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Introduction

The theory of generalized functions as a positive answer to the question of product distributions [37], caused a very important area of research [6, 8, 24, 26] and [30], this theory has been developed and applied in linear and nonlinear partial differential equations with non-smooth coefficients and distributions data by several authors [23], [26] and [30].

Ultradistributions are useful in applications in quantum field theory, partial differential equations, convolution equations, harmonic analysis, pseudo-differential theory, time-frequency analysis, and other areas of analysis, see [28] and [35], so it is necessary to develop a generalized functions type theory in connection with ultradistributions.

Generalized Gevrey ultradistributions of Colombeau type have been defined, but as a side-theme, in the paper [23]. The first paper aiming to construct differential algebras containing ultradistributions is [31]. Let us also mention the interesting approach of the paper [14] to algebras of generalized ultradistributions. However, a Colombeau type theory of generalized Gevrey ultradistributions has been addressed in [3], where was developed the core of a full theory and also introduced a new way of defining differential algebras of generalized Gevrey ultradistributions that makes such a complete theory possible. But, it was not clear in that paper why different Gevrey exponents occurred in the embedding of the spaces of Gevrey ultradistributions. In [2] was given a general construction of algebras of generalized Gevrey ultradistributions and then the microlocal analysis suitable for them. It also highlights the explicit contribution of the mollification in the embedding of ultradistributions into algebras of generalized functions of Colombeau type. In [1] was introduced a new algebras of generalized functions containing

Roumieu ultradistributions.

The first chapter is a brief and minimal introduction to Colombeau algebra of generalized function, in particularly the microlocal analysis and as an application we establish an extension of the well-known Hörmander's theorem.

However in the second chapter we introduce new algebras of generalized function containing Gevrey ultradistributions and then develop a Gevrey microlocal analysis suitable for these algebra.

In the last chapter we give a contribution of generalized Roumieu ultradistribution theory where we develop the microlocal analysis suitable for this algebra.

Chapter 1

Colombeau generalized functions

1.1 Notation

We begin by presenting some basic notations. Let $x = (x_1, ..., x_n)$ be an element of $\mathbb{R}^n$ the $n-$dimensional Euclidean space. The scalar product $x_1\xi_1 + ... + x_n\xi_n$ between x and ξ is denoted by $\langle x, \xi\rangle$ or else $x\xi$ for short; $|x| = \sqrt{x_1^2 + ... + x_n^2}$ is the Euclidean norm in $\mathbb{R}^n$. Let Ω be an open subset of $\mathbb{R}^n$, $K \Subset \Omega$ means that K is relatively compact in Ω. Let $\alpha = (\alpha_1, ..., \alpha_n)$ with $\alpha_j \in \mathbb{Z}_+$ for $j = 1, ..., n$ a multi-index, his length is $|\alpha| = \alpha_1 + ... + \alpha_n$. Moreover $\alpha! = \alpha_1!...\alpha_n!$, and if $\beta \leq \alpha$

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} ... \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} = \frac{\alpha!}{\beta!(\alpha - \beta)!}$$

We write $\partial^\alpha = (\partial/\partial x_1)^{\alpha_1}...(\partial/\partial x_n)^{\alpha_n}$; using the notation $D_{x_j} = -i\partial/\partial x_j$ where i is the imaginary unit, we write also: $D^\alpha = D_{x_1}^{\alpha_1}...D_{x_n}^{\alpha_n}$. Similarly for $x \in \mathbb{R}^n$ we set: $x^\alpha = x_1^{\alpha_1}...x_n^{\alpha_n}$.

1.2 The algebra $\mathcal{G}(\Omega)$

Following [24], the impossibility result of L.Schwartz [37] on the product of distributions can be interpreted as, the space of distributions $\mathcal{D}'(\Omega)$ cannot be embedded into an associative commutative algebra $(\mathcal{A}(\Omega)), +, \circ)$ satisfying:

(i) $\mathcal{D}'(\Omega)$ is linearly embedded into $\mathcal{A}(\Omega)$ and $f(x) \equiv 1$ is the unity in $\mathcal{A}(\Omega)$.

(ii) There exist derivation operators $\partial_i : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega)$, that are linear and satisfy the Leibniz rule.

(iii) $\partial_i/_{\mathcal{D}'(\Omega)}$ is the usual partial derivative.

(iv) $\circ/_{C(\Omega)\times C(\Omega)}$ coincides with the pointwise product of functions.

For example, let $I\ =]0,1]$, the set $(C^\infty(\Omega))^I$ with the following operations, $(u_\varepsilon)_\varepsilon$, $(v_\varepsilon)_\varepsilon \in (C^\infty(\Omega))^I$, $\lambda \in \mathbb{C}$, $\alpha \in \mathbb{Z}_+^n$

1. $(u_\varepsilon)_\varepsilon + (v_\varepsilon)_\varepsilon = (u_\varepsilon + v_\varepsilon)_\varepsilon$.

2. $(u_\varepsilon)_\varepsilon.(v_\varepsilon)_\varepsilon = (u_\varepsilon.v_\varepsilon)_\varepsilon$.

3. $\lambda(u_\varepsilon)_\varepsilon = (\lambda u_\varepsilon)_\varepsilon$.

4. $\partial^\alpha(u_\varepsilon)_\varepsilon = (\partial^\alpha u_\varepsilon)$

is an associative, commutative and differential algebra, with $f(x) = 1$ is unity element. But we have not (iv). However J-F. Colombeau, in [8], succeeded to construct a commutative and associative algebra $(\mathcal{G}(\Omega), +, .)$ satisfying $(i) - (iii)$, but instead of (iv) we have:

$(iv)'$ The restriction $\circ/_{C^\infty(\Omega)\times C^\infty(\Omega)}$ coincide with smooth functions multiplication.

Definition 1.2.1 *(i) The space of moderate elements, denoted $\mathcal{E}_m(\Omega)$, is the space of $(u_\varepsilon)_\varepsilon \in (C^\infty(\Omega))^I$, satisfying for every compact K of Ω, $\forall\alpha \in \mathbb{Z}_+^n$, $\exists m > 0, \exists c > 0$*

$$\exists\varepsilon_0 \in I, \forall\varepsilon \leq \varepsilon_0,\ \sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| \leq c\varepsilon^{-m} \tag{1.1}$$

(ii) The space of null elements, denoted $\mathcal{N}(\Omega)$, is the space of $(u_\varepsilon)_\varepsilon \in (C^\infty(\Omega))^I$, satisfying for every compact K of Ω, $\forall\alpha \in \mathbb{Z}_+^n$, $\forall q > 0, \exists c > 0$,

$$\exists\varepsilon_0 \in I, \forall\varepsilon \leq \varepsilon_0,\ \sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| \leq c\varepsilon^{q} \tag{1.2}$$

Remark 1.2.2 *The estimates (1.1) and (1.2) mean respectively that $\sup_{x\in K} |\partial^\alpha u_\varepsilon| = O(\varepsilon^{-m})$ and $\sup_{x\in K} |\partial^\alpha u_\varepsilon| = O(\varepsilon^{q})$ as $\varepsilon \to 0$.*

Proposition 1.2.3 *i) The space $\mathcal{E}_m(\Omega)$ is subalgebra of $C^\infty(\Omega)^I$ stable by derivation.*

ii) The space $\mathcal{N}(\Omega)$ is an ideal of $\mathcal{E}_m(\Omega)$.

Proof.

i) Let $(u_\varepsilon)_\varepsilon$, $(v_\varepsilon)_\varepsilon \in (C^\infty(\Omega))^I$, $\lambda_1, \lambda_2 \in \mathbb{C}$,

$\forall K \Subset \Omega$, $\forall\alpha \in \mathbb{Z}_+^n$ we have

$$\exists m_1 = m_1(\alpha) > 0, \exists c_1 = c_1(\alpha) > 0, \exists\varepsilon_1 = \varepsilon_1(\alpha) \in I,\ \forall\varepsilon \leq \varepsilon_1, \sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| \leq c_1\varepsilon^{m_1}$$

$$\exists m_2 = m_2(\alpha) > 0, \exists c_2 = c_2(\alpha) > 0, \exists\varepsilon_2 = \varepsilon_2(\alpha) \in I,\ \forall\varepsilon \leq \varepsilon_2, \sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| \leq c_2\varepsilon^{m_2}$$

Then

$$\begin{aligned} \underset{x\in K}{sup}\left|\partial^\alpha(\lambda_1 u_\varepsilon(x)+\lambda_2 v_\varepsilon(x))\right| &\leq |\lambda_1|\left|\partial^\alpha u_\varepsilon(x)\right|+|\lambda_2|\left|\partial^\alpha v_\varepsilon(x)\right| \\ &\leq |\lambda_1|\, c_1\varepsilon^{m_1}+|\lambda_2|\, c_1\varepsilon^{m_2} \end{aligned}$$

Taken $c = Max(c1\,|\lambda_1|\,, c_2\,|\lambda_2|)$, $m = m_1 + m_2$ for $\varepsilon_0 = \min(\varepsilon_1, \varepsilon_2)$ we obtain

$$\underset{x\in K}{sup}\left|\partial^\alpha(\lambda_1 u_\varepsilon(x)+\lambda_2 v_\varepsilon(x)\right| \leq c\varepsilon^{-m}$$

Consequently $\lambda_1(u_\varepsilon)_\varepsilon + \lambda_2(v_\varepsilon)_\varepsilon \in \mathcal{E}_m(\Omega)$. Furthermore, we have

$$\left|\partial^\alpha(u_\varepsilon v_\varepsilon)(x)\right| \leq \sum_{\beta\leq\alpha} C_\alpha^\beta \left|\partial^\beta u_\varepsilon(x)\right| . \left|\partial^{\alpha-\beta} v_\varepsilon(x)\right|$$

Then

$$\begin{aligned} \sup_{x\in K}\left|\partial^\alpha(u_\varepsilon v_\varepsilon)(x)\right| &\leq \sum_{\beta\leq\alpha} C_\alpha^\beta \sup_{x\in K}\left|\partial^\beta u_\varepsilon(x)\right| .\sup_{x\in K}\left|\partial^{\alpha-\beta} v_\varepsilon(x)\right| \\ &\leq \sum_{\beta\leq\alpha} C_\alpha^\beta . c_\beta . \varepsilon^{-m_\beta} . c_{\alpha-\beta} . \varepsilon^{-m_{\alpha-\beta}} \end{aligned}$$

Taken $m = \underset{\beta\leq\alpha}{min}(m_\beta, m_{\alpha-\beta})$, $C = \sum_{\beta\leq\alpha} C_\alpha^\beta c_\beta c_{\alpha-\beta}$, then $(u_\varepsilon v_\varepsilon)_\varepsilon \in \mathcal{E}_m(\Omega)$.

Now, let $K \Subset \Omega$ and $\alpha \in \mathbb{Z}_+^n$, $\beta \in \mathbb{Z}_+^n$ so $\left|\partial^\alpha(\partial^\beta u_\varepsilon)(x)\right| = \left|\partial^{\alpha+\beta} u_\varepsilon(x)\right|$, and by definition

$\exists m_{\alpha+\beta} > 0,\ \exists c_{\alpha+\beta} > 0,\ \exists \varepsilon_{\alpha+\beta} \in I,\ \forall \varepsilon \leq \varepsilon_{\alpha+\beta}$

$$\left|\partial^\alpha(\partial^\beta u_\varepsilon)(x)\right| \leq c_{\alpha+\beta}\varepsilon^{-m_{\alpha+\beta}}$$

We take $c = c_{\alpha+\beta}$, $m = m_{\alpha+\beta}$ then $\left|\partial^\alpha(\partial^\beta u_\varepsilon)(x)\right| \leq c\varepsilon^{-m}$. Hence $\partial^\beta u_\varepsilon \in \mathcal{E}_m(\Omega)$.

ii) Let $(u_\varepsilon)_\varepsilon \in \mathcal{E}_m(\Omega)$, $(v_\varepsilon)_\varepsilon \in \mathcal{N}(\Omega)$, $K \Subset \Omega$ and $\alpha \in \mathbb{Z}_+^n$, we have

$$|\partial^\alpha (u_\varepsilon v_\varepsilon)(x)| \leq \sum_{\beta \leq \alpha} C_\alpha^\beta \left|\partial^\beta u_\varepsilon(x)\right| \left|\partial^{\alpha-\beta} v_\varepsilon(x)\right|$$

By definition $\forall \alpha, \beta \in \mathbb{Z}_+^n, \forall q > 0, \exists c_{\alpha-\beta} > 0, \exists \varepsilon_2 \in I, \forall \varepsilon < \varepsilon_2,$

$$\underset{x\in K}{sup} \left|\partial^{\alpha-\beta} v_\varepsilon(x)\right| \leq c_{\alpha-\beta}\varepsilon^q$$

So

$$\underset{x\in K}{sup} |\partial^\alpha (u_\varepsilon v_\varepsilon)(x)| \leq \sum_{\beta \leq \alpha} C_\alpha^\beta c_\beta \varepsilon^{-m_\beta} c_{\alpha-\beta}\varepsilon^q$$

Let $m' < \underset{\beta\leq\alpha}{min}(m_\beta, q)$, $m_\beta = q - m'$ and $C = \sum_{\beta \leq \alpha} C_\alpha^\beta c_\beta . c_{\alpha-\beta}$. We have

$$\underset{x\in K}{sup} |\partial^\alpha (u_\varepsilon v_\varepsilon)(x)| \leq C\varepsilon^q$$

This shows that $(u_\varepsilon v_\varepsilon)_\varepsilon \in \mathcal{N}(\Omega)$.

□

Proposition 1.2.4 *Let $(u_\varepsilon)_\varepsilon \in \mathcal{E}_m(\Omega)$, we say that $(u_\varepsilon)_\varepsilon$ is negligible if and only if the following condition is satisfied*

$$\forall K \Subset \Omega; \forall m \in \mathbb{N} : \sup_{x\in K} |u_\varepsilon(x)| = O(\varepsilon^m) \ \textit{as} \ \varepsilon \to 0.$$

Definition 1.2.5 *The simplified (or special) Colombeau algebra of generalized functions is de-*

fined as the quotient space

$$\mathcal{G}(\Omega) := \frac{\mathcal{E}_m(\Omega)}{\mathcal{N}(\Omega)}.$$

If $u \in \mathcal{G}(\Omega)$, we write $u = [(u_\varepsilon)_\varepsilon]$ and we say that u is the class of $(u_\varepsilon)_\varepsilon$.

Definition 1.2.6 *Let $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}(\Omega)$ and Ω' an open subset of Ω. The restriction of u to Ω', denote by u/Ω', is the element defined by $[(u_\varepsilon/\Omega')_\varepsilon]$, i.e.*

$$u/\Omega' = (u_{/\Omega'})_\varepsilon + \mathcal{N}(\Omega').$$

So it's clear that $u/\Omega' \in \mathcal{G}(\Omega')$.

Definition 1.2.7 *Let $u \in \mathcal{G}(\Omega)$ and Ω' an open subset of Ω. We say that u is null on Ω', if $u/\Omega' = 0$ in $\mathcal{G}(\Omega')$.*

The following result shows that $\Omega \to \mathcal{G}(\Omega)$ is a sheaf of differential algebras on $\mathbb{R}^n$.

Theorem 1.2.8 *Let $(\Omega_\lambda)_{\lambda\in\Lambda}$ be an open covering of Ω, and let $(u_\lambda)_{\lambda\in\Lambda}$ such that $u_\lambda \in \mathcal{G}(\Omega_\lambda)$, $\lambda \in \Lambda$, then we have the following properties.*

(i) *If $u, v \in \mathcal{G}(\Omega)$ and $u/\Omega_\lambda = v/\Omega_\lambda$, for all $\lambda \in \Lambda$, then $u = v$ on Ω.*

(ii) *If for all $\lambda, \mu \in \Lambda$, $u_\lambda/\Omega_\lambda \cap \Omega_\mu = u_\mu/\Omega_\lambda \cap \Omega_\mu$, with $\Omega_\lambda \cap \Omega_\mu \neq \emptyset$, then there exists a unique element $u \in \mathcal{G}(\Omega)$ such that $u/\Omega_\lambda = u_\lambda$ for all $\lambda \in \Lambda$.*

The property (i) of the last theorem motivate as the following definition.

Definition 1.2.9 *We call support of $u \in \mathcal{G}(\Omega)$, denoted $supp_g u$, the complement of the largest open subset where u is null, i.e.*

$$supp_g u = \Omega \backslash \left(\cup \{ \Omega' \text{ open subset of } \Omega, \; u/\Omega' \equiv 0 \} \right)$$

Definition 1.2.10 *We denote by $\mathcal{G}_c(\Omega)$ the subspace of $\mathcal{G}(\Omega)$ consisting of elements with compact support.*

1.3 Embedding of distributions

The main goal of this section is to give a complete analysis of the various techniques of embedding $\mathcal{D}'(\Omega)$ into Colombeau algebra $\mathcal{G}(\Omega)$.

Local structure of distribution

We first recall two classical results on the local structure of distributions.

Theorem 1.3.1 *For all $T \in \mathcal{D}'(\Omega)$ and all Ω' open subset of $\mathbb{R}^n$ with $\overline{\Omega'} \subset \Omega$, there exists $f \in C^0(\mathbb{R}^n)$ whose support is contained in arbitrary neighborhood of $\overline{\Omega'}$, $\alpha \in \mathbb{Z}_+^n$ such that $T_{|\Omega'} = \partial^\alpha f$.*

Theorem 1.3.2 *For all $T \in E'(\Omega)$, there exists an integer $r \geq 0$, a finite family $(f_\alpha)_{0 \leq |\alpha| \leq r} (\alpha \in \mathbb{Z}_+^n)$ with each $f_\alpha \in C^0(\mathbb{R}^n)$ having its support contained in the same arbitrary neighborhood of the support of T, such that $T = \sum_{0 \leq |\alpha| \leq r} \partial^\alpha f_\alpha$.*

Construction of mollifiers

Take $\rho \in \mathcal{S}(\mathbb{R}^n)$ even such that:

$$\int \rho(x)dx = 1, \int x^m \rho(x)dx = 0,\ \forall m \in \mathbb{N}^n \backslash \{0\}$$

And $\chi \in \mathcal{D}(\mathbb{R}^n)$ such that $0 \leq \chi \leq 1$, $\chi \equiv 1$ on $\overline{B(0,1)}$ and $\chi \equiv 0$ on $\mathbb{R}^n \backslash B(0,2)$.

Define

$$\forall \varepsilon \in]0,1], \quad \forall x \in \mathbb{R}^n, \quad \rho_\varepsilon(x) = \frac{1}{\varepsilon^n}\rho(\frac{x}{\varepsilon})$$

And

$$\forall \varepsilon \in]0,1], \quad \forall x \in \mathbb{R}^n, \quad \theta_\varepsilon(x) = \rho_\varepsilon(x)\chi(|\ln \varepsilon|\, x)$$

Lemma 1.3.3 *We have the following properties:*

- $[(\theta_\varepsilon) - (\rho_\varepsilon)]_\varepsilon \in \mathcal{N}(\Omega)$.
- $\forall k \in \mathbb{N}$, $\int \theta_\varepsilon(x)dx = 1 + o(\varepsilon^k)$ *for* $\varepsilon \to 0$.
- $\forall k \in \mathbb{N}$, $\forall m \in \mathbb{N}^n \backslash \{0\}$, $\int x^m \theta_\varepsilon(x)dx = o(\varepsilon^k)$ *as* $\varepsilon \to 0$

In other words, we have $\left(\int \theta_\varepsilon(x)dx - 1\right)_\varepsilon \in \mathcal{N}(\Omega)$*; and for all* $m \in \mathbb{N}^n \backslash \{0\}$ *we have* $\left(\int x^m \theta_\varepsilon(x)dx\right)_\varepsilon \in \mathcal{N}(\Omega)$

The space $C^\infty(\Omega)$ is canonically embedded into $\mathcal{G}(\Omega)$ by the map

$$\begin{array}{rlcl} i: & C^\infty(\Omega) & \to & \mathcal{G}(\Omega) \\ & f & \to & (f_\varepsilon)_\varepsilon + \mathcal{N}(\Omega) \end{array}$$

Where $f_\varepsilon = f$, $\forall \varepsilon \in I$.

Construction of the embedding i_A

Proposition 1.3.4 *The map i_0 defined by*

$$\begin{array}{lll} i_0: & E'(\Omega) & \to & \mathcal{G}(\Omega) \\ & w & \to & ((w * \rho_\varepsilon)_{|\Omega})_\varepsilon + \mathcal{N}(\Omega) \end{array}$$

is a linear and injective.

Let $(\Omega_\lambda)_{\lambda\in\Lambda}$ be an open covering of Ω such that each $\overline{\Omega_\lambda}$ is compact subset of Ω and let $(\psi_\lambda)_\lambda \subset \mathcal{D}(\Omega)$ such that each $\psi_\lambda \equiv 1$ in a neighborhood of $\overline{\Omega_\lambda}$. For any $\lambda \in \Lambda$, we define the linear map i_λ by

$$\begin{array}{lll} i_\lambda: & \mathcal{D}'(\Omega) & \to & \mathcal{G}(\Omega_\lambda) \\ & T & \to & ((\psi_\lambda T * \rho_\varepsilon)/\Omega_\lambda)_\varepsilon + \mathcal{N}(\Omega), \end{array}$$

i.e. $i_\lambda(T) = i_0(\psi_\lambda T)/\Omega_\lambda$. The linear map $i_\lambda/\mathcal{D}'(\Omega_\lambda)$, from the proposition (1.3.4), is injective. For any $T \in \mathcal{D}'(\Omega)$, we check easily, that $(i_\lambda(T))_\lambda$ is a coherent family of generalized functions, i.e

$$i_\lambda(T)/\Omega_\lambda \cap \Omega_\mu = i_\mu(T)/\Omega_\lambda \cap \Omega_\mu, \qquad \forall \lambda, \mu \in \Lambda$$

If $(\chi_j)_{j=1}^{\infty}$ is a smooth partition of unity subordinate to the covering $(\Omega_\lambda)_{\lambda\in\Lambda}$ we define the linear injective map

$$\begin{array}{lll} i_A: & \mathcal{D}'(\Omega) & \to & \mathcal{G}(\Omega) \\ & T & \to & [T] = cl(\sum_j \chi_j(\psi_{\lambda_j} T * \rho_\varepsilon)) \end{array}$$

Which is independent of the chose of $(\Omega_\lambda)_{\lambda\in\Lambda}$, $(\chi_j)_{j\in\mathbb{N}}$ and (ψ).

Proposition 1.3.5 *We have*

i) $i_{A/E'(\Omega)} = i_0$

ii) $i_{A/C^\infty(\Omega)} = i$

The second item of the above proposition means that the space $C^\infty(\Omega)$ can be embedded into $\mathcal{G}(\Omega)$ by two ways, shown by following commutative diagram

$$\begin{array}{ccc} C^\infty(\Omega) & \rightarrow & \mathcal{D}'(\Omega) \\ & {}_i\searrow & \downarrow i_A \\ & & \mathcal{G}(\Omega) \end{array}$$

Construction of embedding i_s

Proposition 1.3.6 *The map*

$$\begin{array}{rccl} i_s : & \mathcal{D}'(\mathbb{R}^n) & \rightarrow & \mathcal{G}(\mathbb{R}^n) \\ & T & \rightarrow & (T * \theta_\varepsilon)_\varepsilon + \mathcal{N}(\mathbb{R}^n) \end{array}$$

is an injective embedding. Moreover $i_{s|C^\infty(\mathbb{R}^n)} = i$.

Proof. Let K be a compact set of Ω, we know that

$$\forall y \in \mathbb{R}^n, T * \theta_\varepsilon(y) = \langle T, x \mapsto \theta_\varepsilon(y - x)\rangle$$

For $y \in K$ and $x \in \mathbb{R}^n$, we have $\theta_\varepsilon(y-x) \neq 0 \Rightarrow y-x \in B(0, \frac{2}{|\ln \varepsilon|}) \Rightarrow x \in B(y, \frac{2}{|\ln \varepsilon|}) \Rightarrow x \in \Omega$ for ε small enough. Then the function $x \mapsto \theta_\varepsilon(y-x)$ belongs to $\mathcal{D}(\Omega)$ and

$$\langle T, \theta_\varepsilon(y-.)\rangle = \left\langle T_{|\Omega}, \theta_\varepsilon(y-.)\right\rangle$$

Using (1.3.1), we have $T_\Omega = \partial_x^\alpha f$ where $f \in C_0(\Omega)$. Then $\partial^\beta(T * \theta_\varepsilon) = f * \partial^{\alpha+\beta}\theta_\varepsilon$ and

$$\forall y \in K, \partial^\beta(T * \theta_\varepsilon)(y) = \int_\Omega f(x-y).\partial^{\alpha+\beta}\theta_\varepsilon(x)dx$$

And we know that $\theta_\varepsilon \in \mathcal{E}_m(\Omega)$. So $\forall \alpha, \beta \in \mathbb{Z}_+$, $\exists m_{\alpha+\beta} > 0$

$$\forall x \in \mathbb{R}^n, |\partial^\alpha \theta_\varepsilon(x)| \leq \varepsilon^{-m_{\alpha+\beta}}$$

We get

$$\left|\partial^\beta(T * \theta_\varepsilon(y))\right| \leq c \sup_{\xi \in \overline{\Omega}} |f(\xi)| .mes(\overline{\Omega}).\varepsilon^{-m_{\alpha+\beta}}$$

So $\sup_{y \in K} |(T * \theta_\varepsilon)(y)| \leq C\varepsilon^{-m_\beta}$. Consequently $(T * \theta_\varepsilon)_\varepsilon \in \mathcal{E}_m(\Omega)$.
Let us prove that i_s injective, i.e.

$$(T * \theta_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}^n) \Rightarrow T = 0$$

Taking $\varphi \in \mathcal{D}(\mathbb{R}^n)$, we have

$$\langle T * \theta_\varepsilon, \varphi\rangle \to \langle T, \varphi\rangle \ \ then\ (T * \theta_\varepsilon \to T \ \ in\ \mathcal{D}')$$

And $T*\theta_\varepsilon \to 0$ uniformly on $supp\varphi$ since $T*\theta_\varepsilon \in \mathcal{N}(\mathbb{R}^n)$. Then $\langle T*\theta_\varepsilon, \varphi\rangle \to 0$ and $\langle T, \varphi\rangle = 0$.

We will prove the last assertion in the case of dimension one. Let $f \in C^\infty(\mathbb{R})$ and set:

$\Delta = i_s(f) - i(f)$. One representative of Δ is given by

$$\begin{array}{rccl} \Delta_\varepsilon : & \mathbb{R} & \to & \mathcal{E}_m(\Omega) \\ & y & \to & (f*\theta_\varepsilon)(y) - f(y) = \int f(y-x)\theta_\varepsilon(x)dx - f(y) \end{array}$$

Let K a compact subset of $\mathbb{R}$, and $\int \theta_\varepsilon(x)dx = 1 + N_\varepsilon$ with $(N_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R})$, we get

$$\Delta_\varepsilon(y) = \int (f(x-y) - f(y))\theta_\varepsilon(x)dx - N_\varepsilon f(y)$$

According to Taylor's formula we have

$$f(y-x) - f(y) = \sum_{i=1}^{n} \frac{(-x)^i}{i!} f^{(i)}(y) + \frac{(-x)^n}{n!}\int_0^1 f^{(n+1)}(y-ux)(1-u)^n du$$

And

$\Delta_\varepsilon(y) = \sum\limits_{i=1}^{n} \frac{(-1)^i}{i!} f^{(i)}(y) \int_{-\frac{2}{|\ln\varepsilon|}}^{\frac{2}{|\ln\varepsilon|}} x^i\theta_\varepsilon(x)dx + \int_{-\frac{2}{|\ln\varepsilon|}}^{\frac{2}{|\ln\varepsilon|}} \frac{(-x)^n}{n!}\int_0^1 f^{(n+1)}(y-ux)(1-u)^n du\theta_\varepsilon(x)dx - N_\varepsilon.f(y).$

Put $P_\varepsilon(n,y) = \sum\limits_{i=1}^{n} \frac{(-1)^i}{i!} f^{(i)}(y) \int_{-\frac{2}{|\ln\varepsilon|}}^{\frac{2}{|\ln\varepsilon|}} x^i\theta_\varepsilon(x)dx$

And $R_\varepsilon(n,y) = \int_{-\frac{2}{|\ln\varepsilon|}}^{\frac{2}{|\ln\varepsilon|}} \frac{(-x)^n}{n!}\int_0^1 f^{(n+1)}(y-ux)(1-u)^n du\theta_\varepsilon(x)dx - N_\varepsilon.f(y).$

According to Lemma (1.3.3), we have $(\int x^i\theta_\varepsilon(x)dx)_\varepsilon \in \mathcal{N}(\mathbb{R})$ and consequently

$(P_\varepsilon(n,y))_\varepsilon \in \mathcal{N}(\mathbb{R})$.

Using the definition of θ_ε, we have

$$R_\varepsilon(n,y) = \frac{1}{\varepsilon}\int_{-\frac{2}{|\ln\varepsilon|}}^{\frac{2}{|\ln\varepsilon|}} \frac{(-x)^n}{n!}\int_0^1 f^{(n+1)}(y-ux)(1-u)^n du\rho(\frac{x}{\varepsilon})\chi(x\,|\ln\varepsilon|)dx.$$

Setting $v = x/\varepsilon$, we get

$$R_\varepsilon(n, y) = \varepsilon^{n+1} \int_{-\frac{2}{|\ln\varepsilon|}}^{\frac{2}{|\ln\varepsilon|}} \frac{(-v)^n}{n!} \int_0^1 f^{(n+1)}(y - \varepsilon uv)(1-u)^n du \rho(v).\chi(\varepsilon\,|\ln\varepsilon|) dv$$

For $(\varepsilon, v) \in [0,1] \times [-\frac{2}{\varepsilon|\ln\varepsilon|}, y + \frac{2}{\varepsilon|\ln\varepsilon|}]$ we have $y - \varepsilon uv \in [y - \frac{2}{\varepsilon|\ln\varepsilon|}, \frac{2}{\varepsilon|\ln\varepsilon|}]$.

Then for $y \in K$ then $y - \varepsilon uv$ is in a compact set K'. It follows:

$$\begin{aligned} |R_\varepsilon(n, y)| &\leq \frac{\varepsilon^n}{n!} \sup_{\xi \in K'} \left| f^{(n+1)}(\xi) \right| \int_{-\frac{2}{\varepsilon|\ln\varepsilon|}}^{\frac{2}{\varepsilon|\ln\varepsilon|}} |v|^{n+1} |\rho(v)|\, dv \\ &\leq \frac{\varepsilon^n}{n!} \sup_{\xi \in K'} \left| f^{(n+1)}(\xi) \right| \int_{-\infty}^{+\infty} |v|^{n+1} |\rho(v)|\, dv \\ &\leq C\varepsilon^n \qquad (C > 0) \end{aligned}$$

The constant C depend only on the integer n, the compact sets K and K', ρ and f.

