الجمهوريــة الجزائريــة الديمقراطيــة الشعبيــة
The People's Democratic Republic of Algeria
وزارة التعليــم العالي و البحث العلمــي
Ministry of Higher Education and Scientific Research

University of Mustapha Stambouli
Mascara

جامعة مصطفى اسطمبولي
معسكر

Faculty of Economics, Commercial Sciences and Management Sciences
Department: Management Sciences

Lessons Intended for Second-Year LMD Students in Management Sciences

Module :

Statistic 03

Prepared by :

Dr .BECHEROUL Fayssal

University Year : 2024/2025

Table of contents

Introduction

This publication consists of lessons on Statistics 03 according to the new ministerial program Intended fo Second-year students in Management Sciences, The content of this Module are considered a continuation of the lessons Statistics 01 and Statistics 02, allowing them to benefit from the foundation they previously acquired. At the same time, studying Statistics 03 serves as a preparation for studying Statistics 04 in the second year during the fourth semester.

The goal of providing these lessons is to achieve the following skills:

- Understanding the most important laws of discrete and continuous probability distributions
- Providing the student with the ability to apply probability distributions to address and solve economic, administrative, and social problems.
- Understanding discrete and continuous bivariate random variables and their main properties.
- Recognizing distributions with two variables.

This lessons contains four main axes, which are :

- **The First Axis :** the most important laws of discrete probability distributions
- **The Second Axis** : the most important laws of continuous probability Distributions
- **The Third Axis** : approximation of some probability distributions
- **The Fourth Axis :** bivariate random variables

It is our sincere hope that Students find this Lessons to be user-friendly with easy and logical explanations, plethora of examples.

Probability distributions are divided into two main sections:

The first section is the discrete probability distributions and concerns discrete random variables such as the binomial distribution, the Poisson distribution... while the second section is the continuous probability distributions and concerns continuous random variables such as the normal distribution, the Student's distribution, the Fisher distribution and the Chi-square distribution...,

Discrete probability distributions are used in many issues in the field of industrial, commercial, and managerial management, the most common of which are the binomial distribution, Poisson distribution, hypergeometric distribution, multiple geometric distribution...

1-The Bernoulli Distribution

1 –1-Definition

Let us have a random experiment with only two outcomes of success or failure (mutually exclusive events), X is a random variable that takes the value **1** when the success result is achieved, and the value **0** in the case of the failure result is achieved.

For example, when we throw a coin once, success expresses the appearance of the image, while failure represents the non-appearance of the image)

p: probability of success (The probability of the phenomenon under study not occurring)

q : probability of failure (The probability of the phenomenon not occurring)

p+q=1 X={0,1}

1-2- The probability Function

The probability of a random variable X following a bernolli distribution is given by the following relation:

$$f(x) = P(X = x) = p^x(1-p)^{1-x} \quad , \quad x = 0,1 \; ; \quad 0 \leq p \leq 1$$

X: A random variable that represents the success or failure of the rand experiment is called a Bernoulli variable and its probability distribution is a Bernoulli distribution And we write : $X \rightsquigarrow B(1; p)$

1 – 3– Properties of the Bernoulli distribution

- ✓ **Mathematical expectation and variance**

$$\mathbf{E(X)=p} \quad , \quad \mathbf{Var(X)}= \sigma^2 = pq$$

$$\boldsymbol{E}(\mathbf{X}) = \sum \boldsymbol{x_i p_i} = \mathbf{1}.\boldsymbol{p} + \mathbf{0}\boldsymbol{q} = \boldsymbol{p}$$

$$V(\mathbf{X}) = E(\mathbf{X^2}) - \big(E(\mathbf{X})\big)^{\mathbf{2}} = (\mathbf{1^2}.\boldsymbol{p} + \mathbf{0^2}.\boldsymbol{q}) - \mathbf{p^2} = \boldsymbol{p} - \mathbf{p^2} = \boldsymbol{p}(\mathbf{1}-\boldsymbol{p}) = \boldsymbol{pq}$$

$$V(\mathbf{X}) = \boldsymbol{pq}$$

- ✓ **Moment Generating Function**

$$m_x(t) = q + pe^t$$

1-3- Distribution Function

If **X** is a random variable that follows a a Bernoulli distribution with parameter **p**, then the distribution function is given by :

$$\boldsymbol{F(x)} = \begin{cases} 0 & ; \quad \boldsymbol{x < 0} \\ \mathbf{1}-\boldsymbol{p} & ; \quad \mathbf{0} \le \boldsymbol{x} < \mathbf{1} \\ \mathbf{1} & ; \quad \boldsymbol{x} \ge \mathbf{1} \end{cases}$$

Example:

Suppose that one selects a card at random from a pack of 52 cards.
Let success denote the event that an ace is drawn, so that X = 1 if this occurs, and X = 0 otherwise. Find the probability function, distribution function, expectation, and variance of X

SOLUTION

X has a Bernoulli distribution. Its parameter is the probability
of success (picking out an ace) when selecting a card and, since the pack contains 52 cards of which 4 are aces, this probability equals

$$\boldsymbol{p} = \boldsymbol{P}(\boldsymbol{X} = \mathbf{1}) = \frac{\mathbf{4}}{\mathbf{54}} = \frac{\mathbf{1}}{\mathbf{13}} \quad , \boldsymbol{q} = (\mathbf{1}-\boldsymbol{p}) = \mathbf{1} - \frac{\mathbf{1}}{\mathbf{13}} = \frac{\mathbf{12}}{\mathbf{13}}$$

Thus, the probability function is :

$$\boldsymbol{f(x) = \left(\frac{1}{13}\right)^{x}\left(1-\frac{1}{13}\right)^{1-x} \quad , \quad x = 0, 1}$$

while the corresponding distribution function is given by :

$$\boldsymbol{F(x) = } \begin{cases} 0 & ; \quad x < 0 \\ \frac{12}{13} & ; \quad 0 \leq x < 1 \\ 1 & ; \quad x \geq 1 \end{cases}$$

The mean and variance are, respectively,

$$\boldsymbol{E(X) = \left(\frac{1}{13}\right) \quad , \quad Var = \left(\frac{1}{13}\right)\left(\frac{12}{13}\right) = \left(\frac{12}{169}\right)}$$

2 – The Binomial Distribution

2-1-Definition

If Bernoulli's experiment is repeated n times so that these randomized trials are independent and during each trial we get only two outcomes, either success or failure with a constant probability each time the randomized trial is repeated .

X : is a random variable representing the number of successes that follows a certain probability distribution called a binomial distribution.

The probability of success is P and the probability of failure is q=(1-p)

n : represents the number of trials or, in other words, the number of repetitions of the rando experiment

2-2- The probability Function

A random variable X is said to have a binomial distribution based on n trials with success probability p if and only if:

$$f(X) = P(X = x) \begin{cases} c_n^x P^x q^{n-x} & , \ x = 0, 1, 2 \ldots n \\ 0 & , \ \textbf{Otherwise} \end{cases}$$

X : A random variable representing the number of successes that follows a binomial distribution: $X \rightsquigarrow B(n; \mathrm{p})$

n,p : The two parameters of this distribution (success rate and number of repetitions, respectively).

2-3- Properties of the Binomial Distribution

- ✓ The mathematical expectation, variance and symmetry coefficient of a random variable X that follows a binomial distribution
  - $E(X) = np \quad ; \quad \sigma^2 = npq \quad ; q = 1 - p$,

Proof: Consider the random variable X as a set of independent Bernoulli variables, $X = X_1 + X_2 + X_3 + \cdots X_n$ have the same probability of success p, So they all have the same mathematical expectation $E(X_i) = p$

$$E(X) = E(X_1 + X_2 + X_3 + \cdots X_n) = E(p + p + p + \cdots) = np$$

$$E(X) = E(X_1 + X_2 + X_3 + \cdots X_n) = E(p + p + p + \cdots) = np$$
$$V(X) = \sigma^2 = V(X_1 + X_2 + X_3 + \cdots X_n) = VX_1) + V(X_2) \ldots V(X_n)$$
$$= \sum \sigma^2 = (pq + pq + \cdots pq) = npq$$

- skewness coefficient : $\alpha_3 = \frac{\mu^3}{\sigma^3} = 0$
- The distribution is symmetric : $\alpha_3 = 0$, if $p = q = 1/2$ and the distribution approaches symmetry as n grows
- The distribution is right skewed (positive skewness) if: $p < q$
- The distribution is skewed to the left (negative skewness) if: $p > q$

✓ A binomial distribution that satisfies the conditions of the probability mass function: $f(x) \geq 0$, $\sum_{x=0}^{n} f(x) = 1$

$$\sum_{x=0}^{n} c_n^x p^x (1-p)^{n-x} = (p+q)^n = 1$$

✓ **Moment Generating Function**

$$m_x(t) = E(e^{tx}) = \sum_x e^{tx} f(x)$$
$$m_x(t) = (q + pe^t)^n$$

Binary Distribution Conditions of Use :

1) A Bernoulli experiment repeated n times
2) Each trial consists of only two outcomes, success or failure
3) Each trial is independent of the other trial
4) The probability of success (or failure) is constant from one trial (random experiment) to another and does not change (Sampling With Replacement)

2-4- Distribution Function

If **X** is a random variable that follows a binomial distribution with parameter **p**, then the distribution function (distribution function or cumulative distribution function) such that $x \in \mathbb{R}$ is given by :

$$F(x) = \begin{cases} 0 & ; \quad x < 0 \\ \sum_{i=0}^{x} c_n^i P^i q^{n-i} & ; \quad 0 \le x < n \\ 1 & ; \quad x \ge n \end{cases}$$

(The probability values of the binomial distribution and the distribution function are given in special statistical tables)

Example01:

Suppose we toss a coin 10 times such that it is known that the coin is unbalanced and the probability of an heads appearing is **0.4** , and let the random variable X be the number of heads that can appear.

- Calculate the probability of getting five heads when flipping this coin.

Solution :

The random experiment in this example is a Bernoulli experiment repeated 10 times with n=10

So **X** : is a random variable that follows the probability distribution of the binomial and we write:

p=0.4. ; n =15 $X \rightsquigarrow B(10; 0.4)$

✓ **the probability P(X=4) using the binomial distribution relation**

$$f(x) = \begin{cases} c_{10}^x P^x q^{10-x} & , \quad x = 0, 1, 2 \ldots 10 \\ 0 & , \quad \text{otherwise} \end{cases}$$

$$P(X = 5) = c_{10}^5 (0.4)^4 (0.6)^{10-5} = \frac{10!}{(10-5)!\,5!} (0.4)^5 (0.6)^{10} = 0.0156$$

Example02:

A box contains 05 black and 10 red balls, three balls are draw at random by return , find the probabilities that:

1) one black ball appearing ?
2) one or more black balls appearing?

Solution :

X : A random variable representing the number of black balls drawn following the binomial distribution

p : Probability of success (black ball) p=5/15=1/3

q: Probability of failure (no black ball) q=1-p=2/3

1) **the probability of one black ball will appear among the three drawn balls**

$$P(X = 1) = c_3^1(1/3)^1(2/3)^{3-1} = \frac{4}{9}$$

2) **The probability that one or more black balls will appear among the three drawn balls**

$$P(X \geq 1) = P(X = 1) + P(X = 2) + P(X = 3) = \frac{19}{27}$$

p=1/3. ; n =03 $X \rightsquigarrow B(03; 1/3)$

Example03:

Thirty people, who are of the same age and the same health status, are insured with the same insurance company. Using life tables, the company estimates that the probability for a randomly chosen person among these 30 to be alive in 15 years from now is 80%.

1) What is the probability that not all 30 people will be alive in 15 years' time?
2) What is the probability that at least one person will be alive?
3) Every insured person who will be alive after 15 years will receive 100 000 DA. Find the expected amount of money the company will have to pay at the end of the 15-year period and the standard deviation of that amount.

Solution:

Let X be the number of individuals who will be alive at the end of the 15-year period. Since lives of different people are assumed to be independent, X has a binomial distribution with parameters n = 30 and p = 0.8
We want P(X < 30). This is :

P(X < 30) = 1 − P(X = 30) = 1 − (30 30) (0.8)30(0.2)0 = 1 − (0.8)30

= 0.998 766 ≅ 99.9%.

$P(X < 30) = 1 - P(X = 3) = 1 - c_{30}^{30}(0.8)^{30}(0.2)^{30-30} = 0.99876$

1) Here we seek P(X > 0), which is equal to :

$$P(X \geq 1) = 1 - P(X < 1) = 1 - P(X = 0) = 1 - c_{30}^{0}(0.8)^{0}(0.2)^{30-0}$$
$$= 1 - 1.1^{-21} \cong 1$$

2) The total amount the insurance company will have to pay at the end of the15-year period is represented by the random variable Y = 100 000X

$E(X) = np = 30(0.8) = 24$; $Var(X) = \sigma^2 = npq = 30(0.8)(0.2) = 4.8$

we get :

$E(Y) = E(100\,000\,X) = 100\,000\,E(X) = 100\,000\,(24) = 2400\,000$

$Var(Y) = Var(100\,000\,X) = 100\,000^2 Var(X) = 4.8\,10^{10}$

so that the standard deviation of the total payment equals :

$\sigma_Y^2 = \sqrt{Var(Y)} = \sqrt{4.8\,10^{10}} = 219\,089.023$

3- The Poisson Distribution

3-1- Definition

The Poisson distribution is the law of rare events or the law of small probabilities, this law is an extended distribution of the binomial law when "**n "**tends to infinity and "**p"** tends to **zero**, so Poisson's distribution is a special case of the binomial distribution when the probability of success **p** is small for very large n.

The Poisson distribution is used in matters where phenomena occur in specific time intervals (seconds, minutes, hours, days...), It is also used in the case of phenomena that occur in specific areas where the area may be a page of a book, a square meter of space.

For example, the number of phone calls received by the customer service department of an organization during an hour, The number of car accidents on a particular road during a weekday, the number of people entering the post office in half-hour, The number of cars that pass through the payment center on a highway during every quarter-hour …

3-2-The probability Function

Let x be a discrete random variable that follows a Poisson probability distribution with parameter $\lambda > 0$ which is denoted by: $X \rightsquigarrow P(\lambda)$, The Poisson's probability function is given by:

$$P(X = x) \begin{cases} \dfrac{e^{-\lambda}\ \lambda^{x}}{x!} & ;\quad x = 0, 1, 2 \ldots \infty \\ 0 & ;\ \textbf{otherwise} \end{cases}$$

Note :

- ✓ The Poisson distribution is related to the parameter λ
- ✓ The statistical table for the Poisson distribution gives the probabilities $P(X = x)$ and there is another table for the cumulative probabilities $\sum_{i=1}^{x} P(X = i)$

3-3- Properties of the Poisson distribution

If **X** follows a Poisson distribution, then :

- ✓ **E(X)=Var(X)= λ** , $\lambda>0$

The Poisson distribution is the only distribution where the expectation is equal to the variance

- ✓ Let $\boldsymbol{X_1, X_2, \ldots . X_n}$ be independent random variables such that $X_i \rightsquigarrow P(\boldsymbol{\lambda}_i)$ with $\boldsymbol{\lambda}_i > 0$ For i=1,2…n, let:

$$\sum_{i=1}^{n} \boldsymbol{X_i} \rightsquigarrow P(\boldsymbol{\lambda}) \qquad ; \boldsymbol{\lambda} = \sum_{i=1}^{n} \boldsymbol{\lambda_i}$$

- ✓ **Moment Generating Function**

$$m_x(t) = \boldsymbol{e}^{\lambda(e^t-1)}$$

3-4- Distribution Function (cumulative distribution function)

If **X** is a random variable that follows a Poisson distribution with parameter **λ>0** , the distribution function of this probability law **F(X) = P(X ≤ x)** is given by :

$$\boldsymbol{F(x)} = \begin{cases} \boldsymbol{0} & ; \ \boldsymbol{x < 0} \\ \displaystyle\sum_{\boldsymbol{i=0}}^{[x]} \boldsymbol{f(x)} = \sum_{\boldsymbol{i=0}}^{[x]} \frac{\boldsymbol{e^{-\lambda}} \ \boldsymbol{\lambda^i}}{\boldsymbol{i}\,!} & ; \forall \ \boldsymbol{x \geq 0} \end{cases}$$

($[x]$ is the largest integer less than or equal to x)

3-5- Calculating the probability of a number of events occurring in "t" units of time

For an amount or units of time t, we have Poisson's law of the form :

$$\boldsymbol{P(X = x)} = \frac{\boldsymbol{e^{-\lambda t}} \ (\lambda t)^x}{\boldsymbol{x!}} \qquad ; \quad \boldsymbol{x = 0, 1, 2 \ldots \infty}$$

Example01:

Let x be a random variable representing the number of cars refuelling in an hour, at a services station, Find :

1) The type and form of the probability function for the random variable **X** ?
2) The expected value and variance of this distribution ?
3) The probability that three cars are refuelled in one hour.?
4) The probability that at most three cars will be fuelled in an hour?

Solution:

1) **The formula for the probability function for this distribution**

λ=6

$$\boldsymbol{P(X=x)} = \frac{\boldsymbol{e^{-6}}\ \ 6^{x}}{\boldsymbol{x!}} \quad ; \quad \boldsymbol{x = 0, 1, 2 \ldots \infty}$$

2) **The expected value and variance of this distribution**

E(X)=Var(X)=$\lambda = 6$

3) **The probability that at most three cars will be fuelled in an hour**

$$P(X=3) = \frac{e^{-6}\ \ 6^{3}}{3!} = 0.08928$$

4) **The probability that at most three cars will be fuelled in an hour**

$$P(X \leq 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$$
$$= 0.00248 + 0.01488 + 0.04464 + 0.08928$$
$$= 0.15128$$

Example: 02

Assuming that the number of phone calls to a civil protection emergency centre in a city follows a Poisson distribution with a mean of λ = 5 per minute

-Find the probability of receiving seven calls in **one and a half minutes** at this center.

Solution:

$$\boldsymbol{P(X=x)=\frac{e^{-\lambda t}\ (\lambda t)^{x}}{x!}}$$

$$\boldsymbol{P(X=7)=\frac{e^{-5(1.5)}\ [(5)(1.5)]^{7}}{7!}=\frac{e^{-5(1.5)}\ [(5)(1.5)]^{7}}{7!}\ 0.1464}$$

Example : 03

If the moment generating function for a random variable X is :

$$\boldsymbol{m_x(t)=e^{4(e^t-1)}}$$

- Find the probability **P(x=3)**

Solution:

The function $\boldsymbol{m_x(t)=e^{4(e^t-1)}}$ is the moment generating function of a random variable that follows a Poisson distribution with parameter λ=4 so that its probability mass function can be deduced as:

$$\boldsymbol{f(x)=}\begin{cases}\dfrac{e^{-4}\ \ 4^{x}}{x!} & ;\quad \boldsymbol{x=0,1,2\ldots\infty}\\ 0 & ;\quad \text{otherwise}\end{cases}$$

$$P(X=3)=\frac{e^{-4}\ \ 4^{3}}{3!}=0.195$$

4- The Geometric Distribution

4-1- Definition

A Geometric Distribution is an extended distribution of a random variable that follows a binomial distribution. Let **X** denote **the number of trials** until the first success occurs in a sequence of independent Bernoulli trials, where the probability of success in each trial equals **p** (the failure with probability **q**) We then say that the distribution of **X** is a geometric distribution with parameter p, which is denoted by: : $\boldsymbol{X \rightsquigarrow G(p)}$.

Example, Tossing a coin several times until the first head appears, Throwing the die until the number 1 appears, Drawing with Replacement productive pieces from a box of defective and non-defective pieces until the first defective piece appears (the assumptions are the same as with a binomial distribution)

For example, when Tossing a coin until the first heads comes up, we have the event probability for the first success {X=x} such that there are x-1 failures in **n** independent trials, so the event { X=x } has probability $\boldsymbol{q^{x-1}p}$ **,** and we write

$$\boldsymbol{PPP \dots \dots PF = q^{x-1}p}$$

P: Tails **F:** Heads

4-2- Probability Function

If a random variable X follows the geometric distribution $\rightsquigarrow G(p)$, then the probability function of X is given by:

$$\boldsymbol{P(X=x)} \quad \begin{cases} \boldsymbol{q^{x-1}p} & ; \quad \boldsymbol{x \in N^*} \quad \boldsymbol{x = 1, 2 \dots \infty} \\ 0 & ; \ \textbf{otherwise} \end{cases}$$

4-3- Properties of a Geometric Distribution

If **X** is a random variable that follows a geometric distribution, then:

- ✓ mathematical expectation $\boldsymbol{\mu = 1/p}$
- ✓ Variance $\boldsymbol{\sigma^2 = q/p^2}$

✓ **Moment Generating Function**

$$m_x(t) = E(e^{tx}) = \sum_x e^{tx}\, f(x) = \sum_x e^{tx}\ \boldsymbol{q^{x-1}p}$$

$$m_x(t) = \frac{\boldsymbol{p}e^t}{(\mathbf{1}-\boldsymbol{q}e^t)}$$

4-4- Distribution Function

If X is a random variable that follows a geometric distribution, the distribution function for this random variable (the cumulative distribution function)

F(x)=P(X≤ x) is given by :

$$\mathbf{F(x)} = \begin{cases} \mathbf{0} & ; \quad \boldsymbol{x} < \mathbf{1} \\ \mathbf{1} - \boldsymbol{q^x} & ; \quad \boldsymbol{x} = \mathbf{1}, \mathbf{2} \ldots. \\ \mathbf{1} & ; \ \mathbf{x} \longrightarrow \infty \end{cases}$$

Note : Unlike the binomial probability law, if X follows a geometric distribution, the dimension of X is unbounded, taking positive values x=1,2 ....∞

Example01 :

If we throw a die until number six appears for the first time, what is the probability that this will happen

1) in the 6th throw?
2) At least on the 10th throw?

Solution

Let X be the number of throws needed until the first appearance of a number six. Then, the distribution of X is geometric with parameter p = 1⁄6, q=5⁄6

f(x) = P(X = x) = $q^{x-1}p = \left(\frac{5}{6}\right)^{x-1} \left(\frac{1}{6}\right)$, $x = 1,2 \ldots$

3) **The probability P(X = 6)**

$$P(X = 6) = \left(\frac{5}{6}\right)^{6-1}\left(\frac{1}{6}\right) = 0.066$$

1) **Find the probability $P(X \leq 10)$**

Using the formula for the geometric probability function, we find that:

$$F(x) = P(X \leq x) = \sum_{i=1}^{10} P(X = i) = \mathbf{1} - \boldsymbol{q^x}$$

$$P(X \leq 10) = \mathbf{1} - \boldsymbol{q^x} = \mathbf{1} - \left(\frac{5}{6}\right)^{10} = \mathbf{0.838}$$

Example02:

In a factory producing an appliances, it is known that the probability of a defective product is p=*2⁄3*. Let's consider a random variable X that represents the number of units selected (with replacement) until the first defective unit appears.

1) Find the The probability function, expected value and variance of the random variable X?
2) Find the distribution function F(X)?
3) Find the probability of the first defective unit of this product appearing in the fifth draw?

Solution:

X is a random variable that follows a geometric distribution such that:

$$\boldsymbol{q = 1 - p = {}^{1}\!/_{3}} \quad ; \quad \boldsymbol{p = {}^{2}\!/_{3}}$$

And we write : X⇝G($\boldsymbol{{}^{2}\!/_{3}}$).

1) The probability function for the random variable X is given as follows:

$$\boldsymbol{f(x)} = \begin{cases} \left({}^{1}\!/_{3}\right)^{x-1}\left({}^{2}\!/_{3}\right) & ; \quad x \in N^* \\ 0 & ; \text{otherwise} \end{cases}$$

$$\mathbf{Var(X)} = {}^{q}/_{p^2} = {}^{3}/_{4} \quad , \quad \mathbf{E(X)} = {}^{1}/_{P} = {}^{3}/_{2}$$

2) The Distribution Function F(X)

$$\mathbf{F(X)} = \begin{cases} 0 & ; \quad x < 1 \\ \mathbf{1} - ({}^{1}/_{3}{}^{x}) & ; \ 1 \le x < \infty \\ 1 & ; \quad x \longrightarrow \infty \end{cases}$$

3) The Probability $\boldsymbol{P(X = 5)}$

$$\boldsymbol{f(5) = P(X = 5) = ({}^{1}/_{3})^{5-1}({}^{2}/_{3}) = ({}^{2}/_{243}) = 0.0082}$$

Example: 03

If the probability of a candidate succeeding in any PhD competition at a university institution is $1/3$, find :

1) The probability mass function for the number of attempts needed before the candidate succeeds in this competition?

2) Find the average number of attempts needed before the candidate succeeds?

3) What is the probability that this candidate will be successful after two attempts?

4) What is the probability that this candidate will be successful after at most three attempts?

Solution:

Let X be a random variable that represents the number of attempts needed before this candidate succeeds and follows a geometric distribution such that The candidate's probability of success in the competition is $\mathrm{p} = {}^{1}/_{3}$, $\boldsymbol{q = 1 - p = {}^{2}/_{3}}$

We write : $\mathrm{X} \rightsquigarrow \mathrm{G}({}^{1}/_{3})$

The probability mass function for the number of attempts needed before the candidate succeeds in this competition for the first time (on the first attempt)

1) **The probability mass function**

$$f(x) = \left(2/3\right)^{x-1}\left(1/3\right) \quad ; \quad x = 1,2, \dots$$

2) **The average number of attempts needed before the candidate succeeds**

$$\mathrm{E(X)} = 1/P = \frac{1}{1/3} = 3$$

3) **The probability that this candidate will be successful after two attempts**

$$P(X = 2) = \left(2/3\right)^{2-1}\left(1/3\right) = \left(2/9\right)$$

4) **The probability that this candidate will be successful after at most three attempts**

We have a cumulative distribution function:

$$\boldsymbol{F(x) = P(X \leq x) = 1 - q^{x}}$$

$$P(X \leq 3) = \quad 1 - \left(1/3\right)^{3} = 19/27 = 0.703$$

5- The Hypergeometric Distribution

5-1-Definition

Consider a population of size N divided into two groups a and b according to a certain characteristic such as gender (male and female), the presence of defects in the produced parts (defective and non-defective), if group "**a"** has a certain characteristic, group **"b"** does not have that characteristic, we choose from this population a random sample **without replacement**.

Let x represents the number of elements that contain a property in the selected sample of size n from a population of size **N** such that **N=a+b**, the question is : what is the probability that in the selected sample of size **n** there are **x** elements from group **a** ?

- ✓ The number of samples of size "**n"** that can be selected is a harmonic $\boldsymbol{C_N^n}$ such that : **N=a+b**
- ✓ A random sample of size n includes an x element from the group **"a"**that has a particular property, so there are $\boldsymbol{C_a^x}$ possibilities selections.
- ✓ n-x elements remain from the group **"b"**, so there is $\boldsymbol{C_b^{n-x}}$ possibilities

The required probability can be obtained by the classical law of probability, which is :

$$\frac{\textbf{The number of Possible cases}}{\textbf{The number of total cases}}$$

By using combinations, which is the law of The Hypergeometric distribution

5-2-Probability function:

Let X be a random variable that follows a Hypergeometric Distribution $X \rightsquigarrow H(N; n; a)$ with the following probability law:

$$\boldsymbol{P(X = x)} \begin{cases} \dfrac{\boldsymbol{C_a^x C_b^{n-x}}}{\boldsymbol{C_N^n}} & ; \boldsymbol{x = 0, 2 \ldots n} \\ 0 & ; \textbf{otherwise} \end{cases}$$

This probabilistic law is associated with three parameters:

N : The size of the population

n : The size of the sample selected without replacement

a : Number of elements with a particular characteristic

5-3-Properties of the Hypergeometric distribution

- ✓ $\mathbf{E}(\mathrm{X}) = np$
- ✓ $\mathrm{Var}(\mathrm{X}) = np\left(\frac{N-n}{N-1}\right)$

$\left(\frac{N-n}{N-1}\right)$ **:** The return coefficient

Notes:

1) A random variable X is a set of non-independent Bernoulli random variables
2) The coefficient of return tends of 1 if N is large compared to n ,and the variance of the variable X tends to follow a binomial distribution.

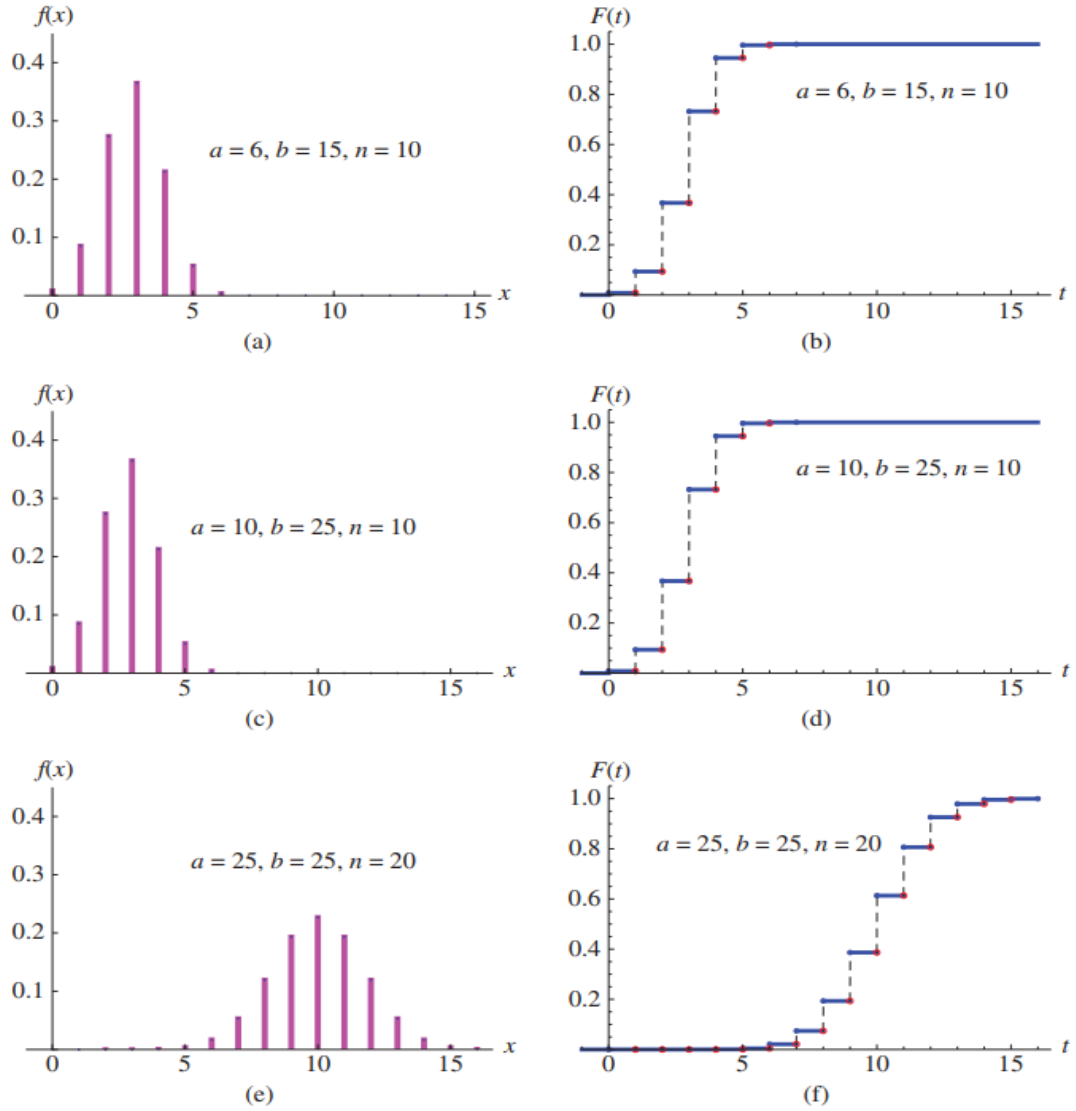
3-4- Distribution Function (cumulative distribution function)

The distribution function of Hypergeometric Distribution

$X \rightsquigarrow H(N; n; a)$, **F(X) = P(X ≤ x)** is given by :

$$\boldsymbol{F(x)} = \begin{cases} \mathbf{0} & , \quad \boldsymbol{x < max\{0, n-b\}} \\ \sum_{\boldsymbol{i=max\{0,n-b\}}}^{[x]} \frac{\boldsymbol{C_a^i C_b^{n-i}}}{\boldsymbol{C_N^n}} & , \quad \boldsymbol{ma\{0, n-b\} \leq x < min\{n, a\}} \\ \mathbf{1} & , \quad \boldsymbol{x \geq min\{n, a\}} \end{cases}$$

Figure 01:The probability function (left) and the cumulative distribution function (right) of the hypergeometric distribution for various choices of *n*, *a*, and *b*



Example01:

In a factory, out of 50 machines produced during a day, 8 are defective having an operational error. An engineer selects six machines at random to examine whether they have this error or not.

- What is the probability that at least two of the machines selected are defective?

Solution

Let X denote the number of defective machines in the sample of the six machines selected by the engineer. Then, and since these 6 machines were selected out of the 50 without replacement,

X : the number of non-defective machines among the 50 ones produced that day , it has a hypergeometric distribution with parameters n = 6, a = 8, and b = 42. Hence, the probability function of X is :

$$\boldsymbol{P(X=x)=\frac{C_8^x C_{42}^{6-x}}{C_{50}^6}} \quad ; \boldsymbol{x=0,2\ldots6}$$

We want to find the probability P(X ≥ 2)

$$\boldsymbol{P(X\geq 2)=1-P(X<2)=1-P(X=0)-P(X=1)=}$$
$$\boldsymbol{1-\frac{C_8^0 C_{42}^{6-0}}{C_{50}^6}-\frac{C_8^1 C_{42}^{6-1}}{C_{50}^6}}=0.24162$$

Example02:

During a recruitment competition, 30 people with the same educational level applied, including 17 females and 13 males, and a random sample was selected without Replacement from 08 individuals, Find the probability that in this sample:

1) 05 females and 3 males ?
2) No males?

