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جامعة مصطفى اسطمبولي
معسكر

Faculty of Economic, Commercial and Management Sciences
Department: Business Administration

Lecture Handout for First-Year Master's Students in Business Administration

In the course of:

Quantitative Methods in Management

Prepared by Professor: Ammar Hamioud

Academic Year: 2024/2025

__Lectures on quantitative methods in management__

These lectures are directed to students:

- Level: First-year Master's
- Specialization: Management
- Branch: Business Administration
- Field: Economic Sciences, Commercial Sciences, and Management Sciences

According to the scheduled program for aligning with the Master's academic training offer, it has been divided into the following content:

" Axis 1: introduction to quantitative methods in management and quantitative analysis methodology in decision-making

Axis 2: Linear programming (decision tree, assignment problems, queuing Theory).

Axis 3: analyzing costs for the purpose of making decisions.

Axis 4: break-even point.

Axis 5: game theory.

Axis 6 :Simulation Modeling.

Axis 7: statistical methods in quality control.

Axis 8: dynamic programming" (Faculty of economic sciences, commercial sciences and sanagement Sciences, 2023/2024, p. 44)

Lectures will be conducted as listed in the following schedule:

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Introduction of the book

Quantitative methods are among the most essential scientific tools relied upon in economics, business, and management sciences to analyze phenomena and support rational decision-making. In light of the increasing complexity facing modern institutions, mastering quantitative competencies and the ability to apply analytical models has become a crucial necessity for every student and practitioner in the field of management. Based on this awareness, this educational book serves as an academic contribution aimed at enabling first-year Master's students in Business Administration-as well as students from similar or related disciplines—to gain a comprehensive understanding of the foundations and applications of quantitative models in the business environment.

In this work, we have strived to present a scientific and methodical content that covers the key components of the Quantitative Methods in Management course in an integrated and structured manner, blending both theoretical and practical aspects. We drew upon extensive teaching experience in this subject and other similar courses such as Operations Research, Statistical Modeling, and Decision-Making Models. The result is the outcome of continuous effort that spanned weeks of preparation, review, and refinement—an attempt to distill the essence of our academic expertise and pedagogical insight in this field.

Preparing this book was far from a simple task. It demanded substantial cognitive and methodological effort, involving careful topic selection, simplification of concepts, logical formatting, and reinforcement with illustrative examples and precise explanations—all while maintaining strict adherence to academic integrity, scientific organization, and the intellectual level of graduate students.

The book includes fourteen comprehensive lectures covering a range of topics such as linear programming, queuing theory, game theory, decision trees, break-even analysis, simulation, and other quantitative models that contribute to more effective managerial decision-making in dynamic and complex environments.

We hope that this book will serve as a valuable academic resource, supporting students in their academic journey, enhancing their ability to understand, analyze, and apply quantitative tools, and fostering a scientific mindset in the pursuit of rational and evidence-based decisions.

Learning Objectives of the Course

By the end of this course, the student is expected to have acquired the following competencies:

- A solid understanding of the fundamental theoretical concepts of quantitative methods, and the ability to grasp quantitative analysis methodology as a scientific tool to support decision-making in modern administrative environments.

- The ability to effectively apply linear programming in modeling allocation and resource distribution problems, and to utilize decision trees and queuing models in analyzing various administrative scenarios.

- The capacity to integrate cost analysis with break-even calculations within a unified analytical framework that enables the evaluation of available alternatives and the formulation of financial and planning decisions based on precise quantitative criteria.

- The application of the principles and rules of game theory to understand the interactive dynamics between actors in markets or organizations, and to develop decision-making strategies under competitive or cooperative conditions.
- The design and implementation of simulation models to address complex administrative problems, and to test the impact of various variables on decision outcomes before actual implementation.
- The use of modern statistical methods in quality control and deviation analysis, thereby contributing to improved organizational performance and enhanced operational outcomes.
- A comprehensive understanding of dynamic programming principles and their application in solving problems that require sequential decision-making across changing and interrelated time periods.
- The development of critical analytical skills in selecting and applying appropriate quantitative models, with the ability to interpret their results and assess their relevance to organizational realities.

Lecture 1: introduction to quantitative methods in management and quantitative analysis methodology in decision-making

Introduction

In light of the growing challenges facing modern organizations, there is an increasing need to adopt tools and techniques that enable decision-making based on systematic analysis and quantitative data. Relying solely on personal experience or intuition is no longer sufficient to address the complex and ever-changing administrative issues, especially in an environment marked by fierce competition and economic and technological volatility. As a result, quantitative methods have emerged as one of the most important scientific alternatives, drawing upon mathematical models and statistical analysis to provide accurate, objective solutions that enhance the efficiency of managerial decisions.

This lecture aims to shed light on the general concept of quantitative methods in management, trace their historical development, explain the stages of model building, and highlight the key motivations behind their use in organizations. Additionally, it explores the most widely used quantitative techniques in decision-making and planning, emphasizing how these tools can contribute to better administrative outcomes and strategic performance.

Lecture Objectives

- Define the concept of quantitative methods and understand their role in managerial decision-making.
- Trace the historical development of quantitative methods and identify major scientific contributors.
- Explain the stages of model building within the quantitative approach to problem-solving.
- Identify the main motivations and practical reasons for using quantitative methods in management.
- Recognize the most commonly applied quantitative techniques and their relevance to solving administrative problems.
- ✓ Evaluate how quantitative methods contribute to rational, objective, and efficient decision-making in organizational contexts.

1. The Concept of Quantitative Methods in Management

In the ever-evolving field of management, the need for rational and evidence-based decision-making has become increasingly essential. This necessity has led to the development and application of quantitative methods, which provide managers with objective tools to analyze problems, evaluate alternatives, and optimize outcomes. By relying on mathematical models, statistical techniques, and data-driven logic, quantitative methods offer a scientific framework for addressing complex managerial challenges. The following discussion presents a comprehensive definition of these methods and highlights their significance in guiding decision-making processes across various administrative functions.

Quantitative methods in management are “a set of techniques, formulas, equations, and models that help solve problems on a rational basis... or they are the mechanisms through which the quantitative approach is implemented. They are tools that rely on quantification and the objective measurability of problem variables and decision criteria, using mathematical methods and models to solve the problem. The quantitative approach represents the framework within which these methods are used.” One researcher points out that the quantitative approach to management requires that decision problems be well-defined and subject to scientific, systematic, logical, and realistic analysis and resolution, based on data, facts, and reason rather than on guesswork or personal bias (Nedjm, 2008, p. 11).

Some have defined quantitative methods in management as “a set of tools or methods used by the decision maker to address a specific problem in order to rationalize the managerial decision made regarding a certain situation. This assumes the availability of sufficient data related to the problem. For example, in the field of production management, the requirements of raw materials, labor, and any other inputs for the production process are determined, along with the nature of the outputs. On the one hand, this involves identifying the resources and data, and on the other, it requires applying them to formulate hypotheses and identify both direct and indirect influencing factors.” (Said, 2007, p. 15)

In summary, quantitative methods serve as a vital foundation for modern management practices, enabling decision-makers to move beyond intuition toward structured, measurable, and rational solutions. Through the use of mathematical tools and empirical data, these methods enhance the quality of administrative decisions and support more efficient resource allocation. As organizations face increasingly complex environments, the role of quantitative methods in achieving strategic clarity and operational effectiveness continues to grow, reinforcing their position as indispensable components of scientific management.

2. Development of Quantitative Methods in Management

The evolution of management science has been deeply influenced by the integration of quantitative methods. These methods, rooted in mathematical modeling and statistical analysis, emerged as essential tools for solving complex administrative and operational problems. The historical development of quantitative techniques in management reflects a gradual shift from intuitive decision-making toward data-driven and analytical approaches, shaped by the contributions of scientists, global events, and institutional support. This paper explores the historical milestones and key scientific contributions that have defined the growth of quantitative methods in the field of management.

Quantitative methods in management had their early beginnings during World War II, when the need arose for precise scientific tools to solve military and logistical problems. In response, Britain formed scientific teams consisting of experts in mathematics, statistics, and physics, whose work later became known as Operations Research. One of the most prominent pioneers of this approach was the British scientist P.M.S. Blackett. After the war, this quantitative approach spread to the United States, where it gained wide interest among researchers and practitioners, most notably B. James and V. Banenvar, who contributed to integrating these methods into civilian and industrial management. (Ratoul, 2006, pp. 4-7), (Nedjm, 2008, p. 12)

Since 1942, applied studies increasingly focused on production management, inventory, and operational processes. A key milestone was the establishment of the first Operations Research Society in the United Kingdom in 1948. This was followed in 1950 by the commissioning of the RAND Corporation in the U.S. to conduct administrative research using quantitative methods, significantly enhancing decision-making and problem-solving methodologies. The establishment of the first scientific journal specialized in operations research in 1952 and 1953 crowned this progression and firmly established quantitative methods as a cornerstone of modern management thinking. (Ratoul, 2006)

This development was accompanied by major contributions from several scientists who laid the foundations for quantitative models and guided management thinking toward mathematically-based solutions. Among them was Gantt, who developed the Gantt chart for planning and scheduling tasks, and A.K. Erlang, who established the foundations of queuing theory. E. Borel and J.V. Newman made important mathematical contributions between 1921 and 1928. In 1949, G. Dantzig developed the simplex method for solving linear programming problems, while Kantrovich created the first mathematical model for economic planning in the Soviet Union in 1939. Vogel, Kooper, and A. Charnes contributed to developing allocation and transportation models. Meanwhile, J. Kelly and Walker introduced the Critical Path Method (CPM) in 1957 in the United States—an approach that became central in project planning. These contributions laid a solid foundation that firmly established quantitative methods as one of the pillars of modern scientific management. (Ratoul, 2006)

The historical development of quantitative methods in management illustrates a transformative journey from military applications during wartime to widespread adoption in civilian and industrial settings. This evolution was driven by the growing complexity of organizational challenges and the increasing need for structured, data-based decision-making processes. With foundational contributions from leading scientists and institutions, quantitative methods have become indispensable tools in modern management, offering precision, efficiency, and strategic insight. As these methods continue to evolve, they remain central to advancing the science and practice of effective decision-making in an increasingly data-driven world.

3. Stages of Model Building in Quantitative Methods in Management

Model building aims to solve decision-making problems using the scientific method, which serves as a systematic approach to finding the best and most optimal solutions—either by developing existing models or by creating new ones. The scientific method forms the foundation upon which quantitative methods are based. According to (Nedjm, 2008, pp. 14-18), building a model using this approach goes through several key stages:

- Identifying Symptoms This stage depends on the decision-maker’s ability to observe events and changes within the organization, such as a decrease in customers, declining sales, or excess inventory. These signs indicate an undesirable situation that requires investigation and analysis to minimize or overcome it.

- Problem Definition At this stage, the aim is to provide a clear description of the problem, including the variables involved and the relationships between them. This involves collecting relevant data and identifying the root causes and determining the key variables contributing to the problem.

- Model Formulation A practical model of the problem is developed by accurately identifying the variables and elements of the model. A distinction is made between variables that can be controlled or influenced to mitigate the problem, and those that cannot be managed. The relationship between the objective of the model and these variables is also established.

- Model Analysis and Solution The goal in this phase is to find the best or optimal solution to the problem using the developed model. Two main approaches are used:

  - The first is a simple trial approach, in which the model is tested as it is without deriving optimal solutions. Related data is collected, necessary adjustments are proposed, and the analyst subjectively estimates the model's effectiveness.
  - The second approach involves deriving the optimal solution analytically from the model.

- Solution Implementation In this stage, the theoretical model is translated into practical application through field implementation to solve the problem. This stage is critical and demands significant effort from those involved, despite challenges such as the model's limited variables compared to real-world complexity, the level of support from management, and the organization's experience with quantitative methods. Another important consideration is balancing the effort and costs of model building and implementation with the results achieved.

- Improvement through Feedback After implementation, the process doesn't end with evaluating the model's effectiveness. Instead, it includes identifying further steps and improvements to make the model more efficient and better suited to solving the intended problem. Feedback allows for refining the model based on performance and real-world outcomes.

Model building in quantitative management methods represents a vital process that bridges theory and practice. By following a logical sequence—from identifying symptoms and defining the problem to formulating, solving, and implementing the model—organizations can make informed and efficient decisions. The final stage of feedback ensures that models evolve and adapt to changing conditions, reinforcing their relevance and effectiveness. As such, the ability to construct and refine models is not just a technical skill but a cornerstone of strategic, data-driven management in today's dynamic business environment.

4. Motivations and Reasons for Using Quantitative Methods

In today's business environment—characterized by increasing complexity and growing reliance on data—decision-makers face challenges that require objective, rational, and effective solutions. The intricate nature of modern organizations and the volatility of markets have paved the way for the adoption of quantitative methods as essential tools in administrative decision-making. These methods offer a scientific and systematic approach to solving problems, especially when traditional or intuitive methods fall short. What follows is a presentation of the key reasons and motivations behind this growing tendency to employ quantitative methods in management: (Nedjm, 2008, pp. 25-28)**:**

- There is often no better alternative than quantitative methods for addressing many of the complex problems faced by organizations and companies.
- These methods offer quantitative interpretation and analysis that introduce objectivity and generate broader acceptance from stakeholders such as shareholders, governments, and others.
- In many cases, decision-makers lack prior experience with newly emerging problems. Quantitative methods, in such situations, provide valuable tools that may lead to the best possible decisions.
- Quantitative methods help reduce administrative burden by providing systematic solutions to recurring problems, which can then be addressed by simply updating data inputs.
- Using quantitative approaches allows organizations to benefit from modern technologies such as computers, leading to lower costs, reduced effort and time, and more efficient and accurate decision-making.

The adoption of quantitative methods in management is not merely a trend but a necessity in today's decision-making landscape. From enhancing objectivity to reducing administrative burden and enabling technological integration, these methods provide a robust foundation for addressing both familiar and novel organizational challenges. Their continued use reflects their undeniable value in supporting rational, data-informed decisions.

5. Quantitative Methods Used in Managerial Decision-Making

A wide range of quantitative methods are used in organizations and companies to support sound decision-making and identify optimal solutions to both routine and exceptional problems. According to (Nedjm, 2008, pp. 26-28) and (Said, 2007, p. 17), the most commonly used methods include:

- Linear and nonlinear programming
- Dynamic programming
- Integer programming
- Simulation and computer-based simulation
- Critical Path Method (CPM) and Program Evaluation and Review Technique (PERT)
- Queuing theory (waiting lines)
- Transportation problems
- Assignment problems
- Game theory
- Inventory planning
- Forecasting
- Statistical analysis (probability theory, regression, sampling, hypothesis testing)
- Break-even analysis
- Markov chains

The diversity and applicability of quantitative methods make them powerful assets in modern management. From linear programming to forecasting and simulation, each technique contributes to informed and effective decision-making. As organizations strive for precision, efficiency, and strategic advantage, mastering these tools becomes a critical component of managerial competence.

Conclusion

The study of quantitative methods in management goes beyond being a technical discipline; it represents an analytical approach that reflects the profound transformation in contemporary managerial practices. By adopting these methods, decision-makers are equipped to handle problems scientifically—exploring alternatives, analyzing variables, and predicting outcomes in ways that move beyond randomness and reduce subjective bias.

This lecture has shown that the development of quantitative methods is the result of decades of accumulated scientific and experimental knowledge, making them indispensable tools in the modern business environment. Their diversity and wide-ranging applications allow organizations to adapt to market fluctuations and achieve higher levels of efficiency and agility. As technological advancement accelerates, the importance of these methods is expected to grow, calling on both researchers and practitioners to continue learning and evolving in this field to support more accurate and reality-aligned decision-making.

Lecture 2: linear programming and graphical solution

Introduction

Linear programming is considered one of the most important quantitative methods in decision-making, as it provides a precise mathematical framework for analyzing problems that involve the allocation of limited resources to achieve the best possible outcome—whether maximizing profit or minimizing cost. It is widely used across various fields such as economics, management, engineering, and industrial planning. Linear programming relies on mathematical models that express the relationships between variables using linear equations and inequalities. These models are built on key components such as the objective function and constraints and are often solved using graphical methods or algorithms like the Simplex method. This lecture aims to introduce students to the concepts of linear programming, its core components, and the steps involved in constructing and solving such models, supported by real-world examples that reinforce both theoretical understanding and practical application.

Learning Objectives

By the end of this lecture, students are expected to be able to:

- Define linear programming and explain its nature and applications.
- Identify the components of a linear programming model, including the objective function, decision variables, and constraints.
- Formulate a mathematical model from a real-world problem by translating the given data into variables and equations.
- Apply the graphical method to solve linear programming problems involving two variables.
- Analyze results and determine the optimal solution, while interpreting constraints and understanding the economic meaning of the outcomes.
- Develop analytical thinking and modeling skills as essential tools for decision-making.

1. Definition of Definition of linear program

Linear programming is a mathematical approach used to model and solve decision-making problems, where the relationships between variables are expressed through linear equations and inequalities. The goal of this model is to achieve the best possible outcome—whether maximizing a benefit or minimizing a cost—while adhering to a set of predefined constraints. This type of model is characterized by clearly defined data and fixed coefficients, making it an effective tool for addressing planning and resource allocation problems across various fields, especially in economics. (Ratoul, 2006, p. 9)

2.Components of Linear Programming

Mathematical models used in linear programming are based on a set of variables that represent the decision elements to be determined. The relationships among these variables must be of the first degree, meaning that the coefficients are constant numbers and do not involve powers, roots, or products between variables. This condition ensures the model's simplicity and facilitates its analysis using mathematical tools.

A linear programming model is built upon a set of core components that define its structure and guide its solution process. These components include:

2.1. Objective Function

This represents the ultimate goal of the model, whether it is to maximize profit, minimize cost, or achieve another target. The objective function is formulated as a sum or difference of variables multiplied by numerical coefficients that reflect the value of each unit.

2.2. Constraints

These express the limitations imposed on the variables, such as resource availability, production capacity, or other operational limits. Constraints are written as equations or inequalities defining the relationships among variables.

2.3. Non-Negativity Constraints

These constraints require that all variables take non-negative values (i.e., zero or positive). This reflects the reality that the quantities represented—such as production, time, or resources—cannot be negative. These are typically written as: $x_i \geq 0 \quad \forall i$

Linear programming is a powerful tool for decision-making when multiple alternatives exist under certain restrictions. It is characterized by its clear structure and ability to deal with real-world problems in a systematic and organized manner, making it one of the most important quantitative analysis methods in fields such as management, economics, and engineering.

3. Steps for Constructing a Mathematical Program

Mathematical programming is an effective tool for analyzing problems that involve decision-making under specific conditions and constraints. The construction of a mathematical model begins with a clear understanding of the given problem and a careful analysis of the provided data. From this, the essential elements are identified, such as the objective of the solution, available resources, and factors influencing the decision.

The next step is to define the variables that represent the decisions to be made, followed by the formulation of the objective function, which expresses the aim of the model—whether to maximize profits or minimize costs. Then, a set of constraints is developed to represent the limits imposed on decisions, such as resource boundaries or technical conditions. Non-negativity constraints are also included to ensure that the decision variables take on only positive or zero values, as they usually represent real-world quantities.

To facilitate formulation and interpretation, it is preferable to organize all model components into a table that summarizes the variables, objective function, and constraints in a clear and structured manner. This tabular representation not only enhances the precision of model building but also simplifies its analysis and subsequent solution. The following examples illustrate this.

3.1. Example 1: (Profit Maximization Problem) A Company Producing Soap and Ghee

A company is engaged in producing two products: soap and ghee. Each unit of soap generates a profit of 40 DZD, while each unit of ghee yields a profit of 60 DZD. The company has a weekly limit of 200 working hours. Producing one unit of soap requires 2 hours of labor, whereas producing one unit of ghee takes 4 hours. Additionally, production constraints limit the weekly output to a maximum of 80 units of soap and 60 units of ghee. Given these parameters, the company seeks to determine the optimal quantities of each product to maximize profit without exceeding available resources or production limits.

Table 2.1. Data Organized in a Table

Item	Soap (X)	Ghee (Y)	Available Limit
Profit per unit	40 DZD	60 DZD	-
Labor hours per unit	2 hours	4 hours	200 hours
Maximum production limit	≤ 80	≤ 60	As specified

Source : Author's own work

Mathematical Formulation of the Problem

- Decision Variables:
  - X: Number of soap units produced per week.
  - Y: Number of ghee units produced per week.
- Objective Function (Profit):
  - Maximize: $Z = 40X + 60Y$
- Constraints:

$$\begin{cases} 2X + 4Y \leq 200 & \text{(Time constraint)} \\ X \leq 80, \quad Y \leq 60 & \text{(Production capacity constraints)} \\ X \geq 0, \quad Y \geq 0 & \text{(Non-negativity)} \end{cases}$$

3.2. Example 2*:* Minimizing Treatment Cost under Nutritional Constraints

A pharmacist aims to formulate a treatment composed of two medicinal components (Drug 1 and Drug 2). Each drug contains certain amounts of vitamin A and vitamin B, and the final treatment must contain no less than:

- 30 units of vitamin A
- 20 units of vitamin B

Table 2.2. Drug Data

Drug	Vitamin A (units/dose)	Vitamin B (units/dose)	Cost (USD/dose)
Drug 1	3	1	2
Drug 2	1	2	1

Source : Author's own work

Objective:

Determine the number of doses of each drug such that:

- The minimum required amount of each vitamin is met.

- The total cost of the treatment is as low as possible.

Decision Variables:

Let:

- X_1: Number of doses of Drug 1
- X_2: Number of doses of Drug 2

Table 2.3.Analytical Data Table for the Linear Programming Model

Component / Substance	Vitamin A (units/dose)	Vitamin B (units/dose)	Cost (USD/dose)
Drug 1 (x_1)	3	1	2
Drug 2 (x_2)	1	2	1
Minimum Requirement	≥ 30 units	≥ 20 units	-

Source : Author's own work

Objective Function: Minimize total cost:

$$\text{Min}Z = 2x_1 + x_2$$

Constraints:

To ensure vitamin requirements are satisfied:

$$\begin{cases} 3x_1 + x_2 \geq 30 & \text{(Vitamin A)} \\ x_1 + 2x_2 \geq 20 & \text{(Vitamin B)} \\ x_1 \geq 0, \quad x_2 \geq 0 & \text{(Non-negativity)} \end{cases}$$

4. Graphical Solution of the Linear Programming Problem

The graphical solution is one of the fundamental methods in linear programming, particularly suited for problems involving only two variables. This approach allows for the graphical representation of linear constraints on a coordinate plane, facilitating the identification of the feasible region formed by the intersection of these constraints. The objective function is then evaluated within this region to determine the point that yields the optimal value, whether minimizing cost or maximizing profit.

As a visual and intuitive approach, the graphical method provides an accessible entry point to understanding key concepts in linear optimization, such as the influence of constraints on the solution space and the fact that the optimal solution often lies at one of the corner points of the feasible region. While not practical for high-dimensional problems, it serves as a valuable pedagogical tool for building foundational intuition before advancing to more sophisticated analytical methods like the Simplex algorithm or numerical techniques used in higher-dimensional optimization models.

4.1. Example 1

Problem Statement

Maximize: $Z = 40X + 60Y$

Subject to the constraints:

$$\begin{cases} 2X + 4Y \le 200 & \text{(Time constraint)} \\ X \le 80, \quad Y \le 60 & \text{(Production capacity constraints)} \\ X \ge 0, \quad Y \ge 0 & \text{(Non-negativity)} \end{cases}$$

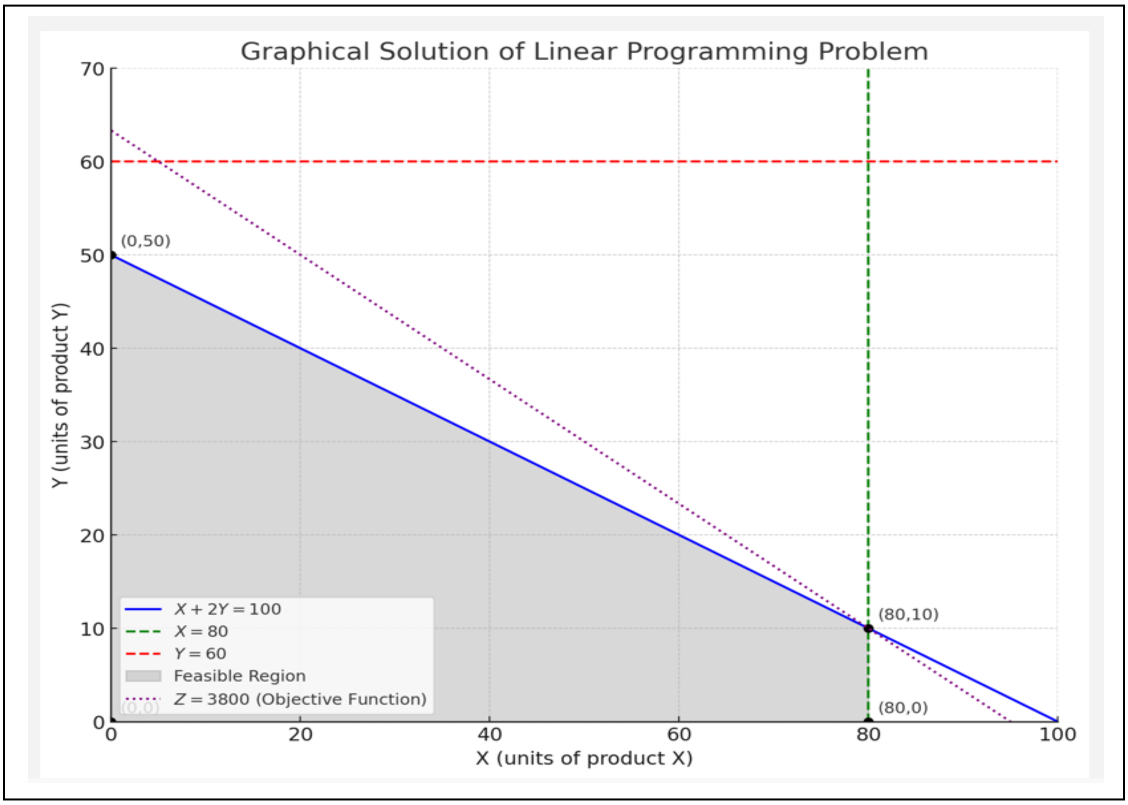
Step 1: Identify and Graph the Feasible Region

We start by rewriting the constraints in a form suitable for graphing:

- $2X + 4Y \le 200 \rightarrow$ Divide by 2: $X + 2Y \le 100$
  - Intercepts:
    - If $X = 0 \rightarrow Y = 50$
    - If $Y = 0 \rightarrow X = 100$
  - Plot this line: $X + 2Y = 100$
- $X \le 80$: vertical line at $X = 80$
- $Y \le 60$: horizontal line at $Y = 60$
- $X \ge 0,\ Y \ge 0$: restrict the solution to the first quadrant

The feasible region is the intersection of all these inequalities in the first quadrant. It is a closed polygon bounded by these constraints.

Fig 2.1



Source : Author's own work

Step 2: Determine the Corner Points of the Feasible Region

We find the coordinates of the corner points (vertices) of the feasible region by solving the equations at the intersections of the boundary lines:

- Intersection of $X = 0$ and $Y = 0 \rightarrow (0,0)$
- Intersection of $X = 80$ and $Y = 0 \rightarrow (80,0)$
- Intersection of $X = 80$ and $X + 2Y = 100$:

$$80 + 2Y = 100 \Rightarrow 2Y = 20 \Rightarrow Y = 10 \Rightarrow (80,10)$$

- Intersection of $Y = 60$ and $X + 2Y = 100$:

$$X + 2(60) = 100 \Rightarrow X + 120 = 100 \Rightarrow X = -20 \quad \text{(not feasible)}$$

→ This point is outside the feasible region.

- Intersection of $X = 0$ and $X + 2Y = 100$:

$$0 + 2Y = 100 \Rightarrow Y = 50 \Rightarrow (0,50)$$

Feasible corner points:

$$(0,0), \quad (80,0), \quad (80,10), \quad (0,50)$$

Step 3: Evaluate the Objective Function at Each Corner Point

We now compute $Z = 40X + 60Y$ at each feasible point:

Table 2.4

Point	Calculation	Z Value
(0;0)	$40(0) + 60(0) = 0$	0
(80;0)	$40(80) + 60(0) = 3200$	3200
(80;10)	$40(80) + 60(10) = 3200 + 600 = 3800$	3800 (Maximum)
(0;50)	$40(0) + 60(50) = 3000$	3000

Source : Author's own work

Step 4: Conclude the Optimal Solution

- The maximum value of Z is **3800**
- It occurs at the point $(80,10)$

$$\boxed{\text{Optimal Solution: } X = 80, \quad Y = 10, \quad \text{Maximum } Z = 3800}$$

4.2. Solution of Example 2

Objective:

$$\min Z = 2X_1 + X_2$$

Subject to:

$$\begin{cases} 3X_1 + X_2 \geq 30 & \text{(Vitamin A constraint)} \\ X_1 + 2X_2 \geq 20 & \text{(Vitamin B constraint)} \\ X_1 \geq 0, \quad X_2 \geq 0 & \text{(Non-negativity constraints)} \end{cases}$$

Step 1: Convert Constraints into Line Equations

- From $3X_1 + X_2 = 30$:

$$X_2 = 30 - 3X_1$$

  - Intercepts: (0,30) and (10,0)
- From $X_1 + 2X_2 = 20$:

$$X_2 = \frac{20 - X_1}{2}$$

  - Intercepts: (0,10) and (20,0)

Step 2: Find Intersection of Constraints

Solving:

$$\begin{cases} 3X_1 + X_2 = 30 \\ X_1 + 2X_2 = 20 \end{cases} \Rightarrow X_1 = 8, \quad X_2 = 6$$

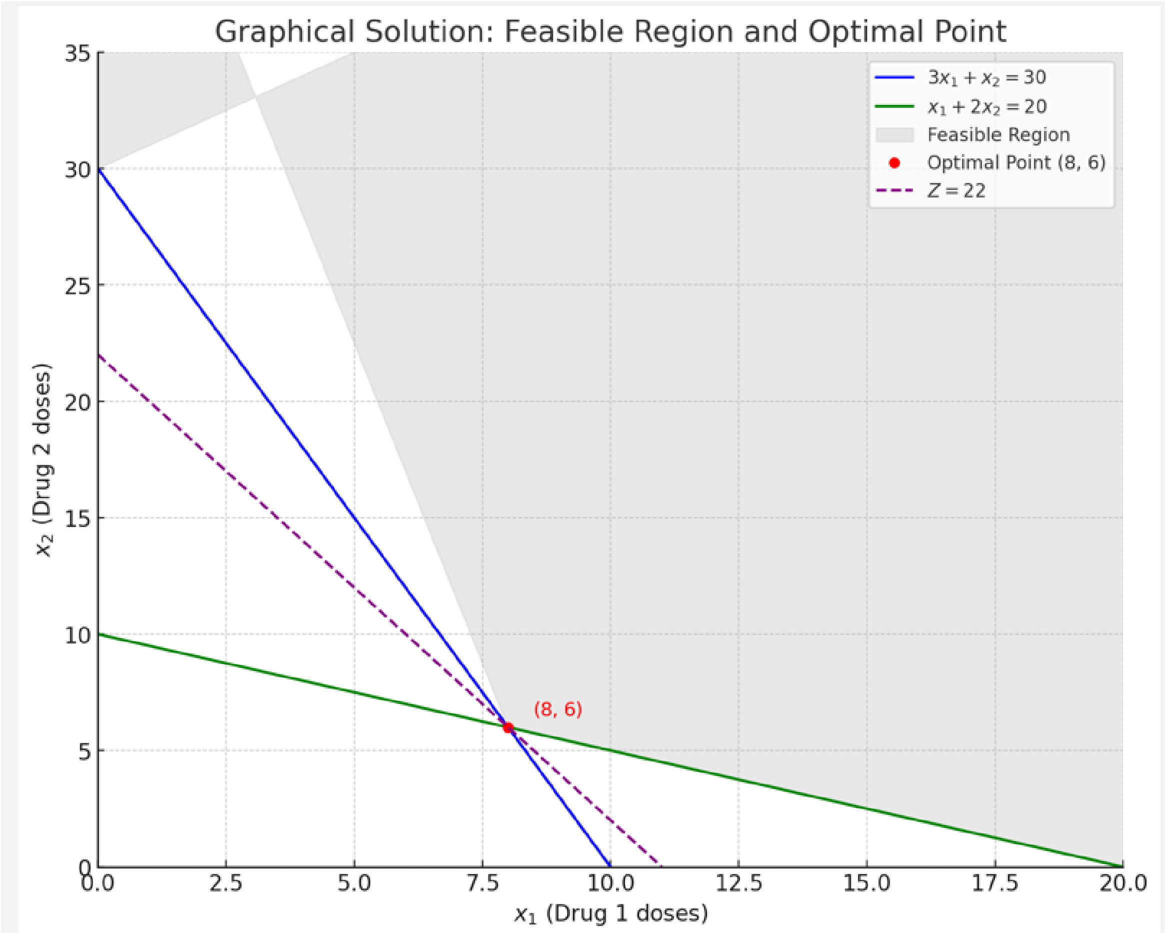
This point lies within the feasible region.

Step 3: Define the Feasible Region

The feasible region is bounded by:

- The area above the line $3X_1 + X_2 = 30$
- The area above the line $X_1 + 2X_2 = 20$
- The first quadrant only ($X_1 \geq 0,\ X_2 \geq 0$)
- This region is shaded in light gray on the graph Fig. 2.2.

