,j=1}^m h\big(R^{N^n}\big(d\varphi(e_i), d\varphi(e_j)\big)d\varphi(e_j), d\varphi(e_i)\big),
\end{aligned} \tag{4.58}$$

where $|\nabla d\varphi|$ is given by

$$|\nabla d\varphi|^2 = \sum_{i,j=1}^m h\big(\nabla d\varphi(e_i, e_j), \nabla d\varphi(e_i, e_j)\big),$$

and $\langle d\varphi, \nabla^{\varphi}\tau(\varphi)\rangle$ is defined by

$$\langle d\varphi, \nabla^{\varphi}\tau(\varphi)\rangle = \sum_{i=1}^{m} h\big(d\varphi(e_i), \nabla^{\varphi}_{e_i}\tau(\varphi)\big),$$

since the map φ is L-harmonic, we have

$$\tau_L(\varphi) = L'_{\varphi}\,\tau(\varphi) + d\varphi(\operatorname{grad}^{M^m} L'_{\varphi}) - (\operatorname{grad}^{N^n} L)\circ\varphi = 0,$$

and $L'_{\varphi} > 0$ on M, we obtain

$$\tau(\varphi) = -\frac{1}{L'_{\varphi}} d\varphi(\operatorname{grad}^{M^m} L'_{\varphi}) + \frac{1}{L'_{\varphi}}(\operatorname{grad}^{N^n} L)\circ\varphi, \tag{4.59}$$

we get the following

$$\begin{aligned}\langle d\varphi, \nabla^{\varphi}\tau(\varphi)\rangle &= \frac{1}{{L'_{\varphi}}^2}|d\varphi(\operatorname{grad}^{M^m} L'_{\varphi})|^2 - \frac{1}{L'_{\varphi}}\langle d\varphi, \nabla^{\varphi} d\varphi(\operatorname{grad}^{M^m} L'_{\varphi})\rangle \\ &\quad -\frac{1}{{L'_{\varphi}}^2} h(d\varphi(\operatorname{grad}^{M^m} L'_{\varphi}), (\operatorname{grad}^{N^n} L)\circ\varphi) \\ &\quad +\frac{1}{L'_{\varphi}}\langle d\varphi, \nabla^{\varphi}(\operatorname{grad}^{N^n} L)\circ\varphi\rangle, \end{aligned}\tag{4.60}$$

by the lemma 5.1.1, the second term on the left-hand side of (5.13) is

$$\begin{aligned}-\frac{1}{L'_{\varphi}}\langle d\varphi, \nabla^{\varphi} d\varphi(\operatorname{grad}^{M^m} L'_{\varphi})\rangle &= -\frac{1}{2L'_{\varphi}}\big(\operatorname{grad}^{M^m} L'_{\varphi}\big)\big(|d\varphi|^2\big) \\ &\quad - \frac{1}{L'_{\varphi}}\langle d\varphi, d\varphi(\nabla^{M^m} \operatorname{grad}^{M^m} L'_{\varphi})\rangle. \end{aligned}\tag{4.61}$$

The Lemma 4.4.2 follows by (5.12), (5.13) and (5.14). □

Proof. (of Theorem 4.4.3) By lemma 4.4.2, we get

$$\begin{aligned}\frac{1}{2}L'_{\varphi}\Delta^{M^m}|d\varphi|^2 &= L'_{\varphi}|\nabla d\varphi|^2 + \frac{1}{L'_{\varphi}}|d\varphi(\operatorname{grad}^{M^m} L'_{\varphi})|^2 - \frac{1}{2}\big(\operatorname{grad}^{M^m} L'_{\varphi}\big)\big(|d\varphi|^2\big) \\ &\quad -\langle d\varphi, d\varphi(\nabla^{M^m} \operatorname{grad}^{M^m} L'_{\varphi})\rangle - \frac{1}{L'_{\varphi}} d\varphi(\operatorname{grad}^{M^m} L'_{\varphi})(L) \\ &\quad +\langle d\varphi, \nabla^{\varphi}(\operatorname{grad}^{N^n} L)\circ\varphi\rangle + L'_{\varphi}\sum_{i=1}^{m} h\big(d\varphi\big(\operatorname{Ricci}^{M^m} e_i\big), d\varphi(e_i)\big) \\ &\quad -L'_{\varphi}\sum_{i,j=1}^{m} h\big(R^{N^n}\big(d\varphi(e_i), d\varphi(e_j)\big)d\varphi(e_j), d\varphi(e_i)\big), \end{aligned}$$

we denote $\Delta_L^{M^m}\rho \equiv L'_\varphi \Delta^{M^m}\rho + \big(\operatorname{grad}^{M^m} L'_\varphi\big)(\rho)$ for all $\rho \in C^\infty(M^m)$, we have

$$\begin{aligned}\frac{1}{2}\Delta_L^{M^m}|d\varphi|^2 &= L'_\varphi|\nabla d\varphi|^2 + \frac{1}{L'_\varphi}|d\varphi(\operatorname{grad}^{M^m} L'_\varphi)|^2 - \big\langle d\varphi, d\varphi(\nabla^{M^m}\operatorname{grad}^{M^m} L'_\varphi)\big\rangle \\ &\quad -\frac{1}{L'_\varphi}d\varphi(\operatorname{grad}^{M^m} L'_\varphi)(L) + \big\langle d\varphi, \nabla^\varphi(\operatorname{grad}^{N^n} L)\circ\varphi\big\rangle \\ &\quad +L'_\varphi\sum_{i=1}^m h\big(d\varphi\big(\operatorname{Ricci}^{M^m} e_i\big), d\varphi(e_i)\big) \\ &\quad -L'_\varphi\sum_{i,j=1}^m h\big(R^{N^n}\big(d\varphi(e_i), d\varphi(e_j)\big)d\varphi(e_j), d\varphi(e_i)\big),\end{aligned}$$

since $\operatorname{Sect}^{N^n} \leq 0$, $\operatorname{Ricci}^{M^m} \geq 0$, $\operatorname{Hess}^{N^n} L \geq 0$, $\operatorname{Hess}^{M^m} L'_\varphi \leq 0$ and $d\varphi(\operatorname{grad}^{M^m} L'_\varphi)(L) \leq 0$ by (5.15) we obtain the following inequality

$$\frac{1}{2}\Delta_L^{M^m}|d\varphi|^2 \geq L'_\varphi\,|\nabla d\varphi|^2, \tag{4.62}$$

since $\frac{1}{2}\Delta_L^{M^m}|d\varphi|^2 = |d\varphi|\Delta_L^{M^m}|d\varphi| + L'_\varphi\,|\operatorname{grad}^{M^m}|d\varphi||^2$, by (5.16) and the Kato's inequality [6], we get the following

$$|d\varphi|\Delta_L^{M^m}|d\varphi| \geq L'_\varphi\big(|\nabla d\varphi|^2 - |\operatorname{grad}^{M^m}|d\varphi||^2\big) \geq 0. \tag{4.63}$$

