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GENERALIZED ROUMIEU ULTRADISTRIBUTIONS AND THEIR MICROLOCAL ANALYSIS

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Abstract. In this paper we introduce a new algebra of generalized functions containing Roumieu ultradistributions and their microlocal analysis suitable for them.

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1. Introduction

The theory of generalized functions as a positive answer to the question of product distributions [16], caused a very important area of research [4, 5, 9, 10] and [13], this theory has been developed and applied in linear and nonlinear partial differential equations with non-smooth coefficients and distributions data by several authors [8], [10] and [13].

Ultradistributions are useful in applications in quantum field theory, partial differential equations, convolution equations, harmonic analysis, pseudo-differential theory, time-frequency analysis, and other areas of analysis, see [12] and [15], so it is necessary to develop a generalized functions type theory in connection with ultradistributions.

Generalized Gevrey ultradistributions of Colombeau type have been defined, but as a side-theme, in the paper [8]. The first paper aiming to construct differential algebras containing ultradistributions is [14]. Let us also mention the interesting approach of the paper [6] to algebras of generalized ultradistributions. However, a Colombeau type theory of generalized Gevrey ultradistributions has been addressed in [3], where was developed the core of a full theory and also introduced a new way of defining differential algebras of generalized Gevrey ultradistributions that makes such a complete theory possible. But, it was not clear in that paper why different Gevrey exponents occurred in the embedding of the spaces of Gevrey ultradistributions. In [2], the authors gave a general construction of algebras of generalized Gevrey ultradistributions and then the microlocal analysis suitable for them. It also highlights the explicit contribution of the mollification in the embedding of ultradistributions into algebras of generalized functions of Colombeau type. In [1] the

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authors introduced new algebras of generalized functions containing Roumieu ultradistributions.

The aim of this paper is to develop a microlocal analysis suitable for our algebras defined in [1] by introducing a notion of generalized regularity which coincides with ultradifferentiability.

2. Roumieu ultradistribution

Let $(M_p)_{p\in\mathbb{Z}_+}$ be a sequence of real positive numbers, recall the following properties.

$(H1)$ Logarithmic convexity:

$$M_p^2 \le M_{p-1}M_{p+1}, \quad \forall p \ge 1$$

$(H2)$ Stability under ultradifferentiation:

$$\exists A>0, \exists H>0, M_{p+q} \le AH^{p+q}M_pM_q, \forall p\ge 0, \forall q \ge 0.$$

$(H2)$' Stability under differentiation:

$$\exists A>0, \exists H>0, M_{p+1} \le AH^{p}M_p, \forall p\ge 0$$

$(H3)$' Non-quasi-analyticity:

$$\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < +\infty$$

The associated function of the sequence $(M_p)_{p\in\mathbb{Z}_+}$ is the function defined by

$$M(t) = \underset{p}{sup} \ln \frac{t^p}{M_p}, t \in \mathbb{R}_+^*$$

Proposition 2.1. *A sequence $(M_p)_{p\in\mathbb{Z}_+}$ of positive numbers satisfies condition (H1) if and only if*

$$M_p = M_0 \sup_{t>0}[t^p \exp(-M(t))],\ p \in \mathbb{Z}_+$$

Proposition 2.2. *Let the sequence $(M_p)_{p\in\mathbb{Z}_+}$ satisfy condition $(H1)$, then it satisfies $(H2)$ if and only if $\exists A>0$, $\exists H>0$, $\forall t>0$,*

$$2M(t) \le M(Ht) + \ln(AM_0).$$

The class of ultradifferentiable functions of class M, denoted $E^M(\Omega)$, is the space of all $f \in C^\infty(\Omega)$ satisfying for every compact subset K of Ω, $\exists c>0$, $\forall \alpha \in \mathbb{Z}_+^n$,

$$\sup_{x\in K} |\partial^\alpha f(x)| \le c^{|\alpha|+1} M_{|\alpha|} \tag{2.1}$$

This space is also called the space of Donjoy-Carleman.

A differential operator of infinite order $P(D) = \sum_{\gamma\in\mathbb{Z}_+^n} a_\gamma D^\gamma$ is called an ultradifferential operator of class $(M_p)_{p\in\mathbb{Z}_+}$, if for every $h > 0$ there exist $c > 0$ such that $\forall\gamma \in \mathbb{Z}_+^n$,

$$|a_\gamma| \leq c\frac{k^{|\gamma|}}{M_{|\gamma|}} \tag{2.2}$$

The basic properties of the space $E^M(\Omega)$ are summarized in the following proposition.

Proposition 2.3. *Let the sequence $(M_p)_{p\in\mathbb{Z}_+}$ satisfy condition $(H1)$, then the space $E^M(\Omega)$ is an algebra moreover, if $(M_p)_{p\in\mathbb{Z}_+}$ satisfies $(H2)'$, then $E^M(\Omega)$ is stable by differential operators of finite order with coefficients in $E^M(\Omega)$, and if $(M_p)_{p\in\mathbb{Z}_+}$ satisfies $(H2)$ then any ultradifferential operator of class M operates also as a sheaf homomorphism.*

The space $\mathcal{D}^M(\Omega) = E^M(\Omega) \cap \mathcal{D}(\Omega)$ is not trivial if and only if the sequence $(M_p)_{p\in\mathbb{Z}_+}$ satisfies $(H3)'$.

Definition 2.4. The strong dual of $\mathcal{D}^M(\Omega)$, denoted $\mathcal{D}'^M(\Omega)$, is called the space of Roumieu ultradistributions.

3. Generalized Roumieu ultradistributions

To consider the algebra of generalized Roumieu ultradistributions, we first introduce the algebra of moderate elements and its ideal of null elements. Let Ω be a non void open set of $\mathbb{R}^n$ and $I =]0, 1]$.

We will always suppose that the sequence $(M_p)_{p\in\mathbb{Z}_+}$ satisfies the conditions $(H1)$, $(H2)$, $(H3)'$ and $M_0 = 1$.

Definition 3.1. The space of moderate elements, denoted $\mathcal{E}_m^M(\Omega)$, is the space of $(f_\varepsilon)_\varepsilon \in C^\infty(\Omega)^I$ satisfying for every compact K of Ω, $\forall\alpha \in \mathbb{Z}_+^n$, $\exists k > 0$, $\exists c > 0$,$\exists\varepsilon_0 \in I$, $\forall\varepsilon \leq \varepsilon_0$,

$$\sup_{x\in K} |\partial^\alpha f_\varepsilon(x)| \leq c \exp(M(\frac{k}{\varepsilon})) \tag{3.1}$$

The space of null elements, denoted $\mathcal{N}^M(\Omega)$, is the space of $(f_\varepsilon)_\varepsilon \in C^\infty(\Omega)^I$ satisfying for every compact K of Ω, $\forall\alpha \in \mathbb{Z}_+^n$, $\forall k > 0$, $\exists c > 0$, $\exists\varepsilon_0 \in I$, $\forall\varepsilon \leq \varepsilon_0$,

$$\sup_{x\in K} |\partial^\alpha f_\varepsilon(x)| \leq c \exp(-M(\frac{k}{\varepsilon})) \tag{3.2}$$

The main properties of the spaces $\mathcal{E}_m^M(\Omega)$ and $\mathcal{N}^M(\Omega)$ are given in the following proposition.

Proposition 3.2. *1. The space of moderate elements $\mathcal{E}_m^M(\Omega)$ is an algebra stable by derivation.*

2. The space $\mathcal{N}^M(\Omega)$ is an ideal of $\mathcal{E}_m^M(\Omega)$.

Definition 3.3. The algebra of generalized Roumieu ultradistributions of class $(M_p)_{p\in\mathbb{Z}_+}$, denoted $\mathcal{G}^M(\Omega)$, is the quotient algebra

$$\mathcal{G}^M(\Omega) = \frac{\mathcal{E}_m^M(\Omega)}{\mathcal{N}^M(\Omega)}.$$

4. Embedding of Roumieu ultradistributions with compact support

Let $N = (N_p)_{p\in\mathbb{Z}_+}$ be a sequence satisfying the conditions $(H1)$, $(H2)$, $(H3)'$ and $N_0 = 1$, the space $\mathcal{S}^N(\mathbb{R}^n)$ is the space of functions $\varphi \in C^\infty(\mathbb{R}^n)$ such that $\forall b > 0$, we have

$$\|\varphi\|_{b,N} = \underset{\alpha,\beta\in\mathbb{Z}_+^n}{sup} \int \frac{|x|^{|\beta|}}{b^{\alpha+\beta} N_{|\alpha|} N_{|\beta|}} |\partial^\alpha \varphi(x)| \, dx < \infty \tag{4.1}$$

Define Σ^N as the set of functions $\phi \in \mathcal{S}^N(\mathbb{R}^n)$ satisfying

$$\int \phi(x)dx = 1 \ \textit{and} \int x^\alpha \phi(x)dx = 0, \quad \forall \alpha \in \mathbb{Z}_+^n \setminus \{0\}.$$

Definition 4.1. The net $\phi_\varepsilon = \varepsilon^{-n}\phi(./\varepsilon),\ \varepsilon \in I$, where $\phi \in \Sigma^N$ is called a $N-$ mollifier net.