Finally, as $(\Delta_\varepsilon)_\varepsilon \in \mathcal{E}_m(\mathbb{R})$ and $\sup_{x \in K} \Delta_\varepsilon(y) = o(\varepsilon^n)$ for all $n > 0$ and $K \Subset \mathbb{R}$, we conclude that $(\Delta_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R})$ □

For the same choice of ρ, we have $i_A = i_S$

1.4 Generalized numbers and point values

Generalized numbers are defined by point values of a generalized function $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}(\Omega)$ at a point $x \in \Omega$. For a fixed $x \in \Omega$, if we take the sequence $(u_\varepsilon(x))_\varepsilon$ a new object, called generalized number, appears. Set

$$\mathcal{E}_m(\mathbb{K}) = \{(z_\varepsilon)_\varepsilon \in \mathbb{K}^I, \exists m \in \mathbb{Z}_+, |z_\varepsilon| = O(\varepsilon^{-m}), \varepsilon \to 0\},$$

And

$$\mathcal{N}(\mathbb{K}) = \{(z_\varepsilon)_\varepsilon \in \mathbb{K}^I, \forall q \in \mathbb{Z}_+, |z_\varepsilon| = O(\varepsilon^q), \varepsilon \to 0\},$$

where $\mathbb{K}$ design $\mathbb{C}$ or $\mathbb{R}$.

Clearly the space $\mathcal{E}_m(\mathbb{K})$ is an subalgebra of $\mathbb{K}^I$ and $\mathcal{N}(\Omega)$ is an ideal of $\mathcal{E}_m(\mathbb{K})$.

Definition 1.4.1 *The Colombeau algebra of generalized complex (resp.real) numbers, denoted* $\widetilde{\mathbb{C}} = \dfrac{\mathcal{E}_m(\mathbb{C})}{\mathcal{N}(\mathbb{C})}, \left(\textit{resp } \widetilde{\mathbb{R}} = \dfrac{\mathcal{E}_m(\mathbb{R})}{\mathcal{N}(\mathbb{R})} \right)$

Proposition 1.4.2 *The field $\mathbb{K}$ is canonically embedded into the ring $\widetilde{\mathbb{K}}$ by the following map*

$$\begin{array}{ccl} \mathbb{K} & \to & \widetilde{\mathbb{K}} \\ z & \to & [z] = (z)_\varepsilon + \mathcal{N}[\mathbb{K}]. \end{array}$$

Remark 1.4.3 *The algebra $\widetilde{\mathbb{K}}$ is not a field, is just a ring.*

Definition 1.4.4 *Let $\in \mathcal{G}(\Omega)$ and $x_0 \in \Omega$, the point value of u at x_0, denoted $f(x_0)$, is the generalized number represented by $(u_\varepsilon(x_0))_\varepsilon$, where $(u_\varepsilon)_\varepsilon$ is a representative of u.*

The generalized number $u(x_0)$ does not depend of the choice of representative $(u_\varepsilon)_\varepsilon$ of u.

Example 1.4.5 *We know that $x\delta \equiv 0$ in $\mathcal{D}'(\mathbb{R})$, but $i(x)i(\delta) \neq 0$ in $\mathcal{G}(\mathbb{R})$, however, every point value of this generalized function is null.*

The last example show that generalized functions are not determined by their point values. To solve this problem, we introduce the notion of generalized points, see [34].

Let Ω be an open set of $\mathbb{R}^n$, on

$$\Omega_M = \{(x_\varepsilon)_\varepsilon \in \Omega^I, \exists N \in \mathbb{Z}_+, \exists \eta > 0, \forall \varepsilon \leq \eta, |x_\varepsilon| < \varepsilon^{-m}\},$$

We define an equivalence relation by

$$(x_\varepsilon)_\varepsilon \sim (y_\varepsilon)_\varepsilon \Leftrightarrow \forall m \in \mathbb{Z}_+, \exists \eta > 0, \forall \varepsilon < \eta, |x_\varepsilon - y_\varepsilon| < \varepsilon^q$$

Definition 1.4.6 *We call the set of generalized points of Ω, denoted $\widetilde{\Omega}$, the quotient set defined by $\widetilde{\Omega} = \Omega_M / \sim$.*

Definition 1.4.7 *The set $\widetilde{\Omega_c}$ defined by*

$$\widetilde{\Omega_c} = \{\widetilde{x} \in \widetilde{\Omega}, \exists (x_\varepsilon)_\varepsilon \textit{ a representative of } \widetilde{x}, \exists K \textit{ a compact of } \Omega, \exists \eta > 0, x_\varepsilon \in K \textit{ if } 0 < \varepsilon < \eta\}$$

is called the set of generalized compactly supported points.

Proposition 1.4.8 *Let $u \in \mathcal{G}(\Omega)$ and $\widetilde{x} \in \widetilde{\Omega_c}$, then the generalized point value of u at $\widetilde{x} = [(x_\varepsilon)_\varepsilon]$, is $u(\widetilde{x}) = [u_\varepsilon(x_\varepsilon))_\varepsilon]$, which is well-defined element of $\widetilde{\mathbb{C}}$.*

The following theorem gives a characterization of generalized function by their point values.

Theorem 1.4.9 *Let Ω be an open set of $\mathbb{R}^n$, then*

$$u = 0 \textit{ in } \mathcal{G}(\Omega) \Leftrightarrow u(\widetilde{x}) = 0 \textit{ in } \widetilde{\mathbb{C}}, \forall \widetilde{x} \in \widetilde{\Omega_c}.$$

Proof. *See [34]* □

1.5 Notion of association

In this section we are going to introduce an association relationship by which we can identify in $\mathcal{G}(\Omega)$ the same elements of $\mathcal{D}'(\Omega)$.

Definition 1.5.1 1) *An element u of $\mathcal{G}(\Omega)$ is called associated with 0 (denoted by $u \approx 0$) if*

$$\lim_{\varepsilon \to 0} \int_{\Omega} u_{\varepsilon}(x)\varphi(x)dx = 0, \qquad \forall \varphi \in \mathcal{D}(\Omega)$$

The definition is independent of the chosen representative $(u_{\varepsilon})_{\varepsilon}$ of u.

2) *Let $u, v \in \mathcal{G}(\Omega)$ are associated with each other if and only if $u - v \approx 0$*

Let us take a closer look at the interplay between distributions and equivalence classes in $\mathcal{G}(\Omega)$ with respect to association

Definition 1.5.2 *Let $u \in \mathcal{G}(\Omega)$ and $w \in \mathcal{D}'(\Omega)$ and suppose that $u \approx i(w)$ then u is said to admit w as associated distribution and w is also called distributional shadow of u. In this case we simply write $u \approx w$.*

The distributional shadow of u is uniquely determined (if it exists)

Proposition 1.5.3 *If $w \in \mathcal{D}'(\Omega)$ and $i(w) \approx 0$ then $w = 0$.*

Example 1.5.4 *(i) We have $x\delta = 0$ in $\mathcal{D}'(\mathbb{R})$, but $x\delta(x) \neq 0$ in $\mathcal{G}(\mathbb{R})$ and that all points values of $x\delta(x)$ vanish. To round off this picture, we show that: $x\delta(x)(= xi(\delta)) \approx 0$. Indeed, if $\varphi \in \mathcal{D}(\mathbb{R})$ then:*

$$\int x\rho_{\varepsilon}(x)\varphi(x)dx = \varepsilon \int y\rho(y)\varphi(\varepsilon y)dy \underset{\varepsilon \to 0}{\to} 0$$

(ii) There are elements of $\mathcal{G}(\Omega)$ that do not have any shadow. Taking for instance $\delta^2 \in \mathcal{G}(\mathbb{R})$ we have

$$\int \rho_\varepsilon^2(x)\varphi(x)dx = \frac{1}{\varepsilon}\int \rho^2(x)\varphi(\varepsilon y)dy \underset{\varepsilon\to 0}{\to} \infty$$

Lemma 1.5.5 *(i) If $f \in C^\infty(\Omega)$ and $w \in \mathcal{D}'(\Omega)$ then*

$$i(f)i(w) \approx i(fw).$$

(ii) If $u, v \in \mathcal{G}(\Omega)$ and $u \approx v$ then

- ♦ $\partial^\alpha u \approx \partial^\alpha v, \qquad \forall \alpha \in \mathbb{N}_0^n.$
- ♦ $i(f)u \approx i(f)v, \qquad \forall f \in C^\infty(\Omega).$

Lemma 1.5.6 *The product of generalized function $\frac{1}{x}$ and δ satisfy*

$$\frac{1}{x}\delta \approx -\frac{1}{2}\delta'$$

Proof. For any net of molifiers $(\rho_\varepsilon)_\varepsilon$ and any $\varphi \in \mathcal{D}(\mathbb{R})$ we can write

$$\left\langle \rho_\varepsilon(\frac{1}{x} * \rho_\varepsilon), \varphi \right\rangle = \left\langle \frac{1}{x}, \check{\rho}_\varepsilon * \varphi\rho_\varepsilon \right\rangle$$

If we write $\varphi(x) = \varphi(0) + \varphi'(0)x + x^2\psi(x)$, then $\psi(0) = \underset{x\to 0}{lim}\psi(x) = 2^{-1}\varphi''(0)$. $\underset{x\to 0}{lim} x^{-1}(\psi(x) - \psi(0)) = (3!)^{-1}\varphi'''(0)$ and so on. Since $\check{\rho}_\varepsilon * \rho_\varepsilon$ is an even function, $\left\langle \frac{1}{x}, \check{\rho}_\varepsilon * \rho_\varepsilon \right\rangle$ vanishes. Put $\alpha_\varepsilon = \check{\rho}_\varepsilon * x\rho_\varepsilon$ then $\check{\alpha}_\varepsilon = \rho_\varepsilon * (-x)\check{\rho}_\varepsilon = (-x)(\rho_\varepsilon * \check{\rho}_\varepsilon) + (x\rho_\varepsilon) * \check{\rho}_\varepsilon$, and so $\alpha_\varepsilon - \check{\alpha}_\varepsilon = x(\rho_n * \check{\rho})$ and therefore

$$\left\langle \frac{1}{x}, \alpha_\varepsilon \right\rangle = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{1}{x}(\alpha_\varepsilon(x) - \check{\alpha}_\varepsilon(x))dx = \frac{1}{2}\int_{-\infty}^{+\infty} \rho_\varepsilon * \check{\rho}_\varepsilon(x)dx \to \frac{1}{2}$$

and thus the second term tends $\frac{1}{2}\varphi'(0) = \left\langle -\frac{1}{2}\delta', \varphi \right\rangle$ Now it follows from Itano [20] that the third term tends to 0 if ε tend to 0 $\square$

An important result has obtained by Damyanov [12]

Proposition 1.5.7 *For an arbitrary p in $\mathbb{N}_0^n$, let*

$$x_+^p = \begin{cases} x^p & \textit{for } x \geq 0 \\ 0 & \textit{elsewhere} \end{cases}$$

and

$$x_-^p = \begin{cases} x^p & \textit{for } x \leq 0 \\ 0 & \textit{elsewhere} \end{cases}$$

Then it holds

$$x_+^p.\delta^{(p)}(x) \approx (-1)^{|p|}p!2^{-n}\delta(x)$$

$$x_-^p.\delta^{(p)}(x) \approx p!2^{-n}\delta$$

Remark 1.5.8 *The equation*

$$x^p.\delta^{(p+q)}(x) = \frac{(-1)^p(p+q)!}{q!}\delta^{(q)}(x) \qquad (p, q \in \mathbb{N}_0)$$

is easily show to hold in $\mathcal{D}'(\mathbb{R})$ in view of the identity $x^p = x_+^p + (-1)^p x_-^p$.

Proposition 1.5.9 *The product of the generalized function x^{-p} and $\delta^{(p)}$ for $p = 1, 2, 3, ...$ and*

$q = 0, 1, 2...$ *in* $\mathcal{G}(\mathbb{R})$ *admits associated distributions and it holds:*

$$\frac{1}{x^p}.\delta^{(q)}(x) \approx \frac{(-1)^p q! \delta^{(p+q)}(x)}{2(p+q)!}$$

Proof. We have: $x^{-1}.\delta \approx -\frac{1}{2}.\delta'(x)$ So

$\partial^{p+q-1}(x^{-1}.\delta(x)) \approx \partial^{p+q-1}(-\frac{1}{2}.\delta'(x))$

$\sum\limits_{k=0}^{p+q-1} C_k^{p+q-1}(x^{-1})^{(k)}.\delta^{(p+q-1-k)}(x) \approx \frac{-1}{2}\delta^{(p+q)}(x)$

$\sum\limits_{k=0}^{p+q-1} \dfrac{(p+q-1)!}{k!(p+q-1-k)!} \dfrac{(-1)^k k!}{x^{k+1}}.\delta^{(p+q-k-1)}(x) \approx \dfrac{-1}{2}\delta^{(p+q)}(x)$

$\sum\limits_{k=0}^{p+q-1} \dfrac{(p+q-1)!(-1)^k}{(p+q-1-k)!}.\dfrac{x^{(p-k-1)}}{x^{k+1}.x^{p-k-1}}.\delta^{p+q-k-1}(x) \approx \dfrac{-1}{2}\delta^{(p+q)}(x)$ *such that* $p \geq 1$

Using

$$x^P.\delta^{(p+q)}(x) \approx \frac{(-1)^p(p+q)!}{q!}\delta^{(q)}(x) \qquad (p, q \in \mathbb{N}_0)$$

we found

$\sum\limits_{k=0}^{p+q-1} \dfrac{(p+q-1)!(-1)^k}{(p+q-1-k)!x^p} \dfrac{(-1)^{p-k-1}(p+q-k-1)!}{q!}.\delta^{(q)}(x) \approx \dfrac{-1}{2}\delta^{(p+q)}(x)$

$\sum\limits_{k=0}^{p+q-1} \dfrac{(p+q-1)!.(-1)^{p-1}}{q!}.\dfrac{1}{x^p.\delta^{(q)}(x)} \approx \dfrac{-1}{2}\delta^{(p+q)}(x)$

$\dfrac{(p+q)!.(-1)^{p-1}}{q!}.\dfrac{1}{x^p\delta^{(q)}(x)} \approx \dfrac{-1}{2}\delta^{(p+q)}(x)$

Then

$$\frac{1}{x^p}.\delta^{(q)}(x) \approx \frac{(-1)^p q! \delta^{(p+q)}(x)}{2(p+q)!}$$

□

Next, to extend this result in $\mathcal{G}(\mathbb{R}^n)$ we need the following lemma

Lemma 1.5.10 *Let* u *and* v *be distribution in* $\mathcal{D}'(\mathbb{R}^n)$ *such that*

$u(x) = \prod\limits_{i=1}^{n} u^i(x_i)$, $v(x) = \prod\limits_{i=1}^{n} v^i(x_i)$ *with each* u^i *and* v^i *in* $\mathcal{D}'(\mathbb{R})$, *and suppose that their embedding in* $\mathcal{G}(\mathbb{R})$ *satisfy* $\tilde{u}^i.\tilde{v}^i \approx w^i$, *for* $i = 1, ..., n$. *Then* $\tilde{u}.\tilde{v} \approx w$, *where* $w = \prod_{i=1}^{n} w^i(x_i)$

Theorem 1.5.11 *The product of the generalized functions x^p and $\delta^{(p)}(x)$ for $k = 1, 2, ...$ and $p = 0, 1, 2, ...$ in $\mathcal{G}(\mathbb{R}^n)$ admits associated distributions and it holds:*

$$\frac{1}{x^p}.\delta^{(q)}(x) \approx \frac{(-1)^{|p|}q!\delta^{(p+q)}(x)}{2^n(p+q)!}$$

Proof. In this case, due to the tensor product structure of the distributions that are considered, we can apply previous Lemma and we have:

$$\begin{aligned} x^p.\delta^{(q)}(x) &= \prod_{i=1}^n x_i^{-p_i}.\delta^{(q_i)}(x_i) \approx \prod_{i=1}^n \frac{(-1)^{p_i}q_i!\delta^{(p_i+q_i)}(x_i)}{2(p_i+q_i)!} \\ &= \frac{(-1)^{|p|}q!\delta^{(p+q)}(x)}{2^n(p+q)!} \end{aligned}$$

□

1.6 Local and microlocal analysis in $\mathcal{G}(\Omega)$

Set

$$\mathcal{E}_m^\infty(\Omega) = \{(u_\varepsilon)_\varepsilon \in (C^\infty(\Omega))^I, \forall K \Subset \Omega, \exists m \in \mathbb{Z}_+, \forall \alpha \in \mathbb{Z}_+^n, \sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{-m}) \text{ as } \varepsilon \to 0\}$$

Proposition 1.6.1 *(i) The space $\mathcal{E}_m^\infty(\Omega)$ is an subalgebra of $\mathcal{E}_m(\Omega)$.*

(ii) The space $\mathcal{N}(\Omega)$ is an ideal of $\mathcal{E}_m^\infty(\Omega)$.

Definition 1.6.2 *The quotient*

$$\mathcal{G}^\infty(\Omega) = \frac{\mathcal{E}_m^\infty(\Omega)}{\mathcal{N}(\Omega)}$$

is called algebra of regular generalized functions, or Oberguggenberger algebra.

The algebra of regular generalized functions $\mathcal{G}^\infty(\Omega)$ plays, in $\mathcal{G}(\Omega)$, the same role as $C^\infty(\Omega)$ in $\mathcal{D}'(\Omega)$ as $\mathcal{G}^\infty(\Omega) \cap \mathcal{D}'(\Omega) = C^\infty(\Omega)$ proved by Oberguggenberger [32]. The algebra $\mathcal{G}^\infty(\Omega)$ is a subsheaf of $\mathcal{G}(\Omega)$, which give as the definition of generalized singular support of an element of $\mathcal{G}(\Omega)$.

Definition 1.6.3 *The generalized singular support of $u \in \mathcal{G}(\Omega)$, denoted $singsupp_g(u)$, is the complement of the largest open set $\Omega' \subset \Omega$ where u is $\mathcal{G}^\infty$.*

Proposition 1.6.4 *Let $\omega \in \mathcal{D}'(\Omega)$, then*

$$singsupp_g(i(\omega)) = singsupp(\omega)$$

Microlocal analysis in Colombeau algebra has been initiated in [13], [30] as natural extension of its distribution theoretic analogue.

Using the Paley-Wiener theorem of Colombeau generalized functions see [29], [30], we have a characterization of element $\mathcal{G}^\infty$ by its Fourier transform as following definition

Definition 1.6.5 *A generalized function $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}(\Omega)$ is said to be $\mathcal{G}^\infty$-microlocally regular at $(x_0, \xi_0) \in \Omega \times \mathbb{R}^n \backslash \{0\}$ if there exist a relatively compact open neighborhood U of x_0, a conic neighborhood $\Gamma \subset \mathbb{R}^n \backslash \{0\}$ of ξ_0, a function $\phi \in \mathcal{D}(U)$ such that $\phi(x_0) = 1$ and natural number N such that for all $\alpha \in \mathbb{Z}_+^n$,*

$$|\xi^\alpha| \, . \, |\mathcal{F}(\phi u_\varepsilon)(\xi)| = O(\varepsilon^{-N}) \text{ as } \varepsilon \to 0.$$

The generalized wave front set of u, denoted $WF_g(u)$, *is the complement of the set of points* (x_0, ξ_0) *where u is* $\mathcal{G}^\infty$*-microlocally regular.*

Proposition 1.6.6 *Let* $T \in \mathcal{D}'(\Omega)$ *then*

$$WF_g(T) = WF(T)$$

Proof. See [17] □

The main properties of the generalized wave front set WF_g are resumed in the following proposition

Proposition 1.6.7 *Let* $f \in \mathcal{G}(\Omega)$, *then*

1) The projection of $WF_g(f)$ *on* Ω *is the* $singsupp_g(f)$.

2) $\forall \alpha \in \mathbb{Z}_+^n, WF_g(\partial^\alpha f) \subset WF_g(f)$.

3) $\forall g \in \mathcal{G}^\infty(\Omega), WF_g(gf) \subset WF_g(f)$.

Corollary 1.6.8 *Let* $P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$ *be a partial differential operator with* $\mathcal{G}^\infty(\Omega)$ *coefficients then*

$$WF_g(P(x, D)f) \subset WF_g(f),\ \forall f \in \mathcal{G}(\Omega)$$

The reverse inclusion, studied in [13], gives a generalized microlocal hypoellipticity of linear partial differential operators with regular generalized coefficients, which are micro-elliptic. The case of generalized G^∞-microlocal hypoellipticity of generalized micro-hypoelliptic linear partial differential operator has been studied recently in [19]. The generalized G^∞-microlocal hypoellipticity of micro-elliptic generalized pseudodifferential operators has been tackled in [22], and lastly in [21] this result was extended to the level of basic functionals in the dual of Colombeau algebra.

1.7 Generalized Hörmander's theorem

To extend the Hörmander's result on the wave front set of the product, define $WF_g(f) + WF_g(f)$, where $f, g \in \mathcal{G}(\Omega)$, as the set

$$\{(x, \xi + \eta) \in WF_g(f), (x, \eta) \in WF_g(g)\}$$

The principal result of this section is the following theorem.

Theorem 1.7.1 *Let $f, g \in \mathcal{G}(\Omega)$, such that $\forall x \in \Omega$,*

$$(x, 0) \notin WF_g(f) + WF_g(g) \tag{1.3}$$

Then

$$WF_g(f.g) \subseteq (WF_g(f) + WF_g(g)) \cup WF_g(f) \cup WF_g(g).$$

Proof. See [19] □

Chapter 2

Algebras of generalized Gevrey ultradistributions

A Colombeau type theory of generalized Gevrey ultradistributions has been addressed in [3], where we recovered a whole list of important result known for the usual Colombeau theory in the setting of generalized Geverey ultradistributions.

This chapter is aimed at giving first a general construction of algebras of generalized Gevrey Ultradistributions and then the microlocal analysis suitable for them. Finally, we give an application through an extension of the well-know Hörmander's theorem on the wave front of the product of two distributions.

2.1 Generalized Gevrey ultradistributions

We first introduce the algebra of moderate elements and it's ideal of null elements depending on the order $\tau > 0$.

Definition 2.1.1 *The space of moderate elements, denoted $\mathcal{E}_m^{\tau}(\Omega)$, is the space of $(f_\varepsilon)_\varepsilon \in C^\infty(\Omega)^I$, satisfying for every compact K of Ω, $\forall\alpha \in \mathbb{Z}_+^n$, $\exists k > 0$, $\exists c > 0$, $\exists\varepsilon_0 \in I$,*

$$\sup_{x\in K} |\partial^\alpha f_\varepsilon(x)| \le c\exp(k\varepsilon^{-\frac{1}{\tau}}), \qquad \forall\varepsilon \le \varepsilon_0$$

The space of null elements, denoted $\mathcal{N}^\tau(\Omega)$, is the space of $(f_\varepsilon)_\varepsilon \in C^\infty(\Omega)^I$ satisfying for every compact K of Ω, $\forall\alpha \in \mathbb{Z}_+^n$, $\forall k > 0$, $\exists c > 0$, $\exists\varepsilon_0 \in I$,

$$\sup_{x\in K} |\partial^\alpha f_\varepsilon(x)| \le c\exp(-k\varepsilon^{-\frac{1}{\tau}}), \qquad \forall\varepsilon \le \varepsilon_0$$

Proposition 2.1.2 *1) The space of moderate elements $\mathcal{E}_m^{\tau}(\Omega)$ is an algebra stable by derivation.*

2) The space $\mathcal{N}^\tau(\Omega)$ is an ideal of $\mathcal{E}_m^{\tau}(\Omega)$.

Proof. Let $(u_\varepsilon)_\varepsilon$, $(v_\varepsilon)_\varepsilon \in \mathcal{E}_m^{\tau}(\Omega)$ and K be a compact subset of Ω. Then $\forall\beta \in \mathbb{Z}_+^n$; $\exists k_1 = k_1(\beta) > 0$, $\exists c_1 = c_1(\beta) > 0$, $\exists\varepsilon_{1\beta} \in I$, $\forall\varepsilon \le \varepsilon_{1\beta}$,

$$\sup_{x\in K} \left|\partial^\beta u_\varepsilon(x)\right| \le c_1\exp(k_1\varepsilon^{-\frac{1}{\tau}})$$

$\forall\beta \in \mathbb{Z}_+^n$, $\exists k_2 = k_2(\beta) > 0$, $\exists c_2 = c_2(\beta) > 0$, $\exists\varepsilon_{2\beta} \in I$, $\forall\varepsilon \le \varepsilon_{2\beta}$

$$\sup_{x\in K} \left|\partial^\beta v_\varepsilon(x)\right| \le c_2\exp(k_2\varepsilon^{-\frac{1}{\tau}})$$

1) Let $\lambda_1, \lambda_2 \in \mathbb{R}$, and $x \in K$ then

$$\begin{aligned} |\partial^\alpha(\lambda_1 u_\varepsilon + \lambda_2 v_\varepsilon)(x)| &\le |\lambda_1| \, |\partial^\alpha u_\varepsilon(x)| + |\lambda_2| \, |\partial^\alpha v_\varepsilon(x)| \\ &\le c_1 \, |\lambda_1| \exp(k_1 \varepsilon^{-\frac{1}{\tau}}) + c_2 \, |\lambda_2| \exp(k_2 \varepsilon^{-\frac{1}{\tau}}) \end{aligned}$$

For $c = max(c_1 \, |\lambda_1| \, , c_2 \, |\lambda_2|)$, $k = k_1 + k_2$, $\varepsilon \le \min(\varepsilon_{1\beta}, \varepsilon_{2\beta}, |\beta| \le |\alpha|)$.

We have $|\partial^\alpha(\lambda_1 u_\varepsilon + \lambda_2 v_\varepsilon)(x)| \le c \exp(k \varepsilon^{-\frac{1}{\tau}})$, i.e. $(\lambda_1 u_\varepsilon + \lambda_2 v_\varepsilon)_\varepsilon \in \mathcal{E}^\tau_m(\Omega)$.

Let $\alpha \in \mathbb{Z}^n_+$, then

$$\begin{aligned} |\partial^\alpha(u_\varepsilon v_\varepsilon)(x)| &\le \sum_{\beta=0}^{\alpha} C^\alpha_\beta \left|\partial^{\alpha-\beta} u_\varepsilon(x)\right| . \left|\partial^\beta v_\varepsilon(x)\right| \\ &\le \sum_{\beta=0}^{\alpha} C^\alpha_\beta c_1(\alpha-\beta) \exp(k_1 \varepsilon^{-\frac{1}{\tau}}) \times c_{2\beta} \exp(k_2 \varepsilon^{-\frac{1}{\tau}}) \\ &\le c(\alpha) \exp(k \varepsilon^{-\frac{1}{\tau}}) \end{aligned}$$

Where $k = max(k_1(\alpha-\beta) + k_2(\beta) : \beta \le \alpha)$, $\varepsilon \le \min(\varepsilon_{1\beta}, \varepsilon_{2\beta}; |\beta| \le |\alpha|)$,

$c(\alpha) = \sum_{\beta}^{\alpha} C^\alpha_\beta c_1(\alpha-\beta) c_2(\beta)$. i.e. $(u_\varepsilon v_\varepsilon)_\varepsilon \in \mathcal{E}^\tau_m(\Omega)$.

It is clear that for every compact subset K of Ω, $\forall \beta \in \mathbb{Z}^n_+$, $\exists k_1 = k_1(\beta+\alpha) > 0$,

$\exists c_1 = c_1(\beta+\alpha) > 0$, $\exists \varepsilon_{1\beta} \in I$ such that $\forall x \in K$, $\forall \varepsilon \le \varepsilon_{1\beta}$.

$$\left|\partial^\beta(\partial^\alpha f_\varepsilon(x))\right| \le c_1 \exp(k_1 \varepsilon^{-\frac{1}{\tau}})$$

i.e. $(\partial^\alpha f_\varepsilon)_\varepsilon \in \mathcal{E}^\tau_m(\Omega)$.

2) If $(v_\varepsilon)_\varepsilon \in \mathcal{N}^\tau(\Omega)$, for every compact subset K of Ω. $\forall \beta \in \mathbb{Z}^n_+$, $\forall k_2 > 0$, $\exists c_2 = c_2(\beta, k_2) > 0$,

$\exists \varepsilon_{2\beta} \in I$,

$$|\partial^\alpha v_\varepsilon(x)| \le c_2 \exp(-k_2 \varepsilon^{-\frac{1}{\tau}}) \; \forall x \in K, \forall \varepsilon \le \varepsilon_{2\beta}$$

Let $\alpha \in \mathbb{Z}_+^n$ and $k > 0$, then

$$|\partial^\alpha (u_\varepsilon v_\varepsilon)(x)| \leq \sum_{\beta=0}^{\alpha} C_\beta^\alpha \left|\partial^{\alpha-\beta} u_\varepsilon(x)\right| . \left|\partial^\beta v_\varepsilon(x)\right|$$

Let $k_2 = \max(k_1(\beta), \beta \leq \alpha) + k$ and $\varepsilon \leq \min(\varepsilon_{1\beta}, \varepsilon_{2\beta}; \beta \leq \alpha)$, then $\forall x \in K$

$$\begin{aligned} |\partial^\alpha (u_\varepsilon v_\varepsilon)(x)| &\leq \sum_{\beta=0}^{\alpha} C_\beta^\alpha c_1(\alpha-\beta) c_2(\beta, k_2) \exp(-k\varepsilon^{-\frac{1}{\tau}}) \\ &\leq c(\alpha, k) \exp(-k\varepsilon^{-\frac{1}{\tau}}) \end{aligned}$$

Which show that $(f_\varepsilon g_\varepsilon)_\varepsilon \in \mathcal{N}^\tau(\Omega)$.

□

Definition 2.1.3 *The algebra of generalized Gevrey ultradistributions of order* $\tau \geq 1$*, denoted* $\mathcal{G}^\tau(\Omega)$*, is the quotient algebra*

$$\mathcal{G}^\tau(\Omega) = \frac{\mathcal{E}_m^\tau(\Omega)}{\mathcal{N}^\tau(\Omega)}$$

Proposition 2.1.4 *Let* $(u_\varepsilon)_\varepsilon \in \mathcal{E}_m^\tau(\Omega)$*, then* $(u_\varepsilon)_\varepsilon \in \mathcal{N}^\tau(\Omega)$ *if and only if for every compact subset* K *of* Ω*,* $\forall k > 0$*,* $\exists c > 0$*,* $\exists \varepsilon_0 \in I$*,* $\forall \varepsilon \leq \varepsilon_0$*,*

$$\sup_{x \in K} |u_\varepsilon(x)| \leq c \exp(-k\varepsilon^{-\frac{1}{\tau}}) \tag{2.1}$$

Proof. Let $(u_\varepsilon)_\varepsilon \in \mathcal{E}_m^\tau(\Omega)$ satisfy (2.1) we will show that $(\partial_i u_\varepsilon)_\varepsilon$ also satisfy (2.1) when $i = 1, ..., n$ and then it will follow by induction that $(u_\varepsilon)_\varepsilon \in \mathcal{N}^\tau(\Omega)$.