Let $\rho : M^m \longrightarrow \mathbb{R}$ be a smooth function with compact support, then

$$\begin{aligned}\rho^2|d\varphi|\Delta_L^{M^m}|d\varphi| &= \rho^2|d\varphi|\operatorname{div}^{M^m}\big(L'_\varphi \operatorname{grad}^{M^m}|d\varphi|\big) \\ &= \operatorname{div}^{M^m}\big(\rho^2|d\varphi|L'_\varphi \operatorname{grad}^{M^m}|d\varphi|\big) - L'_\varphi\rho^2|\operatorname{grad}^{M^m}|d\varphi||^2 \\ &\quad - 2L'_\varphi\rho|d\varphi|g(\operatorname{grad}^{M^m}\rho, \operatorname{grad}^{M^m}|d\varphi|),\end{aligned} \tag{4.64}$$

by (5.17), (5.18) and the Stokes Theorem, we deduce

$$\begin{aligned}0 \leq &-\int_{M^m} L'_\varphi\rho^2|\operatorname{grad}^{M^m}|d\varphi||^2 v^g \\ &- 2\int_{M^m} L'_\varphi\rho|d\varphi|g(\operatorname{grad}^{M^m}\rho, \operatorname{grad}^{M^m}|d\varphi|)v^g,\end{aligned} \tag{4.65}$$

using the Young inequality [48], we have

$$\begin{aligned}-2g(|d\varphi|\operatorname{grad}^{M^m}\rho, \rho\operatorname{grad}^{M^m}|d\varphi|) \leq &\frac{1}{\epsilon}|d\varphi|^2|\operatorname{grad}^{M^m}\rho|^2 \\ &+ \epsilon\rho^2|\operatorname{grad}^{M^m}|d\varphi||^2,\end{aligned} \tag{4.66}$$

for any $\epsilon > 0$, substituting (5.20) in (5.19), we obtain

$$0 \leq -\int_{M^m} L'_\varphi\rho^2|\operatorname{grad}^{M^m}|d\varphi||^2 v^g + \frac{1}{\epsilon}\int_{M^m} L'_\varphi|d\varphi|^2|\operatorname{grad}^{M^m}\rho|^2 v^g$$

$$+\epsilon \int_{M^m} L'_\varphi \rho^2 |\operatorname{grad}^{M^m} |d\varphi||^2 v^g,$$

the last inequality is equivalent to

$$(1-\epsilon)\int_{M^m} L'_\varphi \rho^2 |\operatorname{grad}^{M^m} |d\varphi||^2 v^g \leq \frac{1}{\epsilon}\int_{M^m} L'_\varphi |d\varphi|^2 |\operatorname{grad}^{M^m} \rho|^2 v^g. \tag{4.67}$$

Choose the smooth cut-off $\rho = \rho_R$ i.e $\rho \leq 1$ on M^m, $\rho = 1$ on the ball $B(0,R)$, $\rho = 0$ on $M^m \backslash B(0,2R)$ and $|\operatorname{grad}^{M^m} \rho| \leq \frac{2}{R}$. Let $0 < \epsilon < 1$, replacing $\rho = \rho_R$ in (5.21) we obtain

$$0 \leq (1-\epsilon)\int_{M^m} L'_\varphi \rho^2 |\operatorname{grad}^{M^m} |d\varphi||^2 v^g \leq \frac{4}{\epsilon R^2}\int_{M^m} L'_\varphi |d\varphi|^2 v^g, \tag{4.68}$$

since $\int_{M^m} L'_\varphi |d\varphi|^2 v^g < \infty$, when $R \longrightarrow \infty$, we have

$$\frac{4}{\epsilon R^2}\int_{M^m} L'_\varphi |d\varphi|^2 v^g \longrightarrow 0.$$

Thus, by (4.68), we have $|\operatorname{grad}^{M^m} |d\varphi|| = 0$ i.e $|d\varphi| = c$ constant. If $c > 0$,

$$\frac{c^2}{2}\int_{M^m} L'_\varphi v^g < \infty,$$

but $\int_{M^m} L'_\varphi v^g = \infty$ then $c = 0$, that is φ is constant map. □

If $L(x,y,r) = r$ for all $(x,y,r) \in M^m \times N^n \times \mathbb{R}_+$, we recover the following classical result.

Corollary 4.4.2. *[14] Let (M^m, g) be a complete noncompact Riemannian manifold of infinite volume with positive Ricci curvature and (N^n, h) a Riemannian manifold with non-positive sectional curvature. Consider an harmonic map $\varphi : (M^m, g) \longrightarrow (N^n, h)$ with finite energy*

$$E(\varphi) = \frac{1}{2}\int_{M^m} |d\varphi|^2 \, v^g < \infty.$$

Then φ is constant.

Let $f : M^m \longrightarrow (0, \infty)$ be a smooth function. If $L(x,y,r) = f(x)r$ for all $(x,y,r) \in M^m \times N^n \times \mathbb{R}_+$. We recover the following result.

Corollary 4.4.3. *[38, 46] Let (M^m, g) be a complete noncompact Riemannian manifold with positive Ricci curvature, (N^n, h) a Riemannian manifold with non-positive sectional curvature and let f be a smooth positive function on M^m with non-positive Hessien $\operatorname{Hess}^{M^m} f \leq 0$. Consider an f-harmonic map $\varphi : (M^m, g) \longrightarrow (N^n, h)$ with finite f-energy*

$$E_f(\varphi) = \frac{1}{2}\int_{M^m} f \, |d\varphi|^2 \, v^g < \infty.$$

If $\operatorname{Vol}_f(M^m) = \int_{M^m} f \, v^g = \infty$. Then φ is constant.

Chapter 5

(p, f)-harmonic maps

Let $\varphi : (M, g) \to (N, h)$ be a smooth map between two Riemannian manifolds, its p-energy is defined by

$$E_p(\varphi; D) = \frac{1}{p}\int_D |d\varphi|^p v^g \qquad (p \geq 2). \tag{5.1}$$

where D is a compact subset of M. We say that φ is a p-harmonic map if it is a critical point of the p-energy functional, that is to say, if it satisfies the Euler-Lagrange equation of the functional (5.1), that is,

$$\tau_p(\varphi) \equiv \operatorname{div}^M(|d\varphi|^{p-2}d\varphi) = 0. \tag{5.2}$$

Let $\tau(\varphi)$ the tension field of φ given by:

$$\tau(\varphi) = \operatorname{trace}_g \nabla d\varphi = \nabla^{\varphi}_{e_i} d\varphi(e_i) - d\varphi(\nabla^M_{e_i} e_i), \tag{5.3}$$

where ∇^M is the Levi-Civita connection of (M, g), ∇^{φ} denote the pull-back connection on $\varphi^{-1}TN$ and $\{e_i\}$ is an orthonormal frame on (M, g) (see [1], [14], [44]). If $|d_x\varphi| \neq 0$, for all $x \in M$, then φ is p-harmonic if and only if (see [5]):

$$|d\varphi|^{p-2}\tau(\varphi) + (p-2)|d\varphi|^{p-3}d\varphi(\operatorname{grad}^M |d\varphi|) = 0. \tag{5.4}$$

For more details on the concept of p-harmonic maps see [2, 5, 16].