Let $(L_p)_{p\in\mathbb{Z}_+}$ satisfying $(H1)$, $(H2)$, $(H3)'$,the space $\mathcal{E}^L(\Omega)$ is embedded into $\mathcal{G}^M(\Omega)$ by the standard canonical injection

$$\begin{array}{llll} I: & E^L(\Omega) & \to & \mathcal{G}^M(\Omega) \\ & f & \to & [f] = cl(f_\varepsilon) \end{array} \tag{4.2}$$

Where $f_\varepsilon = f$, $\forall \varepsilon \in I$.

And by [1] we have the following result gives the embedding of Roumieu ultradistributions into $\mathcal{G}^M(\Omega)$. Let M and N be two sequences satisfying $(H1)$, $(H2)$, $(H3)'$ with $M_0 = N_0 = 1$,$M_p > N_p$, $\forall p \in \mathbb{Z}^+$ and $\phi \in \Sigma^N$

Theorem 4.2. *The map*

$$\begin{array}{llll} J_0: & E'_{MN}(\Omega) & \to & \mathcal{G}^M(\Omega) \\ & T & \to & [T] = cl((T * \phi_\varepsilon)_{/\Omega}) \end{array} \tag{4.3}$$

is an embedding.

Notation 4.3. If $M = (M_p)_{p\in\mathbb{Z}_+}$ and $N = (N_p)_{p\in\mathbb{Z}_+}$ are two sequences, then $MN^{-1} := (M_p N_p^{-1})_{p\in\mathbb{Z}_+}$

In order to show the commutativity of the following diagram of embeddings

$$\begin{array}{ccc} \mathcal{D}^{MN^{-1}p!}(\Omega) & \rightarrow & \mathcal{G}^{M}(\Omega) \\ & \searrow & \uparrow \\ & & \mathcal{E}'_{MN}(\Omega) \end{array}$$

We have the following fundamental result [1].

Proposition 4.4. *Let $f \in \mathcal{D}^{MN^{-1}p!}(\Omega)$ and $\phi \in \Sigma^N$, then*

$$(f - (f * \phi_\varepsilon)_{/\Omega})_\varepsilon \in \mathcal{N}^M(\Omega).$$

5. Regular generalized Roumieu ultradistributions

Definition 5.1. The space of N-ultraregular moderate elements of class M, denoted $\mathcal{E}_m^{M,N,+\infty}(\Omega)$, is the space of $(f_\varepsilon)_\varepsilon \in C^\infty(\Omega)$ satisfying $\forall K \Subset \Omega,\ \exists k > 0,\ \exists c > 0,\ \exists \varepsilon_0 \in]0,1],\ \forall \alpha \in \mathbb{Z}_+^n$

$$\sup_{x \in K} |\partial^\alpha f_\varepsilon(x)| \leq c^{|\alpha|+1} N_{|\alpha|} \exp(M(\frac{k}{\varepsilon}))$$

The space of null elements is defined as $\mathcal{N}^{M,N,+\infty}(\Omega) := \mathcal{N}^M(\Omega) \cap \mathcal{E}_m^{M,N,+\infty}(\Omega)$.

The main properties of these two spaces are given in the following proposition.

Proposition 5.2.

1) *The space $\mathcal{E}_m^{M,N,+\infty}(\Omega)$ is an algebra stable by the action of N-ultradifferential operators.*

2) *The space $\mathcal{N}^{M,N,+\infty}(\Omega)$ is an ideal of $\mathcal{E}_m^{M,N,+\infty}(\Omega)$.*

Proof. 1) Let $(f_\varepsilon)_\varepsilon, (g_\varepsilon)_\varepsilon \in \mathcal{E}_m^{M,N,+\infty}(\Omega)$ and K be a compact subset of Ω, then $\exists k_1 > 0,\ \exists c_1 > 0,\ \exists \varepsilon_1 \in]0,1],\ \forall \alpha \in \mathbb{Z}_+^n,\ \forall \varepsilon \leq \varepsilon_1$,

$$\sup_{x \in K} |\partial^\alpha f_\varepsilon(x)| \leq c_1^{|\alpha|+1} N_{|\alpha|} \exp(M(\frac{k_1}{\varepsilon}))$$

We have also $\exists k_2 > 0,\ \exists c_2 > 0,\ \exists \varepsilon_2 \in]0,1],\ \forall \alpha \in \mathbb{Z}_+^n,\ \forall \varepsilon \leq \varepsilon_2$,

$$\sup_{x \in K} |\partial^\alpha g_\varepsilon(x)| \leq c_2^{|\alpha|+1} N_{|\alpha|} \exp(M(\frac{k_2}{\varepsilon}))$$

let $\alpha \in \mathbb{Z}_+^n,\ \lambda_1, \lambda_2 \in \mathbb{Z}_+^n$, it's clear that $\exists c = max(c_1, c_2),\ \exists k = (\lambda_1 + \lambda_2) max(k_1, k_2),\ \exists \varepsilon_0 = min(\varepsilon_1, \varepsilon_2)$ such that $\forall \varepsilon \leq \varepsilon_0$,

$$|\partial^\alpha(\lambda_1 f_\varepsilon(x) + \lambda_2 g_\varepsilon(x))| \leq c^{|\alpha|+1} N_{|\alpha|} \exp(M(\frac{k}{\varepsilon}))$$

So, $(\lambda_1 f_1 + \lambda_2 f_2) \in \mathcal{E}_m^{M,N,+\infty}(\Omega)$.
And we have

$$\begin{aligned}
&|\partial^\alpha (f_\varepsilon g_\varepsilon)(x)| \\
&\le \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} \left|\partial^{\alpha-\beta} f_\varepsilon(x)\right| . \left|\partial^\beta g_\varepsilon(x)\right| \\
&\le \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} c_1^{|\alpha-\beta|+1} . c_2^{|\beta|+1} . N_{|\alpha-\beta|} . N_{|\beta|} \exp(M(\frac{k_1}{\varepsilon}) + M(\frac{k_2}{\varepsilon}))
\end{aligned}$$

then $\exists A > 0,\ \exists H > 0, \forall t > 0$

$$2M(t) \le M(Ht) + ln(A).$$

$t = \frac{1}{\varepsilon} max(k_1, k_2) = \frac{k}{\varepsilon}$, $C = max(c_1, c_2)$.

$$\begin{aligned}
|\partial^\alpha (f_\varepsilon . g_\varepsilon)(x)| &\le \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} . A . C^{|\alpha|+1} N_{|\alpha|} . \exp(M(\tfrac{Hk}{\varepsilon})) \\
&\le C^{|\alpha|+1} . N_{|\alpha|} . \exp(M(\tfrac{k}{\varepsilon}))
\end{aligned}$$

Then $(f_\varepsilon . g_\varepsilon)_\varepsilon \in \mathcal{E}_m^{M,N,\infty}(\Omega)$.
Let now $P(D) = \Sigma a_\gamma D^\gamma$ be an N-ultradifferential operator, then $\forall h > 0$, $\exists b > 0$, such that

$$\begin{aligned}
&\frac{\exp(-M(\frac{k_1}{\varepsilon}))}{N_{|\alpha|}} |\partial^\alpha (P(D) f_\varepsilon(x))| \\
&\le \exp(-M(\frac{k_1}{\varepsilon})) \sum_{\gamma \in \mathbb{Z}_+^n} b \frac{h^{|\gamma|}}{N_{|\gamma|} . N_{|\alpha|}} \left|\partial^{\alpha+\gamma} f_\varepsilon(x)\right| \\
&\le b \exp(-M(\frac{k_1}{\varepsilon})) \sum_{\gamma \in \mathbb{Z}_+^n} \frac{A(H)^{|\alpha+\gamma|} h^{|\gamma|}}{N_{|\alpha+\gamma|}} \left|\partial^{\alpha+\gamma} f_\varepsilon(x)\right| \\
&\le b \sum_{\gamma \in \mathbb{Z}_+^n} A(H)^{|\alpha+\gamma|} h^{|\gamma|}
\end{aligned}$$

hence, for $Hh < \frac{1}{2}$ we have

$$\exp(-M(\frac{k_1}{\varepsilon})) \frac{1}{N_{|\alpha|}} |\partial^\alpha (P(D) f_\varepsilon(x))| \le c' H^{|\alpha|}$$

which shows that $(P(D) f_\varepsilon)_\varepsilon \in \mathcal{E}_m^{M,N,\infty}(\Omega)$

2) The fact that $\mathcal{N}^{M,N,\infty}(\Omega) = \mathcal{N}^M(\Omega) \cap \mathcal{E}_m^{M,N,\infty}(\Omega) \subset \mathcal{E}_m^{M,N,\infty}(\Omega)$, and that $\mathcal{N}^M(\Omega)$ is an ideal of $\mathcal{E}_m^M(\Omega)$, imply that $\mathcal{N}^{M,N,\infty}$ is an ideal of $\mathcal{E}_m^{M,N,\infty}(\Omega)$

□

Definition 5.3. The algebra of $N-$ultraregular generalized functions of class $M = (M_p)_{p\in\mathbb{Z}_+}$, denoted $\mathcal{G}_M^{N,\infty}(\Omega)$, is the quotient algebra

$$\mathcal{G}_N^{M,\infty}(\Omega) = \frac{\mathcal{E}_m^{M,N,\infty}(\Omega)}{\mathcal{N}^{M,N,\infty}(\Omega)}$$

The basic properties of $\mathcal{G}_N^{M,\infty}(\Omega)$ are given by the following result.