Suppose that u_ε has real values, Let K be a compact subset of Ω.

For $\delta = \min(1, dist(K, \partial\Omega))$, set $L = K + \overline{B(0, \frac{\delta}{2})}$. Then $K \Subset L \Subset \Omega$. By the moderateness of $(u_\varepsilon)_\varepsilon$, we have $\exists k_1 > 0$, $\exists c_1 > 0$, $\exists \varepsilon_1 \in I$, $\forall \varepsilon \leq \varepsilon_1$.

$$\sup_{x\in L} \left|\partial_i^2 u_\varepsilon(x)\right| \leq c_1 \exp(k_1 \varepsilon^{-\frac{1}{\tau}}) \tag{2.2}$$

By (2.1), $\forall k > 0$, $\exists c_2 > 0$, $\exists \varepsilon_2 \in I$, $\forall \varepsilon \leq \varepsilon_2$.

$$\sup_{x\in L} |u_\varepsilon(x)| \leq c_2 \exp(-(k + k_1)\varepsilon^{-\frac{1}{\tau}}) \tag{2.3}$$

Let $x \in K$, ε sufficiently small and $r = \exp(-(k + k_1)\varepsilon^{-\frac{1}{\tau}}) < \frac{\delta}{2}$. By Taylor's formula, we have

$$\partial_i u_\varepsilon(x) = \frac{u_\varepsilon(x + re_i) - u_\varepsilon(x)}{r} - \frac{1}{2}\partial_i^2 u_\varepsilon(x + \theta re_i)r,$$

Where e_i is i^{th} vector of the canonical base of $\mathbb{R}^n$ hence $(x + \theta re_i) \in L$, and then

$$|\partial_i u_\varepsilon(x)| \leq |u_\varepsilon(x + re_i) - u_\varepsilon(x)|\, r^{-1} + \frac{1}{2}\left|\partial_i^2 u_\varepsilon(x + \theta re_i\right| r$$

From (2.2) and (2.3) $|u_\varepsilon(x + re_i) - u_\varepsilon(x)|\, r^{-1} \leq c_2 \exp(-k\varepsilon^{-\frac{1}{\tau}})$ and

$|\partial_i^2 u_\varepsilon(x + \theta re_i|\, r \leq c_1 \exp(-k\varepsilon^{-\frac{1}{\tau}})$ so $|\partial_i u_\varepsilon(x)| \leq c \exp(-k\varepsilon^{-\frac{1}{\tau}})$ which complete the proof. $\square$

2.2 Generalized point values

The ring of Gevrey generalized complex numbers, denoted $\mathcal{C}^{\tau}$, is defined by the quotient

$$\mathcal{C}^{\tau} = \frac{\mathcal{E}_0^{\tau}}{\mathcal{N}_0^{\tau}}$$

Where

$$\mathcal{E}_0^{\tau} = \{(a_\varepsilon)_\varepsilon \in \mathbb{C}^I, \exists k > 0, \exists c > 0, \exists \varepsilon_0 \in I,\ \textit{such that: } \forall \varepsilon \leq \varepsilon_0;\ |a_\varepsilon| \leq c\exp(k\varepsilon^{-\frac{1}{\tau}})\}$$

And

$$\mathcal{N}_0^{\tau} = \{(a_\varepsilon)_\varepsilon \in \mathbb{C}^I, \forall k > 0, \exists c > 0 \exists \varepsilon_0 \in I\ \textit{such that}\ \forall \varepsilon \leq \varepsilon_0, |a_\varepsilon| \leq c\exp(-k\varepsilon^{-\frac{1}{\tau}})\}$$

It is clear that $\mathcal{E}_0^{\tau}$ is an algebra and $\mathcal{N}_0^{\tau}$ is an ideal of $\mathcal{E}_0^{\tau}$.

Proposition 2.2.1 *If $u \in \mathcal{G}^{\tau}(\Omega)$ and $x \in \Omega$, then the element $u(x)$ represented by $(u_\varepsilon(x))_\varepsilon$ is an element of $\mathcal{C}^{\tau}$ independent of the representative $(u_\varepsilon)_\varepsilon$ of u.*

A generalized Gevrey ultradistribution is not defined by its point values.

Example 2.2.2 *We give here an example of a generalized Gevrey ultradistribution $f = [(f_\varepsilon)]_\varepsilon \notin \mathcal{N}^{\tau}(\mathbb{R})$, but $[(f_\varepsilon(x))_\varepsilon] \in \mathcal{N}_0^{\tau}$ for every $x \in \mathbb{R}$. Let $\varphi \in \mathcal{D}(\mathbb{R})$ such that $\varphi(0) \neq 0$. For $\varepsilon \in I$, define*

$$f_\varepsilon(x) = x\exp(-\varepsilon^{-\frac{1}{\tau}})\varphi(\frac{x}{\varepsilon});\ x \in \mathbb{R}$$

It is clear that $(f_\varepsilon)_\varepsilon \in \mathcal{E}_m^\tau(\mathbb{R})$. *Let* K *be a compact neighborhood of* 0, *then:*

$$\sup_K |f'(x)| \leq |f'_\varepsilon(0)| = \exp(-\varepsilon^{-\frac{1}{\tau}})\, |\varphi(0)|$$

which shows that $(f_\varepsilon)_\varepsilon \notin \mathcal{N}^\tau(\mathbb{R})$.

For any $x_0 \in \mathbb{R}$, *there exists* ε_0 *such that* $\varphi(\frac{x_0}{\varepsilon}) = 0$, $\forall \varepsilon \leq \varepsilon_0$, *i.e.* $f(x_0) \in \mathcal{N}_0^\tau$.

In order to give a solution to this situation, set

$$\Omega_M^\tau = \{(x_\varepsilon)_\varepsilon \in \Omega^I, \exists k > 0, \exists c > 0, \exists \varepsilon_0 > 0, \forall \varepsilon \leq \varepsilon_0, |x_\varepsilon| \leq c \exp(k\varepsilon^{-\frac{1}{\tau}})\}$$

Define in Ω_M^τ the equivalence relation $\sim$ by $x_\varepsilon \sim y_\varepsilon \Leftrightarrow \forall k > 0, \exists c > 0, \exists \varepsilon_0 > 0, \forall \varepsilon \leq \varepsilon_0,$

$$|x_\varepsilon - y_\varepsilon| \leq c \exp(-k\varepsilon^{-\frac{1}{\tau}})$$

Definition 2.2.3 *The set* $\widetilde{\Omega}^\tau = \Omega_M^\tau/_\sim$ *is called the set of generalized Gevrey point. The set of compactly supported Gevrey points is defined by*

$$\widetilde{\Omega}_c^\tau = \{\widetilde{x} = [(x_\varepsilon)_\varepsilon] \in \widetilde{\Omega}^\tau : \exists K \text{ a compact set of } \Omega, \exists \varepsilon_0 > 0, \forall \varepsilon \leq \varepsilon_0, x_\varepsilon \in K\}$$

Proposition 2.2.4 *Let* $f \in \mathcal{G}^\tau(\Omega)$ *and* $\tilde{x} = [(x_\varepsilon)_\varepsilon] \in \widetilde{\Omega}_c^\tau$, *then the generalized Gevrey point value of* f *at* $\tilde{x}$, *i.e.*

$$f(\tilde{x}) = [(f_\varepsilon(x_\varepsilon))_\varepsilon].$$

is a well-defined element of the algebra of generalized Gevrey complex numbers.

Proof. Let $f \in \mathcal{G}^\tau(\Omega)$ and $\tilde{x} = [(x_\varepsilon)_\varepsilon] \in \tilde{\Omega}_c^\tau$, there exists a compact subset K of Ω such that $x_\varepsilon \in K$ for ε small, then $\exists k > 0$, $\exists c > 0$, $\exists \varepsilon_0 > 0$, $\forall \varepsilon \leq \varepsilon_0$,

$$|f_\varepsilon(x_\varepsilon)| \leq \sup_{x\in K} |f_\varepsilon(x)| \leq c \exp(k\varepsilon^{-\frac{1}{\tau}})$$

Therefore $(f_\varepsilon(x_\varepsilon))_\varepsilon \in \mathcal{E}_0^\tau$, and it is clear that if $f \in \mathcal{N}^\tau(\Omega)$; then $(f_\varepsilon(x_\varepsilon))_\varepsilon \in \mathcal{N}_0^\tau$,i.e, $f(\tilde{x})$ does not depend on the choice of the representative $(f_\varepsilon)_\varepsilon$.

Let now $\tilde{x} = [(x_\varepsilon)_\varepsilon] \sim \tilde{y} = [(y_\varepsilon)_\varepsilon]$, then $\forall k > 0$, $\exists c > 0$, $\exists \varepsilon_0 > 0$, $\forall \varepsilon \leq \varepsilon_0$,

$$|x_\varepsilon - y_\varepsilon| \leq c \exp(-k\varepsilon^{-\frac{1}{\tau}})$$

Since $(f_\varepsilon)_\varepsilon \in \mathcal{E}^\tau(\Omega)$, so for every compact subset K of Ω,

$\forall j \in \{1, m\}$, $\exists k_j > 0$, $\exists c_j > 0$, $\exists \varepsilon_j > 0$, $\forall \varepsilon \leq \varepsilon_0$

$$\sup_{x\in K} \left| \frac{\partial}{\partial x_j} f_\varepsilon(x) \right| \leq c_j \exp(k_j \varepsilon^{-\frac{1}{\tau}})$$

We have

$$|f_\varepsilon(x_\varepsilon) - f_\varepsilon(y_\varepsilon)| \leq |x_\varepsilon - y_\varepsilon| \sum_{j=1}^{m} \int_0^1 \left| (\frac{\partial}{\partial x_j} f_\varepsilon)(x_\varepsilon + t(y_\varepsilon - x_\varepsilon))dt \right|$$

and $x_\varepsilon + t(y_\varepsilon - x_\varepsilon)$ remains within some compact subset K of Ω for $\varepsilon \leq \varepsilon'$.

Let $k' > 0$ then for $k + k' = \sup_j k_j$ and $\varepsilon \leq \min(\varepsilon', \varepsilon_0, \varepsilon_j : j = 1, m)$ we have

$$|f_\varepsilon(x_\varepsilon) - f_\varepsilon(y_\varepsilon)| \leq c \exp(-k'\varepsilon^{-\frac{1}{\tau}}),$$

which gives $(f_\varepsilon(x_\varepsilon) - f_\varepsilon(y_\varepsilon))_\varepsilon \in \mathcal{N}_0^\tau$. □

Theorem 2.2.5 *Let $f \in \mathcal{G}^\tau(\Omega)$. Then $f = 0$ in $\mathcal{G}^\tau(\Omega) \Leftrightarrow f(\tilde{x}) = 0$ in $\mathcal{C}^\tau$ for all $\tilde{x} \in \widetilde{\Omega}_c^\tau$*

Proof. If $f \in \mathcal{N}^\tau(\Omega)$, then $f(\tilde{x}) \in \mathcal{N}_0^\tau$, $\forall \tilde{x} \in \widetilde{\Omega}_c^\tau$. Suppose that $f \neq 0$ in $\mathcal{G}^\tau(\Omega)$. Then by the characterization of $\mathcal{N}^\tau(\Omega)$ we have there exists a compact subset K of Ω, $\exists k > 0$, $\forall c > 0$, $\forall c > 0$, $\forall \varepsilon_0 > 0$, $\exists \varepsilon \leq \varepsilon_0$,

$$\sup_{x \in K} |f_\varepsilon(x)| > c \exp(-k\varepsilon^{-\frac{1}{\tau}})$$

So there exists a sequence $\varepsilon_n \to 0$ and $x_n \in K$ such that $\forall m \in \mathbb{Z}^+$

$$|f_{\varepsilon_n}(x_n)| > \exp(-k\varepsilon_n^{-\frac{1}{\tau}}) \tag{2.4}$$

For $\varepsilon > 0$ we set $x_\varepsilon = x_n$ when $\varepsilon_{n+1} \leq \varepsilon \leq \varepsilon_n$, we have $(x_\varepsilon)_\varepsilon \in \Omega_M^\tau$ with values in K, so $\tilde{x} = [(x_\varepsilon)_\varepsilon] \in \widetilde{\Omega}_c^\tau$ and (2.4) means that $(f_\varepsilon(x_\varepsilon))_\varepsilon \notin \mathcal{N}_0^\tau$ i.e $f(\tilde{x}) \neq 0$ in $\mathcal{C}^\tau$. □

2.3 Sheaf properties of $\mathcal{G}^\tau$

Let Ω' be an open subset of Ω and let $f = (f_\varepsilon)_\varepsilon + \mathcal{N}^\tau(\Omega) \in \mathcal{G}^\tau(\Omega)$, the restriction of f to Ω', denoted $f_{/\Omega'}$, is defined as

$$(f_{\varepsilon/_{\Omega'}})_\varepsilon + \mathcal{N}^\tau(\Omega') \in \mathcal{G}^\tau(\Omega')$$

Theorem 2.3.1 *The functor $\Omega \to \mathcal{G}^\tau(\Omega)$ is a sheaf of differential algebras on $\mathbb{R}^n$.*

Proof. Let Ω be a non void open of $\mathbb{R}^n$ and $(\Omega_\lambda)_{\lambda \in \Lambda}$ be an open covering of Ω. we have to show the properties

(S1) If $f, g \in \mathcal{G}^\tau(\Omega)$ such that $f_{/\Omega_\lambda} = g_{/\Omega_\lambda}, \forall\lambda \in \Lambda$, then $f = g$.

(S2) If for each $\lambda \in \Lambda$, we have $f_\lambda \in \mathcal{G}^\tau(\Omega_\lambda)$, such that

$$f_{\lambda/\Omega_\lambda\cap\Omega_\mu} = f_{\mu/\Omega_\lambda\cap\Omega_\mu} \textit{ for all } \lambda, \mu \in \Lambda \textit{ with: } \Omega_\lambda \cap \Omega_\mu \neq \emptyset$$

then there exists a unique $f \in \mathcal{G}^\tau(\Omega)$ with $f_{/\Omega_\varepsilon} = f_\lambda, \forall\lambda \in \Lambda$

To show $(S1)$, take K a compact subset of Ω. Then there exists compact set K_1, K_2,..., K_n and indice $\lambda_1, \lambda_2, ..., \lambda_n \in \Lambda$ such that:

$$K \subset \bigcup_{i=1}^{n} K_i \textit{ and } K_i \subset \Omega_{\lambda_i},$$

Where $(f_\varepsilon - g_\varepsilon)_\varepsilon$ satisfies the $\mathcal{N}^\tau-$estimate on each K_i. Then it satisfies the $\mathcal{N}^\tau-$estimate on K which means $(f_\varepsilon - g_\varepsilon)_\varepsilon \in \mathcal{N}^\tau(\Omega)$.

To show $(S2)$, let $(\chi_j)_{j=1}^\infty$ be a $C^\infty-$partition of unity subordinate to the covering $(\Omega_\lambda)_{\lambda\in\Lambda}$. Set

$$f := (f_\varepsilon)_\varepsilon + \mathcal{N}(\Omega),$$

where $f_\varepsilon = \sum\limits_{j=1}^{\infty} \chi_j f_{\lambda_j\varepsilon}$ and $(f_{\lambda_j\varepsilon})_\varepsilon$ is a representative of f_{λ_j}. Moreover, we set $f_{\lambda_j\varepsilon} = 0$ on $\Omega\backslash\Omega_{\lambda j}$, so that $\chi_j f_{\lambda_j\varepsilon}$ is C^∞ on all of Ω. First let K be a compact subset of Ω. Then $K_j = K \cap supp\chi_j$ is a compact subset of Ω_{λ_j} and $(f_{\lambda_j\varepsilon})_\varepsilon \in \mathcal{E}_m^\tau(\Omega_{\lambda j})$. Then $(\chi_j f_{\lambda_j\varepsilon})$ satisfies $\mathcal{E}_m^\tau-$estimate on each K_j, and $\chi_j(x) \equiv 0$ on K except for a finite number of j, i.e, $\exists N > 0$, such that

$$\sum_{j=1}^{\infty} \chi_j f_{\lambda_j\varepsilon}(x) = \sum_{j=1}^{N} \chi_j f_{\lambda_j\varepsilon}(x), \forall x \in K$$

So $\left(\sum \chi_j f_{\lambda_j\varepsilon}\right)$ satisfies $\mathcal{E}_m^\tau-$estimates on K, which means $(f_\varepsilon)_\varepsilon \in \mathcal{E}_m^\tau(\Omega)$. It remains to show that $f_{\Omega_\lambda} = f_\lambda$, $\forall\lambda \in \Lambda$.

Let K be a compact subset of Ω_λ, choose $N > 0$ in such a way that $\sum\limits_{j=1}^{N} \chi_j(x) = 1$ on a neighborhood Ω' of K with $\overline{\Omega'}$ compact in Ω_λ. For $x \in K$,

$$f_\varepsilon(x) - f_{\lambda\varepsilon}(x) = \sum_{j=1}^{N} \chi_j(x)(f_{\lambda_j\varepsilon}(x) - f_{\lambda\varepsilon}(x))$$

Since $(f_{\lambda_j\varepsilon} - f_{\lambda\varepsilon}) \in \mathcal{N}^\tau(\Omega_{\lambda_j} \cap \Omega_\lambda)$ and $K_j = K \cap supp\chi_j$ is compact subset of $\Omega \cap \Omega_{\lambda_j}$, then $\left(\sum_{j=1}^{N} \chi_j(f_{\lambda_j\varepsilon} - f_{\lambda\varepsilon})\right)$ satisfies the $\mathcal{N}^\tau-$estimate on K. The uniqueness of such $f \in \mathcal{G}^\tau(\Omega)$ follows from $(S1)$. □

Definition 2.3.2 *The support of $f \in \mathcal{G}^\tau(\Omega)$, denoted $supp_g^\tau f$, is the complement of the largest open set U such that $f_{/U} = 0$.*

2.4 Embedding of Gevrey ultradistributions

Definition 2.4.1 *A function $f \in E^\tau(\Omega)$, if $f \in C^\infty(\Omega)$ and for every compact subset K of Ω, $\exists c > 0$, $\forall\alpha \in \mathbb{Z}_+^n$,*

$$\sup_{x\in K} |\partial^\alpha f(x)| \leq c^{|\alpha|+1}(\alpha!)^\tau$$

Denote by $\mathcal{D}^\tau(\Omega)$ the space $E^\tau(\Omega) \cap C_0^\infty(\Omega)$. Then $\mathcal{D}^\tau(\Omega)$ is nontrivial if and only if $\tau > 1$.

The topological dual of $\mathcal{D}^\tau(\Omega)$, denoted $\mathcal{D}'(\Omega)$, is called the space of Gevrey ultradistributions of order τ.

The space $E'_\tau(\Omega)$ is the topological dual of $E^\tau(\Omega)$ and is identified with the space of Gevrey ultradistributions with compact support.

Definition 2.4.2 *A differential operator of infinite order*

$$P(D) = \sum_{\gamma \in \mathbb{Z}_+^n} a_\gamma D^\gamma$$

is called a $\tau-$*ultradifferential operator, if for every* $h > 0$ *there exist* $c > 0$ *such that* $\forall \gamma \in \mathbb{Z}_+^n$,

$$|a_\gamma| \leq c \frac{h^{|\gamma|}}{(\gamma!)^\tau}$$

Proposition 2.4.3 *Let* $T \in E'_\tau(\Omega)$, $\tau > 1$ *and* $suppT \subset K$ *then there exist a* $\tau-$*ultradifferential operator* $P(D) = \sum_{\gamma \in \mathbb{Z}_+^n} a_\gamma D^\gamma$ *and* $M > 0$ *and continuous functions* $f_\gamma \in C_0(K)$ *such that*

$$\sup_{\substack{\gamma \in \mathbb{Z}_+^n \\ x \in K}} |f_\gamma(x)| \leq M$$

$$T = \sum_{\gamma \in \mathbb{Z}_+^n} a_\gamma D^\gamma f_\gamma$$

The space $\mathcal{S}^{(\sigma)}(\mathbb{R}), \sigma > 1$ is the space of functions $\varphi \in C^\infty(\mathbb{R}^n)$ such that $\forall b > 0$, we have

$$\|\varphi\|_{b,\sigma} = \sup_{\alpha, \beta \in \mathbb{Z}_+^n} \int \frac{|x|^{|\beta|}}{b^{|\alpha+\beta|} \alpha!^\sigma \beta!^\sigma} |\partial^\alpha \varphi(x)| \, dx \leq \infty$$

There exists $\phi \in \mathcal{S}^{(\sigma)}(\mathbb{R}^n)$ satisfying

$$\int \phi(x)dx = 1 \text{ and } \int x^\alpha \phi(x)dx = 0, \ \forall \alpha \in \mathbb{Z}_+^n \backslash \{0\} \tag{2.5}$$

Definition 2.4.4 *The net* $\phi_\varepsilon = \varepsilon^{-n} \phi(\frac{\cdot}{\varepsilon})$, $\varepsilon \in I$, *where* ϕ *satisfies the condition* (2.5), *is called a* *net of mollifiers.*

The space $E^t(\Omega)$ is embedded into $\mathcal{G}^\tau(\Omega)$ by the standard canonical injection

$$\begin{array}{rccl} I: & E^t(\Omega) & \to & \mathcal{G}^\tau(\Omega) \\ & f & \to & [f] = cl(f_\varepsilon) \end{array}$$

where $f_\varepsilon = f$, $\forall \varepsilon \in I$.

Theorem 2.4.5 *The map*

$$\begin{array}{rccl} J_0: & E'_{\tau+\sigma} & \to & \mathcal{G}^\tau(\Omega) \\ & T & \to & [T] = cl((T * \phi_\varepsilon)/_\Omega) \end{array}$$

is an embedding.

Proof. Let $T \in E'_{\tau+\sigma}(\Omega)$ with $suppT \subset K$, then there exists an $(\tau+\sigma)-$ultradifferential operator $P(D) = \sum_{\gamma \in \mathbb{Z}_+^n} a_\gamma D^\gamma$ and continuous functions f_γ with $suppf_\gamma \subset K$, $\forall \gamma \in \mathbb{Z}_+^n$, and $\sup_{\substack{x \in K \\ \gamma \in \mathbb{Z}_+^n}} |f_\gamma(x)| \leq M$, such that $T = \sum_{\gamma \in \mathbb{Z}_+^n} a_\gamma D^\gamma f_\gamma$. Let $\alpha \in \mathbb{Z}_+^n$, then

$$|\partial^\alpha (T * \phi_\varepsilon)(x)| \leq \sum_{\gamma \in \mathbb{Z}_+^n} a_\gamma \frac{1}{\varepsilon^{|\gamma+\alpha|}} \int |f_\gamma(x + \varepsilon y)| \left| D^{\gamma+\alpha} \phi(y) \right| dy$$

We have $\forall h > 0$, $\exists c > 0$, such that

$$|\partial^\alpha (T * \phi_\varepsilon)(x)| \leq \sum_{\gamma \in \mathbb{Z}_+^n} c \frac{h^{|\gamma|}}{\gamma!^{\tau+\sigma} \varepsilon^{|\gamma+\alpha|}} \int |f_\gamma(x + \varepsilon y)| \left| D^{\gamma+\alpha} \phi(y) \right| dy$$

And from the inequality $(\beta+\alpha)!^t \leq 2^{t|\alpha+\beta|}\alpha!^t\beta!^t$, $\forall t \geq 1$.

$$\begin{aligned} |\partial^\alpha(T*\phi_\varepsilon)(x)| &\leq \sum_{\gamma\in\mathbb{Z}_+^n} c\alpha!^{\tau+\sigma}\frac{2^{(\tau+\sigma)|\gamma+\alpha|}h^{|\gamma|}}{(\gamma+\alpha)^\tau}\frac{1}{\varepsilon^{|\gamma+\alpha|}}b^{|\gamma+\alpha|} \\ &\quad \times\int |f_\gamma(x+\varepsilon y)|\frac{|D^{\gamma+\alpha}\phi(y)|}{b^{|\gamma+\alpha|}(\gamma+\alpha)!^\tau}dy \end{aligned}$$

then for $h > \frac{1}{2}$

$$\begin{aligned} |\partial^\alpha(T*\phi_\varepsilon)(x)| &\leq \|\phi\|_{b,\sigma} Mc\alpha!^{\tau+\sigma}\sum_{\gamma\in\mathbb{Z}_+^n} 2^{-|\gamma|}\frac{(2^{\tau+\sigma+1}bh)^{|\gamma+\alpha|}}{(\gamma+\alpha)!^\tau}\frac{1}{\varepsilon^{|\gamma+\alpha|}} \\ &\leq c(\alpha)\exp(k_1\varepsilon^{-\frac{1}{\tau}}), \end{aligned}$$

where $k_1 = \tau(2^{\tau+\sigma+1}bh)^{\frac{1}{\tau}}$.

Suppose that $(T*\phi_\varepsilon)_\varepsilon \in \mathcal{N}^\tau(\Omega)$, then for every compact L of Ω, $\exists c>0$, $\forall k>0$, $\exists \varepsilon_0 \in I$,

$$|T*\phi_\varepsilon(x)| \leq c\exp(-k\varepsilon^{-\frac{1}{\tau}}),\ \forall x\in L, \varepsilon\leq\varepsilon_0$$

Let $\chi \in \mathcal{D}^{\tau+\sigma}(\Omega)$ and $\chi = 1$ in a neighborhood of K, then $\forall\psi \in E^{\tau+\sigma}(\Omega)$,

$$\langle T,\psi\rangle = \langle T,\psi\chi\rangle = \lim_{\varepsilon\to 0}\int (T*\phi_\varepsilon)(x)\chi(x)\psi(x)dx = 0$$

i.e $T=0$. □

In order to show the commutativity of the following diagram of embedding

$$\begin{array}{ccc} \mathcal{D}^{\tau-\sigma+1} & \rightarrow & \mathcal{G}^{\tau} \\ & \searrow & \uparrow \\ & & E'_{\tau+\sigma} \end{array}$$

We have to prove the following fundamental result:

Proposition 2.4.6 *Let* $f \in \mathcal{D}^{\tau-\sigma+1}(\Omega)$ *then:* $(f - (f * \phi_{\varepsilon})_{/\Omega}) \in \mathcal{N}^{\tau}(\Omega)$

Proof. Let $f \in \mathcal{D}^{\tau-\sigma+1}(\Omega)$, then there exists a constant $c > 0$, such that

$$|\partial^{\alpha} f(x)| \leq c^{|\alpha|+1} \alpha!^{\tau-\sigma+1}, \ \forall \alpha \in \mathbb{Z}_{+}^{n}, \forall x \in \Omega$$

Let $\alpha \in \mathbb{Z}_{+}^{n}$, the Taylor's formula and the properties of ϕ_{ε} give

$$\partial^{\alpha}(f * \phi_{\varepsilon} - f)(x) = \sum_{|\beta|=N} \int \frac{(\varepsilon y)^{\beta}}{\beta!} \partial^{\alpha+\beta} f(\xi) \phi(y) dy$$

Where $x \leq \xi \leq x + \varepsilon y$. Consequently, for $b > 0$, we have

$$\begin{aligned} |\partial^{\alpha}(f * \phi_{\varepsilon} - f)(x)| &\leq \varepsilon^{N} \sum_{|\beta|=N} \int \frac{|y|^{N}}{\beta!} \left|\partial^{\alpha+\beta} f(\xi)\right| |\phi(y)| \, dy \\ &\leq \varepsilon^{N} \sum_{|\beta|=N} \frac{b^{|\beta|} \beta!^{\sigma} (\alpha+\beta)!^{\tau-\sigma+1}}{\beta!} \times \frac{\left|\partial^{\alpha+\beta} f(\xi)\right|}{(\alpha+\beta)!^{\tau-\sigma+1}} \frac{|y|^{|\beta|}}{b^{|\beta|} \beta!^{\sigma}} |\phi(y)| \, dy \\ &\leq \varepsilon^{N} \sum_{|\beta|=N} \frac{b^{|\beta|} \beta!^{\sigma} (2^{\tau-\sigma+1})^{|\alpha+\beta|} \alpha!^{\tau-\sigma+1} \beta!^{\tau-\sigma+1}}{\beta!} c^{|\alpha+\beta|+1} . \|\phi\|_{b,\sigma} \\ &\leq \varepsilon^{N} \sum_{|\beta|=N} b^{|\beta|} \beta!^{\tau} \alpha!^{\tau-\sigma+1} (c 2^{\tau-\sigma+1})^{|\alpha+\beta|} \|\phi\|_{b,\sigma} \\ &\leq \varepsilon^{N} \alpha!^{\tau-\sigma+1} \|\phi\|_{b,\sigma} \sum_{|\beta|=N} b^{|\beta|} \beta!^{\tau} (c 2^{\tau-\sigma+1})^{|\alpha+\beta|} \end{aligned}$$

Let $k > 0$ and $T > 0$, then

$$\leq \quad (\alpha!)^{\tau-\sigma+1}.\, \|\phi\|_{b,\sigma}\, (\varepsilon N^{\tau})^{N}(k^{\tau}T)^{-N}(c2^{\tau-\sigma+1})^{|\alpha|} \sum_{|\beta|=N}(bk^{\tau}T)^{|\beta|}(c2^{\tau-\sigma+1})^{|\beta|}$$

Hence, taking $2^{\tau-\sigma+1}k^{\tau}bTc < \frac{1}{2a}$, with $a > 1$, we obtain

$$|\partial^{\alpha}(f * \phi_{\varepsilon} - f)(x)| \leq c\alpha!^{\tau-\sigma+1}(\varepsilon N^{\tau})^{\tau}(k^{\tau}T)^{-N}\, \|\phi\|_{b,\sigma}\, (c2^{\tau-\sigma+1})^{|\alpha|}a^{-N} \sum_{|\beta|=N}(\tfrac{1}{2})^{|\beta|}$$

$$\leq \quad c^{|\alpha|+1}\alpha!^{\tau-\sigma+1}(\varepsilon N^{\tau})^{N}.(k^{\tau}T)^{-N}\, \|\phi\|_{b,\sigma} \tag{2.6}$$

Let $\varepsilon_0 \in]0,1]$ such that $\varepsilon_0^{\frac{1}{\tau}}\frac{\ln a}{k} < 1$ and take $T > 2^{\tau}$ then $(T^{\frac{1}{\tau}} - 1) > 1 > \varepsilon^{\frac{1}{\tau}}\dfrac{\ln a}{k}, \varepsilon \leq \varepsilon_0$

In particular, we have

$$\left(\frac{\ln a}{k}\varepsilon^{\frac{1}{\tau}}\right)^{-1} T^{\frac{1}{\tau}} - \left(\frac{\ln a}{k}\varepsilon^{\frac{1}{\tau}}\right)^{-1} > 1$$

Then, there exists $N = N(\varepsilon) \in \mathbb{Z}^{+}$, such that

$$\left(\frac{\ln a}{k}\varepsilon^{\frac{1}{\tau}}\right)^{-1} < N < \left(\frac{\ln a}{k}\varepsilon^{\frac{1}{\tau}}\right)^{-1} T^{\frac{1}{\tau}}$$

which gives $a^{-N} \leq \exp(-k\varepsilon^{-\frac{1}{\tau}})$ and $\dfrac{\varepsilon N^{\tau}}{k^{\tau}T} \leq \left(\dfrac{1}{\ln a}\right)^{\tau} < 1$

If we choose $\ln a > 1$. Finally, from (2.6) we have

$$|\partial^{\alpha}(f * \phi_{\varepsilon} - f)(x)| \leq c\exp(-k\varepsilon^{-\frac{1}{\tau}})$$

i.e $f * \phi_{\varepsilon} - f \in \mathcal{N}^{\tau}(\Omega)$. □

Now we construct the embedding of $\mathcal{D}'_{\tau+\sigma}(\Omega)$ into $\mathcal{G}^{\tau}(\Omega)$ using the sheaf properties of $\mathcal{G}^{\tau}$.