Solution

Let X be a random variable representing the number of females selected Under the assumptions of the previous example, X follows a hypergeometric distribution $X \rightsquigarrow H(30;8;17)$

X=(0,1,2....8) , a=17: Number of females , b=13: Number of males

The probability law for variable X is given by :

$$\boldsymbol{f(x)=P(X=x)} \begin{cases} \boldsymbol{\frac{C_{17}^x C_{13}^{8-x}}{C_{30}^8}} & ; \quad \boldsymbol{x=0,2\ldots8} \\ 0 & ; \text{otherwise} \end{cases}$$

1) the probability that the selected sample includes 05 females and 3 males

$$\boldsymbol{f(5) = P(X = 5)} = \frac{C_{17}^{5} C_{13}^{8-5}}{C_{30}^{8}} = 0.302$$

2) the probability that there are no males in the selected sample

$$\boldsymbol{f(5) = P(X = 8)} = \frac{C_{17}^{8} C_{13}^{8-8}}{C_{30}^{8}} = 0.004$$

1- The Normal Distribution (Laplace-Gauss Distribution)

1-1- Definition

The normal distribution is one of the most important probability distributions Because many social and economic phenomena, ... Naturally distributed, This importance is also due to the fact that many probability distributions (discrete or continuous) converge to the normal distribution through the central limit theorem.

1-2- Probability Density Function

Let x be a random variable that follows a normal distribution with parameters μ and σ if its probability density function it is given as :

$$\boldsymbol{f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}} \qquad -\infty < x < +\infty$$
$$\sigma > 0 \quad \sigma \in R^+ ; \quad \mu \in R$$

And we write : X follows a normal distribution with parameters $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$:
$X \rightsquigarrow N(\boldsymbol{\mu}\,;\boldsymbol{\sigma})$

So that: $E(X) = \mu \quad ; \quad Var(X) = \boldsymbol{\sigma}^2$

1-3- Distribution Function

If X is a random variable that follows a normal distribution, the distribution function of this random variable F(X)=P(X≤x) is given by

$$\mathbf{F(X)=P(X\leq x)} = \boldsymbol{\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx}$$

It should be noted that this integral is not easy to interpret mathematically or calculate, as there are tables that give the values of the distributive function

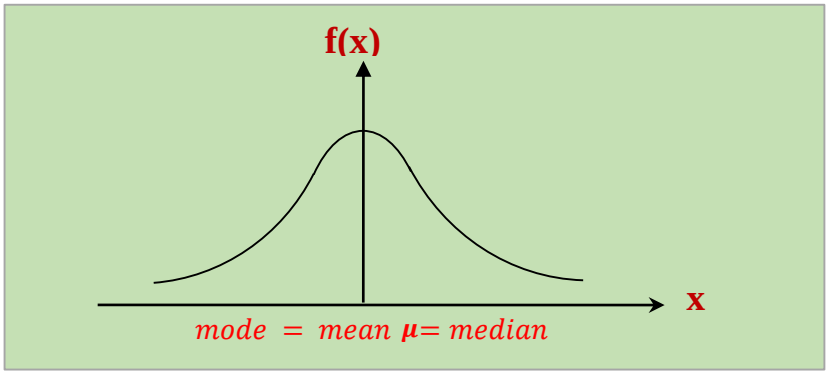
1-4- Normal Distribution Properties

- ✓ Mean and variance of the normal distribution

$$E(X) = \mu \quad ; \quad Var(X) = \boldsymbol{\sigma}^2$$

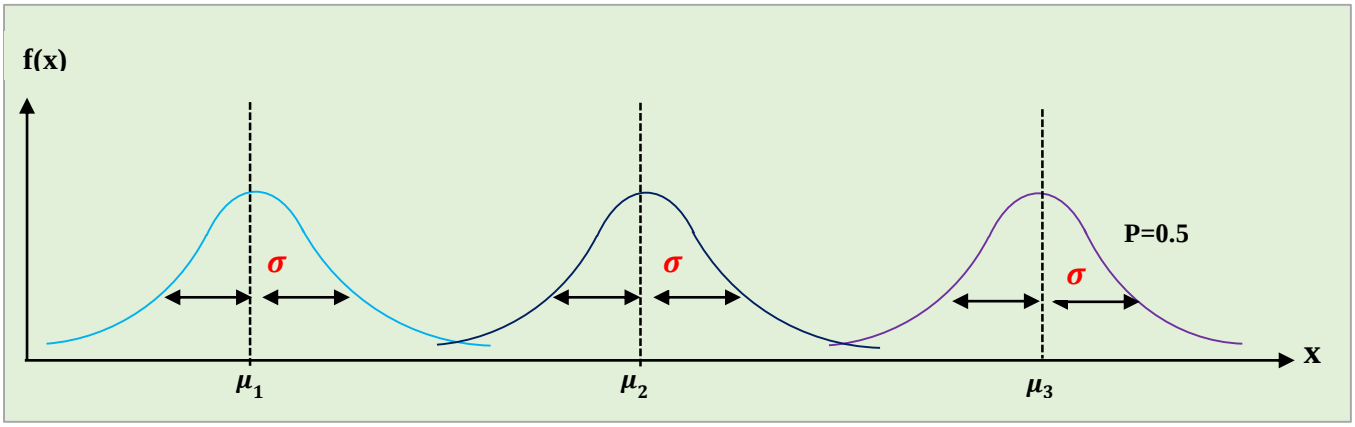
- ✓ $m_x(t) = E(e^{tx}) = \boldsymbol{e^{\mu t + \frac{1}{2}\sigma^2 t^2}}$, $-\infty \leq t \leq \infty$
- ✓ The highest point of the normal distribution curve corresponds on the horizontal axis to the mean μ, which is equal to the median and mode of this distribution**.**

Figure 02 : The Normal Distribution Curve



- ✓ The normal distribution is symmetrical and takes the shape of a bell, The shape of the curve to the left of the mean μ corresponds to the shape of the curve to the right of the mean **μ**, and the tail of the curve continues indefinitely and never touches the horizontal axis (the distribution is symmetrical if the skewness coefficient is zero $\boldsymbol{\alpha_3 = 0}$).
- ✓ $\boldsymbol{\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0}$
- ✓ The probability of a normal random variable is given by the area under the curve so that the total area under the normal curve is equal to 1, the area on the right is equal to the area on the left and equal to 0.5
- ✓ There are several forms of the normal distribution (families) that differ in terms of mean μ and Standard Deviation σ or different from both

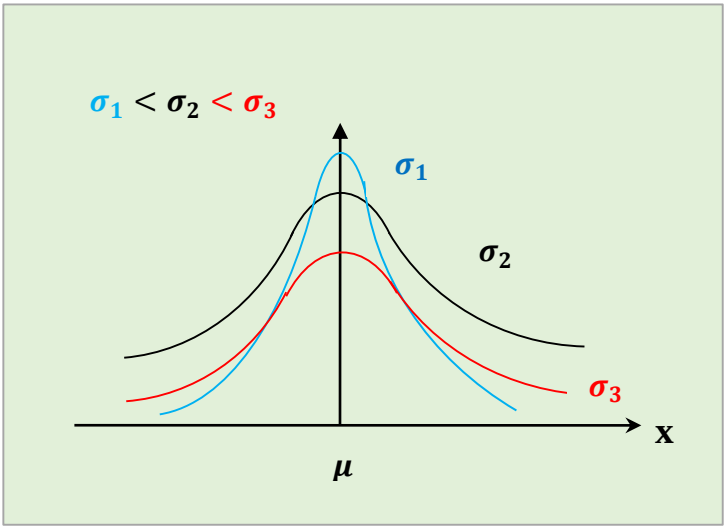
Figure 03 : infinite number of the normal distribution curves with same standard deviations σ and different means μ



✓ From the figure above, we observe that there are an infinite number of symmetrical curves that have the same standard deviation σ and differ in the value of the mean μ

✓ As for the figure below, we observe that there are an infinite number of the normal distribution curves with different standard deviations σ and the same means μ. The standard deviation of this distribution determines the degree of kurtosis of the curve. When the standard deviation is large, it means that the kurtosis is large and therefore the curve is wide and flattened, which means greater dispersion of the data**.** If the variance is less kurtosis, the curve will be less kurtosis, so there will be a higher peak of the curve and more convergence of the two ends of the curve**.**

Figure 04 : infinite number of the normal distribution curves with different standard deviations σ and same means μ



In general :

- 68.3% of the values of the natural random variable are in the range$[\boldsymbol{\mu - \sigma; \mu + \sigma}\ ]$

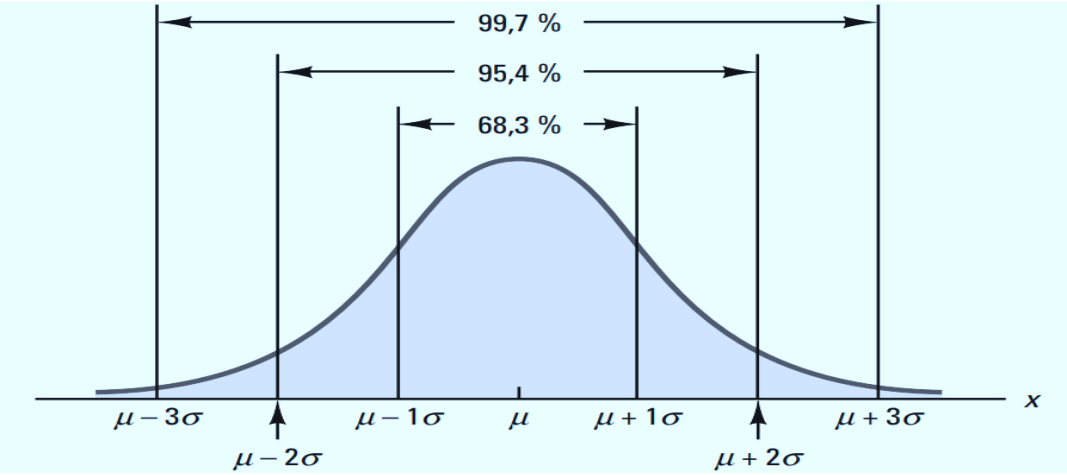
- 95.4% of the values of the natural random variable are in the range

$[\boldsymbol{\mu - 2\sigma; \mu + 2\sigma}\ ]$

- 99.7% of the values of the natural random variable are confined to the domain $[\boldsymbol{\mu - 3\sigma; \mu + 3\sigma}\ ]$

The figure below illustrates the above property:

Figure 05 : Area under the normal distribution curve



1-5- Standard Normal Distribution

If X is a random variable that follows the normal distribution $X \rightsquigarrow N(\boldsymbol{\mu}\,;\boldsymbol{\sigma}^2)$then :

The standard normal distribution is a continuous probability distribution with mean $\boldsymbol{\mu = 0}$ equal to zero and variance $\boldsymbol{\sigma}^2 = 1$. It is a normal distribution that is transformed into a standard normal distribution by replacing the random variable X with the standard random variable Z.

This distribution has the same properties as the previous normal distribution with some differences (including the mean and standard deviation).

$$\boldsymbol{Z = \frac{X - \mu}{\sigma}}$$

1-6- The probability density function of a standard normal distribution

Let Z be a standard random variable that follows a standard normal distribution with parameters $\boldsymbol{\mu = 0}$ and $\boldsymbol{\sigma = 1}$, and its probability density function is given by**:**

$$\boldsymbol{f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z)^2}} \qquad -\infty < z < +\infty$$

And we write : $Z \rightsquigarrow N(\mathbf{0}\,;1)$

1-7- Distribution Function (Cumulative Distribution Function)

If **Z** is a random variable that follows a standard normal distribution, the distribution function (cumulative probability) of this random variable F(z)=P(Z≤z) is given as :

$$\mathbf{F(z) = P(Z \le z) = } \frac{\mathbf{1}}{\sqrt{\mathbf{2\pi}}} \int_{-\infty}^{\mathbf{z}} \boldsymbol{e}^{-\frac{\mathbf{1}}{\mathbf{2}}(z)^2}\, \boldsymbol{dz}$$

- ✓ Like other continuous random variables, the probabilities of the normal distribution are obtained by the area under the normal curve of the probability density function. For the standard normal distribution, the areas under the curve are found in special statistical tables containing the cumulative probability for values of Z less than or equal to (or in other tables, greater than or equal to)**.**
- ✓ Assuming that $X \rightsquigarrow N(\boldsymbol{\mu}\,;\boldsymbol{\sigma}^2)$ and we want to calculate F(X)=P(X≤c) where c is a real number, the accumulated probability can be modelled by the Cumulative Distribution Function as follows:

$$\mathbf{F(X) = P(X \le c) = } \boldsymbol{P}\left(\frac{\boldsymbol{X-\mu}}{\boldsymbol{\sigma}} < \frac{\boldsymbol{c-\mu}}{\boldsymbol{\sigma}}\right)$$

$$= \boldsymbol{P}\left(\boldsymbol{Z} < \frac{\boldsymbol{X-\mu}}{\boldsymbol{\sigma}}\right) \rightsquigarrow \boldsymbol{Z} = \frac{\boldsymbol{X-\mu}}{\boldsymbol{\sigma}} \rightsquigarrow N(\mathbf{0}\,;1)$$

- ✓ Properties of the standard normal distribution include **:**

1) From the property of symmetry **:**

$$P(X < -c) = P(X > c)\ ;\ \ P(X < c) = P(X > -c)$$

2) The sum of the probabilities is equal to one **:**

$$P(X > c) + P(X \le c) = 1$$

$$\Rightarrow P(X > c) = 1 - P(X \le c)$$

3) $P(a < X < b) = P(a < X \le b) = P(a \le X < b)$

$$P(a \le X \le b) = \int_a^b f(x)\,dx = F(b) - F(a)$$

- ✓ **Moment Generating Function**

$$m_x(t) = e^{-\frac{1}{2}(t)^2}$$

1-8- The Normal Distribution Table (Cumulative Probability Table):

There are several situations in which it is important to know how to find the corresponding probability from statistical tables:

$\mathbf{P}(\boldsymbol{Z \leq a})$; $\mathbf{P}(\boldsymbol{Z > b})$; $\mathbf{P(c \leq Z \leq d)}$

Example 01:

-Find the Probability $P(Z \leq 1)$

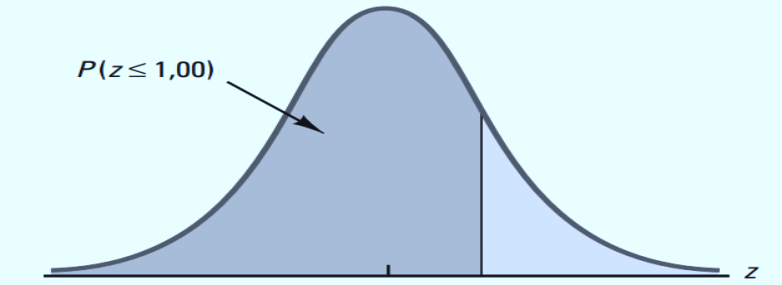
- Find the probability P(Z>1.58)

- Find the probability $P(-0.5 \leq Z \leq 1.25)$

Solution :

$\mathbf{P}(\boldsymbol{Z \leq 1}) =$ **?**

In order to find the probability of a standard random variable Z that follows the standard normal distribution to be less than 1, the accumulated probability corresponds to the area under the curve of the standard normal distribution to the left of the value 1 as shown in the following figure:



$\boldsymbol{\mu = 0}$ **1**

From the statistical table of the standard normal distribution **(Cumulative Probability Table)**, we find the probability corresponding to $\mathbf{P}(\boldsymbol{Z \leq 1})$ corresponding to the intersection of the column containing the value 1.0 with the line containing the value 0.00, where this intersection results in a probability value of 0.84134 so **:**

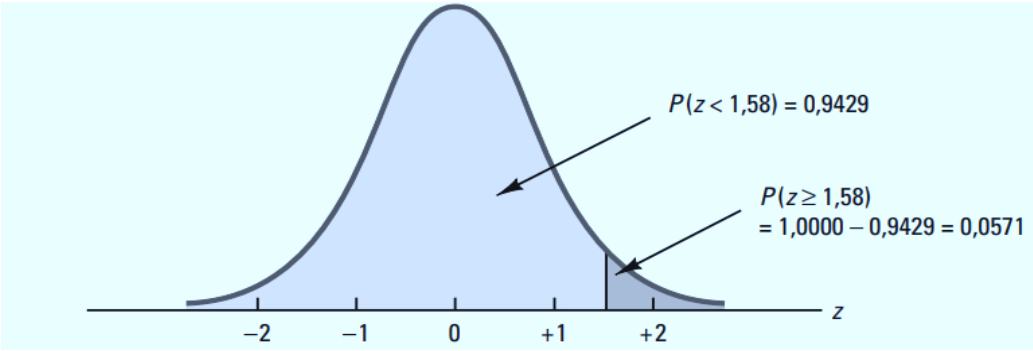
$$\mathbf{P}(\boldsymbol{Z \leq 1}) = \mathbf{0.8413}$$

Table 01: The Standard Normal Distribution Table

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
.										
.										
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
.										
.										
$\mathbf{P(Z \leq 1) = 0.8413}$										

P(Z>1.58)= ?

The required probability corresponds to the area under the standard normal distribution curve to the right of the value 1.58 as shown in the following figure :



✓ We know that : $P(Z \geq 1.58) + P(Z < 1.58) = 1$

$$P(Z \geq 1.58) = 1 - P(Z < 1.58)$$
$$= 1 - 0.9429 = 0.0571$$

From the statistical table of the normal distribution, we look for the probability value P(Z≤0.5) that corresponds to the intersection of the line corresponding to the value 1.5 and the column corresponding to the value 0.08 to get the probability **0.94295**

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
.										
.										
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	**0.94295**	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
.										
.										
									$\mathbf{P(Z \leq 1.58)}$	

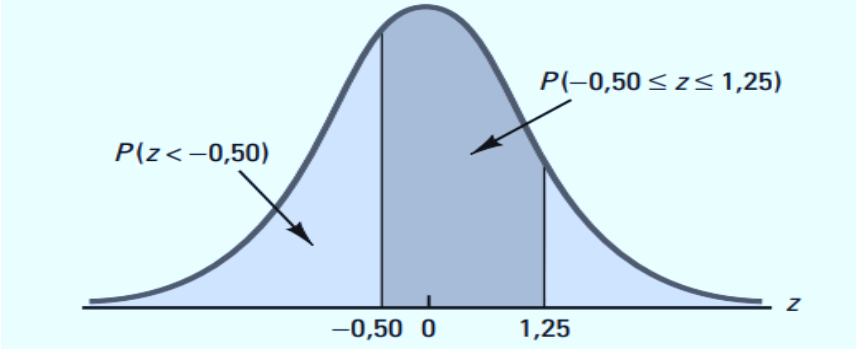
Note:

Only in the case of continuous distributions are the following probabilities equal:

$\mathrm{P}(Z \geq a) = \mathrm{P}(Z > a)$ and $\mathrm{P}(Z < a) = \mathrm{P}(Z \leq a)$

$\boldsymbol{P(-0.5 \leq Z \leq 1.25)}$ **= ?**

The probability of the random variable Z being between 0.5 and 1.25 corresponds to the area under the curve of the normal distribution between these values as shown in the following figure



- ✓ To find the value of the above probability, we subtract the area to the left of Z=1.25 from the area to the left of Z=0.5 so that
- ✓ By the symmetry property : $\mathrm{P}(Z \leq -0.5) = \mathrm{P}(Z \geq 0.5)$
- ✓ We also know that : $1 - P(Z \leq 0.5) = \mathrm{P}(Z > 0.5)$

So : $P(0.5 \leq Z \leq 1.25) = P(Z \leq 1.25) - P(Z \leq -0.5)$
$= P(Z \leq 1.25) - [1 - P(Z \leq 0.5)]$
$= 0.89435 - [1 - 0.69146] = 0.58581$

Example02:

Let X be a random variable that follows a normal distribution such that **:**

$\boldsymbol{X \rightsquigarrow N(0.6\,;4)}$

Find the following probabilities:

1) $P(X \leq -0.22)$

2) $P(X > 1.86)$

Solution:

$\boldsymbol{\sigma^2 = 4 \quad ; \sigma = 2 \qquad \mu = 0.6}$

$$\boldsymbol{Z = \frac{X - \mu}{\sigma} \rightsquigarrow N(0\,;1)}$$

$$P(X < -0.22) = P\left(\boldsymbol{Z < \frac{-0.22 - 0.6}{2}}\right) = P(\boldsymbol{Z < -0.41})$$
$$= P(\boldsymbol{Z > 0.41}) = 1 - P(\boldsymbol{Z \leq 0.41})$$
$$1 - 0.6591 = \boldsymbol{0.3409}$$

$$P(X > 1.86) = P\left(\boldsymbol{Z > \frac{X - 0.6}{2}}\right) = P\left(\boldsymbol{Z > \frac{1.86 - 0.6}{2}}\right)$$
$$= P(\boldsymbol{Z > 0.63}) = 1 - P(\boldsymbol{Z \leq 0.63})$$

$$1 - 0.7356 = \boldsymbol{0.2644}$$

Example03:

It is known that the lamp life of a brand follows a normal distribution with mean $\mu = 100\ h$ and standard deviation $\sigma = 8$ h , what is the probability that the lifetime of a randomly selected lamp is between 110 and 120 hours?

Solution:

Let X be a random variable that follows the normal distribution representing the lifetime of lamps such that $\boldsymbol{X \rightsquigarrow N(100\,;64)}$

$$\boldsymbol{\sigma^2 = 8^2 = 64} \qquad \boldsymbol{\sigma = 8h}$$

$$\boldsymbol{Z = \frac{X-\mu}{\sigma} \rightsquigarrow N(0\,;1)}$$

$$P(110 \le X \le 120) = P\left(\frac{\mathbf{110-100}}{\mathbf{8}} \le \boldsymbol{Z} \le \frac{\mathbf{120-100}}{\mathbf{8}}\right)$$
$$= P(\mathbf{1.25} < \boldsymbol{Z} < \mathbf{2.5})$$
$$= P(\boldsymbol{Z} \le 2.5) - P(\boldsymbol{Z} \le 1.25)$$
$$= 0.9937 - 0.8943 = 0.103$$

Example 04:

If it is known that the mean weight of 500 students is 151 pounds with a standard deviation of 15, assuming that the weights are normally distributed. Find the proportion and number of students whose weight is:

1) between 119.5 and 155.5 pounds?
2) More than 185.5 pounds ?

Solution:

$$X \rightsquigarrow N(151\,;15^2)$$

1) The percentage and number of students who weigh between 119.5 and 155.5 pounds:

$$P(119.5 \le X \le 155.5) = P\left(\frac{119.5-151}{15} \le Z \le \frac{155.5-151}{15}\right)$$
$$= P(-2.1 \le Z \le 0.3)$$
$$= P(Z \le 0.3) - P(Z \le -2.1)$$
$$= P(Z \le 0.3) - [1 - P(Z \le 2.1)]$$
$$= P(Z \le 0.3) - [1 - P(Z \le 2.1)]$$
$$= 0.6179 - [1 - 0.9821]$$

$$P(119.5 \leq X \leq 155.5) = 0.6$$

Sixty percent (%60) of the total number of students (500 students) weigh between 119.5 and 155.5 pounds.

Their number is: (500) (0.60005) =300

2) The percentage and number of students who weigh more than 185.5 pounds:

$$P(X > 185.5) = P\left(Z > \frac{185.5 - 151}{15}\right) = P(Z > 2.3)$$

$$= [1 - P(Z \leq 2.3)]$$

$$= [1 - 0.9892] = 0.0108$$

$$P(X > 185.5) = 0.0108$$

Sixty percent (1%) of the total number of students (500 students) weigh more than 185.5 pounds.

Their number is: (500) (0.0108) =5.4≈5

2- The Uniform Distribution

2-1- Definition

Suppose that a continuous random variable X takes values in a finite interval [a, b]. In many cases, it will be reasonable to assume that, any subinterval of [a, b] with a fixed length, has the same probability that X falls into it. we say that "X takes any value in [a, b] with the same probability, So, we define The uniform distribution as a continuous probability law defined over the interval [a-b]∈R and characterized by a constant probability density function for all real values.

Although the uniform distribution appears to be the simplest continuous distribution, it is not a widely used distribution for modeling real-life phenomena since it is rare in practice to have an unknown quantity that takes values in a finite interval and assume that its probability density is constant. In fact, the normal distribution is the most widely used probability distribution in many fields of application.

If no other information about a random variable X is known ,The uniform probability distribution is useful as a "first guess", Also, is used in real-world problems that have uniform behavior in a given interval.

2-2- Probability Density Function

Let X be a continuous random variable with density:

$$f(x) = \begin{cases} \dfrac{1}{b-a} & ; \quad a \le x \le b \\ 0 & ; \ \textbf{Otherwise} \end{cases}$$

We then say that X has the uniform distribution over the interval [a, b]. We denote this $X \rightsquigarrow U([a-b])$

2-3- Distribution Function

Let X is a random variable that follows a uniform distribution such that $X \rightsquigarrow U([a-b])$, the distribution function of this random variable F(x)=P(X≤x) is given as follows:

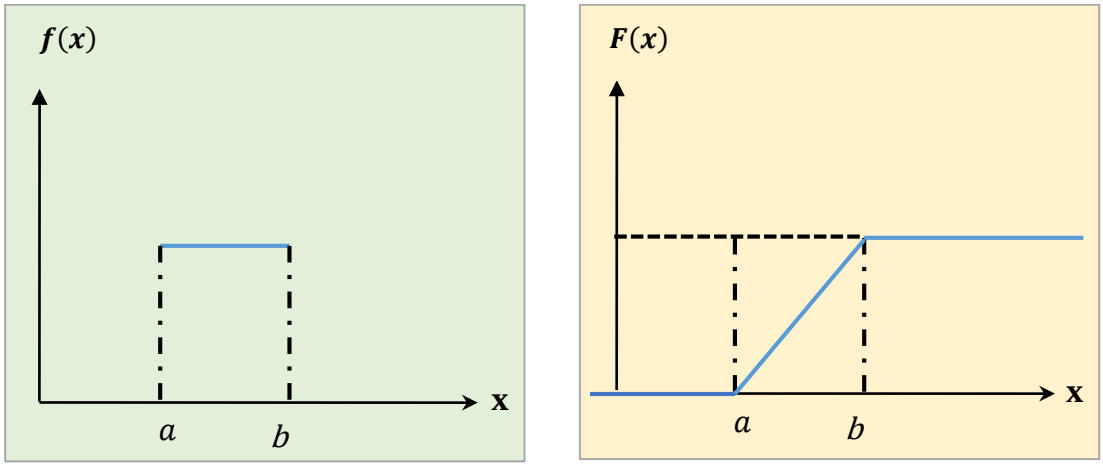
$$\mathbf{F(x)} = \boldsymbol{P(X \le x)} = \begin{cases} 0 & ; \quad x < 0 \\ \frac{x-a}{b-a} & ; \quad \boldsymbol{a \le x < b} \\ 1 & ; \quad \boldsymbol{x \ge b} \end{cases}$$

2-4- Uniform Distribution Properties

✓ $\boldsymbol{E(X) = \frac{b+a}{2}}$, $\boldsymbol{V(X) = \frac{(b-a)^2}{12}}$

✓ The probability density function f(x) and the distribution function F(x) of a random variable that follows a uniform distribution

Figure 06 : The probability density function and the distribution function



a) Two random variables (independent or not) that follow a uniform distribution on the interval $[a - b]$ whose sum does not follow a uniform distribution

✓ **Moment Generating Function**

$$m_x(t) = E(e^{tx}) = \int_b^a e^{tx} \frac{\mathbf{1}}{\boldsymbol{b-a}} dx = \frac{\mathbf{1}}{\boldsymbol{b-a}} \left[\frac{e^{bt} - e^{at}}{\boldsymbol{t}}\right]$$

$$m_x(t) = \frac{e^{bt} - e^{at}}{(b-a)t} \quad , \quad \boldsymbol{t \neq 0}$$

Example 01:

Let X be a random variable representing the duration of a flight in minutes between Oran and Algiers in the interval [120-140], and let all domains with the same amplitude and belonging to the previous intervals have the same probability , Find:

a) The type of probability distribution for X and its mathematical formula.

b) The probability that the duration of the flight between Oran and Algiers is between 128 and 136 minutes and then between 120 and 130 minutes?

c) The expectation and variance of the duration of the flight?

Solution :

a) The type of probability distribution for X and its mathematical formul

The appropriate probability distribution for the random variable X is the uniform distribution(continuous distribution) because all Secondary intervals (in minutes) with the same amplitude and belonging to the interval[120-140] have the same probability , Then $\boldsymbol{X \rightsquigarrow U([120-140])}$ and the formula for the probability density function is given as follows:

$$f(x) = \begin{cases} \dfrac{1}{20} & ; \quad 120 \leq x \leq 140 \\ 0 & ; \quad \text{Otherwise} \end{cases}$$

b) The probability that the duration of the flight between Oran and Algiers is between :

- ✓ **128 and 136 minutes**

$$P(c \leq X \leq d) = \int_c^d f(x)dx = \int_c^d \frac{1}{b-a}\,dx = \left[\frac{1}{b-a}(d-c)\right]$$

$$P(128 \leq X \leq 136) = \left[\frac{1}{20}(136-128)\right] = 0.4$$

- **120 and 130 minutes**

$$P(120 \leq X \leq 130) = \left[\frac{1}{20}(130 - 120)\right] = 0.5$$

c) The expectation and variance for the duration of a flight

$$\boldsymbol{E(X)} = \frac{b+a}{2} = \frac{120+140}{2} = 130 \quad , \quad \boldsymbol{V(X)} = \frac{\boldsymbol{(b-a)^2}}{\boldsymbol{12}} = \frac{(140-120)^2}{12} = 33.33$$

Example02

It is known that the number of customers arriving at the checkout counter of a store follows a Poisson distribution, such that in a given interval of 30 minutes, one customer comes to the checkout counter.

- ✓ Find the probability that a customer comes to the checkout counter during the last 05 minutes of a 30-minute period.

Solution :

Let X be a random variable representing the time customers come to the checkout counter that follows a uniform distribution over the range Then $\boldsymbol{X \rightsquigarrow U([0-30])}$

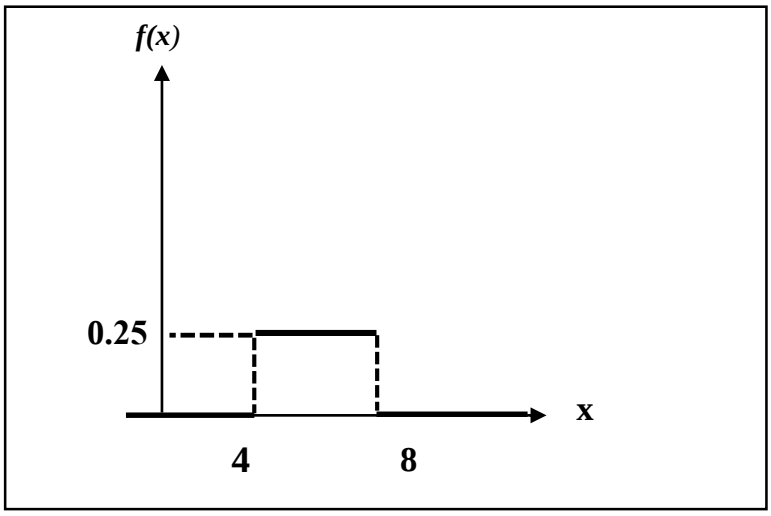
$$\boldsymbol{f(x)} = \begin{cases} \dfrac{\boldsymbol{1}}{\boldsymbol{30}} & ; \quad \boldsymbol{x \in [0-30]} \\ \boldsymbol{0} & \boldsymbol{;\ Otherwise} \end{cases}$$

$$P(25 \leq X \leq 30) = \int_{25}^{30} f(x)dx = \int_{25}^{30} \frac{1}{30}\,dx = \left[\frac{1}{30}(30-25)\right] = \frac{1}{6}$$

So the probability of a customer arriving within an interval of 05 minutes is $\frac{1}{6}$

Example03

Let X be a random variable with a probability density function represented in the accompanying figure.



a) From the above figure, deduce the probability distribution followed by the random variable x and specify the mathematical formula for this probability distribution?

b) Find the cumulative distribution function, then find the expected value and variance of the random variable X ?

c) Calculate the following probabilities:

$$P(X > 5) \quad ; P(X \leq 6)$$
$$P(X \leq 5); \quad ; P\ (4 \leq X \leq 6\ )$$

Solution :

a) Type and form of the probability distribution

The appropriate distribution is the uniform distribution because for all real values in the range [4-8], the probability is constant (the probability density function is constant) , Then $\boldsymbol{X \rightsquigarrow U([4-8])}$ and the probability density function is given by:

$$f(x) = \begin{cases} \frac{1}{4} & ; \quad 4 \leq x \leq 8 \\ 0 & ; \text{ Otherwise} \end{cases}$$

b) The distribution function, expected value, and variance for the random variable x

$$\mathbf{F(X)} = \boldsymbol{P(X \le x)} = \begin{cases} \mathbf{0} & ; \quad \boldsymbol{x < 0} \\ \frac{\boldsymbol{x-a}}{\boldsymbol{b-a}} & ; \quad \boldsymbol{a \le x < b} \\ \mathbf{1} & ; \quad \boldsymbol{x \ge b} \end{cases}$$

$$\mathbf{F(X)} = \boldsymbol{P(X \le x)} = \begin{cases} \mathbf{0} & ; \quad \boldsymbol{x < 0} \\ \frac{\boldsymbol{x-4}}{\boldsymbol{4}} & ; \quad \boldsymbol{4 \le x < 8} \\ \mathbf{1} & ; \quad \boldsymbol{x \ge 8} \end{cases}$$

$$\mathbf{F(X)} = P(X \le x) = \int_{-\infty}^{x} f(x)dx$$

$$\mathrm{F(X)} = P(X \le x) = \int_4^x \frac{1}{4}dx \int_4^x \frac{1}{4}dx$$

$$= \left[\frac{1}{4}x\right]_4^x = \frac{1}{4}x - \frac{4}{4} = \frac{x-4}{4}$$

$$\boldsymbol{E(X)} = \frac{b+a}{2} = \frac{8+4}{2} = \mathbf{6} \quad ; \quad \boldsymbol{V(X)} = \frac{\boldsymbol{(b-a)^2}}{\mathbf{12}} = \frac{(8-4)^2}{12} = \frac{\mathbf{16}}{\mathbf{12}}$$

c) Calculating the probability values:

$$\boldsymbol{P(X \le 5)} = \int_4^5 \frac{\mathbf{1}}{\mathbf{4}}dx = \left[\frac{\mathbf{1}}{\mathbf{4}}\boldsymbol{x}\right]_4^5 = \frac{\mathbf{1}}{\mathbf{4}}(\mathbf{5}) - \frac{\mathbf{1}}{\mathbf{4}}(\mathbf{4}) = \frac{\mathbf{1}}{\mathbf{4}}$$

Or directly through the Distribution Function, we find:

$$P(X \le 5) = \frac{\boldsymbol{x-4}}{\mathbf{4}} = \frac{\mathbf{5-4}}{\mathbf{4}} = \frac{\mathbf{1}}{\mathbf{4}}$$

$$\mathbf{P(}\boldsymbol{X} > \mathbf{5)} = 1 - P(X \le 5) = \ 1 - \frac{1}{4} = \frac{3}{4} = 0.75$$

$$\mathbf{P(}\boldsymbol{X} \le \mathbf{6)} = \frac{\boldsymbol{x-4}}{\mathbf{4}} = \frac{\mathbf{6-4}}{\mathbf{4}} = \frac{\mathbf{2}}{\mathbf{4}} = \frac{\mathbf{1}}{\mathbf{2}}$$

$$\mathbf{P(4 \le X \le 6)} = \int_4^6 \frac{\mathbf{1}}{\mathbf{4}}\, dx = \left[\frac{\mathbf{1}}{\mathbf{4}}\boldsymbol{x}\right]_4^6 = \frac{\mathbf{1}}{\mathbf{4}}(\mathbf{6}) - \frac{\mathbf{1}}{\mathbf{4}}(\mathbf{4}) = \frac{\mathbf{2}}{\mathbf{4}} = \frac{\mathbf{1}}{\mathbf{2}}$$

Or directly through the distribution function, we find:

$$\mathbf{P(4 \le X \le 6)} = P(X \le 6) - P(X \le 4) = \frac{\mathbf{6-4}}{\mathbf{4}} - \frac{\mathbf{4-4}}{\mathbf{4}} = \frac{\mathbf{2}}{\mathbf{4}} = \frac{\mathbf{1}}{\mathbf{2}}$$

Example04

University Student goes to his University class every weekday by using a train that leaves at 8:00 a.m. We assume that the duration of the journey, in minutes is a random variable following the uniform distribution in the interval [58, 63]. Further, suppose that from the train platform he needs a 15-minute walk to enter the classroom and that his class starts precisely at 9:15 a.m, Let X The duration of the train journey. Find :

a) The Probability Density Function and The Distribution Function of X ?
b) The probability that the student arrives in time for his class ?
c) The probability that the student arrives for his class at least two minutes after lesson starts?
d) The expected time that the student arrives for his class?