Proposition 5.4. *The space $\mathcal{G}_N^{M,\infty}(\Omega)$ is a sheaf subalgebra of $\mathcal{G}^M(\Omega)$.*

This motivates the following definition.

Definition 5.5. We define the $\mathcal{G}_N^{M,\infty}-$singular support of a generalized ultradistribution $f \in \mathcal{G}^M(\Omega)$, denoted by $N - singsupp_g(f)$ as the complement of the largest open set Ω' such that $f \in \mathcal{G}_N^{M,\infty}(\Omega')$

The following result is Paley-Wiener type characterization of $\mathcal{G}_N^{M,\infty}(\Omega)$.

Proposition 5.6. *Let $f = cl(f_\varepsilon)_\varepsilon \in \mathcal{G}_c^M(\Omega)$, then f is $N-$ultraregular if and only if $\exists k_1 > 0$, $\exists k_2 > 0$, $\exists c > 0$, $\exists \varepsilon_1 > 0$, $\forall \varepsilon \leq \varepsilon_1$, such that*

$$|\mathcal{F}(f_\varepsilon)(\xi)| \leq c\exp(M(\frac{k_1}{\varepsilon}) - N(k_2\,|\xi|)), \qquad \forall \xi \in \mathbb{R}^n. \tag{5.1}$$

Proof. Suppose that $f = cl(f_\varepsilon) \in \mathcal{G}_c^M(\Omega) \cap \mathcal{G}_N^{M,\infty}(\Omega)$ then $\exists k_1 > 0$, $\exists c > 0$, $\exists \varepsilon_1 > 0$, $\forall \varepsilon \leq \varepsilon_1$, $\forall \alpha \in \mathbb{Z}_+^n$,

$$|\partial^\alpha f_\varepsilon(x)| \leq c^{|\alpha|+1}.N_{|\alpha|}.\exp(M(\frac{k_1}{\varepsilon}))$$

Consequently, we have $\forall \xi \in \mathbb{R}^n$ $\forall \alpha \in \mathbb{Z}_+^n$,

$$|\xi^\alpha|\,.\,|\mathcal{F}(f_\varepsilon)(\xi)| \leq \left|\int_K \exp(-ix\xi)\partial^\alpha f_\varepsilon(x)dx\right|.$$

Then

$$|\xi^\alpha|\,.\,|\mathcal{F}(f_\varepsilon)(\xi)| \leq mes(K)c^{|\alpha|+1}.N_{|\alpha|}.\exp(M(\frac{k}{\varepsilon}))$$

$$\begin{aligned}
|\mathcal{F}(f_\varepsilon)(\xi)| &\leq c^{|\alpha|+1}.mes(K).\frac{N_{|\alpha|}}{|\xi|^{|\alpha|}}.\exp(M(\frac{k}{\varepsilon})) \\
&\leq c.mes(K).\inf_\alpha(\frac{c^{|\alpha|}N_{|\alpha|}}{|\xi|^{|\alpha|}}).\exp(M(\frac{k}{\varepsilon})) \\
&\leq c.mes(K).\frac{1}{\sup_\alpha(\frac{|\xi|^{|\alpha|}}{c^{|\alpha|}N_{|\alpha|}})}.\exp(M(\frac{k}{\varepsilon})) \\
&\leq c.mes(K)\frac{1}{\exp(\ln(\sup_\alpha(\frac{|\xi|^{|\alpha|}}{c^{|\alpha|}N_{|\alpha|}})))}.\exp(M(\frac{k}{\varepsilon}))
\end{aligned}$$

Take $k_2 = \frac{1}{c}$, $C = c.mes(K)$, $\forall\varepsilon \leq \varepsilon_0$

$$|\mathcal{F}(f_\varepsilon)(\xi)| \leq c\exp(-N(k_2 |\xi|)).\exp(M(\frac{k_1}{\varepsilon}))$$

So we have (5.1).

Suppose now that (5.1) is valid. Then $\forall\varepsilon \leq \varepsilon_0$,

$$\begin{aligned}
&|\partial^\alpha f_\varepsilon(x)| \\
&\leq c\left|\int_{\mathbb{R}^n} \exp(ix\xi)\xi^\alpha \mathcal{F}(f_\varepsilon)(\xi)d\xi\right| \\
&\leq c\exp(M(\frac{k_1}{\varepsilon}))\int_{\mathbb{R}^n} |\xi^\alpha|.\exp(-N(k_2|\xi|))dx \\
&\leq c\exp(M(\frac{k_1}{\varepsilon}))\sup_{|\xi|}(|\xi^\alpha|\exp(-N(k_2|\xi|))) \\
&\leq C^{|\alpha|+1}.N_{|\alpha|}.\exp(M(\frac{k_1}{\varepsilon})),
\end{aligned}$$

with $C = man(c, \frac{1}{k_2})$, i.e. $f_\varepsilon \in \mathcal{G}_N^{M,\infty}(\Omega)$. □

Remark 5.7. Let $f = cl(f_\varepsilon) \in \mathcal{G}_c^M(\Omega)$, then $\exists k_1 > 0$, $\exists c > 0$, $\exists\varepsilon_0 > 0$, $\forall k_2 > 0$, $\forall\varepsilon \leq \varepsilon_0$,

$$|\mathcal{F}(f_\varepsilon)(\xi)| \leq c\exp(M(\frac{k_1}{\varepsilon}) + N(k_2|\xi|)), \qquad \forall\xi \in \mathbb{R}^n. \tag{5.2}$$

The algebra $\mathcal{G}_N^{M,\infty}(\Omega)$ plays the same role as the Oberguggenberger subalgebra of regular elements $\mathcal{G}^\infty(\Omega)$ in the Colombeau algebra $\mathcal{G}(\Omega)$.

Theorem 5.8. *We have*

$$\mathcal{G}_{MN^{-1}p!}^{M,\infty}(\Omega) \cap \mathcal{D}'_{MN}(\Omega) = E^{MN^{-1}p!}(\Omega)$$

Proof. Let $S \in \mathcal{G}_{MN^{-1}p!}^{M,\infty}(\Omega) \cap \mathcal{D}'_{MN}(\Omega)$. For any fixed $x_0 \in \Omega$, we take $\psi \in \mathcal{D}^{MN}(\Omega)$, with $\psi \equiv 1$ on a neighborhood U of x_0. Then, $T = \psi S \in E'_{MN}(\Omega)$. Let ϕ_ε be a net mollifiers with $\check{\phi} = \phi$ and let $\chi \equiv 1$ on $K = supp\psi$. and $\chi \in \mathcal{D}^{MN^{-1}p!}(\Omega)$, As $[T] \in \mathcal{G}_{MN^{-1}p!}^{M,\infty}(\Omega)$, $\exists k_1 > 0$, $\exists k_2 > 0$, $\exists c_1 > 0$, $\exists\varepsilon_1 > 0$, $\forall\varepsilon \leq \varepsilon_1$,

$$|\mathcal{F}(\chi(T * \phi_\varepsilon))(\xi)| \leq c_1\exp(M(\frac{k_1}{\varepsilon}) - MN^{-1}p!(k_2|\xi|))$$

$$\begin{aligned}
&|\mathcal{F}(\chi(T * \phi_\varepsilon))(\xi) - \mathcal{F}(T)(\xi)| \\
&= |\mathcal{F}(\chi(T * \phi_\varepsilon))(\xi) - \mathcal{F}(\chi T)(\xi)| \\
&= \left|\left\langle T(x), (\chi(x)e^{-i\xi x}) * \phi_\varepsilon(x) - (\chi(x)e^{-i\xi x})\right\rangle\right|
\end{aligned}$$

As $E'_{MN}(\Omega) \subset E'_{MN^{-1}p!}(\Omega)$, then $\exists L \Subset \Omega$ such that $\forall h > 0, \exists c > 0$

$$\begin{aligned} & |\mathcal{F}(\chi(T * \phi_\varepsilon))(\xi) - \mathcal{F}(T)(\xi)| \\ & \leq \; c \sup_{\alpha\in\mathbb{Z}^n_+,\, x\in L} \frac{h^{|\alpha|}}{\frac{M_{|\alpha|}}{N_{|\alpha|}}|\alpha|!} \left|\partial_x^\alpha(\chi(x)e^{-ix\xi} * \phi_\varepsilon(x) - \chi(x)e^{-i\xi x})\right| \end{aligned}$$