First, choose some covering $(\Omega_{\lambda})_{\lambda\in\Lambda}$ of Ω. Let $(\psi_{\lambda})_{\lambda\in\Lambda}$ be a family of elements of $\mathcal{D}^{\tau}(\Omega) \subset \mathcal{D}^{\tau+\sigma}$

with $\psi_\lambda \equiv 1$ in some neighborhood of $\overline{\Omega_\lambda}$. For each λ we define

$$\begin{array}{llll} J_\lambda : & \mathcal{D}'_{\tau+\sigma}(\Omega) & \rightarrow & \mathcal{G}^\tau(\Omega) \\ & T & \rightarrow & [T]_\lambda = cl((\psi_\lambda T * \phi_\varepsilon)_{/\Omega_\lambda})_\varepsilon \end{array}$$

We have $[T]_\lambda = cl((\psi_\lambda T * \phi_\varepsilon)_{/\Omega_\lambda})_\varepsilon \in \mathcal{E}^\tau_m(\Omega)$ and the family $(J_\lambda(T))_{\lambda\in\Lambda}$ is coherent, i.e.

$$J_\lambda(T)_{/\Omega_\lambda\cap\Omega_\mu} = J_\mu(T)_{/\Omega_\lambda\cap\Omega_\mu}, \forall\lambda, \mu \in \Lambda$$

Then if $(\chi_j)_{j=1}^\infty$ is a smooth partition of unity subordinate to $(\Omega_\lambda)_{\lambda\in\Lambda}$, the preceding theorem allows the embedding

$$\begin{array}{llll} J_\sigma : & \mathcal{D}'_{\tau+\sigma}(\Omega) & \rightarrow & \mathcal{G}^\tau(\Omega) \\ & T & \rightarrow & [T] = cl\left(\sum\limits_{j=1}^\infty \chi_j(\psi_{\lambda_j} T * \phi_\varepsilon)\right) \end{array}$$

We can also embedded directly $\mathcal{D}'_{\tau+\sigma}(\Omega)$ into $\mathcal{G}^\tau(\Omega)$. Indeed, let $\phi \in \mathcal{D}^\sigma(B(0,2))$, $0 \leq \phi \leq 1$, $\phi \equiv 1$ on $B(0,1)$, and take $\phi \in S^{(\sigma)}$, define the function ρ_ε by

$$\rho_\varepsilon(x) = \left(\frac{1}{\varepsilon}\right)^n \phi(\frac{x}{\varepsilon})\varphi(x\,|\ln(\varepsilon)|)$$

We have $\exists c > 0$, such that $\forall\alpha \in \mathbb{Z}^n_+$,

$$\sup_{x\in\mathbb{R}^n} |\partial^\alpha \rho_\varepsilon(x)| \leq c^{|\alpha|+1}\alpha!^\sigma\varepsilon^{-n-|\alpha|}$$

Define the injective map

$$\begin{array}{rlcl} J: & \mathcal{D}'_{\tau+\sigma}(\Omega) & \to & \mathcal{G}^\tau(\Omega) \\ & T & \to & [T] = cl(T * \rho_\varepsilon)_\varepsilon \end{array}$$

Proposition 2.4.7 *The map J coincides on $E'_{\tau+\sigma}(\Omega)$ with J_0.*

The sheaf properties of $\mathcal{G}^\tau$ and the precedent proposition show that the embedding J_σ coincides with the embedding J. We have the following commutative diagram

$$\begin{array}{ccc} E^\tau(\Omega) & \to & \mathcal{G}^\tau(\Omega) \\ \downarrow & \nearrow & \\ \mathcal{D}'_{\tau+\sigma}(\Omega) & & \end{array}$$

Definition 2.4.8 *The space of elements of $\mathcal{G}^\tau(\Omega)$ with compact support is denoted $\mathcal{G}^\tau_c(\Omega)$.*

Proposition 2.4.9 *The space $\mathcal{G}^\tau_c(\Omega)$ is the space of elements f of $\mathcal{G}^\tau(\Omega)$ satisfying there exist a representative $(f_\varepsilon)_{\varepsilon\in I}$ and a compact subset K of Ω such that $\mathrm{supp} f_\varepsilon \subset K$, $\forall \varepsilon \in I$.*

2.5 Equalities in $\mathcal{G}^\tau(\Omega)$

In $\mathcal{G}^\tau(\Omega)$, we have the strong equality, denoted $=$, between two elements $f = [(f_\varepsilon)_\varepsilon]$ and $g = [(g_\varepsilon)_\varepsilon]$ which means that

$$(f_\varepsilon - g_\varepsilon) \in \mathcal{N}^\tau(\Omega)$$

We define the equality in the sense of ultradistributions, denoted $\overset{t}{\sim}$, where $t \in [\tau - \sigma + 1, \tau + \sigma]$, by

$$f \overset{t}{\sim} g \Leftrightarrow \left(\int (f_\varepsilon(x) - g_\varepsilon(x))\phi(x)dx \right)_\varepsilon \in \mathcal{N}_0^t, \forall \phi \in \mathcal{D}^t(\Omega)$$

and we say that f equals g in the sense of ultradistributions. We say that $f = [(f_\varepsilon)_\varepsilon]$ is associated to $g = [(g_\varepsilon)_\varepsilon]$, denoted $f \approx g$, if

$$\lim_{\varepsilon \to 0} \int (f_\varepsilon - g_\varepsilon)(x)\psi(x)dx = 0, \forall \psi \in \mathcal{D}^{\tau+\sigma}(\Omega)$$

In particular, we say that $f = [(f_\varepsilon)_\varepsilon] \in \mathcal{G}^\tau(\Omega)$ is associated to the Gevrey ultradistribution $T \in E'_{\tau+\sigma}(\Omega)$, denoted $f \approx T$, if

$$\lim_{\varepsilon \to 0} \int f_\varepsilon(x)\psi(x)dx = \langle T, \psi \rangle, \forall \psi \in \mathcal{D}^{\tau+\sigma}(\Omega)$$

The main relationship between these inequalities is given by the following results.

Proposition 2.5.1 *Let $f, g \in \mathcal{G}^\tau(\Omega)$, $T \in E'_{\tau+\sigma}(\Omega)$, and $t \in [\tau - \sigma + 1, \tau + \sigma]$. Then*

(1) $f = g \Rightarrow f \overset{t}{\sim} g \Rightarrow f \overset{\tau}{\sim} g \Rightarrow f \approx g$

(2) $T \approx 0$ *in* $\mathcal{G}^{\tau-\sigma+1}(\Omega) \Rightarrow T = 0$ *in* $E'_{\tau+\sigma}(\Omega)$

2.6 Regular generalized ultradistributions

To define the algebra of regular generalized ultradistributions, we needs first to define these regular moderate elements and these null elements.

Definition 2.6.1 *The space of* $\sigma-$*regular elements denoted* $\mathcal{E}_m^{\tau,\sigma,\infty}(\Omega)$, *is the space of* $(f_\varepsilon)_\varepsilon \in (C^\infty(\Omega))^I$ *satisfying, for every compact* K *of* Ω, $\exists k>0$, $\exists c>0$, $\exists \varepsilon_0 \in I$, $\forall \alpha \in \mathbb{Z}_+^n$, $\forall \varepsilon \leq \varepsilon_0$

$$\sup_{x\in K} |\partial^\alpha f_\varepsilon(x)| \leq c^{|\alpha|+1} \alpha!^\sigma \exp(k\varepsilon^{-\frac{1}{\tau}})$$

Proposition 2.6.2 *1) The space* $\mathcal{E}_m^{\tau,\sigma,\infty}(\Omega)$ *is an algebra stable under the action of* $\sigma-$*ultradifferential operators.*

2) The space $\mathcal{N}_m^{\tau,\sigma,\infty}(\Omega) := \mathcal{N}^\tau(\Omega) \cap \mathcal{E}_m^{\tau,\sigma,\infty}(\Omega)$ *is an ideal of* $\mathcal{E}_m^{\tau,\sigma,\infty}(\Omega)$.

Proof.

1) Let $(f_\varepsilon)_\varepsilon$, $(g_\varepsilon)_\varepsilon \in \mathcal{E}_m^{\tau,\sigma,\infty}(\Omega)$ and K be a compact subset of Ω. Then $\exists k_1 > 0, \exists c_1 > 0, \exists \varepsilon_1 \in I$ such that $\forall x \in K$, $\forall \alpha \in \mathbb{Z}_+^n$, $\forall \varepsilon \leq \varepsilon_1$,

$$|\partial^\alpha f_\varepsilon(x)| \leq c_1^{|\alpha|+1} \alpha!^\sigma \exp(k_1\varepsilon^{-\frac{1}{\tau}})$$

We have also $\exists k_2 > 0, \exists c_2 > 0, \exists \varepsilon_2 \in I$ such that $\forall x \in K$, $\forall \alpha \in \mathbb{Z}_+^n$, $\forall \varepsilon \leq \varepsilon_2$,

$$|\partial^\alpha g_\varepsilon(x)| \leq c_2^{|\alpha|+1} \alpha!^\sigma \exp(k_2\varepsilon^{-\frac{1}{\tau}})$$

Let $\alpha \in \mathbb{Z}_+^n$. Then

$$\begin{aligned}
|\partial^\alpha (f_\varepsilon g_\varepsilon)(x)| &\leq \sum_{\beta=0}^{\alpha} C_\beta^\alpha \left|\partial^{\alpha-\beta} f_\varepsilon(x)\right| \left|\partial^\beta g_\varepsilon(x)\right| \\
&\leq \sum_{\beta=0}^{\alpha} C_\alpha^\beta c_1^{|\alpha-\beta|+1} (\alpha-\beta)!^\sigma \exp(k_1 \varepsilon^{-\frac{1}{\tau}}) c_2^{|\beta|+1} \beta!^\sigma \exp(k_2 \varepsilon^{-\frac{1}{\tau}}) \\
&\leq c^{|\alpha|+1} \alpha!^\sigma (\tfrac{1}{2})^{\sigma|\alpha|} \exp(k\varepsilon^{-\frac{1}{\tau}}) \\
&\leq c^{|\alpha|+1} \alpha!^\sigma \exp(k\varepsilon^{-\frac{1}{\tau}})
\end{aligned}$$

i.e. $(f_\varepsilon g_\varepsilon)_\varepsilon \in \mathcal{E}_m^{\tau,\sigma,\infty}(\Omega)$. Now let $P(D) = \sum a_\gamma D^\gamma$ be a ultradifferential operator, then $\forall h > 0$, $\exists b > 0$, such that

$$\begin{aligned}
|\partial^\alpha P(D) f_\varepsilon(x)| &\leq \sum_{\gamma \in \mathbb{Z}_+^n} b \frac{h^{|\gamma|}}{\gamma!^\sigma} \left|\partial^{\alpha+\gamma} f_\varepsilon(x)\right| \\
&\leq \sum_{\gamma \in \mathbb{Z}_+^n} b \frac{h^{|\gamma|}}{\gamma!^\sigma} c^{|\alpha+\gamma|+1} (\alpha+\gamma)!^\sigma \exp(k\varepsilon^{-\frac{1}{\tau}}) \\
&\leq \sum_{\gamma \in \mathbb{Z}_+^n} b \left|h\right|^{|\gamma|} c^{|\alpha+\gamma|+1} 2^{\sigma|\alpha+\gamma|} \alpha!^\sigma \exp(k\varepsilon^{-\frac{1}{\tau}}) \\
&\leq (2^\sigma bc)^{|\alpha|+1} \alpha!^\sigma \sum_{\gamma \in \mathbb{Z}_+^n} h^{|\gamma|} (2^\sigma c)^{|\gamma|}
\end{aligned}$$

Hence, for $2^\sigma hc \leq \frac{1}{2}$, we have

$$|\partial^\alpha (P(D) f_\varepsilon(x))| \leq c^{|\alpha|+1} \alpha!^\sigma \exp(k\varepsilon^{-\frac{1}{\tau}})$$

Which shows that $(P(D) f_\varepsilon)_\varepsilon \in \mathcal{E}_m^{\tau,\sigma,\infty}(\Omega)$.

(2) The fact that $\mathcal{N}^{\tau,\sigma,\infty}(\Omega) = \mathcal{N}^\tau(\Omega) \cap \mathcal{E}_m^{\tau,\sigma,\infty}(\Omega) \subset \mathcal{E}_m^\tau(\Omega)$ and $\mathcal{N}^\tau(\Omega)$ is an ideal $\mathcal{E}_m^\tau(\Omega)$ implies that $\mathcal{N}^{\tau,\sigma,\infty}(\Omega)$ is an ideal of $\mathcal{E}_m^{\tau,\sigma,\infty}(\Omega)$

□

Now, we define the Gevrey regular elements of $\mathcal{G}^{\tau}(\Omega)$.

Definition 2.6.3 *The algebra of regular generalized Gevrey ultradistributions of order* $\sigma > 0$, *denoted* $\mathcal{G}^{\tau,\sigma,\infty}(\Omega)$, *is the quotient algebra*

$$\mathcal{G}^{\tau,\sigma,\infty}(\Omega) = \frac{\mathcal{E}_m^{\tau,\sigma,\infty}(\Omega)}{\mathcal{N}^{\tau,\sigma,\infty}(\Omega)}$$

Proposition 2.6.4 $\mathcal{G}^{\tau,\sigma,\infty}(\Omega)$ *is a subsheaf of* $\mathcal{G}^{\tau}$.

Definition 2.6.5 *We define the* $\mathcal{G}^{\tau,\sigma,\infty}$ *singular support of a generalized Gevrey ultradistribution* $f \in \mathcal{G}^{\tau}(\Omega)$, *denoted* $\sigma - singsupp_g(f)$, *as the complement of the largest open set* Ω' *such that* $f \in \mathcal{G}^{\tau,\sigma,\infty}(\Omega)$.

The following result is a Paley-Weiner type characterization of $\mathcal{G}^{\tau,\sigma,\infty}(\Omega)$.

Proposition 2.6.6 *Let* $f = cl(f_\varepsilon) \in \mathcal{G}_c^{\tau}(\Omega)$. *Then* f *is* $\sigma-$*regular if and only if* $\exists k_1 > 0$, $\exists k_2 > 0$, $\exists c > 0$, $\exists \varepsilon_1 \in I$, $\forall \varepsilon \leq \varepsilon_1$, *such that*

$$|\mathcal{F}(f_\varepsilon)(\xi)| \leq c \exp(k_1 \varepsilon^{-\frac{1}{\tau}} - k_2 |\xi|^{\frac{1}{\sigma}}),\ \forall \xi \in \mathbb{R}^n \tag{2.7}$$

Proof. Suppose that $f = cl(f_\varepsilon) \in \mathcal{G}_c^{\tau}(\Omega) \cap \mathcal{G}^{\tau,\sigma,\infty}(\Omega)$. Then $\exists k_1 > 0$, $\exists c_1 > 0$, $\exists \varepsilon_1 > 0$, $\forall \alpha \in \mathbb{Z}_+^n$

$$|\partial^\alpha f_\varepsilon(x)| \leq c_1^{|\alpha|+1} \alpha!^\sigma \exp(k_1 \varepsilon^{-\frac{1}{\tau}})$$

Consequently we have, $\forall \alpha \in \mathbb{Z}_+^n$

$$|\xi^\alpha| \, |\mathcal{F}(f_\varepsilon)(\xi)| \leq \left| \int \exp(-ix\xi) \partial^\alpha f_\varepsilon(x) dx \right|$$

Then $\exists c > 0, \forall \varepsilon \leq \varepsilon_1$,

$$|\xi^\alpha| \, |\mathcal{F}(f_\varepsilon)(\xi)| \leq c^{|\alpha|+1} \alpha!^\sigma \exp(k_1 \varepsilon^{-\frac{1}{\tau}})$$

For $\alpha \in \mathbb{Z}_+^n$, $\exists N \in \mathbb{Z}_+$ such that

$$\frac{N}{\sigma} < |\alpha| < \frac{N}{\sigma} + 1,$$

So

$$\begin{aligned} |\xi^\alpha|^{\frac{N}{\sigma}} \, |\mathcal{F}(f_\varepsilon)(\xi)| &\leq c^{|\alpha|+1} \, |\alpha|^{|\alpha|} \exp(k_1 \varepsilon^{-\frac{1}{\tau}}) \\ &\leq c^{N+1} N^N \exp(k_1 \varepsilon^{-\frac{1}{\tau}}) \end{aligned}$$

Hence $\exists c > 0, \forall N \in \mathbb{Z}^+$,

$$|\mathcal{F}(f_\varepsilon)(\xi)| \leq C^{N+1} \, |\xi|^{-\frac{N}{\sigma}} \, N! \exp(k_1 \varepsilon^{-\frac{1}{\tau}})$$

Which gives

$$|\mathcal{F}(f_\varepsilon)(\xi)| \exp(\frac{1}{2c} \, |\xi|^{\frac{1}{\sigma}}) \leq c \exp(k_1 \varepsilon^{-\frac{1}{\tau}}) \sum 2^{-N}$$

Or

$$|\mathcal{F}(f_\varepsilon)(\xi)| \leq c' \exp(k_1 \varepsilon^{-\frac{1}{\tau}} - k_2 \, |\xi|^{\frac{1}{\sigma}})$$

i.e, we have (2.7).

Suppose now that (2.7) is valid. Then $\forall \varepsilon \leq \varepsilon_0$,

$$|\partial^\alpha f_\varepsilon(x)| \leq c_1 \int_K \exp(k_1 \varepsilon^{-\frac{1}{\tau}} - k_2 \, |\xi|^{\frac{1}{\sigma}}) \, |\xi|^{|\alpha|} \, d\xi$$

Due to inequality $t^N \leq N! \exp(t), \forall t > 0$, then $\exists c_2 = c(k_2)$ such that

$$|\xi^\alpha| \, . \exp(-\frac{k_2}{2} |\xi|^{\frac{1}{\sigma}}) \leq c_2 \alpha!^\sigma.$$

Then $|\partial^\alpha f_\varepsilon(x)| \leq c \exp(k_1 \varepsilon^{-\frac{1}{\tau}}) \alpha!^\sigma$ i.e. $f \in \mathcal{G}^{\tau,\sigma,\infty}(\Omega)$. □

Remark 2.6.7 *Let* $f = cl(f_\varepsilon)_\varepsilon \in \mathcal{G}_c^\tau(\Omega)$, *then* $\exists k_1 > 0$, $\exists c > 0$, $\forall k_2 > 0$, $\varepsilon_0 \in I$, $\forall \varepsilon \leq \varepsilon_0$:

$$|\mathcal{F}(f_\varepsilon)| \leq c \exp(k_1 \varepsilon^{-\frac{1}{\tau}} + k_2 |\xi|^{\frac{1}{\sigma}}), \forall \xi \in \mathbb{R}^n$$

Theorem 2.6.8 *We have*

$$\mathcal{G}^{\tau,\tau-\sigma+1,\infty}(\Omega) \cap \mathcal{D}'_{\tau+\sigma}(\Omega) = E^{\tau-\sigma+1}(\Omega)$$

Proof. Let $S \in \mathcal{G}^{\tau,\tau-\sigma+1,\infty}(\Omega) \cap \mathcal{D}'_{\tau+\sigma}(\Omega)$. For any fixed $x_0 \in \Omega$ we take $\psi \in \mathcal{D}^{\tau+\sigma}(\Omega)$ with $\psi \equiv 1$ on neighborhood U of x_0. Then $T = \psi S \in E'_{\tau+\sigma}(\Omega)$.

Let ϕ_ε be a net of mollifiers, with $\check{\phi} = \phi$ and $\psi \in \mathcal{D}^{\tau-\sigma+1}(\Omega)$ such that $\chi \equiv 1$ on $K = supp\psi$. As $[T] \in \mathcal{G}^{\tau,\tau-\sigma+1,\infty}(\Omega)$, $\exists k_1 > 0$, $\exists k_2 > 0$, $\exists c_1 > 0$, $\exists \varepsilon_1 > 0$, $\forall \varepsilon \leq \varepsilon_1$

$$|\mathcal{F}(T)(\xi)| \leq c \exp(k_1 \varepsilon^{-\frac{1}{\tau}} - k_2 |\xi|^{\frac{1}{\tau-\sigma+1}})$$

Then

$$\begin{aligned} |\mathcal{F}(\chi(T * \phi_\varepsilon))(\xi) - \mathcal{F}(T)(\xi)| &= |\mathcal{F}(\chi(T * \phi_\varepsilon))(\xi) - \mathcal{F}(\chi T)(\xi)| \\ &= \left| \left\langle T(x), (\chi(x) e^{-i\xi x}) * \phi_\varepsilon(x) - (\chi(x) e^{-i\xi x}) \right\rangle \right| \end{aligned}$$

As $E'_{\tau+\sigma}(\Omega) \subset E'_{\tau-\sigma+1}(\Omega)$, $\exists L$ a compact subset of Ω such that $\forall h > 0$, $\exists c > 0$ and

$$|\mathcal{F}(\chi(T * \phi_\varepsilon))(\xi) - \mathcal{F}(T)(\xi)| \leq c \sup_{\substack{x \in L \\ \alpha \in \mathbb{Z}_+^n}} \frac{h^{|\alpha|}}{\alpha!^{\tau-\sigma+1}} \left| \partial_x^\alpha ((\chi(x) e^{-i\xi x}) * \phi_\varepsilon(x) - (\chi(x) e^{-i\xi x})) \right|$$

We have $e^{-i\xi}\psi \in \mathcal{D}^{\tau-\sigma+1}(\Omega)$, so $\forall k_3 > 0$, $\exists c_2 > 0$, $\exists \eta > 0$, $\forall \varepsilon \leq \eta$

$$\sup_{\substack{x \in L \\ \alpha \in \mathbb{Z}_+^n}} \frac{h^{|\alpha|}}{\alpha!^{\tau-\sigma+1}} \left| \partial_x^\alpha ((\chi(x) e^{-i\xi x}) * \phi_\varepsilon(x) - (\chi(x) e^{-i\xi x})) \right| \leq c_2 \exp(-k_3 \varepsilon^{-\frac{1}{\tau}})$$

So there exists $c' = c'(k_3) > 0$, such that

$$|\mathcal{F}(T)(\xi) - \mathcal{F}(\chi(T * \phi_\varepsilon))(\xi)| \leq c' \exp(-k_3 \varepsilon^{-\frac{1}{\tau}})$$

Let $\varepsilon \leq \min(\eta, \varepsilon_1)$, then

$$\begin{aligned} |\mathcal{F}(T)(\xi)| &\leq |\mathcal{F}(T)(\xi) - \mathcal{F}(\chi(T * \phi_\varepsilon))(\xi)| + |\mathcal{F}(\chi(T * \phi_\varepsilon))(\xi)| \\ &\leq c' \exp(-k_3 \varepsilon^{-\frac{1}{\tau}}) + c_1 \exp(k_1 \varepsilon^{-\frac{1}{\tau}} - k_2 |\xi|^{\frac{1}{\tau-\sigma+1}}) \end{aligned}$$

Take $c = \max(c', c_1)$, $\varepsilon = \left(\dfrac{k_1}{(k_2 - r) |\xi|^{\frac{1}{\tau-\sigma+1}}} \right)^\tau$, $r \in]0, k_2[$ and $k_3 = \dfrac{k_1 r}{k_2 - r}$, then $\exists \delta > 0$, $\exists c > 0$ such that

$$|\mathcal{F}(T)(\xi)| \leq c \exp(-\delta |\xi|^{\frac{1}{\tau-\sigma+1}}),$$

Which means $T = \psi S \in E^{\tau-\sigma+1}(\Omega)$. As $\psi \equiv 1$ on the neighborhood U of x_0, consequently $S \in E^{\tau-\sigma+1}(\Omega)$, which proves $\mathcal{G}^{\tau,\tau-\sigma+1,\infty}(\Omega) \cap \mathcal{D}'_{\tau+\sigma}(\Omega) \subset E^{\tau-\sigma+1}(\Omega)$. We have $E^{\tau-\sigma+1}(\Omega) \subset E^{\tau+\sigma}(\Omega) \subset \mathcal{D}'_{\tau+\sigma}(\Omega)$ and $E^{\tau-\sigma+1}(\Omega) \subset \mathcal{G}^{\tau,\tau-\sigma+1,\infty}(\Omega)$ then

$E^{\tau-\sigma+1}(\Omega) \subset \mathcal{G}^{\tau,\tau-\sigma+1,\infty}(\Omega) \cap \mathcal{D}'_{\tau+\sigma}(\Omega)$ Consequently we have $\mathcal{G}^{\tau,\tau-\sigma+1,\infty}(\Omega) \cap \mathcal{D}'_{\tau+\sigma}(\Omega) = E^{\tau-\sigma+1}(\Omega)$

□

2.7 Generalized Gevrey wave front

Definition 2.7.1 *We define $\sum_g^{\tau,\sigma}(f) \subset \mathbb{R}^n \backslash \{0\}$, $f \in \mathcal{G}_c^{\tau,\sigma}(\Omega)$, as the complement of the set of points having a conic neighborhood Γ such that: $\exists k_1 > 0$, $\exists k_2 > 0$, $\exists c > 0$, $\exists \varepsilon_0 \in I$, $\forall \xi \in \Gamma$, $\forall \varepsilon \leq \varepsilon_0$:*

$$|\mathcal{F}(f_\varepsilon)(\xi)| \leq c \exp(k_1 \varepsilon^{-\frac{1}{\tau}} - k_2 |\xi|^{\frac{1}{\sigma}})$$

Proposition 2.7.2 *For every $f \in \mathcal{G}_c^{\tau,\sigma}(\Omega)$, we have*

(1) The set $\sum_g^{\tau,\sigma}(f)$ is a closed cone.

(2) $\sum_g^{\tau,\sigma}(f) = \phi \Leftrightarrow f \in \mathcal{G}^{\tau,\sigma,\infty}(\Omega)$.

(3) $\sum_g^{\tau,\sigma}(\psi f) \subset \sum_g^{\tau,\sigma}(f), \forall \psi \in E^\sigma(\Omega)$.

Proof. One can easily, from the definition (2.7.1) and proposition (2.6.6), prove the assertions (1) and (2).

Let us suppose that $\xi_0 \notin \sum_g^{\tau,\sigma}(f)$, then $\exists \Gamma$ a conic neighborhood of ξ_0, $\exists k_1 > 0$, $\exists c_1 > 0$, $\exists \varepsilon_1 \in I$, $\forall \xi \in \Gamma$, $\forall \varepsilon \leq \varepsilon_1$,

$$|\mathcal{F}(f_\varepsilon)(\xi)| \leq c_1 \exp(k_1 \varepsilon^{-\frac{1}{\tau}} - k_2 |\xi|^{\frac{1}{\sigma}})$$

Let $\chi \in \mathcal{D}^\sigma(\Omega)$, $\chi \equiv 1$ on a neighborhood of *suppf*, so $\chi\psi \in \mathcal{D}^\sigma(\Omega)$, hence $\exists k_3 > 0$, $\exists c_2 > 0$, $\forall \xi \in \mathbb{R}^n$

$$|\mathcal{F}(\chi\psi)(\xi)| \leq c_2 \exp(-k_3 |\xi|^{\frac{1}{\sigma}})$$

Let Λ be a conic neighborhood of ξ_0 such that $\bar{\Lambda} \subset \Gamma$. We have, for a fixed $\xi \in \Lambda$

$$\begin{aligned} \mathcal{F}(\psi f_\varepsilon)(\xi) &= \mathcal{F}(\chi\psi f_\varepsilon)(\xi) \\ &= \int_A \mathcal{F}(f_\varepsilon)(\eta).\mathcal{F}(\chi\psi)(\eta-\xi)d\eta + \int_B \mathcal{F}(f_\varepsilon)(\eta).\mathcal{F}(\chi\psi)(\eta-\xi)d\eta \end{aligned}$$

Where $A = \{\eta : |\xi - \eta| \leq \delta(|\xi| + |\eta|)\}$ and $B = \{\eta : |\xi - \eta| > \delta(|\xi| + |\eta|)\}$ Take δ sufficient small such that $\dfrac{|\xi|}{2^\sigma} < |\eta| < 2^\sigma |\xi|$, $\forall \eta \in A$, then $\exists c > 0$, $\forall \varepsilon \leq \varepsilon_1$

$$\left| \int_A \mathcal{F}(f_\varepsilon)(\eta)\mathcal{F}(\chi\psi)(\eta-\xi)d\eta \right| \leq c_1.c_2.exp(k_1\varepsilon^{-\frac{1}{\tau}} - \frac{k_2}{2}|\xi|^{\frac{1}{\sigma}}) \times \int_A \exp(-k_3|\eta-\xi|^{\frac{1}{\sigma}})d\eta$$

then $\exists c > 0$, $\exists k_2' > 0$

$$\left| \int_A \mathcal{F}(f_\varepsilon)(\eta)\mathcal{F}(\chi\psi)(\eta-\xi)d\eta \right| \leq c\exp(k_1\varepsilon^{-\frac{1}{\tau}} - k_2'|\xi|^{\frac{1}{\sigma}})$$

As $f \in \mathcal{G}_c^\tau(\Omega)$, from remark (2.6.7), $\exists c_3 > 0$, $\exists \mu_1 > 0$, $\exists \varepsilon_2 > 0$, $\forall \mu_2 > 0$, $\forall \xi \in \mathbb{R}^n$, $\forall \varepsilon \leq \varepsilon_2$, such that

$$|\mathcal{F}(f_\varepsilon)(\xi)| \leq c_3 \exp(\mu_1\varepsilon^{-\frac{1}{\tau}} + \mu_2|\xi|^{\frac{1}{\sigma}})$$

Hence, for $\varepsilon \leq min(\varepsilon_1, \varepsilon_2)$, we have

$$\begin{aligned} \left|\int_B \mathcal{F}(f_\varepsilon)(\eta).\mathcal{F}(\chi\psi)(\eta-\xi)d\eta\right| &\leq c_2.c_3.\exp(\mu_1\varepsilon^{-\frac{1}{\tau}})\int_B \exp(\mu_2|\eta|^{\frac{1}{\sigma}} - k_3|\eta-\xi|^{\frac{1}{\sigma}})d\eta \\ &\leq c.\exp(\mu_1\varepsilon^{\frac{1}{\sigma}})\int_B \exp(\mu_2|\eta|^{\frac{1}{\sigma}} - k_3\delta(|\xi|^{\frac{1}{\sigma}} + |\eta|^{\frac{1}{\sigma}})d\eta \end{aligned}$$

Then taking $\mu_2 < k_3\delta$, we obtain

$$\left|\int_B \mathcal{F}(f_\varepsilon)(\eta).\mathcal{F}(\chi\psi)(\eta-\xi)d\eta\right| \leq c\exp(\mu_1\varepsilon^{-\frac{1}{\tau}} - k_3\delta\,|\xi|^{\frac{1}{\sigma}}) \tag{2.8}$$

Consequently $\xi_0 \notin \sum_g^{\tau,\sigma}(\psi f)$. $\square$

Definition 2.7.3 *Let $f \in \mathcal{G}^\tau(\Omega)$ and $x_0 \in \Omega$, the cone of $\sigma-$singular directions of f at x_0, denoted $\sum_{g,x_0}^{\tau,\sigma}(f)$, is*

$$\Sigma_{g,x_0}^{\tau,\sigma}(f) = \bigcap\{\Sigma_g^{\tau,\sigma}(\varphi f) : \varphi \in \mathcal{D}^\tau(\Omega) \textit{ and } \varphi \equiv 1 \textit{ on a neighborood of } x_0\}$$

Lemma 2.7.4 *Let $f \in \mathcal{G}^\tau(\Omega)$, then*

$$\Sigma_{g,x_0}^{\tau,\sigma}(f) = \emptyset \Leftrightarrow x_0 \notin \sigma - singsupp_g(f)$$

Proof. Let $x_0 \notin \sigma - singsupp_g(f)$, i.e. $\exists U \subset \Omega$ an open neighborhood of x_0 such that $f \in \mathcal{G}^{\tau,\sigma,\infty}(U)$, let $\phi \in \mathcal{D}^\tau(U)$ such that $\phi \equiv 1$ on a neighborhood of x_0, then $\phi f \in \mathcal{G}^{\tau,\sigma,\infty}(\Omega)$. Hence, from the proposition (2.6.6), $\sum_g^{\tau,\sigma}(\phi f) = \emptyset$, i.e: $\sum_{g,x_0}^{\tau,\sigma}(f) = \emptyset$.