Fig 2.2



Source : Author's own work

Step 4: Evaluate Objective Function at the Intersection Point

At point (8,6):

$$Z = 2(8) + 6 = 16 + 6 = 22$$

Final Result:

$$X_1 = 8, \quad X_2 = 6, \quad Z_{\min} = 22$$

Conclusion

In conclusion, linear programming highlights the importance of mathematical modeling as an effective tool for solving complex problems in a structured and scientific manner. Throughout this lecture, we explored the fundamental principles of linear programming, its components, and how to formulate and solve models using graphical methods, supported by practical, real-life examples. Linear programming serves as the gateway to more advanced mathematical programming models, including integer and nonlinear programming. The ability to construct and analyze such models is a core competency in fields like management, economics, and engineering, and a crucial step toward improving decision-making efficiency in environments characterized by resource scarcity and constraint complexity.

Lecture 3: : linear programming (Simplex method)

Introduction

Linear programming is considered one of the most important mathematical tools used to support decision-making, especially in areas related to maximizing profit or minimizing cost under certain constraints. Among the most effective methods for solving linear programming models, the Simplex Method stands out as a systematic approach that efficiently solves such problems. This lecture aims to introduce students to the fundamental principles of the Simplex method, how to convert linear programming models into standard form, and how to apply the procedural steps for solving using tabular techniques.

Learning Objectives:

By the end of this lecture, students will be able to:

- Understand the basic principles of the Simplex method as a tool for solving linear programming problems.
- Distinguish between the two objective types in linear programming (maximization and minimization) and their general forms.
- Convert linear programming models into standard form using auxiliary variables (slack, surplus, artificial).
- Identify the rules for converting constraints into equality equations depending on the problem and constraint type.
- Construct the initial Simplex table and identify basic and non-basic variables.
- Apply the procedural steps of the Simplex method, including selecting the pivot column and pivot row, and performing iterative operations until reaching the optimal solution.
- Analyze the final table and extract the optimal values of decision variables and the objective function.

1. Overview of the Simplex Approach

The Simplex algorithm is a systematic procedure designed to solve linear programming problems efficiently. It focuses on identifying the optimal value of an objective function—whether to maximize profit or minimize cost—while adhering to a set of linear constraints. The process involves navigating from one feasible vertex (corner point) of the solution space to another, guided by improving values of the objective function, until the optimal point is located. (Ratoul, 2006, p. 41)

2. General Form of a Linear Programming Problem

Linear programming problems can generally be formulated in matrix notation. Depending on the objective, two standard forms are recognized: (Ratoul, 2006)

For Maximization Problems:

Max: $Z=C^TX$

Subject to: $AX\leq B$, $X\geq 0$

For Minimization Problems:

Min: $Z=C^TX$

Subject to: $AX\geq B$, $X\geq 0$

Where:

- Z denotes the objective value to be optimized.
- X is the vector of decision variables.
- C contains the coefficients associated with each variable in the objective function.
- A is the matrix representing the coefficients of the constraints.
- B is the vector of right-hand-side constants in the constraints.

3. Converting the Model into Standard Form

Before applying the Simplex method, the problem must be restructured to comply with a set of standard conditions:

Purpose:

The transformation ensures the model is ready for algorithmic treatment by converting all constraints into equalities and guaranteeing all variables remain non-negative.

3.1. Procedure

- Constraints expressed as “less than or equal to” ($\leq$) are converted into equalities by introducing slack variables that absorb the unused portion of the resources.
- Constraints of the “greater than or equal to” ($\geq$) type require subtracting surplus variables, and because this transformation can disrupt feasibility, artificial variables are added to maintain solvability in the initial phase.
- Equality constraints may necessitate artificial variables directly, depending on their role in defining a feasible starting solution.
- The objective function must also be augmented to reflect these structural modifications, typically by appending zero coefficients for newly introduced variables or penalizing artificial variables using specific strategies such as the Big M method or the two-phase approach.

the following is a summary of how to convert a linear programming problem into standard form in all possible cases:

Table 3.1 Conversion Rules for Linear Programming Models into Standard Form

Case No.	Objective Function Type	Constraint Type	Conversion to Equality Form	Adjustment to Objective Function	Notes
1	Maximization (Max)	$\leq$	Add a positive slack variable $+X_s$	Add $+0X_s$	using simplex method
2	Maximization (Max)	$=$	Add artificial variable $+X_a$	Add $-MX_a$	using Big M or Two-Phase method
3	Maximization (Max)	$\geq$	Subtract $-X_s$, then add artificial variable $+X_a$	Add $+0X_s - MX_a$	using Big M or Two-Phase method
4	Minimization (Min)	$\leq$	Add a positive slack variable $+X_s$	Add $+0X_s$	using simplex method
5	Minimization (Min)	$=$	Add artificial variable $+X_a$	Add $+MX_a$	using Big M or Two-Phase method
6	Minimization (Min)	$\geq$	Subtract $-X_s$, then add artificial variable $+X_a$	Add $+0X_s + MX_a$	using Big M or Two-Phase method

Source: Author's own work

3.2. Examples

Example 1

Original Form:

$$\text{Maximize } Z = 3X_1 + 5X_2$$

$$\text{Subject to: } \begin{cases} 2X_1 + X_2 \leq 10 \\ X_1 + X_2 = 6 \\ 3X_1 + 4X_2 \geq 12 \\ X_1, X_2 \geq 0 \end{cases}$$

Standard Form:

$$Z = 3X_1 + 5X_2 + 0X_3 - MX_4 + 0X_5 - MX_6$$

$$\text{Subject to: } \begin{cases} 2X_1 + X_2 + X_3 = 10 \\ X_1 + X_2 + X_4 = 6 \\ 3X_1 + 4X_2 - X_5 + X_6 = 12 \\ X_1, X_2, X_3, X_4, X_5, X_6 \geq 0 \end{cases}$$

Example 2

Original Form:

$$\text{Maximize } Z = 4X_1 + 2X_2 + X_3$$

$$\text{Subject to:} \begin{cases} X_1 + X_2 + X_3 \leq 7 \\ 2X_1 + 3X_2 = 8 \\ X_2 + X_3 \geq 5 \\ X_1, X_2, X_3 \geq 0 \end{cases}$$

Standard Form:

$$Z = 4X_1 + 2X_2 + X_3 + 0X_4 - MX_5 + 0\,\mathrm{X}_6 - MX_7$$

$$\text{Subject to:} \begin{cases} X_1 + X_2 + X_3 + X_4 = 7 \\ 2X_1 + 3X_2 + X_5 = 8 \\ X_2 + X_3 - X_6 + X_7 = 5 \\ X_1, X_2, X_3, X_4, X_5, X_6, X_7 \geq 0 \end{cases}$$

Example 3

Original Form:

$$\text{Minimize } Z = 6X_1 + 4X_2$$

$$\text{Subject to:} \begin{cases} X_1 + 2X_2 \geq 10 \\ 3X_1 - X_2 = 5 \\ 2X_2 \leq 8 \\ X_1, X_2 \geq 0 \end{cases}$$

Standard Form:

$$Z = 6X_1 + 4X_2 + 0X_3 + MX_4 + MX_5 + 0X_6$$

$$\text{Subject to:} \begin{cases} X_1 + 2X_2 - X_3 + X_4 = 10 \\ 3X_1 - X_2 + X_5 = 5 \\ \quad 2 + X_2 + X_6 = 8 \\ X_1, X_2, X_3, X_4, X_5, X_6 \geq 0 \end{cases}$$

Example 4

Original Form:

$$\text{Minimize } Z = 5X_1 + X_2 + 3X_3$$

$$\text{Subject to:} \begin{cases} X_1 + X_2 + X_3 = 6 \\ X_1 + 3X_2 \leq 12 \\ 2X_1 + X_3 \geq 9 \\ X_1, X_2, X_3 \geq 0 \end{cases}$$

Standard Form:

$$Z = 5X_1 + X_2 + 3X_3 + MX_4 + 0X_5 + 0X_6 + MX_7$$

$$\text{Subject to:}\begin{cases}X_1 + X_2 + X_3 + X_4 = 6\\ X_1 + 3X_2 + X_5 = 12\\ 2X_1 + X_3 - X_6 + X_7 = 9\\ X_1, X_2, X_3, X_4, X_5, X_6, X_7 \geq 0\end{cases}$$

<u>Example 5</u>

Original Form:

$$\text{Maximize } Z = 7X_1 + 2X_2$$

$$\text{Subject to:}\begin{cases}X_1 + 4X_2 \leq 16\\ X_1 - X_2 = 3\\ 2X_2 \geq 4\\ X_1, X_2 \geq 0\end{cases}$$

Standard Form:

$$Z = 7X_1 + 2X_2 + 0X_3 - MX_4 + 0X_5 - MX_6$$

$$\text{Subject to:}\begin{cases}X_1 + 4X_2 + X_3 = 16\\ X_1 - X_2 + X_4 = 3\\ 2X_2 - X_5 + X_6 = 4\\ X_1, X_2, X_3, X_4, X_5, X_6 \geq 0\end{cases}$$

<u>Example 6</u>

Original Form:

$$\text{Minimize } Z = 2X_1 + 6X_2 + 4X_3$$

$$\text{Subject to:}\begin{cases}X_1 + X_3 \geq 5\\ X_2 + X_3 = 7\\ 3X_1 + 2X_2 \leq 18\\ X_1, X_2, X_3 \geq 0\end{cases}$$

Standard Form:

$$Z = 2X_1 + 6X_2 + 4X_3 + 0X_4 + MX_5 + MX_6 + 0X_7$$

$$\text{Subject to:}\begin{cases}X_1 + X_3 - X_4 + X_5 = 5\\ X_2 + X_3 + X_6 = 7\\ 3X_1 + 2X_2 + X_7 = 18\\ X_1, X_2, X_3, X_4, X_5, X_6, X_7 \geq 0\end{cases}$$

4. Solution using the Simplex Method

Below is an overview of the steps for solving using the Simplex method, along with an example.

4.1.Steps to Solve Using the Simplex Method:

<u>Step1:</u> Convert the Linear Program to Standard Form:

- Objective Function: Ensure that the objective function is in the correct form (whether maximization or minimization).

- Constraints: Convert all constraints into equalities by adding slack variables (for ≤) or subtracting surplus variables (for ≥), and add artificial variables when needed.
- Non-Negativity Constraints: Ensure that all variables (including decision variables, slack, surplus, and artificial variables) are non-negative.

<u>Step2:</u> Set Up the Initial Simplex Table:

- Create the initial table using:
  - Coefficients of the objective function in the first row.
  - Coefficients of the constraints in the subsequent rows, including slack, surplus, and artificial variables as necessary.
  - Place the right-hand side values (b-values) of the constraints in the last column.
- Initial Basic Feasible Solution (BFS): Identify which variables are basic (those representing slack, surplus, and artificial variables) and which are non-basic (those corresponding to the original decision variables).

<u>Step3:</u> Check for Optimality:

- Examine the Objective Function Row (Z-row):
  - For maximization: If all coefficients in the Z-row are non-negative, the current solution is optimal.
  - For minimization: If all coefficients in the Z-row are non-positive, the current solution is optimal.

<u>Step4:</u> Perform Pivoting:

- Choose the Pivot Column:
  - For maximization: Select the column with the most negative coefficient in the Z-row (indicating the greatest potential improvement in the objective function).
  - For minimization: Select the column with the most positive coefficient in the Z-row.
- Choose the Pivot Row:
  - Calculate the ratio of the right-hand side value to the corresponding pivot column element for each row (only consider positive elements in the pivot column).
  - The row with the smallest non-negative ratio is the pivot row.
- Execute the Pivot Operation:
  - Pivot around the selected element to make it equal to 1 and set all other elements in the pivot column to 0.
  - Update the remaining elements in the table based on the new pivot.

<u>Step5:</u> Repeat the Process:

- After each pivot, recheck the Z-row to determine if the solution is optimal.
- If the solution is not optimal, continue pivoting until all coefficients in the Z-row are non-negative (for maximization) or non-positive (for minimization).

Step6: Interpret the Final Table:

- The solution to the linear program is found in the final table:
  - The values of the decision variables correspond to the non-basic variables' values in the right-hand side column.
  - If a variable is part of the basic solution, its value will be the value in the right-hand side column; otherwise, its value will be zero.
  - The optimal value of the objective function is given by the right-hand side of the Z-row.

4.2. Example

Max Z= $X_1 + \ 9X_2 + X_3$

subject to $\begin{cases} X_1 + 2X_2 + 3X_3 & \leq & 9 \\ 3X_1 + 2X_2 + 2X_3 & \leq & 15 \\ X_1, X_2, X_3 \geq 0 \end{cases}$

Converting to Standard Form:
Max Z= $X_1 + \ 9X_2 + X_3 + +0X_4 + 0X_5$

subject to $\begin{cases} X_1 + 2X_2 + 3X_3 + \ X_4 & = & 9 \\ 3X_1 + 2X_2 + 2X_3 + \ X_5 & = & 15 \\ X_1, X_2, X_3, X_4, X_5 \geq 0 \end{cases}$

Based on the standard form, the initial basic solution table is constructed as follows:

Table. 3.2

T_0	Coefficients	1	9	1	0	0	0		
Coefficients	Basic Variable	X_1	X_2	X_3	X_4	X_5	RHS (Right-Hand Side)	$\frac{\text{RHS}}{\text{Pivot Column}}$	Determining the Pivot Row
0	X_4	1	**2**	3	1	0	9	9/2	Pivot Row
0	X_5	3	2	2	0	1	15	15/2	
	ΔZ	1	9	1	0	0	0		
	Determining the Pivot Column		Pivot Column						

Source : Author's own work

Since row Z contains positive values, the table T_0 does not provide the optimal solution. Therefore, the solution must be improved by identifying the pivot column, which is the one containing the highest positive value in row Z. Next, the values in the RHS column are divided by the corresponding positive (non-negative and non-zero) values in the pivot column. This yields the ratio column, from which the smallest value is selected; the corresponding row is the pivot row.

The pivot element results from the intersection of the pivot column and the pivot row. In this table, the pivot element is the value : 2.

Accordingly, the solution is improved by calculating the remaining values of the table using the following formula:

$$\text{The new value} = \text{the old value} - \frac{\text{the corresponding value in the pivot column} * \text{the corresponding value in the pivot row}}{\text{the pivot element}}$$

With the variable X_4 leaving the basis column and the variable X_2 entering the basis column.

The value in row Z is calculated as the coefficient in the objective function (as indicated in the objective function coefficients) minus the sum of the products of the values in the column and their corresponding coefficients in the basis column. for instance, the first value in Z row can be calculated as follows:

1- [(9*1/2)+(0*2)]=-7/2

Thus, we obtain the next updated table:

Table. 3.3

T_1	Coefficients	1	9	1	0	0	0
Coefficients	Basic Variable	X_1	X_2	X_3	X_4	X_5	RHS (Right-Hand Side)
9	X_2	1/2	1	3/2	1/2	0	9/2
0	X_5	2	0	-1	-1	1	6
	ΔZ	-7/2	0	-25/2	-9/2	0	-81/2

Source : Author's own work

Since all the values in row ΔZ are less than or equal to zero, table T1 provides the optimal solution, which is as follows:

$X_2 = 9/2$

$X_5 = 6$

$X_1 = X_3 = X_4 = 0$

Z=81/2

Conclusion:

In this lecture, we have explored the Simplex method as a powerful tool for solving linear programming problems. We explained the systematic steps for converting a model into its standard form, constructing the initial Simplex table, and then executing the iterative steps until the optimal solution is found.

Lecture 4: : linear programming (big M and mixed formulation)

Introduction

Linear programming is a powerful mathematical tool used to make optimal decisions in various fields such as production, transportation, distribution, and finance. Among the advanced methods for solving complex linear programming problems—especially those involving ≥ or = constraints—the Big M method stands out as a direct and effective approach. This method introduces artificial variables associated with a large constant denoted by M, enabling the transformation of the problem into a standard form that can be solved using the Simplex Method.

In this lecture, we will explore the Big M method, apply it to examples with mixed constraint formulations, and demonstrate how artificial variables are progressively removed from the basis to reach the optimal solution.

Learning Objectives

By the end of this lecture, students will be able to:

- Understand the core principle of the Big M method and distinguish it from the standard Simplex approach.
- Convert any linear programming problem with a mixed formulation into its standard form using artificial variables.
- Apply the Big M method in tabular form to solve maximization and minimization problems.
- Analyze solution tables and interpret the final Simplex table, particularly the elimination of artificial variables and its implication for optimality.
- Solve practical problems using the Big M method in diverse contexts, such as business and engineering.

1. Overview of the big M method

The Big M method is a tabular approach that follows the same steps as the Simplex method but differs in its use of artificial variables associated with a large constant MMM, from which the method derives its name. The problem is solved, and the optimal solution is reached, if all the values in the delta Z row are less than or equal to zero in a table that no longer contains the MMM term. In other words, the optimal solution table should not include any artificial variables, as they are completely removed each time they are eliminated from the basis column.

2. Mixed formulation

If we have a mixed formulation, meaning there are constraints with different types such as greater than or equal to, less than or equal to, or in the form of an equation, whether in the case of maximization or minimization, the same conditions and the same steps are applied based on the case, either maximization or minimization.

3. Example1 (the big M method)

$$MinZ = 10X_1 + 20X_2 + 30X_3$$

$$\text{Subject to:}\begin{cases} 2X_1 + X_2 + 2X_3 \ \geq 10 \\ \qquad 2X_2 + 3X_3 \geq 8 \\ X_2 + 3X_2 \qquad\quad \geq 6 \\ X_1, X_2, X_3 \geq 0 \end{cases}$$

<u>The solution using the Big M Method</u>

Converting to Standard Form:

$$MinZ = 10X_1 + 20X_2 + 30X_3 + \ 0X_4 + 0X_5 + 0X_6 + \text{M}X_7 + \text{M}X_8 + \text{M}X_9$$

$$\text{Subject to:}\begin{cases} 2X_1 + X_2 + 2X_3 \quad + X_4 \qquad\qquad - X_7 \qquad\qquad \geq 10 \\ \qquad 2X_2 + 3X_3 \qquad + X_5 \qquad\qquad - X_8 \qquad \geq 8 \\ X_2 + 3X_2 \qquad\qquad\qquad + X_6 \qquad\qquad - X_9 \quad \geq 6 \\ X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8, X_9 \geq 0 \end{cases}$$

<u>Note:</u> If any constraint in the standard form includes both a slack variable and an artificial variable, priority is given to the artificial variable to enter the basis column in the initial basic solution table.

Based on the standard form, the initial basic solution table is constructed as follows:

Table. 4.1

T_0	Coefficients	10	20	30	0	0	0	M	M	M			Determining the Pivot Row
Coefficients	Basic Variable	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	RHS	RHS/ Pivot column	
M	X_7	2	1	2	-1	0	0	1	0	0	10	10/1	
M	X_8	0	2	3	0	-1	0	0	1	0	8	8/2	
M	X_9	1	**3**	0	0	0	-1	0	0	1	6	6/3	Pivot Row
	ΔZ	10-3M	20-6M	30-5M	0+M	0+M	0+M	0+0	0+0	0+0	0-24M		
Determining the Pivot Column			Pivot Column										

Source : Author's own work

Since row Z contains nigative values, the table T_0 does not provide the optimal solution. Therefore, the solution must be improved by identifying the pivot column, which is the one containing The most negative value in row Z. Next, the values in the RHS column are divided by the corresponding positive (non-negative and non-zero) values in the pivot column. This yields the ratio column, from which the smallest value is selected; the corresponding row is the pivot row.

The pivot element results from the intersection of the pivot column and the pivot row. In this table, the pivot element is the value : 3.

Accordingly, the solution is improved by calculating the remaining values of the table using the following formula:

$$\text{The new value} = \text{the old value} - \frac{\text{the corresponding value in the pivot column} * \text{the correspon ding value in the pivot row}}{\text{the pivot element}}$$

With the variable X_9leaving the basis column and the variable X2 entering the basis column, and since X_9 is an artificial variable, it is removed entirely from the table and does not appear at all in the next table.

The value in row Z is calculated as the coefficient in the objective function (as indicated in the objective function coefficients) minus the sum of the products of the values in the column and their corresponding coefficients in the basis column. for instance, the first value in Z row can be calculated as follows:

10- [(M*2)+(M*0)+(M*1)]=10-3M

Thus, we obtain the next updated table:

Table. 4.2

T_1	Coefficients	10	20	30	0	0	0	M	M			Determining the Pivot Row
Coefficients	Basic Variable	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	RHS	RHS/ Pivot column	
M	X_7	5/6	0	2	-1	0	1/3	1	0	8	8/2	
M	X_8	-2/3	0	3	0	-1	2/3	0	1	4	4/3	Pivot Row
20	X_2	1/3	1	0	0	0	-1/3	0	0	2		
	ΔZ	10/3-1/6M	0+0	30-5M	0+M	0+M	20/3-M	0+0	0+0	-40-12M		
Determining the Pivot Column				Pivot Column								

Source : Author's own work

Since row Z contains negative values, table T_1 does not provide the optimal solution. By repeating the same steps, we obtain the next table:

Table. 4.2

T_2	Coefficients	10	20	30	0	0	0	M			Determining the Pivot Row
Coefficients	Basic Variable	X_1	X_2	X_3	X_4	X_5	X_6	X_7	RHS	RHS/ Pivot column	
M	X_7	23/18	0	0	-1	2/3	-1/9	1	16/3	96/23	Pivot Row
30	X_3	-2/9	0	1	0	-1/3	2/9	0	4/3		
20	X_2	1/3	1	0	0	0	-1/3	0	2	6	
	ΔZ	10-23M/18	0+0	0+0	0+M		10-2M/3	0+M/9	0+0	-80-16M/3	
Determining the Pivot Column		Pivot Column									

Source : Author's own work

Since row Z contains negative values, table T_2 does not provide the optimal solution. By repeating the same steps, we obtain the next table:

Table. 4.3

T_3	Coefficients	10	20	30	0	0	0	
Coefficients	Basic Variable	X_1	X_2	X_3	X_4	X_5	X_6	RHS
10	X_1	1	0	0	-18/23	12/23	-2/23	96/23
30	X_3	0	0	1	-4/23	-5/23	14/69	52/23
20	X_2	0	1	0	6/23	-4/23	-7/23	14/23
	ΔZ	0	0	0	180/23	110/23	20/23	-2800/23

Source : Author's own work

Since all the values in row ΔZ are greater than or equal to zero, table T_3 provides the optimal solution, which is as follows:

$X_1 = 96/23$

$X_3 = 52/23$

$X_2 = 14/23$

Z=2800/23

4. Example2 (mixed formulation)

$$MaxZ = 3X_1 - 2X_2 - 3X_3 + 3X_4$$

$$\text{Subject to:} \begin{cases} 2X_1 - X_2 - X_3 + X_4 \leq 4 \\ X_1 + X_2 + X_3 - X_4 \geq 5 \\ 2X_1 - 3X_2 + 3X_3 - 3X_4 = 6 \\ X_1, X_2, X_3, X_4 \geq 0 \end{cases}$$

Adabted from: (Ratoul, 2006, p. 80)

Converting to standard form:

$$MaxZ = 3X_1 - 2X_2 - 3X_3 + 3X_4 + 0X_5 + 0X_6 - \mathrm{M}X_7 - \mathrm{M}X_8$$

$$\text{Subject to:} \begin{cases} 2X_1 - X_2 - X_3 + X_4 + X_5 = 4 \\ X_1 + X_2 + X_3 - X_4 - X_6 + X_7 = 5 \\ 2X_1 - 3X_2 + 3X_3 - 3X_4 + X_8 = 6 \\ X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8 \geq 0 \end{cases}$$

We apply the same steps as in the case of maximization, taking into account the presence of artificial variables, and solve this program starting from the construction of the initial basic solution table as follows:

Table. 4.4

T_0	Coefficients	3	-2	-3	3	0	0	-M	-M			Determining the Pivot Row
Coefficients	Basic Variable	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	RHS	RHS/Pivot column	
0	X_5	2	1	-1	1	1	0	0	0	4		
-M	X_7	1	1	1	-1	0	-1	1	0	5	5/1	
-M	X_8	2	-3	3	-3	0	0	0	1	6	6/3	Pivot Row
	ΔZ	3+3M	-2-2M	-3+4M	3-4M	0+0	0-M	0+0	0+0	+11M		
Determining the Pivot Column				Pivot Column								

Source : Author's own work

ince row Z contains positive values, table T0 does not provide the optimal solution. By repeating the same steps as in the maximization case, we obtain the next table:

Table. 4.5

T_1	Coefficients	3	-2	-3	3	0	0	-M			Determining the Pivot Row
Coefficients	Basic Variable	X_1	X_2	X_3	X_4	X_5	X_6	X_7	RHS	RHS/Pivot column	
0	X_5	8/3	-2	0	0	1	0	0	6		
-M	X_7	1/3	2	0	0	0	-1	1	3	3/2	Pivot Row
-3	X_3	2/3	-1	1	-1	0	0	0	2		
	ΔZ	5+M/3	-5+2M	0+0	+00	0+0	0-M	0+0	6+3M		
Determining the Pivot Column			Pivot Column								

Source : Author's own work

Since row Z contains positive values, table T0 does not provide the optimal solution. By repeating the same steps as in the maximization case, we obtain the next table:

Table. 4.6

T_2	Coefficients	3	-2	-3	3	0	0			Determining the Pivot Row
Coefficients	Basic Variable	X_1	X_2	X_3	X_4	X_5	X_6	RHS	RHS/Pivot column	
0	X_5	3	0	0	0	1	-1	9	9/3	Pivot Row
-2	X_7	1/6	1	0	0	0	-1/2	3/2	9	
-3	X_3	5/6	0	1	-1	0	-1/2	7/2	21/5	
	ΔZ	35/6	0	0	0	0	-5/2	27/2		
Determining the Pivot Column		Pivot Column								

Source : Author's own work

Since row Z contains positive values, table T0 does not provide the optimal solution. By repeating the same steps as in the maximization case, we obtain the next table:

Table. 4.7

T_3	Coefficients	3	-2	-3	3	0	0	
Coefficients	Basic Variable	X_1	X_2	X_3	X_4	X_5	X_6	RHS
3	X_1	1	0	0	0	1/3	-1/3	3
-2	X_7	0	1	0	0	-1/18	-4/9	1
-3	X_3	0	0	1	-1	-5/18	-2/9	1
	ΔZ	0	0	0	0	-35/18	-5/9	-4

Source : Author's own work

Since all the values in row ΔZ are less than or equal to zero, table T1 provides the optimal solution, which is as follows:

$X_3 = 1$

$X_5 = 3$

$X_7 = 1$

Z=4.

***5. Important note*:** If any variable is non-positive (i.e., negative or free), we make the following adjustments:

If the variable is negative, we set $X = -X^{'}$, where $X^{'} \geq 0$.

If X is free($X \in \mathbb{R}$), we set $X = X_1 - X_2$, where $X_1 \geq 0$ and $X_2 \geq 0$.

Conclusion

The Big M method is a fundamental technique in linear programming, allowing the resolution of problems with complex constraints by introducing temporary artificial variables to facilitate the process. Successful application of this method requires accuracy in transforming the problem into standard form and a solid understanding of the Simplex tables and result interpretation. Mastering this technique is a key step toward gaining practical competence in linear programming applications across economic, engineering, and managerial domains.

Lecture 5: Linear programming (Dual program)

Introduction

Linear programming is one of the most widely used quantitative techniques for solving decision-making problems related to the optimal allocation of resources. A central concept within this field is duality, which describes a mathematical relationship between two equivalent formulations: the primal and the dual problem. Grasping this relationship not only enhances theoretical understanding of linear models but also facilitates detailed analysis of solutions and their sensitivity to changes in input data. This lecture is designed to explain how to systematically convert a linear program from its primal to its dual form, addressing both standard and mixed-form cases. It also explores how to derive the dual's optimal solution table based on that of the primal, and vice versa.

Learning Objectives

By the end of this session, students are expected to be able to:

- Clarify the conceptual and mathematical distinctions between the primal and dual forms of linear programs.
- Convert a primal linear program into its dual using a structured and logical process.
- Interpret the economic meaning of dual variables in relation to the primal constraints.
- Handle mixed-form linear programs that involve various types of constraints and variable restrictions.
- Construct the optimal solution table of the dual program by analyzing the optimal solution of the primal program, and understand the bidirectional relationship.
- Validate derived results by directly solving the dual problem and comparing them with the interpreted values from the primal.

1.Definition of dual program

For each linear program containing the variablesX_1, X_2,…,X_n, there is a corresponding dual program that contains the variables Y_1, Y_2,……..,Y_m where:

- M is the number of constraints in the primal (original) program.
- N is the number of variables in the dual program.

The coefficients of the objective function in each program are the values on the right-hand side of the constraints in the other program. (Ratoul, 2006, p. 81)

2. Mathematical Formulation

The mathematical formulation includes two forms as follows: (Ratoul, 2006, p. 82)

2.1. Mathematical Formulation of the Primal Program

The general form of the primal program can be written as follows:

Maximize:

$$Z = c_1X_1 + c_2X_2 + \cdots + c_NX_N$$

Subject to the constraints:

$$a_{11}X_1 + a_{12}X_2 + \cdots + a_{1N}X_N \le b_1$$

$$a_{21}X_1 + a_{22}X_2 + \cdots + a_{2N}X_N \le b_2$$

$$\vdots$$

$$a_{M1}X_1 + a_{M2}X_2 + \cdots + a_{MN}X_N \le b_M$$

Where $X_1, X_2, \dots, X_N \ge 0$

2.2. Mathematical Formulation of the Dual Program

The corresponding dual program can be written as follows:

Minimize:

$$W = b_1Y_1 + b_2Y_2 + \cdots + b_MY_M$$

Subject to the constraints:

$$a_{11}Y_1 + a_{21}Y_2 + \cdots + a_{M1}Y_M \ge c_1$$

$$a_{12}Y_1 + a_{22}Y_2 + \cdots + a_{M2}Y_M \ge c_2$$

$$\vdots$$

$$a_{1N}Y_1 + a_{2N}Y_2 + \cdots + a_{MN}Y_M \ge c_N$$

Where:

- Y_1, Y_2,……..,Y_m>=0 are the dual variables corresponding to the constraints in the primal program.

- The primal program aims to maximize the objective function, subject to a set of constraints.
- The dual program aims to minimize the objective function, where the coefficients of the objective function in the dual program are derived from the right-hand side values of the constraints in the primal program.

3. Steps of converting original program to dual program

The following are the steps to convert a linear program from its primal form to its dual form: (Ratoul, 2006, pp. 81-82)

- Define the Objective Function:
  - If the objective function in the primal program is Maximization, the objective function in the dual program will be Minimization.
  - The coefficients of the objective function in the dual program will be the right-hand side values from the constraints in the primal program.
- Define the Variables:

- For each constraint in the primal program, there is a dual variable in the dual program.
- If there are M constraints in the primal program, there will be M dual variables in the dual program.

- Convert the Constraints:
  - "Less than or equal" (≤) constraints in the primal program become "Greater than or equal" (≥) constraints in the dual program.
  - Variables in the primal program (such as $X_1, X_2, \dots, X_N$) that must be greater than or equal to zero (non-negative) in the primal, become dual variables in the dual program.
- Convert the Coefficients of the Constraints:
  - The coefficients of the constraints in the primal program (such as a_{ij}) become the coefficients of the objective function in the dual program.
- Objective Function of the Dual Program:
  - The objective function in the dual program is made up of the coefficients from the primal program's right-hand side, multiplied by the corresponding dual variables.
- Write the Dual Program:
  - After determining all the constraints, dual variables, and the objective function, the dual program is written using the appropriate equations.

Note: Based on the optimal solution table of the primal program, the optimal solution table of the dual program can be derived, and vice versa. That is, from the optimal solution table of the dual program, one can infer the optimal solution table of the primal program, starting from the optimal tables of both programs.

4. Examples

4.1. Example 1: Primal Program in Maximization Case

Primal Program:

$$\text{Max } Z = 4X_1 + 3X_2$$

$$\text{Subject to:}\begin{cases} X_1 + X_2 \leq 5 \\ 2X_1 + X_2 \leq 8 \\ X_1 \geq 0, X_2 \geq 0 \end{cases}$$

Dual Program*:*

$$\text{Min } W = 5Y_1 + 8Y_2$$

$$\text{Subject to:}\begin{cases} Y_1 + 2Y_2 \geq 4 \\ Y_1 + Y_2 \geq 3 \\ Y_1, Y_2 \geq 0 \end{cases}$$

4.2.Example 2: Primal Program in Minimization Case

Primal Program:

$$\text{Min } Z = 3X_1 + 2X_2$$

$$\text{Subject to:}\begin{cases} X_1 + 2X_2 \geq 6 \\ X_1 + X_2 \geq 4 \\ X_1 \geq 0, X_2 \geq 0 \end{cases}$$

Dual Program:

$$\text{Max } W = 6Y_1 + 4Y_2$$

$$\text{Subject to:}\begin{cases} Y_1 + Y_2 \leq 3 \\ 2Y_1 + Y_2 \leq 2 \\ Y_1, Y_2 \geq 0 \end{cases}$$

5.The dual program for mixed form

The dual program for mixed form linear programming provides an alternative way of solving optimization problems. It involves transforming the primal problem with a combination of inequalities and equations into a dual problem, offering valuable insights into the relationships between variables and constraints.

5.1.Cases

In the case of a linear program in mixed form, where the objective function is either maximization or minimization, and the constraints are a mix of "greater than or equal to", "less than or equal to", and equations, or at least two of these cases, with the possibility of having non-positive variables, the following steps are applied to handle the situation: (Ratoul, 2006, p. 89)

First case: If a constraint in the primal program is in the form of an equation, the corresponding variable extracted in the dual program will be free, meaning it belongs to the set of real numbers.