Solution

The duration of the train journey is the only source of uncertainty , we define a random variable X, which represents the number of minutes that this journey lasts

Then $\boldsymbol{X \rightsquigarrow U([58-63])}$,

a) The Probability Density Function and The Distribution Function

X has density function is given by:

$$\boldsymbol{f(x)} = \begin{cases} \dfrac{\mathbf{1}}{\mathbf{63-58}} & ; \quad \mathbf{58 \le x < 63} \\ \mathbf{0} & \mathbf{;\ Otherwise} \end{cases}$$

while the corresponding distribution function of X is given by :

$$\mathbf{F(x)} = \boldsymbol{P(X \le x)} = \begin{cases} \mathbf{0} & ; \quad \boldsymbol{x < 58} \\ \frac{\boldsymbol{x-58}}{\mathbf{63-58}} = \frac{\boldsymbol{x-58}}{\mathbf{5}} & ; \quad \mathbf{58 \le x < 63} \\ \mathbf{1} & ; \quad \boldsymbol{x \ge 63} \end{cases}$$

b) The probability that the student arrives in time for his class

For the student to be in time (i.e. before 9:15 a.m.) for his class, he must arrive at the platform by 9:00 a.m., which means that the train journey lasts for 60 minutes at most. The probability for this is :

$$P(X \le 60) = \frac{60 - 58}{5} = \frac{2}{5}$$

c) The probability that the student arrives for his class at least two minutes after lesson starts

we seek the probability that Simon arrives at the classroom after 9:17 a.m., which in turn implies that he arrives at the platform after 9:02 a.m. This happens if the train journey takes 62 minutes or more, with associated probability

$$P(X > 62) = 1 - P(X \le 62) = 1 - F(62) = 1 - \frac{62 - 58}{5} = \frac{1}{5}$$

There is a **20%** chance that he arrives at least two minutes late for the class

d) The expected time that the student arrives for his class

$$\boldsymbol{E(X)} = \frac{63 + 58}{2} = 60.5$$

This implies that Simon is expected to arrive at the train platform of the University station half a minute after 9:00 a.m. and, as a result, his expected arrival time at the classroom is half a minute after the beginning of the class (9:15 a.m.).

3- The Exponential Distribution

3−1− Definition

If the occurrence of events follows a Poisson distribution, then the waiting time for an event to occur and the time between two events follow an **exponential distribution**. For variables that are assumed to take only positive values, the exponential distribution is the most commonly used. The exponential law corresponds to the time measured between independent events that follow a Poisson distribution.

This type of probability distribution is often used in in matters related to the waiting time for an event to occur, such as the amount of time a customer waits in a restaurant or bank before being served, the time it takes for a certain load to be loaded on a truck, Or in other words The time required for the next bus to arrive at the bus station, the amount of time a machine or device remains operational before it malfunctions, the time between the arrival of two customers at a store or bank, the duration of a phone call, the time it takes for a patient to recover from a surgery…

3-2- Probability density function

Let X be a random variable follow an exponential distribution where its probability density function is given by:

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & ; \quad x \geq 0 \\ 0 & ; \quad x < 0 \end{cases}$$

For λ>0 we say that X follows an exponential distribution with parameter λ and we write : $X \rightsquigarrow EXP(\lambda)$

3-3- Distribution function

X is a random variable that follows an exponential distribution with parameter **λ** such that $X \rightsquigarrow EXP(\lambda)$, the distribution function of this random variable F(x)=P(X≤x) is given as follows:

$$\mathbf{F(x)} = \boldsymbol{P(X \le x)} \begin{cases} \mathbf{0} & ; \quad \boldsymbol{x < 0} \\ \mathbf{1 - e^{-\lambda x}} & ; \ \boldsymbol{x \ge 0} \end{cases}$$

Proof :

$$\mathbf{F(X)} = \int_{-\infty}^{x} \boldsymbol{f(x)dx}$$

$$\mathbf{F(X)} = \int_{0}^{x} \boldsymbol{\lambda\, e^{-\lambda x}dt} = -\int_{0}^{x} \boldsymbol{-\lambda\, e^{-\lambda x}}\, dt = [-e^{-\lambda x}]_0^x = (1 - e^{-\lambda x})$$

f(x)= $\grave{u}\boldsymbol{e^{u(x)}}$ F(x)= $\boldsymbol{e^{u(x)}}$

3-4- Properties of Exponential Distribution

- ✓ $E(X) = \frac{1}{\lambda}$, $V(X) = \frac{1}{\lambda^2}$

- ✓ **Moment Generating Function**

$$m_x(t) = E(e^{tx}) = \int_0^{\infty} e^{tx}\, \boldsymbol{\lambda\, e^{-\lambda x}}\, dx = \boldsymbol{\lambda} \int_0^{\infty} e^{-x(\lambda - t)}\ dx$$

$$\boldsymbol{m_x(t)} = \left(\mathbf{1} - \frac{\boldsymbol{t}}{\boldsymbol{\lambda}}\right)^{-1} = \left(\frac{\boldsymbol{\lambda}}{\boldsymbol{\lambda - t}}\right) , \qquad \boldsymbol{t < \lambda}$$

- ✓ **Relationship between the Poisson and Exponential Distributions**

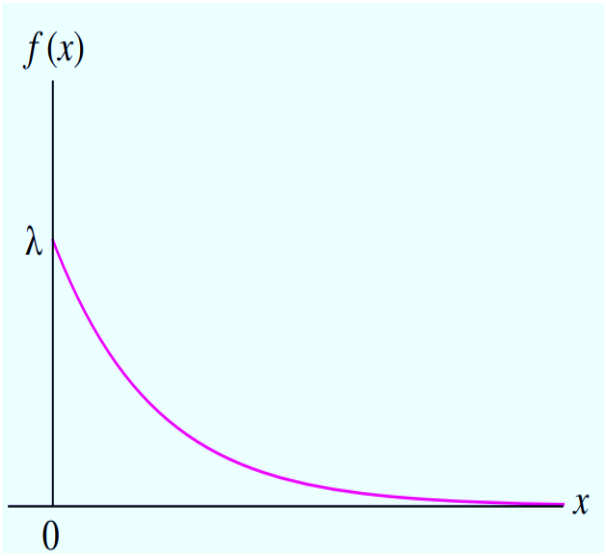
Assuming that the number of cars that come to the car wash follows a Poisson distribution with an average of 10 cars per hour, the probability density function that indicates x cars coming in during the hour is:

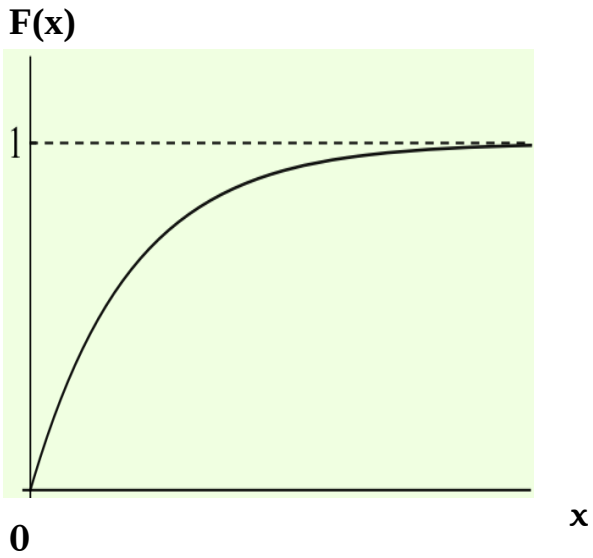
$$\boldsymbol{f(x)} = \frac{\boldsymbol{e^{-10}}\ \ 10^x}{\boldsymbol{x!}} \quad ; \quad \boldsymbol{x = 0, 1, 2 \ldots \infty}$$

Because the average number of cars entering the wash station is 10 cars in an hour, λ = 10 , Therefore, the average time between the arrival of one car and another car is equal to:

$$E(X) = \frac{\mathbf{1}}{\boldsymbol{\lambda}} = \frac{\mathbf{1}}{\mathbf{10}} = \mathbf{0.1}$$

Figure 07 : The probability density function and the distribution function of the exponential distribution





The exponential probability distribution describing the time between the arrival of one car and another has a mean of **1/10** hour per car and is given by the probability density function of the form :

$$\boldsymbol{f(x) = 10\, e^{-10x}} \qquad ; \quad \boldsymbol{x \geq 0}$$

Example01:

Assuming that the time required to load a given commodity onto a truck at a factory following an exponential distribution, if the average time required to load the commodity in the truck is 15 minutes.

1) Find the formula for the probability density function f(x) for the time to load the commodity in the truck ?
2) Find the distribution function F(x)?
3) Find the probability of loading the commodity in the truck in a time of :

- At most 06 minutes?
- At most 18 minutes?
- Between 06 and 18 minutes?

Solution:

Let X be a random variable representing the time required to load the good onto the truck

1) The probability density function

$$E(X) = \frac{1}{\lambda} = 15 \Rightarrow \lambda = \frac{1}{15}$$

$$f(x) = \begin{cases} \frac{1}{15} e^{-\frac{1}{15}x} & ; \quad x \geq 0 \\ 0 & ; \quad x < 0 \end{cases}$$

2) The distribution function F(x)

$$F(x) = P(X \leq x) \begin{cases} 0 & ; \quad x < 0 \\ 1 - e^{-\frac{1}{15}x} & ; \quad x \geq 0 \end{cases}$$

3) The probability of loading the commodity onto the truck in a time :

- At most 06 minutes

To calculate the probabilities, we use the previous cumulative distribution function to obtain the probability of any value less than a given value

$$P(X \leq 6) = 1 - e^{-\frac{1}{15}(6)} = 0.3297$$

- **At most 18 minutes**

$$P(X > 18) = 1 - \big(P(X < 18)\big) = 1 - \left(1 - e^{-\frac{1}{15}(18)}\right)$$
$$= 1 - (0.3297) = 0.6703$$

- **Between 06 and 18 minutes**

$$P(6 \leq X \leq 18) = P(X \leq 18) - P(X \leq 6)$$
$$= 0.6988 - 0.3297 = 0.3691$$

Note :

If events or successes follow a Poisson distribution, the probability of the first event occurring within a certain time interval t can be found, in the sense that :

$$P(T < t) = \mathbf{1} - \boldsymbol{e}^{-\lambda x}$$

λ: is the average number of occurrences during the time interval "t"

$$\boldsymbol{E(T)} = \frac{\mathbf{1}}{\boldsymbol{\lambda}} \quad , \quad \boldsymbol{V(X)} = \frac{\mathbf{1}}{(\boldsymbol{\lambda})^{\mathbf{2}}}$$

- ✓ We know that the formula for Poisson's law is of the form :

$$\boldsymbol{f(x)} = \frac{\boldsymbol{e}^{-\lambda}\ \lambda^{x}}{\boldsymbol{x}!} \quad ; \ \boldsymbol{x = 0, 1, 2 \ldots \infty}$$

- ✓ The probability of a single event x=1 occurring in time t is:

$$\boldsymbol{f(1)} = \boldsymbol{e}^{-\lambda}\ \lambda$$

The probability that a customer arrives within time t, for example, within one hour is equal to:

A single event x=1 in time t is : $\boldsymbol{P(T < t) = \ 1 - \ e^{-\lambda}}$

Example02:

Assuming that the length of a telephone call in the reception center of a commercial organization is a random variable X that follows an exponential distribution

$$\boldsymbol{f(x)} = \begin{cases} \mathbf{0} & ; \ \boldsymbol{x < 0} \\ \boldsymbol{C\, e^{-\frac{1}{10}x}} & ; \ \boldsymbol{x \geq 0} \end{cases}$$

If a call to the call center requires a certain waiting time before the call can be made, Find

1) The value of the constant **"c "** ?
2) The distribution function **F(x)** ?
3) The probability of someone waiting more than 10 minutes, between 10 and 20 minutes?
4) The expectation and variance of the waiting time **E(X)** , **Var(X)** ?

Solution:

1) the value of the constant c

Since a probability density function f(x) satisfies the property :

$$\int_{-\infty}^{\infty} \boldsymbol{f(x)dx = 1}$$

$$\int_{-\infty}^{\infty} f(x)dx = \int_{0}^{\infty} c\, e^{-\frac{1}{10}x}dx = -10c\left[e^{\frac{1}{10}x}\right]_{0}^{\infty} = -10c[e^{-\infty} - e^{0}] = 10c = 1$$

$$\boldsymbol{c = \frac{1}{10}}$$

$$\boldsymbol{f(x)} \;=\; \begin{cases} 0 & ; \quad x < 0 \\ \boldsymbol{\frac{1}{10}\, e^{-\frac{1}{10}x}} & ; \; x \geq 0 \end{cases}$$

1) **The distribution function F(x)**

$$\boldsymbol{F(x) = P(X < x) = 1 - \, e^{-\lambda x}}$$

$$\boldsymbol{F(x) = 1 - \, e^{-\frac{1}{10}x}}$$

2) **The probability of someone waiting more than 10 minutes, between 10 and 20 minutes**

$$P(X \geq 10) = \int_{10}^{\infty} \frac{1}{10}\, e^{-\frac{1}{10}x}dx = \left[-e^{\frac{1}{10}x}\right]_{10}^{\infty} = -(e^{\infty} - e^{1}) = -e^{1} = 0.3679$$

Or We have a distributive function relationship of the form :

We have : $\boldsymbol{P(X \leq x) = 1 - \, e^{-\lambda x}}$

$$\boldsymbol{P(X > x) + P(X \leq x) = 1}$$
$$\boldsymbol{\Rightarrow P(X > x) = 1 - P(X \leq x)}$$
$$\boldsymbol{P(X > 10) = 1 - \, P(X \leq 10)}$$
$$\boldsymbol{= 1 - \left[1 - \, e^{-\frac{1}{10}(10)}\right]}$$
$$= e^{-1} = 0.3678$$

$$\boldsymbol{P(10 \le X \le 20)} = \int_{10}^{20} \frac{1}{10}\, e^{-\frac{1}{10}x} dx = \left[-e^{\frac{1}{10}x}\right]_{10}^{20} = -(e^2 - e^1) = 0.2326$$
$$= 0.3679$$

Or, using the distribution function, we find:

$$\boldsymbol{F(x) = P(X \le x) = 1 - e^{-\lambda x}}$$

$$\mathbf{P(a \le X \le b)} = \int_a^b f(x)\, dx = F(b) - F(a)$$
$$= P(\mathrm{X} \le b) - P(\mathrm{X} \le a)$$
$$= \left[\mathbf{1} - e^{-\lambda b}\right] - \left[\mathbf{1} - e^{-\lambda a}\right]$$

So:

$$P(10 \le X \le 20) = F(20) - F(10)$$
$$= P(\mathrm{X} \le 20) - P(\mathrm{X} \le 10)$$
$$= P(\mathrm{X} \le 20) - P(\mathrm{X} \le 10)$$
$$= \left[\mathbf{1} - e^{-\frac{1}{10}(\mathbf{20})}\right] - \left[\mathbf{1} - e^{-\frac{1}{10}(\mathbf{10})}\right]$$
$$= -0.1353 + 0.3678 = 0.2325$$

$$P(10 \le X \le 20) = 0.2325$$

3) **The expectation and variance of the waiting time E(X) , Var(X)**

$$\boldsymbol{E(X)} = \frac{1}{1/10} = 10 \quad , \quad \mathbf{Var(X)} = \frac{1}{(1/10)^2} = 100$$

Example03:

Assuming that x is a random variable representing the time in hours required to repair a production machine, which follows an exponential distribution, and it is known that the average time required to repair the production machine is 0.2 hours. Find :

a) The probability density function f(x) ?

b) The cumulative distribution function F(x) ?

c) The probability that the repair time of the production machine **exceeds Two hours**?

d) The probability that the repair time of the production machine is **less than one hour**?

Solution:

a) The probability density function f(x)

X follows the exponential distribution $\boldsymbol{X \rightsquigarrow EXP(\lambda)}$ where its probability density function is given as:

$$\boldsymbol{E(X) = 2 = \frac{1}{\lambda} \Longrightarrow \lambda = \frac{1}{2}}$$

$$\boldsymbol{f(x)} = \begin{cases} \boldsymbol{\frac{1}{2}\, e^{-\frac{1}{2}x}} & ; \quad \boldsymbol{x \geq 0} \\ 0 & ; \quad \boldsymbol{x < 0} \end{cases}$$

b) The cumulative distribution function F(x)

$$\mathbf{F(X)} = \begin{cases} \mathbf{0} & ; \quad \boldsymbol{x < 0} \\ \mathbf{1 - e^{-\frac{1}{2}x}} & ; \quad \boldsymbol{x \geq 0} \end{cases}$$

c)

$$P(X > 2) = 1 - P(X \leq 2) = 1 - \left(1 - e^{-\lambda x}\right) = \mathbf{1} - \left(1 - \boldsymbol{e^{-\frac{1}{2}(2)}}\right) = \boldsymbol{e^{-1}}$$
$$= \mathbf{0.3678}$$

Or in another form:

$$P(X > 2) = \int_2^{\infty} \boldsymbol{\frac{1}{2} e^{-\frac{1}{2}x} dx} = \left[-\boldsymbol{e^{-\frac{1}{2}x}}\right]_2^{\infty} = -\left(0 - \boldsymbol{e^{-\frac{1}{2}(2)}}\right)$$
$$= (\boldsymbol{e^{-1}}) = \mathbf{0.3678}$$

d)

$$P(X \leq 1) = \left(1 - \boldsymbol{e^{-\frac{1}{2}(1)}}\right) = \mathbf{1} - \left(\boldsymbol{e^{-\frac{1}{2}}}\right)$$
$$= 1 - 0.6065 = \mathbf{0.3934}$$

3- *The Gamma Distribution*

4-1-Probability Density Function

A random variable X has the Gamma distribution with parameters $\alpha > 0$ and $\lambda > 0$ if its density function is given by :

$$f(x) = \begin{cases} \dfrac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} & ; \ x \geq 0 \\ 0 & ; \ x < 0 \end{cases}$$

And we write : $X \rightsquigarrow \gamma(\alpha;\lambda)$

For $\alpha > 0$ **The Gamma Function $\Gamma(\alpha)$** is given by :

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x}\, dx$$

4-3- Distribution function

X is a random variable that follows an Gamma Distribution with parameters α , λ such that, the distribution function of this random variable F(X)=P(X≤x) is given as follows:

$$\mathrm{F(x)} = P(X \leq x) = \int_0^{x} f(x)dx$$

$$\mathbf{F(x)} = \boldsymbol{P(X \leq x)} \begin{cases} 0 & ; \ X < 0 \\ 1 - e^{-\lambda x} \sum_{i=1}^{\alpha} \dfrac{(\lambda x)^{i-1}}{\Gamma(i)} & ; \ x \geq 0 \end{cases}$$

4-2- Properties of the Gamma Distribution

- ✓ **The expectation and variance**

$$E(X) = \frac{\alpha}{\lambda} \quad , \quad V(X) = \frac{\alpha}{(\lambda)^2}$$

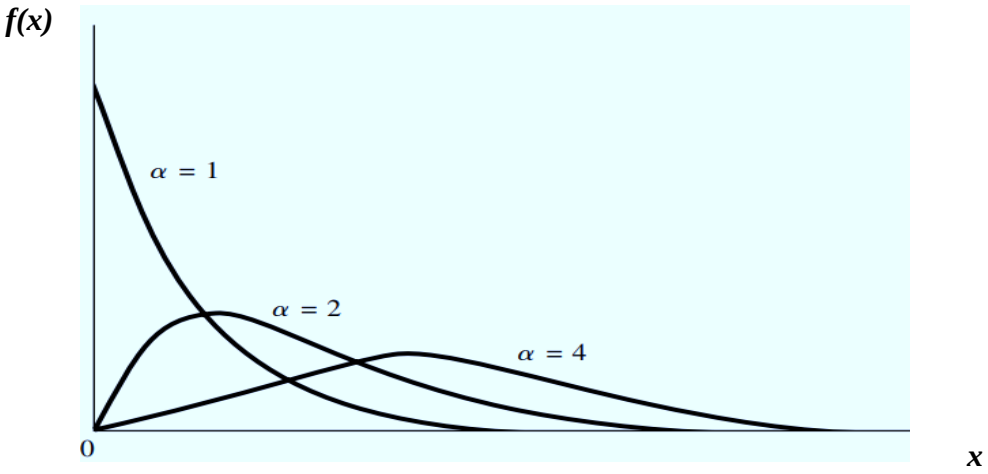
- ✓ **Moment Generating Function**

$$m_x(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} \ \frac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$$

$$m_x(t) = \left(1 - \frac{t}{\lambda}\right)^{-\alpha} = \left(\frac{\lambda}{\lambda - t}\right)^{\alpha} \quad , t < \lambda$$

- ✓ The parameter **α** is the shape parameter and if **α=1** then the exponential distribution is a special case of the ***gamma*** distribution. For different values of α (α=1, α=2, α=4, and λ=1), shown in the figure below, the shape of the density function of ***gamma*** varies for different values of α (α is sometimes called the shape parameter of the ***gamma*** distribution and $\frac{1}{\lambda}$ is called the scale parameter.

Figure08 : The density function of the Gamma distribution for different values of α with , λ = 1



- ✓ In a Poisson chain of events, if the time between successive failures in a given system follows an exponential distribution, then the accumulated time for λ failures follows the ***Gamma*** Distribution **X⇝ Γ (α;λ)**
- ✓ For different values of **α** (an integer) we find the values of the γ-function corresponding to different values of α by using the integral by parts of the ***gamma*** function.

$$\Gamma(1) = \int_0^{\infty} x^{1-1}\, e^{-x}\, dx = [-e^{-x}]_0^{\infty} = 1$$

$$\Gamma(\alpha + 1) = \int_0^{\infty} x^{\alpha}\, e^{-x}\, dx = \int_0^{\infty} x^{\alpha} \left(\frac{d(-e^{-x})}{dx}\right) dx$$

$$= [-\boldsymbol{x^{\alpha}e^{-x}}]_{\boldsymbol{0}}^{\boldsymbol{\infty}} - \int_{\boldsymbol{0}}^{\boldsymbol{\infty}} \boldsymbol{\alpha x^{\alpha-1}}\,(-\boldsymbol{e^{-x}})\,\boldsymbol{dx} = \boldsymbol{0} + \boldsymbol{\alpha}\int_{\boldsymbol{0}}^{\boldsymbol{\infty}} \boldsymbol{x^{\alpha-1}}\,\boldsymbol{e^{-x}}\,\boldsymbol{dx}$$

To summarize, we find :

$$\boldsymbol{\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha)} \quad ;\ \boldsymbol{\alpha > 0}$$

Using the previous formula, we get*:*

$$\boldsymbol{\Gamma}(2) = \boldsymbol{\Gamma}(1+1) = \boldsymbol{1}\,\boldsymbol{\Gamma}(1) = \boldsymbol{1}$$

$$\boldsymbol{\Gamma}(3) = \boldsymbol{\Gamma}(2+1) = \boldsymbol{2}\,\boldsymbol{\Gamma}(2) = \boldsymbol{2.1} = \boldsymbol{2}!$$

$$\boldsymbol{\Gamma}(4) = \boldsymbol{\Gamma}(3+1) = \boldsymbol{3}\,\boldsymbol{\Gamma}(3) = \boldsymbol{3.2.1} = \boldsymbol{3}\,!$$

Therefore, for every non-integer for α, we find :

$$\boldsymbol{\Gamma}(\alpha+1) = \boldsymbol{\alpha}\,\boldsymbol{\Gamma}(\alpha) = \alpha(\alpha-1)(\alpha-2)\ldots.\boldsymbol{2.1} = \alpha! \quad ;\ \alpha > 0$$

$$\boldsymbol{\Gamma}(\alpha+1) = \alpha!$$

We also find that :

$$\boldsymbol{\Gamma(\alpha) = (\alpha-1)\,\Gamma(\alpha-1)!}$$

And if : $\boldsymbol{\alpha}$ is an integer then : $\boldsymbol{\Gamma(\alpha) = (\alpha-1)!}$

It is difficult to calculate the values of $\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1}\,e^{-x}\,dx$

In order for the values of α to be **an no integer**, we use the formula

$\Gamma(\alpha+1) = \alpha!$ so that : $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$$\boldsymbol{\Gamma}(3/2) = \boldsymbol{\Gamma}\left(\frac{1}{2}+1\right) = \frac{1}{2}\boldsymbol{\Gamma}\left(\frac{1}{2}\right) = \frac{1}{2}\boldsymbol{\sqrt{\pi}}$$

$$\boldsymbol{\Gamma}(5/2) = \boldsymbol{\Gamma}\left(\frac{3}{2}+1\right) = \frac{3}{2}\boldsymbol{\Gamma}\left(\frac{3}{2}\right) = \frac{3}{2}\boldsymbol{\Gamma}\left(\frac{1}{2}+1\right) = \frac{3}{2}\frac{1}{2}\boldsymbol{\Gamma}\left(\frac{1}{2}\right) = \frac{3}{4}\boldsymbol{\Gamma}\left(\frac{1}{2}\right) = \boldsymbol{\frac{1}{2}\sqrt{\pi}}$$

Example01:

If the lifespan (in months) of light lamps produced in a factory has the following probability density function:

$$\boldsymbol{f(x) = C\,x\,e^{-2x}} \quad ; \quad \boldsymbol{x \geq 0}$$

1) Determine the type of probability distribution for the random variable X and then find the values of the constant C ?

2) If one lamp produced by this factory is randomly selected, what is the probability that the the lifespan of this lamp is less than 03 months?

Solution:

a) the values of the constant C with the type of probability distribution of the random variable X

It is clear that X is a random variable that follows the ***gamma*** distribution **$X \rightsquigarrow \Gamma(\alpha;\lambda)$** with parameters α=2, λ=2 and therefore:

$$f(x) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$$

$$f(x) = C\, x\, e^{-2x}$$

$$C = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} = \frac{2^2}{\Gamma(2)} \quad ; \quad \Gamma(2) = 1 \quad ; \quad \Gamma(1) = 1$$

$$C = \frac{2^2}{1} = 4$$

so the density function will become :

$$f(x) = 4\, x\, e^{-2x}$$

b) The probability that the lifespan of this lamp is less than 03 months

$$P(X \leq 3) = \int_0^3 4\, x\, e^{-2x}\, dx = -2 \int_0^3 x(-2)^{-2x}\, dx$$

$$= -2\left([x\, e^{-2x}]_0^3 - \int_0^3 1(e^{-2x})\, dx \right)$$

$$= [-2x\, e^{-2x}]_0^3 - 2\left[\frac{1}{2} e^{-2x}\right]_0^3$$

$$= 0.9832$$

Example02:

The daily consumption of aviation fuel in millions of gallons at a certain airport can be treated as a **gamma random variable** with **α = 3, λ = 1**

a) What is the probability that on a given day the fuel consumption will be less than million gallons?

b) Suppose the airport can store only 2 million gallons of fuel. What is the probability that the fuel supply will be inadequate on a given day?

Solution:

a) Let **X** be the fuel consumption in millions of gallons on a given day at a certain airport. Then **$X \rightsquigarrow \Gamma(\alpha=3;\lambda=1)$**

$$f(x) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} = \frac{1^3}{\Gamma(3)} x^{3-1} e^{-1x} = \frac{1}{2} x^2 e^{-x} \;;\; x > 0$$

using integration by parts, we obtain :

$$P(X \leq 1) = \frac{1}{2}\int_0^1 x^2 e^{-x}\, dx = 1 - \frac{5}{2e} = 0.08025$$

This probability can also be calculated directly using the distribution function:

$$F(x) = P(X \leq x) = 1 - e^{-\lambda x} \sum_{i=1}^{\alpha} \frac{(\lambda x)^{i-1}}{\Gamma(i)}$$

$$P(X \leq 1) = 1 - e^{-1(1)} \sum_{i=1}^{3} \frac{(\lambda x)^{i-1}}{\Gamma(\alpha)} = 1 - e^{-1(1)} \left(\frac{[1(1)]^{1-1}}{\Gamma(1)} + \frac{[1(1)]^{2-1}}{\Gamma(2)} + \frac{[1(1)]^{3-1}}{\Gamma(3)} \right)$$

$$P(X \leq 1) = 1 - e^{-1}\left(1 + 1 + \frac{1}{2}\right) = 1 - \frac{5}{2e} = 0.08025$$

There is about an 8. 025 % chance that on a given day the fuel consumption will be less than 1 million gallons

b) The probability that the fuel supply will be inadequate on a given day

$$P(X > 2) = \frac{1}{2}\int_2^{\infty} x^2 e^{-x}\, dx = 0.677$$

Because the airport can store only 2 million gallons, the fuel supply will be inadequate if the fuel consumption **X** is greater than **2**

5- The Beta Distribution

5-1- Definition

The **Beta Distribution** is characterized by its high flexibility according to the values of its parameters so that it is used to calculate the Student-Fisher distribution, the negative binomial distribution..., and it is used to represent some variables whose values are between 0 and 1 such as a ratio such as the percentage of spoiled production or the percentage of sales...

5-2- Probability Density Function:

We say that a random variable X has a beta distribution if its probability density function takes the following form:

$$f(x) = \begin{cases} \dfrac{1}{B(\alpha;\beta)} x^{\alpha-1}(1-x)^{\beta-1} & ; \ 0 < x < 1 \\ 0 & ; \ \text{Otherwise} \end{cases}$$

Where :

Where : $B(\alpha;\beta) = B(\beta;\alpha) = \int_0^1 (x)^{\alpha-1}(1-x)^{\beta-1}dx$

$$B(\alpha;\beta) = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

And we write : $X \rightsquigarrow B(\alpha;\beta)$ $\alpha > 0$; $\beta > 0$

So that : $B(\alpha;\beta)$ is a beta function **eulérienne,**

The formula for the probability density function of the variable X varies according to the values of the parameters **α** and . Through the relationship of the beta function $B(\alpha;\beta)$ to the gamma function, the probability density function of the beta distribution can be rewritten as:

$$f(x) = \begin{cases} \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} & ; \ 0 < x < 1 \\ 0 & ; \ \text{Otherwise} \end{cases}$$

5-3- The Distribution Function

The cumulative distribution function for the beta random variable is commonly called the incomplete beta function and is denoted by :

$$\mathbf{F(x)} \;=\; \int_{-\infty}^{x} \boldsymbol{f(x)\,dx} \;=\; \int_{0}^{x} \frac{\mathbf{1}}{\boldsymbol{B(\alpha;\beta)}}\, \boldsymbol{x}^{\,\alpha-1}\,(\mathbf{1}-\boldsymbol{t})^{\beta-1}\,\boldsymbol{dx} = \boldsymbol{I_x(\alpha;\beta)}$$

A tabulation of $\boldsymbol{I_x(\alpha;\beta)}$ is given in Tables of the Incomplete Beta Function (Pearson,1968). When α and β are both positive integers, $\boldsymbol{I_x(\alpha;\beta)}$ is related to the binomial probability function. Integration by parts can be used to show that for $\mathbf{0} < \boldsymbol{x} < \mathbf{1}$ and α and β both integers :

$$\mathbf{F(x)} \;=\; \begin{cases} \mathbf{0} & ;\ \boldsymbol{x} < \mathbf{0} \\ \sum_{\boldsymbol{i}=\boldsymbol{\alpha}}^{\boldsymbol{n}} \boldsymbol{C_n^i}\ (\boldsymbol{x})^{\boldsymbol{i}}\,(\mathbf{1}-\boldsymbol{x})^{\boldsymbol{n-1}} & ;\ \mathbf{0} \le \boldsymbol{x} < \mathbf{1} \\ 1 & ;\ \boldsymbol{x} \ge \mathbf{1} \end{cases}$$

where $\mathbf{n = \alpha + \beta - 1}$, Notice that the sum on the right-hand side of this expression is just the sum of probabilities associated with a binomial random variable with n = α + β − 1 and p = y. The binomial cumulative distribution function is presented in Appendix (Table 01).