Suppose now $\sum_{g,x_0}^{\tau,\sigma}(f) = \emptyset$, $\forall \xi \in \mathbb{R}^n\backslash\{0\}$, $\exists V_\xi \in \mathcal{V}(x_0)$, $\exists w_\xi \in \mathcal{V}(\xi)$ a conic neighborhood of ξ. $\exists k_1 > 0$, $\exists k_2 > 0$, $\exists c > 0$, $\exists \varepsilon_0 > 0$, $\forall \xi \in W_\xi$, $\forall \varepsilon \leq \varepsilon_0$, $\forall \phi_\xi \in \mathcal{D}^\tau(\Omega)$.

$$\left|\mathcal{F}(\phi_\xi f_\varepsilon)(\xi)\right| \leq c.\exp(k_1\varepsilon^{-\frac{1}{\tau}}) - k_2\,|\xi|^{\frac{1}{\sigma}})$$

Since the unit sphere $|\xi| = 1$ is a compact set, then one can find finite points $\xi_j, j = 1, ..., n$ in $\mathbb{R}^n$, $W_j \in \xi_|$ and $\phi_j \in \mathcal{D}^M(\Omega)$, $\phi_j(x) = 1$ in V_j, $k_1 > 0$, $\exists k_2 > 0$, $\exists c > 0$, $\varepsilon_0 > 0$, $\forall \varepsilon \leq \varepsilon_0$

$$\left|\mathcal{F}(\phi_j f_\varepsilon)(\xi)\right| \leq c. \exp(k_1 \varepsilon^{-\frac{1}{\tau}} - k_2 |\xi|^{\frac{1}{\sigma}}), \qquad \xi \in W_j$$

Taking $V = \bigcap_j V_j$ and $W = \bigcup_j W_j$, $\varphi = \phi_1 ... \phi_n$, we have $\varphi \in \mathcal{D}^\tau(\Omega)$ and $\varphi(x) = 1$ on V.

$$|\mathcal{F}(\varphi f_\varepsilon)(\xi)| \leq c. \exp(k_1 \varepsilon^{-\frac{1}{\tau}} - k_2 |\xi|^{\frac{1}{\sigma}}), \qquad \xi \in W$$

Consequently, $(\varphi f_\varepsilon) \notin \mathcal{G}_c^{\tau,\sigma,\infty}$ where: $x_0 \in \sigma - singsupp_g(f)$ □

Definition 2.7.5 *A point $(x_0, \xi_0) \notin WF_g^{\tau,\sigma}(f) \subset \Omega \times \mathbb{R}^n \backslash \{0\}$. If $\xi_0 \notin \sum_{g,x_0}^{\tau,\sigma}(f)$, i.e: there exists $\phi \in \mathcal{D}^\tau(\Omega), \phi(x) = 1$ neighborhood of x_0, and conic neighborhood Γ of ξ_0, $\exists k_1 > 0$, $\exists k_2 > 0$, $\exists c > 0$, $\exists \varepsilon_0 > 0$ such that $\forall \xi \in \Gamma$, $\forall \varepsilon \leq \varepsilon_0$,*

$$|\mathcal{F}(\phi f_\varepsilon)(\xi)| \leq c \exp(k_1 \varepsilon^{-\frac{1}{\tau}} - k_2 |\xi|^{\frac{1}{\sigma}})$$

The main properties of the generalized Roumieu wave front $WF_g^{\tau,\sigma}$ are subsumed in the following proposition

Proposition 2.7.6 *Let $f \in \mathcal{G}^\tau(\Omega)$, then*

(1) The projection of $WF_g^{\tau,\sigma}(f)$ on Ω is $\sigma - sinsupp_g(f)$.

(2) If $f \in \mathcal{G}_c^\tau(\Omega)$, The projection of $WF_g^{\tau,\sigma}(f)$ on $\mathbb{R}^n \backslash \{0\}$ is $\sum_g^{\tau,\sigma}(f)$.

(3) $\forall g \in \mathcal{G}^{\tau,\sigma,\infty}(\Omega)$, $WF_g^{\tau,\sigma}(gf) \subset WF_g^{\tau,\sigma}(f)$.

(4) $\forall \alpha \in \mathbb{Z}_+^n$, $WF_g^{\tau,\sigma}(\partial^\alpha f) \subset WF_g^{\tau,\sigma}(f)$.

Proof. (1) and (2) hold from the definition, Proposition (2.6.6) and lemma (2.7.4).

(3) Let $(x_0, \xi_0) \notin WF_g^{\tau,\sigma}(f)$ then $\exists \phi \in \mathcal{D}^\tau(\Omega)$, $\phi \equiv 1$ on a neighborhood of x_0, $\xi_0 \notin \sum_g^{\tau,\sigma}(\phi f)$ by proposition (2.6.6), for $g \in \mathcal{G}^{\tau,\sigma,\infty}(\Omega)$, we have $\xi_0 \notin \sum_g^{\tau,\sigma}(g\phi f)$ which proves: $(x_0, \xi_0) \notin WF_g^{\tau,\sigma}(gf)$.

(4) Let $(x_0, \xi_0) \notin WF_g^{\tau,\sigma}(f)$, then $\exists \phi \in \mathcal{D}^\tau(\Omega)$, $\phi \equiv 1$ on a neighborhood $\bar{U}$ of x_0, there exist a conic neighborhood Γ of ξ_0, $\exists k_1 > 0$, $\exists k_2 > 0$, $\exists c_1 > 0$, $\exists \varepsilon_0 \in]0, 1]$,such that $\forall \xi \in \Gamma$, $\varepsilon \leq \varepsilon_0$,

$$|\mathcal{F}(\phi f_\varepsilon)(\xi)| \leq c_1 \exp(k_1 \varepsilon^{-\frac{1}{\tau}} - k_2 |\xi|^{\frac{1}{\sigma}}) \tag{2.9}$$

We have for $\psi \in \mathcal{D}^\tau(U)$ such that $\psi(x_0) = 1$.

$$\begin{aligned} |\mathcal{F}(\psi \partial f_\varepsilon)(\xi)| &= |\mathcal{F}(\partial(\psi f_\varepsilon))(\xi) - \mathcal{F}(\partial\psi . f_\varepsilon(\xi))| \\ &\leq |\xi| \, |\mathcal{F}(\psi\phi f_\varepsilon)(\xi)| + |\mathcal{F}((\partial\psi)\phi f_\varepsilon)(\xi)| \end{aligned}$$

As $WF_g^{\tau,\sigma}(\psi f) \subset WF_g^{\tau,\sigma}(f)$, (2.9) holds for both $|\mathcal{F}(\psi\phi f_\varepsilon)(\xi)|$ and $|\mathcal{F}((\partial\psi)\phi f_\varepsilon)(\xi)|$.

So

$$\begin{aligned} |\xi| \, |\mathcal{F}(\psi\phi f_\varepsilon)(\xi)| &\leq c \, |\xi| \exp(k_1 \varepsilon^{-\frac{1}{\tau}} - k_2 |\xi|^{\frac{1}{\sigma}}) \\ &\leq c' \exp(k_1 \varepsilon^{-\frac{1}{\tau}} - k_3 |\xi|^{\frac{1}{\sigma}}) \end{aligned}$$

With $c' > 0$, $k_3 > 0$, such that $|\xi| \leq c' \exp((k_2 - k_3) |\xi|^{\frac{1}{\sigma}})$ which prove $(x_0, \xi_0) \notin WF_g^{\tau,\sigma}(\partial f)$. □

Corollary 2.7.7 *Let* $P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$ *be a partial differential operator with* $\mathcal{G}^{\tau,\sigma,\infty}(\Omega)$ *coefficient, then:* $WF_g^{\tau,\sigma}(P(x, D)f) \subset WF_g^{\tau,\sigma}(f)$, $\forall f \in \mathcal{G}^\tau(\Omega)$.

Lemma 2.7.8 *Let $\varphi \in \mathcal{D}^{\sigma}(B(0.2))$, $0 \leq \varphi \leq 1$, and $\varphi \equiv 1$ on $B(0,1)$ and let $\phi \in S^{\sigma}$, then* $\exists c > 0$, $\exists v > 0$, $\exists \varepsilon_0 > 0$, $\forall \varepsilon \in]0, \varepsilon_0]$, $\forall \xi \in \mathbb{R}^n$,

$$\left|\hat{\theta}_{\varepsilon}(\xi)\right| \leq c\varepsilon^{-n} e^{-v\varepsilon^{\frac{1}{\sigma}}|\xi|^{\frac{1}{\sigma}}}$$

Where $\theta_{\varepsilon}(x) = (\frac{1}{\varepsilon})^n.\phi(\frac{x}{\varepsilon}).\varphi(x\,|\ln(\varepsilon)|)$, and $\hat{\theta}$ denoted the Fourier transform of θ.

Proof. We have, for ε sufficiently small, $\varepsilon \leq |\ln \varepsilon|^{-n} \leq 1$

Let $\xi \in \mathbb{R}^n$, then

$$\begin{aligned}
\hat{\theta}_{\varepsilon}(\xi) &= \frac{1}{\varepsilon^n} \textstyle\int \hat{\phi}(\varepsilon(\xi - \eta)).\frac{1}{|\ln \varepsilon|^n}.\hat{\varphi}(\frac{\eta}{|\ln \varepsilon|})d\eta \\
&= |\ln \varepsilon|^{-n} \left[\textstyle\int_A \hat{\phi}(\varepsilon(\xi - \eta))\hat{\varphi}(\frac{\eta}{|\ln \varepsilon|})d\eta + \int_B \hat{\phi}(\varepsilon(\xi - \eta))\hat{\varphi}(\frac{\eta}{|\ln \varepsilon|})d\eta\right]
\end{aligned}$$

Where $A = \{\eta : |\xi - \eta|^{\frac{1}{\sigma}} \leq \delta^{\frac{1}{\sigma}}(|\xi|^{\frac{1}{\sigma}} + |\eta|^{\frac{1}{\sigma}})\}$ and $B = \{\eta : |\xi - \eta|^{\frac{1}{\sigma}} > \delta^{\frac{1}{\sigma}}(|\xi|^{\frac{1}{\sigma}} + |\eta|^{\frac{1}{\sigma}})\}$ we choose δ sufficiently small such that $\frac{|\xi|}{2^{\sigma}} < |\eta| < 2^{\sigma}|\xi|$, $\forall \eta \in A$. Since $\varphi \in \mathcal{D}^{\sigma}(\Omega)$, $\phi \in S^{\sigma}$ then $\exists k_1, k_2 > 0, \exists c_1, c_2 > 0$, $\forall \xi \in \mathbb{R}$,

$$|\hat{\varphi}(\xi)| \leq c_1 \exp(-k_1 |\xi|^{\frac{1}{\sigma}})$$

And

$$\left|\hat{\phi}(\xi)\right| \leq c_2 \exp(-k_2 |\xi|^{\frac{1}{\sigma}})$$

So

$$\begin{aligned}
I_1 &= |\ln \varepsilon|^{-n} \left|\textstyle\int_A \hat{\phi}(\varepsilon(\xi - \eta))\hat{\varphi}(\frac{\eta}{|\ln \varepsilon|})d\eta\right| \\
&\leq c_1 c_2 \exp\left(-\frac{k_2 |\ln \varepsilon|^{-\frac{1}{\sigma}} |\xi|^{\frac{1}{\sigma}}}{2}\right) \textstyle\int \exp(-k_1 |\varepsilon(\eta - \xi)|^{\frac{1}{\sigma}})d\eta
\end{aligned}$$

Let $z = \varepsilon(\eta - \xi)$, then

$$\begin{aligned} I_1 &\leq c\varepsilon^{-n} \exp(-\tfrac{k_2}{2} |\ln \varepsilon|^{-\frac{1}{\sigma}} |\xi|^{\frac{1}{\sigma}}) \int \exp(-k_1 |z|^{\frac{1}{\sigma}}) dz \\ &\leq c\varepsilon^{-n} \exp(-v\varepsilon^{\frac{1}{\sigma}} |\xi|^{\frac{1}{\sigma}}) \end{aligned}$$

For I_2 we have

$$\begin{aligned} I_2 &= |\ln \varepsilon|^{-n} \left| \int_B \hat{\phi}(\varepsilon(\xi - \eta)) \hat{\varphi}(\tfrac{\eta}{|\ln \varepsilon|}) d\eta \right| \\ &\leq c_1 c_2 \int_B \exp(-k_1 \varepsilon^{\frac{1}{\sigma}} |\xi - \eta|^{\frac{1}{\sigma}} - k_2 |\ln \varepsilon|^{-\frac{1}{\sigma}} |\eta|^{\frac{1}{\sigma}}) d\eta \\ &\leq c \exp(-k_1 \delta \varepsilon^{\frac{1}{\sigma}} |\xi|^{\frac{1}{\sigma}}). \int_B \exp(-k_1 \delta \varepsilon^{\frac{1}{\sigma}} |\eta|^{\frac{1}{\sigma}} - k_2 |\ln \varepsilon|^{-\frac{1}{\sigma}} |\eta|^{\frac{1}{\sigma}}) d\eta \\ &\leq c \exp(-k_1 \delta \varepsilon^{\frac{1}{\sigma}} |\xi|^{\frac{1}{\sigma}}). \int_B \exp(-k \varepsilon^{\frac{1}{\sigma}} |\eta|^{\frac{1}{\sigma}}) d\eta \\ &\leq c\varepsilon^{-n} \exp(-v\varepsilon^{\frac{1}{\sigma}} |\xi|^{\frac{1}{\sigma}}) \end{aligned}$$

Consequently, $\exists c > 0$, $\exists v > 0$, $\exists \varepsilon_0 > 0$, $\forall \varepsilon \leq \varepsilon_0$ such that

$$\left| \hat{\theta}_\varepsilon(\xi) \right| \leq c\varepsilon^{-n} \exp\left(-v\varepsilon^{\frac{1}{\sigma}} |\xi|^{\frac{1}{\sigma}} \right), \qquad \forall \xi \in \mathbb{R}^n$$

□

Theorem 2.7.9 *Let $T \in \mathcal{D}'_{\tau+\sigma}(\Omega) \cap \mathcal{G}^\tau(\Omega)$; then $WF_g^{\tau,\tau-\sigma+1}(T) = WF^{\tau-\sigma+1}(T)$.*

Proof. Put $\rho = \tau - \sigma + 1$. Let $S \in E'_{\tau+\sigma}(\Omega) \subset E'_\rho(\Omega)$ and $\psi \in \mathcal{D}^\rho(\Omega)$, we have $|\mathcal{F}(\psi(S * \phi_\varepsilon))(\xi) - \mathcal{F}(\psi S)(\xi)| = \left| \left\langle S(x), (\psi(x)e^{-i\xi x}) * \check{\phi}_\varepsilon(x) - (\psi(x)e^{-ix\xi}) \right\rangle \right|$ then $\exists L$ a compact of Ω such that $\forall h > 0$, $\exists c > 0$,

$$|\mathcal{F}(\psi(S * \phi_\varepsilon))(\xi) - \mathcal{F}(\psi S)(\xi)| \leq c \sup_{\alpha \in \mathbb{Z}_+^n; x \in L} \frac{h^{|\alpha|}}{\alpha!^\rho} \left| \partial_x^\alpha (\psi(x)e^{-i\xi x} * \check{\phi}_\varepsilon(x) - \psi(x)e^{-i\xi x}) \right|$$

We have $e^{-i\xi}\psi \in \mathcal{D}^{\rho}(\Omega)$, then, $\exists c_2, \forall k_0 > 0$, $\exists \eta > 0$, $\forall \varepsilon \leq \eta$,

$$\underset{\alpha\in\mathbb{Z}_+^n;x\in L}{sup} \frac{c_2^{|\alpha|}}{\alpha!} \left|\partial_x^{\alpha}(\psi(x)e^{-i\xi x} * \check{\phi}_{\varepsilon}(x) - \psi(x)e^{-i\xi x})\right| \leq c_2 e^{-k_0\varepsilon^{-\frac{1}{\tau}}};$$

So there exist $c' > 0$, $\forall k_0 > 0$, $\exists \eta > 0$, $\forall \varepsilon \leq \eta$, such that

$$|\mathcal{F}(\psi S)(\xi) - \mathcal{F}(\psi(S * \phi_{\varepsilon})(\xi))| \leq c' e^{-k_0\varepsilon^{-\frac{1}{\tau}}} \tag{2.10}$$

Let $T \in \mathcal{D}'_{\tau+\sigma}(\Omega) \cap \mathcal{G}^{\tau}(\Omega)$ and $(x_0, y_0) \notin WF_g^{\tau,\rho}(T)$, then there exist $\chi \in \mathcal{D}^{\rho}(\Omega)$, $\chi(x) = 1$ in a neighborhood of x_0, and a conic neighborhood Γ of ξ_0, $\exists k_1 > 0$, $\exists k_2 > 0$, $\exists c_1 > 0$, $\exists \varepsilon_0 \in]0, 1[$, such that: $\forall \xi \in \Gamma$, $\forall \varepsilon \leq \varepsilon_0$,

$$|\mathcal{F}(\chi(T * \theta_{\varepsilon}))(\xi)| \leq c_1 e^{k_1\varepsilon^{-\frac{1}{\tau}} - k_2|\xi|^{\frac{1}{\rho}}} \tag{2.11}$$

Let $\psi \in \mathcal{D}^{\rho}(\Omega)$ equal to 1 in neighborhood of x_0 such that for sufficiently small ε we have $\chi \equiv 1$ on $supp\psi + B(0, \frac{2}{|\ln \varepsilon|})$, and let $\varphi \in \mathcal{D}^{\rho}(B(0, 2))$, $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ on $B(0, 1)$, then there exist $\varepsilon_0 \leq 1$, such that $\forall \varepsilon < \varepsilon_0$,

$$\psi(T * \theta_{\varepsilon})(x) = \psi(\chi T * \theta_{\varepsilon})(x).$$

Where $\theta_{\varepsilon}(x) = \frac{1}{\varepsilon^n}\varphi(x\,|\ln \varepsilon|)\phi(\frac{x}{\varepsilon})$. As $\chi T \in E'_{\tau+\sigma}(\Omega)$, then

$$\psi(T * \theta_{\varepsilon})(x) = \psi(\chi T * \theta_{\varepsilon})(x) = \psi(\chi T * \phi_{\varepsilon})(x)$$

Let $\varepsilon \leq min(\eta, \varepsilon_0)$ and $\xi \in \Gamma$, we have

$$\begin{aligned}|\mathcal{F}(\psi T)(\xi)| &\leq |\mathcal{F}(\psi T)(\xi) - \mathcal{F}(\psi(T * \theta_\varepsilon))(\xi)| + |\mathcal{F}(\chi(T * \theta_\varepsilon))(\xi)| \\ &\leq |\mathcal{F}(\psi \chi T)(\xi) - \mathcal{F}(\psi(\chi T * \phi_\varepsilon))(\xi)| + |\mathcal{F}(\chi(T * \theta_\varepsilon))(\xi)|\end{aligned}$$

Then by (2.10) and (2.11), we obtain

$$|\mathcal{F}(\psi T)(\xi)| \leq c' e^{-k_0 \varepsilon^{-\frac{1}{\tau}}} + c_1 e^{k_1 \varepsilon^{-\frac{1}{\tau}} - k_2 |\xi|^{\frac{1}{\rho}}}.$$

Take $c = max(c_1, c')$, $\varepsilon = \left(\dfrac{k_1}{(k_2 - r)\,|\xi|^{\frac{1}{\rho}}} \right)^\tau$, $r \in]0, k_2[$ and $k_0 = \dfrac{k_1 r}{k_2 - r}$, then $\exists \delta > 0$, $\exists c > 0$ such that

$$|\mathcal{F}(\chi T)(\xi)| \leq c e^{-\delta |\xi|^{\frac{1}{\rho}}},$$

Which proves that $(x_0, \xi_0) \notin WF^\rho(T)$.So $WF^\rho(T) \subset WF_g^{\tau,\rho}(T)$.

Suppose that $(x_0, \xi_0) \notin WF^\rho(T)$, then there exist $\chi \in \mathcal{D}^\rho(\Omega)$, $\chi(x) = 1$ in a neighborhood of x_0, a conic neighborhood Γ of ξ_0, $\exists \lambda > 0$, $\exists c_1 > 0$, such that $\forall \xi \in \Gamma$

$$|\mathcal{F}(\chi T)(\xi)| \leq c_1 e^{-\lambda |\xi|^{\frac{1}{\rho}}}. \tag{2.12}$$

Let $\psi \in \mathcal{D}^\rho(\Omega)$ equals 1 in neighborhood of x_0 such that for sufficiently small ε we have $\chi \equiv 1$ on $supp\psi + B(0, \frac{2}{|\ln \varepsilon|})$, then there exist $\varepsilon_0 < 1$, such that $\forall \varepsilon < \varepsilon_0$,

$$\psi(T * \theta_\varepsilon)(x) = \psi(\chi T * \theta_\varepsilon)(x).$$

We have

$$\mathcal{F}(\psi(T*\theta_\varepsilon))(\xi) = \int \mathcal{F}(\psi)(\xi-\eta).\mathcal{F}(\chi T)(\eta).\mathcal{F}(\theta_\varepsilon)(\eta)d\eta.$$

Let Λ be a conic neighborhood of ξ_0 such that, $\bar{\Lambda} \subset \Gamma$. For a fixed $\xi \in \Lambda$, we have

$$\mathcal{F}(\psi(\chi T*\theta_\varepsilon))(\xi) = \int_A \mathcal{F}(\psi)(\xi-\eta).\mathcal{F}(\chi T)(\eta).\mathcal{F}(\theta_\varepsilon)(\eta)d\eta + \int_B \mathcal{F}(\psi)(\xi-\eta).\mathcal{F}(\chi T)(\eta).\mathcal{F}(\theta_\varepsilon)(\eta)d\eta$$

Where $A = \{\eta : |\xi-\eta|^{\frac{1}{\rho}} \leq \delta(|\xi|^{\frac{1}{\rho}} + |\eta|^{\frac{1}{\rho}})\}$ and $B = \{\eta : |\xi-\eta|^{\frac{1}{\rho}} > \delta(|\xi|^{\frac{1}{\rho}} + |\eta|^{\frac{1}{\rho}})\}$. We choose δ sufficiently small such that $A \subset \Gamma$ and $\frac{|\xi|}{2^\rho} < |\eta| < 2^\rho |\xi|$. Since $\psi \in \mathcal{D}^\rho(\Omega)$, then $\exists \mu > 0$, $\exists c_2 > 0$, $\forall \xi \in \mathbb{R}^n$,

$$|\mathcal{F}(\psi)(\xi)| \leq c_2 \exp(-\mu |\xi|^{\frac{1}{\rho}}),$$

Then $\exists c > 0$, $\exists \varepsilon_0 \in]0,1[$, $\forall \varepsilon \leq \varepsilon_0$,

$$\left| \int_A \mathcal{F}(\psi)(\xi-\eta).\mathcal{F}(\chi T)(\eta).\mathcal{F}(\theta_\varepsilon)(\eta)d\eta \right| \leq c \exp(-\frac{\lambda}{2}|\xi|^{\frac{1}{\rho}}) \times \left| \int_A \exp(-\mu |\eta-\xi|^{\frac{1}{\rho}}).\mathcal{F}(\theta_\varepsilon)(\eta)d\eta \right|$$

From lemma (2.7.8), $\exists c_3 > 0$, $\exists v > 0$, $\exists \varepsilon_0 > 0$, such that

$$|\mathcal{F}(\theta_\varepsilon)(\xi)| \leq c_3 \varepsilon^{-n} e^{-v\varepsilon^{\frac{1}{\sigma}}|\xi|^{\frac{1}{\rho}}} \ \forall \xi \in \mathbb{R}^n$$

then $\exists c > 0$, such that

$$\left| \int_A \mathcal{F}(\psi)(\xi-\eta).\mathcal{F}(\chi T)(\eta).\mathcal{F}(\theta_\varepsilon)(\eta)d\eta \right| \leq c \quad \varepsilon^{-n} \exp(-\tfrac{\lambda}{2}|\xi|^{\frac{1}{\rho}}) \times \left| \int_A \exp(-\mu |\eta-\xi|^{\frac{1}{\rho}}).\exp(-v\varepsilon^{\frac{1}{\sigma}} |\eta|^{\frac{1}{\sigma}})d\eta \right|$$

We have $\exists k > 0$, $\forall \varepsilon \in]0, \varepsilon_0[$,

$$\varepsilon^{-n} \exp(-v\varepsilon^{\frac{1}{\sigma}} |\eta|^{\frac{1}{\sigma}}) \leq \exp(k\varepsilon^{-\frac{1}{\tau}}), \tag{2.13}$$

So

$$\left| \int_A \mathcal{F}(\psi)(\xi - \eta).\mathcal{F}(\chi T)(\eta).\mathcal{F}(\theta_\varepsilon)(\eta) d\eta \right| \leq c \exp(k\varepsilon^{\frac{1}{\tau}} - \frac{\lambda}{2} |\xi|^{\frac{1}{\rho}}) \tag{2.14}$$

As $\psi T \in E'_{\tau+\sigma}(\Omega) \subset E'_\rho(\Omega)$, then $\forall l > 0$, $\exists c > 0$, $\forall \xi \in \mathbb{R}^n$,

$$|\mathcal{F}(\chi T)(\xi)| \leq c \exp(l |\xi|^{\frac{1}{\rho}})$$

Hence, we have

$$\begin{aligned} \left| \int_B \mathcal{F}(\psi)(\xi - \eta)\mathcal{F}(\chi T)(\eta)\mathcal{F}(\theta_\varepsilon)(\eta) d\eta \right| &\leq c \int_B \exp(l |\eta|^{\frac{1}{\rho}} - \mu |\xi - \eta|^{\frac{1}{\rho}}). |\mathcal{F}(\theta_\varepsilon)| \, d\eta \\ &\leq c' \varepsilon^{-n}. \exp(-\mu\delta |\xi|^{\frac{1}{\rho}}) \\ &\quad \int_B \exp((l - \mu\delta) |\eta|^{\frac{1}{\rho}} - v\varepsilon^{\frac{1}{\sigma}} |\eta|^{\frac{1}{\sigma}}) d\eta, \end{aligned}$$

Then, taking $l - \mu\delta = -a < 0$ and using (2.13), we obtain for a constant $c > 0$

$$\left| \int_B \mathcal{F}(\psi)(\xi - \eta).\mathcal{F}(\chi T)(\eta).\mathcal{F}(\theta_\varepsilon)(\eta) d\eta \right| \leq c \exp(k_1 \varepsilon^{-\frac{1}{\tau}} - \mu\delta |\xi|^{\frac{1}{\rho}})$$

Which gives that $(x_0, \xi_0) \notin WF_g^{\tau,\rho}(T)$, so $WF_g^{\tau,\rho}(T) \subset WF^\rho(T)$.

□

2.8 Generalized Hörmander's theorem

To extend the generalized Hörmander's result on the wave front set of the product, define $WF_g^{\tau,\sigma}(f)+WF_g^{\tau,\sigma}(f)$, where $f,g\in\mathcal{G}^\tau(\Omega)$, as the set:

$$\{(x,\xi+\eta)\in WF_g^{\tau,\sigma}(f),(x,\eta)\in WF_g^{\tau,\sigma}(g)\}$$

Lemma 2.8.1 *Let Σ_1, Σ_2 be closed cones in $\mathbb{R}^n\backslash\{0\}$, such that: $0\notin\Sigma_1+\Sigma_2$, then:*

i) $\overline{\Sigma_1+\Sigma_2}^{\mathbb{R}^n\backslash\{0\}}=(\Sigma_1+\Sigma_2)\cup\Sigma_1\cup\Sigma_2$.

ii) For any open conic neighborhood Γ of $\Sigma_1+\Sigma_2$ in $\mathbb{R}^n\backslash\{0\}$, one can find open conic neighborhood of Γ_1, Γ_2 in $\mathbb{R}^n\backslash\{0\}$ of respectively Σ_1, Σ_2 such that:

$$\Gamma_1+\Gamma_2\subset\Gamma$$

The principal result of this section is the following theorem.