5.1 Main results

5.1.1 The first variation of the (p, f)-energy

Definition 5.1.1. *Let $\varphi : (M, g) \to (N, h)$ be a smooth map between two Riemannian manifolds, the (p, f)-energy is defined by*

$$E_{p,f}(\varphi; D) = \frac{1}{p}\int_D f(x)|d\varphi|^p v^g, \tag{5.5}$$

where $p \geq 2$, f is a smooth positive function on M, and D is a compact subset of M.

Remark 5.1.1.
- The (p,f)-energy functional (5.5) includes as a special case ($f=1$) the p-energy functional, and a special case ($p=2$) the f-energy functional (see [9, 11, 29, 33]).
- We call (p,f)-harmonic (or generalized p-harmonic) a smooth map φ which is a critical point of the (p,f)-energy functional for any compact domain D.

Theorem 5.1.1. *[41] Let $\varphi:(M,g)\longrightarrow(N,h)$ be a smooth map between two Riemannian manifolds, and $\{\varphi_t\}_{t\in(-\epsilon,\epsilon)}$ a smooth variation of φ to support in $D\subset M$. Then*

$$\frac{d}{dt}E_{p,f}(\varphi_t;D)\Big|_{t=0}=-\int_D h(v,\tau_{p,f}(\varphi))v_g, \tag{5.6}$$

where $\tau_{p,f}(\varphi)$ is the (p,f)-tension field of φ given by

$$\tau_{p,f}(\varphi)\equiv\operatorname{div}^M(f|d\varphi|^{p-2}d\varphi)=f\tau_p(\varphi)+|d\varphi|^{p-2}d\varphi(\operatorname{grad}^M f), \tag{5.7}$$

and $v=\frac{d\varphi_t}{dt}\Big|_{t=0}$ denotes the variation vector field of $\{\varphi_t\}_{t\in(-\epsilon,\epsilon)}$.

Proof. Let $\phi: M\times(-\epsilon,\epsilon)\longrightarrow N$ be a smooth map defined by $\phi(x,t)=\varphi_t(x)$, we have $\phi(x,0)=\varphi(x)$, and the variation vector field $v\in\Gamma(\varphi^{-1}TN)$ associated to the variation $(\varphi_t)_{t\in(-\epsilon,\epsilon)}$ is given by $v(x)=d_{(x,0)}\phi(\frac{\partial}{\partial t})$, for all $x\in M$. Let $\{e_i\}$ be an orthonormal frame with respect to g on M, such that $\nabla^M_{e_j}e_i=0$ at $x\in M$ for all $i,j=1,...,m$. We compute

$$\frac{d}{dt}E_{p,f}(\varphi_t;D)\Big|_{t=0} \;=\; \frac{1}{p}\int_D f(x)\frac{\partial}{\partial t}|d\varphi_t|^p\Big|_{t=0}v_g. \tag{5.8}$$

First, note that

$$\begin{aligned}
\frac{\partial}{\partial t}|d\varphi_t|^p &= \frac{\partial}{\partial t}\left(|d\varphi_t|^2\right)^{\frac{p}{2}}\\
&= \frac{p}{2}\left(|d\varphi_t|^2\right)^{\frac{p}{2}-1}\frac{\partial}{\partial t}(|d\varphi_t|^2)\\
&= p|d\varphi_t|^{p-2}h(\nabla^\phi_{\frac{\partial}{\partial t}}d\phi(e_i,0),d\phi(e_i,0)).
\end{aligned}$$

Substituting the last formula in (5.8), and using $\nabla^\phi_{\frac{\partial}{\partial t}}d\phi(e_i,0)=\nabla^\phi_{(e_i,0)}d\phi(\frac{\partial}{\partial t})$, we obtain the following equation

$$\begin{aligned}
\frac{d}{dt}E_{p,f}(\varphi_t;D)\Big|_{t=0} &= \int_D f|d\varphi|^{p-2}h(\nabla^\phi_{(e_i,0)}d\phi(\frac{\partial}{\partial t}),d\phi(e_i,0))\Big|_{t=0}v_g\\
&= \int_D h(\nabla^\varphi_{e_i}v,f|d\varphi|^{p-2}d\varphi(e_i))v_g.
\end{aligned} \tag{5.9}$$

Let $\omega\in\Gamma(T^*M)$ defined by

$$\omega(X)=h(v,f|d\varphi|^{p-2}d\varphi(X)),\quad \forall X\in\Gamma(TM).$$

So that, the divergence of ω at x, is given by

$$\operatorname{div}^M \omega = e_i \left[h(v, f|d\varphi|^{p-2} d\varphi(e_i)) \right]. \tag{5.10}$$

By the equations (5.9), (5.10), we get

$$\frac{d}{dt} E_{p,f}(\varphi_t; D)\Big|_{t=0} = \int_D (\operatorname{div}^M \omega) v_g - \int_D h(v, \nabla^{\varphi}_{e_i} f |d\varphi|^{p-2} d\varphi(e_i)) v_g. \tag{5.11}$$

The Theorem 5.1.1 follows from (5.11), and the divergence Theorem. □

From Theorem 5.1.1, we deduce:

Theorem 5.1.2. *Let $\varphi : (M,g) \to (N,h)$ be a smooth map between Riemannian manifolds. Then, φ is (p,f)-harmonic if and only if $\tau_{p,f}(\varphi) = 0$.*

Example 5.1.1. According to Theorem 5.1.2, the inversion map

$$\varphi : \mathbb{R}^n \backslash \{0\} \longrightarrow \mathbb{R}^n \backslash \{0\}, \quad x \longmapsto \frac{x}{|x|^2},$$

is (p,f)-harmonic, for all $p \geq 2$, where $f(x) = |x|^{2(p-n)}$, for all $x \in \mathbb{R}^n \backslash \{0\}$.