Second case: If one of the variables in the primal program is non-positive (i.e., belongs to the set of real numbers), then the corresponding constraint extracted in the dual program will be in the form of an equation.

Third case: If a constraint in the primal program does not align with the objective function according to the canonical form, for example, a constraint in the form of "greater than or equal to" with an objective function in the case of maximization, the constraint should be changed to "less than or equal to" by multiplying both sides of the inequality by -1. This will convert the inequality to the "less than or equal to" form, which aligns with the maximization objective function in the canonical form. The same action should be taken in the case of a minimization objective function and a constraint in the "less than or equal to" form. This is done to avoid having the variable extracted from that constraint to be negative in the dual program.

5.2.Note

It is important to note that if there is a negative variable in the primal program, it should be made positive by replacing it with a variable equal to it with a negative sign, as previously mentioned in the end of the last lecture.

5.3.Examples

Example 1: Case of a Constraint as an Equation in the Primal Program

Primal Program:

$$\text{Max } Z = 3X_1 + 2X_2$$

$$\text{Subject to:}\begin{cases} X_1 + X_2 = 5 \\ 2X_1 + X_2 \leq 8 \\ X_1 \geq 0, X_2 \geq 0 \end{cases}$$

Dual Program:

$$\text{Min } W = 5Y_1 + 8Y_2$$

$$\text{Subject to:}\begin{cases} Y_1 + 2Y_2 \geq 3 \\ Y_1 + Y_2 \geq 2 \\ Y_1 \in \mathbb{R}, Y_2 \geq 0 \end{cases}$$

Example 2: Case of a Non-Positive Variable in the Primal Program

Primal Program:

$$\text{Min } Z = 4X_1 + 3X_2$$

$$\text{Subject to:}\begin{cases} X_1 + X_2 \geq 6 \\ -X_1 + X_2 \geq 5 \\ X_1 \geq 0, X_2 \leq 0 \end{cases}$$

We set X=$-X_2'$, where $X_2' \geq 0$

We substitute in the primal program and obtain the following:

$$\text{Min } Z = 4X_1 - 3X_2'$$

$$\text{Subject to:}\begin{cases} X_1 - X_2' \geq 6 \\ -X_1 - X_2' \geq 5 \\ X_1 \geq 0, X_2' \geq 0 \end{cases}$$

Dual Program:

$$\text{Max } W = 6Y_1 + 5Y_2$$

$$\text{Subject to:}\begin{cases} Y_1 - Y_2 \leq 4 \\ -Y_1 - Y_2 \leq 3 \\ Y_1, Y_2 \geq 0 \end{cases}$$

Example 3: Free variable and a constraint that does not align with the objective function in the canonical form.

Primal Program:

$$\text{Max } Z = 5X_1 + 4X_2 + X_3$$

$$\text{Subject to:}\begin{cases} X_1 + X_2 - X_3 \geq 8 \\ 2X_1 - X_2 + X_3 \leq 12 \\ X_1, X_2 \geq 0, X_3 \in \mathbb{R} \end{cases}$$

Before converting to the dual program, we first transform the first constraint to be in the "less than or equal to" form, which will align with the objective function in the maximization case in the legal form. This is done by multiplying both sides of the inequality by -1. The primal program then becomes as follows:

$$\text{Max } Z = 5X_1 + 4X_2 + X_3$$

$$\text{Subject to:}\begin{cases} -X_1 - X_2 + X_3 \leq -8 \\ 2X_1 - X_2 + X_3 \leq 12 \\ X_1, X_2 \geq 0, X_3 \in \mathbb{R} \end{cases}$$

Dual Program:

Thus, the dual program becomes as follows:

$$\text{Min } W = -8Y_1 + 12Y_2$$

$$\text{Subject to:}\begin{cases} -Y_1 + 2Y_2 \geq 5 \\ -Y_1 - Y_2 \geq 4 \\ Y_1 + Y_2 = 1 \\ Y_1, Y_2 \geq 0 \end{cases}$$

6. Deriving the Optimal Solution Table for the Dual Program

In the following example, the process of deriving the optimal solution table for the dual program from the optimal solution table of the primal program will be explained.

$$MaxZ = 3X_1 - 2X_2 - 3X_3 + 3X_4$$

$$\text{Subject to:}\begin{cases} 2X_1 - X_2 - X_3 + X_4 \leq 4 \\ X_1 + X_2 + X_3 - X_4 \geq 5 \\ 2X_1 - 3X_2 + 3X_3 - 3X_4 = 6 \\ X_1, X_2, X_3, X_4 \geq 0 \end{cases}$$

Adabted from: (Ratoul, 2006, p. 80)

6.1.Converting to Dual program

First, the direction of the second constraint will be changed to align with the objective function in the maximization case as follows:

$$MaxZ = 3X_1 - 2X_2 - 3X_3 + 3X_4$$

$$\text{Subject to:}\begin{cases}2X_1 - X_2 - X_3 + X_4 \le 4\\ -X_1 - X_2 - X_3 + X_4 \le -5\\ 2X_1 - 3X_2 + 3X_3 - 3X_4 = 6\\ X_1, X_2, X_3, X_4 \ge 0\end{cases}$$

We obtain the following dual program:

$$MinW = 4Y_1 - 5Y_2 + 6Y_3$$

$$\text{Subject to:}\begin{cases}2Y_1 - Y_2 + 2Y_3 \ge 3\\ -Y_1 - Y_2 - 3Y_3 \ge -2\\ -Y_1 - Y_2 + 3Y_3 \ge -3\\ Y_1 + Y_2 - 3Y_3 \ge 3\\ Y_1, Y_2 \ge 0, Y_3 \in \mathbb{R}\end{cases}$$

We set $Y_3 = Y_3' - Y_3''$, where $Y_3' - Y_3'' \ge 0$

We obtain the following program:

$$MinW = 4Y_1 - 5Y_2 + 6Y_3' - 6Y_3''$$

$$\text{Subject to:}\begin{cases}2Y_1 - Y_2 + 2Y_3' - 2Y_3'' \ge 3\\ -Y_1 - Y_2 - 3Y_3' + 3Y_3'' \ge -2\\ -Y_1 - Y_2 + 3Y_3' - 3Y_3'' \ge -3\\ Y_1 + Y_2 - 3Y_3' + 3Y_3'' \ge 3\\ Y_1, Y_2, Y_3', Y_3'' \ge 0\end{cases}$$

$$MinW = 4Y_1 - 5Y_2 + 6Y_3' - 6Y_3''$$

$$\text{Subject to:}\begin{cases}2Y_1 - Y_2 + 2Y_3' - 2Y_3'' \ge 3\\ Y_1 + Y_2 + 3Y_3' - 3Y_3'' \le 2\\ Y_1 + Y_2 - 3Y_3' + 3Y_3'' \le 3\\ Y_1 + Y_2 - 3Y_3' + 3Y_3'' \ge 3\\ Y_1, Y_2, Y_3', Y_3'' \ge 0\end{cases}$$

Converti ng to standard program

$$MinW = 4Y_1 - 5Y_2 + 6Y_3' - 6Y_3'' + 0Y_4 + 0Y_5 + 0Y_6 + 0Y_7 + \mathrm{M}Y_8 + \mathrm{M}Y_9$$

$$\text{Subject to:}\begin{cases}2Y_1 - Y_2 + 2Y_3' - 2Y_3'' - \boldsymbol{Y_4} + Y_8 = 3 \ldots\ldots \boldsymbol{X_1}\\ Y_1 + Y_2 + 3Y_3' - 3Y_3'' + \boldsymbol{Y}_5 = 2 \ldots\ldots \boldsymbol{X_2}\\ Y_1 + Y_2 - 3Y_3' + 3Y_3'' + \boldsymbol{Y_6} = 3 \ldots\ldots \boldsymbol{X_3}\\ Y_1 + Y_2 - 3Y_3' + 3Y_3'' - \boldsymbol{Y_7} + Y_9 = 3 \ldots\ldots \boldsymbol{X_4}\\ Y_1, Y_2, Y_3', Y_3'', Y_4, Y_5, Y_6, Y_7, Y_8, Y_9 \ge 0\end{cases}$$

And we have the standard form of the primal program as follows:

$$MaxZ = 3X_1 - 2X_2 - 3X_3 + 3X_4 + 0X_5 + 0X_6 - MX_7 - MX_8$$

$$\text{Subject to:} \begin{cases} 2X_1 - X_2 - X_3 + X_4 + \boldsymbol{X_5} = 4 \ldots \boldsymbol{Y_1} \\ X_1 + X_2 + X_3 - X_4 - \boldsymbol{X_6} + X_7 = 5 \ldots \boldsymbol{Y_2} \\ 2X_1 - 3X_2 + 3X_3 - 3X_4 + X_8 = 6 \ldots . \boldsymbol{Y_3} \\ X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8 \geq 0 \end{cases}$$

We observe that each basic variable in the primal program corresponds to a slack (or artificial) variable in the dual program, and likewise, each basic variable in the dual program corresponds to a slack (or artificial) variable in the primal program. Based on this, we obtain the following table.

6.2. Deriving the Optimal Solution Table

Based on the optimal solution table of the primal program, which was previously solved in an earlier lecture as follows:

Table. 5.1

T_3	Coefficients	3	-2	-3	3	0	0	
Coefficients	Basic Variable	X_1	X_2	X_3	X_4	X_5	X_6	RHS
3	X_1	1	0	0	0	1/3	-1/3	3
-2	X_7	0	1	0	0	-1/18	-4/9	1
-3	X_3	0	0	1	-1	-5/18	-2/9	1
	ΔZ	0	0	0	0	-35/18	-5/9	-4

Source : Author's own work

We can derive the optimal solution table of the dual program as follows:

Table. 5.2

T_*	Coefficients	$\boldsymbol{X_5}$	$\boldsymbol{X_6}$	$\boldsymbol{X_1}$	$\boldsymbol{X_3}$	$\boldsymbol{X_4}$	
Coefficients	Basic Variable	Y_1	Y_2	Y_4	Y_6	Y_7	RHS
$\boldsymbol{X_5}$	Y_1	1	0	-1/3	5/18	0	35/18
$\boldsymbol{X_6}$	Y_2	0	1	1/3	2/9	0	5/9
$\boldsymbol{X_4}$	Y_7	0	0	0	1	1	0
	ΔW	0	0	3	1	0	-4

Source : Author's own work

There is an exchange between the shaded values in the Z-row and the RHS column, such that the values transferred from the RHS column in the primal program to the Z-row in the dual program retain their positive sign. In contrast, the values transferred in the opposite direction—that is, from the Z-row in the primal program to the RHS column in the dual program—change from negative to positive. As for the remaining values inside the table, their signs are reversed during the transfer, while maintaining their respective cell positions according to the correspondence of the X-values, as illustrated in the table.

6.3. Verification of the Solution

To verify that the derivation is correct, the dual program will be solved as follows.

Converti ng to standard program

$$MinW = 4Y_1 - 5Y_2 + 6Y_3{}' - 6Y_3{}''$$

$$\text{Subject to:} \begin{cases} 2Y_1 - Y_2 + 2Y_3{}' - 2Y_3{}'' \geq 3 \\ Y_1 + Y_2 + 3Y_3' - 3Y_3{}'' \leq 2 \\ Y_1 + Y_2 - 3Y_3' + 3Y_3{}'' \leq 3 \\ Y_1 + Y_2 - 3Y_3{}' + 3Y_3{}'' \geq 3 \\ Y_1, Y_2, Y_3{}', Y_3{}'' \geq 0 \end{cases}$$

Thus: $MinW = 4Y_1 - 5Y_2 + 6Y_3' - 6Y_3'' + 0Y_4 + 0Y_5 + 0Y_6 + 0Y_7 + \text{M}Y_8 + \text{M}Y_9$

$$\text{Subject to:} \begin{cases} 2Y_1 - Y_2 + 2Y_3' - 2Y_3'' - Y_4 + Y_8 = 3 \\ Y_1 + Y_2 + 3Y_3' - 3Y_3'' + Y_5 = 2 \\ Y_1 + Y_2 - 3Y_3' + 3Y_3'' + Y_6 = 3 \\ Y_1 + Y_2 - 3Y_3' + 3Y_3'' - Y_7 + Y_9 = 3 \\ Y_1, Y_2, Y_3', Y_3'', Y_4, Y_5, Y_6, Y_7, Y_8, Y_9 \geq 0 \end{cases}$$

Based on the standard form, the initial basic solution table is constructed as follows:

Table. 5.3

T_0	Coefficients	4	-5	6	-6	-6	0	0	0	M	M		RHS/ Pivot column	Determining the Pivot Row
Coefficients	Basic Variable	Y_1	Y_2	Y_3'	Y_3''	Y_4	Y_5	Y_6	Y_7	Y_8	Y_9	RHS		
M	Y_8	2	-1	2	-2	-1	0	0	0	1	0	3	3/2	Pivot Row
0	Y_5	1	1	3	-3	0	1	0	0	0	0	2	2	
0	Y_6	1	1	-3	3	0	0	1	0	0	0	3	3	
M	Y_9	1	1	-3	3	0	0	0	-1	0	1	3	3	
	ΔW	4-3M	-5+0	6+M	-6-M	0+M	0+0	0+0	0+M	0+0	0+0	-6M		
Determining the Pivot Column		Pivot Column												

Source : Author’s own work

Since row ΔW contains negative values, table T_0 does not provide the optimal solution. By repeating the same steps, we obtain the next table:

Table. 5.4

T_1	Coefficients	4	-5	6	-6	-6	0	0	0	M		RHS/ Pivot column	Determining the Pivot Row
Coefficients	Basic Variable	Y_1	Y_2	Y_3'	Y_3''	Y_4	Y_5	Y_6	Y_7	Y_9	RHS		
4	Y_1	1	-1/2	1	-1	-1/2	0	0	0	0	3/2		
0	Y_5	0	3/2	2	-2	½	1	0	0	0	½		
0	Y_6	0	3/2	-4	4	½	0	1	0	0	3/2	3/8	
M	Y_9	0	3/2	-4	4	1/2	0	0	-1	1	3/2	3/8	Pivot Row
	ΔW	0+0	-3-3M/2	2+4M	10-4M	2-M/2	0+0	0+0	0+M	0+0	-8/2-3M/2		
Determining the Pivot Column					Pivot Column								

Source : Author's own work

Since row ΔW contains negative values, table T_1 does not provide the optimal solution. By repeating the same steps, we obtain the next table:

Table. 5.5

T_2	Coefficients	4	-5	6	-6	-6	0	0	0		RHS/ Pivot column	Determining the Pivot Row
Coefficients	Basic Variable	Y_1	Y_2	Y_3'	Y_3''	Y_4	Y_5	Y_6	Y_7	RHS		
4	Y_1	1	-1/8	0	0	-3/8	0	0	-1/4	15/8		
0	Y_5	0	9/4	0	0	3/4	1	0	-1/2	5/4	5/9	Pivot Row
0	Y_6	0	0	0	0	0	0	1	1	0		
-6	Y_3''	0	3/8	-1	1	1/8	0	0	-1/4	3/8	1	
	ΔW	0	-9/4	0	0	9/4	0	0	-1/2	-21/4		
Determining the Pivot Column			Pivot Column									

Source : Author's own work

Since row ΔW contains negative values, table T_2 does not provide the optimal solution. By repeating the same steps, we obtain the next table:

Table. 5.6

T_3	Coefficients	4	-5	6	-6	-6	0	0	0			Determining the Pivot Row
Coefficients	Basic Variable	Y_1	Y_2	Y_3'	Y_3''	Y_4	Y_5	Y_6	Y_7	RHS	RHS/Pivot column	
4	Y_1	1	0	0	0	-1/3	1/18	0	-5/18	35/18		
-5	Y_2	0	1	0	0	3/9	4/9	0	-2/9	5/9		
0	Y_6	0	0	0	0	0	0	1	1	0	0	Pivot Row
-6	Y_3''	0	0	-1	1	0	-1/6	0	-1/6	1/6		
	Y_1	0	0	0	0	3	1	0	-1	1/6		
Determining the Pivot Column									Pivot Column			

Source : Author’s own work

Since row ΔW contains negative values, table T_3 does not provide the optimal solution. By repeating the same steps, we obtain the next table:

Table. 5.7

T_4	Coefficients	4	-5	6	-6	-6	0	0	0	
Coefficients	Basic Variable	Y_1	Y_2	Y_3'	Y_3''	Y_4	Y_5	Y_6	Y_7	RHS
4	Y_1	1	0	0	0	-1/3	1/18	5/18	0	35/18
-5	Y_2	0	1	0	0	1/3	4/9	2/9	0	5/9
0	Y_7	0	0	0	0	0	0	1	1	0
-6	Y_3''	0	0	-1	1	0	-1/6	0	-1/6	1/6
0	ΔW	0	0	0	0	3	1	1	0	-4

Source : Author’s own work

Since all the values in row ΔW are greater than or equal to zero, table T_4 provides the optimal solution, which is as follows:

$Y_1 = 35/18$

$Y_2 = 5/9$

$Y_3'' = 1/6$

$Y_7 = 0$

W = 4.

Conclusion

The principle of duality serves as a key theoretical framework for a deeper understanding of linear programming. Throughout this lecture, we explored the logical steps for transforming primal problems into dual form, addressed exceptional cases that arise in practical applications, and examined how optimal solutions are linked between the two formulations. Understanding these interconnections enriches a student's analytical toolkit and enhances their ability to generate adaptable and accurate solutions in real-world optimization scenarios.

Lecture 6: decision tree

Introduction

Decision-making is a fundamental pillar in both management and economics, as institutions are constantly faced with situations that require choosing the most appropriate alternative from a set of available options. In this context, decision analysis tools—such as the decision matrix and decision tree—emerge as systematic methods that support the decision-maker, especially in environments characterized by uncertainty or risk. This lecture aims to introduce students to the basic concepts of decision-making, explain both quantitative and qualitative decision-making techniques, and focus on practical applications involving criteria such as Expected Monetary Value (EMV), Expected Opportunity Loss (EOL), and the Expected Value of Perfect Information (EVPI). Real-world examples are included to reinforce understanding and enhance analytical and decision-making skills.

Learning Objectives

By the end of this lecture, students will be able to:

- Define the concept of decision-making and its types under conditions of certainty, uncertainty, and risk.
- Construct decision matrices and decision trees to represent possible paths and outcomes.
- Apply various quantitative decision-making criteria such as EMV, EOL, and EVPI to evaluate alternatives.
- Analyze real-world decision problems using decision analysis tools.
- Determine when perfect information is beneficial and assess the value of acquiring it.

1. Basic concept of decision

The decision matrix is a vital analytical tool used to support the decision-making process, especially in situations where multiple alternatives exist and various possible states of nature may occur. This matrix allows the researcher or decision-maker to systematically organize the available data in a way that clarifies the relationship between each possible alternative and each potential state, along with the resulting outcomes. It also facilitates the comparison of different options based on quantitative or probabilistic criteria, thereby enhancing the objectivity of the decision and reducing the influence of personal bias. This tool serves as a fundamental approach for analyzing decisions under conditions of certainty, risk, and uncertainty

1.1. Concept of decision

A decision is a selective process through which a comparison is made between a set of available alternatives, based on a prior analysis of the potential outcomes of each option, with the aim of identifying the one that best contributes to achieving the desired objectives. (Beladjoz, 2010, p. 99)

1.2. Types of decision (Decision-making situations)

Decisions can be classified according to various criteria, such as their nature, the judgment standards applied, the tools used, the organizational level, the environment and conditions under which the decision is made, among others. (Beladjoz, 2010, pp. 101-104) In our case, we will focus on the types of decisions based on the conditions under which they are made. These include: decision-making under conditions of certainty, decision-making under conditions of uncertainty, and decision-making under conditions of risk. The classification is as follows: (Beladjoz, 2010, pp. 110-115)

1.2.1. Decision-Making Environment Under Conditions of Certainty

The state of certainty refers to a situation in which the future is entirely clear and well-defined for the decision-maker, as all necessary information is fully available and accurate. In such cases, expected outcomes are known in advance, and no probabilities are considered for the occurrence of changes or surprises. Each available alternative leads to a single, predetermined result, since only one unchanging state of nature exists.

1.2.2. Decision-Making Environment Under Conditions of Risk

When the decision-maker operates in an environment marked by a certain degree of uncertainty but supported by sufficient historical quantitative data, this is referred to as a condition of risk. In this context, future events are uncertain, and it is not possible to know in advance which one will occur. However, it is possible to assign objective probabilities based on historical frequencies and statistical records. The availability of such data enables the decision-maker to use quantitative analytical tools to evaluate alternatives and select the most appropriate option.

1.2.3. Decision-Making Environment Under Conditions of Uncertainty

Conditions of uncertainty are more complex, as the decision-making environment is largely ambiguous and lacks precise probability estimates for future states of nature. Within this category, two levels can be distinguished:

- Complete Ignorance: This arises when the decision-maker has no information that would allow for the estimation of probabilities related to future events. As a result, it becomes impossible to build any probability model for those events.
- Partial Ignorance: In this situation, the decision-maker has access to some knowledge or experience that allows for the construction of probability estimates. However, these estimates are not based on empirical data but rather on subjective judgments, personal insights, or intuition. Such probabilities, often referred to as subjective probabilities, are common in environments where precise information is lacking.

From the above, we can conclude that both conditions of risk and conditions of uncertainty are characterized by the absence of complete and certain knowledge about the future. The key distinction between them lies in how probabilities are handled. Under risk, probabilities are clearly defined based on objective, data-driven sources, which supports well-founded, analytical decision-making. In contrast, under uncertainty, probabilities may be entirely absent or based on personal judgment, making outcomes harder to predict and decisions more reliant on intuition and individual perception.

1.3. Decision matrix

A decision matrix is a table that compiles all available information related to the problem at hand, aiming to support the process of making an appropriate decision. This matrix consists of a set of alternatives and possible states of nature—one or more of which may occur—along with the outcomes resulting from selecting each alternative under each state of nature. Additionally, it may include the probabilities associated with the occurrence of these states. The matrix is illustrated in the following table:

Table 6.1. Decision matrix

States of Nature / Alternatives	S_1	S_2	.	.	.	.	S_n
D_1	R_{11}	R12	.	.	.	.	R_{1m}
D_2	R_{21}	R22	.	.	.	.	R_{2m}
.	.	.	.	.	.	.	.

.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
D_n	R_{2n}	R_{2m}	.	.	.	.	R_{nm}
Probabilities	$P(S_1)$	$P(S_2)$	.	.	.	.	$P(S_m)$

Source: (Beladjoz, 2010, p. 181)

2. Basic concept of the decision tree

A decision tree is an analytical tool that helps represent various alternatives and the possible outcomes associated with a particular decision, especially under uncertainty. It visually and sequentially displays the available options, making it easier for the decision-maker to understand the different decision paths and their associated results, whether gains or losses. This tool is widely used in fields such as management, planning, and economics, and supports selecting the optimal alternative based on expectations and probabilities.

2.1. concept of the decision tree

A decision tree is a graphical and analytical tool used to represent various decision-making paths and the potential outcomes associated with each choice. It is particularly valuable when decisions must be made under uncertainty, as it helps to visualize alternatives, the probabilities of different states of nature, and the corresponding payoffs. Unlike traditional payoff tables, the decision tree offers a dynamic and sequential structure that illustrates how decisions unfold over time. (Nedjm, 2008, p. 96)

This structure resembles a branching tree, where each branch represents a possible course of action or event. Decision trees are especially useful when dealing with sequential decisions, where one decision leads to another, depending on the outcome of the preceding stage. Nevertheless, they are also applicable in single-decision scenarios, where the analysis involves evaluating different alternatives based on expected results. (Nedjm, 2008, p. 96)

2.2. Key components of a decision tree

A decision tree consists of several basic elements that form its analytical structure: (Nedjm, 2008, p. 96)

- Decision node: Represented by a square (□), it indicates a point where a decision-maker must choose between two or more available options or strategies.
- Chance node: Represented by a circle (○), this node reflects a situation involving uncertainty, where the outcome depends on external conditions or probabilities that cannot be precisely predicted.
- Decision branch: This is a line extending from a decision node, representing a specific choice or action available to the decision-maker. Each branch leads to either another decision node or a chance node.
- Chance branch: A line that emerges from a chance node, indicating one of the possible outcomes that could occur based on the given probabilities of different events.
- Terminal branch (or end node): These are the endpoints of the tree, where no further decisions or outcomes follow. Each terminal branch reflects the final result of a specific decision path—typically quantified as a payoff, cost, or benefit.

2.3.The Basic Steps for Constructing a Decision Tree

A decision tree is one of the most effective visual tools for analyzing complex decisions, especially under uncertainty and when multiple alternatives are available. Its construction relies on a set of structured steps that help clarify possible paths and the expected outcomes for each option. Below is a brief overview of the basic steps for constructing a decision tree: (Taama, 2008, p. 274)

- Identify the decision point: Determine where a choice between alternatives must be made.
- List available alternatives: Draw branches for all possible options from the decision node.
- Define states of nature: Connect each alternative to potential uncertain outcomes.
- Assign probabilities: Allocate a probability to each state based on available data or estimates.
- Estimate outcomes: Attach expected payoffs or costs to each outcome.
- Perform backward analysis: Calculate the expected value for each alternative by working backwards.
- Select the best option: Choose the alternative with the most favorable expected result.

3. Decision tree under certainty and uncertainty

3.1. Decision tree under certainty

This case can be illustrated through the following example

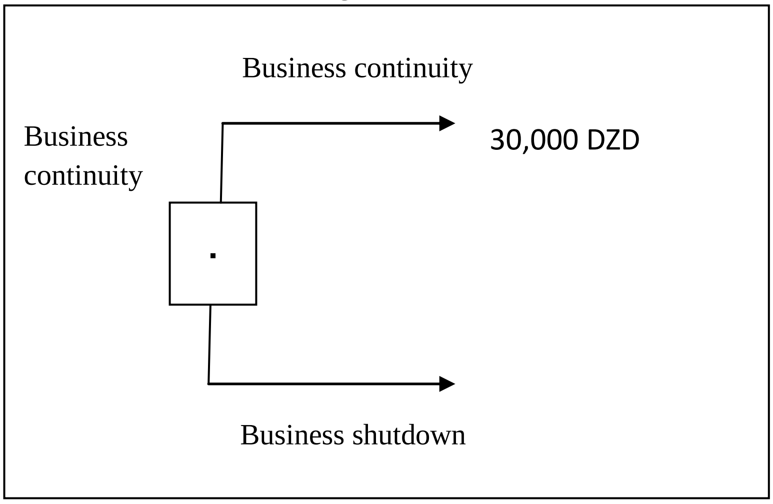
Exmple1:

A restaurant manager knows that the cost of preparing a specific meal is 500 DZD, and it is sold daily at a fixed price of 800 DZD. The daily demand is stable, and the restaurant consistently sells 100 meals per day.

Daily revenue = 800 × 100 = 80,000 DZD, Daily cost = 500 × 100 = 50,000 DZD, Confirmed daily profit = 80,000 – 50,000 = 30,000 DZD

Since the price, quantity, and cost are all known and constant, the decision to continue offering this meal is made under a case of certainty.

Fig 6.1



Source: Author’s own work

3.2. Decision-Making Criteria Under Uncertainty

When accurate information about the probabilities of future states of nature is not available, the decision-maker operates under uncertainty. In such situations, quantitative probabilistic analysis becomes inapplicable, and decision-makers resort to behavioral decision-making criteria, which help them select the best alternative based on their judgment, attitude toward risk, or subjective assessments.

3.2.1. Maximin Criterion

This criterion assumes a cautious attitude, where the decision-maker expects the worst possible scenario. It follows these steps:

- Identify the minimum payoff for each alternative.
- Compare the worst outcomes across all alternatives.
- Select the alternative with the highest minimum payoff.

This criterion aims to protect the decision-maker from substantial losses under the most adverse conditions.

Example2:

Table 6.2

Alternative	Worst Outcome
A	30
B	20
C	40

Source : Author’s own work

From The table, The selected strategy is C, as it offers the highest minimum value.

3.2.2.. Maximax Criterion

This criterion assumes an optimistic stance, where the decision-maker anticipates the best possible outcome. It proceeds as follows:

- Identify the maximum payoff for each alternative.
- Compare these maximum values.
- Select the alternative with the highest possible return.

This approach aims to maximize potential gains if the best-case scenario occurs.

Example3:

Table 6.3

Alternative	Best Outcome
A	70
B	90
C	80

Source : Author’s own work

From The table, The best choice is B, which yields the highest return (90).

3.2.3. Laplace Criterion

Based on the assumption that all states of nature are equally likely, this criterion is considered neutral. It is implemented through the following steps:

- Sum the payoffs of each alternative.
- Calculate the average (mean) payoff.
- Choose the alternative with the highest average.

This method is used when there is no reason to prefer one outcome over another.

Example4:

Table 6.4

Alternative	Payoffs	Average
A	40, 60, 80	60
B	30, 90, 60	60
C	50, 50, 50	50

Source : Author’s own work

From The table, A and B are equally preferred.

3.2.4. Minimax Regret Criterion

This approach focuses on minimizing the feeling of regret that could result from not choosing the best alternative. It proceeds through the following steps:

- Construct a regret matrix by calculating the difference between each alternative's outcome and the best possible outcome for each state.
- Identify the maximum regret for each alternative.
- Choose the alternative with the lowest maximum regret.

This criterion seeks to limit the psychological cost of making a wrong decision.

Example5:

Table 6.5

State of Nature	A	B	C	Best Outcome
S1	60	40	50	60
S2	30	90	70	90

Source : Author’s own work

From Table 6.5, we can create the regret matrix as follows:

Table 6.6. Regret Matrix:

State	A	B	C
S1	0	20	10
S2	60	0	20

Source : Author’s own work

From regret mateix, Maximum Regret is:

- A = 60
- B = 20
- C = 20

As a result, Strategy B or C would be chosen.

We can summarize a comparison between uncertainty criterion in the following table:

Table 6.7. Summary Comparison of Criteria:

Criterion	Decision Style	Key Steps	Goal
Maximin	Pessimistic	Select the highest among the lowest	Reduce risk of major loss
Maximax	Optimistic	Select the highest possible return	Maximize potential gain
Laplace	Neutral	Choose based on highest average	Balanced decision without probability data
Minimax Regret	Cautious	Minimize the worst regret	Avoid future disappointment

Source : Author’s own work

Example 6: Practical Example Using a Decision Tree

A company is considering three marketing strategies:

- S1, S2, S3

Market conditions (states of nature) could be an active market, a moderate market, or a weak market.

Table 6.8. Payoff Table (in thousand DZD):

Strategy	Active	Moderate	Weak
S_1	80	60	20
S_2	100	30	10
S_3	40	40	40

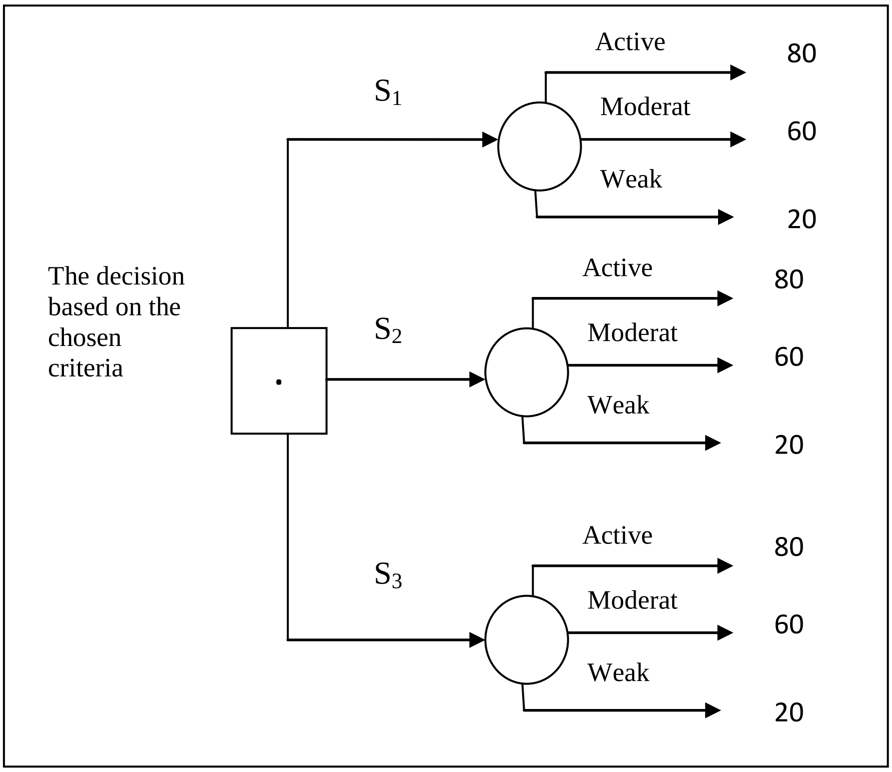
Source : Author’s own work

Required: analysis under Uncertainty with a decision tree diagram

The solution

1.Drawing the Decision Tree

Fig 6.2



Source : Author’s own work

2.The decision Under Uncertainty (No probabilities available):

- Maximin:
  - S1 → 20
  - S2 → 10
  - S3 → 40 → ❖ Select S3
- Maximax:
  - S1 → 80
  - S2 → 100 → ❖ Select S2
  - S3 → 40

- Laplace:
  - S1: (80+60+20)/3 = 53.33
  - S2: (100+30+10)/3 = 46.67
  - S3: (40+40+40)/3 = 40 → ❖ Select S1
- Minimax Regret:

Table 6.9

State	S1	S2	S3
Active	20	0	60
Moderate	0	30	20
Weak	20	30	0

Source : Author's own work

→ Max regrets:

- S1 = 20
- S2 = 30
- S3 = 60 → Select S1

4. Decision tree under risk

In risk-based decision situations, the decision-maker possesses sufficient information regarding the possible outcomes of each alternative, as well as the associated probabilities of the states of nature. This availability of data allows for the application of structured, quantitative methods in evaluating and selecting the most appropriate course of action.