Theorem 2.8.2 *Let $f,g\in\mathcal{G}^\tau(\Omega)$, such that: $\forall x\in\Omega$,*

$$(x,0)\notin WF_g^{\tau,\sigma}(f)+WF_g^{\tau,\sigma}(g) \tag{2.15}$$

Then:

$$WF_g^{\tau,\sigma}(f.g)\subseteq(WF_g^{\tau,\sigma}(f)+WF_g^{\tau,\sigma}(g))\cup WF_g^{\tau,\sigma}(f)\cup WF_g^{\tau,\sigma}(g).$$

Proof. Let $(x_0,\xi_0)\notin(WF_g(f)+WF_g(g))\cup WF_g(f)\cup WF_g(g)$, then: $\exists\phi\in\mathcal{D}(\Omega)$; $\phi(x_0)=1$, $\xi_0\notin(\Sigma_g(\phi f)+\Sigma_g(\phi g))\cup\Sigma_g(\phi f)\cup\Sigma_g(\phi g)$ From (2.15) we have $0\notin\Sigma_g(\phi f)+\Sigma_g(\phi g)$ then by lemma (2.8.1) i), we have

$$\xi_0 \notin (\Sigma_g(\phi f)+\Sigma_g(\phi g))\cup\Sigma_g(\phi f)\cup\Sigma_g(\phi g)=\overline{\Sigma_g(\phi f)+\Sigma_g(\phi g)}^{\mathbb{R}^n\backslash\{0\}}$$

Let Γ_0 be an open conic neighborhood of $\Sigma_g(\phi f)+\Sigma_g(\phi g)$ in $\mathbb{R}^n\backslash\{0\}$ such that: $\xi_0\notin\overline{\Gamma_0}$ then, from lemma (2.8.1) ii), there exist open cones Γ_1 and Γ_2 in $\mathbb{R}^n\backslash\{0\}$ such that

$$\Sigma_g(\phi f)\subset\Gamma_1;\quad \Sigma_g(\phi g)\subset\Gamma_2$$

And

$$\Gamma_1+\Gamma_2\subset\Gamma_0$$

Define $\Gamma=\mathbb{R}^n\backslash\Gamma_0$, so

$$\Gamma\cap\Gamma_2=\emptyset\, and\,(\Gamma-\Gamma_2)\cap\Gamma_1=\emptyset \tag{2.16}$$

Let $\xi\in\Gamma$ and $\varepsilon\in I$.

$$\begin{aligned}\mathcal{F}(\phi f_\varepsilon\phi g_\varepsilon)(\xi) &= (\mathcal{F}(\phi f_\varepsilon)*\mathcal{F}(\phi g_\varepsilon))(\xi)\\ &= \int_{\Gamma_2}\mathcal{F}(\phi f_\varepsilon)(\xi-\eta).\mathcal{F}(\phi g_\varepsilon)(\eta)d\eta+\\ &\quad \int_{\Gamma_2^c}\mathcal{F}(\phi f_\varepsilon)(\xi-\eta).\mathcal{F}(\phi g_\varepsilon)(\eta)d\eta=I_1(\xi)+I_2(\xi)\end{aligned}$$

From (2.16), $\exists c_1>0$, $\exists k_1,k_2>0$, $\exists\varepsilon_1>0$, such that: $\forall\varepsilon\le\varepsilon_1$, $\forall\eta\in\Gamma_2$,

$$|\mathcal{F}(\phi f_\varepsilon)(\xi-\eta)|\le c_1\exp(k_1\varepsilon^{-\frac{1}{\tau}}-k_2\,|\xi-\eta|^{\frac{1}{\sigma}})$$

And from remark (2.6.7), $\exists c_2 > 0$, $\exists k_3 > 0$, $\forall k_4 > 0$, $\exists \varepsilon_2 > 0$, $\forall \eta \in \mathbb{R}^n$, $\forall \varepsilon \leq \varepsilon_2$,

$$|\mathcal{F}(\phi g_\varepsilon)(\eta)| \leq c_2 \exp(k_3 \varepsilon^{-\frac{1}{\tau}} + k_4 |\eta|^{\frac{1}{\sigma}})$$

Let $\gamma > 0$ sufficiently small such that:

$$|\xi - \eta|^{\frac{1}{\sigma}} \geq \gamma(|\xi|^{\frac{1}{\sigma}} + |\eta|^{\frac{1}{\sigma}}), \quad \forall \eta \in \Gamma_2$$

Hence for $\varepsilon \leq min(\varepsilon_1, \varepsilon_2)$,

$$|I_1(\xi)| \leq c_1.c_2 \exp((k_1 + k_3)\varepsilon^{-\frac{1}{\tau}} - k_2 \gamma |\xi|^{\frac{1}{\sigma}}) \int \exp(-k_2 \gamma |\eta|^{\frac{1}{\sigma}} + k_4 |\eta|^{\frac{1}{\sigma}}) d\eta$$

Take $k_4 < k_2 \gamma$, then:

$$|I_1(\xi)| \leq c' \exp(k'_1 \varepsilon^{-\frac{1}{\tau}} - k'_2 |\xi|^{\frac{1}{\sigma}})$$

Let $r > 0$,

$$\begin{aligned} I_2(\xi) &= \int_{\Gamma_2^c \cap \{|\eta| \leq r|\xi|\}} \mathcal{F}(\phi f_\varepsilon)(\xi - \eta).\mathcal{F}(\phi g_\varepsilon)(\eta) d\eta + \int_{\Gamma_2^c \cap \{|\eta| \geq r|\xi|\}} \mathcal{F}(\phi f_\varepsilon)(\xi - \eta).\mathcal{F}(\phi g_\varepsilon)(\eta) d\eta \\ &= I_{21}(\xi) + I_{22}(\xi). \end{aligned}$$

Choose r sufficiently small such that $\{|\eta|^{\frac{1}{\sigma}} \leq r |\xi|^{\frac{1}{\sigma}}\} \Rightarrow \xi - \eta \notin \Gamma_1$. Then

$|\xi - \eta|^{\frac{1}{\sigma}} \geq (1 - r) |\xi|^{\frac{1}{\sigma}} \geq (1 - 2r) |\xi|^{\frac{1}{\sigma}} + |\eta|^{\frac{1}{\sigma}}$, Consequently $\exists c_3 > 0$, $\exists \lambda_1, \lambda_2, \lambda_3 > 0$, $\exists \varepsilon_3 > 0$

such that $\forall \varepsilon \leq \varepsilon_1$;

$$
\begin{aligned}
|I_{21}(\xi)| &\leq c_3 \exp(\lambda_1 \varepsilon^{-\frac{1}{\tau}}) \int \exp(-\lambda_2 |\xi - \eta|^{\frac{1}{\sigma}} - \lambda_3 |\eta|^{\frac{1}{\sigma}}) d\eta \\
&\leq c_3 \exp(\lambda_1 \varepsilon^{-\frac{1}{\tau}} - \lambda_2' |\xi|^{\frac{1}{\sigma}}) \int \exp(-\lambda_3' |\eta|^{\frac{1}{\sigma}}) d\eta \\
&\leq c_3' \exp(\lambda_1 \varepsilon^{-\frac{1}{\tau}} - \lambda_2' |\xi|^{\frac{1}{\sigma}})
\end{aligned}
$$

If $|\eta|^{\frac{1}{\sigma}} \geq r |\xi|^{\frac{1}{\sigma}}$, we have $|\eta|^{\frac{1}{\sigma}} \geq \dfrac{|\eta|^{\frac{1}{\sigma}} + r |\xi|^{\frac{1}{\sigma}}}{2}$, and then $\exists c_4 > 0$, $\exists \mu_1, \mu_3 > 0$, $\forall \mu_2 > 0$, $\exists \varepsilon_4 > 0$ such that $\forall \varepsilon \leq \varepsilon_4$,

$$
\begin{aligned}
|I_{22}(\xi)| &\leq c_4 \exp(\mu_1 \varepsilon^{-\frac{1}{\tau}}) \int \exp(\mu_2 |\xi - \eta|^{\frac{1}{\sigma}} - \mu_3 |\eta|^{\frac{1}{\sigma}}) d\eta \\
&\leq c_4 \exp(\mu_1 \varepsilon^{-\frac{1}{\tau}}) \int \exp(\mu_2 |\xi - \eta|^{\frac{1}{\sigma}} - \mu_3' |\eta|^{\frac{1}{\sigma}} - \mu_3' r |\xi|^{\frac{1}{\sigma}}) d\eta
\end{aligned}
$$

If we take μ_2 sufficiently small we obtain

$$
|I_{22}| \leq c_4' \exp(k_3' \varepsilon^{-\frac{1}{\tau}} - \mu_3" |\xi|^{\frac{1}{\sigma}})
$$

Which finishes the proof. □

Chapter 3

Generalized Roumieu ultradistributions

The aim of this chapter is to introduce and to study a new classes of generalized functions containing the space of Roumieu ultradistributions introduced by Komatsu [27] as natural generalization of Schwartz distributions. The problem of multiplication of ultradistributions is still posed, so it's natural to search for algebras of generalized functions containing the space of ultradistributions where we recovered a whole list of important result know in generalized Geverey ultradistribution theory [3],[2].

3.1 Roumieu ultradistributions

Let $(M_p)_{p\in\mathbb{Z}_+}$ be a sequence of reel positive numbers, recall the following properties:

$(H1)$ Logarithmic convexity:

$$M_p^2 \leq M_{p-1}M_{p+1}, \quad \forall p \geq 1$$

$(H2)$ Stability under ultradifferentiation:

$$\exists A > 0, \exists H > 0, M_{p+q} \leq AH^{p+q} M_p M_q, \forall p \geq 0, \forall q \geq 0.$$

$(H2)$' Stability under differentiation:

$$\exists A > 0, \exists H > 0, M_{p+1} \leq AH^{p} M_p, \forall p \geq 0$$

$(H3)$' Non-quasi-analyticity:

$$\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < +\infty$$

The associated function of the sequence $(M_p)_{p\in\mathbb{Z}_+}$ is the function defined by

$$M(t) = \sup_{p} \ln \frac{t^p}{M_p}, t \in \mathbb{R}_+^*$$

Example 3.1.1 *The Gevrey sequence* $(M_p)_{p\in\mathbb{Z}_+} = (p!^{\sigma})_{p\in\mathbb{Z}_+}$, $\sigma > 0$, *has associated function equivalent to the function* $M(t) = t^{\frac{1}{\sigma}}$.

Proposition 3.1.2 *A positives sequence* $(M_p)_{p\in\mathbb{Z}_+}$ *satisfies condition (*$H1$*) if and only if*

$$M_p = M_0 \sup_{t>0} [t^p \exp(-M(t))], \ p \in \mathbb{Z}_+$$

Proposition 3.1.3 *Let the sequence* $(M_p)_{p\in\mathbb{Z}_+}$ *satisfy condition* $(H1)$, *then it satisfies* $(H2)$ *if* *and only if* $\exists A > 0$, $\exists H > 0$, $\forall t > 0$,

$$2M(t) \leq M(Ht) + \ln(AM_0).$$

The class of ultradifferential functions of class M, denoted $E^M(\Omega)$, is the space of all $f \in C^\infty(\Omega)$ satisfying for every compact subset K of Ω, $\exists c > 0$, $\forall \alpha \in \mathbb{Z}_+^n$,

$$\sup_{x \in K} |\partial^\alpha f(x)| \leq c^{|\alpha|+1} M_{|\alpha|} \tag{3.1}$$

This space is also called the space of Donjoy-Carleman.

Example 3.1.4 *If* $(M_p)_{p\in\mathbb{Z}_+} = (p!^\sigma)_{p\in\mathbb{Z}_+}$ *we obtain* $E^\sigma(\Omega)$ *the Gevrey space of order* σ*, and* $\mathcal{A}(\Omega) := E^1(\Omega)$ *is the space of real analytic functions on the open set* Ω*.*

A differential operator of infinite order $P(D) = \sum_{\gamma\in\mathbb{Z}_+^n} a_\gamma D^\gamma$ is called an ultradifferential operator of class $(M_p)_{p\in\mathbb{Z}_+}$, if for every $h > 0$ there exist $c > 0$ such that $\forall \gamma \in \mathbb{Z}_+^n$,

$$|a_\gamma| \leq c\frac{h^{|\gamma|}}{M_{|\gamma|}} \tag{3.2}$$

The basic properties of the space $E^M(\Omega)$ are summarized in the following proposition.

Proposition 3.1.5 *Let the sequence* $(M_p)_{p\in\mathbb{Z}_+}$ *satisfy condition* $(H1)$*, then the space* $E^M(\Omega)$ *is an algebra moreover, if* $(M_p)_{p\in\mathbb{Z}_+}$ *satisfies* $(H2)'$*, then* $E^M(\Omega)$ *is stable by differential operators of finite order with coefficients in* $E^M(\Omega)$*, and if* $(M_p)_{p\in\mathbb{Z}_+}$ *satisfies* $(H2)$ *then any ultradifferential operator of class* M *operates also as a sheaf homomorphism.*

The space $\mathcal{D}^M(\Omega) = E^M(\Omega) \cap \mathcal{D}(\Omega)$ is not trivial if and only if the sequence $(M_p)_{p\in\mathbb{Z}_+}$ satisfies $(H3)'$. To develop a local and microlocal analysis with respect to a "good" space of regular elements.

Remark 3.1.6 *The sequence $(p!^\sigma)_{p\in\mathbb{Z}_+}$ satisfies $(H3)'$ if and only if $\sigma > 1$*

Definition 3.1.7 *The strong dual of $\mathcal{D}^M(\Omega)$, denoted $\mathcal{D}'^M(\Omega)$, is called the space of Roumieu ultradistributions.*

3.2 Generalized Roumieu ultradistributions

To consider the algebra of generalized Roumieu ultradistributions, we first introduce the algebra of moderate elements and its ideal of null elements. Let Ω be a non void open set of $\mathbb{R}^n$ and $I =]0,1]$.

We will always suppose that the sequence $(M_p)_{p\in\mathbb{Z}_+}$ satisfies the conditions $(H1)$, $(H2)$, $(H3)'$ and $M_0 = 1$.

Definition 3.2.1 *The space of moderate elements, denoted $\mathcal{E}_m^M(\Omega)$, is the space of $(f_\varepsilon)_\varepsilon \in C^\infty(\Omega)^I$ satisfying for every compact K of Ω, $\forall\alpha \in \mathbb{Z}_+^n$, $\exists k > 0$, $\exists c > 0$, $\exists\varepsilon_0 \in I$, $\forall\varepsilon \leq \varepsilon_0$,*

$$\sup_{x\in K} |\partial^\alpha f_\varepsilon(x)| \leq c \exp(M(\frac{k}{\varepsilon})) \tag{3.3}$$

The space of null elements, denoted $\mathcal{N}^M(\Omega)$, is the space of $(f_\varepsilon)_\varepsilon \in C^\infty(\Omega)^I$ satisfying for every *compact* K *of* Ω, $\forall\alpha\in\mathbb{Z}_{+}^{n}$, $\forall k>0$, $\exists c>0$, $\exists\varepsilon_0\in I$, $\forall\varepsilon\leq\varepsilon_0$,

$$\underset{x\in K}{sup}\left|\partial^{\alpha}f_{\varepsilon}(x)\right|\leq c\exp(-M(\frac{k}{\varepsilon})) \tag{3.4}$$

The main properties of the spaces $\mathcal{E}_{m}^{M}(\Omega)$ and $\mathcal{N}^{M}(\Omega)$ are given in the following proposition.

Proposition 3.2.2 *1) The space of moderate elements* $\mathcal{E}_{m}^{M}(\Omega)$ *is an algebra stable by derivation.*

2) The space $\mathcal{N}^{M}(\Omega)$ *is an ideal of* $\mathcal{E}_{m}^{M}(\Omega)$*.*

Proof.

1) Let $(f_{\varepsilon})_{\varepsilon}$, $(g_{\varepsilon})_{\varepsilon}\in\mathcal{E}_{m}^{M}(\Omega)$ and K be the compact of Ω, then

$\forall\beta\in\mathbb{Z}_{+}^{n}$, $\exists k_1=k_1(\beta)>0$, $\exists c_1=c_1(\beta)>0$, $\exists\varepsilon_{1\beta}\in I$, $\forall\varepsilon\leq\varepsilon_{1\beta}$,

$$\sup_{x\in K}\left|\partial^{\beta}f_{\varepsilon}(x)\right|\leq c_1\exp M(\frac{k_1}{\varepsilon}), \tag{3.5}$$

$\forall\beta\in\mathbb{Z}_{+}^{n}$, $\exists k_2=k_2(\beta)>0$, $\exists c_2=c_2(\beta)>0$, $\exists\varepsilon_{2\beta}\in I$, $\forall\varepsilon\leq\varepsilon_{2\beta}$,

$$\sup_{x\in K}\left|\partial^{\beta}f_{\varepsilon}(x)\right|\leq c_2\exp M(\frac{k_2}{\varepsilon}), \tag{3.6}$$

Let $\alpha\in\mathbb{Z}_{+}^{n}$, then

$$\left|\partial^{\alpha}(f_{\varepsilon}g_{\varepsilon})(x)\right|\leq\sum_{\beta=0}^{\alpha}\binom{\alpha}{\beta}\left|\partial^{\alpha-\beta}f_{\varepsilon}(x)\right|\left|\partial^{\beta}g_{\varepsilon}(x)\right|$$

From proposition (3.1.3), we have $\exists A > 0$, $\exists H > 0$, $\forall t > 0$,

$$2M(t) \leq M(Ht) + \ln(AM_0) \tag{3.7}$$

For $k = H(\max\{k_1(\beta), k_2(\beta) : \beta \leq \alpha\})$, $\varepsilon \leq \min\{\varepsilon_{1\beta}, \varepsilon_{2\beta}; \beta \leq \alpha\}$ and $x \in K$, we have for $t = \dfrac{k}{\varepsilon}$

$$\begin{aligned}
\exp\left(-M\left(\tfrac{k}{\varepsilon}\right)\right) |\partial^\alpha (f_\varepsilon g_\varepsilon)(x)| &\leq \exp\left(\ln(AM_0)\right) \sum_{\beta=0}^{\alpha} \tbinom{\alpha}{\beta} \exp\left(-M\left(\frac{k_1}{\varepsilon}\right)\right) \\
&\quad \times \left|\partial^{\alpha-\beta} f_\varepsilon(x)\right| \exp\left(-M(\frac{k_2}{\varepsilon})\right) \left|\partial^\beta g_\varepsilon(x)\right| \\
&\leq A \sum_{\beta=0}^{\alpha} \tbinom{\alpha}{\beta} c_1(\alpha-\beta) c_2(\beta) = c(\alpha)
\end{aligned}$$

i.e. $(f_\varepsilon g_\varepsilon)_\varepsilon \in \mathcal{E}_m^M(\Omega)$. It is clear that for every compact K of Ω,

$\forall \beta \in \mathbb{Z}_+^n$, $\exists k_1 = k_1(\beta+1) > 0$, $\exists c_1 = c_1(\beta+1) > 0$, $\exists \varepsilon_{1\beta} \in I$ such that $\forall x \in K$, $\forall \varepsilon \leq \varepsilon_{1\beta}$,

$$\left|\partial^\beta (\partial f_\varepsilon)(x)\right| \leq c_1 \exp(M(\frac{k_1}{\varepsilon})),$$

i.e. $(\partial f_\varepsilon)_\varepsilon \in \mathcal{E}_m^M(\Omega)$.

2) If $(g_\varepsilon)_\varepsilon \in \mathcal{N}^M(\Omega)$, for every K compact of Ω, $\forall \beta \in \mathbb{Z}_+^n$, $\forall k_2 > 0$, $\exists c_2 = c_2(\beta, k_2) > 0$, $\exists \varepsilon_{2\beta} \in I$,

$$|\partial^\alpha g_\varepsilon(x)| \leq c_2 \exp(-M(\frac{k_2}{\varepsilon})), \ \forall x \in K, \forall \varepsilon \leq \varepsilon_{2\beta}$$

Let $\alpha \in \mathbb{Z}_+^n$ and $k > 0$ then

$$\exp\left(M\left(\tfrac{k}{\varepsilon}\right)\right)|\partial^\alpha(f_\varepsilon g_\varepsilon)(x)| \leq \exp\left(M\left(\tfrac{k}{\varepsilon}\right)\right)\sum_{\beta=0}^{\alpha}\tbinom{\alpha}{\beta}\left|\partial^{\alpha-\beta}f_\varepsilon(x)\right|\left|\partial^\beta g_\varepsilon(x)\right|$$

Let $k_2 = H.\max\{k_1(\beta), k; \beta \leq \alpha\}$ and $\varepsilon \leq \min\{\varepsilon_{1\beta}, \varepsilon_{2\beta}; \beta \leq \alpha\}$, then $\forall x \in K$, we have for $t = \frac{k_2}{\varepsilon}$ in (3.7)

$$\begin{aligned}\exp\left(M\left(\tfrac{k}{\varepsilon}\right)\right)|\partial^\alpha(f_\varepsilon g_\varepsilon)(x)| &\leq A\sum_{\beta=0}^{\alpha}\tbinom{\alpha}{\beta}\exp\left(-M\left(\frac{k_1}{\varepsilon}\right)\right)\left|\partial^{\alpha-\beta}f_\varepsilon(x)\right| \\ &\quad\times\exp\left(M(\frac{k_2}{\varepsilon})\right)\left|\partial^\beta g_\varepsilon(x)\right| \\ &\leq A\sum_{\beta=0}^{\alpha}\tbinom{\alpha}{\beta}c_1(\alpha-\beta)c_2(\beta,k_2) = c(\alpha,k)\end{aligned}$$

Which shows that $(f_\varepsilon g_\varepsilon)_\varepsilon \in \mathcal{N}^M(\Omega)$

□

Definition 3.2.3 *The algebra of generalized Roumieu ultradistributions of class* $(M_p)_{p\in\mathbb{Z}_+}$*, denoted* $\mathcal{G}^M(\Omega)$*, is the quotient algebra*

$$\mathcal{G}^M(\Omega) = \frac{\mathcal{E}_m^M(\Omega)}{\mathcal{N}^M(\Omega)}.$$

Definition 3.2.4 *If* $(M_p)_{p\in\mathbb{Z}_+} = (p!^\sigma)_{p\in\mathbb{Z}_+}$ *we obtain* $\mathcal{G}^\sigma(\Omega)$ *the algebra of generalized Gevrey ultradistributions*

3.3 Embedding of Roumieu ultradistributions

Let $N = (N_p)_{p\in\mathbb{Z}_+}$ be a sequence satisfying the conditions $(H1)$, $(H2)$, $(H3)'$ and $N_0 = 1$, the space $\mathcal{S}^N(\mathbb{R}^n)$ is the space of functions $\varphi \in C^\infty(\mathbb{R}^n)$ such that $\forall b > 0$, we have

$$\|\varphi\|_{b,N} = \underset{\alpha,\beta\in\mathbb{Z}_+^n}{sup} \int \frac{|x|^{|\beta|}}{b^{\alpha+\beta} N_{|\alpha|} N_{|\beta|}} |\partial^\alpha \varphi(x)| \, dx < \infty \tag{3.8}$$

Define Σ^N as the set of functions $\phi \in \mathcal{S}^N(\mathbb{R}^n)$ satisfying

$$\int \phi(x)dx = 1 \ and \int x^\alpha \phi(x)dx = 0, \quad \forall \alpha \in \mathbb{Z}_+^n \diagdown \{0\}.$$

Definition 3.3.1 *The net $\phi_\varepsilon = \varepsilon^{-n}\phi(./\varepsilon)$, $\varepsilon \in I$, where $\phi \in \Sigma^N$ is called a $N-$ mollifier net.*

Let $(L_p)_{p\in\mathbb{Z}_+}$ satisfying $(H1)$, $(H2)$, $(H3)'$,the space $E^L(\Omega)$ is embedded into $\mathcal{G}^M(\Omega)$ by the standard canonical injection

$$\begin{array}{lccc} I: & E^L(\Omega) & \rightarrow & \mathcal{G}^M(\Omega) \\ & f & \rightarrow & [f] = cl(f_\varepsilon) \end{array} \tag{3.9}$$

Where $f_\varepsilon = f, \forall \varepsilon \in I$.

The following theorem gives the embedding of Roumieu ultradistributions into $\mathcal{G}^M(\Omega)$. Let M and N two sequences satisfying $(H1)$, $(H2)$, $(H3)'$ with $M_0 = N_0 = 1$,$M_p > N_p$, $\forall p \in \mathbb{Z}^+$ and $\phi \in \Sigma^N$.

Theorem 3.3.2 *The map*

$$\begin{array}{llll} J_0: & E'_{MN}(\Omega) & \to & \mathcal{G}^M(\Omega) \\ & T & \to & [T] = cl((T*\phi_\varepsilon)_{/\Omega}) \end{array} \tag{3.10}$$

is an embedding.

Proof. Let $T \in E'_{MN}(\Omega)$ with $suppT \subset K$, then there exists $P(D) = \sum\limits_{\gamma\in\mathbb{Z}_+^n} a_\gamma D^\gamma$ an ultradifferential operator of class $(M_pN_p)_{p\in\mathbb{Z}_+}$, $C>0$, and continuous functions f_γ with $suppf_\gamma \subset K$, $\forall\gamma\in\mathbb{Z}_+^n$, and $\sup\limits_{\gamma\in\mathbb{Z}_+^n, x\in K} |f_\gamma(x)| \leq C$, such that

$$T = \sum_{\gamma\in\mathbb{Z}_+^n} a_\gamma D^\gamma f_\gamma.$$

We have

$$T*\phi_\varepsilon(x) = \sum_{\gamma\in\mathbb{Z}_+^n} a_\gamma \frac{(-1)^{|\gamma|}}{\varepsilon^{|\gamma|}} \int f_\gamma(x+\varepsilon y) D^\gamma\phi(y)dy.$$

Let $\alpha\in\mathbb{Z}_+^n$, then

$$|\partial^\alpha(T*\phi_\varepsilon)(x)| \leq \sum_{\gamma\in\mathbb{Z}_+^n} a_\gamma \frac{1}{\varepsilon^{|\gamma+\alpha|}} \int |f_\gamma(x+\varepsilon y)| \left|D^{\gamma+\alpha}\phi(y)\right| dy.$$

From (3.2) and condition $(H2)$, we have $\exists A>0$, $\exists H>0$, $\forall h>0$, $\exists c>0$, such that

$$\begin{aligned} |\partial^\alpha(T*\phi_\varepsilon)(x)| &\leq c\sum_{\gamma\in\mathbb{Z}_+^n} \frac{h^{|\gamma|}}{M_{|\gamma|}N_{|\gamma|}} \frac{1}{\varepsilon^{|\gamma+\alpha|}} \int |f_\gamma(x+\varepsilon y)| \left|D^{\gamma+\alpha}\phi(y)\right| dy \\ &\leq c\sum_{\gamma\in\mathbb{Z}_+^n} \frac{h^{|\gamma|}b^{|\gamma+\alpha|}N_{|\gamma+\alpha|}}{M_{|\gamma|}N_{|\gamma|}} \frac{1}{\varepsilon^{|\gamma+\alpha|}} \int |f_\gamma(x+\varepsilon y)| \frac{|D^{\gamma+\alpha}\phi(y)|}{b^{|\gamma+\alpha|}N_{|\gamma+\alpha|}} dy \\ &\leq cA\sum_{\gamma\in\mathbb{Z}_+^n} C\frac{h^{|\gamma|}b^{|\gamma+\alpha|}H^{|\gamma+\alpha|}M_{|\alpha|}N_{|\alpha|}}{M_{|\gamma+\alpha|}} \frac{1}{\varepsilon^{|\gamma+\alpha|}} c\left\|\phi\right\|_{b,N} \end{aligned}$$

then,

$$\begin{aligned} \frac{(2h)^{|\alpha|}}{M_{|\alpha|}N_{|\alpha|}} \left|\partial^{\alpha}(T*\phi_{\varepsilon})(x)\right| &\leq cCA\left\|\phi\right\|_{b,N} \sum_{\gamma\in\mathbb{Z}_{+}^{n}} 2^{-|\gamma|}\frac{(2hHb)^{|\gamma+\alpha|}}{M_{|\gamma+\alpha|}}\frac{1}{\varepsilon^{|\gamma+\alpha|}} \\ &\leq c\exp\left(M\left(\tfrac{k}{\varepsilon}\right)\right) \end{aligned}$$

i.e.

$$\left|\partial^{\alpha}(T*\phi_{\varepsilon})(x)\right| \leq c(\alpha)\exp\left(M\left(\frac{k}{\varepsilon}\right)\right) \tag{3.11}$$

Where $k = 2hHb$.

Suppose that $(T*\phi_{\varepsilon})_{\varepsilon} \in \mathcal{N}^{M}(\Omega)$, then for every compact L of Ω, $\exists c > 0$, $\forall k > 0$, $\exists \varepsilon_0 \in I$,

$$|T*\phi_{\varepsilon}(x)| \leq c\exp\left(-M(\frac{k}{\varepsilon})\right), \forall x \in L, \forall \varepsilon \leq \varepsilon_0 \tag{3.12}$$

Let $\chi \in \mathcal{D}^{MN}(\Omega)$ and $\chi = 1$ in a neighborhood of K, then $\forall \psi \in E^{MN}(\Omega)$,

$$\langle T,\psi\rangle = \langle T,\chi\psi\rangle = \lim_{\varepsilon\to 0}\int (T*\phi_{\varepsilon})(x)\chi(x)\psi(x)dx$$

Consequently, from (3.12), we obtain

$$\left|\int (T*\phi_{\varepsilon})(x)\chi(x)\psi(x)dx\right| \leq c\exp\left(-M\left(\frac{k}{\varepsilon}\right)\right), \forall \varepsilon \leq \varepsilon_{0},$$

Which gives $\langle T,\psi\rangle = 0$

□

Notation 3.3.3 *If $M = (M_p)_{p\in\mathbb{Z}_+}$ and $N = (N_p)_{p\in\mathbb{Z}_+}$ are two sequences, then*

$MN^{-1} := (M_p N_p^{-1})_{p\in\mathbb{Z}_+}$

In order to show the commutativity of the following diagram of embeddings

$$\begin{array}{ccc} \mathcal{D}^{MN^{-1}p!}(\Omega) & \rightarrow & \mathcal{G}^{M}(\Omega) \\ & \searrow & \uparrow \\ & & E'_{MN}(\Omega) \end{array}$$

We have the following fundamental result

Proposition 3.3.4 *Let* $f \in \mathcal{D}^{MN^{-1}p!}(\Omega)$ *and* $\phi \in \Sigma^{N}$*, then*

$$(f - (f * \phi_{\varepsilon})_{/\Omega})_{\varepsilon} \in \mathcal{N}^{M}(\Omega).$$

Proof. Let $f \in \mathcal{D}^{MN^{-1}p!}(\Omega)$, then there exists a constant $c > 0$, such that

$$|\partial^{\alpha} f(x)| \leq c^{|\alpha|+1} \frac{M_{|\alpha|}}{N_{|\alpha|}} \alpha!, \forall \alpha \in \mathbb{Z}_{+}^{n}, \forall x \in \Omega.$$

Let $\alpha \in \mathbb{Z}_{+}^{n}$, the Taylor's formula and the properties of ϕ_{ε} give

$$\partial^{\alpha}(f * \phi_{\varepsilon} - f)(x) = \sum_{|\beta|=N} \int \frac{(\varepsilon y)^{\beta}}{\beta!} \partial^{\alpha+\beta} f(\xi) \phi(y) dy,$$

Where $x \leq \xi \leq x + \varepsilon y$. Consequently, for any $b > 0$, we have

$$\begin{aligned}
|\partial^\alpha(f*\phi_\varepsilon - f)(x)| &\leq \varepsilon^N \sum_{|\beta|=N} \int \frac{|y|^\beta}{\beta!}\partial^{\alpha+\beta} f(\xi)\phi(y)dy \\
&\leq \varepsilon^N \sum_{|\beta|=N} \frac{b^{|\beta|}N_{|\beta|}M_{|\alpha+\beta|}(\alpha+\beta)!}{\beta! N_{|\alpha+\beta|}} \\
&\quad \times \int \frac{N_{|\alpha+\beta|}}{M_{|\alpha+\beta|}(\alpha+\beta)!}\left|\partial^{\alpha+\beta} f(\xi)\right| \frac{|y|^{|\beta|}}{b^{|\beta|}N_{|\beta|}} |\phi(y)|\, dy \\
&\leq A\,\|\phi\|_{b,N}\, c\frac{c^{|\alpha|}H^{|\alpha|}M_{|\alpha|}\alpha!}{N_{|\alpha|}}\varepsilon^N \sum_{|\beta|=N} b^{|\beta|}H^{|\beta|}M_{|\beta|}c^{|\beta|}
\end{aligned}$$

Let $k > 0$ and $T > 0$, then

$$|\partial^\alpha(f*\phi_\varepsilon - f)(x)| \leq c(\alpha)\varepsilon^N M_N (kT)^{-N} \sum_{|\beta|=N} (kTbHc)^{|\beta|}$$

Where $c(\alpha) = A\,\|\phi\|_{b,N}\, c\dfrac{c^{|\alpha|}H^{|\alpha|}M_{|\alpha|}\alpha!}{N_{|\alpha|}}$. Taking $kTbHc \leq \dfrac{1}{2a}$, with $a > 1$, we obtain

$$\begin{aligned}
|\partial^\alpha(f*\phi_\varepsilon - f)(x)| &\leq c(\alpha)\varepsilon^N M_N (kT)^{-N} a^{-N} \sum_{|\beta|=N} \left(\frac{1}{2}\right)^{|\beta|} \\
&\leq c(\alpha)\varepsilon^N M_N (kT)^{-N} a^{-N}.
\end{aligned}$$

Let $\varepsilon_0 \in I$ such that $\varepsilon_0 M_1 < 1$ and take $T \geq \dfrac{M_{p-1}M_{p+1}}{M_p^2}$, $\forall p \geq 1$.