5- 4- Properties of the Beta Distribution

- ✓ $\boldsymbol{E(X) = \frac{\alpha}{\alpha+\beta}}$, $\boldsymbol{V(X) = \frac{\alpha\,\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}}$
- ✓ **Moment Generating Function**

$$m_x(t) = E(e^{tx}) = 1 + \sum_{n=1}^{\infty}\left(\prod_{k=0}^{n-1}\frac{\alpha+k}{\alpha+\beta+k}\right)\frac{t^n}{n!}$$

- ✓ The ratio between two variables that follow a gamma distribution is a variable that follows a beta distribution
- ✓ The beta distribution is associated with two parameters and is used to describe many positive random phenomena such as waiting time, product lifetime…

Figure 09: The density function of the Beta distribution for different choices of the parameters α and β.

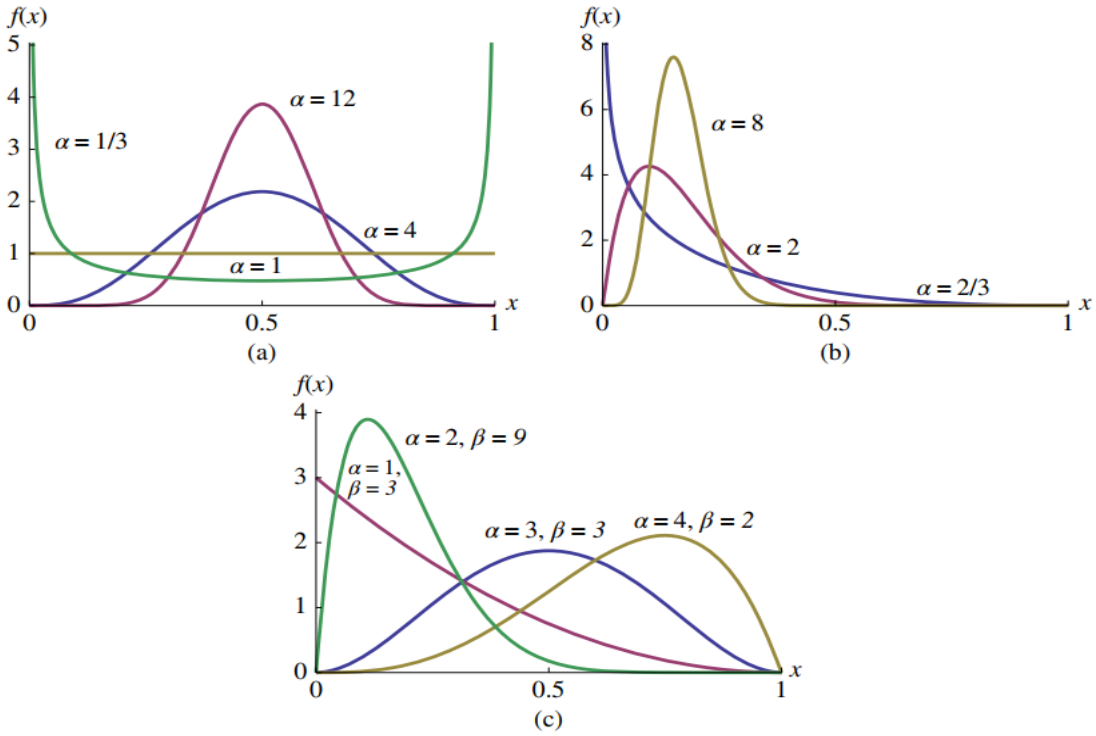


Figure (a) presents the plot of the density of the Beta distribution when $\alpha = \beta$. As can be seen easily, for $\alpha = \beta$ the Beta distribution is symmetric around the point 1⁄2. Note that for $\alpha = \beta = 1$, the Beta distribution reduces to the Uniform distribution

Figure (b) shows the graph of three Beta densities for which the mean $\alpha/(\alpha + \beta)$ is kept fixed. In all cases, we have $\alpha < \beta$ and we see that the distribution is **skewed to the right**; that is, the density function has a peak at a point to **the left** of the midpoint $x = \frac{1}{2}$ and large values are less likely. When $\alpha > \beta$, the situation is reversed and the Beta distribution **is skewed to the left**.

Figure (c) presents (in the same system of coordinates) the density function of the Beta distribution for several choices of the parameters α **and** β

Example01:

Calculate the following:

1) $B(3;2)$, $B\left(4;\frac{1}{2}\right)$

2) $\int_0^1 x\,(1-x)\,dx$

3) $\int_0^1 x^4\,(1-x)^3\,dx$

Solution:

1) $\boldsymbol{B(\alpha;\beta)} = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}$

$$\boldsymbol{B(3;2)} = \frac{\Gamma(3)\,\Gamma(2)}{\Gamma(3+2)} = \frac{2!\;\;1!}{4!} = \frac{2}{24} = \frac{1}{12}$$

$$\boldsymbol{B\left(4;\frac{1}{2}\right)} = \frac{\Gamma(4)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(4+\frac{1}{2}\right)} = \frac{3!\;\sqrt{\pi}}{\Gamma(4.5)}$$

$$\Gamma(4.5) = \Gamma(3.5+1) = 3.5\Gamma(3.5) = 3.5(2.5)\,\Gamma(2.5)$$
$$= 3.5(2.5)\,(1.5)(0.5)\Gamma(0.5) = 6.5625\sqrt{\pi}$$

$$B\left(4;\frac{1}{2}\right) = \frac{3!\;\sqrt{\pi}}{\Gamma(3.5+1)} = \frac{3!\;\;\sqrt{\pi}}{6.5625\sqrt{\pi}} = 0.9142 = \frac{1}{12}$$

2) $\boldsymbol{\int_0^1 x\,(1-x)\,dx}$

We know that:

$$\beta(\alpha;\theta) = \int_0^1 (x)^{\alpha-1}(1-x)^{\boldsymbol{\beta}-1}dx \;= \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

$$\int_0^1 x\,(1-x)\,dx = \beta(2;2) = \frac{\Gamma(2)\,\Gamma(2)}{\Gamma(2+2)} = \frac{(2-1)!\,(2-1)!}{(4-1)!} = \frac{1}{6}$$

3) $\boldsymbol{\int_0^1 x^4\,(1-x)^3\,dx}$

$$\int_0^1 x^4\,(1-x)^3\,dx = \beta(5;4) = \frac{\Gamma(5)\,\Gamma(4)}{\Gamma(5+4)} = \frac{(5-1)!\,(4-1)!}{(9-1)!} = \frac{1}{280}$$

Example02:

If the proportion of spoiled production X in a factory has a probability density function :

$$f(x) = \quad a\,x^2\,(1-x) \quad ; \quad 0 < x < 1$$

1) Determine the type of probability distribution for X and then find the value of the constant term" **a "** .

2) Find the probability that the percentage of spoiled production is less than 60% $P(X \leq 0.6)$?

3) Find the expectation and variance of the random variable X ?

Solution:

1) The type of probability distribution for X and the value of the constant term" a "

It is clear that the variable X follows a **Beta Distribution** with parameters **α, β** such that :

$$f(x) = \quad 12\,x^2\,(1-x) \quad ; \quad 0 \leq x \leq 1 \qquad \text{....(01)}$$

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)}\,x^{\alpha-1}\,(1-x)^{\beta-1} \quad ; \quad 0 < x < 1 \qquad \text{.....(02)}$$

By conformity relations 01 and 02, we find :

$$\beta - 1 = 1 \Rightarrow \beta = 2$$

$$\alpha - 1 = 2 \Rightarrow \alpha = 3$$

$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)} = \frac{\Gamma(3+2)}{\Gamma(3)\,\Gamma(2)} = \frac{4!}{2!\;1!} = 12$$

$$a = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)} = 12$$

$$a = 12$$

The Probability Density Function becomes of the form :

$$f(x) \begin{cases} \mathbf{12\, x^2\, (1-x)} & ;\ \mathbf{0 \le x \le 1} \\ \mathbf{0} & ;\ \mathbf{Otherwise} \end{cases}$$

1) The percentage of spoiled production is less than 60%

$$P(X \le 0.6) = \int_0^{0.6} 12\, x^2\, (1-x)\, dx = [4x^3 - 3x^4]_0^{0.6} = 0.475$$

2) The expectation and variance of the random variable X

$$\boldsymbol{E(X) = \frac{\alpha}{\alpha+\beta} = \frac{3}{3+2} = \frac{3}{5}}$$

$$\boldsymbol{Var(X) = \frac{\alpha\,\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{1}{25}}$$

Example 03:

A gasoline wholesaler has large storage tanks that contain a steady supply and are filled on one day of each week. Of interest to the wholesaler is the proportion of this inventory that is sold during the week. Over many weeks of observation, the distributor has found that this proportion can be modeled by a **beta distribution** with $\boldsymbol{\alpha = 4}$ and $\boldsymbol{\beta = 2}$

1) Find the probability density function for the proportion of inventory sold during the week.

2) Find the probability that the wholesaler sells at least 90% of this inventory in a given week , does he have a high chance of selling this stock of gasoline?

Solution:

1) The probability density function for the proportion of inventory sold during the week.

If X refers to the percentage sold during the week, then :

$$f(x) = \begin{cases} \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)}\, x^{\alpha-1}\,(1-x)^{\beta-1} & ;\ \mathbf{0 < x < 1} \\ \mathbf{0} & ;\ \mathbf{Otherwise} \end{cases}$$

$$f(x) = \begin{cases} \dfrac{\Gamma(6)}{\Gamma(4)\,\Gamma(2)} x^3\,(1-x) = \ 20\ (x^3 - x^4) \ ; \ 0 < x < 1 \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad ; \text{ Otherwise} \end{cases}$$

$$P(X > 0.9) = \int_{0.9}^{\infty} f\,(x)\,dx = \int_{0.9}^{\infty} 20\ (x^3 - x^4)\,dx = 20\left[\frac{x^4}{4} - \frac{x^5}{5}\right]_{0.9}^{1}$$
$$= 0.08$$

It is not likely that 90% of the inventory will be sold in a given week because the resulting probability is very small

Example 04:

The maximum time allowed for an exam is three hours. The time required X, as a proportion of this maximum duration by a student to complete the exam (that is, if a student completes the exam in t hours, X takes the value t/3) has a Beta distribution with parameters α = 5 and β = 2.

a) Find the density function and the Distribution Function of X ?
b) Obtain the mean and the standard deviation of X ?
c) What proportion of students complete the exam within two hours ?

Solution:

a) The density function of X is given by :

$$f(x) = \begin{cases} \dfrac{1}{B(5;2)} x^{5-1}\,(1-x)^{2-1} \quad ; \ 0 < x < 1 \\ 0 \qquad\qquad\qquad\qquad ; \quad \text{Otherwise} \end{cases}$$

we see that the value of B(5, 2) is :

$$B(5;2) = \frac{\Gamma(5)\,\Gamma(2)}{\Gamma(5+2)} = \frac{4!\ 1!}{6!} = \frac{1}{30}$$

Thus, the density takes the form :

$$\boldsymbol{f(x)} = \begin{cases} \mathbf{30\, x^4\, (1-x)} & ; \quad \mathbf{0 < x < 1} \\ \mathbf{0} & ; \quad \textbf{Otherwise} \end{cases}$$

For the distribution function, we have:

$$\mathbf{F(X)} = \int_{-\infty}^{x} \boldsymbol{f(x)\, dx} = \int_{0}^{x} 30\, x^4\, (1-x)\, \boldsymbol{dx}$$

$$\mathbf{F(X)} = \begin{cases} \mathbf{0} & ; \quad \boldsymbol{x < 0} \\ \mathbf{6\, x^5 - 5\, x^6} & ; \quad \mathbf{0 \le x < 1} \\ \mathbf{1} & ; \quad \boldsymbol{x \ge 1} \end{cases}$$

the mean and variance

$$\boldsymbol{E(X)} = \frac{\alpha}{\alpha+\beta} = \frac{5}{5+2} = \mathbf{\frac{5}{7}}$$

$$\boldsymbol{Var(X)} = \frac{\alpha\, \beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \mathbf{\frac{10}{392}}$$

As explained in the statement, if a student completes the exam in t hours, the associated value of X is $\frac{\boldsymbol{t}}{\mathbf{3}}$. Thus, in order to find the probability that a student completes the exam within two hours (or, the proportion of students who complete the exam within two hours, which is the same), we need to find $P(X \le \frac{2}{3})$. This can be obtained easily upon substituting $\boldsymbol{t = \frac{2}{3}}$ in the distribution function F. Thus, we see that :

$$\mathrm{F(X)} = P\left(X \le \frac{2}{3}\right) = \mathrm{F}\left(\frac{2}{3}\right) = \mathbf{6}\left(\frac{2}{3}\right)^{\mathbf{5}} - \mathbf{5}\left(\frac{2}{3}\right)^{\mathbf{6}} = \frac{256}{729} \cong 0.3512$$

so about 35% of the students complete the exam within two hours.

6- The Chi -Square Distribution

6-1-Definition

The Chi-squared distribution is among the most well-known continuous probability distributions and has many uses in inferential statistics, including tests of goodness of fit, homogeneity, and independence of variables.

This distribution can be defined as follows:

If $X_1, X_2 \dots X_v$ are random variables that follow the standard normal distribution and are independent of each other $X_i \rightsquigarrow N(0;1)$, then the random variable $\boldsymbol{X = \sum_{i=1}^{v} X_i^2 = X_1^2 + X_2^2 + \cdots + X_v^2}$ follows a **chi-square distribution** $\boldsymbol{\chi_v^2}$ with degree of freedom " **v**" and we write : $X \sim \chi_v^2$

6.2 – The Probability Density Function

The probability density function for this distribution with degree of freedom v is given by the following formula:

$$\boldsymbol{f(x) = \begin{cases} 0 & ; \quad x < 0 \\ \dfrac{1}{(2)^{\frac{v}{2}}\,\Gamma\left(\frac{v}{2}\right)} (x)^{\frac{v}{2}-1} (e)^{-\frac{x}{2}} & ; \quad 0 \le x < \infty \end{cases}}$$

$\boldsymbol{\Gamma(\alpha)}$: It represents the function Gamma

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1}\, e^{-x} dt \qquad , \alpha > 0$$

The values of this distribution can be found using the table for this distribution .

6-3- The Distribution Function

X is a random variable that follows a chi-squared distribution $X \sim \chi_v^2$ so that the distribution function for this random variable F(X)=P(X≤x) is given as follows:

$$\boldsymbol{F(X) = P(X \le x) = \begin{cases} 0 & ; \ x < 0 \\ \frac{1}{(2)^{\frac{v}{2}}\,\Gamma\left(\frac{v}{2}\right)} \int_0^x (x)^{\frac{v}{2}-1} (e)^{-\frac{x}{2}}\, dx & ; \quad 0 \le x < \infty \\ 1 & ; \ x \to \infty \end{cases}}$$

6-4- Properties of the Chi-Squared Distribution

✓ **The expectation and variance :**

E(X)=v ; Var(X)=2v

By the definition of the chi-square distribution $X_i \rightsquigarrow N(0;1)$ we have:

$$E(X_i^2) = Var(X_i) = 1, \quad ou\ E\left(\sum_{i=1}^{v} X_i^2\right) = E(X) = v$$

$$Var(X_i^2) = E(X_i^4) - \left(E\left(X_i^2\right)\right)^2 = \mu_4 - 1\ ;\ \mu_4 = 3$$

$$Var(X_i^2) = 2$$

$$Var\left(\sum_{i=1}^{v} X_i^2\right) = Var(X) = 2v$$

So by the definition of the chi-square distribution $\boldsymbol{X_i \rightsquigarrow N(0;1)}$ we conclude that :

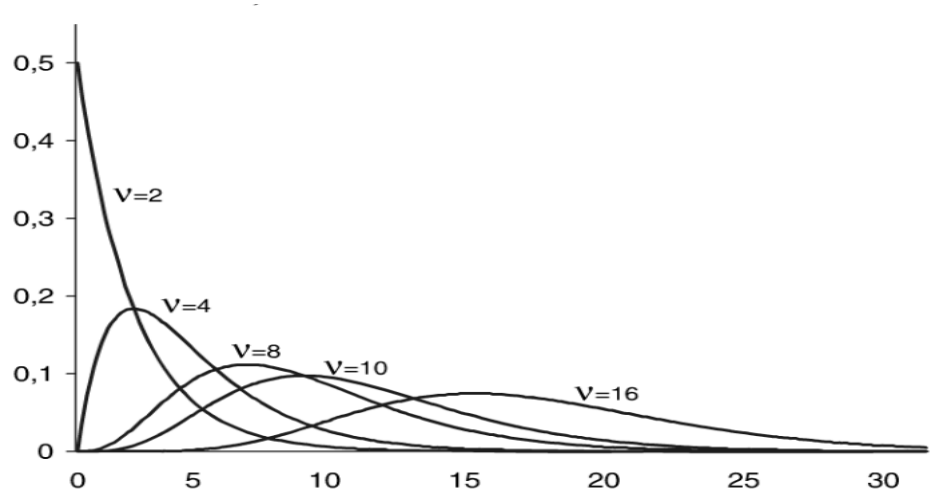
$$\boldsymbol{E(X) = v \quad ; \quad Var(X) = 2v}$$

✓ **Moment Generating Function**

$$m_x(t) = (1-2t)^{-\frac{v}{2}}$$

✓ According to the values of degrees of freedom **v**, the chi-square distribution has different shapes:

Figure 10 : Density Function of the chi-square distribution for different values of the parameter v.



✓ If X is replaced by 2y, the probability density function takes the form

$$f(2y) = \begin{cases} 0 & ; \quad y < 0 \\ \dfrac{1}{\Gamma\left(\frac{v}{2}\right)} (x)^{\frac{v}{2}-1} (e)^{-y} & , \quad 0 \le y < \infty \end{cases}$$

This relationship represents the probability density function of the gamma distribution

- ✓ The Chi-square distribution represents a special case of the Gamma distribution with two parameters: λ=1/2 , α=v/2
- ✓ If X is a random variable that follows a gamma distribution with parameters α and λ, then :

$$X = \frac{2X}{\lambda} \sim \chi^2_{(2\alpha)}$$

✓ If v=2, the probability density function takes the form

$$f(x) = \begin{cases} 0 & ; \quad x < 0 \\ \dfrac{1}{2} (e)^{-\frac{x}{2}} & , \quad 0 \le x < \infty \end{cases}$$

It expresses the probability density function of the exponential distribution with **λ=1/2**

- ✓ The values of this distribution can be found using the statistical table for this distribution
- ✓ The curve of the probability density function is skewed to the right but becomes symmetrical as the value of the degree of freedom v increases

✓ If we have a random variable X where $X \rightsquigarrow N(\mu, \sigma^2)$ then

$Z = \frac{\bar{X}-\mu}{\sigma} \rightsquigarrow N(0,1)$, We define the probability distribution of Z squares as a chi-square distribution with a degree of freedom **v=1**, denoted by χ^2

(where the degree of freedom is the parameter of this distribution) and express it by the following relation:

$$Z^2 = \left(\frac{\overline{X} - \mu}{\sigma}\right)^2 \rightsquigarrow \chi^2_{(1)}$$

- ✓ If $\boldsymbol{U_1, U_2, U_3 \ldots U_n}$ are independent random variables that follow a chi-square distribution with degrees of freedom $v_1, v_2, v_3 \ldots v_n$ respectively, then :

$$\sum_{i=1}^{n} \boldsymbol{U_i} \sim \boldsymbol{\chi^2_{(v)}}$$

With: $v = \sum_{i=1}^{n} v_i$

- ✓ If $\boldsymbol{X_1, X_2, X_3 \ldots X_n}$ is a random sample from a normal population with the mean μ and variance $\boldsymbol{\sigma^2}$, then

$$\frac{(n-1)S^2}{\sigma^2} = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{\sigma^2} \rightsquigarrow \chi^2{}_{(n-1)}$$

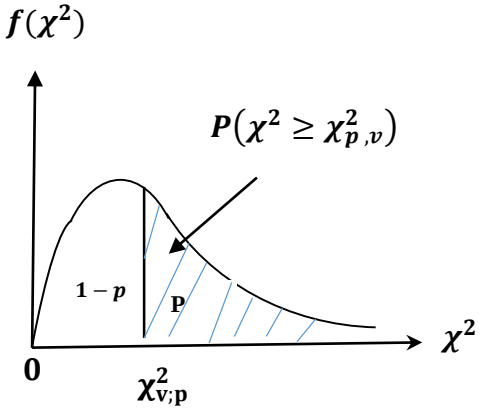
has a chi-square distribution with (n - 1) degrees of freedom.
S^2 : variance random n : size sample

6-5- The Chi-square Distribution Table

As with a normal distribution, to calculate the cumulative probability of a chi-square distribution, we use statistical tables where the values of the **chi-square variable** corresponding to the degree of freedom v and the probability p, which represents the area to the **left** of the Chi-square value under the curve .

And sometimes through other reference**s** the point of the Chi-square is defined in terms of the area to the right of the Chi-square value under the curve
$\mathrm{P} = \mathrm{P}\left(\mathrm{X} \geq \chi^2_{\mathrm{v;p}}\right)$ as shown in the following figure :

Figure 10 : the probability in the density function Curve of the chi-square distribution



From the chi-square statistical table, we can find the probability **p** such that the chi-square statistic is greater than the value $\boldsymbol{\chi^2_{p,v}}$ In the sense that :

$\mathbf{P}(\boldsymbol{\chi^2} \geq \boldsymbol{\chi^2_{p,v}}) = \int_{\chi^2_{p,v}}^{\infty} \boldsymbol{f(x)dx} = \mathbf{p}$ by intersecting the probability **column "P"** with the degree of freedom **line "v"** to produce the critical chi-square statistic $\boldsymbol{\chi^2_{p,v}}$ and this probability is shown in the shaded area in the figure above.

Note:

Degrees of freedom are defined as the number n of independent observations in the sample minus the number **k** representing the **population parameters** (arithmetic mean, standard deviation...), thus **v=n-k**. For example, in the case of the Student's distribution, **v=n-1**, on the basis that the arithmetic mean of this distribution must be known, and therefore the number of estimated parameters is **1**.

Example01:

In the first column for the **degree of freedom "v"** and the first row for **the probability "p"**, we find a chi-square statistic such as :

- ✓ $P(\chi^2 \geq \chi^2_{0.95;\,15}) = 0.95$

  At degree of freedom v=15 and P=0.95 we find:

  $$P(\chi^2 > 7.261) = 0.95$$

- ✓ $P(\chi^2 > \chi^2_{0.05\,;11}) = 0.05$

At degree of freedom v=11 and P=0.05 we find: $P(\chi^2 > 19.675) = 0.05$

Example02:

If χ^2 is a random variable that follows a chi-square distribution with degree of freedom v, from the chi-square distribution statistical table $\chi^2_{p,v}$, find :

a) The critical value of the chi-square statistic at v=9 and p=0.05; $P(\chi^2 \geq \chi^2_{0.05;9})$ = 0.05 ?

b) Critical value of the chi-square statistic when v=9 and p=0.975

$P(\chi^2 \geq \chi^2_{0.975\,;9}) = 0.975$?

c) The Probability $P(\chi^2 \geq 18.307)$ at v=10 ?

d) The Probability $P(\chi^2 < 18.307)$ at v=10 ?

Solution:

a) The critical value of the chi-square statistic $\boldsymbol{P(\chi^2 \geq \chi^2_{0.05\,,9})}$

From the chi-square statistical table, in the line corresponding to the degree of freedom v=9 and the column corresponding to p=0.05, we find that

$\chi^2_{0.05\,,9} = 16.919$ so: $P(\chi^2 \geq 16.919) = 0.05$

b) The critical value of the chi-square statistic $\boldsymbol{P(\chi^2 \geq \chi^2_{0.975\,,4}) = 0.975}$

From the chi-square statistical table, the line corresponding to the degree of freedom v=4 and the column corresponding to p=0.975, we find that

$\boldsymbol{\chi^2_{0.975\,,4} = 0.484}$, so : $\boldsymbol{P(\chi^2 \geq 0.484) = 0.975}$

c) The probability $\boldsymbol{P(\chi^2 \geq 18.307)}$

From the statistical table, we can see that the intersection of the statistical critical value of the Cui-square $\chi^2_{p\,,10} = 18.307$ With degree of freedom v=10, the probability p=0.05 results : $P(\chi^2 \geq 18.307) = \boldsymbol{0.05}$

d) The probability $\boldsymbol{P(\chi^2 < 18.307)}$

$$P(\chi^2 < 18.307) = 1 - P(\chi^2 \geq 18.307) = 1 - 0.05 = 0.95$$

Example03:

A fruit-drink company wants to know the variation, as measured by the standard deviation, of the amount of juice in 16-ounce cans. From past experience, it is known that $\boldsymbol{\sigma^2 = 2}$. The company statistician decides to take a sample of **25** cans from the production line and compute the sample variance. Assuming that the sample values may be viewed as a random sample from a normal population, find a value of **b** such that $\boldsymbol{P(s^2 < b) = 0.05}$

Solution:

To find the necessary probability, use the fact that :

$$\frac{(n-1)S^2}{\sigma^2} \rightsquigarrow \chi^2{}_{(n-1)} \quad \text{, with n = 25}$$

$$P(s^2 > b) = 0.05$$

$$= P\left(\frac{(25-1)S^2}{2} > \frac{(25-1)b}{2}\right)$$

$$= P\left(\frac{(25-1)S^2}{2} > \frac{24\,b}{2}\right)$$

$$= P(\chi^2 > c)$$

$$c = \frac{24\,b}{2}$$

From the chi-square table we obtain, c = 36.4151.

Hence, $c = \frac{24b}{2} \Longrightarrow b = \frac{2}{24}c = \frac{2}{24}(36.4151) = 3.03$

$$\boldsymbol{P(s^2 > 3.03) = 0.05}$$

7- The Student Distribution

7-1- Probability Density Function

Let the random variables Y and Z be such that : $Y \sim \chi_v^2$, $Z \sim N(0;1)$ and $(Z \in R, Y \in R^+, T \in R)$, The variable $\mathbf{T} = \frac{Z}{\sqrt{\frac{Y}{v}}}$ It follows a Student's distribution and has a **probability density function**:

$$\boldsymbol{f(t)} = \frac{\Gamma\left(\frac{v+1}{2}\right)}{\sqrt{v\pi}\ \Gamma\left(\frac{v}{2}\right)}\left(1 + \frac{t^2}{v}\right)^{-\left(\frac{v+1}{2}\right)} \quad ; \ -\infty < \mathbf{T} < \infty \quad ;$$

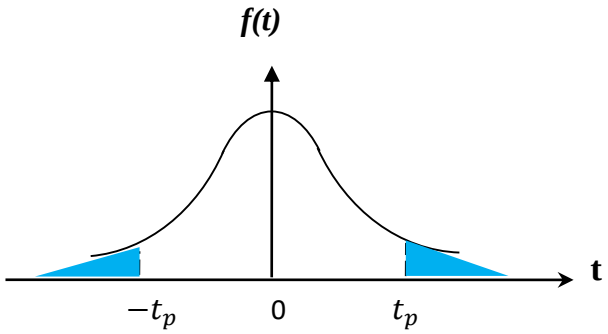
$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1}\, e^{-x}\, dx \ ; \ \alpha > 0$$

We say that X follows a Student's distribution with degree of freedom V and write: $X \sim t_v^2$

7-2- Properties of the Student distribution

- ✓ **Expectation and variance** $E(T) = 0 \quad v \geq 2 \quad , \quad V(T) = \frac{v}{v-2} \quad v \geq 3$
- ✓ $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$
- ✓ The probability density function f(x) reaches its maximum value when x=0, as shown by the Student's distribution curve if the mean E(x)=0 corresponds to the greatest probability
- ✓ The Student's distribution curve is symmetrical (symmetrical) (every negative value of the Student's statistic corresponds to a positive value) in the sense of $t_{p;v} = -t_{1-p;v}$, and as v increases, the Student's distribution curve f(t) approaches the normal distribution curve .

Figure 11 : The symmetry in the Student's distribution



- ✓ Because the probability density function of the Student's distribution is symmetrical to the mathematical expectation value (equal to 0), the distribution function satisfies :

$F_x(0) = P(T < 0) = 0.5$

$P(T < -t) = 1 - P(T < t)$, In other words : $F_x(-x) = 1 - F_x(x)$

- ✓ $P(|T| \geq t) = 1 - P(|T| \leq t) = 1 - P(-t \leq T \leq t)$

$= 1 - P(T \leq t) + P(T \leq -t)$

- ✓ In statistical tables for the Student's t distribution, the value of "**t"** is given by the degree of freedom **v** and the probability **p** to the right of t under the curve and we write t_(p;v) (sometimes the value of t is given by the area function to the left of "t ").
- ✓ The following table shows the values of t and the associated probabilities p such that : P(T≥t)=p, the probabilities p appear at the top of the first row of the table (0.50 to 0.0005), the first column of this table shows the degrees of freedom of the Student's distribution, and in the center of the table, through the intersection of probabilities and degrees of freedom, the values of the Student's t statistic appear as shown in part of the statistical table of the following Student's distribution:

Table 02:Statistical Table for The Student's Distribution

p / v	0.50	0.25	0.20	0.15	0.10	0.05	0.025	0.01	0.005	0.001	0.0005
1	0.000	1.000	1.376	1.963	3.078	6.314	12.71	31.82	63.66	318.31	636.62
2	0.000	0.816	1.061	1.386	1.886	2.920	4.303	6.965	9.925	22.327	31.599
3	0.000	0.765	0.978	1.250	1.638	2.353	3.182	4.541	5.841	10.215	12.924
4	0.000	0.741	0.941	1.190	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	0.000	0.727	0.920	1.156	1.476	2.015	2.571	3.365	4.032	5.893	6.869
6	0.000	0.718	0.906	1.134	1.440	1.943	2.447	3.143	3.707	5.208	5.959
7	0.000	0.711	0.896	1.119	1.415	1.895	2.365	2.998	3.499	4.785	5.408
8	0.000	0.706	0.889	1.108	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	0.000	0.703	0.883	1.100	1.383	1.833	2.262	2.821	3.250	4.297	4.781
10	0.000	0.700	0.879	1.093	1.372	1.812	2.228	2.764	3.169	4.144	4.587

Example01:

Using the statistical tables of critical values for the Student distribution, find the following probabilities:

a) $P(T \geq 1.812) = \ ?\ \ n = 11$
b) $P(T \leq -2.131) = \ ?\ \ n = 16$
c) $P(T < 1.325) = ?\ \ v=20$

Solution:

a) At degree of freedom v=n-1=11-1=10 , The corresponding probability can be found : $P(T \geq 1.812) = 0.05$
b) At degree of freedom v= 16-1=15 .
Since the student distribution is symmetric, then:
$(T \leq -2.131) = (T \geq 2.131) = 0.025$
c) At degree of freedom v= 20,
$P(T < 1.325) = 1 - P(T \geq 1.325) = 1 - 0.1 = 0.99$

Example02 :

At degree of freedom v, and probability P , find the values of the critical Student's t statistic $t_{v;\,p}$ that corresponds to :

a) $P(T \geq t_{9;0.05})$?
b) $P(T < t_{9;0.99})$?
c) $P(-t_{0.005} < T < t_{0.025})$?

Solution:

a) The area on the right-hand side corresponding to the p-value 0.05 and v=9 corresponds the value of the Student's t-statistic. 1.833, meaning
$P(T \geq 1.833) = 0.05$
b) The area on the left-hand side corresponding to the p-value 0.99 and v=9 , so the misleading area (probability) is equal to p=1-0.99=0.01, so the area corresponds the value of the Student's t-statistic 2.821 , meaning
$P(T < 2.821\ ) = 0.99$
c) The area on the right of $t_{0.25}$ is 0.025 and the area on the left of $-t_{0.005}$ is 0.0025, so the area between them is :

$P(-t_{0.005} < T < t_{0.25}) = 1 - (0.025 + 0.005) = 0.97$

8- The Fisher Distribution

8-1- Definition:

Given two independent random variables $U \sim \chi^2_{v_1}$ and $Y \sim \chi^2_{v_2}$, the random variable $F = \frac{U/v_1}{Y/v_2} \sim F(v_1;\ v_2)$ follows a Fisher distribution and is defined on $\mathbb{R}^+$, The Fisher distribution defined on $\mathbb{R}^+$ is the ratio of two random variables that follow a chi-square distribution, where it is never negative.

$\boldsymbol{v_1}$, $\boldsymbol{v_1}$: represent the degrees of freedom of the Fisher distribution

6-3-2-The probability density function:

Let the random variable X follow a Fisher distribution such that v_1 , $v_1 \in \mathbb{N}^*$ degrees of freedom and $X \in \mathbb{R}^+$ if its probability density function is defined as:

$$\boldsymbol{f(x) = \begin{cases} 0 & ;\ x \le 0 \\ \dfrac{\Gamma\left(\frac{v_1+v_2}{2}\right)}{\Gamma\left(\frac{v_1}{2}\right)\Gamma\left(\frac{v_2}{2}\right)}\left(\dfrac{v_1}{v_2}\right)^{\frac{v_1}{2}} \dfrac{x^{\frac{v_1-2}{2}}}{\left(1+\frac{v_1}{v_2}x\right)^{\frac{v_1+v_2}{2}}} & ;\ x > 0 \end{cases}}$$

$\Gamma(\alpha)$: the function gamma

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1}\, e^{-x} dt \qquad , \alpha > 0$$

The values of this distribution can be found using the table for this distribution

6-3-3- *Properties of the Fisher Distribution*

✓ ***If*** $v_2 \ge 3$ ***then :*** $E(x) = \frac{v_2}{v_2-2}$,

If: $v_2 \ge 5$ ***then:*** : $V(x) = \frac{2v_2^2(v_1+v_2-2)}{v_1(v_2-2)^2 v_2-4}$

$$F_{p\ ;v_1;\ v_2} \neq F_{p\ ;\ v_2;\ v_1}$$

✓ ***Let*** $X \sim F(v_1;\ v_2)$ ***) such that if*** $v_1 = 1$ **,** ***then*** $F(1;\ v_2) = t_v^2$

That is, $F(1;\ v_2)$ and T^2 have the same distribution.

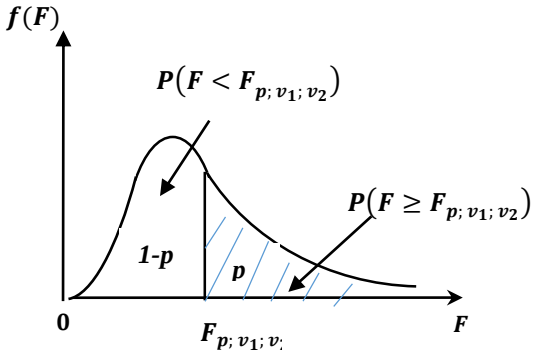
Due to the importance of the Fisher distribution, Fisher's statistical values $F_{p;\,v_1;\,v_2}$ are tabulated according to the two degrees of freedom $\boldsymbol{v_1;\ v_2}$ respectively, where the critical Fisher statistic is calculated by the probability value p (which represents the area under the curve to the right of the F statistic) at the intersection of the two degrees of freedom $v_1;\ v_2$.