We have $e^{-i\xi}\chi \in \mathcal{D}^{MN^{-1}p!}(\Omega)$ and by [4], we obtain $\forall k_3 > 0,\ \exists c_2 > 0,\ \exists \eta > 0, \forall \varepsilon \leq \eta,$

$$\sup_{\alpha\in\mathbb{Z}^n_+,\, x\in L} \frac{h^{|\alpha|}}{\frac{M_{|\alpha|}}{N_{|\alpha|}}|\alpha|!} \left|\partial_x^\alpha(\chi(x)e^{-ix\xi} * \phi_\varepsilon(x) - \chi(x)e^{-i\xi x})\right| \leq c_2 \exp(-M(\frac{k_3}{\varepsilon}))$$

So there exists $c' = c'(k_3) > 0$, such that

$$|\mathcal{F}(\chi(T * \phi_\varepsilon))(\xi) - \mathcal{F}(T)(\xi)| \leq c'.\exp(-M(\frac{k_3}{\varepsilon}))$$

Let $\varepsilon \leq min(\eta, \varepsilon_1)$, then

$$\begin{aligned} |\mathcal{F}(T)(\xi)| &\leq |\mathcal{F}(T)(\xi) - \mathcal{F}(\chi(T * \phi_\varepsilon))| + |\mathcal{F}(\chi(T * \phi_\varepsilon))| \\ &\leq c'.\exp(-M(\tfrac{k_3}{\varepsilon})) + c_1 \exp(M(\tfrac{k_1}{\varepsilon}) - MN^{-1}p!(k2\,|\xi|)) \end{aligned}$$

Take $c = max(c_1, c')$, $\varepsilon = \dfrac{k_1 p!^{\frac{1}{p}}}{(k2 - r)\,|\xi|\, N_p^{\frac{1}{p}}}$, $r \in]0, k_2[$ and $k_3 = \dfrac{k_1 r}{k_2 - r}$, then $\exists \delta > 0,\ \exists c > 0$ such that

$$|\mathcal{F}(T)(\xi)| \leq c\exp(-MN^{-1}p!(\delta\,|\xi|)),$$

Which means $T = \psi S \in E^{MN^{-1}p!}(\Omega)$. As $\psi \equiv 1$ on the neighborhood U of x_0, Consequently $S \in E^{MN^{-1}p!}(\Omega)$. Which proves

$$\mathcal{G}^{M,\infty}_{MN^{-1}p!}(\Omega) \cap \mathcal{D}'_{MN}(\Omega) \subset E^{MN^{-1}p!}(\Omega).$$

We have $E^{MN^{-1}p!}(\Omega) \subset E^{MN}(\Omega) \subset \mathcal{D}'_{MN}(\Omega)$, $E^{MN^{-1}p!}(\Omega) \subset \mathcal{G}^{M,\infty}_{MN^{-1}p!}(\Omega)$, then $E^{MN^{-1}p!}(\Omega) \subset \mathcal{G}^{M,\infty}_{MN^{-1}p!}(\Omega) \cap \mathcal{D}'_{MN}(\Omega)$.

Consequently we have

$$\mathcal{G}^{M,\infty}_{MN^{-1}p!}(\Omega) \cap \mathcal{D}'_{MN}(\Omega) = E^{MN^{-1}p!}(\Omega).$$

□

6. Generalized Roumieu wave front

The aim of this section is to introduce the generalized Roumieu wave front of generalized Roumieu ultradistribution and to give its main properties.

Definition 6.1. We define $\sum_g^{M,N}(f) \subset \mathbb{R}^n \backslash \{0\}$, $f \in \mathcal{G}_c^M(\Omega)$, as the complement of the set of points having a conic neighborhood Γ such that $\exists k_1 > 0$, $\exists k_2 > 0$, $\exists c > 0$, $\exists \varepsilon_0 \in I$, $\forall \xi \in \Gamma$, $\forall \xi \in \Gamma$, $\forall \varepsilon \leq \varepsilon_0$,

$$|\mathcal{F}(f_\varepsilon)(\xi)| \leq c \exp(M(\frac{k_1}{\varepsilon}) - N(k_2 |\xi|))$$

The following essential properties of $\sum_g^{M,N}(f)$ are sufficient to define later the generalized Roumieu wave front of generalized Roumieu ultradistribution

Proposition 6.2. *For every $f \in \mathcal{G}_c^M(\Omega)$ we have*

1. *The Set $\sum_g^{M,N}(f)$ is closed cone.*
2. $\sum_g^{M,N}(f) = \emptyset \Longleftrightarrow f \in \mathcal{G}^{M,N,\infty}$.
3. $\sum_g^{M,N}(\psi f) \subset \sum_g^{M,N}(f)$, $\forall \psi \in E^N(\Omega)$.

Proof. One can easily, from Definition (6.1) and Proposition (5.6), prove the assertion 1 and 2.
Let suppose that $\xi_0 \notin \sum_g^{M,N}(f)$, then $\exists \Gamma$ a conic neighborhood of ξ_0, $\exists k_1 > 0$, $\exists k_2 > 0$, $\exists c_1 > 0$, $\exists \varepsilon_1 > 0$, $\forall \xi \in \Gamma$, $\forall \varepsilon \in \varepsilon_1$,

$$|\mathcal{F}(f_\varepsilon)(\xi)| \leq c. \exp(M(\frac{k_1}{\varepsilon}) - N(k_2 |\xi|))$$

Let $\chi \in \mathcal{D}^N(\Omega)$, $\chi \equiv 1$ on neighborhood of $supp(f)$, so $\chi\psi \in \mathcal{D}^N(\Omega)$, $\forall \psi \in E^N(\Omega)$ hence from [11] $\exists k_3 > 0$, $\exists c_2 > 0$, $\forall \xi \in \mathbb{R}^n$,

$$|\mathcal{F}(\chi\psi)(\xi)| \leq c. \exp(-N(k_3 |\xi|))$$

Let Λ be a conic neighborhood of ξ_0 such that $\bar{\Lambda} \subset \Gamma$ we have for a fixed $\xi \in \Lambda$,

$$\begin{aligned} &\mathcal{F}(\psi f_\varepsilon)(\xi) \\ &\quad = \mathcal{F}(\chi\psi f_\varepsilon)(\xi) \\ &\quad = \int_A \mathcal{F}(f_\varepsilon)(\eta).\mathcal{F}(\chi\psi)(\eta - \xi) d\eta + \int_B \mathcal{F}(f_\varepsilon)(\eta).\mathcal{F}(\chi\psi)(\eta - \xi) d\eta, \end{aligned}$$

where $A = \{\eta : |\xi - \eta| \leq \delta(|\xi| + |\eta|)\}$ and $B = \{\eta : |\xi - \eta| > \delta(|\xi| + |\eta|)\}$

Take δ sufficient small such that $\frac{|\xi|}{2} < |\eta| < 2|\xi|$, $\forall \eta \in A$, then $\exists c > 0$, $\forall \varepsilon \leq \varepsilon_1$,

$$\begin{aligned} &\left| \int_A \mathcal{F}(f_\varepsilon)(\eta)\mathcal{F}(\chi\psi)(\eta - \xi) d\eta \right| \\ &\quad \leq c_1.c_2.exp(M(\frac{k_1}{\varepsilon}) - N(k_2 \frac{|\xi|}{2})) \times \int_A \exp(-N(k_3 |\eta - \xi|)) d\eta \end{aligned}$$

Then $\exists c > 0$, $\exists k_2' > 0$

$$\left| \int_A \mathcal{F}(f_\varepsilon)(\eta)\mathcal{F}(\chi\psi)(\eta - \xi)d\eta \right| \le c\exp(M(\frac{k_1}{\varepsilon}) - N(k_2' |\xi|)) \tag{6.1}$$

As $\mathcal{G}_c^M(\Omega)$, from Remark (5.7), $\exists c_2 > 0$, $\exists \mu_1 > 0$, $\exists \varepsilon_2 > 0$, $\forall \mu_2 > 0$, $\forall \xi \in \mathbb{R}^n$, $\forall \varepsilon \le \varepsilon_2$, such that

$$|\mathcal{F}(f_\varepsilon)(\xi)| \le c\exp(M(\frac{\mu_1}{\varepsilon}) + N(\mu_2 |\xi|))$$

Hence, for $\varepsilon \le min(\varepsilon_1, \varepsilon_2)$, we have

$$\begin{aligned} &\left| \int_B \mathcal{F}(f_\varepsilon)(\eta).\mathcal{F}(\chi\psi)(\eta - \xi)d\eta \right| \\ &\le \quad c_2.c_3.\exp(M(\frac{\mu_1}{\varepsilon})) \left| \int_B \exp(N(\mu_2 |\eta|) - N(k_3 |\eta - \xi|))d\eta \right| \\ &\le \quad c.\exp(M(\frac{\mu_1}{\varepsilon})) \left| \int_B \exp(N(\mu_2 |\eta|) - N(k_3\delta(|\xi| + |\eta|)d\eta \right| \end{aligned}$$