Remark 5.1.2. In particular, we note that every harmonic map with constant energy density $\frac{1}{2}|d\varphi|^2$ is (p,f)-harmonic if and only if $\operatorname{grad}^M f \in \ker d\varphi$. The previous example prove the following results; There is no equivalence between the p-harmonicity of smooth map $\varphi : (M,g) \to (N,h)$ and the (p,f)-harmonicity of φ. There are (p,f)-harmonic maps that are neither p-harmonic nor harmonic.

5.1.2 A Liouville type Theorem for (p,f)-harmonic maps

Liouville type theorems for harmonic maps between complete smooth Riemannian manifolds have been done by many authors. Liu [21] proved the Liouville-type theorem for p-harmonic maps with free boundary values.
The purpose of this section is to provide a proof of the Liouville type theorem for (p,f)-harmonic maps from complete noncompact Riemannian manifold (M,g) with positive Ricci curvature into a Riemannian manifold (N,h) with non-positive sectional curvature.

Theorem 5.1.3. *[41] Let (M,g) be a complete non-compact Riemannian manifold with positive Ricci curvature $\operatorname{Ricci}^M \geq 0$, and (N,h) be a Riemannian manifold with non-positive sectional curvature $\operatorname{Sect}^N \leq 0$. Consider an (p,f)-harmonic map $\varphi : (M,g) \longrightarrow (N,h)$, where $f \in C^\infty(M)$ is a smooth positive function, and $p \geq 3$. Suppose that*

$$\operatorname{Hess}^M f \leq 0, \quad E_{p,f}(\varphi) < \infty, \quad \int_M f\, v_g = \infty.$$

Then φ is constant.

We will need the following lemma to prove the Theorem 5.1.3.

Lemma 5.1.1 ([12, 38])**.** *Let $\varphi : (M, g) \longrightarrow (N, h)$ a smooth mapping between Riemannian manifolds and let $f \in C^\infty(M)$, then*

$$\big\langle d\varphi, \nabla^\varphi d\varphi(\operatorname{grad}^M f)\big\rangle = \frac{1}{2}\big(\operatorname{grad}^M f\big)\big(|d\varphi|^2\big) + \big\langle d\varphi, d\varphi(\nabla^M \operatorname{grad}^M f)\big\rangle.$$

Here $\langle,\rangle$ denote the inner product on $T^*M \otimes \varphi^{-1}TN$.

Proof of Theorem 5.1.1. We start recalling the standard Bochner formula for the smooth map φ. Let $\{e_i\}$ be a orthonormal frame on (M, g), we have

$$\begin{aligned}\frac{1}{2}\Delta^M|d\varphi|^2 &= |\nabla d\varphi|^2 + \big\langle d\varphi, \nabla^\varphi\tau(\varphi)\big\rangle + h\big(d\varphi\big(\operatorname{Ricci}^M e_i\big), d\varphi(e_i)\big) \\ &\quad - h\big(R^N\big(d\varphi(e_i), d\varphi(e_j)\big)d\varphi(e_j), d\varphi(e_i)\big),\end{aligned} \tag{5.12}$$

where $|\nabla d\varphi|^2$ and $\big\langle d\varphi, \nabla^\varphi\tau(\varphi)\big\rangle$ are given by

$$\begin{aligned}|\nabla d\varphi|^2 &= h\big(\nabla d\varphi(e_i, e_j), \nabla d\varphi(e_i, e_j)\big), \\ \big\langle d\varphi, \nabla^\varphi\tau(\varphi)\big\rangle &= h\big(d\varphi(e_i), \nabla^\varphi_{e_i}\tau(\varphi)\big).\end{aligned}$$

Since the map φ is (p, f)-harmonic, we have

$$f|d\varphi|^{p-2}\tau(\varphi) + (p-2)f|d\varphi|^{p-3}d\varphi(\operatorname{grad}^M |d\varphi|) + |d\varphi|^{p-2}d\varphi(\operatorname{grad}^M f) = 0.$$

Let $\theta_1, \theta_2, \theta_3 \in \Gamma(T^*M)$ defined by

$$\begin{aligned}\theta_1(X) &= h(f|d\varphi|^{p-2}d\varphi(X), \tau(\varphi)), \\ \theta_2(X) &= |d\varphi|^{p-2}h(d\varphi(X), d\varphi(\operatorname{grad}^M f)), \\ \theta_3(X) &= (p-2)f|d\varphi|^{p-3}h(d\varphi(X), d\varphi(\operatorname{grad}^M |d\varphi|)),\end{aligned}$$

where $X \in \Gamma(TM)$. By using the (p, f)-harmonic condition of φ, we obtain

$$\begin{aligned}\operatorname{div}^M \theta_1 &= f|d\varphi|^{p-2}\big\langle d\varphi, \nabla^\varphi\tau(\varphi)\big\rangle, \\ \operatorname{div}^M \theta_2 &= -\frac{1}{f}|d\varphi|^{p-2}|d\varphi(\operatorname{grad}^M f)|^2 + |d\varphi|^{p-2}\big\langle d\varphi, \nabla^\varphi d\varphi(\operatorname{grad}^M f)\big\rangle.\end{aligned}$$

Note that by the (p, f)-harmonic condition of φ, we have $\theta_1 + \theta_2 + \theta_3 = 0$. From the last equations, and Lemma 5.1.1, we find that

$$\begin{aligned}\operatorname{div}^M \theta_3 &= -f|d\varphi|^{p-2}\big\langle d\varphi, \nabla^\varphi\tau(\varphi)\big\rangle + \frac{1}{f}|d\varphi|^{p-2}|d\varphi(\operatorname{grad}^M f)|^2 \\ &\quad -\frac{1}{2}|d\varphi|^{p-2}\big(\operatorname{grad}^M f\big)\big(|d\varphi|^2\big) - |d\varphi|^{p-2}\big\langle d\varphi, d\varphi(\nabla^M \operatorname{grad}^M f)\big\rangle.\end{aligned}$$

By using the Bochner formula (5.12), with $\text{Ricci}^M \geq 0$, $\text{Sect}^N \leq 0$, and $\text{Hess}^M f \leq 0$, we have the following inequality

$$\begin{aligned} \operatorname{div}^M \theta_3 &\geq f|d\varphi|^{p-2}|\nabla d\varphi|^2 - \frac{1}{2}|d\varphi|^{p-2}\Delta^M|d\varphi|^2 \\ &\quad -\frac{1}{2}|d\varphi|^{p-2}\big(\operatorname{grad}^M f\big)\big(|d\varphi|^2\big). \end{aligned} \tag{5.13}$$

We set $\Delta_f^M|d\varphi|^2 = f\Delta^M|d\varphi|^2 + (\operatorname{grad}^M f)(|d\varphi|^2)$. So, the inequality (5.13) becomes

$$\operatorname{div}^M \theta_3 \geq f|d\varphi|^{p-2}|\nabla d\varphi|^2 - \frac{1}{2}|d\varphi|^{p-2}\Delta_f^M|d\varphi|^2. \tag{5.14}$$