Among the most widely used criteria under conditions of risk are:

- Expected Monetary Value (EMV)
- Expected Opportunity Loss (EOL)
- Value of Perfect Information (VPI)

For each of these criteria, the decision tree serves as a central analytical tool. It visually represents the alternatives, the states of nature, their probabilities, and the resulting payoffs or losses. The decision tree facilitates a clear, step-by-step evaluation of outcomes, helping the decision-maker apply the chosen criterion effectively.

The following sections provide a detailed explanation of each criterion along with how it is implemented using a decision tree.

4.1. Decision Tree under Expected Monetary Value (EMV)

In a decision-making environment characterized by risk, where probabilities of different states of nature are known, the Expected Monetary Value (EMV) criterion is used to evaluate the available alternatives. This approach involves estimating the expected outcome of each alternative by multiplying the payoff associated with each state of nature by the probability of its occurrence, and then summing these values to obtain the EMV. This method is particularly useful when represented through a decision tree, as it allows for a clear comparison between alternatives based on both their potential outcomes and the likelihood of their realization. (Beladjoz, 2010, p. 180)

$$\text{EMV}(A_i) = \sum_{j=1}^{n} (R_{ij} \times P(S_j))$$

Where:

- $EMV(A_i)$Expected monetary value of alternative i.
- R_{ij}: Payoff of alternative i under state of nature j.
  $P(S_j)$Probability of occurrence of state of nature j.

- n: Total number of possible states of nature. (Beladjoz, 2010, p. 181)

4.2. Decision Tree under Expected Opportunity Loss (EOL)

Opportunity loss represents the amount of loss resulting from not choosing the best alternative among the available options under each state of nature (Beladjoz, 2010, p. 185). It is calculated as follows:

Opportunity Loss (OL) = Maximum payoff under the state of nature − All payoffs under the same state

By applying this rule, we obtain the opportunity loss matrix. It should be noted that in cost-based decisions, the calculation is reversed: the minimum cost is identified, and all other costs under the same state are subtracted from it, taking the absolute values of the results.

Afterward, the expected opportunity loss (EOL) is calculated using the following formula:

$$EOL(\mathrm{A}_i) = \sum_{j=1}^{n} (\mathrm{OL}_{\mathrm{ij}} \times P(S_j))$$

Finally, the alternative with the lowest expected opportunity loss is selected.

4.3. Decision Tree under Conditions of Perfect Information

Expected Value of Perfect Information (EVPI) represents the additional value that a decision-maker could gain if they had complete information about the future before making a decision. It is calculated as the difference between the expected payoff with perfect information (EVwPI) and the maximum expected monetary value (EMV) obtained without such information. (Beladjoz, 2010, p. 182)

The process involves the following steps: (Beladjoz, 2010, p. 183)

- Calculate the Expected Value with Perfect Information (EMV/PI): For each state of nature, select the best possible payoff across all alternatives, and multiply it by the probability of that state. Then sum the results:

$$\text{EMV/PI} = \sum_{j=1}^{m} (\max_j(R_{ij}) \times P(S_i))$$

2. Calculate the Expected Monetary Value (EMV): Identify the alternative that yields the highest EMV without perfect information:

$$\text{EMV(d*)} = \max_i \left(\sum_{i=1}^{n} R_{ij} \times P(S_i) \right)$$

3. Compute the Expected Value of Perfect Information (EVPI): Subtract EMV from EVwPI:

EMVPI= EMV/PI-EMV(d*)

This value indicates the maximum amount a decision-maker should be willing to pay to obtain perfect information before making a choice.

4.4. Example

A petroleum company wishes to make a decision on whether to drill a well or not, as it bears a drilling cost of 2 million monetary units, with an extraction cost of 10 monetary units per extracted barrel. The selling price of one barrel is 20 monetary units. It is possible that the well is dry, meaning it contains no oil, with a probability of 0.1, or it contains 2 million barrels with a probability of 0.35, or it contains 4 million barrels with a probability of 0.25, or it contains 6 million barrels with a probability of 0.3.

In front of this company, the following options are available: drilling the well, or leasing it to another company in return for 2 million monetary units per month for a period of 6 months, or selling it to another company in return for 30 million monetary units.
Required:

- Construct the decision matrix for this company.
- Find the optimal decision using the Expected Monetary Value (EMV) criterion with drawing the decision tree in this case.
- Find the optimal alternative using the Expected Opportunity Loss (EOL) criterion with drawing the decision tree in this case.
- If there is a petroleum exploration consulting office offering to obtain complete information in return for a cost of 2 million monetary units, determine whether the company should accept this offer using the criterion of the value of perfect information, with drawing the decision tree in this case.

The solution:
1. Building the decision matrix

Table 6.10 (Million monetary units)

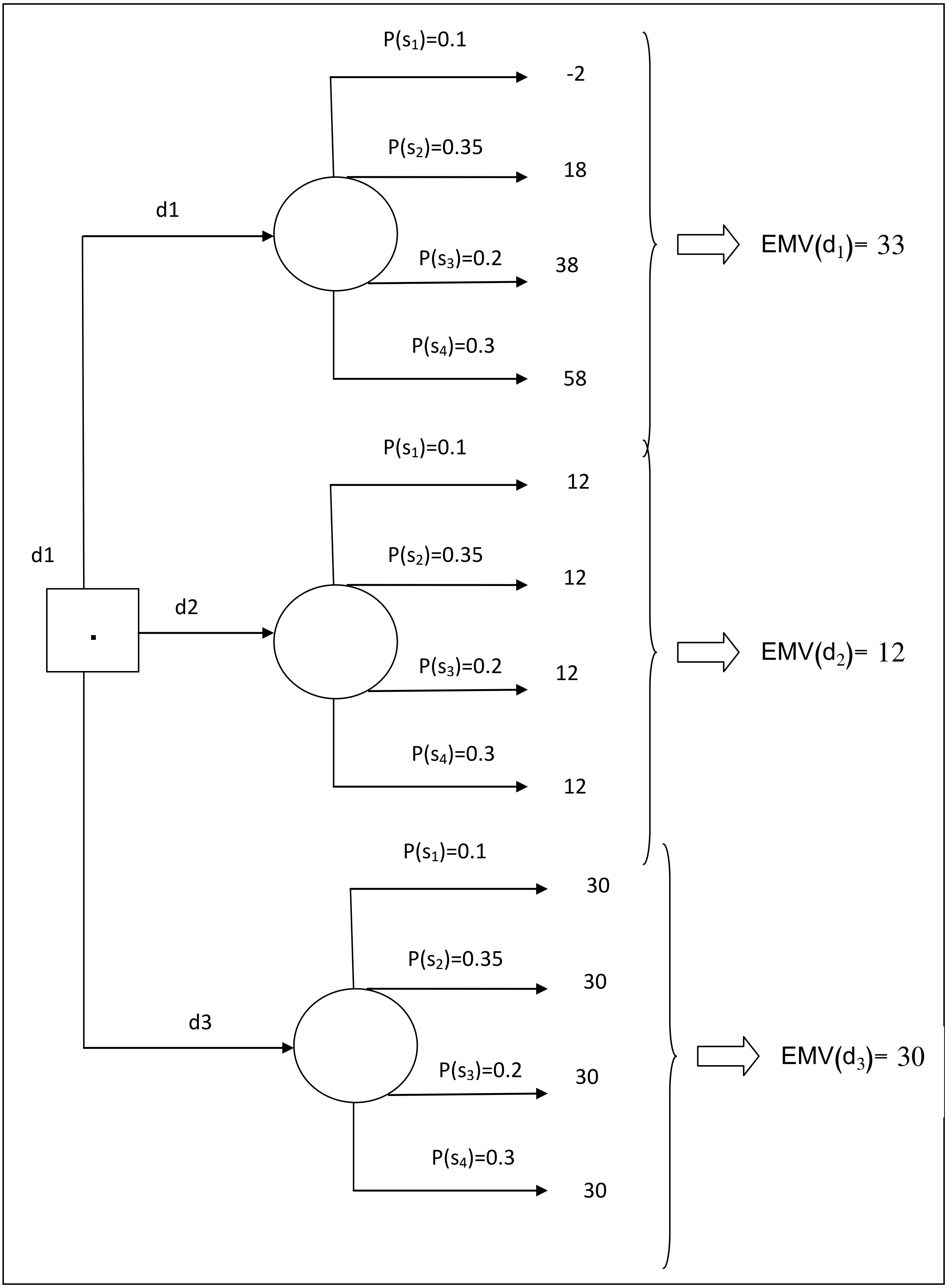
States of Nature (sj) / Strategies (di)	The well is dry (S_1)	The well contains 2 million barrels (S_2)	The well contains 4 million barrels (S_3)	The well contains 6 million barrels (S_4)
Drill the well (d_1)	-2	18	38	58
Rent the well (d_2)	12	12	12	12
Sell the well (d_3)	30	30	30	30
Probabilities $P(S_j)$	0.1	0.35	0.25	0.3

Source : Author's own work

2. Determining the optimal decision using the Expected Monetary Value (EMV) criterion with drawing the decision tree in this case
2.1. Determining the optimal alternative
EMV(d1) = (-2 * 0.1) + (18 * 0.35) + (38 * 0.25) + (58 * 0.3) = 33
EMV(d2) = (12 * 0.1) + (12 * 0.35) + (12 * 0.25) + (12 * 0.3) = 12
EMV(d3) = (30 * 0.1) + (30 * 0.35) + (30 * 0.25) + (30 * 0.3) = 30
Since EMV(d1) > EMV(d3) > EMV(d2) Hence, the optimal alternative is d1, i.e., the optimal decision is to drill the well.

2.2.Drawing the Decision Tree

Fig 6.3



Source : Author’s own work

3.1. Determining the optimal alternative

First, the Expected Opportunity Loss matrix is constructed:

Table. 6.11

States of Nature (sj) / Strategies (di)	The well is dry (s1)	The well contains 2 million barrels (s2)	The well contains 4 million barrels (s3)	The well contains 6 million barrels (s4)
Drill the well (d1)	32	12	0	0
Rent the well (d2)	18	18	26	46
Sell the well (d3)	0	0	8	28
Probabilities	0.1	0.35	0.25	0.3

Source : Author's own work

Thus, the Expected Opportunity Loss (EOL) can be calculated as follows

EOL(d1) = (32 * 0.1) + (12 * 0.35) + (0 * 0.25) + (0 * 0.3) = 7.4

EOL(d2) = (18 * 0.1) + (18 * 0.35) + (26 * 0.25) + (46 * 0.3) = 28.4

EOL(d3) = (0 * 0.1) + (0 * 0.35) + (8 * 0.25) + (28 * 0.3) = 10.4

Since EOL(d1) < EOL(d3) < EOL(d2), the optimal alternative is d1, which corresponds to the first decision: drilling the well

3.2. Drawing the decision tree (see Fig. 6.4)

4. Determining whether the company accepts the offer with drawing the decision tree

4.1. Calculating the value of perfect information

First, the expected value of perfect information is calculated as follows:

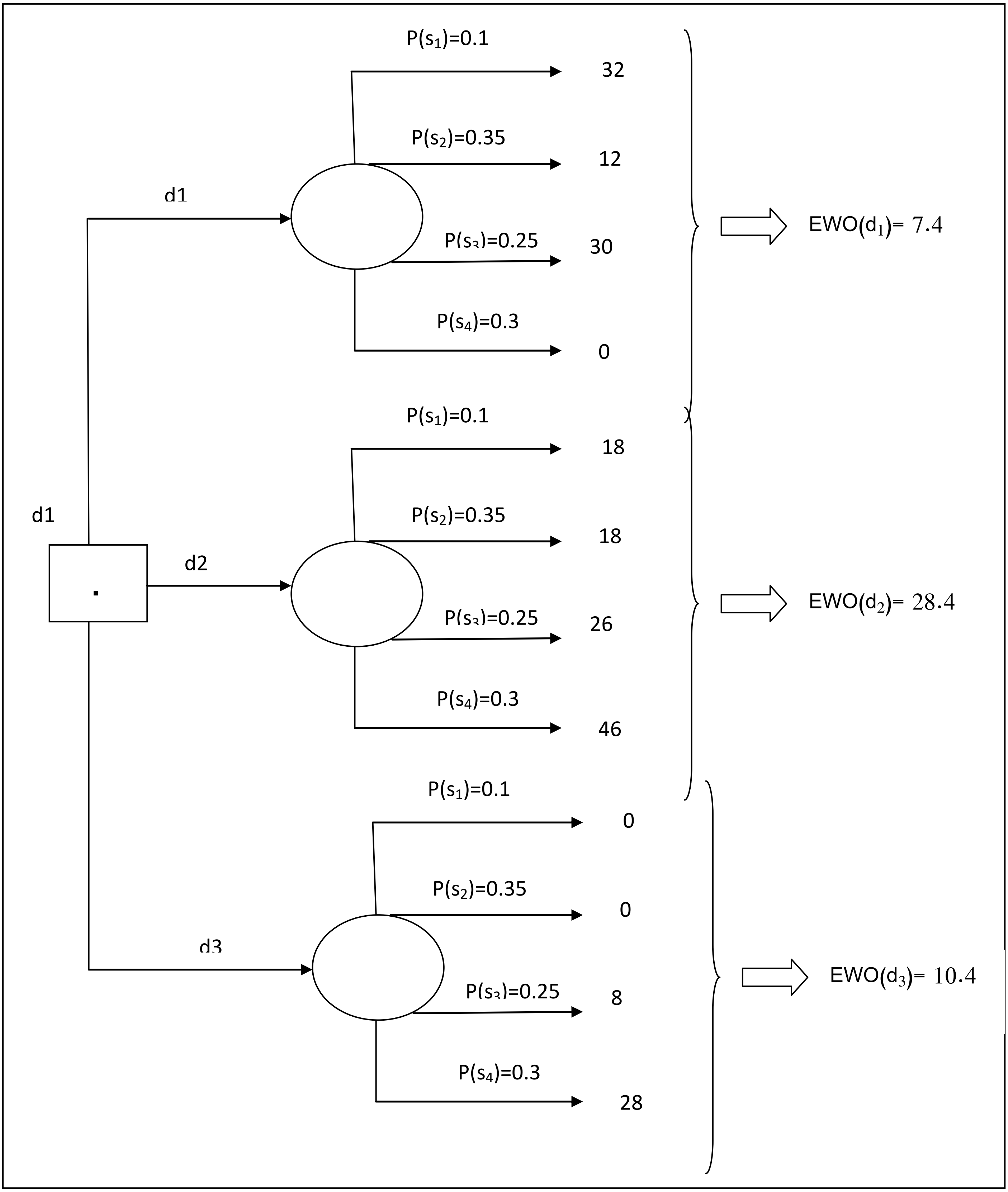
EVPI = (30 * 0.1) + (30 * 0.35) + (38 * 0.25) + (58 * 0.3) = 40.4

Hence, the value of perfect information is PIV = EVPI - EMV(d1) = 40.4 - 33 = 7.4

Since the value of perfect information is 7.4 million monetary units, which is greater than the cost of obtaining perfect information (2 million monetary units), the company accepts the offer of the petroleum exploration consulting office to obtain perfect information.

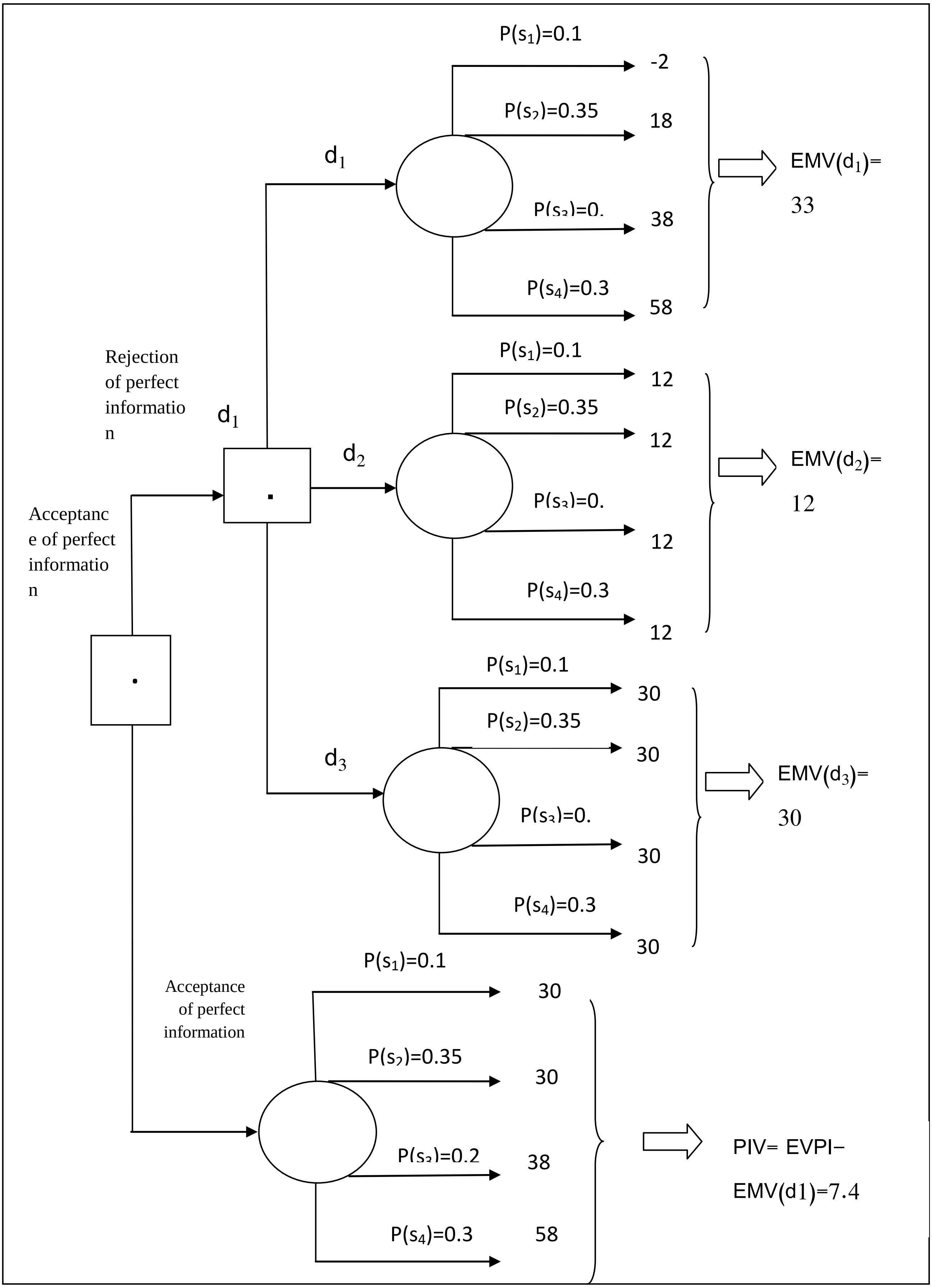
4.2. Drawing the decision tree (see Fig. 6.5)

Fig. 6.4



Source : Author's own work

Fig. 6.5



Source: Author’s own work

Conclusion

This lecture addressed both the theoretical and practical aspects of decision-making, emphasizing the quantitative tools that assist in evaluating available alternatives under different conditions. By constructing decision matrices and decision trees, decision-makers can approach complexity and uncertainty in a structured and informed way. The case examples highlighted the importance of understanding the context in which decisions are made and selecting the appropriate tool and criterion. Mastering these skills enhances the quality and effectiveness of decisions while minimizing potential risks in the business environment.

Lecture 7: Bayes' Theorem and Decision Tree

Introduction

Decision-making under uncertainty is one of the most critical challenges facing managers and analysts. Traditional models rely heavily on prior information, which may become outdated or insufficient when new evidence emerges. In this lecture, we explore Bayes' Theorem—a powerful statistical tool that allows for the dynamic updating of probabilities based on new information. We also introduce the decision tree, a structured and visual representation of decision-making paths, integrating probabilistic reasoning with monetary outcomes. Together, these tools provide a comprehensive framework for rational decision-making in uncertain environments.

Learning Objectives

By the end of this lecture, students will be able to:

- Define Bayes' Theorem and explain its role in updating prior beliefs using new information.
- Distinguish between prior, conditional, and posterior probabilities.
- Apply Bayes' Theorem to practical decision-making problems involving uncertainty.
- Calculate the Expected Monetary Value (EMV) using both prior and posterior probabilities.
- Construct a decision tree diagram that incorporates the results of Bayesian analysis.
- Evaluate alternative strategies using decision criteria under imperfect information.

1. Definition of Bayes' Theorem

Bayes' Theorem is a fundamental statistical tool used to support decision-making in environments characterized by uncertainty. At its core, the theorem is based on a straightforward yet powerful concept: updating probabilities in light of new information. Initially, decision-makers rely on prior probabilities, which are estimates grounded in previous knowledge or historical data. However, these estimates may not accurately reflect current conditions, especially when additional insights become available—such as expert evaluations or study findings—often represented through conditional probabilities, which indicate the likelihood of an outcome given a specific state of nature. By integrating this new evidence with prior beliefs, Bayes' Theorem enables the derivation of posterior probabilities, offering more precise and credible assessments of a situation. As such, the theorem serves not only as a method for statistical inference, but also as a structured framework that guides rational decision-making in complex, uncertain contexts. (Taama, 2008, pp. 253-254)

2. The Mathematical Form of Bayes' Theorem

The general form of Bayes' Theorem is expressed as follows: (GeeksforGeeks, 2025)

$$P(E_i \mid A) = \frac{P(E_i) \cdot P(A \mid E_i)}{\sum_{k=1}^{n} P(E_k) \cdot P(A \mid E_k)}$$

Where:

- $P(E_i \mid A)$ is the posterior probability of event E_i given that event A has occurred.
- $P(E_i)$ is the prior probability of event E_i.
- $P(A \mid E_i)$ is the conditional probability of observing event A given that E_i is true.

- $\sum_{k=1}^{n} P(E_k) \cdot P(A \mid E_k)$ represents the total probability of event A across all mutually exclusive events E_1 to E_n.

3. Example: (adapted from (Taama, 2008, pp. 254-257))

A commercial company specializing in the marketing of various products is considering two strategic alternatives under three possible natural states. The decision environment and estimated monetary payoffs (in thousand dinars) are as shown in the following table:

Payoff Table:

Table. 7.1

States of Nature (Pj)	$P_1 = 0.3$	$P_2 = 0.5$	$P_3 = 0.2$
Alternatives (Si)			
S_1	60	90	50
S_2	70	30	100

Source : Author's own work

The company has commissioned a consulting expert to conduct a study to provide additional information that may aid in the decision-making process. The results of the consultant's study provided conditional probabilities for two possible outcomes (favorable and unfavorable), based on each state of nature, as follows:

Consultant Study Results (Conditional Probabilities):

Table. 7.2

Study Outcome	P_1	P_2	P_3
Favorable	0.7	0.2	0.4
Unfavorable	0.3	0.8	0.6

Source : Author's own work

Required:
1. Determine the optimal investment alternative using the Expected Monetary Value (EMV) criterion based on prior probabilities.
2. Determine the best investment alternative using posterior probabilities in light of the consultant's study results (i.e., if the result is favorable or unfavorable).

The solution:

1.Determining the Best Alternative Using Expected Monetary Value (EMV) with Prior Probabilities

EMV Calculations:

For Alternative S_1: $EMV(S_1) = 60(0.3) + 90(0.5) + 50(0.2) = 18 + 45 + 10 = 73$

For Alternative S_2: $EMV(S_2) = 70(0.3) + 30(0.5) + 100(0.2) = 21 + 15 + 20 = 56$

Decision: Alternative S_1 is preferred, as it yields the highest EMV.

2. Using Posterior Probabilities Based on Study Results

A. Posterior Probabilities in the Case of Favorable Results

Joint Probabilities (prior × conditional):

Table. 7.3

State	P_1	P_2	P_3
Prior	0.3	0.5	0.2
Conditional (Favorable)	0.7	0.2	0.4
Joint	0.21	0.10	0.08

Source : Author's own work

Sum of joint probabilities = 0.39

Posterior Probabilities:

Table. 7.4

State	Posterior = Joint / 0.39
P_1	0.21 / 0.39 ≈ 0.54
P_2	0.10 / 0.39 ≈ 0.26
P_3	0.08 / 0.39 ≈ 0.20

Source : Author's own work

Recalculate EMV with Posterior Probabilities:

For S_1:

EMV(S_1 / Favorable Case) = 60(0.54) + 90(0.26) + 50(0.20) = 32.4 + 23.4 + 10 = 65.8

For S_2:
EMV(S_2 / Favorable Case) = 70(0.54) + 30(0.26) + 100(0.20) = 37.8 + 7.8 + 20 = 65.6

Decision: Choose S_1, as it still provides the highest EMV.

B. Posterior Probabilities in the Case of Unfavorable Results

Joint Probabilities:

Table. 7.5

State	P_1	P_2	P_3
Prior	0.3	0.5	0.2
Conditional (Unfavorable)	0.3	0.8	0.6
Joint	0.09	0.40	0.12

Source : Author's own work

Sum of joint probabilities = 0.61

Posterior Probabilities:

Table. 7.6

State	Posterior = Joint / 0.61
P_1	0.09 / 0.61 ≈ 0.15
P_2	0.40 / 0.61 ≈ 0.66
P_3	0.12 / 0.61 ≈ 0.19

Source : Author's own work

EMV with Posterior Probabilities (Unfavorable Case):

For S_1:

EMV(S_1 / Unfavorable Case) = 60(0.15) + 90(0.66) + 50(0.19) = 9 + 59.4 + 9.5 = 77.9

For S_2:
EMV(S_2 / Unfavorable Case) = 70(0.15) + 30(0.66) + 100(0.19) = 10.5 + 19.8 + 19 = 49.3

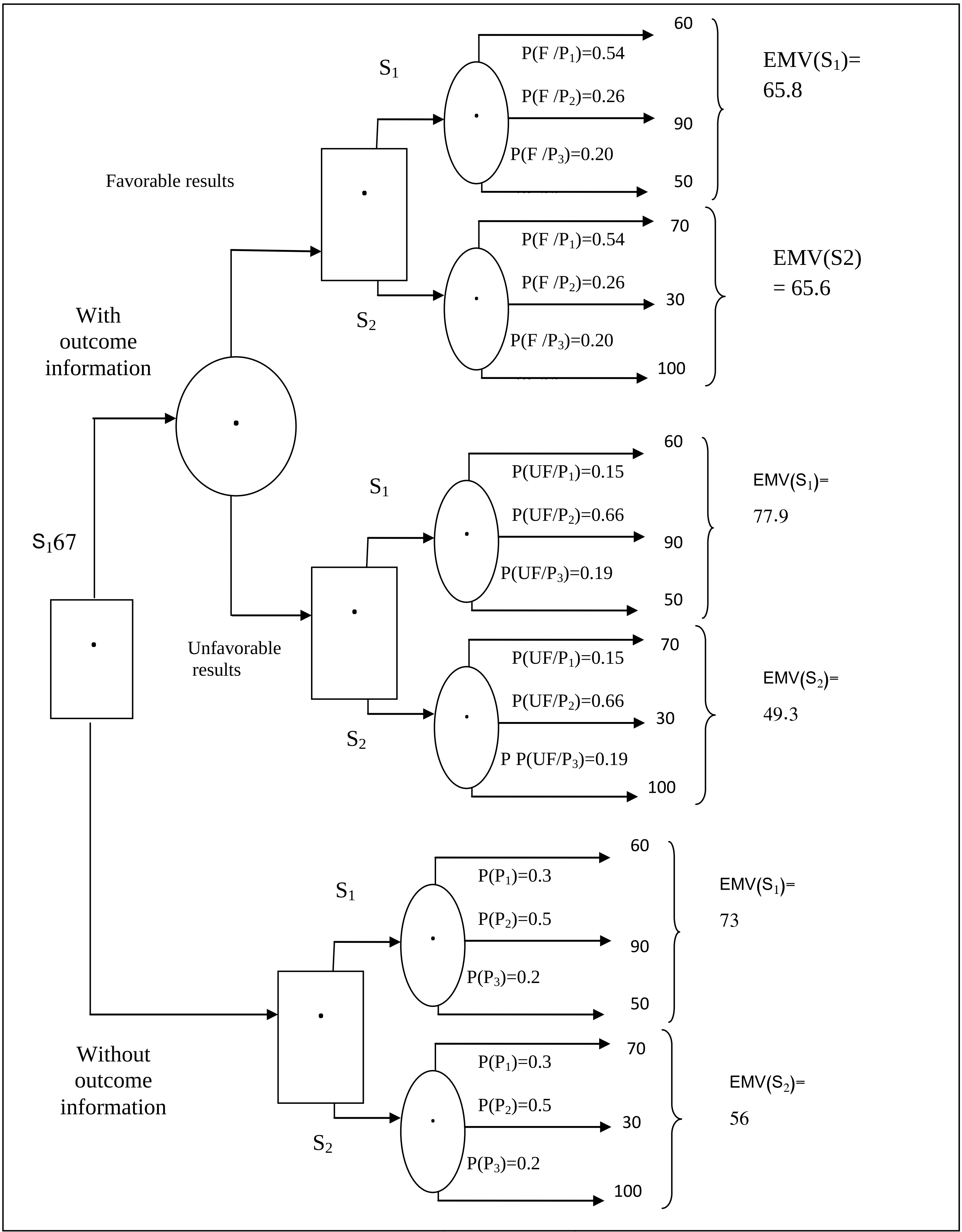
Decision: Again, alternative S_1 is superior under unfavorable conditions.

Thus, S_1 is the optimal investment decision.

4. Decision Tree Diagram

Based on the previous example, the decision tree can be constructed using Bayes' Theorem as follows:

Fig. 7.1



Source: Author's own work

Conclusion

Bayes' Theorem offers a structured approach to incorporating new evidence into existing beliefs, transforming the quality and precision of decision-making in uncertain contexts. When combined with decision trees, this methodology not only enhances the analytical clarity of alternatives but also supports more informed and rational strategic choices. As uncertainty becomes an inherent part of modern management and operations, mastering these techniques is essential for students, professionals, and researchers alike.

Lecture 8: Assignment problems

Introduction

Assignment models are among the key quantitative models within the framework of linear programming. They are used to solve problems related to the allocation of limited resources—such as individuals, machines, or projects—to specific tasks or jobs, where each resource is assigned to exactly one task, and vice versa, with an equal number of resources and tasks. These models are applied in various fields, including production planning, work scheduling, project distribution, and transportation allocation, as well as in other applications that require precise and efficient decision-making. The importance of assignment models lies in their ability to achieve one of two primary economic objectives: either minimizing costs to the lowest possible level or maximizing profits to the highest possible level, all while considering the specific constraints and conditions of the assignment process.

Assignment problems can be solved using three main analytical methods: the probabilistic enumeration method, the Hungarian method, and the transportation method. Each of these methods provides a distinct approach to solving the problem—whether by exhaustively listing all possible permutations, progressively modifying a cost matrix to reach the optimal solution, or by simulating the classical transportation model in a simplified form. Mastery of these methods is essential for understanding how to distribute resources in a mathematical and systematic manner, thereby enhancing decision-making efficiency in complex production and administrative environments.

Objectives of the Lecture

- To define assignment models and highlight their role within linear programming and operations research.
- To clarify the different assignment cases, whether focused on cost minimization or profit maximization.
- To present the mathematical formulation of the assignment problem and its structural constraints.
- To explain and apply three major methods for solving assignment problems:
  - The Probabilistic Enumeration Method
  - The Hungarian Method
  - The Transportation Method
- To analyze real-world examples illustrating how to make optimal decisions based on the outcome of each method.