Then taking $\mu_2 < k_3\delta$, we obtain

$$\left| \int_B \mathcal{F}(f_\varepsilon)(\eta).\mathcal{F}(\chi\psi)(\eta - \xi)d\eta \right| \le c\exp(M(\frac{\mu_1}{\varepsilon}) - N(k_3\delta |\xi|)) \tag{6.2}$$

Consequently, (6.1) and (5.6) give $\xi_0 \notin \sum_g^{M,N}(\psi f)$. □

Definition 6.3. Let $f \in \mathcal{G}^M(\Omega)$ and $x_0 \in \Omega$, the cone of $N-$singular directions of f at x_0, denoted $\sum_{g,x_0}^{M,N}(f)$, is

$$\Sigma_{g,x_0}^{M,N}(f) = \bigcap \{\Sigma_g^{M,N}(\varphi f) : \varphi \in \mathcal{D}^M(\Omega) \text{ and } \varphi \equiv 1 \text{ on a neighborood of } x_0\}$$

Lemma 6.4. *Let $f \in \mathcal{G}^M(\Omega)$, then*

$$\Sigma_{g,x_0}^{M,N}(f) = \emptyset \Leftrightarrow x_0 \notin N - singsupp_g(f)$$

Proof. Let $x_0 \notin N - singsupp_g(f)$, i.e. $\exists U \subset \Omega$ an open neighborhood of x_0 such that $f \in \mathcal{G}_N^{M,\infty}(U)$, let $\phi \in \mathcal{D}^M(U)$ such that $\phi \equiv 1$ on a neighborhood of x_0, then $\phi f \in \mathcal{G}_N^{M,\infty}(\Omega)$. Hence, from Proposition (6.2), $\sum_g^{M,N}(\phi f) = \emptyset$, i.e. $\sum_{g,x_0}^{M,N}(f) = \emptyset$.

Suppose now $\sum_{g,x_0}^{M,N}(f) = \emptyset$, $\forall \xi \in \mathbb{R}^n \backslash \{0\}$, $\exists V_\xi \in \mathcal{V}(x_0)$, $\exists w_\xi \in \xi$ conical neighborhood. $\exists k_1 > 0$, $\exists k_2 > 0$, $\exists c > 0$, $\exists \varepsilon_0 > 0$, $\forall \xi \in W_\xi$, $\forall \varepsilon \le \varepsilon_0$, $\forall \phi_\xi \in \mathcal{D}^M(\Omega)$.

$$|\mathcal{F}(\phi_\xi f_\varepsilon)(\xi)| \le c.\exp(M(\frac{k_1}{\varepsilon}) - N(k_2 |\xi|))$$

Since the unit sphere $|\xi| = 1$ is a compact set, then one can find finite points $\xi_j, j = 1, ..., n$ in $\mathbb{R}^n$, $W_j \in \xi_|$ and $\phi_j \in \mathcal{D}^M(\Omega)$, $\phi_j(x) = 1$ in V_j, $k_1 > 0$, $\exists k_2 > 0$, $\exists c > 0$, $\varepsilon_0 > 0$, $\forall \varepsilon \leq \varepsilon_0$

$$|\mathcal{F}(\phi_j f_\varepsilon)(\xi)| \leq c. \exp(M(\frac{k_1}{\varepsilon}) - N(k_2 |\xi|)), \qquad \xi \in W_j$$

Taking $V = \bigcap_j V_j$ and $W = \bigcup_j W_j$, $\varphi = \phi_1 ... \phi_n$, we have $\varphi \in \mathcal{D}^M(\Omega)$ and $\varphi(x) = 1$ on V.

$$|\mathcal{F}(\varphi f_\varepsilon)(\xi)| \leq c. \exp(M(\frac{k_1}{\varepsilon}) - N(k_2 |\xi|)), \qquad \xi \in W$$

Consequently, $(\varphi f_\varepsilon) \notin \mathcal{G}_{M,c}^{N,\infty}$, where $x_0 \in N - singsupp_g(f)$ □

Definition 6.5. A point $(x_0, \xi_0) \notin WF_g^{M,N}(f) \subset \Omega \times \mathbb{R}^n \backslash \{0\}$ if $\xi_0 \notin \sum_{g,x_0}^{M,N}(f)$, i.e. there exists $\phi \in \mathcal{D}^M(\Omega), \phi(x) = 1$ neighborhood of x_0, and conic neighborhood Σ of ξ_0, $\exists k_1 > 0$, $\exists k_2 > 0$, $\exists c > 0$, $\exists \varepsilon_0 > 0$ such that $\forall \xi \in \Gamma$, $\forall \varepsilon \leq \varepsilon_0$,

$$|\mathcal{F}(\phi f_\varepsilon)(\xi)| \leq c \exp(M(\frac{k_1}{\varepsilon}) - N(k_2 |\xi|))$$

The main proprieties of the generalized Roumieu wave front $WF_g^{M,N}$ are subsumed in the following proposition.

Proposition 6.6. *Let $f \in \mathcal{G}^M(\Omega)$, then*

(1) The projection of $WF_g^{M,N}(f)$ on Ω is $N - sinsupp_g(f)$.

(2) If $f \in \mathcal{G}_c^M(\Omega)$, The projection of $WF_g^{M,N}(f)$ on $\mathbb{R}^n \backslash \{0\}$ is $\sum_g^{M,N}(f)$.

(3) $\forall \alpha \in \mathbb{Z}_+^n$, $WF_g^{M,N}(\partial^\alpha f) \subset WF_g^{M,N}(f)$.

(4) $\forall g \in \mathcal{G}_N^{M,\infty}(\Omega)$, $WF_g^{M,N}(gf) \subset WF_g^{M,N}(f)$.

Proof. (1) and (2) hold from the definition, Proposition (6.2) and Lemma (6.4).

(3) Let $(x_0, \xi_0) \notin WF_g^{M,N}(f)$, then $\exists \phi \in \mathcal{D}^M(\Omega)$, $\phi \equiv 1$ on a neighborhood $\bar{U}$ of x_0, there exists a conic neighborhood Γ of ξ_0, $\exists k_1 > 0$, $\exists k_2 > 0$, $\exists c_1 > 0$, $\exists \varepsilon_0 \in]0, 1]$, such that $\forall \xi \in \Gamma$, $\varepsilon \leq \varepsilon_0$,

$$|\mathcal{F}(\phi f_\varepsilon)(\xi)| \leq c_1 exp(M(\frac{k_1}{\varepsilon}) - N(k_2 |\xi|)) \tag{6.3}$$

We have for $\psi \in \mathcal{D}^M(U)$ such that $\psi(x_0) = 1$.

$$\begin{aligned} |\mathcal{F}(\psi \partial f_\varepsilon)(\xi)| &= |\mathcal{F}(\partial(\psi f_\varepsilon))(\xi) - \mathcal{F}(\partial \psi) f_\varepsilon(\xi)| \\ &\leq |\xi| \, |\mathcal{F}(\psi \phi f_\varepsilon)(\xi)| + |\mathcal{F}((\partial \psi) \phi f_\varepsilon)(\xi)| \end{aligned}$$

As $WF_g^{M,N}(\psi f) \subset WF_g^{M,N}(f)$, (6.3) holds for both $|\mathcal{F}(\psi \phi f_\varepsilon)(\xi)|$ and $|\mathcal{F}((\partial \psi) \phi f_\varepsilon)(\xi)|$.

So

$$\begin{array}{rcl} |\xi|\,|\mathcal{F}(\psi\phi f_\varepsilon)(\xi)| & \leq & |\xi|\exp(M(\frac{k_1}{\varepsilon})-N(k_2\,|\xi|)) \\ & \leq & c'\exp(M(\frac{k_1}{\varepsilon})-N(k_3\,|\xi|)) \end{array}$$

With $c'>0$, $k_3>0$, such that $|\xi|\leq c'\exp(M(k_2\,|\xi|)-M(k_3\,|\xi|))$. This proves $(x_0,\xi_0)\notin WF_g^{M,N}(\partial f)$.

(4) Let $(x_0,\xi_0)\notin WF_g^{M,N}(f)$ then $\exists\phi\in\mathcal{D}^M(\Omega)$, $\phi\equiv 1$ on a neighborhood of x_0, $\xi_0\notin\sum_g^{M,N}(\phi f)$ by Proposition (6.2), for $g\in\mathcal{G}_M^{N,\infty}(\Omega)$, we have $\xi_0\notin\sum_g^{M,N}(g\phi f)$, which proves $(x_0,\xi_0)\notin WF_g^{M,N}(gf)$. □

Corollary 6.7. *Let $P(x,D)=\sum\limits_{|\alpha|\leq m}a_\alpha(x)D^\alpha$ be a partial differential operator with $\mathcal{G}_N^{M,\infty}(\Omega)$ coefficient, then $WF_g^{M,N}(P(x,D)f)\subset WF_g^{M,N}(f)$, $\forall f\in\mathcal{G}^M(\Omega)$.*

Lemma 6.8. *Let $\varphi\in\mathcal{D}^M(B(0.2))$, $0\leq\varphi\leq 1$, and $\varphi\equiv 1$ on $B(0,1)$ and let $\phi\in S^M$, then $\exists c>0$, $\exists v>0$, $\exists\varepsilon_0>0$, $\forall\varepsilon\in]0,\varepsilon_0]$, $\forall\xi\in\mathbb{R}^n$,*

$$\left|\hat{\theta}_\varepsilon(\xi)\right|\leq c\varepsilon^{-n}e^{-M(v\varepsilon|\xi|)},$$

where $\theta_\varepsilon(x)=(\frac{1}{\varepsilon})^n.\phi(\frac{x}{\varepsilon}).\varphi(x\,|\varepsilon|)$, and $\hat{\theta}$ denoted the Fourier transform of θ.