By using the following formula

$$\frac{1}{2}\Delta_f^M|d\varphi|^2 = |d\varphi|\Delta_f^M|d\varphi| + f\,|\operatorname{grad}^M|d\varphi||^2, \tag{5.15}$$

and inequality (5.14), we have the following

$$\operatorname{div}^M \theta_3 \geq f|d\varphi|^{p-2}|\nabla d\varphi|^2 - |d\varphi|^{p-1}\Delta_f^M|d\varphi| - f|d\varphi|^{p-2}|\operatorname{grad}^M|d\varphi||^2.$$

From the Kato's inequality $|\nabla d\varphi|^2 - |\operatorname{grad}^M|d\varphi||^2 \geq 0$, and the last inequality, we get

$$\operatorname{div}^M \theta_3 \geq -|d\varphi|^{p-1}\Delta_f^M|d\varphi|. \tag{5.16}$$

Let $\rho : M \longrightarrow \mathbb{R}$ be a smooth function with compact support. Multiplying the inequality (5.16) by ρ^2, with $\Delta_f^M|d\varphi| = \operatorname{div}^M\big(f\operatorname{grad}^M|d\varphi|\big)$, we conclude that

$$\begin{aligned} \operatorname{div}^M(\rho^2\theta_3) &- 2(p-2)\rho f|d\varphi|^{p-3}h(d\varphi(\operatorname{grad}^M\rho), d\varphi(\operatorname{grad}^M|d\varphi|)) \\ &\geq -\operatorname{div}^M\big(\rho^2 f|d\varphi|^{p-1}\operatorname{grad}^M|d\varphi|\big) \\ &\quad +2\rho f|d\varphi|^{p-1}g(\operatorname{grad}^M\rho, \operatorname{grad}^M|d\varphi|) \\ &\quad +(p-1)\rho^2 f|d\varphi|^{p-2}|\operatorname{grad}^M|d\varphi||^2. \end{aligned} \tag{5.17}$$

By the Young inequality, we have

$$\begin{aligned} &-2(p-2)\rho f|d\varphi|^{p-3}h(d\varphi(\operatorname{grad}^M\rho), d\varphi(\operatorname{grad}^M|d\varphi|)) \\ &\qquad \leq \epsilon_1(p-2)\rho^2 f|d\varphi|^{p-2}|\operatorname{grad}^M|d\varphi||^2 + \frac{p-2}{\epsilon_1}f|d\varphi|^p|\operatorname{grad}^M\rho|^2, \end{aligned} \tag{5.18}$$

and the following inequality

$$\begin{aligned} &-2\rho f|d\varphi|^{p-1}g(\operatorname{grad}^M\rho, \operatorname{grad}^M|d\varphi|) \\ &\qquad \leq \epsilon_2\rho^2 f|d\varphi|^{p-2}|\operatorname{grad}^M|d\varphi||^2 + \frac{1}{\epsilon_2}f|d\varphi|^p|\operatorname{grad}^M\rho|^2, \end{aligned} \tag{5.19}$$

for any $\epsilon_1, \epsilon_2 > 0$. Substituting (5.18) and (5.19) in (5.17) we obtain

$$\begin{aligned} \operatorname{div}^M(\rho^2\theta_3) &+ \left(\frac{p-2}{\epsilon_1} + \frac{1}{\epsilon_2}\right) f|d\varphi|^p|\operatorname{grad}^M \rho|^2 \\ &\geq -\operatorname{div}^M\left(\rho^2 f|d\varphi|^{p-1}\operatorname{grad}^M |d\varphi|\right) \\ &\quad + [p-1-\epsilon_1(p-2)-\epsilon_2]\,\rho^2 f|d\varphi|^{p-2}|\operatorname{grad}^M |d\varphi||^2. \end{aligned} \tag{5.20}$$

By using the divergence Theorem, with $\epsilon_1 = 1$ and $\epsilon_2 = \frac{1}{2}$, we deduce

$$p\int_M f|d\varphi|^p|\operatorname{grad}^M \rho|^2 v_g \geq \frac{1}{2}\int_M \rho^2 f|d\varphi|^{p-2}|\operatorname{grad}^M |d\varphi||^2 v_g. \tag{5.21}$$

Choose the smooth cut-off $\rho = \rho_R$ on M, i.e. $\rho \leq 1$ on M, $\rho = 1$ on the geodesic ball $B(x, R)$, $\rho = 0$ on $M\backslash B(x, 2R)$ and $|\operatorname{grad}^M \rho| \leq \frac{2}{R}$, where $x \in M$. Replacing $\rho = \rho_R$ in (5.21), we obtain

$$\frac{4p}{R^2}\int_{B(x,2R)} f|d\varphi|^p v_g \geq \frac{1}{2}\int_{B(x,R)} f|d\varphi|^{p-2}|\operatorname{grad}^M |d\varphi||^2 v_g.$$

Since $\int_M f\,|d\varphi|^p\,v_g < \infty$, when $R \to \infty$, we have

$$\int_M f|d\varphi|^{p-2}|\operatorname{grad}^M |d\varphi||^2 = 0.$$

Thus, if $|d\varphi| \neq 0$ on M, we have $|\operatorname{grad}^M |d\varphi|| = 0$, i.e. $|d\varphi|$ is a positive constant on M. So that

$$E_{p,f}(\varphi) = \frac{|d\varphi|^p}{p}\int_M f v_g < \infty.$$

But $\int_M f\,v_g = \infty$. Hence φ is constant on M. □

From Theorem 5.1.3, we deduce:

Corollary 5.1.1 ([30, 32])**.** *Let (M, g) be a complete non-compact Riemannian manifold with positive Ricci curvature, (N, h) be Riemannian manifold with non-positive sectional curvature. If* $\operatorname{Vol}(M)$ *is infinite, then any p-harmonic map of $E_p(\varphi) < \infty$ is constant.*

5.1.3 Stress (p, f)-energy tensor

Let $\varphi : (M, g) \longrightarrow (N, h)$ be a smooth map between two Riemannian manifolds, and $f \in C^\infty(M)$ be a smooth positif function. Consider a smooth one-parameter variation of the metric g, i.e. is a smooth family of metrics $\{g_t\}_{(-\epsilon<t<\epsilon)}$, such that $g_0 = g$. Write

$\delta = \frac{\partial}{\partial t}\Big|_{t=0}$, then $\delta g \in T^*M \odot T^*M$ is a symmetric 2-covariant tensor field on M. Let $\langle , \rangle$ the induced Riemannian metric on $T^*M \otimes T^*M$, we have

$$\delta(v_{g_t}) = \frac{1}{2}\langle g, \delta g\rangle v_g, \quad \delta(\frac{|d\varphi|^p}{p}) = -\frac{1}{2}|d\varphi|^{p-2}\langle \varphi^* h, \delta g\rangle, \quad (p \geq 2) \tag{5.22}$$

where $\varphi^* h$ is the pull-back of the metric h (see [1]).