Example01:

$F_{p;\,v_1;\,v_2} = F_{0.05;6;\,10} = 3.22$ **so:** $P(F_c \geq 3.22) = 0.05$

$F_{p;\,v_1;\,v_2} = F_{0.01;6;\,10} = 5.39$ **so** : $P(F_c \geq 5.39) = 0.01$

Figure 12 : the probability in the density function Curve of the fisher distribution



✓ We have a $X \sim F(v_1;\ v_2)$ So $\frac{1}{X} \sim F(v_2;\ v_1)$ in the sense that

$$\boldsymbol{F_{p\,,v_1;\,v_2} = \frac{1}{F_{1-p\,,v_2;\,v_1}}}$$

✓ Relationship with other distributions :

$$\boldsymbol{F_{p;v} = \frac{\chi^2_{p\,;v}}{v}} \quad ; \quad \boldsymbol{F_{p;v} = t^2_{\frac{p}{2};v}}$$

Example 02:

Test the validity of the following relationships:

a) $\boldsymbol{F_{0.05} = t^2_{0.05}}$

b) $\boldsymbol{F_{0.01;v} = \frac{\chi^2_{p;v}}{v}}$

Solution:

a) $\boldsymbol{F_{0.05} = t^2_{0.025}}$

We have the relationships : $\boldsymbol{F_{p;v} = t^2_{\frac{p}{2};v}}$

We compare the values of Fisher's statistic in the first column $F_{0.05;v}$ at $v_1 = 1$ and different degrees of freedom $\boldsymbol{v}$ $\boldsymbol{(v = v_2)}$

- At $v = 1$ $\quad 161 = (12.71)^2$
- At $v = 2$ $\quad 18.5 = (4.30)^2$
- At $v = 3$ $\quad 10.1 = (3.18)^2$

b) $\boldsymbol{F_{0.01;v} = \frac{\chi^2_{p;v}}{v}}$

We compare the values of the Fisher statistic in the last row of degrees of freedom $\mathbf{v_2}$, and at different degrees of freedom v $\boldsymbol{(v = v_1)}$

$F_{0.01;v}$

- ✓ At $v = 1$: $F_{0.01;v} = \frac{\chi^2_{p;v}}{v} \Rightarrow 6.63 = \frac{6.63}{1}$
- ✓ At $v = 2$ $\quad 4.61 = \frac{9.21}{2}$
- ✓ At $v = 3$ $\quad 3.78 = \frac{11.3}{3}$

c) The Fisher and chi -square distributions are special cases of each other and are both special cases of another distribution, the Gamma distribution, so the Fisher normality can be defined :

$$\boldsymbol{F_{v_1;\, v_2} = \frac{\chi^2_1/v_1}{\chi^2_2/v_2} = \frac{\sigma^2_2}{\sigma^2_1}\,\frac{S^2_1}{S^2_2} = \frac{S^2_1/\sigma^2_1}{S^2_2/\sigma^2_2} \sim F(v_1;\, v_2)}$$

$$v_1 = n_1 - 1\,;\;\; v_2 = n_2 - 1$$

There are cases in which probability distributions can converge with each other, , meaning that the two distributions give a probability value that is close, and this means that when this convergence occurs, two (or more) probability distributions can be used to calculate the same probability for the variable X.

1- *Central Limit Theory*

Let $X_1, X_2 \ldots X_n$ be a random sample such that Xi are independent and identically distributed (i=1,2…n) drawn from an infinite statistical population with mean **μ** and variance $\boldsymbol{\sigma^2}$ such that:

$\boldsymbol{S_n} = X1 + X2 + \cdots Xn = \sum_{n}^{i=1} \boldsymbol{X_i}$, then:

$$\boldsymbol{Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}}$$

The probability distribution of $\boldsymbol{Z_n}$ approaches the standard normal distribution as **n** approaches infinity $\boldsymbol{n \rightsquigarrow \infty}$

And we write:

$$\boldsymbol{\lim_{n\to\infty} p(Z_n \leq z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\Pi}}\, e^{-\frac{z^2}{2}}\, dz}$$

Through the central limit theorem, we conclude that:

✓ The probability related to $\boldsymbol{Z_n}$ can be approximated by the probabilities corresponding to the standard variable when the sample size **n** is large (usually, a value of $n \geq 30$ ensures that the distribution of $\boldsymbol{Z_n}$ can be approximated to the normal distribution).

✓ The Central Limit Theorem assumes that we have a number of independent random variables with the same distribution (discrete or continuous) and with a finite mean **μ** and variance $\boldsymbol{\sigma^2}$, such that the standardized random variable $\boldsymbol{Z = \frac{S_n - n\mu}{\sigma\sqrt{n}}}$ is considered a standardized variable for S_n . This theorem is also valid under more general conditions, such as when the variables $X_1, X_2 \ldots X_n$ are random and independent with the same mean and variance, but it is not necessary for each of them to have the same distribution.

The following figure summarizes the approximation rules between some of the most important discrete and continuous probability distributions.

2- Approximating The Binomial Distribution to The Poisson Distribution

If X is a random variable that follows a binomial distribution with parameters **n** and **p**. For values x=0,1... and large n values **n**→∞ , and the probability of success is close to zero p→0 (the probability of failure q is close to integer one), then the event is rare occurrence.

We know that the random variable X follows a binomial distribution with mean: E(X)=μ=np and the variance $V(X) = \sigma^2 = npq$

- ✓ As a rule, in practical situations, an event is considered rare when **p** is small and **n** is large such that $\mathbf{p \leq 0.1}$ and $\mathbf{n \geq 40}$ and therefore the Poisson approximation should be used
- ✓ In general, another rule for using the Poisson approximation to a binomial distribution is when : $\boldsymbol{np < 5}$, with $E(X) = \mu = np = \lambda$
  It is not possible to use statistical tables for the binomial distribution when **n is large**, so calculating any probability in this case requires approximating the binomial distribution with a Poisson distribution.

Example01 :

A factory has a defective production rate of 10%. We select a random sample of 40 units of the factory's production on a given day.

- Using both the binomial and Poisson distributions, calculate the probability that two units are defective ?

Solution:

- ✓ **Calculate the probability using a binomial distribution**

$P(X) = c_X^{40}(0.9)^X(0.1)^{40-X}$, $X = 0,1,2,\dots 40$

$$\boldsymbol{P(X = 2)} = c_2^{40}(0.1)^2(0.90)^{40-2} = \frac{40!}{(40-2)!\,2!}(0.1)^2(0.90)^{38} = 0.1423$$

- ✓ **Calculate the probability using a Poisson distributions**

Rule Approximation : $\boldsymbol{np = 40(0.1) = 4 < 5}$,

$\mu = np = \lambda$

$\mu = 40(0.1) = 4$

$$\boldsymbol{P(X = x) = \frac{e^{-\lambda}\ \lambda^{x}}{x!}} \quad ; \quad \boldsymbol{x = 0, 1, 2 \ldots n}$$

$$P(X = x) = \frac{e^{-4}\ 4^{2}}{2!} = 0.1465 \quad ; \quad x = 0,1,2 \ldots 40$$

Calculating the probability using the previous two distributions, we can see that the results are close but not identical

Example 02:

It is known that the probability of a person having a rare allergic reaction to a medication is 0.001, so if 2000 people who have taken this medication are randomly selected, in the sample, calculate the probability that:

1) Three person are allergic to this medication?

2) More than three person are allergic to this medication?

Solution :

1) The probability of three person who are allergic this medication in the sample (using a Poisson distributions)

Rule Approximation

$$\boldsymbol{np = 2000(0.001) = 2 < 5}$$

$$\mu = np = \lambda$$

$$\mu = 2000(0.001\ ) = 2$$

$$\boldsymbol{P(X = x) = \frac{e^{-\lambda}\ \lambda^{x}}{x!}} \quad ; \quad \boldsymbol{x = 0, 1, 2 \ldots n}$$

$$P(X = 3) = \frac{e^{-2}\ 2^{3}}{3!} = 0.1804 \quad ; \quad x = 0,1,2 \ldots 2000$$

2) More than three person are allergic to this medication

$$P(X > 2) = 1 - P(X = 0) - P(X = 1) - P(X = 2)$$
$$= 1 - 5e^{-2} = 0.323$$

3- Approximation of The Hypergeometric Distribution to The Binomial Distribution

If the population size **N** Become ssufficiently large compared to the sample size **n**, there is no difference between sampling with return and sampling without return, in the case

l, there will effectively be little difference in the probabilities obtained by the hypergeometric (sampling without return) and binomial distributions (sampling with return)

of $N \to \infty$ and $\frac{A}{N} \to p$ and $0 < p < 1$:

$$\lim_{a,b\to\infty} \frac{a}{a+b} = p$$

we have that for x = 0,1,2,…,n, :

$$\lim_{a,b\to\infty} \frac{C_a^x C_b^{n-x}}{C_N^n} \to \; c_X^n (1-p)^X (q)^{n-X}$$

The approximation of the hypergeometric distribution by the binomial is good if both parameters **a** and **b** are much **larger** than **n**.

As a rule, according to statisticians, an estimate is good if:

$N > 50$ *and* $\frac{n}{N} \leq 0.1$

4- *Approximation of The Binomial Distribution to The Normal Distribution*

If X is a random variable following the binomial distribution $X \rightsquigarrow B(n, \mathrm{p})$ then the binomial distribution approaches the normal distribution according to the central limit theorem if the sample size is large, which is explained by the following theorem.

4-1-De Moivre-Laplace Theorem

Let X be a random variable following the binomial distribution X⇝B(n,p) with parameters n and p, for real numbers a<b, we have:

$$\lim_{n\to\infty}\left(a < \frac{X-np}{\sqrt{npq}} < b\right) = \frac{1}{\sqrt{2\pi}}\int_a^b -\frac{1}{2}x^2 dx$$

So that: $E(X) = np \quad ; \quad V(X) = npq$

4-2-The approximation rule:

The previous theorem indicates that in the case of a large n and the probability p not close to zero, the normal distribution can be considered a good approximation of the binomial distribution, and the two distributions yield more similar results as the sample size increases.

Generally, as a rule of thumb, the approximation is considered appropriate if:

- **np≥5, n(1-p)≥5**
- There is another approximation rule if: **np≥10**, **nq≥10**, **or npq≥10**

The normal distribution statistic Z (the standard variable) is given with a mean of 0 and a variance of 1 as follows:

$$Z = \frac{X - E(X)}{\sigma} = \frac{X-np}{\sigma} \rightsquigarrow N(0,1)$$

"Yates" found that approximating the binomial distribution (a discrete distribution) to the normal distribution (a continuous distribution) is accurate and realistic if a correction is made to the value of the random variable x by adding or subtracting 0.5 from it. This is called "**Yates' correction** "or the "**continuity correction factor**".As follows:

Table 03.: Rules for Approximating the Binomial Distribution to The Normal Distribution

The cases	approximation
$\boldsymbol{P(X > a)}$	$P\left(Z > \frac{(a+0.5)-np}{\sigma}\right)$
$\boldsymbol{P(X \leq a)}$	$P\left(Z \leq \frac{(a+0.5)-np}{\sigma}\right)$
$\boldsymbol{P(X < a)}$	$P\left(Z < \frac{(a-0.5)-np}{\sigma}\right)$
$\boldsymbol{P(X \geq a)}$	$P\left(Z \geq \frac{(a-0.5)-np}{\sigma}\right)$
$\boldsymbol{P(a \leq X \leq b)}$	$P\left(\frac{(a-0.5)-np}{\sigma} \leq Z \leq \frac{(b+0.5)-np}{\sigma}\right)$
P(X=k) where **k** is a positive integer between **0** and **n**	$P\left(\frac{k-0.5-np}{\sigma} \leq Z \leq\right)\frac{k+0.5-np}{\sigma}$

Example01

Assuming a coin is tossed 15 times, the probability of heads appearing is **0.4**, and let X be the random variable representing the number of heads that can appear.
- By using the binomial distribution formula and then using the normal approximation, calculate the probability **P(X=4)**?

Solution:

$$p=0.4.\ ; \quad n=15 \qquad\qquad X \rightsquigarrow B(15; 0.4)$$

-Calculating the probability: P(X=4)

- ✓ **Using the binomial distribution relationship :**

$$P(X) = \ c_X^{15}(0.4)^X(0.6)^{15-X} \quad , \quad X = 0,1,2, \dots 15$$

$$\boldsymbol{P(X=4)} = c_4^{15}(0.4)^4(0.6)^{15-4} = \frac{15!}{(15-4)!\,4!}(0.4)^4(0.6)^{11} = 0.1267$$

- ✓ **Using the binomial distribution approximation to the normal distribution, then:**

$$E(X) = np = (15)(0.4) = 6$$

$$\sigma^2 = npq = (15)(0.4)(0.6) = 3.6 \ , \sigma = 1.8973$$

$$P(X = 4) \approx P(4 - 0.5 \leq X \leq 4 + 0.5)$$

$$P\left(\frac{(4-0.5)-np}{\sigma} \le Z \le \frac{(4+0.5)-np}{\sigma}\right)$$

$$P\left(\frac{3.5-6}{1.897} \le Z \le \frac{4.5-6}{1.897}\right)$$

$$P(-1.3176 \le Z \le -0.7905)$$

$$P(Z \le -0.790) - P(Z \le -1.318)$$

$$=(1-0.78524)-(1-0.90490) = 0.11966$$

Example02

Let us have $\boldsymbol{X \rightsquigarrow B(20;0.3)}$

- Find : $P(X) = 8$, $P(X) \le 8$?

Solution:

Using the binomial distribution formula:

$$\boldsymbol{P(X=8)} = c_8^{20}(0.3)^8(0.7)^{20-8} = 0.1144$$

Using the approximation of the binomial distribution to the normal distribution:

$$np = 6 \ ; \ nq = 14$$

Since the conditions for approximation are met, a binomial distribution can be approximated to a normal distribution.

$$E(X) = np = (20)(0.3) = 6$$

$$\sigma^2 = npq = (20)(0.3)(0.7) = 4.2 \quad ; \ \sigma = \sqrt{4.2}$$

$$P(X=8) \approx P(8-0.5 \le X \le 8+0.5)$$

$$P\left(\frac{7.5-np}{\sigma} \le Z \le \frac{8.5-np}{\sigma}\right)$$

$$= P\left(\frac{7.5-6}{\sqrt{4.2}} \le Z \le \frac{8.5-6}{\sqrt{4.2}}\right)$$

$$= P(0.73 \le Z \le 1.22) = 0.888 - 0.7673 = 0.1215$$

$$\boldsymbol{P(X \le 8)} = P\left(Z \le \frac{8+0.5-np}{\sigma}\right)$$

$$= P\left(Z \le \frac{8.5-6}{\sqrt{4.2}}\right) = P(Z \le 1.22) = 0.8888$$

5- ***Approximation of The Poisson Distribution To The Normal Distribution***

Let X be a random variable following the Poisson probability distribution with parameter λ>0, with the probability density function:

$$P(X = x)\begin{cases} \frac{e^{-\lambda}\ \lambda^{x}}{x!} & ;\quad x = 0, 1, 2 \ldots \propto \\ 0 & \text{, Otherwise} \end{cases}$$

$$E(X) = \mu = \lambda \quad ; \quad V(X) = \sigma^2 = \lambda$$

The standard random variable Z: $Z = \frac{X-\mu}{\sigma} == \frac{X-\lambda}{\sqrt{\lambda}}$

Therefore, the Poisson distribution approaches the normal distribution as λ→∞, meaning:

$$\lim_{\lambda\to\infty}\left(a \le \frac{X-\lambda}{\sqrt{\lambda}} \le b\right) = \frac{1}{\sqrt{2\pi}}\int_a^b -\frac{1}{2}x^2 dx$$

Statisticians consider that the **Poisson distribution** approaches **the normal distribution when λ is large**, specifically **when λ≥18**.

Example:

The average defective production in a computer manufacturing plant over a month was 20 units.

✓ Find the probability of having 03 defective units among the production of a randomly selected sample during a month using the normal distribution as an approximation to the Poisson distribution.

Solution:

The Poisson distribution can be approximated by the normal distribution because the approximation rule is satisfied with **λ=20**.

$\mu = 20 \quad ; \sigma^2 = 20$

$$P(X \leq 3) = P\left[Z \leq \frac{(3 + 0.5) - \lambda}{\sqrt{\lambda}}\right] = P\left[Z \leq \frac{3.5 - 20}{\sqrt{20}}\right]$$

$$= P(Z \leq 3.68) = 1 - 0.99987 = 0.00013$$

6- ***Approximating The Chi-Squared Distribution to The Normal Distribution***

If the random variable X follows a chi-squared distribution with V degrees of freedom, $X \sim \chi_v$ is a chi-squared distribution.

E(X) = v ; Var(X) = 2v

If the number of degrees of freedom **v** $\rightarrow \infty$, then:

$$\frac{\boldsymbol{X - \mathrm{v}}}{\sqrt{\boldsymbol{2\mathrm{V}}}} \underset{\boldsymbol{law}}{\longrightarrow} \boldsymbol{N(0,1)}$$

- However, for small values of ν , we obtain a better approximation by using convergence as the result:

$\sqrt{\boldsymbol{2\chi^2}} - \sqrt{\boldsymbol{2v-1}} \underset{\boldsymbol{law}}{\longrightarrow} N(0,1)$ follows the standard normal distribution.

Example:

The order fractile 0.025 of the chi-square distribution with 50 degrees of freedom has the value $\boldsymbol{\chi^2 = 71.420}$ **(Table No. 5 in the appendix) ,** With the first approximation, we write:

$$71.420 = P\left(\frac{X-50}{\sqrt{2(50)}} \geq \frac{u-\mathrm{v}}{\sqrt{2\mathrm{V}}}\right)$$

$$1 - 0.025 = 0.975 = P\left(\frac{X-\mathrm{V}}{\sqrt{2\mathrm{V}}} \leq \frac{u-\mathrm{v}}{\sqrt{2\mathrm{V}}}\right)$$

The order fractile 0.975 of the Normal distribution N(0,1) has the value z = 1.96 **(Table No. 3 in the appendix)**

from which we deduce as an approximate value of u:

$$\frac{u-\mathrm{v}}{\sqrt{2\mathrm{V}}} = z \Leftrightarrow \frac{u-50}{\sqrt{2(50)}} = 1.96 \Rightarrow u - 50 = 1.96\left(\sqrt{2(50)}\right) \Rightarrow$$

$$u = 50 + 19.6 = 69.6$$

To use t**he second approximation**, we write:

$$\mathbf{0.025} = \boldsymbol{P}\left(\sqrt{\mathbf{2}\boldsymbol{\chi}^{\mathbf{2}}} - \sqrt{\mathbf{2}\boldsymbol{v} - \mathbf{1}} \geq \sqrt{\mathbf{2}\boldsymbol{u}} - \sqrt{\mathbf{2}\boldsymbol{v} - \mathbf{1}}\right)$$

$$\sqrt{\mathbf{2}\boldsymbol{u}} - \sqrt{\mathbf{2}\boldsymbol{v} - \mathbf{1}} \cong \boldsymbol{z} \ , \quad \boldsymbol{z} = \mathbf{1.96}$$

$$\sqrt{\mathbf{2}\boldsymbol{u}} \cong \boldsymbol{z} + \sqrt{\mathbf{2}\boldsymbol{v} - \mathbf{1}}$$

$$\sqrt{\mathbf{2}\boldsymbol{u}} \cong \mathbf{1.96} + \sqrt{\mathbf{2}(\mathbf{50}) - \mathbf{1}}$$

$\boldsymbol{u} \cong \mathbf{70.93}$ (value closer to the exact value.)

7- *Approximating The Student Distribution to The Normal Distribution*

If the random variable X follows a Student's t-distribution with degrees of freedom **v**, meaning $X \sim t_v$, then the random variable $Z = \frac{X-v}{\sqrt{\frac{v}{v-2}}}$ approaches the standard normal distribution as **v approaches infinity**.

$$\boldsymbol{E(T) = 0} \qquad \boldsymbol{v \geq 2} \qquad , \quad \boldsymbol{V(T) = \frac{v}{v-2}}$$

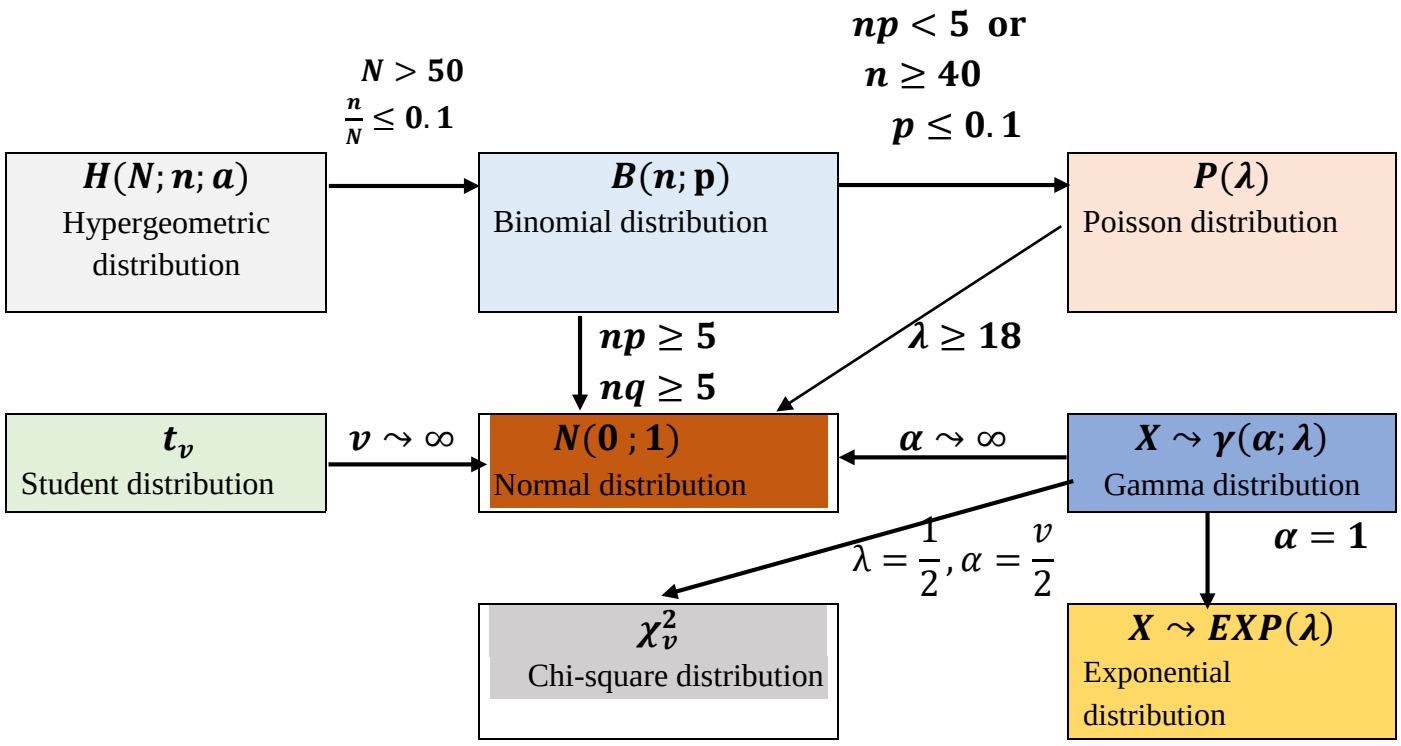
Statisticians have determined that when **v≥30**, the Student's t-distribution approaches the normal distribution.

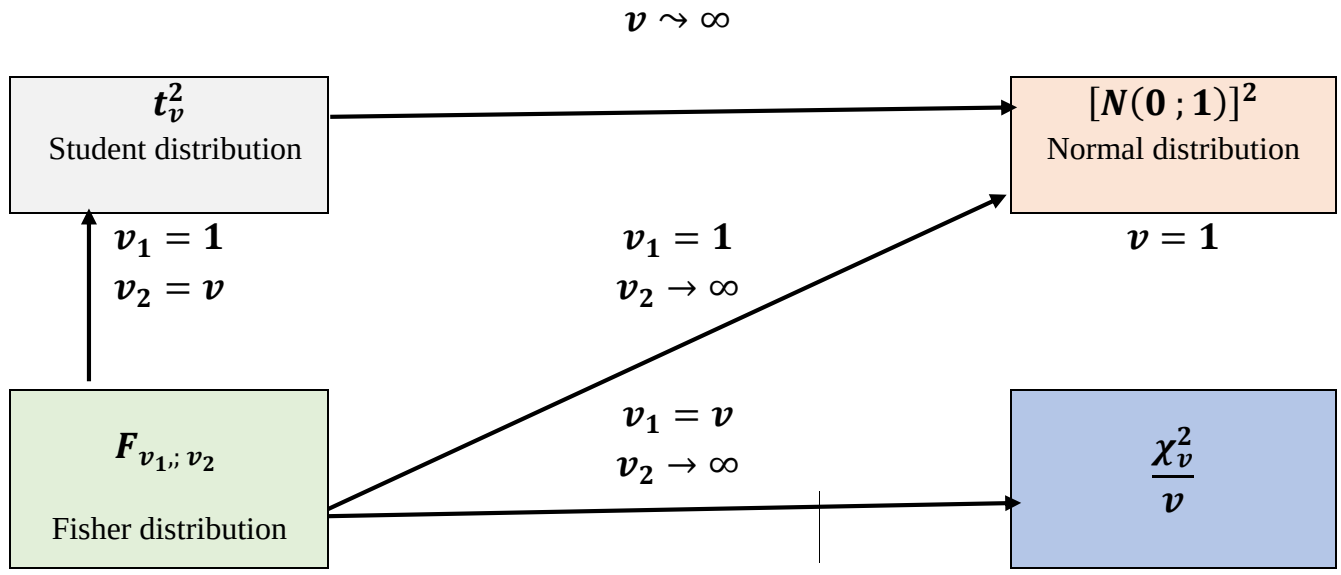
Example:

To get an idea of the approximation of the fractile of order 0.95 by that of the normal distribution which is z =1.645, the table below gives the exact fractiles of the Student distribution as a function of n **(Table No. 4 in the appendix)**

n	30	40	80	100	1000
t	1.697	1.684	1.664	1.660	1.646

Figure 13 : Approximation rules among the most important probability distributions





The study of discrete or continuous random variables, in one dimension, does not allow for the resolution of all issues that require statistical theory.

Indeed, the studied events can be linked to several random variables simultaneously, and hence the need arose to determine the joint, marginal, and conditional probability laws as well as all the associated properties.
All random variables are defined on the same sample space, and in this section, we will limit ourselves to studying two random variables (X, Y) by conducting a study of discrete joint random variables. and continuous And their properties, then studying the properties of discrete and continuous bivariate random functions, with a focus on the case where the variables are independent of each other. Independent of each other.

1- Bivariate Discrete Random Variables

1-1- Definition

The random vector (X,Y) is called a discrete random vector if each of its components is a discrete random variable, meaning that the vector (X,Y) is discrete if its values form a finite and countable (indivisible) set of real pairs X,Y. Therefore, the discrete bivariate random variables consist of two variables X and Y, for which the set of possible values can be written in the form $\{x_i\}_{i\in I}$ and $\{y_j\}_{j\in J}$, where I and J belong to N.

1-2- Bivariate Discrete Distributions and The Marginal Function

As in the case of one-dimensional random variables, the law of discrete bivariate random variables is defined by the set of possible values, let it be:
$\{(x_i, y_j);\ (i,j) \in I \times J\}$ and associated with probabilities.

$$f(X,Y) = p_{ij} = P(X = x_i\ , Y = y_j\ )$$
$$= P\left((X = x_i) \cap (Y = y_j\ )\right) = P(x_i\ , y_j\ )$$

The joint probability distribution for the independent variables (X,Y) is a joint probability density function f(X,Y) for the independent random variables (X,Y), and it gives the probabilities corresponding to different values of (X,Y) for the various values of this function. This function satisfies the conditions of a density function.

$$\begin{cases} f(X,Y) \geq 0 \\ \sum_X \sum_Y f(X,Y) = 1 \end{cases}$$

This function is given in the form of a table called the discrete probability distribution table .

Table 04: Discrete Probability Distribution Table

	y_1	y_2	…	y_m	$f_1(x_i)$
x_1	$f(x_1,y_1)$	$f(x_1,y_2)$	…	$f(x_1,y_m)$	$f(x_1)$
x_2	$f(x_2,y_1)$	$f(x_2,y_2)$	…	$f(x_2,y_m)$	$f(x_2)$
.	.	…	…		.
.	.				.
.	.				.
		…			
x_n	$f(x_n,y_1)$	$f(x_n,y_2)$	…	$f(x_n,y_m)$	$f(x_n)$
$f_2(y_j)$	$f(y_1)$	$f(y_2)$	…	$f(y_m)$	**1**

$f_1(x_i)$: represents the sum of all probabilities in the row corresponding to the values x_i (i=1,2,…n)

$f_2(y_j)$: represents the sum of all probabilities in the row corresponding to the values y_j (j=1,2,…m).

The functions $f_1(x_i)$ and $f_2(y_j)$ are called **Marginal Functions** for both x and y, and it means that one of the variables takes a specific value regardless of all the values that the variable can take, and they are defined as follows:

$$f_1(x) = \sum_j f(x,y) \qquad , f_2(y) = \sum_i f(x,y)$$

So that : $\sum f_1(x) = 1$; $\sum f_2(y) = 1$;

The probability that: X=x is in the form of:

$$P(\mathrm{X} = \mathrm{x}) = f_1(x) = \sum_{j=1}^{m} f(x, y_j)$$

Then $f_1(x)$ represents the probability that the variable **X** takes the value x_i without considering the values taken by the random variable Y,

- ✓ The probability that: Y=y is in the form:

$$P(Y = \mathrm{y}) = f_2(y) = \sum_{i=1}^{n} f(x_i, y)$$

Table 05:The Marginal Distribution Table

X	x_1	x_2	…	x_i	…
$\boldsymbol{f_1(x)}$	$f(y_1)$	$f(y_2)$	…..	$f(y_i)$	.....

1-3- Discrete cumulative distribution function (distribution function):

The cumulative function for the variables (x,y) is given as follows:

$$\mathrm{F}(\mathrm{x},\mathrm{y}) = \mathrm{P}(X \le x, Y \le x)$$
$$= \sum_{X \le x} \sum_{Y \le y} \mathrm{f}(\mathrm{X},\mathrm{Y}) = \sum_{X \le x} \sum_{Y \le y} P_{ij}$$

Example01:

Two dice were rolled once, assuming that X is a random variable representing the sum of the points showing on the upper face of the two dice and Y represents the largest number showing on the upper face of the two dice. Find the joint probability density function for the pair (x,y).

Solution:

The sample space for this random experiment is:

$$S = \{(i,j): i = j = 1,2,3 \ldots 6\}$$
$$X = (i+j) = 2,3,4 \ldots 12$$
$$Y = Max(i,j) = 1,2,3, \ldots 6$$

The probabilities corresponding to the pair of variables (x,y) are calculated as follows:

$$\mathrm{f}(2,1)=\mathrm{P}(\mathrm{X}=2,\ \mathrm{Y}=1)=\mathrm{P}(1,1)=1/36$$

$$\mathrm{f}(2,2) = \mathrm{P}(\mathrm{X}=2, \mathrm{Y}=2) = \mathrm{P}(\Phi)=0$$

$$\mathrm{f}(4,1) = \mathrm{P}(\mathrm{X}=4, \mathrm{Y}=1) = \mathrm{P}(\Phi)=0$$

$$f(4,2) = P(X = 4, Y = 2) = P(2,2) = 1/36$$

$$f(4,3) = P(X = 4, Y = 3) = P[(3,1),(1,3)] = 2/36$$

$$f(4,4) = P(X = 4, Y = 4) = P[(\Phi)] = 0$$

In the same way, the remaining probabilities corresponding to the bivariate variable (x,y) can be calculated, so that this can be summarized in the joint table representing the values of the probability density function f(X,Y).

The Discrete Probability Distribution Table

Y / X	**1**	**2**	**3**	**4**	**5**	**6**	$f_1(x)$
2	1/36	0	0	0	0	0	**1/36**
3	0	2/36	0	0	0	0	**2/36**
4	0	1/36	2/36	0	0	0	**3/36**
5	0	0	2/36	2/36	0	0	**4/36**
6	0	0	1/36	2/36	2/36	0	**5/36**
7	0	0	0	2/36	2/36	2/36	**6/36**
8	0	0	0	1/36	2/36	2/36	**5/36**
9	0	0	0	0	2/36	2/36	**4/36**
10	0	0	0	0	1/36	2/36	**3/36**
11	0	0	0	0	0	2/36	**2/36**
12	0	0	0	0	0	1/36	**1/36**
$f_2(y)$	**1/36**	**3/36**	**5/36**	**7/36**	**9/36**	**11/36**	**1**

Example02:

The electronic device contains nine electrical connections, three of which are white. Two black and four red, with the aim of inspecting this device and ensuring its safety before ramping up production, two units of this device were randomly selected for inspection.

Let X be a random variable representing the number of white links drawn.
Let Y be a random variable representing the number of black links drawn.

a) Find the joint probability distribution table for the random variables X and Y.

b) Find the marginal distributions for each of the separate variables X and Y.

Solution:

a) The Probability Distribution Table

Y \ X	0	1	2
0	f(0,0)	f(0,1)	f(0,2)
1	f(1,0)	f(1,1)	f(1,2)
2	f(2,0)	f(2,1)	f(2,2)

Such that:

$$f(0,0) = P(X = 0, Y = 0) = \frac{C_3^0\ C_2^0 C_4^2}{C_9^2} = \frac{6}{36}$$

$$f(0,1) = P(X = 0, Y = 1) = \frac{C_3^0\ C_2^1 C_4^1}{C_9^2} = \frac{8}{36}$$

$$f(0,1) = P(X = 0, Y = 1) = 0$$

b) The Joint Probability Distribution Table

X \ Y	0	1	2	**The total $f_1(x)$**
0	$\frac{6}{36}$	$\frac{8}{36}$	$\frac{1}{36}$	$^{15}/_{36}$
1	$\frac{12}{36}$	$\frac{6}{36}$	0	$^{18}/_{36}$
0	$\frac{3}{36}$	0	0	$^{3}/_{36}$
The total $f_2(y)$	$^{21}/_{36}$	$^{14}/_{36}$	$^{1}/_{36}$	1

$f_1(x)$ ، $f_2(y)$: They represent the marginal (boundary) functions of the variables x and y, respectively

Example03:

$\mathbf{f(X,Y) = q^2 p^{y-2}}$

$x = 1,2 \ldots . y - 1 \quad ; y = 2,3,4 \ldots .$

$0 \leq p \leq 1 \ , \ q = 1 - p$

a) Make sure that f(X,Y) is a probability function?

b) Find the value of the probability f(2,4)?