Proof. We have, for ε sufficiently small, $\varepsilon\leq|\ln\varepsilon|^{-n}\leq 1$
Let $\xi\in\mathbb{R}^n$, then

$$\begin{array}{rcl} \hat{\theta}_\varepsilon(\xi) & = & \dfrac{1}{\varepsilon^n}\int\hat{\phi}(\varepsilon(\xi-\eta)).\frac{1}{|\ln\varepsilon|^n}.\hat{\varphi}(\frac{\eta}{|\ln\varepsilon|})d\eta \\ & = & |\ln\varepsilon|^{-n}\left[\int_A\hat{\phi}(\varepsilon(\xi-\eta))\hat{\varphi}(\frac{\eta}{|\ln\varepsilon|})d\eta+\int_B\hat{\phi}(\varepsilon(\xi-\eta))\hat{\varphi}(\frac{\eta}{|\ln\varepsilon|})d\eta\right], \end{array}$$

where $A=\{\eta:|\xi-\eta|\leq\delta(|\xi|+|\eta|)\}$ and $B=\{\eta:|\xi-\eta|>\delta(|\xi|+|\eta|)\}$

We choose δ sufficiently small such that $\frac{|\xi|}{2}<|\eta|<2\,|\xi|$, $\forall\eta\in A$.
Since $\varphi\in\mathcal{D}^M(\Omega)$, $\phi\in S^M$ then $\exists k_1,k_2>0,\exists c_1,c_2>0$, $\forall\xi\in$ ‘$\mathbb{R}$,

$$|\hat{\varphi}(\xi)|\leq c_1\exp(-M(k_1\,|\xi|))$$

and

$$\left|\hat{\phi}(\xi)\right|\leq c_2\exp(-M(k_2\,|\xi|)),$$

so,

$$\begin{array}{rcl} I_1 & = & |\ln\varepsilon|^{-n}\left|\int_A\hat{\phi}(\varepsilon(\xi-\eta))\hat{\varphi}(\frac{\eta}{|\ln\varepsilon|})d\eta\right| \\ & \leq & c_1c_2\exp(-M(\frac{k_2}{2}\frac{|\xi|}{|\ln\varepsilon|})) \end{array}$$

Let $z=\varepsilon(\eta-\xi)$, then

$$\begin{array}{rcl} I_1 & \leq & c\varepsilon^{-n}\exp(-M(\frac{k_2}{2}\,|\ln\varepsilon|^{-1}\,|\xi|))\int\exp(-M(k_1\,|z|))dz \\ & \leq & c\varepsilon^{-n}\exp(-M(v\varepsilon\,|\xi|)) \end{array}$$

For I_2 we have

$$\begin{array}{rcl} I_2 & = & |\ln\varepsilon|^{-1}\left|\int_B \hat{\phi}(\varepsilon(\xi-\eta))\hat{\varphi}(\frac{\eta}{|\ln\varepsilon|})d\eta\right| \\ & \leq & c_1c_2\int_B \exp(-M(k_1\varepsilon\,|\xi-\eta|)-M(k_2\frac{|\eta|}{|\ln\varepsilon|}))d\eta \\ & \leq & c\exp(-M(k_1\delta\varepsilon\,|\xi|)).\int_B \exp(-M(k_1\delta\varepsilon\,|\eta|-M(k_2\delta\varepsilon\,|\eta|))d\eta \\ & \leq & c\exp(-M(k_1\delta\varepsilon\,|\xi|)).\int_B \exp(-M(k_2'\varepsilon\,|\eta|))d\eta \\ & \leq & c\varepsilon^{-n}\exp(-M(v\varepsilon\,|\xi|)) \end{array}$$

Consequently, $\exists c>0$, $\exists v>0$, $\exists\varepsilon_0>0$, $\forall\varepsilon\leq\varepsilon_0$ such that

$$\left|\hat{\theta}_\varepsilon(\xi)\right|\leq c\varepsilon^{-n}e^{-M(v\varepsilon|\xi|)}$$

□

We have the following important result.

Theorem 6.9. *Let* $T\in\mathcal{D}'_{MN}(\Omega)\cap\mathcal{G}^M(\Omega)$*; then*

$$WF_g^{M,MN^{-1}}(T)=WF^{MN^{-1}p!}(T).$$

Proof. Let $S\in E'_{MN}(\Omega)\subset E'_{\frac{M}{N}p!}(\Omega)$ and $\psi\in\mathcal{D}^{\frac{M}{N}p!}(\Omega)$, we have

$$|\mathcal{F}(\psi(S*\phi_\varepsilon))(\xi)-\mathcal{F}(\psi S)(\xi)|=\left|\left\langle S(x),(\psi(x)e^{-i\xi x})*\check{\phi}_\varepsilon(x)-(\psi(x)e^{-ix\xi})\right\rangle\right|.$$

Then there exists a compact subset L of Ω such that $\forall h>0$, $\exists c>0$,

$$\begin{array}{l} |\mathcal{F}(\psi(S*\phi_\varepsilon))(\xi)-\mathcal{F}(\psi S)(\xi)| \\ \quad\leq\ c\ \underset{\alpha\in\mathbb{Z}^n_+;x\in L}{sup}\dfrac{h^{|\alpha|}}{\frac{M_{|\alpha|}}{N_{|\alpha|}}\alpha!}\left|\partial_x^\alpha(\psi(x)e^{-i\xi x}*\check{\phi}_\varepsilon(x)-\psi(x)e^{-i\xi x})\right| \end{array}$$

We have $e^{-i\xi}\psi\in\mathcal{D}^{\frac{M}{N}p!}(\Omega)$, then, $\exists c_2,\forall k_0>0$, $\exists\eta>0$, $\forall\varepsilon\leq\eta$,

$$\underset{\alpha\in\mathbb{Z}^n_+;x\in L}{sup}\dfrac{c_2^{|\alpha|}}{\frac{M_{|\alpha|}}{N_{|\alpha|}}\alpha!}\left|\partial_x^\alpha(\psi(x)e^{-i\xi x}*\check{\phi}_\varepsilon(x)-\psi(x)e^{-i\xi x})\right|\leq c_2e^{-M(\frac{k_0}{\varepsilon})};$$

So there exist $c'>0$, $\forall k_0>0$, $\exists\eta>0$, $\forall\varepsilon\leq\eta$, such that

$$|\mathcal{F}(\psi S)(\xi)-\mathcal{F}(\psi(S*\phi_\varepsilon)(\xi))|\leq c'e^{-M(\frac{k_0}{\varepsilon})} \tag{6.4}$$

Let $T\in\mathcal{D}'_{MN}(\Omega)\cap\mathcal{G}^M(\Omega)$ and $(x_0,y_0)\notin WF_g^{M,\frac{M}{N}p!}(T)$, Then there exist $\chi\in\mathcal{D}^{\frac{M}{N}p!}(\Omega)$, $\chi(x)=1$ in a neighborhood of x_0, and a conic neighborhood Γ of ξ_0, $\exists k_1>0$, $\exists k_2>0$, $\exists c_1>0$, $\exists\varepsilon_0\in]0,1[$, such that $\forall\xi\in\Gamma$, $\forall\varepsilon\leq\varepsilon_0$,

$$|\mathcal{F}(\chi(T*\theta_\varepsilon))(\xi)|\leq c_1e^{M(\frac{k_1}{\varepsilon})-\frac{M}{N}p!(k_2|\xi|)} \tag{6.5}$$

Let $\psi \in \mathcal{D}^{\frac{M}{N}p!}(\Omega)$ equal to 1 in neighborhood of x_0 such that for sufficiently small ε we have $\chi \equiv 1$ on $supp\psi + B(0, \frac{2}{|\ln\varepsilon|})$, and let $\varphi \in \mathcal{D}^{\frac{M}{N}p!}(B(0,2))$; $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ on $B(0,1)$, then there exist $\varepsilon_0 \leq 1$, such that $\forall\varepsilon < \varepsilon_0$,

$$\psi(T * \theta_\varepsilon)(x) = \psi(\chi T * \theta_\varepsilon)(x).$$

where $\theta_\varepsilon(x) = \frac{1}{\varepsilon^n}\varphi(x\,|\ln\varepsilon|)\phi(\frac{x}{\varepsilon})$. As $\chi T \in E'_{MN}(\Omega)$, then

$$\psi(T * \theta_\varepsilon)(x) = \psi(\chi T * \theta_\varepsilon)(x) = \psi(\chi T * \phi_\varepsilon)(x)$$