1. Basic concepts

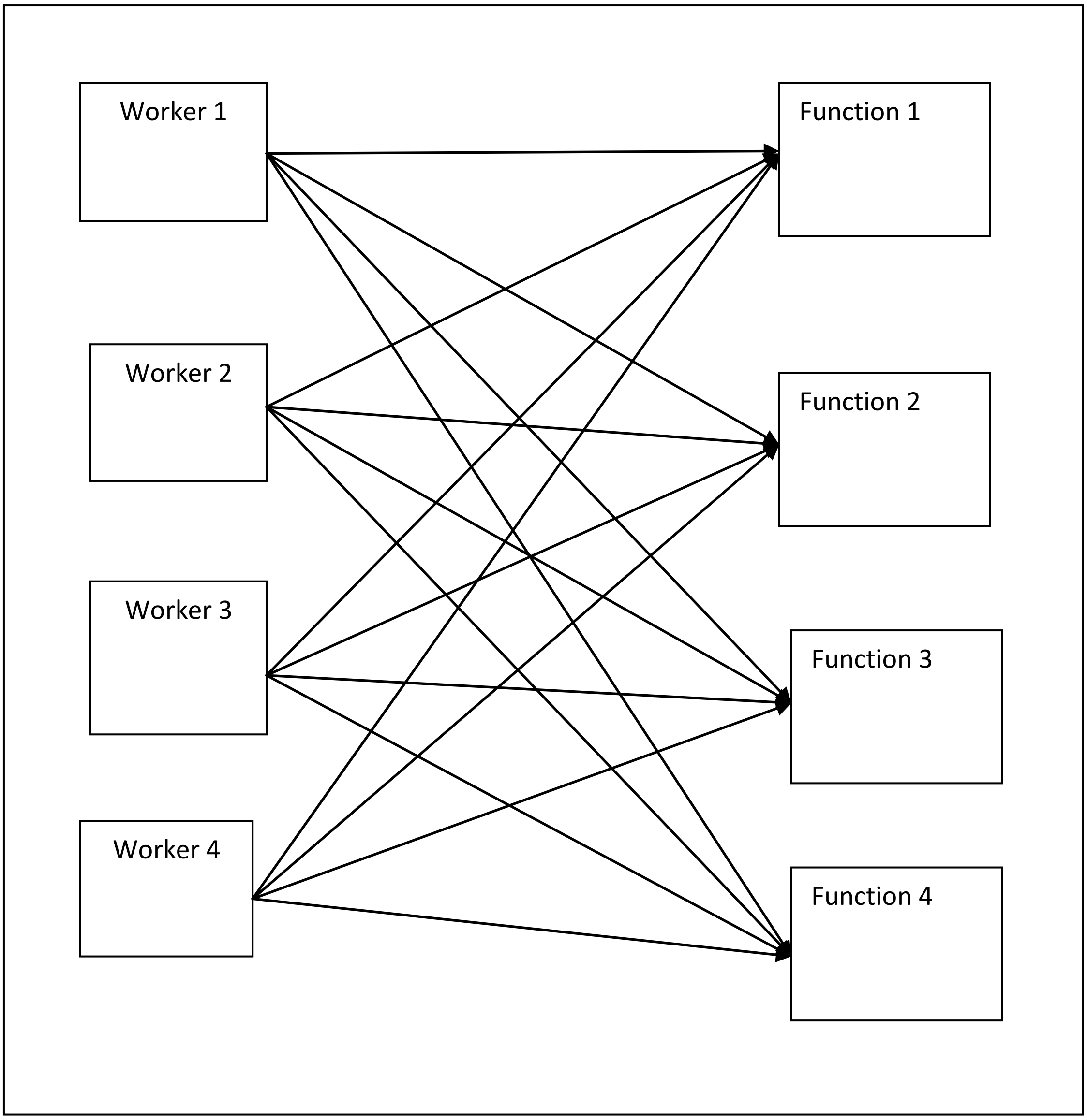
In this part of the lecture, we will address the practical applications of assignment models, followed by a presentation of the general mathematical formulation of this type of model, along with the associated constraints and conditions. We will also highlight three main analytical methods used to solve such models: the probabilistic enumeration method, the Hungarian method, and the transportation method.

1.1. Overview

The issues and questions addressed by linear programming do not only stop at determining the optimal combination of the number of units produced to achieve the greatest profit or to achieve the lowest cost or to determine the optimal transport combination as in transport issues but also to determine the optimal allocation of people, machines and tasks, this is within one of the branches of linear programming, namely assignment issues, so that one worker is assigned to one job or one machine is assigned to one task or one worker is employed to work on one machine.

Assignment issues can be illustrated by the following diagram where we have four workers and four jobs and each worker can perform one of these jobs (see Fig. 8. 1)

Fig. 8. 1



Source: Author's own work

1.2. Uses of the assignment model

We can use the assignment model for solving many problems as follows: (Ratoul, 2006, pp. 155-156)

- Assigning the employees to their tasks, such as the distribution of the workers to their functions in the production workshop.
- Assigning the employees to some machines.

- Assigning a set of machines to a set of tasks.
- We can also use the assignment model in many reality problems such as assigning a set of buses to a set of neighbourhoods or assigning a group of entrepreneurs to a group of projects.

All the previous assignments target to reduce the total cost to the lowest value when the problem is related to the cost (the minimum cost). But when the problem is related to the profit all the previous assignments target to increase the total profit to the highest value (the maximum profit or return).

The basic rule of the assignment model is one employee to one machine, one employee to one function, one machine to one task, etc. In addition, the number of employees, machines, functions, and tasks is the same. That is if we allocate some employees to some jobs, the number of employees must be equal to the number of jobs.

1.3. The mathematic description

Let us assume we have N machines and N tasks with costs of assignment as shown in Table 1 as follows

Table. 8.1

tasks / machines	Task 1	Task 2	....	Task N	Availability
Machine 1	C_{11}	C_{12}	....	C_{1N}	1
Machine 2	C_{21}	C_{22}	....	C_{2N}	1
....	....	....	....	....	....
Machine N	C_{N1}	C_{N2}	....	C_{NN}	1
Requirement	1	1	....	1	N

Source: Prepared based on: (Murthy, 2007, p. 213)

We can write the Mathematical Model as follows: (Murthy, 2007, pp. 213-214)

Minimize Z = $\sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij}\ X_{ij}$ → objective constraint.

Subject to: $X_{ij} = (X_{ij})^2$ i and j = 1 to n

$\sum_{j=1}^{n} X_{ij} = 1\ (d_j)$ and $\sum_{i=1}^{n} X_{ij} = 1\ (b_i)$ → Structural constraints.

For i and j = 1 to n

(Each device is assigned to only one task and each task is assigned to only one device)

$X_{ij} \geq 0$ for all values of j and i → Non-negativity constraint.

1.4. Approach to Solution

Three methods can be used for solving the assignment model, the methods are the Full counting method (different combinations), the Hungarian method and the transport method.

2. Solving the assignment problem by the Hungarian Method

2.1. case of costs:

When we use the Hungarian Method we apply the following steps:

Step1:

We choose the lowest value in each row and subtract it from all values in the same row, thus we find a new table.

Step2:

We choose the lowest value in each column in the table that is found in the previous step and subtract it from all values in the same column.

Step 3:

Based on the table obtained in the previous step we put a frame around the row or column that contains the largest number of zeros. We continue until all the zeros are inside a frame.

Step4:

We check the optimal solution. If the number of frames equals the number of rows or columns the table gives us the optimal solution and we can achieve the assignment process as is explained in Step 8, unless It does not give us the optimal solution and we must continue in other stages to achieve the optimal solution.

Step5:

If the last table which was reached in Step 4 does not give the optimal solution we continue the stages of the solution by choosing the lowest value among the values outside the frames and then subtracting it from them and adding it to the numbers of the intersection points of the frames.

Step6:

We repeat the step number 3. We get a new table.

Step7:

We repeat the step number 4, that is checking the optimal solution.

Step8**:**

If the number of frames equals the number of rows or columns the table gives us the optimal solution and we can achieve the assignment process. we can assign each Machine to one Task whereby selecting the zero that is located in the intersection points of the row of the Machine and the column of the task, then we cross out all zeros that fall along the same row or column. Paying attention to achieving the assigning process with each Machine for a task, that is do not cross out any zero which leads to a Machine for no task, and hence the focus must be paid when making the allocation.

When we assign any zero we put a circle on it.

We can assign by putting a circle and crossing out zeros which did not assign

If not we come back to Step 4 and continue with it and all the steps which come after it.

2.2. case of profits or returns :

In this case, we use all previous steps as in the case of costs, but we do an important process before them. This process involves choosing the highest value in the matrix (table of profits or returns) and subtracting all values from it. we reach a new matrix (new table) on which we apply the previous eight steps.

When we reach the table that gives the optimal solution we come back to the first table which contains profits or returns and we assign through it.

2.2.1. Example 1: (costs case)

Four entrepreneurs have applied for four projects in an organisation, and senior management wants to select each of them for each project, so they have determined the costs (Costs in million monetary units) of allocating each entrepreneur to each project as shown in the table below(Table 2):

Table. 8.2

projects / entrepreneurs	Project1	Project2	Project3	Project4
Entrepreneur1	5	10	8	4
Entrepreneur2	2	10	5	7
Entrepreneur3	3	8	5	1
Entrepreneur4	6	15	7	2

Source: (Ratoul, 2006, p. 179)

Question:

Find the optimal allocation using the Hungarian method

We apply the following steps:

The First step:

We choose the lowest value in each row and subtract it from all values in the same row, for instance, we choose the lowest value in the first row which is 4, and then we subtract 4 from all values in the same row which is the first. The results of subtraction are the following values:

5-4= 1, 10-4= 6, 8-4= 4, 4-4=0. We carry out this calculation for all rest lines. The values in the previous table will change in the following table:

Table. 8.3

projects / entrepreneurs	Project1	Project2	Project3	Project4
Entrepreneur1	1	6	4	0
Entrepreneur2	0	8	3	5
Entrepreneur3	2	7	4	0
Entrepreneur4	4	13	5	0

Source: Author's own work.

The second step:

We choose the lowest value in each column in Table 3 and subtract it from all values in the same column, for instance, we choose the lowest value in the second column which is 6, and then we subtract 6 from all values in the same column which is the second. The results of subtraction are the following values:

6-6=0, 8-6=2, 7-6=1, 13-6=7. We carry out this calculation for all rest columns. The values in the previous table will change in the following table (see Table 4):

Note: when the lowest value in the row or column is 0, the values of this line or column stay the same because the subtraction of the value 0 will not make any difference.

Table. 8.4

projects / entrepreneurs	Project1	Project2	Project3	Project4
Entrepreneur1	1	0	1	0
Entrepreneur2	0	2	0	5
Entrepreneur3	2	1	1	0
Entrepreneur4	4	7	2	0

Source: Author's own work.

The third step:

Based on the last table, we carry out this step. We put a frame around the row or column that contains the largest number of zeros. We continue until all the zeros are inside the frame. We get the following table:

Table. 8.5

projects / entrepreneurs	Project1	Project2	Project3	Project4
Entrepreneur1	1	0	1	0
Entrepreneur2	0	2	0	5
Entrepreneur3	2	1	1	0
Entrepreneur4	4	7	2	0

Source: Author's own work.

The fourth step:

We check the optimal solution. If the number of frames equals the number of rows or columns this table gives us the optimal solution, unless It does not give us the optimal solution and we must continue in other stages to achieve the optimal solution.

From the table above, the number of frames is 3 low than the number of rows or columns which is 4, thus this table does not give the optimal solution and we must continue improving the solution as follows:

The fifth step:

We choose the lowest value among the values outside the frames and then subtract it from them and add it to the numbers of the intersection points of the frames We find these calculations:
The lowest value among the values outside the frames is 1.
The calculations are:
The subtractions:2-1=1, 1-1=0, 1-1=0, 4-1=3, 7-1=6, 2-1=1.
The additions: 0+1=1, 5+1=6.
The previous table will be changed as follows:

Table. 8.6

projects / entrepreneurs	Project1	Project2	Project3	Project4
Entrepreneur1	1	0	1	1
Entrepreneur2	0	2	0	6
Entrepreneur3	1	0	0	0
Entrepreneur4	3	6	1	0

Source: Author's own work.

The Sixth step:

We repeat the step number 3. We get the following table:

Table. 8.7

projects / entrepreneurs	Project1	Project2	Project3	Project4
Entrepreneur1	1	0	1	1
Entrepreneur2	0	2	0	6
Entrepreneur3	1	0	0	0
Entrepreneur4	3	6	1	0

Source: Author's own work.

The seventh step:

We repeat the step number 4, that is checking the optimal solution. If the number of frames equals the number of rows or columns this table gives us the optimal solution, unless It does not give us the optimal solution and we must continue in other stages to achieve the optimal solution.

From the table above, the number of frames is 4 equals the number of rows or columns which is 4, and hence this table gives the optimal solution, thus we can assign each Entrepreneur to one Project.

The eighth step:

we can assign each Entrepreneur to one Project whereby selecting the zero that is located in the intersection points of the row of the entrepreneur and the column of the project, then we cross out all zeros that fall along the same row or column. Paying attention to achieving the assigning process with each entrepreneur for a project, that is do not cross out any zero which leads to an entrepreneur for no project, and hence the focus must be paid when making the allocation.

When we assign any zero we put a circle on it.

We can assign by putting a circle and crossing out zeros which did not assign as follows (see Table 8):

Based on the table. 8.8 we can assign as follows:

Entrepreneur 1 is assigned to project 2 which cost 10 million monetary units.

Entrepreneur 2 is assigned to project 1 which cost 2 million monetary units.

Entrepreneur 3 is assigned to project 3 which cost 5 million monetary units.

Entrepreneur 4 is assigned to project 4 which cost 2 million monetary units.

Thus, the lowest total cost is:

Min (C)= $C_{12}+C_{21}+C_{33}+C_{44}$= 10+2+5+2= 19 million monetary units.

Table. 8.8

projects / entrepreneurs	Project1	Project2	Project3	Project4
Entrepreneur1	1	0	1	1
Entrepreneur2	0	2	0	6
Entrepreneur3	1	0	0	0
Entrepreneur4	3	6	1	0

Source: Author's own work.

Based on the table. 8.8 we can assign as follows:

Entrepreneur 1 is assigned to project 2 which cost 10 million monetary units.
Entrepreneur 2 is assigned to project 1 which cost 2 million monetary units.
Entrepreneur 3 is assigned to project 3 which cost 5 million monetary units.
Entrepreneur 4 is assigned to project 4 which cost 2 million monetary units.

Thus, the lowest total cost is:

Min (C)= $C_{12}+C_{21}+C_{33}+C_{44}$= 10+2+5+2= 19 million monetary units.

2.2.2 Example 2: (profits or returns case)

Assuming that The following table shows the achieved returns from assigning a group of workers to a group of machines in a production company.

The returns in hundreds of monetary units.

Table. 8.9

Machines / Workers	Machine1	Machine2	Machine3	Machine4
Worker1	50	10	20	40
Worker2	70	20	40	60
Worker3	30	40	10	80
Worker4	20	50	30	10

Source: Author's own work.

Question:

Find the optimal allocation using the Hungarian method

The solution:

We apply the following steps:

The first step:

We choose the highest value in the matrix (table of profits or returns) that is 80 hundreds of monetary units and subtracting all values from it. we reach a new matrix (new table) as follows:

Table. 8.10

Machines / Workers	Machine1	Machine2	Machine3	Machine4
Worker1	30	70	60	40
Worker2	10	60	40	20
Worker3	50	40	70	0
Worker4	60	30	50	70

Source: Author’s own work.

The second step:

We choose the lowest value in each row and subtract it from all values in the same row, for instance, we choose the lowest value in the first row which is 30, and then we subtract 30 from all values in the same row which is the first. The results of subtraction are the following values:

30-30= 0, 70-30= 40, 60-30= 30, 40-30=10. We carry out this calculation for all rest lines. The values in the previous table will change in the following table:

Table. 8.11

Machines / Workers	Machine1	Machine2	Machine3	Machine4
Worker1	0	40	30	10
Worker2	0	50	30	10
Worker3	50	40	70	0
Worker4	30	0	20	40

Source: Author’s own work.

The third step:

We choose the lowest value in each column and subtract it from all values in the same column, for instance, we choose the lowest value in the third column which is 20, and then we subtract 20 from all values in the same column which is the third. The results of subtraction are the following values:

30-20=10, 30-20=10, 70-20=50, 20-20=0. We carry out this calculation for all rest columns. The values in the previous table will change as in the following table:

Note: when the lowest value in the row or column is 0, the values of this line or column stay the same because the subtraction of the value 0 will not make any difference.

Table. 8.12

Machines / Workers	Machine1	Machine2	Machine3	Machine4
Worker1	0	40	10	10
Worker2	0	50	10	10
Worker3	50	40	50	0
Worker4	30	0	0	40

Source: Author's own work.

The fourth step:

Based on the last table, we carry out this step. We put a frame around the row or column that contains the largest number of zeros. We continue until all the zeros are inside the frame. We get the following table:

Table. 8.13

Machines / Workers	Machine1	Machine2	Machine3	Machine4
Worker1	0	40	10	10
Worker2	0	50	10	10
Worker3	50	40	50	0
Worker4	30	0	0	40

Source: Author's own work.

The fifth step:

We check the optimal solution. If the number of frames equals the number of rows or columns this table gives us the optimal solution, unless It does not give us the optimal solution and we must continue in other stages to achieve the optimal solution.

From the table above, the number of frames is 3 low than the number of rows or columns which is 4, thus this table does not give the optimal solution and we must continue improving the solution as follows:

The sixth step:

We choose the lowest value among the values outside the frames and then subtract it from them and add it to the numbers of the intersection points of the frames We find these calculations:

The lowest value among the values outside the frames is 10.

The calculations are:

The subtractions: 40-10 =30, 50-10=40, 40-10=30, 10-10=0, 10-10=0, 50-10=40.

The additions: 30+10=40, 40+10=50.

The previous table will be changed as follows:

Table. 8.14

Machines / Workers	Machine1	Machine2	Machine3	Machine4
Worker1	0	30	0	10
Worker2	0	40	0	10
Worker3	50	30	40	0
Worker4	40	0	0	50

Source: Author's own work.

The seventh step:

We repeat the step number 3. We get the following table:

Table. 8.15

Machines / Workers	Machine1	Machine2	Machine3	Machine4
Worker1	0	30	0	10
Worker2	0	40	0	10
Worker3	50	30	40	0
Worker4	40	0	0	50

Source: Author's own work.

The eighth step:

We repeat the step number 5, that is checking the optimal solution. If the number of frames equals the number of rows or columns this table gives us the optimal solution, unless It does not give us the optimal solution and we must continue in other stages to achieve the optimal solution.

From the table above, the number of frames is 4 equals the number of rows or columns which is 4, and hence this table gives the optimal solution, thus we can assign each Entrepreneur to one Project.

The ninth step:

we can assign each worker to one machine whereby selecting the zero that is located in the intersection points of the row of the worker and the column of the machine, then we cross out all zeros that fall along the same row or column. Paying attention to achieving the assigning process with each worker for a machine, that is do not cross out any zero which leads to an worker for no machine, and hence the focus must be paid when making the allocation.

When we assign any zero we put a circle on it.

We can assign by putting a circle and crossing out zeros which did not assign as follows:

Table. 8.16

Machines / Workers	Machine1	Machine2	Machine3	Machine4
Worker1	0	30	0	10
Worker2	0	40	0	10
Worker3	50	30	40	0
Worker4	40	0	0	50

Source: Author's own work.

Based on the table above we can assign as follows:

Worker 1 is assigned to machine 1 which makes profit 50 hundreds of monetary units.
Worker 2 is assigned to machine 3 which makes profit 40 hundreds of monetary units.
Worker 3 is assigned to machine 4 which makes profit 80 hundreds of monetary units.
Worker 4 is assigned to machine 2 which makes profit 50 hundreds of monetary units.

Thus, the lowest total cost is:
Max (Z)= $Z_{12}+Z_{21}+Z_{33}+Z_{44}$= 50+40+80+50= 220 hundreds of monetary units.

3. The probabilistic Enumeration Method

In the probabilistic enumeration method, we identify all possible assignment combinations, where each job is assigned to one person or one machine. In this case, the number of possible assignments is the product of the number of machines and the number of jobs. If the number of jobs is equal to the number of machines, the total number of possible assignments is n! (factorial of the number). After determining all the possibilities, we calculate the cost of each assignment and select the one with the lowest cost. In the case of profits, we choose the assignment that yields the highest possible profit. (Ratoul, 2006, p. 157)

This method is based on the factor law as follows: (Marza, 2010, p. 216)

N! = n(n-1)(n-2)(n-3).....*3*2*1.

Where

N: number of sources.

3.1. The cost case

Example: Assigning Machines to Jobs with Minimizing Cost Using the probabilistic Enumeration Method
Given Data:
We have 4 machines (A1, A2, A3, A4) and 4 jobs (F1, F2, F3, F4). The costs associated with assigning the machines to the jobs are as follows:

Table. 8.17

	F1 (DZD)	F2 (DZD)	F3 (DZD)	F4 (DZD)
A1	5	10	8	6
A2	12	6	9	11

A3	7	9	5	8
A4	8	7	10	4

Source : Author's own work

Requirement: Assign the machines to the jobs in such a way as to minimize the total cost using the probabilistic Enumeration Method.

Soluition:

Steps to Solve the Problem Using the probabilistic Enumeration Method:

1. Possible Assignments:

We have 4 machines and 4 jobs, so the total number of possible assignments is 4! = 4 × 3 × 2 × 1 = 24 assignments.

2. Assignments and Total Cost Calculation:

All possible assignments and calculate the total cost for each assignment in the following table

Table. 8.18

Assignment Number	Assignment (From Machine to Job)	Total Cost (DZD)
1	A1 → F1, A2 → F2, A3 → F3, A4 → F4	5 + 6 + 5 + 4 = 20 DZD
2	A1 → F1, A2 → F2, A3 → F4, A4 → F3	5 + 6 + 8 + 10 = 29 DZD
3	A1 → F1, A2 → F3, A3 → F2, A4 → F4	5 + 9 + 9 + 4 = 27 DZD
4	A1 → F1, A2 → F3, A3 → F4, A4 → F2	5 + 9 + 8 + 7 = 29 DZD
5	A1 → F1, A2 → F4, A3 → F2, A4 → F3	5 + 11 + 9 + 10 = 35 DZD
6	A1 → F1, A2 → F4, A3 → F3, A4 → F2	5 + 11 + 8 + 7 = 31 DZD
7	A1 → F2, A2 → F1, A3 → F3, A4 → F4	10 + 12 + 5 + 4 = 31 DZD
8	A1 → F2, A2 → F1, A3 → F4, A4 → F3	10 + 12 + 10 + 10 = 40 DZD
9	A1 → F2, A2 → F3, A3 → F1, A4 → F4	10 + 9 + 7 + 4 = 30 DZD
10	A1 → F2, A2 → F3, A3 → F4, A4 → F1	10 + 9 + 8 + 8 = 34 DZD
11	A1 → F2, A2 → F4, A3 → F1, A4 → F3	10 + 11 + 7 + 10 = 38 DZD
12	A1 → F2, A2 → F4, A3 → F3, A4 → F1	10 + 11 + 8 + 8 = 37 DZD
13	A1 → F3, A2 → F2, A3 → F1, A4 → F4	8 + 6 + 7 + 4 = 25 DZD
14	A1 → F3, A2 → F2, A3 → F4, A4 → F1	8 + 6 + 8 + 8 = 30 DZD
15	A1 → F3, A2 → F1, A3 → F2, A4 → F4	8 + 12 + 9 + 4 = 33 DZD
16	A1 → F3, A2 → F1, A3 → F4, A4 → F2	8 + 12 + 8 + 7 = 35 DZD
17	A1 → F3, A2 → F4, A3 → F1, A4 → F2	8 + 11 + 7 + 10 = 36 DZD
18	A1 → F3, A2 → F4, A3 → F2, A4 → F1	8 + 11 + 9 + 8 = 36 DZD
19	A1 → F4, A2 → F1, A3 → F3, A4 → F2	6 + 12 + 5 + 7 = 30 DZD
20	A1 → F4, A2 → F1, A3 → F4, A4 → F3	6 + 12 + 10 + 10 = 38 DZD
21	A1 → F4, A2 → F3, A3 → F1, A4 → F2	6 + 9 + 7 + 7 = 29 DZD
22	A1 → F4, A2 → F3, A3 → F4, A4 → F1	6 + 9 + 8 + 8 = 31 DZD
23	A1 → F4, A2 → F4, A3 → F1, A4 → F3	6 + 11 + 7 + 10 = 34 DZD
24	A1 → F4, A2 → F4, A3 → F3, A4 → F1	6 + 11 + 8 + 8 = 33 DZD

Source : Author's own work

3. Result:

After calculating the total cost for all the assignments, we find that the assignment that yields the lowest cost is:

Assignment 1: (A1 → F1, A2 → F2, A3 → F3, A4 → F4) with a total cost = 20 DZD.

3.2 The profit case

Example: Assigning Machines to Jobs with Maximizing Profit Using the probabilistic Enumeration Method

Given Data (with profits instead of costs):

Table. 8.19

	F1 (DZD)	F2 (DZD)	F3 (DZD)	F4 (DZD)
A1	15	10	12	13
A2	8	14	11	9
A3	11	9	16	10
A4	9	13	10	17

Source : Author's own work

Objective: Assign the machines to the jobs in such a way as to maximize the total profit using the probabilistic Enumeration Method.

Solution:

Steps to Solve the Problem Using the probabilistic Enumeration Method:

Assignments and Total Profit:

All the assignments and the profits generated from them are presented in the following table:

Table. 8.20

Assignment Number	Assignment (From Machine to Job)	Total Profit (DZD)
1	A1 → F1, A2 → F2, A3 → F3, A4 → F4	15 + 14 + 16 + 17 = 62 DZD
2	A1 → F1, A2 → F2, A3 → F4, A4 → F3	15 + 14 + 10 + 10 = 49 DZD
3	A1 → F1, A2 → F3, A3 → F2, A4 → F4	15 + 9 + 9 + 17 = 50 DZD
4	A1 → F1, A2 → F3, A3 → F4, A4 → F2	15 + 9 + 16 + 13 = 53 DZD
5	A1 → F1, A2 → F4, A3 → F2, A4 → F3	15 + 13 + 11 + 10 = 49 DZD
6	A1 → F1, A2 → F4, A3 → F3, A4 → F2	15 + 13 + 16 + 13 = 57 DZD
7	A1 → F2, A2 → F1, A3 → F3, A4 → F4	10 + 8 + 16 + 17 = 51 DZD
8	A1 → F2, A2 → F1, A3 → F4, A4 → F3	10 + 8 + 10 + 10 = 38 DZD
9	A1 → F2, A2 → F3, A3 → F1, A4 → F4	10 + 14 + 11 + 17 = 52 DZD
10	A1 → F2, A2 → F3, A3 → F4, A4 → F1	10 + 14 + 10 + 13 = 47 DZD
11	A1 → F2, A2 → F4, A3 → F1, A4 → F3	10 + 13 + 11 + 10 = 44 DZD
12	A1 → F2, A2 → F4, A3 → F3, A4 → F1	10 + 13 + 16 + 13 = 52 DZD
13	A1 → F3, A2 → F2, A3 → F1, A4 → F4	12 + 14 + 11 + 17 = 54 DZD
14	A1 → F3, A2 → F2, A3 → F4, A4 → F1	12 + 14 + 10 + 9 = 45 DZD
15	A1 → F3, A2 → F1, A3 → F2, A4 → F4	12 + 8 + 9 + 17 = 46 DZD
16	A1 → F3, A2 → F1, A3 → F4, A4 → F2	12 + 8 + 16 + 13 = 49 DZD
17	A1 → F3, A2 → F4, A3 → F1, A4 → F2	12 + 13 + 11 + 13 = 49 DZD
18	A1 → F3, A2 → F4, A3 → F2, A4 → F1	12 + 13 + 16 + 9 = 50 DZD
19	A1 → F4, A2 → F1, A3 → F3, A4 → F2	13 + 8 + 16 + 13 = 50 DZD
20	A1 → F4, A2 → F1, A3 → F4, A4 → F3	13 + 8 + 10 + 17 = 48 DZD

Assignment Number	Assignment (From Machine to Job)	Total Profit (DZD)
21	A1 → F4, A2 → F3, A3 → F1, A4 → F2	13 + 9 + 11 + 13 = 46 DZD
22	A1 → F4, A2 → F3, A3 → F4, A4 → F1	13 + 9 + 10 + 9 = 41 DZD
23	A1 → F4, A2 → F4, A3 → F1, A4 → F3	13 + 13 + 11 + 17 = 54 DZD
24	A1 → F4, A2 → F4, A3 → F3, A4 → F1	13 + 13 + 16 + 9 = 51 DZD

Source : Author's own work

Result:

After calculating the total profit for all the assignments, we find that the assignment that yields the highest profit is:

Assignment 1: (A1 → F1, A2 → F2, A3 → F3, A4 → F4) with a total profit = 62 DZD.

4. The Transportation Method.

The transportation method can be used to find the best assignment by constructing a table that includes the costs associated with the assignments. In this table, both the supply quantities and demand quantities for each row and column should be equal to one, making the total supply equal to the total demand, which corresponds to the number of locations. The initial solution used is the minimum cost method, which is the best option, or any other suitable method, as is done in transportation problems. However, care should be taken to avoid situations where many rows or columns become saturated at the same time. In such cases, priority is given to saturating a row or column first. Once the row or column is saturated, we adjust the quantities in the supply and proceed to the next table. In the case of profits, instead of using the minimum cost method, we use the maximum profit method, where we select the highest possible profit at each step. (Ratoul, 2006, p. 165)

4.1. The cost case

Example: Assigning Machines to Jobs with Minimizing Cost Using the Transportation Method

Given Data:

Table. 8.21

	F1	F2	F3	F4
A1	5	10	8	6
A2	12	6	9	11
A3	7	9	5	8
A4	8	7	10	4

Source : Author's own work

Requirement: Assign the machines to the jobs in such a way as to minimize the total cost using the Transportation Method.

Solution:

Steps to Solve the Problem Using the Transportation Method:

1. Setting up the table:

We start by defining costs, supplies, and demands:

- Supplies: Since we have 4 machines, the supply for each machine is 1.
- Demands: Since we have 4 jobs, the demand for each job is 1.

Initial Transportation Table:

Table. 8.22

	F1	F2	F3	F4	Supply
A1	5	10	8	6	1
A2	12	6	9	11	1
A3	7	9	5	8	1
A4	8	7	10	4	1
Demand	1	1	1	1	

Source : Author's own work

2. Step 1: Assign the Lowest Cost:
The lowest cost in the table is 4 in the cell (A4 → F4).

- Assignment: A4 → F4 with cost 4.
- The demand for F4 is now fulfilled, and we remove column F4 from the table.

Table after assigning A4 → F4:

Table. 8.23

	F1	F2	F3	Supply
A1	5	10	8	1
A2	12	6	9	1
A3	7	9	5	1
A4	8	7	10	0
Demand	1	1	1	

Source : Author's own work

3. Step 2: Assign the Lowest Cost in the Remaining Table:
The lowest cost in the remaining table is 5 in the cell (A3 → F3).

- Assignment: A3 → F3 with cost 5.
- The demand for F3 is now fulfilled, and we remove column F3 from the table.

Table after assigning A3 → F3:

Table. 8.24

	F1	F2	Supply
A1	5	10	1
A2	12	6	1
A3	7	9	0
A4	8	7	0
Demand	1	1	

Source : Author's own work

4. Step 3: Assign the Lowest Cost in the Remaining Table:
The lowest cost in the remaining table is 5 in the cell (A1 → F1).

- Assignment: A1 → F1 with cost 5.
- The demand for F1 is now fulfilled, and we remove column F1 from the table.

Table after assigning A1 → F1:

Table. 8.25

	F2	Supply
A1	10	0
A2	6	1
A3	9	0
A4	7	0
Demand	1	

Source : Author's own work

5. Step 4: Assign the Lowest Cost in the Remaining Table:
The lowest cost in the remaining table is 6 in the cell (A2 → F2).

- Assignment: A2 → F2 with cost 6.
- The demand for F2 is now fulfilled, and we remove column F2 from the table.

Table after assigning A2 → F2:

Table. 8.26

	Supply
A1	0
A2	0
A3	0
A4	0
Demand	

Source : Author's own work

6. Final Assignments:

- Final Assignments are:
  - A4 → F4 with cost 4
  - A3 → F3 with cost 5
  - A1 → F1 with cost 5
  - A2 → F2 with cost 6

7. Total Cost:

- Total Cost is the sum of the cost in each assignment:
  - 4 (A4 → F4) + 5 (A3 → F3) + 5 (A1 → F1) + 6 (A2 → F2) = 20.

Conclusion:

- Final Assignments:
  - A4 → F4 with cost 4
  - A3 → F3 with cost 5
  - A1 → F1 with cost 5
  - A2 → F2 with cost 6
- Total Cost = 20

4.2. The profit case

Example: Assigning Machines to Jobs with Maximizing Profit Using the Transportation Method
Given Data (with profits instead of costs):

Table. 8.27

	F1 (DZD)	F2 (DZD)	F3 (DZD)	F4 (DZD)
A1	15	10	12	13

	F1 (DZD)	F2 (DZD)	F3 (DZD)	F4 (DZD)
A2	8	14	11	9
A3	11	9	16	10
A4	9	13	10	17

Source : Author's own work

Objective: Assign the machines to the jobs in such a way as to maximize the total profit using the Transportation Method.

Solution:

Steps to Solve the Problem Using the Transportation Method:

Step 1: Set up the table:

We start by defining the profits, supplies, and demands:

- Supplies: Since we have 4 machines, the supply for each machine is 1.
- Demands: Since we have 4 jobs, the demand for each job is 1.

Initial Transportation Table:

Table. 8.28

	F1	F2	F3	F4	Supply
A1	15	10	12	13	1
A2	8	14	11	9	1
A3	11	9	16	10	1
A4	9	13	10	17	1
Demand	1	1	1	1	

Source : Author's own work

Step 2: First Step – Assign the Highest Profit:

The highest profit in the table is 17 in the cell (A4 → F4).

- Assignment: A4 → F4 with profit 17.
- The demand for F4 is now fulfilled, and we remove column F4 from the table.

Table after assigning A4 → F4:

Table. 8.29

	F1	F2	F3	Supply
A1	15	10	12	1
A2	8	14	11	1
A3	11	9	16	1
A4	9	13	10	0
Demand	1	1	1	

Source : Author's own work

Step 3: Second Step – Assign the Highest Profit in the Remaining Table:

The highest profit in the remaining table is 16 in the cell (A3 → F3).

- Assignment: A3 → F3 with profit 16.
- The demand for F3 is now fulfilled, and we remove column F3 from the table.

Table after assigning A3 → F3:

Table. 8.30

	F1	F2	Supply
A1	15	10	1
A2	8	14	1

	F1	F2	Supply
A3	11	9	0
A4	9	13	0
Demand	1	1	

Source : Author’s own work

Step 4: Third Step – Assign the Highest Profit in the Remaining Table:

The highest profit in the remaining table is 15 in the cell (A1 → F1).

- Assignment: A1 → F1 with profit 15.
- The demand for F1 is now fulfilled, and we remove column F1 from the table.