Solution:

- **Ensuring that f(X,Y) is a Probability Function**

The first condition: We observe that $f(X,Y) \geq 0$ for all values of x, y.

The second condition: $\sum_X \sum_Y f(X,Y) = 1$

From the limits of the variables, we notice that x starts from 1; and y starts from 1+x, thus:

$$\sum_{x=1}^{\infty} \sum_{y=x+1}^{\infty} f(x,y) = q^2 \sum_{x=1}^{\infty} \sum_{y=x+1}^{\infty} p^{y-2}$$

$$= q^2 \sum_{x=1}^{\infty} (p^{x-1} + p^x + p^{x+1} + \cdots)$$

$$= q \sum_{x=1}^{\infty} p^{x-1} = q . \frac{1}{1-p} = 1$$

So we conclude that the function f(X,Y). is a probability density function

- **Calculating the probability f(2,4)**

$$\mathbf{F(x,y) = P(X \leq x, Y \leq y) = \sum_{x=1}^{\infty} \sum_{y=x+1}^{\infty} q^2 p^{y-2}}$$

$$\mathbf{F(2,4) = \sum_{x=1}^{2} \sum_{y=x+1}^{4} q^2 p^{y-2} = F(2,4)}$$

$$\mathbf{= q^2 (1 + 2p + 2p^2)}$$

Example04:
If the probability function for the variables X and Y is given in the following table:

X \ Y	-2	0	5
1	0.15	0.25	0.2
3	0.2	0.05	0.15

- **Find the Marginal Probability Density Functions for the random variables X and Y?**

Solution:
By summing the probabilities horizontally and vertically, we obtain the following table:

X \ **Y**	**-2**	**0**	**5**	$f_1(x)$
1	0.15	0.25	0.20	**0.60**
3	0.20	0.05	0.15	**0.40**
$f_2(y)$	**0.35**	**0.30**	**0.35**	**1**

-The marginal probability density function of the variable X is given as follows:

X	−2	0
$\boldsymbol{f_1(x)}$	0.60	0.40

- The marginal probability density function of the variable Y is given as follows:

Y	-2	0	5
$\boldsymbol{f_2(y)}$	0.35	0.30	0.35

2- Bivariate Continuous Random Variables

2-1- Joint Probability Density Function:

Let X and Y be continuous variables. We say that f(X,Y) is **a Joint Probability Density Function** for the random variables (X,Y) if the following two conditions are satisfied:

$$\begin{cases} \mathbf{f}(\mathbf{x},\mathbf{y}) \geq \mathbf{0} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{f}(\mathbf{x},\mathbf{y})\boldsymbol{dxdy} = \mathbf{1} \end{cases}$$

From the previous definition, we find:

$$P(x_1 \leq X \leq x_2\ , y_1 \leq Y \leq y_2\ ) = \int_{x_1}^{x_2}\int_{y_1}^{y_2} \mathrm{f(x,y)}dxdy$$

The graphical representation of the random vector (X,Y) is obtained by plotting the probability density function f(X,Y) graphically for every value $(\mathrm{X},\mathrm{Y}) \in \mathbb{R}^2$ in the plane.

Thus, the relationship $P(x_1 \leq X \leq x_2\ , y_1 \leq Y \leq y_2\ )$ represents the volume under the curve of the joint probability density function and above the area defined by the lines $x = x_1$ و $x = x_2$, $y = y_1$ و $y = y_2$ in the Cartesian plane.

2−2− The Continuous Marginal Probability Density Function

Through the definition of the joint probability density function for the continuous random variables (X, Y), it is possible to obtain the marginal probability density functions for each of the continuous random variables x and y, respectively.

$$f_1(x) = \int_{-\infty}^{\infty} \mathrm{f(x,y)}dy$$

$$f_2(y) = \int_{-\infty}^{\infty} \mathrm{f(x,y)}dx$$

2-3- Continuous Cumulative Distribution Function:

If **f(X,Y)** is the probability density function for two random variables X and Y, then the **joint cumulative distribution function** for the variables X and Y, denoted by **F(x,y)**, is defined as follows:

$$\mathbf{F(a,b)} = \boldsymbol{P(X \le a, Y \le b)} = \int_{-\infty}^{a}\int_{-\infty}^{b} \mathbf{f(x,y)}\,dxdy$$

Where **a** and **b** represent real numbers of the values of the variables X and Y, respectively.

The distribution function satisfies the following conditions:

$$\mathbf{0 \le F\,(x,y) \le 1}$$

$$\mathrm{F(x,-\infty)} = p(X \le x\,, Y \le -\infty) = 0$$

$$\mathrm{F(-\infty,y)} = p(X \le -\infty\,, Y \le y) = 0$$

$$\mathrm{F(+\infty,+\infty)} = 1$$

$$\mathrm{F(-\infty,-\infty)} = 0$$

Such that this function is non-decreasing, that is:

$$x_1 \le x_2 \Rightarrow \mathrm{F}(x_1, \mathrm{y}) \le \mathrm{F}(x_2, \mathrm{y})$$

$$y_1 \le y_2 \Rightarrow \mathrm{F}(x, y_1) \le \mathrm{F}(x, y_2)$$

If (x,y) are continuous random variables, then the joint probability density function (probability density function) for these variables **f(x,y)** is given as follows:

$$\mathrm{f(x,y)} = \frac{\partial}{\partial x}\frac{\partial}{\partial y}F(x,y)$$

Example 01:

If x and y are independent random variables, their **joint probability density function** is given as follows:

$\mathbf{f(x,y)} = \boldsymbol{D\,x\,y}$; $\mathbf{0 \le x \le y \le 1}$

$\boldsymbol{D \ge 0}$, D : Constant value

Find :

a) The value of the constant D ?
b) The probability $F(1/3, 1/2)$?
c) The marginal probability density function for both X and Y ?

Solution:

a) The value of the constant term c

- We conclude that the condition $\mathbf{f}(\mathbf{X},\mathbf{Y}) \geq 0$ is satisfied for all values of x and y

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{f(X,Y)}dxdy = c\int_0^1 xdx\int_x^1 \mathrm{y}dy = 1$$

$$= c\int_0^1 x\left[\frac{y^2}{2}\right]_x^1 dx \quad = \frac{c}{2}\int_0^1 x(1-x^2)dx$$

$$= \frac{c}{2}\left[\frac{x^2}{2} - \frac{x^2}{4}\right]_0^1 = 1$$

$$\frac{c}{8} = 1 \implies c = 8$$

$$\mathbf{f(x,y) = 8\,x\,y} \quad ; \ \mathbf{0 \leq x \leq y \leq 1}$$

b) The probability $\mathbf{F(1/3, 1/2)}$

$$F(a,b) = \boldsymbol{P(X \leq a, Y \leq b)}$$

$$= \mathbf{8}\int_0^a xdx\int_x^b ydy$$

$$= \frac{8}{2}\int_0^a x(b^2 - x^2)dx$$

$$= \frac{8}{2}\left[b^2\frac{x^2}{2} - \frac{x^4}{4}\right]_0^a$$

$$2a^2b^2 - a^4$$

$$F\left(\frac{1}{3},\frac{1}{2}\right) = 2\left(\frac{1}{3}\right)^2\left(\frac{1}{2}\right)^2 - \left(\frac{1}{3}\right)^4 = \frac{7}{162}$$

c) The Marginal Probability function for both X and Y

$$f_1(x) = \int_{-\infty}^{\infty} \mathrm{f(X,Y)}dy = 8\int_x^1 xy\,dy = 8x\left[\frac{y^2}{2}\right]_x^1 = 4x(1-x^2)$$

$$f_2(y) = \int_{-\infty}^{\infty} \mathrm{f(X,Y)}dx = 8\int_0^y xy\,dx = 8y\left[\frac{x^2}{2}\right]_0^y = 4y^3$$

✓ The Marginal Probability function for X becomes in the form :

$$\boldsymbol{f_1(x)} = \begin{cases} \boldsymbol{4x(1-x^2)} & ; \quad \boldsymbol{0 \le x \le 1} \\ \boldsymbol{0} & ; \ \textbf{Otherwise;} \end{cases}$$

✓ The Marginal Probability function for Y becomes in the form :

$$\boldsymbol{f_2(y)} = \begin{cases} \boldsymbol{4y^3} & ; \quad \boldsymbol{0 \le y \le 1} \\ \boldsymbol{0} & ; \ \textbf{Otherwise} \end{cases}$$

Example 02:

Let X and Y be continuous random variables whose joint density function is given in the following form.

$$\boldsymbol{f(x,y) = \frac{x}{4}(1+3y^2)} \ ; \ \boldsymbol{0 \le x \le 2} \ ; \ \boldsymbol{0 \le y \le 1}$$

a) Prove that f(X,Y) represents a **probability density function**?
b) Find the **marginal density functions** for each of the random variables X and Y?
c) Find the **joint cumulative distribution function** for the random variables (X,Y)?
d) Find the probability : **P(X<1, Y<0.5)** ?

Solution:

a) The previous probability density function can be rewritten as follows:

$$f(x,y) = \begin{cases} \frac{x}{4}(1+3y^2) & ;\ 0<x<2\ ,\quad 0<y<1 \\ 0 & ;\text{Otherwise} \end{cases}$$

No matter the value of both Y and X, f(x,y) ≥ 0.
Also:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{f(x,y)}dxdy = 1$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{f(x,y)}dxdy = \int_{0}^{2}\int_{0}^{1} \frac{x}{4}(1+3y^2)dxdy$$

$$= \frac{1}{4}\int_{0}^{2} x\left[\int_{0}^{1}[(1+3y^2)dy]\right]dx$$

$$= \frac{1}{4}\int_{0}^{2} x[(y+y^3)]_0^1\,dx = \frac{1}{4}\int_{0}^{2} 2xdx = \frac{1}{2}\left[\frac{x^2}{2}\right]_0^2 = 1$$

And thus we conclude that f(x,y) represents the joint probability density function of the variables x and y.

b) The Marginal Density Functions

$$f_1(x) = \int_{-\infty}^{\infty} \mathrm{f(x,y)}dy = \int_{0}^{1} \frac{x}{4}(1+3y^2)dy = \frac{x}{4}[(y+y^3)]_0^1 = \frac{x}{2}$$

$$f_2(y) = \int_{-\infty}^{\infty} \mathrm{f(x,y)}dx = \int_{0}^{2} \frac{x}{4}(1+3y^2)dx = \frac{1+3y^2}{4}\left[\frac{x^2}{2}\right]_0^2 = \frac{1+3y^2}{4}$$

- The marginal density function of the random variable X becomes as follows:

$$f_1(x) = \begin{cases} \frac{x}{2} & ; \quad 0 < x < 2 \\ 0 & ;\text{Otherwise} \end{cases}$$

- The marginal density function of the random variable X becomes as follows:

$$f_2(y) = \begin{cases} \frac{1+3y^2}{2} & ; \quad 0 < y < 1 \\ 0 & ; \text{Otherwise} \end{cases}$$

c) Joint Distribution Function $F(x, y)$

The joint cumulative distribution function (Cumulative) for the variables F(x,y), can be found as follows:

$$F(x,y) = \boldsymbol{P}(\boldsymbol{X} \le \boldsymbol{x}, \boldsymbol{Y} \le \boldsymbol{y}) = \int_{-\infty}^{x}\int_{-\infty}^{y} \mathrm{f(X,Y)}dxdy$$

$$\int_{-\infty}^{x}\int_{-\infty}^{y} \frac{x}{4}x(1+3y^2)dxdy$$

$$= \frac{1}{4}\int_{0}^{x}\int_{0}^{y} x(1+3y^2)dxdy$$

$$= \frac{1}{4}\int_{0}^{x} x[(y+y^3)]_0^y\, dx = \frac{1}{4}\int_{0}^{x} x(y+y^3)dx$$

$$= \frac{1}{4}(y+y^3)\left[\frac{x^2}{2}\right]_0^x = \frac{1}{4}(y+y^3)\frac{x^2}{2} = \frac{1}{8}x^2(y+y^3)$$

$$\boldsymbol{F(x,y) = \frac{1}{8}x^2(y+y^3)}$$

d) The probability: P(X<1, Y<0.5)

$$F(1,0.5) = \boldsymbol{P}(\boldsymbol{X} \leq \mathbf{1}, \boldsymbol{Y} \leq \mathbf{0.5}) = \int_0^1 \int_0^{0.5} \frac{x}{4} x(1+3y^2) dxdy$$

$$= \frac{1}{4}\int_0^1 x[(y+y^3)]_0^{0.5}\, dx = 0.156\left[\frac{x^2}{2}\right]_0^1 = 0.078$$

The probability can also be calculated using the cumulative distribution function.

$$F(1,0.5) = \boldsymbol{P}(\boldsymbol{X} \leq \mathbf{1}, \quad \boldsymbol{Y} \leq \mathbf{0.5}) =$$

$$\frac{1}{8}x^2(y+y^3) = \frac{1}{8}1^2(0.5+0.5^3) = 0.078$$

3- Conditional Probability Distribution

3-1- The conditional probability function of two discrete random variables

If **X** and **Y** are independent random variables, their joint probability density function is f(X,Y).
The conditional probability function of the variable Y given $X = x_i$ given as follows:

$$f(y/X = x_i) = \frac{f(x_i\ ,\ y\ )}{f_1(x_i\ )}\ \ ;\ f(x_i) > 0$$

$\boldsymbol{f_1(x_i\ )}$:The marginal probability function of the variable x And thus:

$$f(y_j\ /\ x_i) = P(Y = y_j\ , X = x_i\ \ ) = \frac{P(X = x_i\ , Y = y_j\ \ )}{P(X = x_i)} = \frac{f(x_i\ ,\ y_j\ \ )}{f_1(x_i\ )}$$

Similarly, the conditional probability function of the variable X given that $\boldsymbol{Y = y_i}$ is as follows:

$$\boldsymbol{f(x/y_j) = f(x/Y = y_j) = \frac{f(x\ , y_j\ \ )}{f_2(y_j)}\ \ ;\ f(y_j) > 0}$$

$f_2(\boldsymbol{y_j})$: The marginal probability function of the variable Y

3-2- The Conditional Probability Function for two continuous random variables

If (X, Y) are continuous random variables such that their probability density function is f(x, y) , then:

- The conditional probability density function of the variable Y given $\boldsymbol{X = x}$ is

$$f(y/x) = f(y/X = x) = \frac{f(x\ ,\ y\ )}{f_1(x\ )} \quad ;\ f_1(x\ ) > 0$$

$f_1(x\ )$: The marginal probability function of the variable X

- The conditional probability density function of the variable x given Y=y is:

$$f\ (x/y) = \frac{f(x\ ,y)}{f_2(y)} \geq 0 \quad , \quad f_2(y) > 0$$

$f_2(y)$: The marginal probability function of the variable Y

The conditional probability function satisfies all the conditions of the known probability distribution, which are:

$$\int_{-\infty}^{\infty} f(x/y)dx = \frac{1}{f_2(y)}\int_{-\infty}^{\infty} f(x,y)dx = \frac{f_2(y)}{f_2(y)} = 1$$

$$\int_{-\infty}^{\infty} f(y/x)dx = \frac{1}{f_1(x)}\int_{-\infty}^{\infty} f(x,y)dx = \frac{f_1(x)}{f_1(x)} = 1$$

Example:

The **joint probability density function of** x and y is given as follows:

$$\mathbf{f(x,y) = 8\,x\,y \quad ; \; 0 \le x \le y \le 1}$$

- Find the marginal probability density function for both X and Y?
- The conditional probability density function of the variables X and Y ?

Solution:

- ✓ **The marginal probability density function for both X and Y**

$$f_1(x) = \int_{-\infty}^{\infty} \mathrm{f(x,y)}dy = 8\int_{x}^{1} xy\,dy = \; 4x(1-x^2)$$

$$f_2(y) = \int_{-\infty}^{\infty} \mathrm{f(x,y)}dx = 8\int_{0}^{y} xy\,dx = 4y^3$$

- ✓ **The conditional probability density function**

$$f\,(x/y) = \frac{f(x\,,y)}{f_2(y)} = \frac{2x}{y^2} \quad ;\; 0 \le x \le \; y$$

$\boldsymbol{f\,(x/y)}$: The conditional probability density function of the variable x

$$f\,(y/x) = \frac{f(x\,,y)}{f_1(x)} = \frac{2y}{1-x^2} \quad ;\; x \le y \le \; 1$$

$\boldsymbol{f\,(y/x)}$: The conditional probability density function of the variable y

4- *Independence of Two Variables*

- The two discrete random variables X and Y are independent if:

$$\boldsymbol{f(x_i, y_j) = f_1(x_i)\, f_2(y_j)}$$

The continuous variables X and Y are independent if their joint probability function **f(x,y)** equals the product of their marginal probability functions **f(x)** and f(y), which means the following relationship holds:

$$f(x, y) = f_1(x)\, f_2(y)$$

If X and Y are independent variables, then the conditional distributions equal the marginal distributions, meaning that:

$$f(x/y) = \frac{f(x\,, y)}{f_2(y)} = \frac{f_1(x)\, f_2(y)}{f_2(y)} = f(x)$$

$$f(y/x) = \frac{f(x\,, y)}{f_1(x)} = \frac{f_1(x)\, f_2(y)}{f_1(x)} = f_2(y)$$

And thus we conclude that:

$$\boldsymbol{f(y/x) = f_2(y)} \qquad \text{و} \qquad \boldsymbol{f(x/y) = f_1(x)}$$

And this is because the independence of the variables means that knowing y does not affect the distribution of X in any way.

Example 01:

Let the joint probability function :

$$f(x, y) = \frac{\lambda^{x+y}\, e^{-2\lambda}}{x!\, y!} \qquad ; x = 0,1,2 \ldots. \;;\; y = 0,1,2 \ldots$$

-Prove that the two variables x and y are independent ?

Solution

$$f(x, y) = \left(\frac{\lambda^x\, e^{-\lambda}}{x!}\right)\left(\frac{\lambda^y\, e^{-\lambda}}{y!}\right) = f_1(x)\, f_2(y)$$

Whereas the two variables follow the Poisson distribution. we conclude that the two variables x and y are independent.

Example 02:

The following table gives the joint probability function for the two variables:

Y / X	2	3	4
1	0.06	0.15	0.09
2	0.14	0.35	0.21

- **Are the variables x and y independent ?**

Solution

Y / **X**	2	3	4	$f_1(x_i)$
1	0.06	0.15	0.09	0.3
2	0.14	0.35	0.21	0.7
$f_2(y_j)$	0.2	0.5	0.3	1

We note that for all values of x and y , for example :

F(1,4)=0.09=f(1) f(4)=(0.3) (0.3)

F(2,3)=0.35=f(2) f(3)=(0.7) (0.5)

So the two variables are independents

Corrected Exercises

1- It is known in a certain city that **10%** of startups achieve profits by the end of the fiscal year. If a random sample of 5 startups is selected, let X be a random variable representing the number of startups that achieve profits by the end of the fiscal year.

a. Determine the type and the mathematical formula for the probability law of X ?

b. In the selected sample, calculate the probability of achieving profits:
- Less than or equal to one startup ?
- Between 01 to 02 startup companies P(1≤X≤2) ?

Solution:

The type and mathematical formula for the probability law of X

$$\boldsymbol{P(X=x)} \begin{cases} \boldsymbol{c_5^x(0.1)^x(0.9)^{5-x}}\ ;\ \boldsymbol{x=0,1,2\ldots5} \\ 0 \quad ;\ \textbf{Otherwise} \end{cases}$$

X follows a binomial distribution $X \rightsquigarrow B(5;0.1)$

a. Probability calculation

$$\boldsymbol{P(X\leq 1)} = P(X=0)+P(X=1)$$
$$= c_5^0(0.1)^0(0.9)^{5-0} + c_5^1(0.1)^1(0.9)^{5-1}$$
$$= 0.5904 + 0.3280 = 0.9184$$

$$\boldsymbol{P(1\leq X\leq 2)} = \boldsymbol{P(X=1)} + \boldsymbol{P(X=2)}$$
$$= c_5^1(0.1)^1(0.9)^{5-1} + c_5^2(0.1)^2(0.9)^{5-2}$$
$$= 0.3280 + 0.0729 = 0.4009$$

2- Assuming that the time required to load a product onto a truck in a factory follows an exponential distribution, if the average time required to load the truck is 5 minutes.

a. Find the probability that this product will be loaded onto the truck in a time greater than 06 minutes?

b. Find the probability that the product will be loaded onto the truck in a time greater than 06 minutes?

Solution:

a. The mathematical formula for the probability density function of the random variable X

X follows an exponential distribution.

$$\boldsymbol{X \rightsquigarrow EXP(\frac{1}{5})}$$

$$\Longrightarrow \boldsymbol{\lambda} = \frac{1}{5}E(X) = 5 = \frac{1}{\boldsymbol{\lambda}}$$

$$\boldsymbol{P(X = x)} = \begin{cases} \boldsymbol{\lambda\, e^{-\lambda x} = \frac{1}{5}\, e^{-\frac{1}{5}x}} & \boldsymbol{; x \geq 0} \\ \boldsymbol{0} & \boldsymbol{;\ x < 0} \end{cases}$$

b. The probability of loading the product onto the truck in a time greater than 6 minutes.

$$\boldsymbol{P(X \leq x) = 1 - e^{-\lambda x}}$$

$$\boldsymbol{P(X > 6) = 1 - P(X \leq 6) = 1 - \left[1 - e^{-\left(\frac{1}{5}\right)(6)}\right] = 0.3011}$$

3- An industrial product is shipped in batches, with each batch consisting of 20 pieces. To determine whether the product is defective or not, factory experts take a random sample without replacement of five pieces from this product (from one of the batches). The batch is rejected for shipment if more than one defective piece is observed.

Let's consider X a random variable representing the number of defective pieces in the selected sample, if the batch contains four defective pieces.

a. Determine the type and mathematical formula of the probability law that X follows?
b. Calculate the probability of rejecting this shipment?
c. What is the expected number and variance of defective items in a sample of 5 pieces

Solution

a. The type and mathematical formula of the probability law for X

X follows the hypergeometric distribution.

$$X \rightsquigarrow H(5; 4\,; 20)$$

$$\boldsymbol{P(X = x)} = \begin{cases} \dfrac{\boldsymbol{C_a^x C_b^{n-x}}}{\boldsymbol{C_N^n}} = \dfrac{\boldsymbol{C_4^x C_{16}^{5-x}}}{\boldsymbol{C_{20}^5}} & ; \boldsymbol{x = 0, 1, 2 \ldots n} \\ \boldsymbol{0} & ; \textbf{ Otherwise} \end{cases}$$

b. The probability of rejecting this shipment.

$$\boldsymbol{P(X > 1)} = P(X = 2) + P(X = 3) + P(X = 4)$$

$$\boldsymbol{P(X > 1)} = 1 - (\boldsymbol{X \leq 1}) = 1 - [P(X = 0) + P(X = 1)]$$

$$= 1 - \frac{C_4^0 C_{16}^{5-0}}{C_{20}^5} - \frac{C_4^1 C_{16}^{5-1}}{C_{20}^5}$$

$$= 1 - 0.2817 - 0.4695 = \boldsymbol{0.2488}$$

c. The expected number and variance of defective items in a sample of size 5

$$\boldsymbol{\mu = np = 5\left(\frac{4}{20}\right) = 1}$$

$$\boldsymbol{\sigma^2 = npq\left(\frac{N-n}{N-1}\right) = 5\left(\frac{4}{20}\right)\left(\frac{16}{20}\right)\left(\frac{20-5}{20-1}\right) = 0.6315}$$

4- Assuming that the duration (measured in minutes) required to display a promotional advertisement for the features of a certain product on the internet for one of the companies follows a uniform distribution over the interval **[4-8]**, during each paid advertisement conducted by this company, Let's consider X a random variable representing the duration of the promotional advertisement for this product.

a. Find the mathematical formula for the probability density function of the random variable X?

b. Find the probability that the advertisement duration exceeds five minutes?

Solution:

a. The mathematical formula for the probability density function of the random variable X

$$\boldsymbol{X \rightsquigarrow U([a-b])}$$

$$\boldsymbol{P(X=x)}= \begin{cases} \dfrac{\mathbf{1}}{\mathbf{4}} \; ; & \boldsymbol{x} \in [4-8] \\ \mathbf{0} \; ; & \textbf{Otherwise } \mathbf{0} \end{cases}$$

b. The probability that the advertisement duration will exceed five minutes

$$P(X>5)=\int_5^8 \frac{\mathbf{1}}{\mathbf{4}}dx=\left[\frac{\mathbf{1}}{\mathbf{4}}\boldsymbol{x}\right]_5^8=\frac{8}{4}-\frac{5}{4}=\frac{3}{4}=\mathbf{0.75}$$

Or we calculate the probability using the cumulative distribution function:

$$\boldsymbol{F(x)=P(X\leq x)=\frac{x-a}{b-a}}$$

$$P(X>5)=1-P(X\leq 5)$$

$$P(X>5)=1-\left[\frac{\mathbf{5-4}}{\mathbf{8-4}}\right]=1-\frac{1}{4}=\frac{3}{4}=\mathbf{0.75}$$

5- Assuming that the number of new annual employees X in public institutions in a city follows a normal probability distribution such that **X→N(60;81)**, if one of the public institutions is randomly selected :

a. Calculate the probability that the number of employees exceeds 65 workers?

b. Calculate the probability that the number of employees is between 55 and 65?

c. Find the number of employees such that : $\boldsymbol{P(X>a)=0.1469}$?

Solution:

a. The probability that the number of employees exceeds 65 workers

$$\mathrm{P(X>65)}=P\left(Z>\frac{65-60}{9}\right)$$

$$=P(Z>0.55)=1-P(Z\leq 0.55)=1-0.7088=\mathbf{0.2912}$$

b. - The probability that the number of employees will be between 55 and 65 workers.

$$P(55 \le X \le 65) = P\left(\frac{55-60}{9} \le X \le \frac{65-60}{9}\right)$$

$$= P(-0.55 \le Z \le 0.55) = P(Z \le 0.55) - P(Z \le -0.55)$$

$$= P(Z \le 0.55) - [1 - P(Z \le 0.55)]$$

$$= 2P(Z \le 0.55) - 1 = 2(0.7088) - 1 = 0.4176$$

c. The number of employees a such that :$P(X > a) = 0.1469$

$$P(X > a) = 0.1469 \Leftrightarrow P\left(Z > \frac{a-60}{9}\right) = 0.1469$$

$$P\left(Z \le \frac{a-60}{9}\right) = 1 - 0.1469 \Leftrightarrow P(Z \le 1.05) = 0.8531$$

$$1.05 = \frac{a-60}{9} \Rightarrow \boldsymbol{a = 69.45}$$

6- It is known that in a factory producing a certain food product, on a given day, **10,000 pieces** are produced with an **average weight of 150 grams** and a **standard deviation of 20 grams**, assuming that the weight of the produced pieces follows a **normal probability distribution**.

a. Find the percentage and number of produced pieces whose weight is between 130 grams and 160 grams?

b. If 69.14% of the produced pieces weigh less than a certain weight, let it be b, find the value of weight b?

Solution

a. The percentage and number of produced pieces whose weight is between 130 grams and 160 grams

$$P(130 \le X \le 160) = P\left(\frac{130-150}{20} \le X \le \frac{160-150}{20}\right)$$

$$= P(-1 \le X \le 0.5) = P(Z \le 0.5) - P(Z \le -1)$$

$$= P(Z \le 0.5) - [1 - P(Z \le 1)]$$

$$= 0.6914 - 1 + 0.8413 = \mathbf{0.5327}$$

The percentage of produced pieces whose weight is between 130 grams and 160 grams is **53.27%**

The number of produced pieces whose weight is between 130 grams and 160 grams is : 5327

0.5327(10000)=5327

b. The value of weight b

$P(Z \leq b) = 0.6914$

$\Leftrightarrow P(Z \leq z) = 0.6914$

$P(Z \leq 0.5) = 0.6914$

$$Z = \frac{X - \mu}{\sigma}$$

$$0.5 = \frac{b - 150}{20} \Rightarrow b = 160$$

7- A university student goes to the college where he studies daily by train, as the train departs from its stop at 08:00 AM. Assuming that X is a random variable representing the duration of the train journey (in minutes) such that it follows a uniform distribution over the interval [50-40] (in minutes),

We also assume that this student needs to walk for a fixed duration of 15 minutes from the train station to the classroom where they study, given that classes in the classroom start at exactly 9:00 AM.

a. Find the probability of this student arriving at the classroom where he studies on time for the start of the lesson?
b. Find the probability that the student arrives at the classroom at least one minute late after the scheduled start time of the lesson ?
c. Find the expected time for this student to arrive at the classroom?

Solution

$$f(x) = \begin{cases} \dfrac{1}{50-40} = \dfrac{1}{10} & ; \quad x \in [40-50] \\ 0 & ; \text{ Otherwise} \end{cases}$$

a. The probability of this student arriving at the classroom where he studies on time for the start of the lesson

$$(X < 45) = \int_{40}^{45} \frac{1}{10}\, dx$$

$$= \left[\frac{1}{10}x\right]_{40}^{45} = \frac{5}{10}$$

b. The probability that the student arrives at the classroom at least one minute late after the scheduled start time of the lesson

$$P(X \geq 46) = 1 - P(X < 46)$$

$$= 1 - \int_{40}^{46} \frac{1}{10}\, dx$$

$$= \left[\frac{1}{10}x\right]_{40}^{46} = \frac{4}{10} = \frac{2}{5}$$

c. The expected time for this student to arrive at the classroom

$$E(X) = \frac{b+a}{2} = \frac{50+40}{2} = 45$$

8- If X is a random variable representing the number of cars entering the gas station in a quarter of an hour, with an **average of three cars** entering this station.

d. Identify the type and formula of the probability law f(x) for the random variable X?

During a quarter of an hour, Find the probability that will enter the gas station:

- One car? - Two cars?

e. Within half an hour, Find the probability of at most one car entering the gas station?

Solution

a. The type and mathematical formula of the probability law for X

X follows the **Poisson distribution**. $X \rightsquigarrow P(3)$

$$f(x) = P(X = x) = \begin{cases} \dfrac{e^{-\lambda}\,\lambda^{x}}{x!} = \dfrac{e^{-3}\,3^{x}}{x!} & ; \quad x = 0, 1, 2 \\ 0 & ; \quad \text{Otherwise} \end{cases}$$

b. The probability that will enter the gas station:

- One car :

$$P(X = 1) = \frac{e^{-3}\,3^{1}}{1!} = 0.149$$

- Two cars :

$$P(X = 2) = \frac{e^{-3}\,3^{2}}{2!} = 0.2240$$

c. The probability of at most one car entering the gas station Within half an hour

$$P(X = x) = \frac{e^{-\acute{\lambda}}\,\acute{\lambda}^{x}}{x!}$$

$$\acute{\lambda} = 2\lambda = 2(3) = 6$$

$$\Rightarrow \acute{\lambda} = 6$$

$$P(X = x) = \frac{e^{-6}\, 6^{x}}{x!}$$

$$P(X \leq 1) = P(X = 0) + P(X = 1)$$

$$= \frac{e^{-6}\, 6^{0}}{0!} + \frac{e^{-6}\, 6^{1}}{1!}$$

$$= 0.0024 + 0.0144$$

$$= 0.0168$$

9- It is known that in a mobile phone manufacturing plant, the defective production rate is 1%.

Let us consider the random variable X, which represents the number of mobile phones selected (with replacement) until the first defective mobile phone is obtained.

a. Identify the type and form of the appropriate probability distribution for the random variable X?

b. Find the probability:

-That the first defective phone selected is during the second draw (the second attempt)?

- the first defective phone selected is during the third draw (third attempt)?

- the first defective phone selected is during at least the second draw?

Solution

a. X follows the **Geometric Distribution**. $X \rightsquigarrow \mathrm{G}(0.01)$

The probability function of X is given by:

$$P(X = x) \begin{cases} q^{x-1}p = (0.99)^{x-1}(0.01) & ; \quad x = 1, 2 \ldots \infty \\ 0 & ; \quad \textbf{Otherwise} \end{cases}$$

b. The probability :

- The first defective phone selected is during the second draw $P(X = 2)$

$$P(X = 2) = (0.99)^{2-1}(0.01) = 0.0099$$

- The first defective phone selected is during the third draw $P(X = 3)$

$$= (0.99)^{3-1}(0.01) = 0.0098$$

- The first defective phone selected is during at least the second draw

$$P(X \geq 2) = 1 - P(X < 2)$$
$$= 1 - P(X = 1)$$
$$= 1 - (0.99)^{1-1}(0.01) = 0.99$$

10- A delivery worker at a commercial establishment travels a fixed distance to deliver goods to customers on every working day. The time period X in minutes required to deliver these goods is a random variable. The probability density function f(x) is given by the following formula:

$$\boldsymbol{f(x)} = \begin{cases} \lambda^2 \boldsymbol{x}\, \boldsymbol{e^{-\lambda x}} & ;\ \boldsymbol{x > 0} \\ \boldsymbol{0} & ;\ \boldsymbol{x \leq 0} \end{cases} \qquad ;\ \lambda > 0$$

a. Deduce the probability distribution type for the random variable X and find the value of the parameter λ, given that The average delivery time to customers is 80 minutes ?

b. Find the cumulative distribution function F(x) ?

c. What is the probability that the delivery time is less than 120 minutes?