Let $\varepsilon \leq min(\eta, \varepsilon_0)$ and $\xi \in \Gamma$, we have

$$\begin{aligned}|\mathcal{F}(\psi T)(\xi)| &\leq |\mathcal{F}(\psi T)(\xi) - \mathcal{F}(\psi(T * \theta_\varepsilon))(\xi)| + |\mathcal{F}(\chi(T * \theta_\varepsilon))(\xi)| \\ &\leq |\mathcal{F}(\psi\chi T)(\xi) - \mathcal{F}(\psi(\chi T * \phi_\varepsilon))(\xi)| + |\mathcal{F}(\chi(T * \theta_\varepsilon))(\xi)|\end{aligned}$$

Then by (6.4) and (6.5), we obtain

$$|\mathcal{F}(\psi T)(\xi)| \leq c' e^{-M(\frac{k_0}{\varepsilon})} + c_1 e^{M(\frac{k_1}{\varepsilon}) - MN^{-1}p!(k_2|\xi|)}$$

Take $c = max(c_1, c')$, $\varepsilon = \dfrac{k_1 p!^{\frac{1}{p}}}{(k2 - r)\,|\xi|\,N_p^{\frac{1}{p}}}$, $r \in]0, k_2[$ and $k_0 = \dfrac{k_1 r}{k_2 - r}$, then $\exists\delta > 0,\ \exists c > 0$ such that

$$|\mathcal{F}(\chi T)(\xi)| \leq c' e^{-\frac{M}{N}p!(\delta|\xi|)},$$

Which proves that $(x_0, \xi_0) \notin WF^{\frac{M}{N}p!}(T)$. So $WF^{\frac{M}{N}p!}(T) \subset WF_g^{M,\frac{M}{N}p!}(T)$.

Suppose that $(x_0, \xi_0) \notin WF^{\frac{M}{N}p!}(T)$, then there exist $\chi \in \mathcal{D}^{\frac{M}{N}p!}(\Omega)$, $\chi(x) = 1$ in a neighborhood of x_0, a conical neighborhood Γ of ξ_0, $\exists\lambda > 0$, $c_1 > 0$, such that $\forall\xi \in \Gamma$

$$|\mathcal{F}(\chi T)(\xi)| \leq c_1 e^{-\frac{M}{N}p!(\lambda|\xi|)}. \tag{6.6}$$

Let $\psi \in \mathcal{D}^{\frac{M}{N}p!}(\Omega)$ equals 1 in neighborhood of x_0 such that for sufficiently small ε we have $\chi \equiv 1$ on $supp\psi + B(0, \frac{2}{|\ln\varepsilon|})$, then there exist $\varepsilon_0 < 1$, such that $\forall\varepsilon < \varepsilon_0$,

$$\psi(T * \theta_\varepsilon)(x) = \psi(\chi T * \theta_\varepsilon)(x).$$

We have

$$\mathcal{F}(\psi(T * \theta_\varepsilon))(\xi) = \int \mathcal{F}(\psi)(\xi - \eta).\mathcal{F}(\chi T)(\eta).\mathcal{F}(\theta_\varepsilon)(\eta)d\eta.$$

Let Λ be a conic neighborhood of ξ_0 such that, $\bar{\Lambda} \subset \Gamma$. For a fixed $\xi \in \Lambda$, we have

$$\begin{aligned}&\mathcal{F}(\psi(\chi T * \theta_\varepsilon))(\xi) \\ &\quad= \int_A \mathcal{F}(\psi)(\xi - \eta).\mathcal{F}(\chi T)(\eta).\mathcal{F}(\theta_\varepsilon)(\eta)d\eta \\ &\qquad+ \int_B \mathcal{F}(\psi)(\xi - \eta).\mathcal{F}(\chi T)(\eta).\mathcal{F}(\theta_\varepsilon)(\eta)d\eta,\end{aligned}$$

where $A=\{\eta:|\xi-\eta|\leq\delta(|\xi|+|\eta|)\}$ and $B=\{\eta:|\xi-\eta|\geq\delta(|\xi|+|\eta|)\}$.

We choose δ sufficiently small such that $A\subset\Gamma$ and $\frac{|\xi|}{2}<|\eta|<2\,|\xi|$. Since $\psi\in\mathcal{D}^M(\Omega)$, then $\exists\mu>0$, $\exists c_2>0$, $\forall\xi\in\mathbb{R}^n$,

$$|\mathcal{F}(\psi)(\xi)|\leq c_2\exp(-\frac{M}{N}p!(\mu\,|\xi|)),$$

Then $\exists c>0$, $\exists\varepsilon_0\in]0,1[$, $\forall\varepsilon\leq\varepsilon_0$,

$$\begin{aligned}&\left|\int_A\mathcal{F}(\psi)(\xi-\eta).\mathcal{F}(\chi T)(\eta).\mathcal{F}(\theta_\varepsilon)(\eta)d\eta\right|\\ &\leq\ c\exp(-\frac{M}{N}p!(\frac{\lambda}{2}\,|\xi|))\times\left|\int_A\exp(-\frac{M}{N}p!(\mu\,|\eta-\xi|).\mathcal{F}(\theta_\varepsilon)(\eta)d\eta\right|\end{aligned}$$

From preceding Lemma, $\exists c_3>0$, $\exists v>0$, $\exists\varepsilon_0>0$, such that

$$|\mathcal{F}(\theta_\varepsilon)(\xi)|\leq c_3\varepsilon^{-n}e^{-N(v\varepsilon|\xi|)}\ \forall\xi\in\mathbb{R}^n$$

then $\exists c>0$, such that

$$\begin{aligned}&\left|\int_A\mathcal{F}(\psi)(\xi-\eta).\mathcal{F}(\chi T)(\eta).\mathcal{F}(\theta_\varepsilon)(\eta)d\eta\right|\\ &\leq\ c\varepsilon^{-n}\exp(-\frac{M}{N}p!(\lambda\,|\xi|))\times\\ &\qquad\left|\int_A\exp(-\frac{M}{N}p!(\mu\,|\eta-\xi|).\exp(-N(v\varepsilon\,|\eta|))d\eta\right|\end{aligned}$$

We have $\exists k>0$, $\forall\varepsilon\in]0,\varepsilon_0[$,

$$\varepsilon^{-m}\exp(-N(v\varepsilon\,|\eta|))\leq\exp(M(\frac{k}{\varepsilon})),\tag{6.7}$$

So

$$\left|\int_A\mathcal{F}(\psi)(\xi-\eta).\mathcal{F}(\chi T)(\eta).\mathcal{F}(\theta_\varepsilon)(\eta)d\eta\right|\leq c\exp(M(\frac{k}{\varepsilon})-\frac{M}{N}p!(\frac{\lambda}{2}\,|\xi|))\tag{6.8}$$

As $\xi T\in E'_{MN}(\Omega)\subset E'_{\frac{M}{N}p!}(\Omega)$, then $\forall l>0$, $\exists c>0$, $\forall\xi\in\mathbb{R}^n$,

$$|\mathcal{F}(\chi T)(\xi)|\leq c\exp(\frac{M}{N}p!(l\,|\xi|))$$

Hence, we have

$$\begin{aligned}&\left|\int_B\mathcal{F}(\psi)(\xi-\eta)\mathcal{F}(\chi T)(\eta)\mathcal{F}(\theta_\varepsilon)(\eta)d\eta\right|\\ &\leq\ c\int_B\exp(\frac{M}{N}p!(l\,|\eta|)-\frac{M}{N}p!(\mu\,|\xi-\eta|)).\,|\mathcal{F}(\theta_\varepsilon)|\,d\eta\\ &\leq\ c'\varepsilon^{-n}.\exp(-\frac{M}{N}p!(\mu\delta\,|\xi|))\\ &\qquad\int_B\exp(\frac{M}{N}p!((l-\mu\delta)\,|\eta|)-N(v\varepsilon\,|\eta|))\end{aligned}$$

Then, taking $l-\mu\delta=-a<0$ and using (6.7), we obtain for a constant $c>0$

$$\left|\int_B \mathcal{F}(\psi)(\xi-\eta).\mathcal{F}(\chi T)(\eta).\mathcal{F}(\theta_\varepsilon)(\eta)d\eta\right| \leq c\exp(M(\frac{k_1}{\varepsilon})-\frac{M}{N}p!(\mu\delta\,|\xi|)),$$

which gives that $(x_0,\xi_0)\notin WF_g^{M,\frac{M}{N}p!}(T)$, so $WF_g^{M,\frac{M}{N}p!}(T)\subset WF^{\frac{M}{N}p!}(T)$. □

7. Generalized Hörmander's theorem

To extend the generalized Hörmander's result on the wave front set of the product, define $WF_g^{M,N}(f)+WF_g^{M,N}(f)$, where $f,g\in\mathcal{G}^M(\Omega)$, as the set

$$\{(x,\xi+\eta)\in WF_g^{M,N}(f),(x,\eta)\in WF_g^{M,N}(g)\}.$$

We recall the following fundamental lemma, see [7] for the proof.