Table after assigning A1 → F1:

Table. 8.31

	F2	Supply
A1	10	0
A2	14	1
A3	9	0
A4	13	0
Demand	1	

Source : Author’s own work

Step 5: Fourth Step – Assign the Highest Profit in the Remaining Table:

The highest profit in the remaining table is 14 in the cell (A2 → F2).

- Assignment: A2 → F2 with profit 14.
- The demand for F2 is now fulfilled, and we remove column F2 from the table.

Table after assigning A2 → F2:

Table. 8.32

	Supply
A1	0
A2	0
A3	0
A4	0
Demand	

Source : Author’s own work

Final Assignments:

- Final Assignments are:
  - A4 → F4 with profit 17
  - A3 → F3 with profit 16
  - A1 → F1 with profit 15
  - A2 → F2 with profit 14

Total Profit:

- Total Profit is the sum of the profit in each assignment:
  - 17 (A4 → F4) + 16 (A3 → F3) + 15 (A1 → F1) + 14 (A2 → F2) = 62 DZD.

Conclusion:

- Final Assignments:
  - A4 → F4 with profit 17
  - A3 → F3 with profit 16

- A1 → F1 with profit 15
- A2 → F2 with profit 14

- Total Profit = 62 DZD

Conclusion

In conclusion, assignment models are not merely mathematical tools for solving numerical problems; they represent systematic and practical frameworks for addressing real-world issues in daily work environments. These models contribute to the efficient use of resources and the strategic alignment of human and material capabilities with institutional objectives.

Their strength lies in their broad applicability—from task distribution among employees, to machine scheduling, to the allocation of transportation units, and even within educational and service-based projects. Throughout this lecture, we presented three analytical methods for solving assignment models. While these methods differ in their approach, they share a common goal: to determine the optimal assignment based on a clear economic criterion—either profit or cost.

What makes these methods valuable is their ability to equip researchers and practitioners with practical tools for making objective, logic-based decisions, free from guesswork or intuition. Therefore, a solid understanding of assignment models and their analytical techniques is a key competency in modern management, and an essential pillar for success in dynamic and complex work environments.

Lecture 9: Queuing Theory

Introduction

Queuing theory is one of the quantitative methods in management. Its importance stems from its broad applicability across diverse service and industrial sectors, such as hospitals, banks, transportation systems, and technical centers. The theory seeks to analyze the random behavior of service requests and evaluate system performance to enhance efficiency, reduce waiting times, and minimize associated costs. This lecture offers a focused overview of the main concepts of queuing theory, introduces key model types, and demonstrates how they support effective operational decision-making.

Learning Objectives

By the end of this lecture, students will be able to:

- Understand the fundamental principles underlying queuing theory.
- Distinguish between major models such as M/M/1 and M/M/s and select the appropriate one based on system characteristics.
- Calculate key performance indicators, such as average number of units in the system and average waiting time.
- Analyze the relationship between service capacity and total cost, and determine the optimal number of service resources.
- Apply mathematical models to real-life scenarios to evaluate and improve system efficiency

1. A General Introduction

A.K. Erlang, a Danish engineer, was the first to study queuing theory. Erland tried out changing demand in telephone traffic in 1909. Eight years later, he wrote a report about the problems with automated dialling technology. After World War II, Erlang's early work was expanded to encompass broader issues and to include practical applications of queuing theory. Queuing theory, which is the study of waiting lines, is one of the oldest and most extensively used ways to do quantitative research. People wait in queue every day to purchase groceries, get petrol, make a bank deposit or wait on the phone for the first available airline reservationist to answer. Waiting lines, often called queues, can also be made up of machinery that need to be fixed, trucks that need to be unloaded, or aeroplanes that are lined up on a runway waiting for clearance to take off. The three main parts of a queuing process are people arriving, the actual waiting line, and the places where people may get help. (Chowdhury, 2013, p. 468)

This historical background demonstrates how a seemingly simple problem related to managing phone calls led to the development of a comprehensive theoretical framework still applied today across diverse fields, including healthcare, transportation, and banking. It highlights the theory's importance in offering practical solutions to recurring operational issues.

2. Basic concepts

In this section, queueing theory will be reviewed, focusing on the study of systems that handle random service requests. It will analyze how queues are formed, determining waiting times and queue lengths. A comprehensive definition of the theory will be provided, along with an overview of the main components of a queuing system, such as the arrival process, the queue itself, service facilities, and the service performance rate.

2.1. Definition of Queueing Theory

Queueing theory is a branch of operations research that focuses on the study of systems in which customer arrivals or service requests occur randomly, and the service times themselves are also irregular. This theory aims to analyze the behavior of such systems in order to understand how queues are formed, estimate performance indicators such as waiting time and queue length, and determine the optimal number of resources required to provide the service efficiently. Queueing theory is applied in various service and industrial sectors to reduce waiting periods, improve resource utilization, and achieve a balance between the cost of operating the system and the cost incurred by customers while waiting. (Nedjm, 2008, p. 132)

2.2. Applications of the Theory

Queuing theory, which looks at waiting lines, is one of the oldest and most extensively used ways to do quantitative research. People wait in queue every day to purchase groceries, get petrol, make a bank deposit or wait on the phone for the first available airline reservationist to answer. Waiting lines, often called queues, can also be made up of machinery that need to be fixed, trucks that need to be unloaded, or aeroplanes that are lined up on a runway waiting for clearance to take off. The three main parts of a queuing process are people arriving, the actual waiting line, and the places where people may get help. (Chowdhury, 2013, p. 468)

Queuing theory emerges as an analytical and practical framework that is closely connected to real-world applications. Its scope extends across various service and industrial domains, from grocery shopping and banking services to the operation of machinery and aircraft. This diversity of examples reflects the comprehensive nature of the theory and its flexibility in addressing different types of waiting entities. Defining the core components of a queuing system—arrival, queue, and service facility—forms the basis for understanding system behavior and analyzing its performance. This, in turn, enables the development of precise quantitative models that help reduce waiting times, enhance resource utilization, and improve the overall quality of service provided.

2.3. Key Characteristics of a Queueing System

Developing an effective queueing model requires identifying a set of core characteristics that define the nature and efficiency of the system. These include: (Nedjm, 2008, p. 132)

2.3.1.Service Discipline

This refers to the rule by which customers are prioritized for service. Common policies include:

- First-Come, First-Served (FCFS)
- Last-Come, First-Served (LCFS)
- Random selection
- Priority based on urgency (e.g., emergency cases in hospitals receive service before regular cases)

2.3.2.System Capacity

This denotes the maximum number of units the system can accommodate, whether waiting in the queue or receiving service. The capacity can be limited, as in small facilities, or theoretically unlimited in certain analytical models.

2.3.3. Arrival Rate Distribution

This describes the pattern by which customers enter the system within a certain time frame. It is commonly assumed that arrivals follow a Poisson distribution, reflecting the random nature of demand flows.

2.3.4. Service Rate Distribution

This represents the amount of time each unit requires to complete service. It is typically modeled using an exponential distribution for simplicity, although more complex distributions may be applied in real-world scenarios.

These characteristics are crucial because they form the foundation on which the analytical model is built. Accurately defining them is essential for selecting the most appropriate model. For instance, the service discipline affects customers' perception of fairness, while system capacity determines flexibility in handling congestion. The arrival and service distributions are used to estimate performance indicators such as waiting time, queue length, and resource utilization rates. Therefore, understanding and analyzing these characteristics contributes to designing more efficient systems and enables data-driven operational decisions that enhance service quality and reduce operating costs.

In addition the caractirisrics of queing system are summurized in the following table:

Table. 9.1. Characteristics of a Queueing System

Arrival Process	- The probability density function that governs client arrivals in the system. -In a messaging system, this pertains to the probability distribution of message arrival.
Service Process	-The probability density function that defines the customer service durations inside the system. -This pertains to the distribution of message transmission times in a messaging system. -The transmission of messages is exactly proportional to their length, hence indicating the distribution of message lengths.
Number of Servers	Quantity of servers accessible for client support. In a messaging system, this denotes the quantity of connections between the source and destination nodes.

Source: (Kumar R. , 2020, p. 6)

These characteristics are crucial because they form the foundation on which the analytical model is built. Accurately defining them is essential for selecting the most appropriate model. For instance, the service discipline affects customers' perception of fairness, while system capacity determines flexibility in handling congestion. The arrival and service distributions are used to estimate performance indicators such as waiting time, queue length, and resource utilization rates. Therefore, understanding and analyzing these characteristics contributes to designing more efficient systems and enables data-driven operational decisions that enhance service quality and reduce operating costs.

2.4.Fundamentals of Queuing Theory

Queuing theory is based on studying the behavior of waiting systems that customers face when requesting service. By analyzing how customers arrive and the service performance, efficiency can be improved, and waiting times reduced in various fields such as telecommunications, transportation, and healthcare. This theory is grounded in the following principles:

2.4.1.Arrival pattern inside the system

Customers either come to a service facility on a set timetable or at random. When arrivals are random, it means that they are independent of each other, and it is not possible to predict exactly when they will occur. The Poisson distribution is a probability distribution that can be used to estimate the number of people who will arrive at a given time (Chowdhury, 2013), and the mathematical expression used is as follows: (Chowdhury, 2013)

$$P(n;t) = \frac{(\lambda t)^{\mathrm{n}} \mathrm{e}^{-\lambda \mathrm{t}}}{n!}$$

Where:

- $P(n;t)$ = Probability of n arrivals
- λ = Average arrival rate
- e = Euler's number (approximately 2.718)
- n = Number of arrivals per unit of time

t = Time period during which arrivals are observed

2.4.2.Service Performance Rate:

It refers to the average number of units that the service center provides service to within a specific time period, such as an hour, a day, a week, or a month. This rate is influenced by several factors, the most important of which are: (Nedjm, 2008, pp. 133-134)

- Determining the size of service centers: Along with the possibility of operating additional centers to improve performance.
- Reorganizing the service delivery method: To increase the rate, including renewing or replacing equipment with more efficient ones.
- Scheduling the service of incoming units: By minimizing wasted time and idle capacity as much as possible.
- Handling incoming units that require more time: Or those that need more service than the average required by other units, in order to avoid disrupting the work in other service centers.

Regarding the issue of waiting, it can be viewed through two main cases:

- Single-server service system (MM1).

- Multiple-server system.

By understanding the arrival patterns and service performance, organizations can adjust their processes to reduce congestion and provide faster responses to customers. Furthermore, applying this theory to multi-server systems enhances the ability to handle large numbers of customers more efficiently, contributing to increased customer satisfaction and operational excellence.

3. Single-server service system (MM1)

In this section, the Single-Server Queue Model (M/M/1)**,** one of the most common and widely used queuing models, will be presented. A definition of this model, which deals with Poisson arrivals and exponential service times, will be provided, along with the basic assumptions upon which the model relies. Following that, the mathematical equations that allow the analysis of system performance will be presented, helping to determine key performance indicators such as waiting times, average number of customers in the system, and in the queue.

3.1. Definition

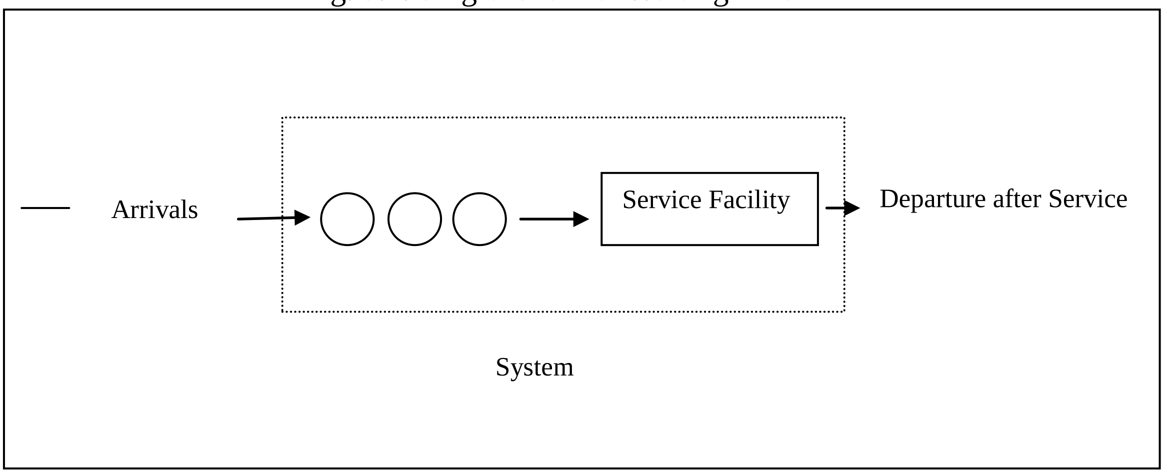
Model for Single-Channel Queuing with Poisson Arrivals and Exponential Service Times (M/M/1): An analytical technique is presented to identify key performance indicators in a typical service system. Once numerical metrics are calculated, cost data may be included to balance service levels with waiting queue expenses. (Chowdhury, 2013, p. 471)

One of the most common and basic queuing models is the single-channel, single-phase model. Assumes conditions escape: (Chowdhury, 2013, p. 471)

1. Arrivals are FIFO (First In, First Out)
2. No one is denied service, regardless of queue length.
3. Arrivals are independent of previous arrivals, yet the average number (arrival rate) remains constant throughout time.
4. Arrivals follow a Poisson distribution and originate from a vast population.
5. While service time varies across passengers, the average rate is known.
6. The number of items in queue and the waiting queue for a certain item are both random variables.
7. Service times follow the negative exponential distribution.
8. Average service rate exceeds arrival rate.
9. Customers have unlimited queuing space.

The following figure explains this type of waiting line:

Fig. 9.1. Single-Channel Waiting Line



Source: (Chowdhury, 2013, p. 471)

3.2. Mathematical Equations

Before presenting the mathematical equations, we must introduce the following symbols and their meanings: (Chowdhury, 2013, p. 471)

λ = average frequency of arrivals per time interval (e.g., per hour)

μ = average quantity of individuals or goods supplied per time interval.
When calculating the arrival rate (λ) and the service rate (μ), it is essential to utilise the same time interval. For instance, if λ represents the average number of arrivals per hour, then μ must denote the average number that can be provided per hour.
The mathematical equations for the single-server service system (M/M/1) are as follows: (Nedjm, 2008, pp. 134-135)

- Probability that the service provider is busy:

$$P = \frac{\lambda}{\mu}$$

- Probability of no units in the system:

$$P_0 = 1 - P$$

- Pobability of having n units in the system (units receiving service + units in the queue):

$$P_n + \left(\frac{\lambda}{\mu}\right)^n * P_0$$

- Average number of units in the system:

$$L = \frac{\lambda}{\mu - \lambda}$$

- Average number of units in the queue:

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)}$$

- Average time spent by a single unit in the system/

$$T = \frac{1}{\mu - \lambda}$$

- Average time spent by a single unit in the queue:

$$T_q = \frac{\lambda}{\mu(\mu - \lambda)}$$

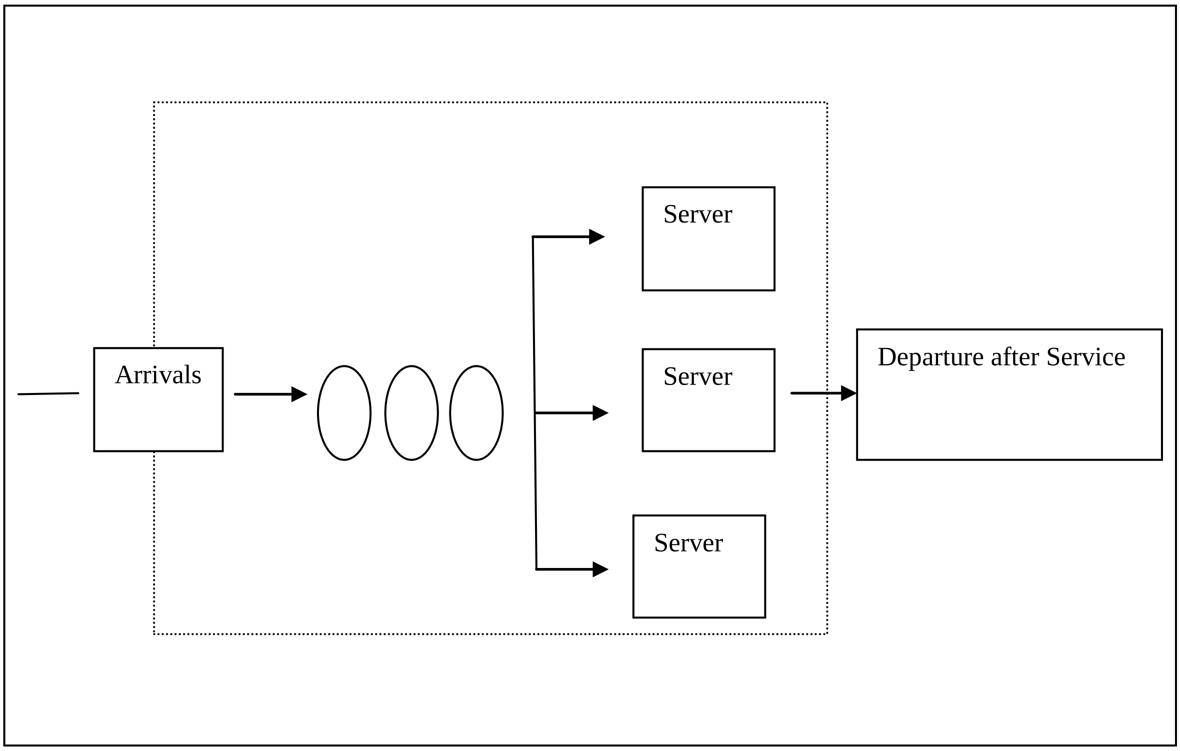
4. Multiple-server system

In this section, the Multiple-Server Queue Model will be introduced, where more than one server or channel is available to serve incoming customers. The model assumes that customers wait in a single queue and proceed to the first available server. Each server operates with an independent and identical exponential service time distribution, with a mean of 1/μ. The arrival process follows a Poisson distribution with a rate of λ, leading to a single line where customers are served on a first-come, first-served basis. We will also explore the mathematical equations that define key performance indicators for this multi-server system, including waiting times, the number of customers in the system, and system utilization.

4.1. Definition

In this system, a unit will enter any available channel (server), and a queue forms when the number of units in the system n exceeds the number of service channels s. This system is characterized by having more than one service channel, and the arrival rate λ must be lower than the service rate μ multiplied by the number of service channels s, meaning $s\mu > \lambda$. Additionally, equilibrium in this system is achieved when the arrival rate λ is less than sμ, i.e., $\lambda < s\mu$. (Nedjm, 2008, p. 136)

Fig. 9.2. Multiple-Channel Waiting Line



Source: (Chowdhury, 2013, p. 473)

4.2. Mathematical Equations

In this section, we will discuss the costs associated with queuing systems and how the number of service facilities affects these costs. We will address two main types of costs: tangible costs related to the operation of service facilities, and intangible costs associated with customer wait times. We will also explore how optimizing the system can balance operational costs and waiting times by determining the optimal number of service facilities.

The mathematical equations for multiple-server system are as follows:

Probability of failure in service channels:

$$P_0 = \left[\sum_{K=0}^{S-1} \frac{a^K}{K!} + \frac{a^S}{S!} \cdot \frac{S\mu}{S\mu - \lambda}\right]^{-1}$$

Average number of units expected in the queue:

$$L_q = \frac{a^S \cdot P}{S! \cdot (1-P)^2} \cdot P_0$$

Average number of units expected in the system:

$$L = L_q + \frac{\lambda}{\mu}$$

Average time a unit spends in the queue:

$$W_q = \frac{L_q}{\lambda}$$

5.Waiting lines and costs

5.1. Overview

Two fundamental categories of expenditures are related with waiting-line concerns. Initially, there are the relatively 'tangible' expenses associated with operating each service site, such as expenditures for equipment, supplies, and labour. The costs, naturally, escalate as the number of operational service facilities increases. Conversely, there are the comparatively 'intangible' expenses linked to requiring clients to endure a waiting period before being attended to—physical discomfort, negative emotional responses, diminished or forfeited purchases, and similar factors. As the quantity of operational service facilities rises, the average wait time for customers diminishes, thereby reducing associated expenses. The accompanying graphic illustrates that the sum of these two fundamental cost kinds reaches a minimum at a particular number of facilities. The optimal number of service facilities to be planned by the operations manager minimises the overall costs associated with both operational service facilities and customer wait times. (Chowdhury, 2013, pp. 474-475)

5.2. Mathematical Equations

The mathematical model for calculating costs in the case of a single channel is as follows: (Nedjm, 2008, p. 139)

Total lost costs :

$$T_c = T_{cn} + T_{cs}$$

Where:

T_{cn} : Lost costs for units accumulated in the queue.

$$T_{cn} = wg_1 * \mathrm{t} * \mathrm{L_q}$$

$$T_{cs} = wg_2 * \mathrm{t}$$

Where:

t: Time unit

wg_1 : Cost per unit in the queue per time unit

wg_2: Cost per unit for the service provider

6. Examples

6.1. Exercise 1: Queuing Analysis at Constantine Post Office (M/M/1)

At Constantine Post Office, there is a single service center for serving customers. 30 customers arrive per hour to complete their postal transactions. The service rate (μ) is 40 customers per hour, where only one employee is working in the office, while the other employees are on vacation.

The waiting cost per customer in the queue is 5 dinars per hour, while the service cost per employee is 10 dinars per hour. The time unit is 1 hour.

Questions:

- Calculate the probability that the employee is busy at the post office.
- Calculate the probability of having no customers in the system.
- Calculate the average number of customers in the system.
- Calculate the average number of customers in the queue.
- Calculate the average time a customer spends in the system.
- Calculate the average time a customer spends in the queue.
- Lost cost calculation:
    - Calculate the lost cost due to units accumulated in the queue.
    - Calculate the lost cost due to the service provider.
    - Calculate the total lost costs.

The solution of Exercise 1: Queuing Analysis in Constantine Post Office (M/M/1)

Given:

- Arrival rate (λ): 30 customers per hour.
- Service rate (μ): 40 customers per hour.
- Cost per unit in the queue (wg_1): 5 dinars per hour.
- Cost per unit for the service provider (wg_2): 10 dinars per hour.
- Time unit (t): 1 hour.
- Number of servers (c): 1 (Only one employee is working while others are on vacation).

Questions and Solutions:

- Probability that the employee is busy (P):

$$P = \frac{\lambda}{\mu}$$

$$P = \frac{30}{40} = 0.75$$

Thus, there is a 75% probability that the employee is busy.

- Probability of no customers in the system (P_0):

$$P_0 = 1 - P = 1 - 0.75 = 0.25$$

Thus, there is a 25% probability that there are no customers in the system.

- Average number of customers in the system (L):

$$L = \frac{\lambda}{\mu - \lambda}$$
$$L = \frac{30}{40 - 30}$$
$$L = \frac{30}{10} = 3$$

Thus, the average number of customers in the system is 3 customers (including customers in the queue and those receiving service).

- Average number of customers in the queue (L_x):

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{30^2}{40(40 - 30)}$$
$$L_q = \frac{900}{400} = 2.25$$

Thus, the average number of customers in the queue is 2.25 customers.

- Average time spent by a customer in the system (T):

$$T = \frac{1}{\mu - \lambda}$$
$$T = \frac{1}{40 - 30} = \frac{1}{10}$$
$$T = 0.1 \text{ hours} = 6 \text{ minutes}$$

Thus, the average time a customer spends in the system is 6 minutes.

- Average time spent by a customer in the queue (T_q):

$$T_q = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{L_q}{\lambda} = \frac{2.25}{30}$$

$$T_q = 0.075 \text{ hours} = 4.5 \text{ minutes}$$

Thus, the average time a customer spends in the queue is 4.5 minutes.

Calculation of Lost Costs:

Given:

- wg_1: Cost per unit in the queue = 5 dinars per hour
- wg_2: Cost per unit for the service provider = 10 dinars per hour
- t: Time unit = 1 hour
- L_x: Average number of customers in the queue = 2.25

- Lost costs due to units accumulated in the queue T_{cn}

$$T_{cn} = wg_1 * \mathrm{t} * \mathrm{L_q} = 5 \cdot 1 \cdot 2.25 = 11.25 \text{ dinars}$$

- Lost costs due to the service provider (T_s):

$$T_{cs} = wg_2 * \mathrm{t} = wg_2 \cdot t = 10 \cdot 1 = 10 \text{ dinars}$$

- Total lost costs (T_n):

$$T_c = T_{cn} + T_{cs} = 11.25 + 10 = 21.25 \text{ dinars}$$

Thus, the total lost costs in the system is 21.25 dinars.

6.2. Exercise 2: Queuing Analysis at Algiers International Airport (M/M/3)

At Algiers International Airport, 60 passengers arrive per hour at the check-in counters. There are 3 servers (employees) at the check-in counters, but due to increased demand during peak hours, all three servers are activated to serve the passengers.

The service rate per server (μ) is 50 passengers per hour. The waiting cost per passenger in the queue is 4 dinars per hour, while the service cost per employee is 8 dinars per hour. The time unit is 1 hour.

Questions:

- Calculate the probability that any server is busy at the airport.
- Calculate the average number of passengers in the system in the case of three servers.
- Calculate the average number of passengers in the queue.
- Calculate the average time a passenger spends in the system.
- Calculate the average time a passenger spends in the queue.
- Lost cost calculation:
    - Calculate the lost cost due to units accumulated in the queue.
    - Calculate the lost cost due to the service provider.
    - Calculate the total lost costs.

The solution of Exercise 2: Queuing Analysis at Algiers International Airport (M/M/3)

Given:

- Arrival rate (λ): 60 passengers per hour.
- Service rate per server (μ): 50 passengers per hour.
- Number of servers (S): 3 servers.
- Cost per passenger in the queue (wg_1): 4 dinars per hour.
- Cost per service provider (wg_2): 8 dinars per hour.
- Time unit (t): 1 hour.

1. Probability of failure in service channels

To calculate this, we use the probability of failure in the service channels (P_0).

We apply the following formula for P_0:

$$P_0 = \left[\sum_{K=0}^{S-1} \frac{a^K}{K!} + \frac{a^S}{S!} \cdot \frac{S\mu}{S\mu - \lambda}\right]^{-1}$$

$$P = \frac{\lambda}{S\mu}$$

$$P = \frac{60}{3 \times 50} = \frac{60}{150} = 0.4$$

Thus: There is a 40% probability that any given server is busy.

2. Probability that there are no customers in the system P_0

We use the general formula for M/M/S:

We put:

$$a = \frac{\lambda}{\mu}$$

$$a = \frac{60}{50} = 1.2$$

$$P_0 = \left[\sum_{K=0}^{S-1} \frac{a^K}{K!} + \frac{a^S}{S!} \cdot \frac{S\mu}{S\mu - \lambda}\right]^{-1}$$

$$\sum_{K=0}^{2} \frac{1.2^K}{K!} = 1 + 1.2 + 0.72 = 2.92$$

$$\frac{1.2^3}{3!} \cdot \frac{150}{90} = \frac{1.728}{6} \cdot \frac{150}{90} \approx 0.288 \cdot 1.6667 \approx 0.48$$

$$P_0 = \frac{1}{2.92 + 0.48}$$

$$P_0 = \frac{1}{3.4} \approx 0.294$$

Answer: $P_0 \approx 0.294$

3. Average number of passengers in the queue L_q:

$$L_q = \frac{a^S \cdot P}{S! \cdot (1 - P)^2} \cdot P_0$$

$$L_q = \frac{1.728 \cdot 0.4}{6 \cdot 0.36^2} \cdot 0.294$$

$$L_q = \frac{0.6912}{6 \cdot 0.1296} \cdot 0.294$$

$$L_q = \frac{0.6912}{0.7776} \cdot 0.294 \approx 0.889 \cdot 0.294 \approx 0.261$$

$L_q \approx 0.261$ passengers

4. Average number of passengers in the system (***L***):

From the previous calculations we can find:

$$L \approx 1.461 \text{ passengers}$$

5. Average time a passenger spends in the system (***T***):

$$T = \frac{L}{\mu} = \frac{1.461}{60} \approx 0.02435 \text{ hours } \approx 1.46 \text{ minutes}$$

Thus: $T \approx 1.46$ minutes

6. Average time a passenger spends in the queue ($\boldsymbol{W_q}$):

$$W_q = \frac{L_q}{\lambda}$$

$$W_q = \frac{0.261}{60} \approx 0.00435 \text{ hours } \approx 0.261 \text{ minutes}$$

Thus: $T_q \approx 0.261$ minutes

Lost Cost Calculations

7. Cost due to waiting ($\boldsymbol{T_{cn}}$):

$$T_{cn} = 4 \times 1 \times 0.261 = 1.044 \text{ dinars}$$

8. Cost due to service ($\boldsymbol{T_{cs}}$):

$$T_{cs} = 8 \times 3 = 24 \text{ dinars}$$

9. Total lost cost ($\boldsymbol{T_c}$):

$$T_c = T_{cn} + T_{cs} = 1.044 + 24 = 25.044 \text{ dinars}$$

Thus. the total cost is more than 25 dinars

Conclusion

Queuing theory stands out as one of the most powerful quantitative methods, effectively combining theoretical depth with practical precision to address congestion problems and enhance workflow across diverse systems. What makes this theory particularly valuable is its ability to mathematically model service and production operations while accounting for fluctuating demand and resource limitations. This allows decision-makers to scientifically analyze system performance, anticipate bottlenecks, and implement proactive solutions aimed at reducing wait times, improving customer satisfaction, and maximizing resource utilization.

Beyond its computational applications, queuing theory offers a conceptual framework that promotes systematic thinking and data-driven decision-making over guesswork. It supports operations managers and researchers in understanding the dynamic interactions within service systems and helps them design more adaptive and responsive mechanisms in the face of sudden changes in demand or capacity. In today's complex and fast-paced business environment, the ability to interpret and solve queuing-related problems is a critical success factor for improving service quality, minimizing costs, and boosting operational efficiency. Thus, mastering queuing theory is not just a technical skill—it is a strategic competency essential for building a resilient and effective management approach.

Lecture 10 : Cost analysis and break-even point

Introduction

In the modern business world, Cost-Volume-Profit (CVP) analysis is a fundamental tool for understanding the relationship between costs, revenues, and sales volume. Among the methods derived from CVP analysis, the break-even point stands out as a crucial tool for businesses. The break-even point is the point at which revenues equal costs, meaning the company neither makes a profit nor incurs a loss at this point. Thus, break-even analysis is one of the practical applications of CVP analysis, as it helps companies determine the optimal sales volume required to achieve a balance between revenues and costs. In this session, we will focus on the break-even point, how to calculate it, and its significant impact on making strategic decisions related to pricing and production.

Lecture Objectives

- Understand the concept of the break-even point and its importance in analyzing the relationship between costs, revenues, and sales volume.
- Learn how to calculate the break-even point using the appropriate formulas.
- Understand the impact of fixed and variable costs on the break-even point and how to determine the required sales volume to achieve a balance between revenues and costs.
- Apply break-even analysis in making strategic decisions related to pricing and production volume.
- Learn how to use graphical representation to show the relationship between revenues, costs, and the break-even point.

1.Cost analysis

In the rapidly changing business world, the ability to make informed decisions based on accurate data is one of the key factors contributing to the success and sustainability of companies. Cost analysis is an important tool that helps businesses assess their expenditures and improve their financial strategies. By studying the costs associated with production and services, companies can understand financial patterns, uncover opportunities for cost savings, and enhance operational efficiency.

1.1. Definition of Cost analysis

Cost analysis entails the systematic evaluation and assessment of a company's expenditures associated with the production of goods and services. It entails monitoring the comprehensive expenses related to operations, encompassing both direct material costs and indirect overhead expenditures. Cost analysis furnishes enterprises with critical insights into their expenditure patterns, enabling them to assess the value derived from their investments, pinpoint possible cost-saving options, and enhance resource allocation. (ePROMIS, 2024)

1.2. Importance of cost analysis

It is important for company executives to make educated judgments to attain success. Comprehending the company's cost structure is essential for establishing budgets, predicting financial results, and formulating long-term growth strategies. Cost analysis is essential in this process since it offers insights into the company's expenditures and identifies areas for performance enhancement. As competition escalates and profit margins diminish, acquiring precise and current information is increasingly vital. Cost analysis assists enterprises in attaining various strategic objectives through the following means: (ePROMIS, 2024)

- Identify Principal Cost Drivers: Ascertain the particular elements that lead to escalating expenses and evaluate whether the organization is receiving adequate value in return.
- Assess Alternative Solutions: Investigate possible savings by examining various suppliers, materials, or methodologies.
- Enhance Budget Planning: Utilize comprehensive cost data to develop more precise budgets and projections.
- Enhance Profitability: Examine expenditure trends to pinpoint inefficiencies and opportunities for cost reduction.
- Augment Competitive Edge: Maintain a competitive advantage by implementing data-driven decisions that enhance operational efficiency and profitability.

Cost analysis is a critical first step in understanding a company's financial position and achieving its strategic objectives. Based on this data, companies can make more accurate decisions to increase profitability and efficiency. In this context, Cost-Volume-Profit (CVP) analysis serves as a vital tool for analyzing the relationship between costs, revenues, and volume, helping to determine the break-even point, where revenues equal costs, resulting in neither profit nor loss. This analysis enhances a company's ability to identify the most suitable strategies for growth and profitability in a changing economic environment. In the next section, we will examine how to apply CVP analysis and the break-even point to improve financial performance and make effective strategic decisions.

2. Cost-Volume-Profit (CVP) analysis

In the modern business world, Cost-Volume-Profit (CVP) analysis is a crucial tool that helps companies understand the relationship between costs, revenues, and sales volume. This analysis is used to assess the impact of activity changes on costs and profits, enabling businesses to make strategic decisions based on accurate information about their costs and revenues. Through CVP analysis, companies can identify the point where revenues equal costs, known as the break-even point, helping them understand how changes in sales volume affect profitability.