Theorem 5.1.4. *Under the notation above we have the following*

$$\frac{d}{dt}E_{p,f}(\varphi; D)\Big|_{t=0} = \frac{1}{2}\int_D \langle S_{p,f}(\varphi), \delta g\rangle v_g,$$

where D is a compact subset of M, and $S_{p,f}(\varphi) \in T^*M \odot T^*M$ is given by

$$S_{p,f}(\varphi) = \frac{f}{p}|d\varphi|^p g - f|d\varphi|^{p-2}\varphi^* h.$$

$S_{p,f}(\varphi)$ is called the stress (p,f)-energy tensor of φ.

Proof. Follows immediately from equations (5.22). □

From Theorem 5.1.4, we deduce:

Theorem 5.1.5. *[41] A non-constant smooth map $\varphi : (M, g) \longrightarrow (N, h)$ is extremal with respect to variations of the metric for (p, f)-energy functional if and only if $\dim M = p$ and φ is weakly conformal.*

Proof. If $S_{p,f}(\varphi) = 0$, taking the trace shows that $\dim M = p$, then comparing with $\varphi^* h = \lambda^2 g$ (where λ is a smooth function on M), shows that φ is weakly conformal, with $\lambda = \frac{|d\varphi|^2}{p}$. □

Theorem 5.1.6. *Let $\varphi : (M, g) \longrightarrow (N, h)$ be a smooth map between Riemannian manifolds, f a smooth positive function in M, and $p \geq 2$. We have*

$$\operatorname{div}^M S_{p,f}(\varphi) = -h(\tau_{p,f}(\varphi), d\varphi) + \frac{|d\varphi|^p}{p}df.$$

Proof. Let $\{e_i\}$ be an orthonormal frame with respect to g on M, such that $\nabla^M_{e_j} e_i = 0$, at $x \in M$ for all $i, j = 1, ..., m$. We compute

$$\begin{aligned}
\left[\operatorname{div}^M S_{p,f}(\varphi)\right](e_j) &= e_i\left[\frac{f}{p}|d\varphi|^p\delta_{ij} - f|d\varphi|^{p-2}h(d\varphi(e_i), d\varphi(e_j))\right] \\
&= \frac{1}{p}e_i(f)|d\varphi|^p\delta_{ij} + \frac{f}{p}e_i(|d\varphi|^p)\delta_{ij} - e_i(f)|d\varphi|^{p-2}h(d\varphi(e_i), d\varphi(e_j)) \\
&\quad - fe_i(|d\varphi|^{p-2})h(d\varphi(e_i), d\varphi(e_j)) - f|d\varphi|^{p-2}h(\nabla^\varphi_{e_i}d\varphi(e_i), d\varphi(e_j))
\end{aligned}$$

$$-f|d\varphi|^{p-2}h(d\varphi(e_i),\nabla^{\varphi}_{e_i}d\varphi(e_j)). \tag{5.23}$$

By the definitions of gradient and $\tau(\varphi)$, with $\nabla^{\varphi}_{e_i}d\varphi(e_j)=\nabla^{\varphi}_{e_j}d\varphi(e_i)$ at x, we get the following

$$\begin{aligned}\left[\operatorname{div}^M S_{p,f}(\varphi)\right](e_j) &= \frac{|d\varphi|^p}{p}g(\operatorname{grad}^M f,e_j)-|d\varphi|^{p-2}h(d\varphi(\operatorname{grad}^M f),d\varphi(e_j))\\ &\quad -(p-2)f|d\varphi|^{p-3}h(d\varphi(\operatorname{grad}^M|d\varphi|),d\varphi(e_j))\\ &\quad -f|d\varphi|^{p-2}h(\tau(\varphi),d\varphi(e_j)).\end{aligned} \tag{5.24}$$

The Theorem 5.1.6 follows from (5.24), and the definition of $\tau_{p,f}(\varphi)$. □

5.1.4 Homothetic vector fields and (p,f)-harmonic maps

A vector field ξ on a Riemannian manifold (M,g) is called a homothetic if $\mathcal{L}_\xi g=2kg$, for some constant k, where $\mathcal{L}_\xi g$ is the Lie derivative of the metric g with respect to ξ, that is:

$$g(\nabla^M_X\xi,Y)+g(\nabla^M_Y\xi,X)=2kg(X,Y),\quad X,Y\in\Gamma(TM). \tag{5.25}$$

If ξ is homothetic, while $k=0$ it is Killing (see [1], [20], [43]).
In the seminal work [29], where we proved that, if (M,g) is a compact Riemannian manifold without boundary, (N,h) is a Riemannian manifold, $\varphi:(M,g)\to(N,h)$ a harmonic map, assume that there is a proper homothetic vector field ξ on (N,h), that is $\mathcal{L}_\xi h=2kh$, for some constant $k\in\mathbb{R}^*$. Then φ is a constant map. We obtain the following results.

Theorem 5.1.7. *[41] Let (M,g) be a complete orientable Riemannian manifold, (N,h) a Riemannian manifold admitting a homothetic vector field ξ with homothetic constant $k\neq 0$, and f a smooth positive function on M. If $\varphi:(M,g)\longrightarrow(N,h)$ is (p,f)-harmonic map, satisfying*

$$\int_M f|d\varphi|^{p-2}|\xi\circ\varphi|^2v^g<\infty.$$

Then φ is constant.

Proof. Let ρ be a smooth function with compact support on M, we set

$$\omega(X)=h(\xi\circ\varphi,\rho^2f|d\varphi|^{p-2}d\varphi(X)),\quad \forall X\in\Gamma(TM),$$

and let $\{e_i\}$ be a normal orthonormal frame at $x\in M$, we have

$$\operatorname{div}^M\omega = e_i\left[h(\xi\circ\varphi,\rho^2f|d\varphi|^{p-2}d\varphi(e_i))\right]. \tag{5.26}$$

By equation (5.26), and (p,f)-harmonicity condition of φ, we get

$$\operatorname{div}^M\omega = \rho^2f|d\varphi|^{p-2}h(\nabla^{\varphi}_{e_i}(\xi\circ\varphi),d\varphi(e_i))+2\rho e_i(\rho)f|d\varphi|^{p-2}h(\xi\circ\varphi,d\varphi(e_i)).$$