Lemma 7.1. *Let Σ_1, Σ_2 be closed cones in $\mathbb{R}^n\backslash\{0\}$, such that $0\notin\Sigma_1+\Sigma_2$, then*

i) $\overline{\Sigma_1+\Sigma_2}^{\mathbb{R}^n\backslash\{0\}}=(\Sigma_1+\Sigma_2)\cup\Sigma_1\cup\Sigma_2.$

ii) For any open conic neighborhood Γ of $\Sigma_1+\Sigma_2$ in $\mathbb{R}^n\backslash\{0\}$, one can find open conic neighborhoods of Γ_1, Γ_2 in $\mathbb{R}^n\backslash\{0\}$ of respectively Σ_1, Σ_2 such that

$$\Gamma_1+\Gamma_2\subset\Gamma$$

The principal result of this section is the following theorem.

Theorem 7.2. *Let $f,g\in\mathcal{G}^M(\Omega)$, such that $\forall x\in\Omega$,*

$$(x,0)\notin WF_g^{M,N}(f)+WF_g^{M,N}(g). \tag{7.1}$$

Then the following holds:

$$WF_g^{M,N}(f.g)\subseteq(WF_g^{M,N}(f)+WF_g^{M,N}(g))\cup WF_g^{M,N}(f)\cup WF_g^{M,N}(g).$$

Proof. Let $(x_0,\xi_0)\notin(WF_g^{M,N}(f)+WF_g^{M,N}(g))\cup WF_g^{M,N}(f)\cup WF_g^{M,N}(g)$, then $\exists\phi\in\mathcal{D}^M(\Omega)$; $\phi(x_0)=1$, $\xi_0\notin(\Sigma_g^{M,N}(\phi f)+\Sigma_g^{M,N}(\phi g))\cup\Sigma_g^{M,N}(\phi f)\cup\Sigma_g^{M,N}(\phi g)$ From (7.1) we have $0\notin\Sigma_g^{M,N}(\phi f)+\Sigma_g^{M,N}(\phi g)$ then by Lemma 7.1 *i*), we have

$$\begin{aligned}\xi_0\notin(\Sigma_g^{M,N}(\phi f)+&\Sigma_g^{M,N}(\phi g))\cup\Sigma_g^{M,N}(\phi f)\cup\Sigma_g^{M,N}(\phi g)\\&=\overline{\Sigma_g^{M,N}(\phi f)+\Sigma_g^{M,N}(\phi g)}^{\mathbb{R}^n\backslash\{0\}}\end{aligned}$$

Let Γ_0 be an open conic neighborhood of $\Sigma_g^{M,N}(\phi f)+\Sigma_g^{M,N}(\phi g)$ in $\mathbb{R}^n\backslash\{0\}$ such that $\xi_0\notin\overline{\Gamma_0}$ then, from Lemma 7.1 *ii*), there exist open cones Γ_1 and Γ_2 in $\mathbb{R}^n\backslash\{0\}$ such that

$$\Sigma_g^{M,N}(\phi f)\subset\Gamma_1;\quad\Sigma_g^{M,N}(\phi g)\subset\Gamma_2$$

and

$$\Gamma_1 + \Gamma_2 \subset \Gamma_0$$

Define $\Gamma = \mathbb{R}^n \backslash \Gamma_0$, so

$$\Gamma \cap \Gamma_2 = \emptyset \, and \, (\Gamma - \Gamma_2) \cap \Gamma_1 = \emptyset \tag{7.2}$$

Let $\xi \in \Gamma$ and $\varepsilon \in I$.

$$\begin{aligned} \mathcal{F}(\phi f_\varepsilon \phi g_\varepsilon)(\xi) &= (\mathcal{F}(\phi f_\varepsilon) * \mathcal{F}(\phi g_\varepsilon))(\xi) \\ &= \int_{\Gamma_2} \mathcal{F}(\phi f_\varepsilon)(\xi - \eta).\mathcal{F}(\phi g_\varepsilon)(\eta) d\eta \\ & \quad \int_{\Gamma_2^c} \mathcal{F}(\phi f_\varepsilon)(\xi - \eta).\mathcal{F}(\phi g_\varepsilon)(\eta) d\eta = I_1(\xi) + I_2(\xi) \end{aligned}$$

By Proposition 5.6, $\exists c_1 > 0$, $\exists k_1, k_2 > 0$, $\exists \varepsilon_1 > 0$, such that $\forall \varepsilon \leq \varepsilon_1$, $\forall \eta \in \Gamma_2$,

$$|\mathcal{F}(\phi f_\varepsilon)(\xi - \eta)| \leq c_1 \exp(M(\frac{k_1}{\varepsilon}) - N(k_2 |\xi - \eta|)),$$

and by Remark 5.7, $\exists c_2 > 0$, $\exists k_3 > 0$, $\forall k_4 > 0$, $\exists \varepsilon_2 > 0$, $\forall \eta \in \mathbb{R}^n$, $\forall \varepsilon \leq \varepsilon_2$,

$$|\mathcal{F}(\phi g_\varepsilon)(\eta)| \leq c_2 \exp(M(\frac{k_3}{\varepsilon}) + N(k_4 |\eta|))$$

Let $\gamma > 0$ be sufficiently small such that

$$|\xi - \eta| \geq \gamma(|\xi| + |\eta|), \quad \forall \eta \in \Gamma_2.$$

Hence for $\varepsilon \leq min(\varepsilon_1, \varepsilon_2)$,

$$|I_1(\xi)| \leq c_1.c_2 \exp(M(\frac{k_1 + k_3}{\varepsilon}) - N(k_2\gamma |\xi|)) \int \exp(-N(k_2\gamma |\eta|) + N(k_4 |\eta|)) d\eta$$

Take $k_4 > k_2\gamma$, then

$$|I_1(\xi)| \leq c' \exp(M(\frac{k_1'}{\varepsilon}) - N(k_2' |\xi|)).$$

Let $r > 0$,

$$\begin{aligned} I_2(\xi) &= \int_{\Gamma_2^c \cap \{|\eta| \leq r|\xi|\}} \mathcal{F}(\phi f_\varepsilon)(\xi - \eta).\mathcal{F}(\phi g_\varepsilon)(\eta) d\eta \\ &\quad + \int_{\Gamma_2^c \cap \{|\eta| \geq r|\xi|\}} \mathcal{F}(\phi f_\varepsilon)(\xi - \eta).\mathcal{F}(\phi g_\varepsilon)(\eta) d\eta \\ &= I_{21}(\xi) + I_{22}(\xi). \end{aligned}$$

Choose r sufficiently small so that $\{|\eta| \leq r |\xi|\} \Rightarrow \xi - \eta \notin \Gamma_1$. Then $|\xi - \eta| \geq (1 - r) |\xi| \geq (1 - 2r) |\xi| + |\eta|$,

Consequently, $\exists c_3 > 0$, $\exists \lambda_1, \lambda_2, \lambda_3 > 0$, $\exists \varepsilon_3 > 0$ such that $\forall \varepsilon \leq \varepsilon_1$,

$$\begin{aligned} |I_{21}(\xi)| &\leq c_3 \exp(M(\tfrac{\lambda_1}{\varepsilon})) \int \exp(-N(\lambda_2 |\xi - \eta|) - N(\lambda_3 |\eta|)) d\eta \\ &\leq c_3 \exp(M(\tfrac{\lambda_1}{\varepsilon}) - N(\lambda_2' |\xi|)) \int \exp(-N(\lambda_3' |\eta|)) \\ &\leq c_3' \exp(M(\tfrac{\lambda_1}{\varepsilon}) - N(\lambda_2' |\xi|)) \end{aligned}$$

If $|\eta| \geq r|\xi|$, we have $|\eta| \geq \dfrac{|\eta| + r|\xi|}{2}$, and then $\exists c_4 > 0$, $\exists \mu_1, \mu_3 > 0$, $\forall \mu_2 > 0$, $\exists \varepsilon_4 > 0$ such that $\forall \varepsilon \leq \varepsilon_4$,

$$\begin{aligned}
&|I_{22}(\xi)| \\
&\leq \quad c_4 \exp(M(\frac{\mu_1}{\varepsilon})) \int \exp(N(\mu_2 |\xi - \eta|) - N(\mu_3 |\eta|)) d\eta \\
&\leq \quad c_4 \exp(M(\frac{\mu_1}{\varepsilon})) \int \exp(N(\mu_2 |\xi - \eta|) - N(\mu_3' |\eta|) - N(\mu_3' |\xi|)) d\eta
\end{aligned}$$

If take $\mu_2 < \frac{\mu_3'}{2}(1 + \frac{1}{r})$, we obtain

$$|I_{22}| \leq c_4' \exp(M(\frac{k_3'}{\varepsilon}) - N(\mu_3' |\xi|)),$$

which finishes the proof. □

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