Solution

a. The type of the probability law for X and the value of the parameter λ

X follows the **Gamma Distribution**. $G \rightsquigarrow P(\alpha; \lambda)$

We have :

$$f(x) = \lambda^2 x\, e^{-\lambda x} \quad ;\ x > 0 \quad \textbf{.....(1)}$$

$$f(x) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1}\, e^{-\lambda x} = \frac{\lambda^{2}}{\Gamma(3)} x^{2-1}\, e^{-\lambda x} \quad ; \; x > 0 \quad \textbf{.....(2)}$$

By comparing the two previous relationships, we obtain : $\boldsymbol{\alpha = 2}$

We have also :

$$E(X) = 80 \quad ; \quad E(X) = \frac{\alpha}{\lambda}$$

We conclude that:

$$\boldsymbol{E(X)} = \frac{\alpha}{\lambda} = \mathbf{80} \Rightarrow \frac{2}{\lambda} = \mathbf{80}$$

$$\Rightarrow \lambda = \frac{2}{80} = \mathbf{\frac{1}{40}}$$

b. The Distribution Function F(x)

- $\mathrm{F(X)} = 0$, if $x \leq 0$
- $\mathrm{F(X)} = P(X \leq x) = 1 - e^{-\lambda x} \sum_{i=1}^{\alpha} \frac{(\lambda x)^{i-1}}{\Gamma(\mathrm{i})}$, if $x \geq 0$

$$\mathbf{F(X)} = 1 - e^{-\frac{1}{40}x} \sum_{i=1}^{2} \frac{\left(\frac{1}{40}x\right)^{i-1}}{\Gamma(\mathrm{i})} = \mathbf{1} - \left[\boldsymbol{e}^{-\frac{\mathbf{1}}{\mathbf{40}}\boldsymbol{x}} \left(\mathbf{1} + \frac{\mathbf{1}}{\mathbf{40}}\boldsymbol{x}\right)\right]$$

$$\mathbf{F(X)} = \boldsymbol{P(X \leq x)} = \begin{cases} \mathbf{0} & ; \; \boldsymbol{x < 0} \\ \mathbf{1} - \left[\boldsymbol{e}^{-\frac{\mathbf{1}}{\mathbf{40}}\boldsymbol{x}} \left(\mathbf{1} + \frac{\mathbf{1}}{\mathbf{40}}\boldsymbol{x}\right)\right] & ; \; \boldsymbol{x \geq 0} \end{cases}$$

c. The probability that the delivery time is less than 120 minutes

$$P(X \leq 120) = \mathbf{1} - \left[\boldsymbol{e}^{-\frac{\mathbf{1}}{\mathbf{40}}} \left(\mathbf{1} + \frac{\mathbf{1}}{\mathbf{40}}(\mathbf{120})\right)\right] = 1 - 0.1991 = 0.8009$$

11- Through the statistical tables, find the **probability values P** and the statistical values of the **Student distribution t** and the **normal distribution Z**.

a. $\boldsymbol{P(Z \leq -1.42) = \cdots ?}$

b. $\boldsymbol{P(Z \leq 1.71) = \cdots ?}$

c. $\boldsymbol{P(Z \leq \cdots) = 0.9632}$

d. $\boldsymbol{P(T_{v=10} \leq -1.093) = \cdots ?}$

e. $\boldsymbol{P(T_{v=7} > -1.895) = \cdots ?}$

f. $\boldsymbol{P(T_{v=12} < \cdots) = 0.9}$

Solution

a. $\boldsymbol{P(Z \leq -1.42) = \cdots}$

$$P(Z \leq -1.42) = P(Z \geq -1.42)$$
$$= 1 - P(Z \leq 1.42)$$
$$= 1 - 0.9222 = 0.0778$$

b. $\boldsymbol{P(Z \leq 1.71) = \cdots}$

$$P(Z \leq 1.71) = 0.9563$$

c. $\boldsymbol{P(Z \leq \cdots) = 0.9632}$

$$P(Z \leq 1.79) = 0.9632$$

d. $\boldsymbol{P(T_{v=10} < -1.093) = \cdots ?}$

$$P(T_{v=10} \leq -1.093) = P(T_{v=10} \geq -1.093)$$
$$= 0.15$$

e. $\boldsymbol{P(T_{v=7} > -1.895) = \cdots ?}$

$$P(T_{v=7} > -1.895) = P(T_{v=7} < 1.895)$$
$$= 1 - P(T_{v=7} \geq 1.895)$$
$$= 1 - 0.05 = 0.95$$

f. $\boldsymbol{P(T_{v=12} < \cdots) = 0.9}$

$$P(T_{v=12} \geq \cdots) = 0.9$$

$1 - P(T_{v=12} \geq \cdots) = 0.9$
$P(T_{v=12} \geq \cdots) = 1 - 0.9$
$P(T_{v=12} \geq \cdots) = 0.1$
$P(T_{v=12} \geq \mathbf{1.356}) = 0.1$

12- A gauging check, carried out for several months on the diameter of parts machined by a machine tool, indicates that the percentage of "defective" parts is equal to 8%.

A sample of 100 parts from production is taken and the diameter of these parts is checked. Let X be the random variable "number of defective parts" in a sample of 100 parts.

Find the probability that there are at least 10 defective parts in the selected sample?

Solution

The variable X follows the binomial distribution B(100 ; 0.08).

Rule for approximating the binomial distribution to the normal distribution:

np ≥ 5, nq ≥ 5

np = 100(0.08) = 8 > 5, nq = 100(0.92) = 92 > 5

It can then be approximated to the normal distribution N(8;2, 712)

E(x)=np=8 , $\sigma = \sqrt{npq} = \sqrt{(100)(0.08)(0.92)} = 2.712$

$$P(X \geq 10) = P\left(Z \geq \frac{(10 - 0.5) - np}{\sigma}\right) = P\left(Z \geq \frac{9.5 - 8}{2.712}\right)$$

$$= P(Z \geq 0.553) = 1 - P(Z < 0.553) = 1 - 0.70884 = 0.2911$$

References :

1) عبد الحميد عبد المجيد البلداوي (2007)، أساليب البحث العلمي والتحليل الاحصائي، التخطيط للبحث وجمع وتحليل البيانات يدويا وباستخدام برنامج SPSS، الطبعة العربية الأولى، الاصدار الثالث، دار الشروق للنشر والتوزيع عمان الأردن.
2) خالد زهدي خواجه (1-2/8/2001)، تطبيقات أساليب المعاينة في الأقطار العربية، اصدار المعهد العربي للتدريب والبحوث الإحصائية (وقائع ندوة تطبيقات أساليب المعاينة في الأقطار العربية دمشق. سوريا)
3) اياد محمد الهوبي (2014)، الاحصاء التطبيقي، الطبعة الأولى الكلية الجامعية للعلوم والتكنولوجيا
4) بوعبد الله صالح، محاضرات الإحصاء الرياضي 2005-2006
5) مبارك أسبر ديب، مبادئ في الاحتمالات والاحصاء، كلية العلوم جامعة تشرين سوريا 2009
6) معتوق أمحمد، الاحصاء الرياضي والنماذج الاحصائية، ديوان المطبوعات الجامعية (بدون سنة نشر)
7) علي عبد السلام العماري، علي حسين العجيلي، الاحصاء والاحتمالات النظرية والتطبيق ، دار الحكمة (بدون سنة نشر)
8) علاء الدين القبانجي و حسم كمامه كمرجي(2012)، الاحتمال والاحصاء، منشورات جامعة دمشق .
9) نبيل جمعة النجار(2015)، الإحصاء التحليلي مع تطبيقات برمجية SPSS ، الطبعة الأولى، دار ومكتبة الحامد للنشر والتوزيع عمان.
10)سمير فهمي حجازي (2004-2003) ، محمود الدريني، الاحصاء التطبيقي، الشركة المصرية لإعادة التأمين. مصر.
11) عزام عبد الرحمن صبري(2015) ، الإحصاء التطبيقي بنظام SPSS، الدار المنهجية للنشر والتوزيع، عمان الأردن.

12) عدنان عباس حميدان، مطانيوس مخول، فريد جاعوني، عمار ناصر آغا (2005.2006) ، الاحصاء التطبيقي، منشورات جامعة دمشق .

13) دومينيك سالفاتور، ترجمة سعدية حافظ منتصر(1982)، ملخصات شوم نظريات ومسائل في الاحصاء والاقتصاد القياسي، الدار الدولية للنشر والتوزيع

14)عماد نشوان (2005)، الدليل العملي لمقرر الإحصاء التطبيقي، جامعة القدس المفتوحة

15) جلال مصطفى الصياد، الاستدلال الاحصائي، دار حافظ للنشر والتوزيع ص 229

16) Weiss, N. A. , Neil A. (2012) Introductory statistics; biographies by Carol A. Weiss. – 9th ed. Pearson Education

17) N. Balakrishnan, Markos V. Koutras, Konstadinos G (2020), Introduction to the Probability Models and Applications, John Wiley & Sons, Inc

18) H. Pishro-Nik, Introduction to Probability, Statistics, and Random Processes consulted on 04/10/2025
https://www.probabilitycourse.com/appendix/some_important_distributions.php

19) Renée Veysseyre (2014), Statistique et Probabilités pour l'ingénieur, 3 ème Edition Dunod, PARIS .

20) Kandethody M. Ramachandran, Chris P.Tsokos (2009), Mathematical Statistics with Applications, Elsevier Academic Press .

21) Michel Lejeune (2010), Statistique La théorie et ses applications, Deuxième édition avec exercices corrigés, Sprin-ger-Verlag Paris France .

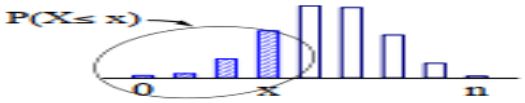
22) CHRISTOPHE HURLIN, VALÉRIE MIGNON (2015), Statistique et Probabilités en Economié-Gestion, Dunod, Paris France.

23) Dennis D. Wackerly , William Mendenhall III, Richard L. Scheaffer (2008) Mathematical Statistics with Applications, Thomson Learning

24) Dominick Salvatore et Georges Loudière (1986), Économétrie et statistique appliquées : cours et problèmes , McGraw-Hill, Auckland

25) Jean-Pierre Lecoutre (2016), Statistique et probabilités : cours et exercices corrigés , 6ème édition , Dunod. , PARIS .

APPENDIX



Tables 01: the Binomial Cumulative Distributio$_n$ $P(X \leq x)$

π		0.1	0.2	0.25	0.3	0.4	0.5	0.6	0.7	0.75	0.8	0.9
n=5	x											
	0	0.5905	0.3277	0.2373	0.1681	0.0778	0.0312	0.0102	0.0024	0.0010	0.0003	0.0000
	1	0.9185	0.7373	0.6328	0.5282	0.3370	0.1875	0.0870	0.0308	0.0156	0.0067	0.0005
	2	0.9914	0.9421	0.8965	0.8369	0.6826	0.5000	0.3174	0.1631	0.1035	0.0579	0.0086
	3	0.9995	0.9933	0.9844	0.9692	0.9130	0.8125	0.6630	0.4718	0.3672	0.2627	0.0815
	4	1.0000	0.9997	0.9990	0.9976	0.9898	0.9688	0.9222	0.8319	0.7627	0.6723	0.4095
	5	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
n=10	x											
	0	0.3487	0.1074	0.0563	0.0282	0.0060	0.0010	0.0001	0.0000	0.0000	0.0000	
	1	0.7361	0.3758	0.2440	0.1493	0.0464	0.0107	0.0017	0.0001	0.0000	0.0000	
	2	0.9298	0.6778	0.5256	0.3828	0.1673	0.0547	0.0123	0.0016	0.0004	0.0001	0.0000
	3	0.9872	0.8791	0.7759	0.6496	0.3823	0.1719	0.0548	0.0106	0.0035	0.0009	0.0000
	4	0.9984	0.9672	0.9219	0.8497	0.6331	0.3770	0.1662	0.0473	0.0197	0.0064	0.0001
	5	0.9999	0.9936	0.9803	0.9527	0.8338	0.6230	0.3669	0.1503	0.0781	0.0328	0.0016
	6	1.0000	0.9991	0.9965	0.9894	0.9452	0.8281	0.6177	0.3504	0.2241	0.1209	0.0128
	7	1.0000	0.9999	0.9996	0.9984	0.9877	0.9453	0.8327	0.6172	0.4744	0.3222	0.0702
	8		1.0000	1.0000	0.9999	0.9983	0.9893	0.9536	0.8507	0.7560	0.6242	0.2639
	9		1.0000	1.0000	1.0000	0.9999	0.9990	0.9940	0.9718	0.9437	0.8926	0.6513
	10				1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
n=15	x											
	0	0.2059	0.0352	0.0134	0.0047	0.0005	0.0000	0.0000				
	1	0.5490	0.1671	0.0802	0.0353	0.0052	0.0005	0.0000	0.0000			
	2	0.8159	0.3980	0.2361	0.1268	0.0271	0.0037	0.0003	0.0000	0.0000		
	3	0.9444	0.6482	0.4613	0.2969	0.0905	0.0176	0.0019	0.0001	0.0000	0.0000	
	4	0.9873	0.8358	0.6865	0.5155	0.2173	0.0592	0.0093	0.0007	0.0001	0.0000	
	5	0.9978	0.9389	0.8516	0.7216	0.4032	0.1509	0.0338	0.0037	0.0008	0.0001	
	6	0.9997	0.9819	0.9434	0.8689	0.6098	0.3036	0.0950	0.0152	0.0042	0.0008	0.0000
	7	1.0000	0.9958	0.9827	0.9500	0.7869	0.5000	0.2131	0.0500	0.0173	0.0042	0.0000
	8	1.0000	0.9992	0.9958	0.9848	0.9050	0.6964	0.3902	0.1311	0.0566	0.0181	0.0003
	9		0.9999	0.9992	0.9963	0.9662	0.8491	0.5968	0.2784	0.1484	0.0611	0.0022
	10		1.0000	0.9999	0.9993	0.9907	0.9408	0.7827	0.4845	0.3135	0.1642	0.0127
	11		1.0000	1.0000	0.9999	0.9981	0.9824	0.9095	0.7031	0.5387	0.3518	0.0556
	12			1.0000	1.0000	0.9997	0.9963	0.9729	0.8732	0.7639	0.6020	0.1841
	13				1.0000	1.0000	0.9995	0.9948	0.9647	0.9198	0.8329	0.4510
	14					1.0000	1.0000	0.9995	0.9953	0.9866	0.9648	0.7941
	15						1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

n		0.1	0.2	0.25	0.3	0.4	0.5	0.6	0.7	0.75	0.8	0.9
n=20	x											
	0	0.1216	0.0115	0.0032	0.0008	0.0000	0.0000					
	1	0.3917	0.0692	0.0243	0.0076	0.0005	0.0000					
	2	0.6769	0.2061	0.0913	0.0355	0.0036	0.0002	0.0000				
	3	0.8670	0.4114	0.2252	0.1071	0.0160	0.0013	0.0000				
	4	0.9568	0.6296	0.4148	0.2375	0.0510	0.0059	0.0003	0.0000			
	5	0.9887	0.8042	0.6172	0.4164	0.1256	0.0207	0.0016	0.0000	0.0000		
	6	0.9976	0.9133	0.7858	0.6080	0.2500	0.0577	0.0065	0.0003	0.0000	0.0000	
	7	0.9996	0.9679	0.8982	0.7723	0.4159	0.1316	0.0210	0.0013	0.0002	0.0000	
	8	0.9999	0.9900	0.9591	0.8867	0.5956	0.2517	0.0565	0.0051	0.0009	0.0001	
	9	1.0000	0.9974	0.9861	0.9520	0.7553	0.4119	0.1275	0.0171	0.0039	0.0006	0.0000
	10	1.0000	0.9994	0.9961	0.9829	0.8725	0.5881	0.2447	0.0480	0.0139	0.0026	0.0000
	11		0.9999	0.9991	0.9949	0.9435	0.7483	0.4044	0.1133	0.0409	0.0100	0.0001
	12		1.0000	0.9998	0.9987	0.9790	0.8684	0.5841	0.2277	0.1018	0.0321	0.0004
	13		1.0000	1.0000	0.9997	0.9935	0.9423	0.7500	0.3920	0.2142	0.0867	0.0024
	14			1.0000	1.0000	0.9984	0.9793	0.8744	0.5836	0.3828	0.1958	0.0113
	15				1.0000	0.9997	0.9941	0.9490	0.7625	0.5852	0.3704	0.0432
	16					1.0000	0.9987	0.9840	0.8929	0.7748	0.5886	0.1330
	17					1.0000	0.9998	0.9964	0.9645	0.9087	0.7939	0.3231
	18						1.0000	0.9995	0.9924	0.9757	0.9308	0.6083
	19						1.0000	1.0000	0.9992	0.9968	0.9885	0.8784
	20							1.0000	1.0000	1.0000	1.0000	1.0000
n=25	x											
	0	0.0718	0.0038	0.0008	0.0001	0.0000						
	1	0.2712	0.0274	0.0070	0.0016	0.0001	0.0000					
	2	0.5371	0.0982	0.0321	0.0090	0.0004	0.0000					
	3	0.7636	0.2340	0.0962	0.0332	0.0024	0.0001	0.0000				
	4	0.9020	0.4207	0.2137	0.0905	0.0095	0.0005	0.0000				
	5	0.9666	0.6167	0.3783	0.1935	0.0294	0.0020	0.0001				
	6	0.9905	0.7800	0.5611	0.3407	0.0736	0.0073	0.0003	0.0000			
	7	0.9977	0.8909	0.7265	0.5118	0.1536	0.0216	0.0012	0.0000			
	8	0.9995	0.9532	0.8506	0.6769	0.2735	0.0539	0.0043	0.0001	0.0000		
	9	0.9999	0.9827	0.9287	0.8106	0.4246	0.1148	0.0132	0.0005	0.0000	0.0000	
	10	1.0000	0.9944	0.9703	0.9022	0.5858	0.2122	0.0344	0.0018	0.0002	0.0000	
	11	1.0000	0.9985	0.9893	0.9558	0.7323	0.3450	0.0778	0.0060	0.0009	0.0001	
	12		0.9996	0.9966	0.9825	0.8462	0.5000	0.1538	0.0175	0.0034	0.0004	
	13		0.9999	0.9991	0.9940	0.9222	0.6550	0.2677	0.0442	0.0107	0.0015	0.0000
	14		1.0000	0.9998	0.9982	0.9656	0.7878	0.4142	0.0978	0.0297	0.0056	0.0000
	15		1.0000	1.0000	0.9995	0.9868	0.8852	0.5754	0.1894	0.0713	0.0173	0.0001
	16			1.0000	0.9999	0.9957	0.9461	0.7265	0.3231	0.1494	0.0468	0.0005
	17				1.0000	0.9988	0.9784	0.8464	0.4882	0.2735	0.1091	0.0023
	18				1.0000	0.9997	0.9927	0.9264	0.6593	0.4389	0.2200	0.0095
	19					0.9999	0.9980	0.9706	0.8065	0.6217	0.3833	0.0334
	20					1.0000	0.9995	0.9905	0.9095	0.7863	0.5793	0.0980
	21					1.0000	0.9999	0.9976	0.9668	0.9038	0.7660	0.2364
	22						1.0000	0.9996	0.9910	0.9679	0.9018	0.4629
	23						1.0000	0.9999	0.9984	0.9930	0.9726	0.7288
	24							1.0000	0.9999	0.9992	0.9962	0.9282
	25							1.0000	1.0000	1.0000	1.0000	1.0000

π		0.1	0.2	0.25	0.3	0.4	0.5	0.6	0.7	0.75	0.8	0.9
n=30	x											
	0	0.0424	0.0012	0.0002	0.0000							
	1	0.1837	0.0105	0.0020	0.0003	0.0000						
	2	0.4114	0.0442	0.0106	0.0021	0.0000						
	3	0.6474	0.1227	0.0374	0.0093	0.0003	0.0000					
	4	0.8245	0.2552	0.0979	0.0302	0.0015	0.0000					
	5	0.9268	0.4275	0.2026	0.0766	0.0057	0.0002					
	6	0.9742	0.6070	0.3481	0.1595	0.0172	0.0007	0.0000				
	7	0.9922	0.7608	0.5143	0.2814	0.0435	0.0026	0.0000				
	8	0.9980	0.8713	0.6736	0.4315	0.0940	0.0081	0.0002				
	9	0.9995	0.9389	0.8034	0.5888	0.1763	0.0214	0.0009	0.0000			
	10	0.9999	0.9744	0.8943	0.7304	0.2915	0.0494	0.0029	0.0000	0.0000		
	11	1.0000	0.9905	0.9493	0.8407	0.4311	0.1002	0.0083	0.0002	0.0000		
	12	1.0000	0.9969	0.9784	0.9155	0.5785	0.1808	0.0212	0.0006	0.0001	0.0000	
	13		0.9991	0.9918	0.9599	0.7145	0.2923	0.0481	0.0021	0.0002	0.0000	
	14		0.9998	0.9973	0.9831	0.8246	0.4278	0.0971	0.0064	0.0008	0.0001	
	15		0.9999	0.9992	0.9936	0.9029	0.5722	0.1754	0.0169	0.0027	0.0002	
	16		1.0000	0.9998	0.9979	0.9519	0.7077	0.2855	0.0401	0.0082	0.0009	
	17		1.0000	0.9999	0.9994	0.9788	0.8192	0.4215	0.0845	0.0216	0.0031	0.0000
	18			1.0000	0.9998	0.9917	0.8998	0.5689	0.1593	0.0507	0.0095	0.0000
	19			1.0000	1.0000	0.9971	0.9506	0.7085	0.2696	0.1057	0.0256	0.0001
	20				1.0000	0.9991	0.9786	0.8237	0.4112	0.1966	0.0611	0.0005
	21					0.9998	0.9919	0.9060	0.5685	0.3264	0.1287	0.0020
	22					1.0000	0.9974	0.9565	0.7186	0.4857	0.2392	0.0078
	23					1.0000	0.9993	0.9828	0.8405	0.6519	0.3930	0.0258
	24						0.9998	0.9943	0.9234	0.7974	0.5725	0.0732
	25						1.0000	0.9985	0.9698	0.9021	0.7448	0.1755
	26						1.0000	0.9997	0.9907	0.9626	0.8773	0.3526
	27							1.0000	0.9979	0.9894	0.9558	0.5886
	28							1.0000	0.9997	0.9980	0.9895	0.8163
	29								1.0000	0.9998	0.9988	0.9576
	30								1.0000	1.0000	1.0000	1.0000

n=40	x											
	0	0.0148	0.0001	0.0000	0.0000							
	1	0.0805	0.0015	0.0001	0.0000							
	2	0.2228	0.0079	0.0010	0.0001							
	3	0.4231	0.0285	0.0047	0.0006	0.0000						
	4	0.6290	0.0759	0.0160	0.0026	0.0000						
	5	0.7937	0.1613	0.0433	0.0086	0.0001						
	6	0.9005	0.2859	0.0962	0.0238	0.0006	0.0000					
	7	0.9581	0.4371	0.1820	0.0553	0.0021	0.0000					
	8	0.9845	0.5931	0.2998	0.1110	0.0061	0.0001					
	9	0.9949	0.7318	0.4395	0.1959	0.0156	0.0003					
	10	0.9985	0.8392	0.5839	0.3087	0.0352	0.0011	0.0000				
	11	0.9996	0.9125	0.7151	0.4406	0.0709	0.0032	0.0000				
	12	0.9999	0.9568	0.8209	0.5772	0.1285	0.0083	0.0001				
	13	1.0000	0.9806	0.8968	0.7032	0.2112	0.0192	0.0004				
	14	1.0000	0.9921	0.9456	0.8074	0.3174	0.0403	0.0012	0.0000			
	15		0.9971	0.9738	0.8849	0.4402	0.0769	0.0034	0.0000			
	16		0.9990	0.9884	0.9367	0.5681	0.1341	0.0083	0.0001			
	17		0.9997	0.9953	0.9680	0.6885	0.2148	0.0189	0.0003	0.0000		
	18		0.9999	0.9983	0.9852	0.7911	0.3179	0.0392	0.0009	0.0000		
	19		1.0000	0.9994	0.9937	0.8702	0.4373	0.0744	0.0024	0.0002	0.0000	
	20		1.0000	0.9998	0.9976	0.9256	0.5627	0.1298	0.0063	0.0006	0.0000	
	21			1.0000	0.9991	0.9608	0.6821	0.2089	0.0148	0.0017	0.0001	
	22			1.0000	0.9997	0.9811	0.7852	0.3115	0.0320	0.0047	0.0003	
	23				0.9999	0.9917	0.8659	0.4319	0.0633	0.0116	0.0010	
	24				1.0000	0.9966	0.9231	0.5598	0.1151	0.0262	0.0029	
	25				1.0000	0.9988	0.9597	0.6826	0.1926	0.0544	0.0079	0.0000
	26					0.9996	0.9808	0.7888	0.2968	0.1032	0.0194	0.0000
	27					0.9999	0.9917	0.8715	0.4228	0.1791	0.0432	0.0001
	28					1.0000	0.9968	0.9291	0.5594	0.2849	0.0875	0.0004
	29					1.0000	0.9989	0.9648	0.6913	0.4161	0.1608	0.0015
	30						0.9997	0.9844	0.8041	0.5605	0.2682	0.0051
	31						0.9999	0.9939	0.8890	0.7002	0.4069	0.0155
	32						1.0000	0.9979	0.9447	0.8180	0.5629	0.0419
	33						1.0000	0.9994	0.9762	0.9038	0.7141	0.0995
	34							0.9999	0.9914	0.9567	0.8387	0.2063
	35							1.0000	0.9974	0.9840	0.9241	0.3710
	36							1.0000	0.9994	0.9953	0.9715	0.5769
	37								0.9999	0.9990	0.9921	0.7772
	38								1.0000	0.9999	0.9985	0.9195
	39								1.0000	1.0000	0.9999	0.9852
	40									1.0000	1.0000	1.0000

Tables02 : The Poisson Cumulative Distribution F(X) =P(X ≤ x)

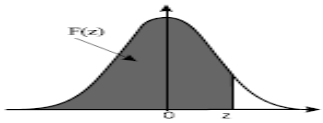
λ=		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.2	1.4	1.6	1.8
x=	0	0.9048	0.8187	0.7408	0.6703	0.6065	0.5488	0.4966	0.4493	0.4066	0.3679	0.3012	0.2466	0.2019	0.1653
	1	0.9953	0.9825	0.9631	0.9384	0.9098	0.8781	0.8442	0.8088	0.7725	0.7358	0.6626	0.5918	0.5249	0.4628
	2	0.9998	0.9989	0.9964	0.9921	0.9856	0.9769	0.9659	0.9526	0.9371	0.9197	0.8795	0.8335	0.7834	0.7306
	3	1.0000	0.9999	0.9997	0.9992	0.9982	0.9966	0.9942	0.9909	0.9865	0.9810	0.9662	0.9463	0.9212	0.8913
	4	1.0000	1.0000	1.0000	0.9999	0.9998	0.9996	0.9992	0.9986	0.9977	0.9963	0.9923	0.9857	0.9763	0.9636
	5	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998	0.9997	0.9994	0.9985	0.9968	0.9940	0.9896
	6	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9994	0.9987	0.9974
	7	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9994
	8	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999
	9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

λ=		2.0	2.2	2.4	2.6	2.8	3.0	3.2	3.4	3.6	3.8	4.0	4.5	5.0	5.5
x=	0	0.1353	0.1108	0.0907	0.0743	0.0608	0.0498	0.0408	0.0334	0.0273	0.0224	0.0183	0.0111	0.0067	0.0041
	1	0.4060	0.3546	0.3084	0.2674	0.2311	0.1991	0.1712	0.1468	0.1257	0.1074	0.0916	0.0611	0.0404	0.0266
	2	0.6767	0.6227	0.5697	0.5184	0.4695	0.4232	0.3799	0.3397	0.3027	0.2689	0.2381	0.1736	0.1247	0.0884
	3	0.8571	0.8194	0.7787	0.7360	0.6919	0.6472	0.6025	0.5584	0.5152	0.4735	0.4335	0.3423	0.2650	0.2017
	4	0.9473	0.9275	0.9041	0.8774	0.8477	0.8153	0.7806	0.7442	0.7064	0.6678	0.6288	0.5321	0.4405	0.3575
	5	0.9834	0.9751	0.9643	0.9510	0.9349	0.9161	0.8946	0.8705	0.8441	0.8156	0.7851	0.7029	0.6160	0.5289
	6	0.9955	0.9925	0.9884	0.9828	0.9756	0.9665	0.9554	0.9421	0.9267	0.9091	0.8893	0.8311	0.7622	0.6860
	7	0.9989	0.9980	0.9967	0.9947	0.9919	0.9881	0.9832	0.9769	0.9692	0.9599	0.9489	0.9134	0.8666	0.8095
	8	0.9998	0.9995	0.9991	0.9985	0.9976	0.9962	0.9943	0.9917	0.9883	0.9840	0.9786	0.9597	0.9319	0.8944
	9	1.0000	0.9999	0.9998	0.9996	0.9993	0.9989	0.9982	0.9973	0.9960	0.9942	0.9919	0.9829	0.9682	0.9462
	10	1.0000	1.0000	1.0000	0.9999	0.9998	0.9997	0.9995	0.9992	0.9987	0.9981	0.9972	0.9933	0.9863	0.9747
	11	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9998	0.9996	0.9994	0.9991	0.9976	0.9945	0.9890
	12	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9998	0.9997	0.9992	0.9980	0.9955
	13	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9993	0.9983
	14	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998	0.9994
	15	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998
	16	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999
	17	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

λ=		6.0	6.5	7.0	7.5	8.0	8.5	9.0	9.5	10.0	11.0	10.0	12.0	14.0	15.0
x=	0	0.0025	0.0015	0.0009	0.0006	0.0003	0.0002	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.0174	0.0113	0.0073	0.0047	0.0030	0.0019	0.0012	0.0008	0.0005	0.0002	0.0005	0.0001	0.0000	0.0000
	2	0.0620	0.0430	0.0296	0.0203	0.0138	0.0093	0.0062	0.0042	0.0028	0.0012	0.0028	0.0005	0.0001	0.0000
	3	0.1512	0.1118	0.0818	0.0591	0.0424	0.0301	0.0212	0.0149	0.0103	0.0049	0.0103	0.0023	0.0005	0.0002
	4	0.2851	0.2237	0.1730	0.1321	0.0996	0.0744	0.0550	0.0403	0.0293	0.0151	0.0293	0.0076	0.0018	0.0009
	5	0.4457	0.3690	0.3007	0.2414	0.1912	0.1496	0.1157	0.0885	0.0671	0.0375	0.0671	0.0203	0.0055	0.0028
	6	0.6063	0.5265	0.4497	0.3782	0.3134	0.2562	0.2068	0.1649	0.1301	0.0786	0.1301	0.0458	0.0142	0.0076
	7	0.7440	0.6728	0.5987	0.5246	0.4530	0.3856	0.3239	0.2687	0.2202	0.1432	0.2202	0.0895	0.0316	0.0180
	8	0.8472	0.7916	0.7291	0.6620	0.5925	0.5231	0.4557	0.3918	0.3328	0.2320	0.3328	0.1550	0.0621	0.0374
	9	0.9161	0.8774	0.8305	0.7764	0.7166	0.6530	0.5874	0.5218	0.4579	0.3405	0.4579	0.2424	0.1094	0.0699
	10	0.9574	0.9332	0.9015	0.8622	0.8159	0.7634	0.7060	0.6453	0.5830	0.4599	0.5830	0.3472	0.1757	0.1185
	11	0.9799	0.9661	0.9467	0.9208	0.8881	0.8487	0.8030	0.7520	0.6968	0.5793	0.6968	0.4616	0.2600	0.1848
	12	0.9912	0.9840	0.9730	0.9573	0.9362	0.9091	0.8758	0.8364	0.7916	0.6887	0.7916	0.5760	0.3585	0.2676
	13	0.9964	0.9929	0.9872	0.9784	0.9658	0.9486	0.9261	0.8981	0.8645	0.7813	0.8645	0.6815	0.4644	0.3632
	14	0.9986	0.9970	0.9943	0.9897	0.9827	0.9726	0.9585	0.9400	0.9165	0.8540	0.9165	0.7720	0.5704	0.4657
	15	0.9995	0.9988	0.9976	0.9954	0.9918	0.9862	0.9780	0.9665	0.9513	0.9074	0.9513	0.8444	0.6694	0.5681
	16	0.9998	0.9996	0.9990	0.9980	0.9963	0.9934	0.9889	0.9823	0.9730	0.9441	0.9730	0.8987	0.7559	0.6641
	17	0.9999	0.9998	0.9996	0.9992	0.9984	0.9970	0.9947	0.9911	0.9857	0.9678	0.9857	0.9370	0.8272	0.7489
	18	1.0000	0.9999	0.9999	0.9997	0.9993	0.9987	0.9976	0.9957	0.9928	0.9823	0.9928	0.9626	0.8826	0.8195
	19	1.0000	1.0000	1.0000	0.9999	0.9997	0.9995	0.9989	0.9980	0.9965	0.9907	0.9965	0.9787	0.9235	0.8752
	20	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998	0.9996	0.9991	0.9984	0.9953	0.9984	0.9884	0.9521	0.9170
	21	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998	0.9996	0.9993	0.9977	0.9993	0.9939	0.9712	0.9469
	22	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9997	0.9990	0.9997	0.9970	0.9833	0.9673
	23	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9995	0.9999	0.9985	0.9907	0.9805
	24	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	1.0000	0.9993	0.9950	0.9888
	25	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	1.0000	0.9997	0.9974	0.9938
	26	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9987	0.9967
	27	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9994	0.9983
	28	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9991
	29	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9996
	30	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998
	31	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999
	32	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 03: Standard Normal Distribution

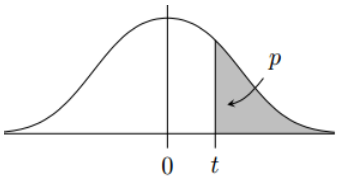
$P(Z \leq z)$, Z = (x - μ) / σ



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.99980	0.99981	0.99981	0.99982	0.99983	0.99983
3.6	0.99984	0.99985	0.99985	0.99986	0.99986	0.99987	0.99987	0.99988	0.99988	0.99989
3.7	0.99989	0.99990	0.99990	0.99990	0.99991	0.99991	0.99992	0.99992	0.99992	0.99992
3.8	0.99993	0.99993	0.99993	0.99994	0.99994	0.99994	0.99994	0.99995	0.99995	0.99995
3.9	0.99995	0.99995	0.99996	0.99996	0.99996	0.99996	0.99996	0.99996	0.99997	0.99997