Since ξ is a homothetic vector field with homothetic constant k, we find that

$$\operatorname{div}^M \omega \quad = \quad k\rho^2 f|d\varphi|^{p-2}h(d\varphi(e_i), d\varphi(e_i)) + 2\rho e_i(\rho) f|d\varphi|^{p-2}h(\xi \circ \varphi, d\varphi(e_i)),$$

is equivalent to the following equation

$$\operatorname{div}^M \omega \quad = \quad k\rho^2 f|d\varphi|^p + 2\rho e_i(\rho) f|d\varphi|^{p-2}h(\xi \circ \varphi, d\varphi(e_i)). \tag{5.27}$$

By the Young's inequality, we have

$$-2\rho e_i(\rho)h(\xi \circ \varphi, d\varphi(e_i)) \leq \epsilon\rho^2|d\varphi|^2 + \frac{1}{\epsilon}e_i(\rho)^2|\xi \circ \varphi|^2,$$

for all $\epsilon > 0$. Multiplying the last inequality by $f|d\varphi|^{p-2}$, we get

$$-2f|d\varphi|^{p-2}\rho e_i(\rho)h(\xi \circ \varphi, d\varphi(e_i)) \leq \epsilon f\rho^2|d\varphi|^p + \frac{1}{\epsilon}f|d\varphi|^{p-2}e_i(\rho)^2|\xi \circ \varphi|^2, \tag{5.28}$$

from (5.27), (5.28), we deduce the following inequality

$$k\rho^2 f|d\varphi|^p - \operatorname{div}^M \omega \quad \leq \quad \epsilon f\rho^2|d\varphi|^p + \frac{1}{\epsilon}f|d\varphi|^{p-2}e_i(\rho)^2|\xi \circ \varphi|^2, \tag{5.29}$$

We assume that $k > 0$, and we set $\epsilon = \frac{k}{2}$. By (5.29), we have

$$\frac{k}{2}\rho^2 f|d\varphi|^p - \operatorname{div}^M \omega \quad \leq \quad \frac{2}{k}f|d\varphi|^{p-2}e_i(\rho)^2|\xi \circ \varphi|^2. \tag{5.30}$$

From (5.30), and the divergence Theorem, we have

$$\frac{k}{2}\int_M \rho^2 f|d\varphi|^p v_g \leq \frac{2}{k}\int_M f|d\varphi|^{p-2}e_i(\rho)^2|\xi \circ \varphi|^2 v_g. \tag{5.31}$$

Now, consider the cut-off smooth function $\rho = \rho_R$ such that $0 \leq \rho \leq 1$ on M, $\rho = 1$ on the geodesic ball $B(x, R)$, $\rho = 0$ on $M \setminus B(x, 2R)$ and $|\operatorname{grad}^M \rho| \leq \frac{2}{R}$, from (5.31) we get

$$\frac{k}{2}\int_{B(x,R)} f|d\varphi|^p v_g \leq \frac{8}{kR^2}\int_{B(x,2R)} f|d\varphi|^{p-2}|\xi \circ \varphi|^2 v^g, \tag{5.32}$$

since $\int_M f|d\varphi|^{p-2}|\xi \circ \varphi|^2 v^g < \infty$, when $R \to \infty$ we obtain:

$$\int_M f|d\varphi|^p v_g \quad = \quad 0. \tag{5.33}$$

Consequently, $|d\varphi| = 0$ that is φ is constant (if $k < 0$, consider the homothetic vector field $\bar{\xi} = -\xi$). □

Corollary 5.1.2. *Let (M, g) be a compact orientable Riemannian manifold without boundary, (N, h) a Riemannian manifold admitting a homothetic vector field ξ with homothetic constant $k \neq 0$, f a smooth positive function on M, and $p \geq 2$. Then, any (p, f)-harmonic map φ from (M, g) to (N, h) is constant.*

Example 5.1.2. Let $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ the Torus. We note that the circle $\mathbb{S}^1$ is compact orientable manifold of dimension 1, and without boundary because $\partial\mathbb{S}^1 = \partial(\partial\mathbb{D}^2) = \emptyset$ where $\mathbb{D}^2$ is the unit disk in $\mathbb{R}^2$. So that the product manifold $\mathbb{S}^1 \times \mathbb{S}^1$ is also compact, without boundary, orientable manifold of dimension 2. In [47], the authors proved that the non-constant map

$$(\mathbb{T}^2, dx_1^2 + dx_2^2) \longrightarrow (\mathbb{S}^2, dy_1^2 + \sin^2 y_1 dy_2^2), \quad (x_1, x_2) \longmapsto (\pi/2, mx_1 + nx_2 + l)$$

is harmonic, where $m, n, l \in \mathbb{R}$. One can verify by direct computations that

$$\varphi : (\mathbb{T}^2, dx_1^2 + dx_2^2) \longrightarrow (\mathbb{S}^2, dy_1^2 + \sin^2 y_1 dy_2^2), \quad (x_1, x_2) \longmapsto (ax_1 + c_1, bx_2 + c_2)$$

is (p, f)-harmonic for all $p \geq 2$, with $f(x_1, x_2) = \delta\, \mathrm{e}^{-\frac{b^2 \cos(2\,ax_1 + 2\,c_1)}{4a^2}}$, where b, c_1, c_2, $\delta \in \mathbb{R}$, and $a \in \mathbb{R}^*$. Thus, the condition of existence of the homothetic vector field with non-zero constant homothetic is necessary to verify the previous Corollary.

Bibliography

[1] P. Baird, J. C. Wood, *Harmonic morphisms between Riemannain manifolds*, Clarendon Press Oxford 2003.

[2] B. Bojarski, and T. Iwaniec, p-Harmonic equation and quasiregular mappings, Partial differential equations (Warsaw, 1984), 25-38, Banach Center Publ., vol. 19. PWN, Warsaw, 1987.

[3] P. Baird, A. Fardoun and S. Ouakkas,*Conformal and semi-conformal biharmonic maps*, Ann Glob Anal Geom (2008) 34, 403-414

[4] P.Buser,Géométrie Riemannienne,2003/2004.

[5] P. Baird, S. Gudmundsson, p-Harmonic maps and minimal submanifolds, Math. Ann. 294 (1992), 611-624.

[6] P. Bérard, *A note on Bochner type theorems for complete manifolds*, Manuscripta Math. **69** (1990) 261-266.

[7] P. Baird, A. Fardoun and S. Ouakkas, *Liouville-type Theorems for Biharmonic Maps between Riemannian Manifolds,* Advances in Calculus of Variations. **3**, Issue 1 (2009), 49-68.

[8] R. Caddeo, S. Montaldo, C. Oniciuc, *Biharmonic submanifolds of* $\mathbb{S}^3$, Int. J. Math., **12** (2001), 867-876.

[9] N. Course, *f-harmonic maps which map the boundary of the domain to one point in the target*, New York Journal of Mathematics, **13**, (2007), 423-435.