Then, see [31], there exists $N = N(\varepsilon) \in \mathbb{Z}_+$, such that

$$1 \leq \frac{\varepsilon}{k}(M_N)^N \leq T$$

Which gives

$$a^{-N} \leq \exp\left(-M\left(\frac{k}{\varepsilon}\right)\right) \;\; \textit{and} \;\; \varepsilon^N M_N (kT)^{-N} < 1$$

if we take $a \geq 2$. Finally we have

$$|\partial^\alpha(f * \phi_\varepsilon - f)(x)| \leq c \exp\left(-M\left(\frac{k}{\varepsilon}\right)\right)$$

i.e $(f * \phi_\varepsilon - f)_\varepsilon \in \mathcal{N}^M(\Omega)$ □

As in [24] and [2], we embed $\mathcal{D}'_{MN}(\Omega)$ into $\mathcal{G}^M(\Omega)$ using the sheaf properties, then we have the following commutative diagram

$$\begin{array}{ccc} E^{\frac{M}{N}p!}(\Omega) & \longrightarrow & G^M(\Omega) \\ \downarrow & \nearrow & \\ D'_{MN}(\Omega) & & \end{array}$$

3.4 Regular generalized Roumieu ultradistributions

Definition 3.4.1 *The space of N-ultraregular moderate elements of class M, denoted $\mathcal{E}_m^{M,N,+\infty}(\Omega)$, is the space of $(f_\varepsilon)_\varepsilon \in C^\infty(\Omega)$ satisfying: $\forall K \Subset \Omega$, $\exists k > 0$, $\exists c > 0$, $\exists \varepsilon_0 \in]0,1]$, $\forall \alpha \in \mathbb{Z}_+^n$*

$$\sup_{x \in K} |\partial^\alpha f_\varepsilon(x)| \leq c^{|\alpha|+1} N_{|\alpha|} \exp(M(\frac{k}{\varepsilon}))$$

The space of null elements is defined as $\mathcal{N}^{M,N,+\infty}(\Omega) := \mathcal{N}^M(\Omega) \cap \mathcal{E}_m^{M,N,+\infty}(\Omega)$.

The main properties of this two spaces are given in the following proposition.

Proposition 3.4.2 *1) The space $\mathcal{E}_m^{M,N,+\infty}(\Omega)$ is an algebra stable by the action of N-ultradifferential operators.*

2) The space $\mathcal{N}^{M,N,+\infty}(\Omega)$ is an ideal of $\mathcal{E}_m^{M,N,+\infty}(\Omega)$.

Proof.

1) Let $(f_\varepsilon)_\varepsilon, (g_\varepsilon)_\varepsilon \in \mathcal{E}_m^{M,N,+\infty}(\Omega)$ and K be a compact of Ω, then

$\exists k_1 > 0,\ \exists c_1 > 0,\ \exists \varepsilon_1 \in]0,1],\ \forall \alpha \in \mathbb{Z}_+^n,\ \forall \varepsilon \leq \varepsilon_1 :$

$$\sup_{x\in K} |\partial^\alpha f_\varepsilon(x)| \leq c_1^{|\alpha|+1} N_{|\alpha|} \exp(M(\frac{k_1}{\varepsilon}))$$

We have also $\exists k_2 > 0,\ \exists c_2 > 0,\ \exists \varepsilon_2 \in]0,1],\ \forall \alpha \in \mathbb{Z}_+^n,\ \forall \varepsilon \leq \varepsilon_2$

$$\sup_{x\in K} |\partial^\alpha g_\varepsilon(x)| \leq c_2^{|\alpha|+1} N_{|\alpha|} \exp(M(\frac{k_2}{\varepsilon}))$$

let $\alpha \in \mathbb{Z}_+^n,\ \lambda_1, \lambda_2 \in \mathbb{Z}_+^n$, it's clear that $\exists c = max(c_1, c_2),\ \exists k = (\lambda_1 + \lambda_2) max(k_1, k_2)$, $\exists \varepsilon_0 = min(\varepsilon_1, \varepsilon_2)$ such that: $\forall \varepsilon \leq \varepsilon_0$

$$|\partial^\alpha(\lambda_1 f_\varepsilon(x) + \lambda_2 g_\varepsilon(x))| \leq c^{|\alpha|+1} N_{|\alpha|} \exp(M(\frac{k}{\varepsilon}))$$

So $(\lambda_1 f_1 + \lambda_2 f_2) \in \mathcal{E}_m^{M,N,+\infty}(\Omega)$.

And we have

$$\begin{aligned} |\partial^\alpha(f_\varepsilon g_\varepsilon)(x)| &\leq \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} \left|\partial^{\alpha-\beta} f_\varepsilon(x)\right| . \left|\partial^\beta g_\varepsilon(x)\right| \\ &\leq \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} c_1^{|\alpha-\beta|+1} . c_2^{|\beta|+1} . N_{|\alpha-\beta|} . N_{|\beta|} \exp(M(\tfrac{k_1}{\varepsilon}) + M(\tfrac{k_2}{\varepsilon})) \end{aligned}$$

then $\exists A > 0,\ \exists H > 0, \forall t > 0$

$$2M(t) \leq M(Ht) + ln(A).$$

$t = \frac{1}{\varepsilon} max(k_1, k_2) = \frac{k}{\varepsilon}$, $C = max(c_1, c_2)$.

$$\begin{aligned}
|\partial^\alpha(f_\varepsilon.g_\varepsilon)(x)| &\leq \sum_{\beta=0}^{\alpha} \tbinom{\alpha}{\beta}.A.C^{|\alpha|+1} N_{|\alpha|}.\exp(M(\tfrac{Hk}{\varepsilon})) \\
&\leq C^{|\alpha|+1}.N_{|\alpha|}.\exp(M(\tfrac{k}{\varepsilon}))
\end{aligned}$$

Then $(f_\varepsilon.g_\varepsilon)_\varepsilon \in \mathcal{E}_m^{M,N,\infty}(\Omega)$.

Let now $P(D) = \Sigma a_\gamma D^\gamma$ be an N-ultradifferential operator, then $\forall h > 0$, $\exists b > 0$, such that

$$\begin{aligned}
\exp(-M(\tfrac{k_1}{\varepsilon}))\frac{1}{N_{|\alpha|}} |\partial^\alpha(P(D)f_\varepsilon(x))| &\leq \exp(-M(\tfrac{k_1}{\varepsilon})) \sum_{\gamma\in\mathbb{Z}_+^n} b \frac{h^{|\gamma|}}{N_{|\gamma|}.N_{|\alpha|}} |\partial^{\alpha+\gamma} f_\varepsilon(x)| \\
&\leq b \exp(-M(\tfrac{k_1}{\varepsilon})) \sum_{\gamma\in\mathbb{Z}_+^n} \frac{A(H)^{|\alpha+\gamma|} h^{|\gamma|}}{N_{|\alpha+\gamma|}} |\partial^{\alpha+\gamma} f_\varepsilon(x)| \\
&\leq b \sum_{\gamma\in\mathbb{Z}_+^n} A(H)^{|\alpha+\gamma|} h^{|\gamma|}
\end{aligned}$$

hence for $Hh < \frac{1}{2}$ we have

$$\exp(-M(\frac{k_1}{\varepsilon}))\frac{1}{N_{|\alpha|}} |\partial^\alpha(P(D)f_\varepsilon(x))| \leq c' H^{|\alpha|}$$

which shows that $(P(D)f_\varepsilon)_\varepsilon \in \mathcal{E}_m^{M,N,\infty}(\Omega)$.

2) The fact that $\mathcal{N}^{M,N,\infty}(\Omega) = \mathcal{N}^M(\Omega) \cap \mathcal{E}_m^{M,N,\infty}(\Omega) \subset \mathcal{E}_m^{M,N,\infty}(\Omega)$.

And $\mathcal{N}^M(\Omega)$ is an ideal of $\mathcal{E}_m^M(\Omega)$, then $\mathcal{N}^{M,N,\infty}$ is an ideal of $\mathcal{E}_m^{M,N,\infty}(\Omega)$

□

Definition 3.4.3 *The algebra of $N-$ultraregular generalized functions of class $M = (M_p)_{p\in\mathbb{Z}_+}$,* *denoted* $\mathcal{G}_M^{N,\infty}(\Omega)$, *is the quotient algebra*

$$\mathcal{G}_N^{M,\infty}(\Omega) = \frac{\mathcal{E}_m^{M,N,\infty}(\Omega)}{\mathcal{N}^{M,N,\infty}(\Omega)}$$

The basic properties of $\mathcal{G}_N^{M,\infty}(\Omega)$ are given by the following result

Proposition 3.4.4 *The space* $\mathcal{G}_N^{M,\infty}(\Omega)$ *is a sheaf subalgebras of* $\mathcal{G}^M(\Omega)$.

This motivates the following definition

Definition 3.4.5 *We define the* $\mathcal{G}_N^{M,\infty}-$*singular support of a generalized ultradistribution* $f \in \mathcal{G}^M(\Omega)$, *denoted by* $N - singsupp_g(f)$ *as the complement of the largest open set* Ω' *such that* $f \in \mathcal{G}_N^{M,\infty}(\Omega')$

The following result is Paley-Wiener type characterization of $\mathcal{G}_N^{M,\infty}(\Omega)$.

Proposition 3.4.6 *Let* $f = cl(f_\varepsilon)_\varepsilon \in \mathcal{G}_c^M(\Omega)$, *then* f *is* $N-$*ultraregular if and only if* $\exists k_1 > 0,\ \exists k_2 > 0,\ \exists c > 0,\ \exists \varepsilon_1 > 0,\ \forall \varepsilon \le \varepsilon_1$, *such that*

$$|\mathcal{F}(f_\varepsilon)(\xi)| \le c \exp(M(\frac{k_1}{\varepsilon}) - N(k_2 |\xi|)), \qquad \forall \xi \in \mathbb{R}^n. \tag{3.13}$$

Proof. Suppose that $f = cl(f_\varepsilon) \in \mathcal{G}_c^M(\Omega) \cap \mathcal{G}_N^{M,\infty}(\Omega)$ then $\exists k > 0,\ \exists c > 0,\ \exists \varepsilon_1 > 0,$ $\forall \varepsilon \le \varepsilon_1,\ \forall \alpha \in \mathbb{Z}_+^n,$

$$|\partial^\alpha f_\varepsilon(x)| \le c^{|\alpha|+1}.N_{|\alpha|}.\exp(M(\frac{k}{\varepsilon}))$$

Consequently we have, $\forall \xi \in \mathbb{R}^n\ \forall \alpha \in \mathbb{Z}_+^n$,

$$|\xi^\alpha| \,.\, |\mathcal{F}(f_\varepsilon)(\xi)| \le \left| \int_K \exp(-ix\xi)\partial^\alpha f_\varepsilon(x)dx \right|$$

Then

$$|\xi^\alpha| . |\mathcal{F}(f_\varepsilon)(\xi)| \leq mes(K)c^{|\alpha|+1}.N_{|\alpha|}. \exp(M(\frac{k}{\varepsilon}))$$

$$\begin{aligned}
|\mathcal{F}(f_\varepsilon)(\xi)| &\leq c^{|\alpha|+1}.mes(K).\frac{N_{|\alpha|}}{|\xi|^{|\alpha|}}. \exp(M(\tfrac{k}{\varepsilon})) \\
&\leq c.mes(K).\inf_\alpha(\frac{c^{|\alpha|}N_{|\alpha|}}{|\xi|^{|\alpha|}}). \exp(M(\tfrac{k}{\varepsilon})) \\
&\leq c.mes(K).\frac{1}{\sup_\alpha(\frac{|\xi|^{|\alpha|}}{c^{|\alpha|}N_{|\alpha|}})}. \exp(M(\tfrac{k}{\varepsilon})) \\
&\leq c.mes(K)\frac{1}{\exp(\ln(\sup_\alpha(\frac{|\xi|^{|\alpha|}}{c^{|\alpha|}N_{|\alpha|}})))}. \exp(M(\tfrac{k}{\varepsilon}))
\end{aligned}$$

take $k_2 = \frac{1}{c}$, $C = cmes(K)$, $\forall\varepsilon \leq \varepsilon_0$

$$|\mathcal{F}(f_\varepsilon)(\xi)| \leq C \exp(-N(k_2 |\xi|)). \exp(M(\frac{k_1}{\varepsilon}))$$

So we have (3.13).

Suppose now that (3.13) is valid. Then $\forall\varepsilon \leq \varepsilon_0$:

$$\begin{aligned}
|\partial^\alpha f_\varepsilon(x)| &\leq c \left|\int_{\mathbb{R}^n} \exp(ix\xi)\xi^\alpha \mathcal{F}(f_\varepsilon)(\xi)d\xi\right| \\
&\leq c \exp(M(\tfrac{k_1}{\varepsilon})) \int_{\mathbb{R}^n} |\xi^\alpha| . \exp(-N(k_2 |\xi|))dx \\
&\leq c \exp(M(\tfrac{k_1}{\varepsilon}))\sup_{|\xi|}(|\xi^\alpha| \exp(-N(k_2 |\xi|))) \\
&\leq C^{|\alpha|+1}.N_{|\alpha|}. \exp(M(\tfrac{k_1}{\varepsilon}))
\end{aligned}$$

With $C = max(c, \frac{1}{k_2})$. i.e: $f_\varepsilon \in \mathcal{G}_N^{M,\infty}(\Omega)$. □

Remark 3.4.7 *Let* $f = cl(f_\varepsilon) \in \mathcal{G}_c^M(\Omega)$, *then* $\exists k_1 > 0$, $\exists c > 0$, $\exists \varepsilon_0 > 0$, $\forall k_2 > 0$, $\forall \varepsilon \leq \varepsilon_0$,

$$|\mathcal{F}(f_\varepsilon)(\xi)| \leq c\exp(M(\frac{k_1}{\varepsilon}) + N(k_2|\xi|)), \qquad \forall \xi \in \mathbb{R}^n. \tag{3.14}$$

The algebra $\mathcal{G}_N^{M,\infty}(\Omega)$ plays the same role as the Oberguggenberger subalgebra of regular elements $\mathcal{G}^\infty(\Omega)$ in the Colombeau algebra $\mathcal{G}(\Omega)$.

Theorem 3.4.8 *We have*

$$\mathcal{G}^{M,\infty}_{MN^{-1}p!}(\Omega) \cap \mathcal{D}'_{MN}(\Omega) = E^{MN^{-1}p!}(\Omega).$$

Proof. Let $S \in \mathcal{G}^{M,\infty}_{MN^{-1}p!}(\Omega) \cap \mathcal{D}'_{MN}(\Omega)$. For any fixed $x_0 \in \Omega$, we take $\psi \in \mathcal{D}^{MN}(\Omega)$, with $\psi \equiv 1$ on a neighborhood U of x_0. Then: $T = \psi S \in E'_{MN}(\Omega)$. Let ϕ_ε be a net mollifiers with $\check{\phi} = \phi$ and let $\chi \equiv 1$ on $K = supp\psi$. and $\chi \in \mathcal{D}^{MN^{-1}p!}(\Omega)$.

As $[T] \in \mathcal{G}^{M,\infty}_{MN^{-1}p!}(\Omega)$, $\exists k_1 > 0$, $\exists k_2 > 0$, $\exists c_1 > 0$, $\exists \varepsilon_1 > 0$, $\forall \varepsilon \leq \varepsilon_1$

$$|\mathcal{F}(\chi(T * \phi_\varepsilon))(\xi)| \leq c_1 \exp(M(\frac{k_1}{\varepsilon}) - MN^{-1}p!(k_2|\xi|))$$

$$\begin{aligned} |\mathcal{F}(\chi(T * \phi_\varepsilon))(\xi) - \mathcal{F}(T)(\xi)| &= |\mathcal{F}(\chi(T * \phi_\varepsilon))(\xi) - \mathcal{F}(\chi T)(\xi)| \\ &= \left|\left\langle T(x), (\chi(x)e^{-i\xi x}) * \phi_\varepsilon(x) - (\chi(x)e^{-i\xi x})\right\rangle\right| \end{aligned}$$

As $E'_{MN}(\Omega) \subset E'_{MN^{-1}p!}(\Omega)$, then $\exists L \Subset \Omega$ such that $\forall h > 0, \exists c > 0$ and

$$|\mathcal{F}(\chi(T * \phi_\varepsilon))(\xi) - \mathcal{F}(T)(\xi)| \leq c \sup_{\alpha \in \mathbb{Z}_+^n,\, x \in L} \frac{h^{|\alpha|}}{\frac{M_{|\alpha|}}{N_{|\alpha|}}|\alpha|!} \left|\partial_x^\alpha (\chi(x)e^{-ix\xi} * \phi_\varepsilon(x) - \chi(x)e^{-i\xi x})\right|$$

We have $e^{-i\xi}\chi \in \mathcal{D}^{MN^{-1}p!}(\Omega)$, we obtain $\forall k_3 > 0,\ \exists c_2 > 0,\ \exists \eta > 0, \forall \varepsilon \leq \eta$,

$$\underset{\alpha\in\mathbb{Z}_+^n,\, x\in L}{sup} \frac{h^{|\alpha|}}{\frac{M_{|\alpha|}}{N_{|\alpha|}}|\alpha|!} \left|\partial_x^\alpha(\chi(x)e^{-ix\xi} * \phi_\varepsilon(x) - \chi(x)e^{-i\xi x})\right| \leq c_2 \exp(-M(\frac{k_3}{\varepsilon}))$$

So there exists $c' = c'(k_3) > 0$, such that

$$|\mathcal{F}(\chi(T * \phi_\varepsilon))(\xi) - \mathcal{F}(T)(\xi)| \leq c'. \exp(-M(\frac{k_3}{\varepsilon}))$$

Let $\varepsilon \leq min(\eta, \varepsilon_1)$, then

$$\begin{aligned} |\mathcal{F}(T)(\xi)| &\leq |\mathcal{F}(T)(\xi) - \mathcal{F}(\chi(T * \phi_\varepsilon))| + |\mathcal{F}(\chi(T * \phi_\varepsilon))| \\ &\leq c'. \exp(-M(\tfrac{k_3}{\varepsilon})) + c_1 \exp(M(\tfrac{k_1}{\varepsilon}) - MN^{-1}p!(k2\,|\xi|)) \end{aligned}$$

Take $c = max(c_1, c')$, $\varepsilon = \dfrac{k_1 p!^{\frac{1}{p}}}{(k2 - r)\,|\xi|\,N_p^{\frac{1}{p}}}$, $r \in]0, k_2[$ and $k_3 = \dfrac{k_1 r}{k_2 - r}$, then $\exists \delta > 0,\ \exists c > 0$ such that

$$|\mathcal{F}(T)(\xi)| \leq c \exp(-MN^{-1}p!(\delta\,|\xi|))$$

Which means $T = \psi S \in E^{MN^{-1}p!}(\Omega)$. As $\psi \equiv 1$ on the neighborhood U of x_0, consequently $S \in E^{MN^{-1}p!}(\Omega)$. Which proves

$$\mathcal{G}^{M,\infty}_{MN^{-1}p!}(\Omega) \cap \mathcal{D}'_{MN}(\Omega) \subset E^{MN^{-1}p!}(\Omega).$$

We have $E^{MN^{-1}p!}(\Omega) \subset E^{MN}(\Omega) \subset \mathcal{D}'_{MN}(\Omega)$, $E^{MN^{-1}p!}(\Omega) \subset \mathcal{G}^{M,\infty}_{MN^{-1}p!}(\Omega)$ then $E^{MN^{-1}p!}(\Omega) \subset \mathcal{G}^{M,\infty}_{MN^{-1}p!}(\Omega) \cap \mathcal{D}'_{MN}(\Omega)$. Consequently we have

$$\mathcal{G}^{M,\infty}_{MN^{-1}p!}(\Omega) \cap \mathcal{D}'_{MN}(\Omega) = E^{MN^{-1}p!}(\Omega).$$

□

3.5 Generalized Roumieu wave front

The aim of this section is to introduce the generalized Roumieu wave front of generalized Roumieu ultradistribution and to give its main properties.

Definition 3.5.1 *We define* $\sum_g^{M,N}(f) \subset \mathbb{R}^n \backslash \{0\}$, $f \in \mathcal{G}_c^M(\Omega)$, *as the complement of the set of points having a conic neighborhood* Γ *such that* $\exists k_1 > 0$, $\exists k_2 > 0$, $\exists c > 0$, $\exists \varepsilon_0 \in I$, $\forall \xi \in \Gamma$, $\forall \xi \in \Gamma$, $\forall \varepsilon \leq \varepsilon_0$,

$$|\mathcal{F}(f_\varepsilon)(\xi)| \leq c \exp(M(\frac{k_1}{\varepsilon}) - N(k_2 |\xi|))$$

The following essential properties of $\sum_g^{M,N}(f)$ are sufficient to define later the generalized Roumieu wave front of generalized Roumieu ultradistribution

Proposition 3.5.2 *For every* $f \in \mathcal{G}_c^M(\Omega)$ *we have*

1. *The Set* $\sum_g^{M,N}(f)$ *is closed cone.*

2. $\sum_g^{M,N}(f) = \emptyset \Longleftrightarrow f \in \mathcal{G}^{M,N,\infty}$.

3. $\sum_g^{M,N}(\psi f) \subset \sum_g^{M,N}(f)$, $\forall \psi \in E^N(\Omega)$.

Proof. One can easily, from definition (3.5.1) and proposition (3.5.2), prove the assertion 1 and 2.

Let suppose that $\xi_0 \notin \sum_g^{M,N}(f)$, then $\exists\Gamma$ a conic neighborhood of ξ_0, $\exists k_1 > 0$, $\exists k_2 > 0$, $\exists c_1 > 0$, $\exists \varepsilon_1 > 0$, $\forall \xi \in \Gamma$, $\forall \varepsilon \in \varepsilon_1$,

$$|\mathcal{F}(f_\varepsilon)(\xi)| \leq c.\exp(M(\frac{k_1}{\varepsilon}) - N(k_2\,|\xi|))$$

Let $\chi \in \mathcal{D}^N(\Omega)$, $\chi \equiv 1$ on neighborhood of $supp(f)$, so $\chi\psi \in \mathcal{D}^N(\Omega)$, $\forall \psi \in E^N(\Omega)$ hence from [27] $\exists k_3 > 0$, $\exists c_2 > 0$, $\forall \xi \in \mathbb{R}^n$,

$$|\mathcal{F}(\chi\psi)(\xi)| \leq c.\exp(-N(k_3\,|\xi|))$$

Let Λ be a conic neighborhood of ξ_0 such that $\bar{\Lambda} \subset \Gamma$ we have for a fixed $\xi \in \Lambda$,

$$\begin{aligned} \mathcal{F}(\psi f_\varepsilon)(\xi) &= \mathcal{F}(\chi\psi f_\varepsilon)(\xi) \\ &= \textstyle\int_A \mathcal{F}(f_\varepsilon)(\eta).\mathcal{F}(\chi\psi)(\eta-\xi)d\eta + \int_B \mathcal{F}(f_\varepsilon)(\eta).\mathcal{F}(\chi\psi)(\eta-\xi)d\eta \end{aligned}$$

Where: $A = \{\eta : |\xi - \eta| \leq \delta(|\xi| + |\eta|)\}$ and $B = \{\eta : |\xi - \eta| > \delta(|\xi| + |\eta|)\}$ Take δ sufficient small such that: $\frac{|\xi|}{2} < |\eta| < 2\,|\xi|\,$, $\forall \eta \in A$, then $\exists c > 0$, $\forall \varepsilon \leq \varepsilon_1$

$$\left|\int_A \mathcal{F}(f_\varepsilon)(\eta)\mathcal{F}(\chi\psi)(\eta-\xi)d\eta\right| \leq c_1.c_2.exp(M(\frac{k_1}{\varepsilon}) - N(k_2\frac{|\xi|}{2})) \times \int_A \exp(-N(k_3\,|\eta-\xi|))d\eta$$

then $\exists c>0$, $\exists k_2'>0$

$$\left|\int_A \mathcal{F}(f_\varepsilon)(\eta)\mathcal{F}(\chi\psi)(\eta-\xi)d\eta\right| \leq c\exp(M(\frac{k_1}{\varepsilon})-N(k_2'\,|\xi|)) \tag{3.15}$$

As $\mathcal{G}_c^M(\Omega)$, from remark (3.14), $\exists c_2>0$, $\exists \mu_1>0$, $\exists \varepsilon_2>0$, $\forall \mu_2>0$, $\forall \xi\in\mathbb{R}^n$, $\forall \varepsilon\leq\varepsilon_2$, such that

$$|\mathcal{F}(f_\varepsilon)(\xi)| \leq c\exp(M(\frac{\mu_1}{\varepsilon})+N(\mu_2\,|\xi|))$$

Hence, for $\varepsilon\leq min(\varepsilon_1,\varepsilon_2)$, we have

$$\begin{aligned}\left|\int_B \mathcal{F}(f_\varepsilon)(\eta).\mathcal{F}(\chi\psi)(\eta-\xi)d\eta\right| &\leq c_2.c_3.\exp(M(\tfrac{\mu_1}{\varepsilon}))\left|\int_B \exp(N(\mu_2\,|\eta|)-N(k_3\,|\eta-\xi|))d\eta\right| \\ &\leq c.\exp(M(\tfrac{\mu_1}{\varepsilon}))\left|\int_B \exp(N(\mu_2\,|\eta|)-N(k_3\delta(|\xi|+|\eta|)d\eta\right|\end{aligned}$$

Then takin: $\mu_2<k_3\delta$, we obtain

$$\left|\int_B \mathcal{F}(f_\varepsilon)(\eta).\mathcal{F}(\chi\psi)(\eta-\xi)d\eta\right| \leq c\exp(M(\frac{\mu_1}{\varepsilon})-N(k_3\delta\,|\xi|)) \tag{3.16}$$

Consequently, (3.15) and (3.16) give $\xi_0\notin\sum_g^{M,N}(\psi f)$. □

Definition 3.5.3 *Let $f\in\mathcal{G}^M(\Omega)$ and $x_0\in\Omega$, the cone of $N-$singular directions of f at x_0, denoted $\sum_{g,x_0}^{M,N}(f)$, is*

$$\Sigma_{g,x_0}^{M,N}(f)=\bigcap\{\Sigma_g^{M,N}(\varphi f):\varphi\in\mathcal{D}^M(\Omega)\ \textit{and}\ \varphi\equiv 1\ \textbf{\textit{on a neighborood of}}\ x_0\}$$

Lemma 3.5.4 *Let $f \in \mathcal{G}^M(\Omega)$, then*

$$\Sigma_{g,x_0}^{M,N}(f) = \emptyset \Leftrightarrow x_0 \notin N - singsupp_g(f)$$

Proof. Let $x_0 \notin N - singsupp_g(f)$, i.e. $\exists U \subset \Omega$ an open neighborhood of x_0 such that $f \in \mathcal{G}_N^{M,\infty}(U)$, let $\phi \in \mathcal{D}^M(U)$ such that $\phi \equiv 1$ on a neighborhood of x_0, then $\phi f \in \mathcal{G}_N^{M,\infty}(\Omega)$. Hence, from the proposition (3.5.2), $\sum_g^{M,N}(\phi f) = \emptyset$, i.e. $\sum_{g,x_0}^{M,N}(f) = \emptyset$.

Suppose now $\sum_{g,x_0}^{M,N}(f) = \emptyset$, $\forall \xi \in \mathbb{R}^n \backslash \{0\}$, $\exists V_\xi \in \mathcal{V}(x_0)$, $\exists w_\xi \in \xi$ a conic neighborhood of ξ. $\exists k_1 > 0$, $\exists k_2 > 0$, $\exists c > 0$, $\exists \varepsilon_0 > 0$, $\forall \xi \in W_\xi$, $\forall \varepsilon \leq \varepsilon_0$, $\forall \phi_\xi \in \mathcal{D}^M(\Omega)$.

$$\left|\mathcal{F}(\phi_\xi f_\varepsilon)(\xi)\right| \leq c.\exp(M(\frac{k_1}{\varepsilon}) - N(k_2 |\xi|))$$

Since the unit sphere $|\xi| = 1$ is a compact set, then one can find finite points $\xi_j, j = 1, ..., n$ in $\mathbb{R}^n$, $W_j \in \xi_|$ and $\phi_j \in \mathcal{D}^M(\Omega)$, $\phi_j(x) = 1$ in V_j, $k_1 > 0$, $\exists k_2 > 0$, $\exists c > 0$, $\varepsilon_0 > 0$, $\forall \varepsilon \leq \varepsilon_0$

$$\left|\mathcal{F}(\phi_j f_\varepsilon)(\xi)\right| \leq c.\exp(M(\frac{k_1}{\varepsilon}) - N(k_2 |\xi|)), \qquad \xi \in W_j$$

Taking $V = \bigcap_j V_j$ and $W = \bigcup_j W_j$, $\varphi = \phi_1 ... \phi_n$, we have $\varphi \in \mathcal{D}^M(\Omega)$ and $\varphi(x) = 1$ on V.

$$|\mathcal{F}(\varphi f_\varepsilon)(\xi)| \leq c.\exp(M(\frac{k_1}{\varepsilon}) - N(k_2 |\xi|)), \qquad \xi \in W$$

Consequently, $(\varphi f_\varepsilon) \notin \mathcal{G}_{N,c}^{M,\infty}$ where: $x_0 \in N - singsupp_g(f)$ □

Definition 3.5.5 *A point $(x_0, \xi_0) \notin WF_g^{M,N}(f) \subset \Omega \times \mathbb{R}^n \backslash \{0\}$ If $\xi_0 \notin \sum_{g,x_0}^{M,N}(f)$, i.e: there exists*

$\phi \in \mathcal{D}^M(\Omega), \phi(x) = 1$ *neighborhood of* x_0, *and conic neighborhood* Γ *of* ξ_0, $\exists k_1 > 0$, $\exists k_2 > 0$, $\exists c > 0$, $\exists \varepsilon_0 > 0$ *such that:* $\forall \xi \in \Gamma$, $\forall \varepsilon \le \varepsilon_0$,

$$|\mathcal{F}(\phi f_\varepsilon)(\xi)| \le c \exp(M(\frac{k_1}{\varepsilon}) - N(k_2 |\xi|))$$

The main proprieties of the generalized Roumieu wave front $WF_g^{M,N}$ are subsumed in the following proposition:

Proposition 3.5.6 *Let* $f \in \mathcal{G}^M(\Omega)$, *then*

(1) The projection of $WF_g^{M,N}(f)$ *on* Ω *is* $N - sinsupp_g(f)$.