2.1. Definition of Cost-Volume-Profit (CVP) analysis

Cost-Volume-Profit (CVP) analysis investigates the correlations between variations in activity and alterations in total sales income, expenses, and profit. It may offer valuable insights, especially for a corporation that is starting operations or encountering challenging economic circumstances. Cost-Volume-Profit (CVP) analysis facilitates the identification of the requisite number of product units that must be sold to achieve a break-even point, wherein all costs, encompassing both fixed and variable expenses, are fully offset by total sales income. It enables the enterprise to evaluate the impact on profits of different alterations in operating costs and revenues, such as a decrease in selling price or an escalation in fixed costs; to ascertain the sales volume necessary to attain a designated profit level, and to identify the extent to which the current sales level can decline before incurring losses. It is essential to acknowledge that CVP research relies on certain assumptions regarding the operational environment of the firm. This article initially outlines the assumptions foundational to comprehending CVP analysis, subsequently defines crucial terminology and enumerates frequently utilized equations, and ultimately closes with a straightforward calculation example. (Kelly, 2025)

2.2. Assumptions of CVP analysis

For CVP analysis to be effective, the underlying assumptions must be acknowledged. These assumptions establish the parameters for analyzing the connections among sales volume, expenses, and profits. The assumptions behind the use of CVP analysis are outlined below: (Kelly, 2025)

-All variables stay unchanged except for volume.

This premise indicates that volume is the sole determinant capable of influencing costs and profits. Factors such as enhanced manufacturing efficiency, altered sales composition, and price fluctuations are excluded from consideration.

-Either a single product is being manufactured or there exists a stable sales mix.

Building on the prior premise, CVP analysis is applicable just when a single product is assessed or, in cases involving several items, when they are consistently offered in same quantities or combinations.

-Total costs and total revenue represent linear functions.

This assumption indicates that the variable cost per unit and the selling price per unit remain constant, unaffected by discounts.

-Profits are determined utilizing variable (marginal) costs.

Variable costing enhances profit analysis by distinguishing between variable and fixed costs, using fixed expenses as a period expenditure instead of allocating them to goods.

-Expenses may be precisely categorized into fixed and variable components.

This constitutes a fundamental prerequisite of variable costing. The recommendation is that semi-variable expenses may be precisely allocated using strategies like the high-low method.

-The analysis is confined to the pertinent range.

The applicable range refers to a sales volume interval (e.g., between 10,000 and 80,000 units) during which the selling price and variable cost per unit stay unchanged. CVP analysis is inapplicable outside the confines of this sales volume range (i.e., sales below 10,000 units or above 80,000 units).

-The analysis is limited to a short-term perspective.

CVP analysis investigates the correlation among sales volume, costs, and profit during a one-year period, during which it is posited that altering selling prices, variable costs, and fixed expenses would be challenging, aligning with the aforementioned assumptions.

CVP analysis is a vital tool for understanding how changes in sales volume impact profits and costs, and it assists businesses in making informed decisions regarding pricing and production strategies. By understanding the assumptions behind this analysis, companies can improve their operational efficiency and achieve their financial objectives. In this context, Break-Even Analysis is one of the key applications of CVP analysis, as it helps determine the sales volume required to balance revenues with costs, which is an essential step towards achieving sustainable profitability. In the next section, we will explore how to apply Break-Even Analysis and CVP analysis to enhance financial performance and make effective strategic decisions.

3. Break-even analysis

In this section, the break-even point will be defined and its concept explained. Additionally, its graphical representation will be discussed, along with the formula used to calculate it and how to determine it accurately using mathematical relationships. The goal is to understand how to calculate the number of units required to achieve a balance between revenues and costs, enabling companies to make informed strategic decisions.

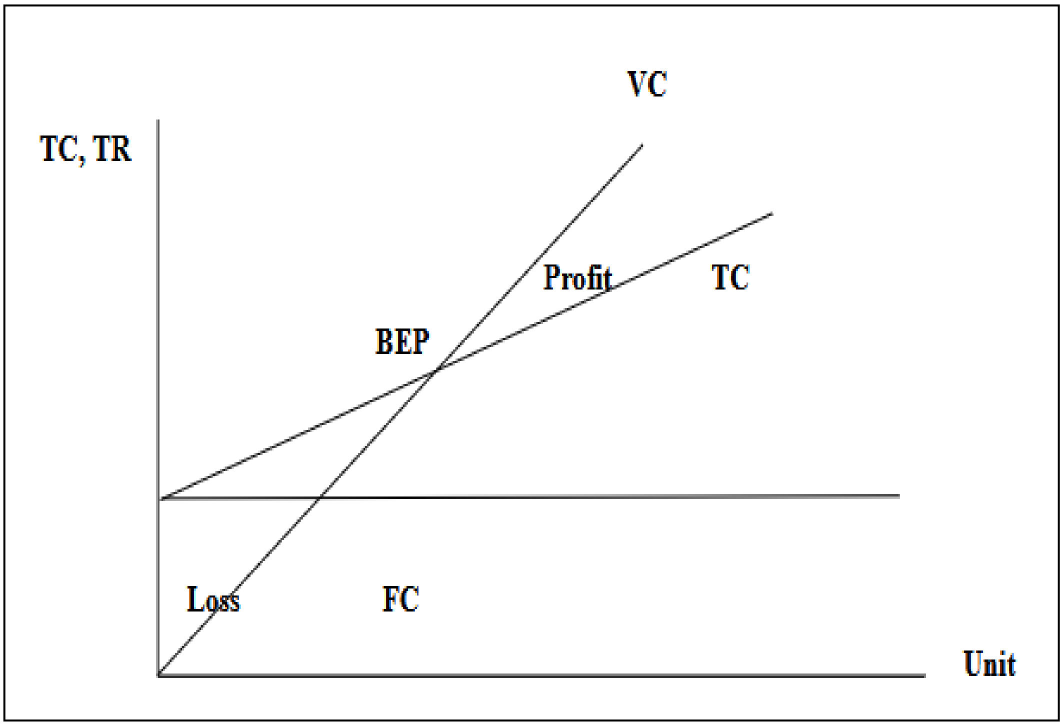
3.1.Definition

Break-even analysis is a fundamental method used to ascertain the juncture at which a corporation attains financial equilibrium, signifying that it neither generates profit nor sustains loss, applicable to a certain product, product line, manufacturing facility, or the entire enterprise. This study differentiates between fixed costs, which remain constant regardless of volume (such as machinery, buildings, or R&D expenditures), and variable costs, which vary with output volume, such as raw material expenses. The break-even point is established by identifying the volume at which total income matches the aggregate of fixed and variable costs, given a constant price. Most firms have a defined profitability target, indicated by the disparity between total revenue and total costs, signifying the objective of attaining a particular profit. (McGee, 2014)

3.2. The graphical representation of the break-even point

The following chart illustrates the break-even point and its relationship with both fixed and variable costs

Fig. 10.1



Source: (Oppusunggu, 2020, p. 214)

The graph provides a clear understanding of when the company begins to generate profit after covering its costs. It also highlights the significant impact of increasing sales volume on profitability, as any further increase in sales beyond the break-even point results in profits. It is important for this type of chart to assist companies in determining the precise sales targets that must be achieved to cover both fixed and variable costs, thereby enabling strategic decisions regarding production and pricing.

3.3. The objective of break-even analysis

The primary objective of break-even analysis extends beyond only identifying the break-even point. The dynamics of costs and profits across varying activity levels are crucial for management, and this insight may be obtained through break-even analysis. The phrase "Cost-Volume-Profit" (CVP) analysis is frequently used due to its emphasis on variations in connections across diverse activity levels. (Collis & Hussey, 1999)

3.4. Break-even Point (BEP) Formula

The purpose of the break-even point is to determine the number of units that must be sold so that revenues equal costs, meaning the company neither makes a profit nor incurs a loss. The following formula is used to calculate the break-even point: (Oppusunggu, 2020, p. 214)

$$Q = \frac{FC}{P - vc}$$

Where:

- Q: The number of units that need to be sold to achieve the break-even point. In other words, it is the number of products that must be sold to cover all costs without making a profit or a loss.
- FC: Fixed costs, which are costs that remain constant regardless of the production or sales level, such as rent and fixed salaries.
- P: Selling price per unit. This is the amount at which the product or service is sold per unit.
- vc: Variable cost per unit. This includes costs that change with the production or sales volume, such as the cost of raw materials or variable wages.
- P - vc: Contribution margin. This represents the difference between the selling price per unit and the variable cost per unit, showing the amount that contributes to covering fixed costs and generating profit.

Break-even analysis provides businesses with vital insights into how changes in sales volume impact profitability. By understanding the contribution margin per unit and using the break-even point formula, companies can determine appropriate strategies to ensure the balance between revenues and costs and achieve profits. This tool helps businesses in financial planning and making informed decisions related to sales, enhancing their ability to respond to economic changes and successfully achieve strategic objectives.

4. Example

A company specializing in selling perfumes sells a single product for 350 Algerian Dinars per unit. The variable cost per unit is 150 Algerian Dinars, and the company's fixed costs are estimated at 1,200,000 Algerian Dinars annually. The company wants to know the break-even point, i.e., the number of units it needs to sell to cover all costs (both fixed and variable) and achieve a balance between revenues and costs, without making a profit or a loss.

Questions:

- What is the break-even point (number of units) that the company must sell to cover all costs?
- If the company achieved sales of 4,500 units, what will be the company's profit?
- What is the effect of increasing the fixed costs to 1,500,000 Algerian Dinars on the break-even point?

Solution:

1. Calculating the break-even point (number of units):
Using the given data:

- Selling price per unit = 350 Algerian Dinars
- Variable cost per unit = 150 Algerian Dinars
- Fixed costs = 1,200,000 Algerian Dinars

We use the following formula to calculate the break-even point:

$$Q = \frac{FC}{P - vc}$$

Where:

- Q: The number of units required to achieve the break-even point.
- FC: Fixed costs.
- P: Selling price per unit.
- vc: Variable cost per unit.

Break-even point:

$$Q = \frac{1{,}200{,}000}{350 - 150} = \frac{1{,}200{,}000}{200} = 6000 \text{ units}$$

So, the break-even point is 6,000 units.

2. If the company sold 4,500 units:

- Sales Revenue = $350 \times 4500 = 1{,}575{,}000$ Algerian Dinars
- Total Costs = $1{,}200{,}000 + (150 \times 4500) = 1{,}200{,}000 + 675{,}000 = 1{,}875{,}000$ Algerian Dinars
- Profit = $1{,}575{,}000 - 1{,}875{,}000 = -300{,}000$ Algerian Dinars

Therefore, if the company sold 4,500 units, it would incur a loss of 300,000 Algerian Dinars.

3. Effect of increasing fixed costs to 1,500,000 Algerian Dinars on the break-even point:

New Fixed Costs = 1,500,000 Algerian Dinars

We recalculate the break-even point:

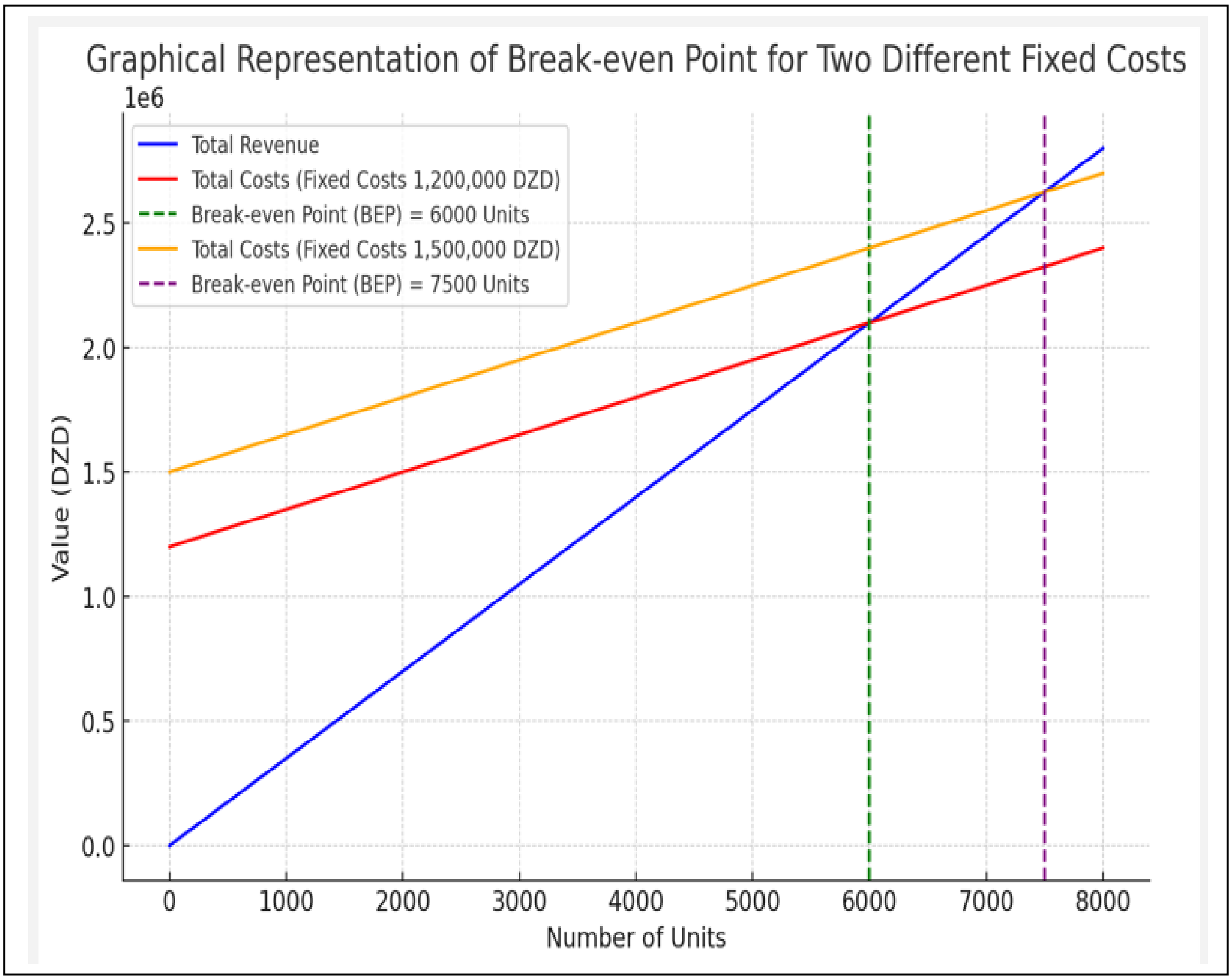
$$Q = \frac{1{,}500{,}000}{350 - 150} = \frac{1{,}500{,}000}{200} = 7{,}500 \text{ units}$$

Therefore, with increased fixed costs of 1,500,000 Algerian Dinars, the company needs to sell 7,500 units to cover the new fixed costs.

3.Graphical Representation:

The following graphs represent the relationship between revenues and costs, with the break-even point shown at 6,000 units. The graph illustrates the effect of increasing unit sales on profits, where profits begin to emerge after the break-even point.

Fig. 10.1



Source: Author's own work.

Conclusion

In conclusion, cost analysis, with a focus on the break-even point, is one of the important methods that help in understanding the relationship between costs, revenues, and sales volume. By determining the break-even point, companies can make precise strategic decisions regarding pricing and production volume, which enhances their ability to achieve profitability. This tool also helps companies improve their financial strategies and set realistic sales targets, thereby enabling them to achieve growth in a competitive business environment.

Lecture 11 : game theory

Introduction

Game theory is a fundamental analytical tool used to understand competitive strategies in various contexts, involving interactions between players who aim to maximize their benefits based on their decisions, while considering the impact of others' choices. Its applications extend to many fields, including economics, sports, and politics, offering a means to understand situations that require strategic decision-making under uncertainty. In this lecture, we will explore the theoretical foundations of game theory, focusing on key concepts such as players, strategies, and payoff matrices, and how different methods are employed to solve these games, including the simplex method and techniques related to mixed strategy games.

Lecture Objectives

- Understanding Game Theory: Familiarize students with the theoretical foundations of game theory and its application in various fields.
- Core Concepts: Grasp essential concepts like players, strategies, and payoff matrices used to analyze strategic interactions between different parties.
- Analyzing Game Strategies: Learn how to determine optimal strategies using mathematical approaches like the Maximin and Minimax criteria.
- Applying the Simplex Method: Understand how to use the simplex method to solve games involving more than two strategies for players.
- Solving Mixed Strategy Games: Explore how linear programming is used to transform games into linear programs and solve them effectively.

1.Background

A number of scholars suggest that the concept of “skills” is not confined to its conventional association with technical or cognitive competence, but rather extends to a competitive dimension that inherently involves strategic interaction among rival parties. From this perspective, game theory emerges as a suitable conceptual framework for analyzing such skills in contexts that require interdependent decision-making, where each player's choice affects the outcome for others. This theory first appeared during World War I, when the French mathematician Émile Borel introduced, in 1921, a mathematical formulation aimed at analyzing decisions of a strategic nature. Later, in 1928, the Hungarian-American mathematician John von Neumann advanced this conceptualization into a more comprehensive mathematical structure, thereby laying the foundation for what would later become known as decision theory. His collaboration with economist Oskar Morgenstern subsequently expanded the application of game theory from abstract mathematics to the domain of applied economics, transforming it into an analytical tool for evaluating the effectiveness of decisions in various competitive scenarios—whether at the level of individuals or institutions. (Said, 2007, p. 270)

Game theory is fundamentally concerned with analyzing interactions between rational players, each of whom aims to maximize their own payoff in environments marked by uncertainty and incomplete information. Understanding a game requires identifying all its components, yet in real-world situations, the ambiguity and complexity of such components often make this task challenging. This necessitates reliance on probabilistic estimates to guide decision-making under uncertainty. As a result, game theory has proven capable of addressing a broad range of strategic problems encountered by athletes, economists, and policymakers alike, particularly those involving competitive conditions in which each player's outcome is influenced by the choices of others. For instance, a more advanced player in a given game has a better chance of prevailing if they are able to employ strategies that enhance their decision-making in a way that optimizes outcomes—especially to the extent that their own gain corresponds with a reduction in the opponent's payoff. This aligns with the logic underpinning zero-sum games, where one player's profit typically mirrors the other's loss. (Said, 2007, p. 270)

2.Basic concepts

Before delving into the practical applications of game theory, it is essential to clarify a set of core concepts and terminologies that form the theoretical foundation of this analytical framework, as follows: (Said, 2007, p. 271)

- ✓ Player: A player refers to any participant in the game, and is often described in academic literature as an "opponent" due to the competitive nature of the setting. The player is considered an independent analytical entity endowed with the cognitive ability to make decisions based on a set of available strategic options, thereby influencing the course and outcomes of the game.
- ✓ Strategy: A strategy is the plan adopted by a player to select a particular action from among various possible alternatives. These alternatives may be finite or open-ended, and are evaluated based on their expected returns. The optimal strategy is chosen with the goal of maximizing profit or minimizing loss, depending on the dynamics of the specific game.
- ✓ Payoff Matrix: The payoff matrix is a mathematical representation of the potential outcomes resulting from the interaction of players' chosen strategies. It consists of rows and columns, with each cell corresponding to a specific outcome determined by the strategic choices of both players. These outcomes are quantified in terms of gains or losses and are typically evaluated relatively—what one player gains is often balanced by what the other loses. The values within the matrix are structured to reflect the direct interaction between players' strategic decisions. In many cases, especially when a player has limited computational capacity, they must rely on estimating the likely strategies of their opponent. This leads to the formulation of various plausible scenarios, which, in turn, inform the player's decision-making process, adding both probabilistic and strategic dimensions to the game.

The following is the general form of the payoff matrix between players A and B: (Beladjoz, 2010, p. 300)

$$\begin{array}{ccccc} & B_1 & B_2 & \cdots & B_n \\ A_1 & g_{11} & g_{12} & \cdots & g_{1n} \\ A_2 & g_{21} & g_{22} & \cdots & g_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_m & g_{m1} & g_{m2} & \cdots & g_{mn} \end{array}$$

Where:

- A_1, A_2, ..., A_m: Represent the strategies of player A (Player A).
- B_1, B_2, ..., B_n: Represent the strategies of player B (Player B).
- g_{ij}: Represents the outcome that player A receives when choosing strategy A_i while player B chooses strategy B_j.

In summary, the key concepts of *Player*, *Strategy*, and *Payoff Matrix* are foundational to understanding game theory. The player, as an independent decision-maker, navigates through a set of strategies aimed at maximizing their own benefits or minimizing potential losses. The strategic choices made by each player are then captured in the payoff matrix, which serves as a crucial tool for visualizing the outcomes of various interactions. Ultimately, these concepts help to frame the competitive and strategic dynamics in games, allowing players to make informed decisions in uncertain and interactive environments. Understanding these principles is essential for analyzing and predicting behavior in competitive situations, where outcomes depend on the interplay between different decision-makers

3. Rules of Games

Strategic games in game theory are governed by a set of basic rules that define the interaction between players. These can be summarized in the following points: (Said, 2007, pp. 271-272)

- Number of Players: Each game involves a specific number of participants (players), where each player represents an independent party making decisions within a competitive environment.

- Variety of Strategies: Each player has a set of possible strategies from which they can choose. These strategies vary depending on the nature and structure of the game and may be limited or open-ended.

- No Communication Between Players: In classical games, it is assumed that there is no form of communication or coordination between players while making their decisions. This means that the choice made by one player regarding a particular strategy is not known to the other player at the time of execution.

- Simultaneous Moves: All decisions are made by players simultaneously, or without a time sequence that allows one player to observe the choice of the other before making their decision. This is known as simultaneous-move games.

- Individual Control Over Decisions: Each player exercises a degree of control over their decisions and is expected to choose the most appropriate strategy based on their evaluation of available options, aiming to maximize their return given their expectations about the opponent's behavior.

- Interdependent Decisions: The outcome achieved by each player is not solely based on their own choices but is also influenced by the decisions of the other player. Therefore, decision-making in this context is not isolated but requires consideration of mutual interaction, while maintaining the independence of each player in choosing their strategy.

4. Solution Methods in Game Theory

There are many types of games based on the number of strategies available to each player and the solution method used. In all cases, the game is initially tested to determine whether it is stable or not. If it is stable, it means that the solution has been reached. If it is unstable, one of the appropriate methods for solving it is chosen.

Based on the number of strategies, there are two-player games of the 2×2 type, where each player has only two strategies. If the game is unstable, it can be solved using the graphical method. This is the same method used if one player has two strategies and the other has more than two strategies, such as in the $M \times 2$ or $2 \times N$ forms. This is after applying the dominance method, ensuring that the number of strategies remains unchanged.

However, if the game is between two players, each having more than two strategies, it is subjected to the dominance method. If it remains unchanged, meaning each player has more than two strategies, which is of the $N \times M$ form, it is then solved using the simplex method for one player, and the optimal solution for the other player is obtained using the dual program.

5. Two-Person Zero-Sum Game

A two-person zero-sum game is based on a set of fundamental assumptions, which can be summarized as follows: (Beladjoz, 2010, p. 303)

- The game involves only two players or teams, where the gain of one player is exactly equal to the loss of the other, meaning that the total sum of gains and losses in each round equals zero.
- Outcomes are directly linked to the performance of both players; if one player earns a certain profit, it represents an equivalent loss to the opponent, and vice versa.
- Each player has a clear objective to pursue, such as maximizing market share or increasing profits.
- Each player has a limited number of available strategies, from which they can choose during the course of the game.
- It is assumed that both players have full knowledge of each other's strategies, implying that the game operates in an environment of complete information.
- The potential payoff for each player depends on the interaction between their own strategy and that of the opponent, meaning that the outcomes are determined by the strategic decisions of both parties.

It can be solved by calculating the following:

5.1. Maximin Criterion

The Maximin approach is based on a conservative strategy. Each player assumes that the opponent will act in a way that minimizes their own payoff. Therefore, the player:

- Identifies the minimum payoff for each of their available strategies.
- Then chooses the strategy with the highest value among these minima.

This method ensures that the player secures the best of the worst-case outcomes.

5.2. Minimax Criterion

Alternatively, the Minimax approach considers the opponent's perspective, assuming they will attempt to maximize their own gain. In this case:

- The maximum payoff is identified for each column in the payoff matrix (i.e., for each strategy of the opponent).
- The player then selects the smallest value among these maxima.

5.3. Saddle Point and Game Value

When the values obtained from the Maximin and Minimax approaches are equal (i.e., Maximin = Minimax), the game reaches what is known as a saddle point. At this point, both players have optimal strategies, and neither has an incentive to deviate. The game is said to be stable, and the common value of both criteria represents the value of the game, which lies between the highest of the minimum payoffs and the lowest of the maximum payoffs.

Example1:

Consider a two-player zero-sum game in which each player has two available strategies:

- Player A chooses between strategies A_1 and A_2
- Player B chooses between strategies B_1 and B_2

The payoff matrix for Player A is given as follows:

	B_1	B_2
A_1	3	2
A_2	4	3

Questions:

- Is the game stable? Does it contain a saddle point?
- What is the value of the game?
- What is the optimal strategy for each player?

The Solution:

Step 1: *Calculating the minimum value for each row (Player A's strategies):*

$Min(A_{1j}) = Min(3, 2) = 2$

$Min(A_{2j}) = Min(4, 3) = 3$

Then calculate:

$$\text{Maximin } A_{1j} = Max\ (2,3)=3$$

Step 2: *Calculating the maximum value for each column (Player B's strategies):*

$Max(B_{1i}) = Max(3, 4) = 4$

$Max(B_{2i}) = Max(2, 3) = 3$

Then calculate:

$$\text{Minimax } B_{1j} = Min\ (4,3)=3$$

Step 3: Test for Stability

Since: Maximin A_{1j} = Minimax B_{1j} The game is stable, and a saddle point exists.

Thus:

- Saddle Point: Occurs at A_2, B_2
- Value of the Game: V=3

- Optimal Strategies:
  - Player A should play strategy A_2
  - Player B should play strategy B_2

6. Solution method of a Game with Mixed Strategies

Mixed strategies are employed in game theory when a game lacks stability—specifically, when there is no identifiable equilibrium point within the pure strategies of the players. In such cases, conventional solution methods are insufficient, and alternative techniques must be applied.

When the game takes the form of $2 \times N$ or $M \times 2$, where one player has only two strategies, the graphical method is typically used. This method relies on visual representation and is suitable for analyzing such simplified structures.

In contrast, for games of the form $M \times N$, where both players have more than two strategies, a more advanced approach is required. Initially, the dominance method is applied to reduce the number of strategies if possible. If this simplification does not occur, the game is then solved using the simplex method, which is a linear programming technique used to determine the optimal strategy for at least one of the players.

6.1. Graphical Method for Solving Games with Mixed Strategies of Type (M×2) or (2×N)

The graphical method for solving games of type $N \times 2$ or $2 \times M$ relies on graphical representation by defining unknown variables representing the probabilities of selecting each strategy among the two available ones. Based on these probabilities, equations are constructed to represent the strategies of the opposing player. These equations take the form of straight lines and are plotted on a graph with a horizontal axis representing probability values ranging from 0 to 1, and two vertical axes — one at probability 0 and the other at probability 1.

The solution region is then determined based on the player being analyzed: MAXMIN for Player A or MINMAX for Player B. The optimal point, representing the best value of the game, is identified by finding the intersection point of the straight lines that define the boundaries of the feasible solution region. This intersection corresponds to the highest value of the game, and the associated probability is obtained by solving the equations of the intersecting lines.

The following example illustrates this process in detail.

Example 2

Suppose we have a two-player game:

- The row player (A) has two strategies: A_1, A_2
- The column player (B) has three strategies: B_1, B_2, B_3

The payoffs for player A are represented in the following payoff matrix:

	B_1	B_2	B_3
A_1	3	4	2
A_2	2	1	5

Required:
1. Determine whether the game is stable or not.

2. Since the game is not stable, use the graphical method to solve it:

a. Define the expected payoff functions for player A as functions of X where X is the probability of choosing strategy A_1

b. Create a table showing the values of these functions at X=0 and X=1

c. Plot the three payoff functions graphically

d. Identify the intersection points of the functions and determine the value of the game from the graph

e. Conclude with the optimal mixed strategy for player A

3. Determine of Player B's Optimal Strategy

The solution

1. Determining whether the game is stable or not
Step1: Calculating Maximin$\boldsymbol{A_{IJ}}$
Min(A_{1J}) = Min(3, 4, 2) = 2
Min(A_{2J}) = Min(2, 1,5) = 1
Then MaximinA_{IJ} $= \text{Max}\{2,1\} = 2$
Step3: Calculaing t Minimax$\boldsymbol{B_{ij}}$
Max(B_{1i}) = Max(3, 2) = 3
Max(B_{2i}) = Max(4, 1) = 4
Max(B_{3i}) = Max(2, 5) = 5
Then calculate:

$$\text{Minimax} = \text{Min}\{3,4,5\} = 3$$

Step 3: Test for Stability
Since: MaximinA_{IJ} $\neq$ MinimaxB_{1i}The game is not stable

2. Since the game is not stable, we use the graphical method to solve it as follows:

a. Defining the expected payoff functions for player A as functions of X where X is the probability of choosing strategy A_1

Table. 11.1

Pure Strategy of Player B	Expected Payoff Function for Player A
B_1	$f_1(X) = 3X + 2(1 - X) = X + 2$
B_2	$f_2(X) = 4X + 1(1 - X) = 3X + 1$
B_3	$f_3(X) = 2X + 5(1 - X) = -3X + 5$

Source: Author's own work.

b. Function Values at $X = 0$ and $X = 1$

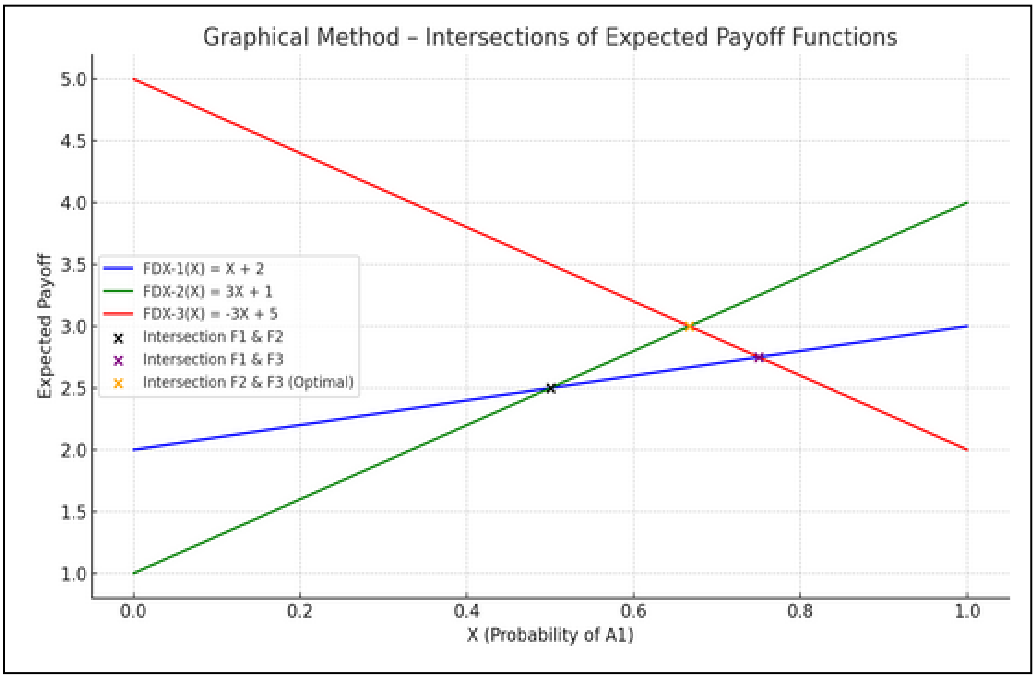
Table. 11.2

X	$f_1(X)$	$f_2(X)$	$f_3(X)$
0	2	1	5
1	3	4	2

Source: Author's own work.

c. The graph

Fig. 11.1



Source: Author's own work.

d. Intersection Points Between the Functions

Table. 11.3

Intersection of Functions	X	Payoff Value
$f_2(X) = f_3(X)$	$\frac{2}{3}$	3.0
$f_1(X) = f_3(X)$	$\frac{3}{4}$	2.75
$f_1(X) = f_2(X)$	$\frac{1}{2}$	2.5

Source: Author's own work.

The optimal solution occurs at X= 2/3, which yields the highest minimum payoff, thus:

- Play A_1 with probability $\frac{2}{3}$
- Play A_2 with probability $\frac{1}{3}$

Value of the game: 3

3. Determination of Player B's Optimal Strategy

Based on the result of Player A that choose optimal solution with Functions 2 and 3, thus we use the strategies 2 and 3 for Determining the propabilities of these strategies as follows:

Table. 11.4

	B_1	B_2	B_3
A_1	3	4	2
A_2	2	1	5
Probabilities		Y_2	Y_3=1- Y_2

Source: Author's own work.

Based on the table above, where the focus was on strategies 2 and 3 to calculate the probabilities of Player A's strategies, these are the strategies chosen by Player B. We will now calculate their corresponding probabilities as follows:

$4Y_2 + 2Y_3 = 4Y_2 + 2(1 - Y_2) = 2Y_2 + 2$

$Y_2 + 5Y_3 = Y_2 + 5(1 - Y_2) = -4Y_2 + 5$

Then $2Y_2 + 2 = -4Y_2 + 5 \rightarrow Y_2 = 1/2$

Thus: $Y_1 = 1/2$

6.2. Dominance Method for Solving Games with Mixed Strategies of Type (M×2) or (2×N)

Dominance strategies rely on eliminating any column for which there exists another column containing values that are all greater than the corresponding values in the first one. Similarly, a row is eliminated if all its values are less than the corresponding values in another row. This process continues until no more dominant rows or columns exist—that is, until there are no rows or columns whose values are strictly greater than (or dominate) the corresponding values in other rows or columns when compared pairwise.