[10] S. Y. Cheng, *Liouville Theorem for Harmonic Maps,* Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, (1979), 147-151, Proc.Sympos. Pure Math., XXXVI, Amer. Math. Soc., Providence, R.I., 1980.

[11] M. Djaa, A.M. Cherif, K. Zagga, S. Ouakkas, *On the generalized of harmonic and bi-harmonic maps*, Int. Electron. J. Geom. 5 **1**, (2012), 90-100.

[12] M. Djaa and A. Mohammed Cherif, *On generalized f-harmonic maps and liouville type theorem*, Konuralp Journal of Mathematics, Volume 4 No. 1 pp. 33-44 (2016)

[13] J. Eells and L. Lemaire, *A report on harmonic maps*, Bull. London Math.Soc. **16** (1978), 1-68.

[14] J. Eells and J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86** (1964), 109-160.

[15] S. Y. Cheng, *Liouville Theorem for Harmonic Maps,* Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, (1979), 147-151, Proc.Sympos. Pure Math., XXXVI, Amer. Math. Soc., Providence, R.I., 1980.

[16] A. Fardoun, On equivariant p-harmonic maps, Ann.Inst. Henri. Poincaré, 15 (1998), 25-72.

[17] S. Gudmundsson, *An Introduction to Riemannian Geometry, Lund University*, Decembre 2001.

[18] R. Howard and S. W. Wei, *Nonexistence of stable harmonic maps to and from certain homogeneous spaces and submanifolds of Euclidean space*, Trans. Amer. Math. Soc., Vol. 294 (1986), 319-331.

[19] S. Kobayashi, *A theorem on the affine transformation group of a Riemannian manifold*, Nagoya Math. J. 9 (1955), 39-41.

[20] W. Kühnel and H. Rademacher, *Conformal transformations of pseudo-Riemannian manifolds, Differential Geom*. Appl. 7 (1997), 237-250.

[21] J. Liu, *Liouville-type Theorems of p-harmonic Maps with free Boundary Values,* Hiroshima Math.**40** (2010), 333-342

[22] P. F. Leung,*On the stability of harmonic maps*, SpringerVerlag Lecture Notes Math. 949 (1982), 122-129.

[23] R. M. Schoen, and S. T. Yau, *Harmonic Maps and the Topology of Stable Hypersurfaces and Manifolds with Non-negative Ricci Curvature,* Comment. Math. Helv. **51** (1976), no.3, 333-341.

[24] A. Lichnerwicz, *Applications harmoniques et variétés khaleriennes*, Symposia Mathematica,vol.III,Academic Press, London,1968-1969,pp.341-402

[25] A. Mohammed Cherif, Géometrie semi-Riemannienne, Notes de cours, Université Mustapha Stambouli-Mascara, 2015.

[26] A. Mohammed. Cherif, M. Djaa, K. Zegga, *Stable f-harmonic maps on sphere*, Commun. Korean Math Soc, **30** **4** (2015) 471-479.

[27] A. Mohammed Cherif, *Some results on harmonic and bi-harmonic maps*, International Journal of Geometric Methods in Modern Physics, Vol. 14, No. 7 (2017).

[28] A. Mohammed Cherif, H. Elhendi and M. Terbeche ,*On Generalized Conformal Maps*, Bulletin of Mathematical Analysis and Applications, 4 Issue 4 (2012), 99-108.

[29] A. Mohammed Cherif and M. Djaa,*Geometry of energy and bienergy variations between Riemannian Manifolds*, Kyungpook Math. J., 55 (2015), 715-730.

[30] D. J. Moon, H. Liu, S. D. Jung, Liouville type theorems for p-harmonic maps, J. Math. Anal. Appl. 342 (2008) 354-360

[31] Y, Ohnita, *Stability of harmonic maps and standard minimal immersions*, Tohoku Math. J., Vol. 38(1986),

[32] N. Nakauchi, A Liouville type theorem for p-harmonic maps , Osaka J. Math. 35 (1998) 303U312 259-267.

[33] S. Ouakkas, R. Nasri and M. Djaa, *On the f-harmonic and f-biharmonic maps*, J. P. Journal. of Geom. and Top., V. **10**, No. **1**, 2010, 11-27.

[34] S. Pigola, M. Rimoldi and A. G. Setti, *Remarks on non-compact gradient Ricci solitons*, Math. Z. 268 (2011) 777-790.

[35] P.Petersen, *Riemannian Geometry*, Second Edition, Mathematics Subject Classification (2000) :53-01.

[36] P. Pansu . Géométrie différentielle, Cours DEA, Laboratoire de Mathéematique d'Orsay . 27 october 2005

[37] O'Neil, *Semi- Riemannian Geometry*, Academic Press, New York, 1983.

[38] M. Rimoldi and G. Veronelli, *f-Harmonic Maps and Applications to Gradient Ricci Solitons*, arXiv:1112.3637, (2011).

[39] E.Remli. and A. M. Cherif,*Some Result on stable f-harmonic maps* ,Commun. Korean Math. Soc. 33 (2018), No. 3, pp. 935-942

[40] E.Remli. and A. M. Cherif,*SOME RESULTS ON f-HARMONIC MAPS AND f-BIHARMONIC SUBMANIFOLDS* ,Acta Math. Univ. Comenianae Vol. LXXXIX, 2 (2020), pp. 299-307

[41] E.Remli. and A. M. Cherif,*On the generalized of p-harmonic and f-harmonic maps*, Preprint in Kyungpook Mathematical Journal.

[42] S. T. Yau, *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math. 28 (1975), 201-228.

[43] K. Yano and T. Nagano, *The de Rham decomposition, isometries and affine transformations in Riemannian space*, Japan. J. Math. 29 (1959), 173-184.

[44] Y. L. Xin, *Geometry of Harmonic Maps*, Birkhäuser Boston, Progress in Nonlinear Differential Equations and Their Aplications, 1996, pp x, 241.

[45] Y.L. Xin, *Some results on stable harmonic maps*, Duke Math. J., Vol. 47 (1980), 609-613.

[46] D. Xu Wang, *Harmonic Maps from Smooth Metric Measure Spaces,* Internat. J. Math. **23** (2012), no. 9, 1250095, 21.

[47] Z. P. Wang, Y. L. Ou, and H.C. Yang, Biharmonic maps from tori into a 2-sphere, Chin. Ann. Math. Ser. B 39(5) (2018), 39-861.

[48] W.C. Young, *On the multiplication of successions of Fourier constants*, Proc. Royal Soc. Lond. 87 (1912), 331-339.

[49] K. Zegga, A. Mohammed Cherif and M. Djaa, *On the f-biharmonic Maps and Submanifolds*, Kyungpook Math. J. 55 (2015), 157-168.