(2) If $f \in \mathcal{G}_c^M(\Omega)$, *The projection of* $WF_g^{M,N}(f)$ *on* $\mathbb{R}^n \backslash \{0\}$ *is* $\sum_g^{M,N}(f)$.

(3) $\forall \alpha \in \mathbb{Z}_+^n$, $WF_g^{M,N}(\partial^\alpha f) \subset WF_g^{M,N}(f)$.

(4) $\forall g \in \mathcal{G}_N^{M,\infty}(\Omega)$, $WF_g^{M,N}(gf) \subset WF_g^{M,N}(f)$.

Proof. (1) and (2) hold from the definition, Proposition (3.5.2) and lemma (3.5.4).

(3) Let $(x_0, \xi_0) \notin WF_g^{M,N}(f)$, then $\exists \phi \in \mathcal{D}^M(\Omega)$, $\phi \equiv 1$ on a neighborhood $\bar{U}$ of x_0, there exist a conic neighborhood Γ of ξ_0, $\exists k_1 > 0$, $\exists k_2 > 0$, $\exists c_1 > 0$, $\exists \varepsilon_0 \in]0, 1]$,such that $\forall \xi \in \Gamma$, $\varepsilon \le \varepsilon_0$,

$$|\mathcal{F}(\phi f_\varepsilon)(\xi)| \le c_1 exp(M(\frac{k_1}{\varepsilon}) - N(k_2 |\xi|)) \tag{3.17}$$

We have for $\psi \in \mathcal{D}^M(U)$ such that $\psi(x_0) = 1$.

$$\begin{aligned} |\mathcal{F}(\psi \partial f_\varepsilon)(\xi)| &= |\mathcal{F}(\partial(\psi f_\varepsilon))(\xi) - \mathcal{F}(\partial \psi) f_\varepsilon(\xi)| \\ &\le |\xi| \, |\mathcal{F}(\psi \phi f_\varepsilon)(\xi)| + |\mathcal{F}((\partial \psi) \phi f_\varepsilon)(\xi)| \end{aligned}$$

As $WF_g^{M,N}(\psi f) \subset WF_g^{M,N}(f)$, (3.17) holds for both $|\mathcal{F}(\psi\phi f_\varepsilon)(\xi)|$ and $|\mathcal{F}((\partial\psi)\phi f_\varepsilon)(\xi)|$.

So

$$\begin{aligned} |\xi|\,|\mathcal{F}(\psi\phi f_\varepsilon)(\xi)| &\le |\xi| \exp(M(\tfrac{k_1}{\varepsilon}) - N(k_2\,|\xi|)) \\ &\le c' \exp(M(\tfrac{k_1}{\varepsilon}) - N(k_3\,|\xi|)) \end{aligned}$$

With $c' > 0$, $k_3 > 0$, such that $|\xi| \le c' \exp(M(k_2\,|\xi|) - M(k_3\,|\xi|))$ Which proves

$(x_0, \xi_0) \notin WF_g^{M,N}(\partial f)$.

(4) Let $(x_0, \xi_0) \notin WF_g^{M,N}(f)$ then $\exists \phi \in \mathcal{D}^M(\Omega)$, $\phi \equiv 1$ on a neighborhood of x_0, $\xi_0 \notin \sum_g^{M,N}(\phi f)$

by proposition (3.5.2), for $g \in \mathcal{G}_M^{N,\infty}(\Omega)$, we have $\xi_0 \notin \sum_g^{M,N}(g\phi f)$

which proves: $(x_0, \xi_0) \notin WF_g^{M,N}(gf)$. □

Corollary 3.5.7 *Let* $P(x,D) = \sum\limits_{|\alpha| \le m} a_\alpha(x) D^\alpha$ *be a partial differential operator with* $\mathcal{G}_N^{M,\infty}(\Omega)$ *coefficient, then:* $WF_g^{M,N}(P(x,D)f) \subset WF_g^{M,N}(f)$, $\forall f \in \mathcal{G}^M(\Omega)$.

Lemma 3.5.8 *Let* $\varphi \in \mathcal{D}^M(B(0.2))$, $0 \le \varphi \le 1$, *and* $\varphi \equiv 1$ *on* $B(0,1)$ *and let* $\phi \in S^M$, *then* $\exists c > 0$, $\exists v > 0$, $\exists \varepsilon_0 > 0$, $\forall \varepsilon \in]0, \varepsilon_0]$, $\forall \xi \in \mathbb{R}^n$,

$$\left|\hat{\theta}_\varepsilon(\xi)\right| \le c\varepsilon^{-n} e^{-M(v\varepsilon|\xi|)}$$

Where: $\theta_\varepsilon(x) = (\frac{1}{\varepsilon})^n.\phi(\frac{x}{\varepsilon}).\varphi(x(|\ln\varepsilon|))$, *and* $\hat{\theta}$ *denoted the Fourier transform of* θ.

Proof. We have, for ε sufficiently small, $\varepsilon \le |\ln\varepsilon|^{-n} \le 1$

Let $\xi \in \mathbb{R}^n$, then

$$\begin{aligned} \hat{\theta}_\varepsilon(\xi) &= \frac{1}{\varepsilon^n} \int \hat{\phi}(\varepsilon(\xi-\eta)).\tfrac{1}{|\ln\varepsilon|^n}.\hat{\varphi}(\tfrac{\eta}{|\ln\varepsilon|})d\eta \\ &= |\ln\varepsilon|^{-n} \left[\int_A \hat{\phi}(\varepsilon(\xi-\eta))\hat{\varphi}(\tfrac{\eta}{|\ln\varepsilon|})d\eta + \int_B \hat{\phi}(\varepsilon(\xi-\eta))\hat{\varphi}(\tfrac{\eta}{|\ln\varepsilon|})d\eta\right] \end{aligned}$$

Where $A = \{\eta : |\xi - \eta| \leq \delta(|\xi|+|\eta|)\}$ and $B = \{\eta : |\xi - \eta| > \delta(|\xi|+|\eta|)\}$ We choose δ sufficiently small such that $\frac{|\xi|}{2} < |\eta| < 2|\xi|$, $\forall\eta \in A$.

Since $\varphi \in \mathcal{D}^M(\Omega)$, $\phi \in S^M$ then: $\exists k_1, k_2 > 0, \exists c_1, c_2 > 0$, $\forall\xi \in$ ‘$\mathbb{R}$,

$$|\hat{\varphi}(\xi)| \leq c_1 \exp(-M(k_1 |\xi|))$$

And

$$\left|\hat{\phi}(\xi)\right| \leq c_2 \exp(-M(k_2 |\xi|))$$

So

$$\begin{aligned} I_1 &= |\ln\varepsilon|^{-n} \left|\int_A \hat{\phi}(\varepsilon(\xi-\eta))\hat{\varphi}(\tfrac{\eta}{|\ln\varepsilon|})d\eta\right| \\ &\leq c_1c_2 \exp(-M(\tfrac{k_2}{2}\tfrac{|\xi|}{|\ln\varepsilon|}))\int \exp(-M(k_1\varepsilon|\xi-\eta|)d\eta \end{aligned}$$

Let $z = \varepsilon(\eta - \xi)$, then

$$\begin{aligned} I_1 &\leq c\varepsilon^{-n} \exp(-M(\tfrac{k_2}{2}|\ln\varepsilon|^{-1}|\xi|))\int \exp(-M(k_1|z|))dz \\ &\leq c\varepsilon^{-n} \exp(-M(v\varepsilon|\xi|)) \end{aligned}$$

For I_2 we have

$$\begin{aligned} I_2 &= |\ln\varepsilon|^{-n} \left|\int_B \hat{\phi}(\varepsilon(\xi-\eta))\hat{\varphi}(\tfrac{\eta}{|\ln\varepsilon|})d\eta\right| \\ &\leq c_1c_2 \int_B \exp(-M(k_1\varepsilon|\xi-\eta|) - M(k_2\tfrac{|\eta|}{|\ln\varepsilon|}))d\eta \\ &\leq c\exp(-M(k_1\delta\varepsilon|\xi|)).\int_B \exp(-M(k_1\delta\varepsilon|\eta|) - M(k_2|\ln\varepsilon|^{-1}|\eta|))d\eta \\ &\leq c\exp(-M(k_1\delta\varepsilon|\xi|)).\int_B \exp(-M(k_2'\varepsilon|\eta|))d\eta \\ &\leq c\varepsilon^{-n} \exp(-M(v\varepsilon|\xi|)) \end{aligned}$$

Consequently, $\exists c>0$, $\exists v>0$, $\exists \varepsilon_0>0$, $\forall \varepsilon \leq \varepsilon_0$ such that

$$\left|\hat{\theta}_\varepsilon(\xi)\right| \leq c\varepsilon^{-n}\exp\left(-M(v\varepsilon\left|\xi\right|)\right), \qquad \forall \xi \in \mathbb{R}^n$$

$\square$

Theorem 3.5.9 *Let $T \in \mathcal{D}'_{MN}(\Omega) \cap \mathcal{G}^M(\Omega)$ then $WF_g^{M,MN^{-1}p!}(T) = WF^{MN^{-1}p!}(T)$*

Proof. Let $S \in E'_{MN}(\Omega) \subset E'_{\frac{M}{N}p!}(\Omega)$ and $\psi \in \mathcal{D}^{\frac{M}{N}p!}(\Omega)$, we have: $|\mathcal{F}(\psi(S*\phi_\varepsilon))(\xi) - \mathcal{F}(\psi S)(\xi)| = \left|\left\langle S(x), (\psi(x)e^{-i\xi x}) * \check{\phi}_\varepsilon(x) - (\psi(x)e^{-ix\xi})\right\rangle\right|$ Then $\exists L$ a compact of Ω such that $\forall h>0$, $\exists c>0$,

$$|\mathcal{F}(\psi(S*\phi_\varepsilon))(\xi) - \mathcal{F}(\psi S)(\xi)| \leq c \sup_{\alpha\in\mathbb{Z}_+^n; x\in L} \frac{h^{|\alpha|}}{\frac{M_{|\alpha|}}{N_{|\alpha|}}\alpha!}\left|\partial_x^\alpha(\psi(x)e^{-i\xi x} * \check{\phi}_\varepsilon(x) - \psi(x)e^{-i\xi x})\right|$$

We have $e^{-i\xi}\psi \in \mathcal{D}^{\frac{M}{N}p!}(\Omega)$, then, $\exists c_2, \forall k_0>0$, $\exists \eta>0$, $\forall \varepsilon \leq \eta$,

$$\sup_{\alpha\in\mathbb{Z}_+^n; x\in L} \frac{c_2^{|\alpha|}}{\frac{M_{|\alpha|}}{N_{|\alpha|}}\alpha!}\left|\partial_x^\alpha(\psi(x)e^{-i\xi x} * \check{\phi}_\varepsilon(x) - \psi(x)e^{-i\xi x})\right| \leq c_2 e^{-M(\frac{k_0}{\varepsilon})};$$

So there exist $c'>0$, $\forall k_0>0$, $\exists \eta>0$, $\forall \varepsilon \leq \eta$, such that

$$|\mathcal{F}(\psi S)(\xi) - \mathcal{F}(\psi(S*\phi_\varepsilon)(\xi))| \leq c' e^{-M(\frac{k_0}{\varepsilon})} \tag{3.18}$$

Let $T \in \mathcal{D}'_{MN}(\Omega) \cap \mathcal{G}^M(\Omega)$ and $(x_0, y_0) \notin WF_g^{M,\frac{M}{N}p!}(T)$, Then there exist $\chi \in \mathcal{D}^{\frac{M}{N}p!}(\Omega)$, $\chi(x)=1$ in a neighborhood of x_0, and a conic neighborhood Γ of ξ_0, $\exists k_1>0$, $\exists k_2>0$, $\exists c_1>0$, $\exists \varepsilon_0 \in]0,1[$, such that: $\forall \xi \in \Gamma$, $\forall \varepsilon \leq \varepsilon_0$,

$$|\mathcal{F}(\chi(T * \theta_\varepsilon))(\xi)| \leq c_1 e^{M(\frac{k_1}{\varepsilon}) - \frac{M}{N}p!(k_2|\xi|)} \tag{3.19}$$

Let $\psi \in \mathcal{D}^{\frac{M}{N}p!}(\Omega)$ equal to 1 in neighborhood of x_0 such that for sufficiently small ε we have $\chi \equiv 1$ on $supp\psi + B(0, \frac{2}{|\ln \varepsilon|})$, and let $\varphi \in \mathcal{D}^{\frac{M}{N}p!}(B(0,2))$; $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ on $B(0,1)$, then there exist $\varepsilon_0 \leq 1$, such that $\forall \varepsilon < \varepsilon_0$,

$$\psi(T * \theta_\varepsilon)(x) = \psi(\chi T * \theta_\varepsilon)(x).$$

Where $\theta_\varepsilon(x) = \frac{1}{\varepsilon^n}\varphi(x\,|\ln \varepsilon|)\phi(\frac{x}{\varepsilon})$. As $\chi T \in E'_{MN}(\Omega)$, then

$$\psi(T * \theta_\varepsilon)(x) = \psi(\chi T * \theta_\varepsilon)(x) = \psi(\chi T * \phi_\varepsilon)(x)$$

Let $\varepsilon \leq min(\eta, \varepsilon_0)$ and $\xi \in \Gamma$, we have

$$\begin{aligned} |\mathcal{F}(\psi T)(\xi)| &\leq |\mathcal{F}(\psi T)(\xi) - \mathcal{F}(\psi(T * \theta_\varepsilon))(\xi)| + |\mathcal{F}(\chi(T * \theta_\varepsilon))(\xi)| \\ &\leq |\mathcal{F}(\psi\chi T)(\xi) - \mathcal{F}(\psi(\chi T * \phi_\varepsilon))(\xi)| + |\mathcal{F}(\chi(T * \theta_\varepsilon))(\xi)| \end{aligned}$$

Then by (3.18) and (3.19), we obtain

$$|\mathcal{F}(\psi T)(\xi)| \leq c' e^{-M(\frac{k_0}{\varepsilon})} + c_1 e^{M(\frac{k_1}{\varepsilon}) - MN^{-1}p!(k_2|\xi|)}$$

Take $c = max(c_1, c')$, $\varepsilon = \dfrac{k_1 p!^{\frac{1}{p}}}{(k2-r)\,|\xi|\,N_p^{\frac{1}{p}}}$, $r \in]0, k_2[$ and $k_0 = \dfrac{k_1 r}{k_2 - r}$, then $\exists\delta > 0,\ \exists c > 0$ such that

$$|\mathcal{F}(\chi T)(\xi)| \leq c' e^{-\frac{M}{N}p!(\delta|\xi|)},$$

Which proves that $(x_0, \xi_0) \notin WF^{\frac{M}{N}p!}(T)$.So $WF^{\frac{M}{N}p!}(T) \subset WF_g^{M,\frac{M}{N}p!}(T)$.

Suppose that $(x_0, \xi_0) \notin WF^{\frac{M}{N}p!}(T)$, then there exist $\chi \in \mathcal{D}^{\frac{M}{N}p!}(\Omega)$, $\chi(x) = 1$ in a neighborhood of x_0, a conic neighborhood Γ of ξ_0, $\exists\lambda > 0$, $c_1 > 0$, such that $\forall\xi \in \Gamma$

$$|\mathcal{F}(\chi T)(\xi)| \leq c_1 e^{-\frac{M}{N}p!(\lambda|\xi|)}. \tag{3.20}$$

Let $\psi \in \mathcal{D}^{\frac{M}{N}p!}(\Omega)$ equals 1 in neighborhood of x_0 such that for sufficiently small ε we have: $\chi \equiv 1$ on $supp\psi + B(0, \frac{2}{|\ln\varepsilon|})$, then there exist $\varepsilon_0 < 1$, such that $\forall\varepsilon < \varepsilon_0$,

$$\psi(T * \theta_\varepsilon)(x) = \psi(\chi T * \theta_\varepsilon)(x).$$

We have

$$\mathcal{F}(\psi(T * \theta_\varepsilon))(\xi) = \int \mathcal{F}(\psi)(\xi - \eta).\mathcal{F}(\chi T)(\eta).\mathcal{F}(\theta_\varepsilon)(\eta)d\eta.$$

Let Λ be a conic neighborhood of ξ_0 such that, $\bar{\Lambda} \subset \Gamma$. For a fixed $\xi \in \Lambda$, we have:

$$\mathcal{F}(\psi(\chi T * \theta_\varepsilon))(\xi) = \int_A \mathcal{F}(\psi)(\xi - \eta).\mathcal{F}(\chi T)(\eta).\mathcal{F}(\theta_\varepsilon)(\eta)d\eta + \int_B \mathcal{F}(\psi)(\xi - \eta).\mathcal{F}(\chi T)(\eta).\mathcal{F}(\theta_\varepsilon)(\eta)d\eta$$

Where $A = \{\eta : |\xi - \eta| \leq \delta(|\xi| + |\eta|)\}$ and $B = \{\eta : |\xi - \eta| \geq \delta(|\xi| + |\eta|)\}$. We choose δ sufficiently small such that $A \subset \Gamma$ and $\frac{|\xi|}{2} < |\eta| < 2\,|\xi|$. Since $\psi \in \mathcal{D}^M(\Omega)$, then $\exists\mu > 0$, $\exists c_2 > 0$,

$\forall \xi \in \mathbb{R}^n$,

$$|\mathcal{F}(\psi)(\xi)| \leq c_2 \exp(-\frac{M}{N} p!(\mu |\xi|)),$$

Then $\exists c > 0$, $\exists \varepsilon_0 \in]0, 1[$, $\forall \varepsilon \leq \varepsilon_0$;

$$\left| \int_A \mathcal{F}(\psi)(\xi - \eta).\mathcal{F}(\chi T)(\eta).\mathcal{F}(\theta_\varepsilon)(\eta) d\eta \right| \leq c \exp(-\frac{M}{N} p!(\frac{\lambda}{2} |\xi|)) \times \left| \int_A \exp(-\frac{M}{N} p!(\mu |\eta - \xi|).\mathcal{F}(\theta_\varepsilon)(\eta) d\eta \right|$$

From Lemma (3.5.8), $\exists c_3 > 0$, $\exists v > 0$, $\exists \varepsilon_0 > 0$, such that:

$$|\mathcal{F}(\theta_\varepsilon)(\xi)| \leq c_3 \varepsilon^{-n} e^{-N(v\varepsilon|\xi|)} \ \forall \xi \in \mathbb{R}^n$$

then $\exists c > 0$, such that

$$\begin{aligned} \left| \int_A \mathcal{F}(\psi)(\xi - \eta).\mathcal{F}(\chi T)(\eta).\mathcal{F}(\theta_\varepsilon)(\eta) d\eta \right| \leq c \quad & \varepsilon^{-n} \exp(-\tfrac{M}{N} p!(\lambda |\xi|)) \times \\ & \left| \int_A \exp(-\tfrac{M}{N} p!(\mu |\eta - \xi|). \exp(-N(v\varepsilon |\eta|)) d\eta \right| \end{aligned}$$

We have $\exists k > 0$, $\forall \varepsilon \in]0, \varepsilon_0[$,

$$\varepsilon^{-m} \exp(-N(v\varepsilon |\eta|)) \leq \exp(M(\frac{k}{\varepsilon})), \tag{3.21}$$

So

$$\left| \int_A \mathcal{F}(\psi)(\xi - \eta).\mathcal{F}(\chi T)(\eta).\mathcal{F}(\theta_\varepsilon)(\eta) d\eta \right| \leq c \exp(M(\frac{k}{\varepsilon}) - \frac{M}{N} p!(\frac{\lambda}{2} |\xi|)) \tag{3.22}$$

As $\psi T \in E'_{MN}(\Omega) \subset E'_{\frac{M}{N}p!}(\Omega)$, then $\forall l > 0$, $\exists c > 0$, $\forall \xi \in \mathbb{R}^n$,

$$|\mathcal{F}(\chi T)(\xi)| \leq c \exp(\frac{M}{N}p!(l\,|\xi|))$$

Hence, we have

$$\begin{aligned}\left|\int_B \mathcal{F}(\psi)(\xi-\eta)\mathcal{F}(\chi T)(\eta)\mathcal{F}(\theta_\varepsilon)(\eta)d\eta\right| &\leq c\int_B \exp(\tfrac{M}{N}p!(l\,|\eta|) - \tfrac{M}{N}p!(\mu\,|\xi-\eta|)).\,|\mathcal{F}(\theta_\varepsilon)|\,d\eta \\ &\leq c'\varepsilon^{-n}.\exp(-\tfrac{M}{N}p!(\mu\delta\,|\xi|))\times \\ &\quad \int_B \exp(\tfrac{M}{N}p!((l-\mu\delta)\,|\eta|) - N(v\varepsilon\,|\eta|))\end{aligned}$$

Then, taking $l - \mu\delta = -a < 0$ and using (3.21), we obtain for a constant $c > 0$

$$\left|\int_B \mathcal{F}(\psi)(\xi-\eta).\mathcal{F}(\chi T)(\eta).\mathcal{F}(\theta_\varepsilon)(\eta)d\eta\right| \leq c\exp(M(\frac{k_1}{\varepsilon}) - \frac{M}{N}p!(\mu\delta\,|\xi|))$$

Which gives that $(x_0,\xi_0) \notin WF_g^{M,\frac{M}{N}p!}(T)$, so $WF_g^{M,\frac{M}{N}p!}(T) \subset WF^{\frac{M}{N}p!}(T)$. □

3.6 Generalized Hörmander's theorem

To extend the generalized Hörmander's result on the wave front set of the product, define $WF_g^{M,N}(f) + WF_g^{M,N}(f)$, where $f, g \in \mathcal{G}^M(\Omega)$, as the set:

$$\{(x,\xi+\eta) \in WF_g^{M,N}(f), (x,\eta) \in WF_g^{M,N}(g)\}$$

The principal result of this section is the following theorem.

Theorem 3.6.1 *Let $f, g \in \mathcal{G}^M(\Omega)$, such that: $\forall x \in \Omega$,*

$$(x, 0) \notin WF_g^{M,N}(f) + WF_g^{M,N}(g) \tag{3.23}$$

Then

$$WF_g^{M,N}(f.g) \subseteq (WF_g^{M,N}(f) + WF_g^{M,N}(g)) \cup WF_g^{M,N}(f) \cup WF_g^{M,N}(g).$$

Proof. Let $(x_0, \xi_0) \notin (WF_g^{M,N}(f) + WF_g^{M,N}(g)) \cup WF_g^{M,N}(f) \cup WF_g^{M,N}(g)$, then: $\exists \phi \in \mathcal{D}^M(\Omega)$; $\phi(x_0) = 1$, $\xi_0 \notin (\Sigma_g^{M,N}(\phi f) + \Sigma_g^{M,N}(\phi g)) \cup \Sigma_g^{M,N}(\phi f) \cup \Sigma_g^{M,N}(\phi g)$ From (3.23) we have $0 \notin \Sigma_g^{M,N}(\phi f) + \Sigma_g^{M,N}(\phi g)$ then by lemma (2.8.1) i), we have

$$\xi_0 \notin (\Sigma_g^{M,N}(\phi f) + \Sigma_g^{M,N}(\phi g)) \cup \Sigma_g^{M,N}(\phi f) \cup \Sigma_g^{M,N}(\phi g) = \overline{\Sigma_g^{M,N}(\phi f) + \Sigma_g^{M,N}(\phi g)}^{\mathbb{R}^n \backslash \{0\}}$$

Let Γ_0 be an open conic neighborhood of $\Sigma_g^{M,N}(\phi f) + \Sigma_g^{M,N}(\phi g)$ in $\mathbb{R}^n \backslash \{0\}$ such that: $\xi_0 \notin \overline{\Gamma_0}$ then, from lemma (2.8.1) ii), there exist open cones Γ_1 and Γ_2 in $\mathbb{R}^n \backslash \{0\}$ such that

$$\Sigma_g^{M,N}(\phi f) \subset \Gamma_1; \quad \Sigma_g^{M,N}(\phi g) \subset \Gamma_2$$

And

$$\Gamma_1 + \Gamma_2 \subset \Gamma_0$$

Define $\Gamma = \mathbb{R}^n \backslash \Gamma_0$, so

$$\Gamma \cap \Gamma_2 = \emptyset \, and \, (\Gamma - \Gamma_2) \cap \Gamma_1 = \emptyset \tag{3.24}$$

Let $\xi \in \Gamma$ and $\varepsilon \in I$.

$$\begin{aligned}\mathcal{F}(\phi f_\varepsilon \phi g_\varepsilon)(\xi) &= (\mathcal{F}(\phi f_\varepsilon) * \mathcal{F}(\phi g_\varepsilon))(\xi) \\ &= \int_{\Gamma_2} \mathcal{F}(\phi f_\varepsilon)(\xi-\eta).\mathcal{F}(\phi g_\varepsilon)(\eta)d\eta \\ &\quad \int_{\Gamma_2^c} \mathcal{F}(\phi f_\varepsilon)(\xi-\eta).\mathcal{F}(\phi g_\varepsilon)(\eta)d\eta = I_1(\xi)+I_2(\xi)\end{aligned}$$

From (3.24), $\exists c_1 > 0$, $\exists k_1, k_2 > 0$, $\exists \varepsilon_1 > 0$, such that: $\forall \varepsilon \leq \varepsilon_1$, $\forall \eta \in \Gamma_2$,

$$|\mathcal{F}(\phi f_\varepsilon)(\xi-\eta)| \leq c_1 \exp(M(\frac{k_1}{\varepsilon}) - N(k_2|\xi-\eta|))$$

And from remark (3.4.7), $\exists c_2 > 0$, $\exists k_3 > 0$, $\forall k_4 > 0$, $\exists \varepsilon_2 > 0$, $\forall \eta \in \mathbb{R}^n$, $\forall \varepsilon \leq \varepsilon_2$,

$$|\mathcal{F}(\phi g_\varepsilon)(\eta)| \leq c_2 \exp(M(\frac{k_3}{\varepsilon}) + N(k_4|\eta|))$$

Let $\gamma > 0$ sufficiently small such that

$$|\xi-\eta| \geq \gamma(|\xi|+|\eta|), \quad \forall \eta \in \Gamma_2$$

Hence for $\varepsilon \leq min(\varepsilon_1, \varepsilon_2)$,

$$|I_1(\xi)| \leq c_1.c_2 \exp(M(\frac{k_1+k_3}{\varepsilon}) - N(k_2\gamma|\xi|)) \int \exp(-N(k_2\gamma|\eta|) + N(k_4|\eta|))d\eta$$

Take $k_4 > k_2\gamma$, then

$$|I_1(\xi)| \leq c' \exp(M(\frac{k_1'}{\varepsilon}) - N(k_2'|\xi|))$$

Let $r > 0$,

$$\begin{aligned} I_2(\xi) &= \int_{\Gamma_2^c \cap \{|\eta| \le r|\xi|\}} \mathcal{F}(\phi f_\varepsilon)(\xi - \eta).\mathcal{F}(\phi g_\varepsilon)(\eta) d\eta + \int_{\Gamma_2^c \cap \{|\eta| \ge r|\xi|\}} \mathcal{F}(\phi f_\varepsilon)(\xi - \eta).\mathcal{F}(\phi g_\varepsilon)(\eta) d\eta \\ &= I_{21}(\xi) + I_{22}(\xi). \end{aligned}$$

Choose r sufficiently small such that $\{|\eta| \le r\,|\xi|\} \Rightarrow \xi - \eta \notin \Gamma_1$. Then

$|\xi - \eta| \ge (1 - r)\,|\xi| \ge (1 - 2r)\,|\xi| + |\eta|$, Consequently: $\exists c_3 > 0$, $\exists \lambda_1, \lambda_2, \lambda_3 > 0$, $\exists \varepsilon_3 > 0$ such that: $\forall \varepsilon \le \varepsilon_1$;

$$\begin{aligned} |I_{21}(\xi)| &\le c_3 \exp(M(\tfrac{\lambda_1}{\varepsilon})) \int \exp(-N(\lambda_2\,|\xi - \eta|) - N(\lambda_3\,|\eta|)) d\eta \\ &\le c_3 \exp(M(\tfrac{\lambda_1}{\varepsilon}) - N(\lambda_2'\,|\xi|)) \int \exp(-N(\lambda_3'\,|\eta|)) \\ &\le c_3' \exp(M(\tfrac{\lambda_1}{\varepsilon}) - N(\lambda_2'\,|\xi|)) \end{aligned}$$

If $|\eta| \ge r\,|\xi|$, we have $|\eta| \ge \dfrac{|\eta| + r\,|\xi|}{2}$, and then $\exists c_4 > 0$, $\exists \mu_1, \mu_3 > 0$, $\forall \mu_2 > 0$, $\exists \varepsilon_4 > 0$ such that $\forall \varepsilon \le \varepsilon_4$,

$$\begin{aligned} |I_{22}(\xi)| &\le c_4 \exp(M(\tfrac{\mu_1}{\varepsilon})) \int \exp(N(\mu_2\,|\xi - \eta|) - N(\mu_3\,|\eta|)) d\eta \\ &\le c_4 \exp(M(\tfrac{\mu_1}{\varepsilon})) \int \exp(N(\mu_2\,|\xi - \eta|) - N(\mu_3'\,|\eta|) - N(\mu_3'\,|\xi|)) d\eta \end{aligned}$$

If take $\mu_2 < \frac{\mu_3'}{2}(1 + \frac{1}{r})$, we obtain

$$|I_{22}| \le c_4' \exp(M(\frac{k_3'}{\varepsilon}) - N(\mu_3'\,|\xi|))$$

Which finishes the proof. $\square$

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