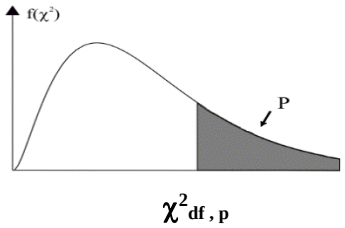
Table 04: Student Distribution

p = P[T ≥ t]



	p										
v	**0.50**	**0.25**	**0.20**	**0.15**	**0.10**	**0.05**	**0.025**	**0.01**	**0.005**	**0.001**	**0.0005**
1	0.000	1.000	1.376	1.963	3.078	6.314	12.71	31.82	63.66	318.31	636.62
2	0.000	0.816	1.061	1.386	1.886	2.920	4.303	6.965	9.925	22.327	31.599
3	0.000	0.765	0.978	1.250	1.638	2.353	3.182	4.541	5.841	10.215	12.924
4	0.000	0.741	0.941	1.190	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	0.000	0.727	0.920	1.156	1.476	2.015	2.571	3.365	4.032	5.893	6.869
6	0.000	0.718	0.906	1.134	1.440	1.943	2.447	3.143	3.707	5.208	5.959
7	0.000	0.711	0.896	1.119	1.415	1.895	2.365	2.998	3.499	4.785	5.408
8	0.000	0.706	0.889	1.108	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	0.000	0.703	0.883	1.100	1.383	1.833	2.262	2.821	3.250	4.297	4.781
10	0.000	0.700	0.879	1.093	1.372	1.812	2.228	2.764	3.169	4.144	4.587
11	0.000	0.697	0.876	1.088	1.363	1.796	2.201	2.718	3.106	4.025	4.437
12	0.000	0.695	0.873	1.083	1.356	1.782	2.179	2.681	3.055	3.930	4.318
13	0.000	0.694	0.870	1.079	1.350	1.771	2.160	2.650	3.012	3.852	4.221
14	0.000	0.692	0.868	1.076	1.345	1.761	2.145	2.624	2.977	3.787	4.140
15	0.000	0.691	0.866	1.074	1.341	1.753	2.131	2.602	2.947	3.733	4.073
16	0.000	0.690	0.865	1.071	1.337	1.746	2.120	2.583	2.921	3.686	4.015
17	0.000	0.689	0.863	1.069	1.333	1.740	2.110	2.567	2.898	3.646	3.965
18	0.000	0.688	0.862	1.067	1.330	1.734	2.101	2.552	2.878	3.610	3.922
19	0.000	0.688	0.861	1.066	1.328	1.729	2.093	2.539	2.861	3.579	3.883
20	0.000	0.687	0.860	1.064	1.325	1.725	2.086	2.528	2.845	3.552	3.850
21	0.000	0.686	0.859	1.063	1.323	1.721	2.080	2.518	2.831	3.527	3.819
22	0.000	0.686	0.858	1.061	1.321	1.717	2.074	2.508	2.819	3.505	3.792
23	0.000	0.685	0.858	1.060	1.319	1.714	2.069	2.500	2.807	3.485	3.768
24	0.000	0.685	0.857	1.059	1.318	1.711	2.064	2.492	2.797	3.467	3.745
25	0.000	0.684	0.856	1.058	1.316	1.708	2.060	2.485	2.787	3.450	3.725
26	0.000	0.684	0.856	1.058	1.315	1.706	2.056	2.479	2.779	3.435	3.707
27	0.000	0.684	0.855	1.057	1.314	1.703	2.052	2.473	2.771	3.421	3.690
28	0.000	0.683	0.855	1.056	1.313	1.701	2.048	2.467	2.763	3.408	3.674
29	0.000	0.683	0.854	1.055	1.311	1.699	2.045	2.462	2.756	3.396	3.659
30	0.000	0.683	0.854	1.055	1.310	1.697	2.042	2.457	2.750	3.385	3.646
40	0.000	0.681	0.851	1.050	1.303	1.684	2.021	2.423	2.704	3.307	3.551
60	0.000	0.679	0.848	1.045	1.296	1.671	2.000	2.390	2.660	3.232	3.460
80	0.000	0.678	0.846	1.043	1.292	1.664	1.990	2.374	2.639	3.195	3.416
100	0.000	0.677	0.845	1.042	1.290	1.660	1.984	2.364	2.626	3.174	3.390
1000	0.000	0.675	0.842	1.037	1.282	1.646	1.962	2.330	2.581	3.098	3.300
z	0.000	0.674	0.842	1.036	1.282	1.645	1.960	2.326	2.576	3.090	3.291
	0%	50%	60%	70%	80%	90%	95%	98%	99%	99.8%	99.9%
	Confidence Level										

$P(\chi^2 \geq \chi^2_{df,p}) = p$

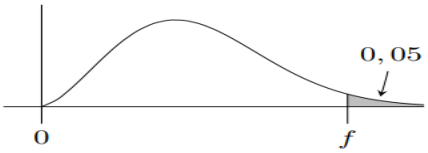


df\p	0.995	0.99	0.975	0.95	0.90	0.10	0.05	0.025	0.01	0.005
1	4×10^{-5}	16×10^{-5}	0.001	0.004	0.016	2.706	3.841	5.024	6.635	7.879
2	0.010	0.020	0.051	0.103	0.211	4.605	5.991	7.378	9.210	10.597
3	0.072	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.345	12.838
4	0.207	0.297	0.484	0.711	1.064	7.779	9.488	11.143	13.277	14.860
5	0.412	0.554	0.831	1.145	1.610	9.236	11.070	12.833	15.086	16.750
6	0.676	0.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812	18.548
7	0.989	1.239	1.690	2.167	2.833	12.017	14.067	16.013	18.475	20.278
8	1.344	1.646	2.180	2.733	3.490	13.362	15.507	17.535	20.090	21.955
9	1.735	2.088	2.700	3.325	4.168	14.684	16.919	19.023	21.666	23.589
10	2.156	2.558	3.247	3.940	4.865	15.987	18.307	20.483	23.209	25.188
11	2.603	3.053	3.816	4.575	5.578	17.275	19.675	21.920	24.725	26.757
12	3.074	3.571	4.404	5.226	6.304	18.549	21.026	23.337	26.217	28.300
13	3.565	4.107	5.009	5.892	7.042	19.812	22.362	24.736	27.688	29.819
14	4.075	4.660	5.629	6.571	7.790	21.064	23.685	26.119	29.141	31.319
15	4.601	5.229	6.262	7.261	8.547	22.307	24.996	27.488	30.578	32.801
16	5.142	5.812	6.908	7.962	9.312	23.542	26.296	28.845	32.000	34.267
17	5.697	6.408	7.564	8.672	10.085	24.769	27.587	30.191	33.409	35.718
18	6.265	7.015	8.231	9.390	10.865	25.989	28.869	31.526	34.805	37.156
19	6.844	7.633	8.907	10.117	11.651	27.204	30.144	32.852	36.191	38.582
20	7.434	8.260	9.591	10.851	12.443	28.412	31.410	34.170	37.566	39.997
21	8.034	8.897	10.283	11.591	13.240	29.615	32.671	35.479	38.932	41.401
22	8.643	9.542	10.982	12.338	14.041	30.813	33.924	36.781	40.289	42.796
23	9.260	10.196	11.689	13.091	14.848	32.007	35.172	38.076	41.638	44.181
24	9.886	10.856	12.401	13.848	15.659	33.196	36.415	39.364	42.980	45.559
25	10.520	11.524	13.120	14.611	16.473	34.382	37.652	40.646	44.314	46.928
26	11.160	12.198	13.844	15.379	17.292	35.563	38.885	41.923	45.642	48.290
27	11.808	12.879	14.573	16.151	18.114	36.741	40.113	43.195	46.963	49.645
28	12.461	13.565	15.308	16.928	18.939	37.916	41.337	44.461	48.278	50.993
29	13.121	14.256	16.047	17.708	19.768	39.087	42.557	45.722	49.588	52.336
30	13.787	14.953	16.791	18.493	20.599	40.256	43.773	46.979	50.892	53.672
40	20.707	22.164	24.433	26.509	29.051	51.805	55.758	59.342	63.691	66.766
50	27.991	29.707	32.357	34.764	37.689	63.167	67.505	71.420	76.154	79.490
60	35.534	37.485	40.482	43.188	46.459	74.397	79.082	83.298	88.379	91.952
70	43.275	45.442	48.758	51.739	55.329	85.527	90.531	95.023	100.425	104.215
80	51.172	53.540	57.153	60.391	64.278	96.578	101.879	106.629	112.329	116.321
90	59.196	61.754	65.647	69.126	73.291	107.565	113.145	118.136	124.116	128.299
100	67.328	70.065	74.222	77.929	82.358	118.498	124.342	129.561	135.807	140.169

df : is the number of degrees of freedom. For df= v > 30, we can assume that the quantity

$\sqrt{2\chi^2} - \sqrt{2V-1}$ follows **the Standard Normal Distribution**

Table 05: Fisher Distribution



	p = 0.05																		
	v_1																		
v_2	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
1	161.4	199.5	215.7	224.6	230.2	234.0	236.8	238.9	240.5	241.9	243.9	245.9	248.0	249.1	250.1	251.1	252.2	253.3	254.3
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.41	19.43	19.45	19.45	19.46	19.47	19.48	19.49	19.50
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57	8.55	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.62	4.56	4.53	4.50	4.46	4.43	4.40	4.36
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74	3.70	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30	3.27	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01	2.97	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79	2.75	2.71
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.85	2.77	2.74	2.70	2.66	2.62	2.58	2.54
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49	2.45	2.40
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38	2.34	2.30
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.53	2.46	2.42	2.38	2.34	2.30	2.25	2.21
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22	2.18	2.13
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40	2.33	2.29	2.25	2.20	2.16	2.11	2.07
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11	2.06	2.01
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06	2.01	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02	1.97	1.92
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.20	2.12	2.08	2.04	1.99	1.95	1.90	1.84
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92	1.87	1.81
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.15	2.07	2.03	1.98	1.94	1.89	1.84	1.78
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.13	2.05	2.01	1.96	1.91	1.86	1.81	1.76
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11	2.03	1.98	1.94	1.89	1.84	1.79	1.73
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.16	2.09	2.01	1.96	1.92	1.87	1.82	1.77	1.71
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.15	2.07	1.99	1.95	1.90	1.85	1.80	1.75	1.69
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20	2.13	2.06	1.97	1.93	1.88	1.84	1.79	1.73	1.67
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.12	2.04	1.96	1.91	1.87	1.82	1.77	1.71	1.65
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18	2.10	2.03	1.94	1.90	1.85	1.81	1.75	1.70	1.64
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.09	2.01	1.93	1.89	1.84	1.79	1.74	1.68	1.62
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.92	1.84	1.79	1.74	1.69	1.64	1.58	1.51
60	4.00	3.15	2.76	2.52	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.84	1.75	1.70	1.65	1.59	1.53	1.47	1.39
120	3.92	3.07	2.68	2.45	2.29	2.17	2.09	2.02	1.96	1.91	1.83	1.75	1.66	1.61	1.55	1.50	1.43	1.35	1.25
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.75	1.67	1.57	1.52	1.46	1.39	1.32	1.22	1.00

	p = 0.005																		
	v_1																		
v_2	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
1	16211	20000	21615	22500	23056	23437	23715	23925	24091	24224	24426	24630	24836	24940	25044	25148	25253	25359	25465
2	198.5	199.0	199.2	199.2	199.3	199.3	199.4	199.4	199.4	199.4	199.4	199.4	199.4	199.5	199.5	199.5	199.5	199.5	199.5
3	55.55	49.80	47.47	46.19	45.39	44.84	44.43	44.13	43.88	43.69	43.39	43.08	42.78	42.62	42.47	42.31	42.15	41.99	41.83
4	31.33	26.28	24.26	23.15	22.46	21.97	21.62	21.35	21.14	20.97	20.70	20.44	20.17	20.03	19.89	19.75	19.61	19.47	19.32
5	22.78	18.31	16.53	15.56	14.94	14.51	14.20	13.96	13.77	13.62	13.38	13.15	12.90	12.78	12.66	12.53	12.40	12.27	12.14
6	18.63	14.54	12.92	12.03	11.46	11.07	10.79	10.57	10.39	10.25	10.03	9.81	9.59	9.47	9.36	9.24	9.12	9.00	8.88
7	16.24	12.40	10.88	10.05	9.52	9.16	8.89	8.68	8.51	8.38	8.18	7.97	7.75	7.65	7.53	7.42	7.31	7.19	7.08
8	14.69	11.04	9.60	8.81	8.30	7.95	7.69	7.50	7.34	7.21	7.01	6.81	6.61	6.50	6.40	6.29	6.18	6.06	5.95
9	13.61	10.11	8.72	7.96	7.47	7.13	6.88	6.69	6.54	6.42	6.23	6.03	5.83	5.73	5.62	5.52	5.41	5.30	5.19
10	12.83	9.43	8.08	7.34	6.87	6.54	6.30	6.12	5.97	5.85	5.66	5.47	5.27	5.17	5.07	4.97	4.86	4.75	4.64
11	12.23	8.91	7.60	6.88	6.42	6.10	5.86	5.68	5.54	5.42	5.24	5.05	4.86	4.76	4.65	4.55	4.44	4.34	4.23
12	11.75	8.51	7.23	6.52	6.07	5.76	5.52	5.35	5.20	5.09	4.91	4.72	4.53	4.43	4.33	4.23	4.12	4.01	3.90
13	11.37	8.19	6.93	6.23	5.79	5.48	5.25	5.08	4.94	4.82	4.64	4.46	4.27	4.17	4.07	3.97	3.87	3.76	3.65
14	11.06	7.92	6.68	6.00	5.56	5.26	5.03	4.86	4.72	4.60	4.43	4.25	4.06	3.96	3.86	3.76	3.66	3.55	3.44
15	10.80	7.70	6.48	5.80	5.37	5.07	4.85	4.67	4.54	4.42	4.25	4.07	3.88	3.79	3.69	3.58	3.48	3.37	3.26
16	10.58	7.51	6.30	5.64	5.21	4.91	4.69	4.52	4.38	4.27	4.10	3.92	3.73	3.64	3.54	3.44	3.33	3.22	3.11
17	10.38	7.35	6.16	5.50	5.07	4.78	4.56	4.39	4.25	4.14	3.97	3.79	3.61	3.51	3.41	3.31	3.21	3.10	2.98
18	10.22	7.21	6.03	5.37	4.96	4.66	4.44	4.28	4.14	4.03	3.86	3.68	3.50	3.40	3.30	3.20	3.10	2.99	2.87
19	10.07	7.09	5.92	5.27	4.85	4.56	4.34	4.18	4.04	3.93	3.76	3.59	3.40	3.31	3.21	3.11	3.00	2.89	2.78
20	9.94	6.99	5.82	5.17	4.76	4.47	4.26	4.09	3.96	3.85	3.68	3.50	3.32	3.22	3.12	3.02	2.92	2.81	2.69
21	9.83	6.89	5.73	5.09	4.68	4.39	4.18	4.01	3.88	3.77	3.60	3.43	3.24	3.15	3.05	2.95	2.84	2.73	2.61
22	9.73	6.81	5.65	5.02	4.61	4.32	4.11	3.94	3.81	3.70	3.54	3.36	3.18	3.08	2.98	2.88	2.77	2.66	2.55
23	9.63	6.73	5.58	4.95	4.54	4.26	4.05	3.88	3.75	3.64	3.47	3.30	3.12	3.02	2.92	2.82	2.71	2.60	2.48
24	9.55	6.66	5.52	4.89	4.49	4.20	3.99	3.83	3.69	3.59	3.42	3.25	3.06	2.97	2.87	2.77	2.66	2.55	2.43
25	9.48	6.60	5.46	4.84	4.43	4.15	3.94	3.78	3.63	3.54	3.37	3.20	3.01	2.92	2.82	2.72	2.61	2.50	2.38
26	9.41	6.54	5.41	4.79	4.38	4.10	3.89	3.73	3.60	3.49	3.33	3.15	2.97	2.87	2.77	2.67	2.56	2.45	2.33
27	9.34	6.49	5.36	4.74	4.34	4.06	3.85	3.69	3.56	3.45	3.28	3.11	2.93	2.83	2.73	2.63	2.52	2.41	2.25
28	9.28	6.44	5.32	4.70	4.30	4.02	3.81	3.65	3.52	3.41	3.25	3.07	2.89	2.79	2.69	2.59	2.48	2.37	2.29
29	9.23	6.40	5.28	4.66	4.26	3.98	3.77	3.61	3.48	3.38	3.21	3.04	2.86	2.76	2.66	2.56	2.45	2.33	2.24
30	9.18	6.35	5.24	4.62	4.23	3.95	3.74	3.58	3.45	3.34	3.18	3.01	2.82	2.73	2.63	2.52	2.42	2.30	2.18
40	8.83	6.07	4.98	4.37	3.99	3.71	3.51	3.35	3.22	3.12	2.95	2.78	2.60	2.50	2.40	2.30	2.18	2.06	1.93
60	8.49	5.79	4.73	4.14	3.76	3.49	3.29	3.13	3.01	2.90	2.74	2.57	2.39	2.29	2.19	2.08	1.96	1.83	1.69
120	8.18	5.54	4.50	3.92	3.55	3.28	3.09	2.93	2.81	2.71	2.54	2.37	2.19	2.09	1.98	1.87	1.75	1.61	1.43
∞	7.88	5.30	4.28	3.72	3.35	3.09	2.90	2.74	2.62	2.52	2.36	2.19	2.00	1.90	1.79	1.67	1.53	1.36	1.00

	p = 0.01																		
	v_1																		
v_2	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
1	4052	4999.5	5403	5625	5764	5859	5928	5982	6022	6056	6106	6157	6209	6235	6261	6287	6313	6339	6366
2	98.50	99.00	99.17	99.25	99.30	99.33	99.36	99.37	99.39	99.40	99.42	99.43	99.45	99.46	99.47	99.47	99,48	99.49	99.50
3	34.12	30.82	29.46	28.71	28.24	27.91	27.67	27.49	27.35	27.23	27.05	26.87	26.69	26.60	26.50	26.41	26.32	26.22	26.13
4	21.20	18.00	16.69	15.98	15.52	15.21	14.98	14.80	14.66	14.55	14.37	14.20	14.02	13.93	13.84	13.75	13.65	13.56	13.46
5	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	10.16	10.05	9.89	9.72	9.55	9.47	9.38	9.29	9.20	9.11	9.02
6	13.75	10.92	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87	7.72	7.56	7.40	7.31	7.23	7.14	7.06	6.97	6.88
7	12.25	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62	6.47	6.31	6.16	6.07	5.99	5.91	5.82	5.74	5.65
8	11.26	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81	5.67	5.52	5.36	5.28	5.20	5.12	5.03	4.95	4.86
9	10.56	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26	5.11	4.96	4.81	4.73	4.65	4.57	4.48	4.40	4.31
10	10.04	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85	4.71	4.56	4.41	4.33	4.25	4.17	4.08	4.00	3.91
11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63	4.54	4.40	4.25	4.10	4.02	3.94	3.86	3.78	3.69	3.60
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30	4.16	4.01	3.86	3.78	3.70	3.62	3.54	3.45	3.36
13	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19	4.10	3.96	3.82	3.66	3.59	3.51	3.43	3.34	3.25	3.17
14	8.86	6.51	5.56	5.04	4.69	4.46	4.28	4.14	4.03	3.94	3.80	3.66	3.51	3.43	3.35	3.27	3.18	3.09	3.00
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89	3.80	3.67	3.52	3.37	3.29	3.21	3.13	3.05	2.96	2.87
16	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78	3.69	3.55	3.41	3.26	3.18	3.10	3.02	2.93	2.84	2.75
17	8.40	6.11	5.18	4.67	4.34	4.10	3.93	3.79	3.68	3.59	3.46	3.31	3.16	3.08	3.00	2.92	2.83	2.75	2.65
18	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60	3.51	3.37	3.23	3.08	3.00	2.92	2.84	2.75	2.66	2.57
19	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52	3.43	3.30	3.15	3.00	2.92	2.84	2.76	2.67	2.58	2.49
20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46	3.37	3.23	3.09	2.94	2.86	2.78	2.69	2.61	2.52	2.42
21	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.40	3.31	3.17	3.03	2.88	2.80	2.72	2.64	2.55	2.46	2.36
22	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35	3.26	3.12	2.98	2.83	2.75	2.67	2.58	2.50	2.40	2.31
23	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.30	3.21	3.07	2.93	2.78	2.70	2.62	2.54	2.45	2.35	2.26
24	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26	3.17	3.03	2.89	2.74	2.66	2.58	2.49	2.40	2.31	2.21
25	7.77	5.57	4.68	4.18	3.85	3.63	3.46	3.32	3.22	3.13	2.99	2.85	2.70	2.62	2.54	2.45	2.36	2.27	2.17
26	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	3.18	3.09	2.96	2.81	2.66	2.58	2.50	2.42	2.33	2.23	2.13
27	7.68	5.49	4.60	4.11	3.78	3.56	3.39	3.26	3.15	3.06	2.93	2.78	2.63	2.55	2.47	2.38	2.29	2.20	2.10
28	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	3.12	3.03	2.90	2.75	2.60	2.52	2.44	2.35	2.26	2.17	2.06
29	7.60	5.42	4.54	4.04	3.73	3.50	3.33	3.20	3.09	3.00	2.87	2.73	2.57	2.49	2.41	2.33	2.23	2.14	2.03
30	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07	2.98	2.84	2.70	2.55	2.47	2.39	2.30	2.21	2.11	2.01
40	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89	2.80	2.66	2.52	2.37	2.29	2.20	2.11	2.02	1.92	1.80
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72	2.63	2.50	2.35	2.20	2.12	2.03	1.94	1.84	1.73	1.60
120	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.56	2.47	2.34	2.19	2.03	1.95	1.86	1.76	1.66	1.53	1.38
∞	6.63	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.41	2.32	2.18	2.04	1.88	1.79	1.70	1.59	1.47	1.32	1.00

v_2	p = 0.025																		
	v_1																		
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**12**	**15**	**20**	**24**	**30**	**40**	**60**	**120**	∞
1	647.8	799.5	864.2	899.6	921.8	937.1	948.2	956.7	963.3	968.6	976.7	984.9	993.1	997.2	1001	1006	1010	1014	1018
2	38.51	39.00	39.17	39.25	39.30	39.33	39.36	39.37	39.39	39.40	39.41	39.43	39.45	39.46	39.46	39.47	39.48	39.49	39.50
3	17.44	16.04	15.44	15.10	14.88	14.73	14.62	14.54	14.47	14.42	14.34	14.25	14.17	14.12	14.08	14.04	13.99	13.95	13.90
4	12.22	10.65	9.98	9.60	9.36	9.20	9.07	8.98	8.90	8.84	8.75	8.66	8.56	8.51	8.46	8.41	8.36	8.31	8.26
5	10.01	8.43	7.76	7.39	7.15	6.98	6.85	6.76	6.68	6.62	6.52	6.43	6.33	6.28	6.23	6.18	6.12	6.07	6.02
6	8.81	7.26	6.60	6.23	5.99	5.82	5.70	5.60	5.52	5.46	5.37	5.27	5.17	5.12	5.07	5.01	4.96	4.90	4.85
7	8.07	6.54	5.89	5.52	5.29	5.12	4.99	4.90	4.82	4.76	4.67	4.57	4.47	4.42	4.36	4.31	4.25	4.20	4.14
8	7.57	6.06	5.42	5.05	4.82	4.65	4.53	4.43	4.36	4.30	4.20	4.10	4.00	3.95	3.89	3.84	3.78	3.73	3.67
9	7.21	5.71	5.08	4.72	4.48	4.32	4.20	4.10	4.03	3.96	3.87	3.77	3.67	3.61	3.56	3.51	3.45	3.39	3.33
10	6.94	5.46	4.83	4.47	4.24	4.07	3.95	3.85	3.78	3.72	3.62	3.52	3.42	3.37	3.31	3.26	3.20	3.14	3.08
11	6.72	5.26	4.63	4.28	4.04	3.88	3.76	3.66	3.59	3.53	3.43	3.33	3.23	3.17	3.12	3.06	3.00	2.94	2.88
12	6.55	5.10	4.47	4.12	3.89	3.73	3.61	3.51	3.44	3.37	3.28	3.18	3.07	3.02	2.96	2.91	2.85	2.79	2.72
13	6.41	4.97	4.35	4.00	3.77	3.60	3.48	3.39	3.31	3.25	3.15	3.05	2.95	2.89	2.84	2.78	2.72	2.66	2.60
14	6.30	4.86	4.24	3.89	3.66	3.50	3.38	3.29	3.21	3.15	3.05	2.95	2.84	2.79	2.73	2.67	2.61	2.55	2.49
15	6.20	4.77	4.15	3.80	3.58	3.41	3.29	3.20	3.12	3.06	2.96	2.86	2.76	2.70	2.64	2.59	2.52	2.46	2.40
16	6.12	4.69	4.08	3.73	3.50	3.34	3.22	3.12	3.05	2.99	2.89	2.79	2.68	2.63	2.57	2.51	2.45	2.38	2.32
17	6.04	4.62	4.01	3.66	3.44	3.28	3.16	3.06	2.98	2.92	2.82	2.72	2.62	2.56	2.50	2.44	2.38	2.32	2.25
18	5.98	4.56	3.95	3.61	3.38	3.22	3.10	3.01	2.93	2.87	2.77	2.67	2.56	2.50	2.44	2.38	2.32	2.26	2.19
19	5.92	4.51	3.90	3.56	3.33	3.17	3.05	2.96	2.88	2.82	2.72	2.62	2.51	2.45	2.39	2.33	2.27	2.20	2.13
20	5.87	4.46	3.86	3.51	3.29	3.13	3.01	2.91	2.84	2.77	2.68	2.57	2.46	2.41	2.35	2.29	2.22	2.16	2.09
21	5.83	4.42	3.82	3.48	3.25	3.09	2.97	2.87	2.80	2.73	2.64	2.53	2.42	2.37	2.31	2.25	2.18	2.11	2.04
22	5.79	4.38	3.78	3.44	3.22	3.05	2.93	2.84	2.76	2.70	2.60	2.50	2.39	2.33	2.27	2.21	2.14	2.08	2.00
23	5.75	4.35	3.75	3.41	3.18	3.02	2.90	2.81	2.73	2.67	2.57	2.47	2.36	2.30	2.24	2.18	2.11	2.04	1.97
24	5.72	4.32	3.72	3.38	3.15	2.99	2.87	2.78	2.70	2.64	2.54	2.44	2.33	2.27	2.21	2.15	2.08	2.01	1.94
25	5.69	4.29	3.69	3.35	3.13	2.97	2.85	2.75	2.68	2.61	2.51	2.41	2.30	2.24	2.18	2.12	2.05	1.98	1.91
26	5.66	4.27	3.67	3.33	3.10	2.94	2.82	2.73	2.65	2.59	2.49	2.39	2.28	2.22	2.16	2.09	2.03	1.95	1.88
27	5.63	4.24	3.65	3.31	3.08	2.92	2.80	2.71	2.63	2.57	2.47	2.36	2.25	2.19	2.13	2.07	2.00	1.93	1.85
28	5.61	4.22	3.63	3.29	3.06	2.90	2.78	2.69	2.61	2.55	2.45	2.34	2.23	2.17	2.11	2.05	1.98	1.91	1.83
29	5.59	4.20	3.61	3.27	3.04	2.88	2.76	2.67	2.59	2.53	2.43	2.32	2.21	2.15	2.09	2.03	1.96	1.89	1.81
30	5.57	4.18	3.69	3.25	3.03	2.87	2.75	2.65	2.57	2.51	2.41	2.31	2.20	2.14	2.07	2.01	1.94	1.87	1.79
40	5.42	4.05	3.46	3.13	2.90	2.74	2.62	2.53	2.45	2.39	2.29	2.18	2.07	2.01	1.94	1.88	1.80	1.72	1.64
60	5.29	3.93	3.34	3.01	2.79	2.63	2.51	2.41	2.33	2.27	2.17	2.06	1.94	1.88	1.82	1.74	1.67	1.58	1.48
120	5.15	3.80	3.23	2.89	2.67	2.52	2.39	2.30	2.22	2.16	2.05	1.94	1.82	1.76	1.69	1.61	1.53	1.43	1.31
∞	5.02	3.69	3.12	2.79	2.57	2.41	2.29	2.19	2.11	2.05	1.94	1.83	1.71	1.64	1.57	1.48	1.39	1.27	1.00

	p = 0.10																		
	v_1																		
v_2	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
1	39.86	49.50	53.59	55.83	57.24	58.20	58.91	59.44	59.86	60.19	60.71	61.22	61.74	62.00	62.26	62.53	62.79	63.06	63.33
2	8.53	9.00	9.16	9.24	9.29	9.33	9.35	9.37	9.38	9.39	9.41	9.42	9.44	9.45	9.46	9.47	9.47	9.48	9.49
3	5.54	5.46	5.39	5.34	5.31	5.28	5.27	5.25	5.24	5.23	5.22	5.20	5.18	5.18	5.17	5.16	5.15	5.14	5.13
4	4.54	4.32	4.19	4.11	4.05	4.01	3.98	3.95	3.94	3.92	3.90	3.87	3.84	3.83	3.82	3.80	3.79	3.78	3.76
5	4.06	3.78	3.62	3.52	3.45	3.40	3.37	3.34	3.32	3.30	3.27	3.24	3.21	3.19	3.17	3.16	3.14	3.12	3.10
6	3.78	3.46	3.29	3.18	3.11	3.05	3.01	2.98	2.96	2.94	2.90	2.87	2.84	2.82	2.80	2.78	2.76	2.74	2.72
7	3.59	3.26	3.07	2.96	2.88	2.83	2.78	2.75	2.72	2.70	2.67	2.63	2.59	2.58	2.56	2.54	2.51	2.49	2.47
8	3.46	3.11	2.92	2.81	2.73	2.67	2.62	2.59	2.56	2.54	2.50	2.46	2.42	2.40	2.38	2.36	2.34	2.32	2.29
9	3.36	3.01	2.81	2.69	2.61	2.55	2.51	2.47	2.44	2.42	2.38	2.34	2.30	2.28	2.25	2.23	2.21	2.18	2.16
10	3.29	2.92	2.73	2.61	2.52	2.46	2.41	2.38	2.35	2.32	2.28	2.24	2.20	2.18	2.16	2.13	2.11	2.08	2.06
11	3.23	2.86	2.66	2.54	2.45	2.39	2.34	2.30	2.27	2.25	2.21	2.17	2.12	2.10	2.08	2.05	2.03	2.00	1.97
12	3.18	2.81	2.61	2.48	2.39	2.33	2.28	2.24	2.21	2.19	2.15	2.10	2.06	2.04	2.01	1.99	1.96	1.93	1.90
13	3.14	2.76	2.56	2.43	2.35	2.28	2.23	2.20	2.16	2.14	2.10	2.05	2.01	1.98	1.96	1.93	1.90	1.88	1.85
14	3.10	2.73	2.52	2.39	2.31	2.24	2.19	2.15	2.12	2.10	2.05	2.01	1.96	1.94	1.91	1.89	1.86	1.83	1.80
15	3.07	2.70	2.49	2.36	2.27	2.21	2.16	2.12	2.09	2.06	2.02	1.97	1.92	1.90	1.87	1.85	1.82	1.79	1.76
16	3.05	2.67	2.46	2.33	2.24	2.18	2.13	2.09	2.06	2.03	1.99	1.94	1.89	1.87	1.84	1.81	1.78	1.75	1.72
17	3.03	2.64	2.44	2.31	2.22	2.15	2.10	2.06	2.03	2.00	1.96	1.91	1.86	1.84	1.81	1.78	1.75	1.72	1.69
18	3.01	2.62	2.42	2.29	2.20	2.13	2.08	2.04	2.00	1.98	1.93	1.89	1.84	1.81	1.78	1.75	1.72	1.69	1.66
19	2.99	2.61	2.40	2.27	2.18	2.11	2.06	2.02	1.98	1.96	1.91	1.86	1.81	1.79	1.76	1.73	1.70	1.67	1.63
20	2.97	2.59	2.38	2.25	2.16	2.09	2.04	2.00	1.96	1.94	1.89	1.84	1.79	1.77	1.74	1.71	1.68	1.64	1.61
21	2.96	2.57	2.36	2.23	2.14	2.08	2.02	1.98	1.95	1.92	1.87	1.83	1.78	1.75	1.72	1.69	1.66	1.62	1.59
22	2.95	2.56	2.35	2.22	2.13	2.06	2.01	1.97	1.93	1.90	1.86	1.81	1.76	1.73	1.70	1.67	1.64	1.60	1.57
23	2.94	2.55	2.34	2.21	2.11	2.05	1.99	1.95	1.92	1.89	1.84	1.80	1.74	1.72	1.69	1.66	1.62	1.59	1.55
24	2.93	2.54	2.33	2.19	2.10	2.04	1.98	1.94	1.91	1.88	1.83	1.78	1.73	1.70	1.67	1.64	1.61	1.57	1.53
25	2.92	2.53	2.32	2.18	2.09	2.02	1.97	1.93	1.89	1.87	1.82	1.77	1.72	1.69	1.66	1.63	1.59	1.56	1.52
26	2.91	2.52	2.31	2.17	2.08	2.01	1.96	1.92	1.88	1.86	1.81	1.76	1.71	1.68	1.65	1.61	1.58	1.54	1.50
27	2.90	2.51	2.30	2.17	2.07	2.00	1.95	1.91	1.87	1.85	1.80	1.75	1.70	1.67	1.64	1.60	1.57	1.53	1.49
28	2.89	2.50	2.29	2.16	2.06	2.00	1.94	1.90	1.87	1.84	1.79	1.74	1.69	1.66	1.63	1.59	1.56	1.52	1.48
29	2.89	2.50	2.28	2.15	2.06	1.99	1.93	1.89	1.86	1.83	1.78	1.73	1.68	1.65	1.62	1.58	1.55	1.51	1.47
30	2.88	2.49	2.28	2.14	2.05	1.98	1.93	1.88	1.85	1.82	1.77	1.72	1.67	1.64	1.61	1.57	1.54	1.50	1.46
40	2.84	2.44	2.23	2.09	2.00	1.93	1.87	1.83	1.79	1.76	1.71	1.66	1.61	1.57	1.54	1.51	1.47	1.42	1.38
60	2.79	2.39	2.18	2.04	1.95	1.87	1.82	1.77	1.74	1.71	1.66	1.60	1.54	1.51	1.48	1.44	1.40	1.35	1.29
120	2.75	2.35	2.13	1.99	1.90	1.82	1.77	1.72	1.68	1.65	1.60	1.55	1.48	1.45	1.41	1.37	1.32	1.26	1.19
∞	2.71	2.30	2.08	1.94	1.85	1.77	1.72	1.67	1.63	1.60	1.55	1.49	1.42	1.38	1.34	1.30	1.24	1.17	1.00