The following example illustrates this process in detail A_1, A_2, A_3 B_1, B_2, B_3

Example 3

Let us assume we have two players:

- Player A has 3 strategies: A_1, A_2, A_3
- Player B has 3 strategies: B_1, B_2, B_3

The payoff matrix for Player A (since it is a zero-sum game) is as follows:

	B_1	B_2	B_3
A_1	4	2	3
A_2	3	1	2
A_3	5	3	4

Step 1: Dominance Check for Rows (Player A's Strategies)

Compare the rows:

- Compare $A_1 = [4,2,3]$ and $A_2 = [3,1,2]$ Since all elements of A_1 are greater than the corresponding elements of A_2, A_1 dominates A_2, so we eliminate A_2.

The reduced matrix becomes:

	B_1	B_2	B_3
A_1	4	2	3
A_3	5	3	4

Step 2: Dominance Check for Columns (Player B's Strategies)
Compare the columns:

- $B_2 = [2,3]$, $B_3 = [3,4]$ Since all elements of B_3 are greater than the corresponding elements of B_2, B_3 dominates B_2, so we eliminate B_2.

The final reduced matrix is:

	B_1	B_3
A_1	4	3
A_3	5	4

We now have a simplified 2×2 game, which can be solved using mixed strategies.

6.3. Solution method of a Game with (M×N) Form Using Linear Programming

In this section, we will explain how to transform the game into a linear program within the context of game theory. Linear programming is a powerful tool for analyzing players' strategies in games that involve competitive interactions between players, where each player makes strategic decisions aimed at maximizing their gains or minimizing their losses based on the strategies of the other.

6.3.1. Assumptions

Let's assume that X_i represents the probabilities of Player A_i, $\forall\, i \in \{1,2,3,\ldots.m\}$
and Y_J represents the probabilities of Player B_j, $\forall\, J \in \{1,2,3,\ldots.n\}$ (Beladjoz, 2010, p. 308)
and the corresponding payoff matrix is as follows:

Table. 11.5

		B			
	probability	Y_1	Y_2	....	Y_m
A	X_1	a_{11}	a_{12}		a_{1m}
	X_2	a_{21}	a_{22}		a_{2m}
	.....	....			
	X_n	A_{n1}	A_{n2}	....	A_{nm}

Source: (Beladjoz, 2010, p. 309)

The solution of mixed strategies based on MAXMIN for player A and MINMAX for player B can be expressed mathematically as follows:

Player A chooses the strategy based on the maximin criterion as follows: (Beladjoz, 2010, p. 309)

$$\underset{x_i}{\text{Max}}\left[MIN\left(\sum_{i=1}^{n} a_{i1}x_i\,, \sum_{i=1}^{n} a_{i2}x_i\,, \ldots\ldots\ldots, \sum_{i=1}^{n} a_{im}x_i\right)\right]$$

Where:

$\sum_{i=1}^{m} x_i = 1\,,\, x_i \geq 0$

It is called the maximin expected payoff.

Player B chooses the strategy based on the minimax criterion as follows:

$$\underset{y_j}{\text{Min}}\left[MAX\left(\sum_{j=1}^{m} a_{i1}y_i\,,\sum_{j=1}^{m} a_{i2}y_i\,,\dots\dots\dots,\sum_{j=1}^{m} a_{in}y_i\right)\right]$$

Where:

$\sum_{i=1}^{m} y_i = 1\,,\, y_i \geq 0$

It is called the minimax expected payoff.

6.3.2. Conditions

To construct the mathematical model for any game, (Beladjoz, 2010, p. 324):

- ✓ Objective Function and Constraints Based on Value $\boldsymbol{V}$:

The objective function in this case is V, representing the game value. The constraints are handled differently based on the value of V, as outlined below:

- Case 1: When $\boldsymbol{V} > 0$:

In this case, the constraints can be simplified by dividing all the terms in the constraints by v. This simplification ensures that the constraints are clearer and easier to solve. Since v is positive, there is no need to reverse the inequality signs.

- Case 2: When $\boldsymbol{V} < 0$:

If the value of v is less than zero, the inequality signs representing the constraints need to be reversed. This means that the constraints, which were originally representing the minimum, are now representing the maximum, reflecting a change in the players' preferences.

- Case 3: When $\boldsymbol{V = 0}$:

If the value of V equals zero, a positive constant k is added to all elements of the payoff matrix. This ensures that the modified game value is always greater than zero. After adjusting the values, the original game value can be retrieved by subtracting k from the modified game value.

- ✓ Players' Behavior Based on Game Value:
- Player A seeks to achieve the game value or greater.
- Player B seeks to achieve the game value or smaller.

- ✓ Constraints in the Linear Program:

When V > 0, the constraints remain as they are in the original equations after applying the simplification

6.3.3. Transforming the Game into a Linear Program

Based on the above, the game is transformed into a linear program as follows:

-Player A attempts to achieve the game value or more based on the **maximin** criterion. Therefore, the constraints are as follows: (Beladjoz, 2010, pp. 324-325)

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_{1n} \geq V$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_{2n} \geq V$$

....

....

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_n \geq V$$

Since the sum of probabilities equals 1, we have the following constraint:

$$x_1 + x_2 + \cdots + x_n = 1$$

By dividing by the game value V, we get the following:

$$a_{11}\frac{x_1}{V} + a_{12}\frac{x_2}{V} + \cdots + a_{1n}\frac{x_{1n}}{V} \geq 1$$

$$a_{21}\frac{x_1}{V} + a_{22}\frac{x_2}{V} + \cdots + a_{21n}\frac{x_{2n}}{V} \geq 1$$

....

....

$$a_{n1}\frac{x_1}{V} + a_{n2}\frac{x_2}{V} + \cdots + a_{nm}\frac{x_n}{V} \geq 1$$

$$\frac{x_1}{V} + \frac{x_2}{V} + \cdots + \frac{x_n}{V} = 1$$

By setting $X_i = \frac{x_1}{V}$, we get:

For Player A:

$$\text{Max}\{= Z = \text{Min}\frac{1}{V} = \text{Min}(X_1 + X_2 + \cdots + X_n)\}$$

Subject to :

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \geq 1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{21n}x_n \geq 1$$

....

....

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{n}x_n \geq 1$$

$$X_i \geq 0, \forall\, i \in \{1,2,3, \dots . n\}$$

For Player B:

$$\text{Min}\{= Z' = \text{Max W} = \text{Min}(Y_1 + Y_2 + \cdots + Y_n)\}$$

Subject to : $a_{11}Y_1 + a_{12}Y_2 + \cdots + a_{1n}Y_m \leq 1$

$$a_{21}Y_1 + a_{22}Y_2 + \cdots + a_{21n}Y_m \leq 1$$

....

....

$$a_{n1}Y_1 + a_{n2}Y_2 + \cdots + a_nY_m \leq 1$$

$$Y_i \geq 0, \forall\, i \in \{1,2,3, \ldots .n\}$$

$$W = \frac{1}{V}, Y_i = \frac{y_i}{V}$$

Example 4

Assume we have the following game between players A and B in a 3x3 matrix as follows:

Table. 11.6 (Beladjoz, 2010, p. 325)

Players		Player B		
	Strategies	B_1	B_2	B_3
Player A	A_1	3	-1	-3
	A_2	-3	3	-3
	A_3	-4	-3	3

Source: (Beladjoz, 2010, p. 325)

The task is to find a solution for this game. (Beladjoz, 2010, pp. 325-326)

Solution

1. Test whether the game is stable or not

Table. 11.7

Players		Player B			Max-Min
	Strategies	B_1	B_2	B_3	
Player A	A_1	3	-1	-3	-3
	A_2	-3	3	-3	-3
	A_3	-4	-3	3	-4
Min-Max	3	3	3		

Source: (Beladjoz, 2010, p. 326)

It is important to note that this game is considered unstable, meaning it does not have a Nash equilibrium. Additionally, the MAXMIN value equals -3, which indicates that the value of the game could be negative or zero. To overcome this issue, we add a constant value to each element in the matrix such that the constant is greater than or equal to 5. This is based on the smallest negative value in the matrix, which is -4. Let K=5. After making this adjustment, (Beladjoz, 2010, p. 326) the matrix becomes as follows:

Table. 11.8 (Beladjoz, 2010, p. 326)

Players		Player B		
	Strategies	B 1	B 2	B 3
Player A	A 1	3	-1	-3
	A 2	-3	3	-3
	A 3	-4	-3	3

(Beladjoz, 2010, p. 326)

Note that from the game matrix, the dominance or control method cannot be applied, and therefore, the game remains as it is.

2. Solve the game using the simplex method.

From the game matrix, the linear program for player B can be derived as follows:

$$MaxW = Y_1 + Y_2 + Y_3$$

$$\text{Subject to:}\begin{cases} 8Y_1 + 4Y_2 \ + 2Y_3 \ \leq 1 \\ 2Y_1 \ + 8Y_2 + 4Y_3 \ \leq 1 \\ Y_1 + \ 2Y_2 + 8Y_3 \ \leq 1 \\ Y_1, Y_2, Y_3 \geq 0 \end{cases}$$

Adabted from: (Ratoul, 2006, p. 80)

Converting to standard form:

$$MaxW = Y_1 + Y_2 + Y_3 + 0Y_4 + 0Y_4 + \ 0Y_6$$

$$\text{Subject to:}\begin{cases} 8Y_1 + 4Y_2 \ + 2Y_3 + \ Y_4 \qquad = 1 \ldots\ldots X_1 \\ 2Y_1 \ + 8Y_2 + 4Y_3 \qquad + Y_5 \ = 1 \ldots\ldots X_2 \\ Y_1 + \ 2Y_2 + 8Y_3 \qquad\qquad Y_6 \ = 1 \ldots\ldots X_3 \\ Y_1, Y_2, Y_3, Y_4, Y_5, Y_6 \ \geq 0 \end{cases}$$

The variables X_i used in the program are for extracting the optimal solution for the dual program, i.e., player A's program

Based on the standard form, the initial basic solution table is constructed as follows:

Table. 11.9

T_0	Coefficients	1	1	1	0	0	0			Determining the Pivot Row
Coefficients	Basic Variable	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	RHS	RHS/Pivot column	
0	Y_4	8	4	2	1	0	0	1	1/8	Pivot Row
0	Y_5	2	8	4	0	1	0	1	1/2	
0	Y_6	1	2	8	0	0	1	1	1	
	ΔW	1	1	1	0	0	0	0		
Determining the Pivot Column		Pivot Column								

Source : Author's own work

Since row Z contains positive values, table T_0 does not provide the optimal solution. By repeating the same steps as in the maximization case, we obtain the next table:

Table. 11.10

T_1	Coefficients	1	1	1	0	0	0			Determining the Pivot Row
Coefficients	Basic Variable	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	RHS	RHS/Pivot column	
1	Y_1	1	1/2	1/4	1/8	0	0	1/8	1/2	
0	Y_5	0	7	7/2	-1/4	1	0	3/4	3/14	
0	Y_6	0	3/2	31/4	-1/8	0	1	7/8	7/62	Pivot Row
	ΔW	0	1/2	3/4	-1/8	0	0			
Determining the Pivot Column				Pivot Column						

Source : Author's own work

Since row Z contains positive values, table T0 does not provide the optimal solution. By repeating the same steps as in the maximization case, we obtain the next table:

Table. 11.11

T_2	Coefficients	1	1	1	0	0	0			Determining the Pivot Row
Coefficients	Basic Variable	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	RHS	RHS/Pivot column	
1	Y_1	1	14/34	0	4/31	0	-1/31	6/62	6/28	
0	Y_5	0	196/31	0	-6/31	1	-14/31	11/31	11/196	Pivot Row
1	Y_3	0	6/31	1	-1/62	0	4/31	7/62	7/12	
	ΔW	0	11/31	0	-7/62	0	-3/31	13/62		
Determining the Pivot Column			Pivot Column							

Source : Author's own work

Since row Z contains positive values, table T0 does not provide the optimal solution. By repeating the same steps as in the maximization case, we obtain the next table:

Table. 11.12

T_3	Coefficients	1	1	1	0	0	0		Dual program (player A)
Coefficients	Basic Variable	Y_1	Y	Y_3	Y_4	Y_5	Y_6	RHS	
1	Y_1	1	0	0	1/7	-1/14	0	14/196=1/14	X_4
1	Y_2	0	1	0	-3/98	31/196	-1/14	11/196	X_5
1	Y_3	0	0	1	-1/98	-3/98	1/7	20/196=5/49	X_6
	ΔW	0	0	0	-10/98	-11/196	-1/14	45/196	
Dual program (player A)					X_1	X_2	X_3		

Source : Author's own work

Note: The shaded part of the table relates to the optimal solution of the dual program, i.e., player A's program.

Since all the values in row ΔZ are less than or equal to zero, table T_3 provides the optimal solution, which is as follows:

$Y_1 = 1/14$

$Y_2 = 11/196$

$Y_3 = 5/49$

W = 45/196.

Therefore, the solution to the original problem is as follows:

$$W = \frac{1}{V} \rightarrow V = \frac{1}{W}$$

And with the addition of K, it becomes as follows:

$$V = \frac{1}{W} - K = \frac{1}{45/196} - 5 = -\frac{29}{45}$$

$$Y_1{}^* = \frac{Y_1}{\mathrm{W}} = \frac{1/14}{45/196} = 14/45$$

$$Y_2{}^* = \frac{Y_2}{\mathrm{W}} = \frac{11/196}{45/196} = 11/45$$

$$Y_3{}^* = \frac{Y_2}{\mathrm{W}} = \frac{5/49}{45/196} = 20/45$$

3. Deriving the optimal solution for player A

For player A, his linear program is as follows:

$$MinZ = X_1 + X_2 + X_3$$

$$\text{Subject to:} \begin{cases} 8X_1 + 2X_2 + X_3 \geq 1 \ldots\ldots y_1 \\ 4X_1 + 8X_2 + 2X_3 \geq 1 \ldots\ldots y_2 \\ 2X_1 + 4X_2 + 8X_3 \geq 1 \ldots.Y_3 \\ X_1, X_2, X_3 \geq 0 \end{cases}$$

The variables Y_j used in the program are for extracting the optimal solution for the dual program, i.e., player A's program.

Based on the dual program, as seen earlier in linear programming, the linear program for player A can be derived as the dual program for player B as follows:

Z= 45/196

$X_1 = \frac{10}{98} = 5/49$

$X_2 = 11/196$

$X_3 = 1/14$

Thus, the original solution for player A's linear program is as follows:

$$Z^* = \frac{1}{Z} = \frac{1}{45/196} = 196/45$$

$$X_1{}^* = \frac{X_1}{\mathrm{Z}} = \frac{5/49}{45/196} = 20/45$$

$$X_2{}^* = \frac{X_2}{\mathrm{Z}} = \frac{11/196}{45/196} = 11/45$$

$$X_3{}^* = \frac{X_3}{\mathrm{Z}} = \frac{1/14}{45/196} = 14/45$$

Conclusion

In conclusion, game theory serves as a powerful tool for understanding strategic interactions in competitive environments. By exploring the core concepts and mathematical models associated with this theory, we can analyze and predict players' behavior in situations involving uncertainty. This lecture highlighted the importance of mathematical techniques such as the simplex method in solving complex games, as well as the role of linear programming in optimizing strategic decisions. Applying these methods improves both academic and practical understanding of strategic decision-making, whether in individual competitions or within larger institutional settings.

Lecture 12 : simulation Modeling

Introduction

In an era defined by complexity and rapid change, simulation has emerged as a strategic necessity rather than a mere analytical tool. Organizations today operate in environments where direct experimentation may be impractical, costly, or even impossible. Simulation offers a robust alternative by allowing systems to be modeled, studied, and improved within a virtual environment that mirrors real-world dynamics. Unlike traditional analysis methods that rely on static assumptions, simulation captures the dynamic interplay of variables over time, making it particularly effective for evaluating scenarios under uncertainty. This lecture provides an in-depth exploration of simulation as a decision-support methodology, highlighting its theoretical foundations, practical applications, and its growing role in contemporary management practices-especially through the use of Discrete Event Simulation and the Monte Carlo Method.

Lecture objectives

- Clarify the general concept of simulation and its significance in system analysis and decision-making.
- Classify simulation models according to their core characteristics such as determinism, dynamics, and type of change.
- Explain the stages of building and implementing a simulation model—from system definition to sensitivity analysis.
- Present practical applications of simulation models, especially Discrete Event Simulation and the Monte Carlo Method.
- Enhance learners' ability to use simulation to assess risks and improve decisions in uncertain environments.

1.Basic concepts

Simulation has become a fundamental analytical approach across various disciplines due to its ability to represent the behavior of complex systems without incurring the risks or expenses associated with real-world experimentation. By developing physical or computational models, simulation enables researchers and decision-makers to test assumptions, explore alternative scenarios, and forecast future outcomes with greater confidence. This work presents a structured overview of simulation, beginning with its definition and key advantages, followed by a classification of simulation models based on core attributes such as determinism, time dependence, and change type. It concludes with a discussion of the main stages involved in implementing a simulation, providing a comprehensive framework for understanding how this tool can be effectively used to analyze systems and support decision-making.

1.1. Definition of simulation

Simulation is a specific methodology for examining models, which is inherently experimental or experiencing. Essentially, simulation resembles doing field testing, with the system of interest substituted by a physical or computational model. Simulation entails constructing a model that replicates specific behaviors of interest; conducting experiments with the model to produce observations of these behaviors; and striving to comprehend, summarize, and generalize these behaviors. In several applications, simulation encompasses the evaluation and comparison of different designs, as well as the validation, elucidation, and substantiation of simulation results and research suggestions. (WHITE & INGALLS, 2017, p. 2)

This definition effectively highlights the experimental nature of simulation as a methodological approach for analyzing system behavior through models rather than direct interaction with real-world systems. Simulation offers a controlled and cost-effective alternative by substituting the actual system with a physical or computational model, enabling observation and analysis without real-life risks. At its core, simulation involves developing a representative model, conducting experiments, and analyzing the resulting behavior to extract insights. Importantly, the definition emphasizes that simulation extends beyond mere observation—it plays a crucial role in comparing design alternatives, validating model accuracy, and supporting research conclusions. As such, simulation is widely regarded as a strategic tool across disciplines including engineering, management, logistics, and complex systems analysis.

1.2. Advantages of simulation

The utilisation of simulation has several advantages, including: (TWI, 2025)

1.2.1. Diminished Financial Risk

Simulation is more cost-effective than real-life experiments. The possible expenses of evaluating theories of real-world systems may encompass expenditures related to transitioning to an unproven procedure, recruiting personnel, or acquiring new equipment. Simulation enables the evaluation of hypotheses and the prevention of expensive errors in reality.

1.2.2. Precise Repetitive Assessment

A simulation enables the repeated testing of various hypotheses and developments under identical conditions. This allows for comprehensive testing and comparison of many concepts without divergence.

1.2.3. Analyse Long-Term Effects

A simulation may be developed to provide a glimpse into the future by precisely simulating the effects of years of usage in few seconds. This enables the assessment of both immediate and prolonged effects, allowing for confident and informed investment decisions that provide advantages over the long run.

1.2.4. Acquire Insights for Process Enhancement

The advantages of simulation are not just realised upon project completion. Enhancements can be incorporated across a whole process by evaluating various hypotheses.

1.2.5. Evaluate Random Occurrences

A simulation can be utilised to evaluate random occurrences, such as unforeseen personnel absences or supply chain disruptions.

1.2.6. Evaluate Non-Standard Distributions

A simulation can accommodate variable and non-standard distributions, rather than being restricted to fixed parameters. For instance, while modelling a supermarket, one may enter several consumer categories who would traverse the store at differing velocities. A young entrepreneur procuring a sandwich will navigate the shop differently than an elderly couple or a mother managing a monthly grocery trip with two children in tow. Incorporating variable parameters enables a simulation to more precisely replicate reality.

1.2.7. Promotes Comprehensive Analysis

The design of a simulation and the identification of various parameters can yield solutions. Thorough contemplation of a process or technique might yield solutions or improvements without necessitating the final simulation.

1.2.8. Enhance Stakeholder Engagement

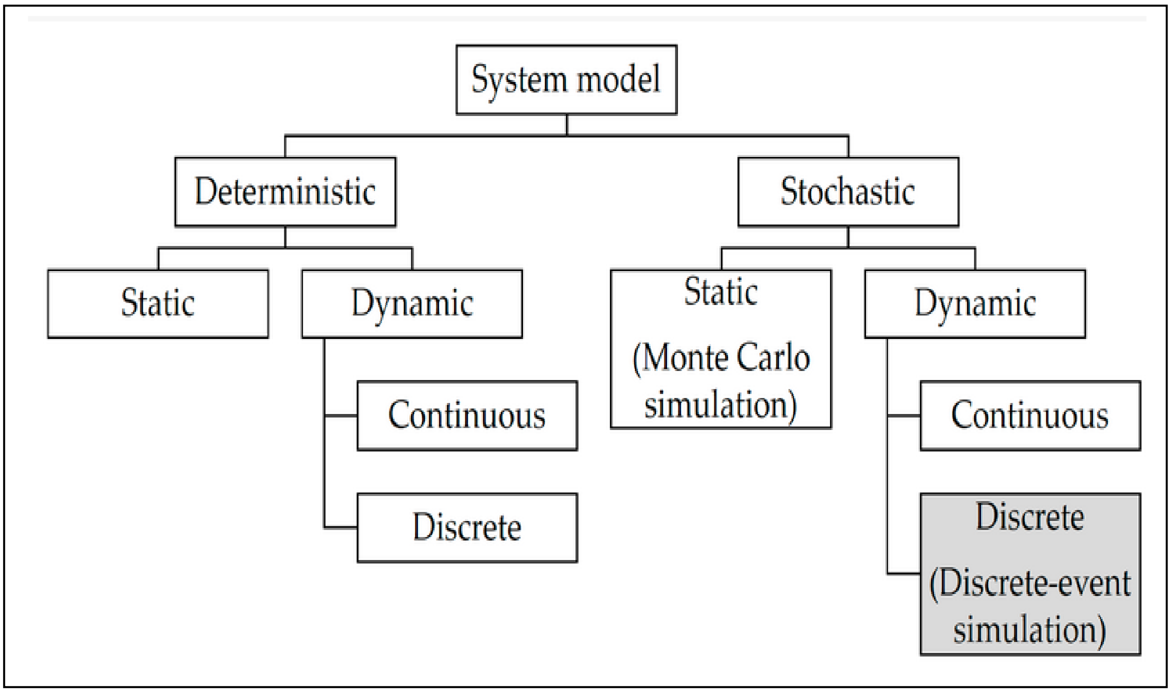
A visual simulation may enhance engagement from partners, associates, and stakeholders. You may graphically illustrate the outcomes of any process modifications and the methods employed, so enhancing interaction with stakeholders or facilitating a simulation-based sales presentation.

1.3. Types of simulation models and Comparison Between Them

Simulation is a valuable analytical tool used to model and understand the behavior of complex systems under varying conditions. It allows for the testing of hypotheses and forecasting outcomes without interfering with the actual system. To effectively develop a suitable simulation model, it is essential to distinguish between the different types of simulation frameworks. These are typically categorized based on three main criteria: whether the model is deterministic or stochastic, static or dynamic, and whether it represents continuous or discrete changes. This classification supports the selection of the most appropriate modeling approach depending on the system's characteristics, the availability of data, and the specific objectives of the analysis—as illustrated in the accompanying figure and table.

Figure 12.1 presents a classification of different simulation model types.

Fig. 12.1. Types of simulation models



Source: (Shishvan & Benndorf, 2017)

The following is a comparison of the different types of simulation model: (Software Solutions Studio, 2021)

1.3.1. Deterministic and Stochastic Models

Deterministic models are employed when the system's behaviour can be accurately forecasted. In these models, a defined set of inputs will provide outcomes that are consistently reproducible. Deterministic models exhibit no randomness, as the values of all input variables are precisely known.

Stochastic models are employed when the input variables cannot be precisely approximated. This simulation addresses data uncertainty through the use of probability distributions. Stochastic models yield varying outcomes at each execution due to the random selection of inputs within a defined distribution.

1.3.2. Static and Dynamic Models

Static models concentrate on representing a process at a particular moment in time. The outputs of these models are contingent solely upon the inputs and internal variables at that moment. These models offer a temporal snapshot of the system and fail to consider temporal variations.

Dynamic models take into account both current inputs and historical results. These models rely on time as a variable, which affects the outcomes. Dynamic models are optimal for processes that evolve over time, such as seasonal variations or degradation, while static models are more appropriate for systems that are constant or assessed at a certain instant.

1.3.3. Discrete and Continuous Models

Discrete models are utilised for systems that undergo changes at distinct, recognisable moments in time. The time of events is clearly delineated, rendering it appropriate for operations characterised by discrete, quantifiable occurrences, such as product acquisitions or customer arrivals.

Continuous models are employed for processes that evolve continuously, wherein the system's state is fluid and cannot be precisely identified at discrete intervals. Instances of continuous processes encompass temperature, velocity, or revenue. Continuous models are superior for systems that endure persistent, uninterrupted changes throughout time, whereas discrete models are more suitable for systems that encounter distinct, separate occurrences.

The difference between the various simulation models can be clarified as follows:

Table. 12.1. Simulation Model Classification

Main Type	Time Category	Change Type	Description	Example Application	Additional Notes
Deterministic	Static	-	Models simulate a process at a specific point in time without time evolution.	Estimating app users based on exact known data.	Outputs remain constant as long as inputs are unchanged.
	Dynamic	Continuous	The process changes continuously over time without randomness.	Water flow in a pipe, temperature variation.	Usually modeled using differential equations.
		Discrete	The process changes at distinct time intervals without randomness.	Number of customers visiting a restaurant each hour.	Events occur at known time steps and are predictable.
	Static	-	Models represent a snapshot with uncertainty in inputs.	Monte Carlo simulation to estimate risks.	Uses probability distributions for unknown inputs.

Main Type	Time Category	Change Type	Description	Example Application	Additional Notes
Stochastic	Dynamic	Continuous	Time-dependent process with continuous and random changes.	Random variation of product demand over time.	Requires accurate representation of probabilistic behavior over time.
		Discrete	Models event-based systems at discrete times with randomness.	Bank queue simulation (Discrete-event simulation).	Common in systems where changes are triggered by events.

Source: Author's own work based on (Software Solutions Studio, 2021)

This table highlights the fundamental distinctions among different types of simulation models based on their nature (deterministic or stochastic), their temporal structure (static or dynamic), and the type of change involved (continuous or discrete). Understanding this classification is essential for selecting the appropriate model according to the characteristics of the system being studied. Deterministic models are applied when all input data is known with certainty and no randomness is present. In contrast, stochastic models are suitable when uncertainty or random variation in inputs is involved. Regarding time, static models represent the system at a single point in time without accounting for its evolution, while dynamic models capture changes over time. Additionally, continuous models describe systems that evolve in a smooth and uninterrupted manner, whereas discrete models focus on systems that change at specific intervals. This classification framework supports informed decision-making when choosing a simulation approach tailored to the behavior and complexity of the target system.

1.4. Simulation Application stages

Applying the simulation approach to a problem of this nature requires going through the following stages: (Beladjoz, 2010, pp. 241-242)

- System Definition: The system to be represented through a model is first defined.
- Model Construction: A model is developed to illustrate the behavior of the system's elements over time.
- Probabilistic Distribution Evaluation: Probabilistic distributions for the factors defined in the model are evaluated.
- Simulation Execution: The simulation is conducted based on the model and the defined probability distributions.
- Sensitivity Analysis: The results of the simulation are analyzed to determine the sensitivity of the system to different factors.

- Comparison of Alternatives: The simulation results for various alternatives are compared and used to identify the preferred alternative. A detailed sensitivity analysis is also performed to identify the factors that require accurate probabilistic distributions.

In conclusion, simulation serves as a versatile and powerful methodology for analyzing, understanding, and improving complex systems. Whether deterministic or stochastic, static or dynamic, continuous or discrete, simulation models offer valuable means for process evaluation, risk assessment, and comparison of alternative strategies. Through the use of probability distributions and repeated experimentation in a controlled environment, simulation generates deep insights into system behavior under uncertainty. The structured application process—from defining the system to conducting sensitivity analysis—ensures a systematic and reliable foundation for drawing meaningful conclusions.

Given the practical relevance of simulation in management contexts, the next section will focus on two of the most widely applied simulation models: Discrete Event Simulation (DES) and the Monte Carlo Method. These approaches are especially valuable for analyzing operational processes, estimating risks, and supporting decision-making in real-world, complex managerial environments.

2. Discrete Event Simulation (DES)

Discrete Event Simulation (DES) is a recognized analytical method employed to analyze dynamic systems whose states fluctuate according to the occurrence of particular events across time. The structural framework utilized in this simulation type is known as the process-oriented model, emphasizing the representation of entity movement within the system and their interactions with resources and activities, (WHITE & INGALLS, 2017, p. 3) and consists of the following structural components: (WHITE & INGALLS, 2017, pp. 3-7)

1. Inputs, Outputs, and State

In DES, inputs are external variables that affect the system and initiate state transitions, such as a customer's arrival or a demand request. These inputs result in transitions of the system state, which denotes the internal status of the system. Outputs are quantifiable indications obtained from the state, utilized to answer the inquiries presented in the simulation investigation.

2. Entities and Attributes

Entities are the dynamic elements inside the system that induce alterations to its state. Each entity has distinct features that delineate distinctive qualities, like arrival time, request type, and waiting period. In increasingly intricate models, entities may signify individuals, products, transactions, or even artificial control mechanisms.

Practical Example: In a call center system, each telephone call is regarded as an entity. The qualities may encompass the specified product kind, arrival time, and waiting duration prior to service commencement.

3. Assets

Resources are finite components that entities must employ to advance within the system. When resources are under use, entities await their turn until access is permitted. Resources may be uncomplicated, such as a one person, or intricate, including machinery, storage facilities, or cars. Certain components, like as high-capacity IVR units, may not be classified as resources if they do not induce bottlenecks within the system.

4. Activities and Events

Activities denote the processes or decisions that entities undergo, including delays, waiting, or routing. Events are specific moments in time when a change in system state transpires, such as the initiation of service or the departure of an entity. Activities are often classified into three categories:
• Deterministic or stochastic delays
• Waiting durations
• Decisions or actions based on logical reasoning
5. Universal Variables
Global variables are accessible during the whole simulation and are utilized to monitor policy efficacy or system dynamics. The metrics may encompass:
• Total count of incoming entities
• Count of rejected or obstructed entities • Aggregate duration entities remained within the system These variables offer a real-time assessment of the system's performance.
6. Stochastic Number Generator
A pseudo-random number generator is crucial for DES, generating numbers within the range of 0 to 1. The values are subsequently converted to adhere to specified probability distributions (e.g., normal, uniform). This method facilitates the modeling of uncertainty in processing durations or arrival timings.
Illustration: If the service time follows a uniform distribution between 10 and 20 minutes, and the random number is 0.7312, then: Delay = 10 + 0.7312 × (20 - 10) = 17.312 minutes.
7. Timekeeping and Chronology
The simulation clock indicates the present time, but the event calendar is an ordered list of forthcoming events arranged chronologically. The calendar regulates the progression of time in the simulation by guaranteeing that events occur in the proper sequence.
8. Statistical Collectors
These instruments collect and evaluate performance data during the simulation. They are often categorized into three types:
• Counters: Monitor event occurrences (e.g., quantity of clients in wait)
• Temporal variables: Observe variations over time (e.g., resource consumption)
• Tallies: Document specific observations (e.g., duration an entity remains in the system)
These measurements are essential for deriving performance insights and pinpointing areas for enhancement.

Discrete Event Simulation provides a comprehensive framework for simulating intricate systems that depend on sequential entity flow and limited resource use. Through the integration of events, actions, entities, resources, and statistical analysis tools, DES empowers analysts and decision-makers to develop realistic, data-driven models that facilitate strategic and operational choices.

Example 1: Customer Reception Center at a Bank Branch

The manager of a bank branch seeks to assess the efficiency of the customer reception point. Only one employee is responsible for providing the service. Customers arrive at irregular intervals and are served in the order of their arrival.

Administrative Objectives of the Simulation:

- Number of customers that can be served within two hours.
- Average customer waiting time.

- The need for an additional employee.

Assumption Details:

- Customers arrive every 6 minutes on average.
- Service time ranges between 4 to 8 minutes.
- Simulation duration: 2 hours (120 minutes).
- No customer is turned away (open queue).

Simulation Data:

Table. 12.2

Customer	Arrival Time (min)	Service Duration (min)	Service Start Time	Service End Time	Waiting Time
1	0	6	0	6	0
2	6	5	6	11	0
3	12	7	12	19	0
4	18	8	19	27	1
5	24	5	27	32	3

Source: Author's own work

How the Data Was Generated:

These numbers are not taken from a real study or academic reference. Rather, they are original educational figures created by the professor solely to illustrate the concept:

- Arrival Time: Assumes each customer arrives 6 minutes after the previous one (0, 6, 12, 18, 24...).
- Service Duration: Selected manually within the range 4–8 minutes, semi-randomly (e.g., 6, 5, 7...).
- Service Start Time: Equals arrival time (if the employee is free) or the previous customer's end time (if the employee is busy).
- Waiting Time = Service Start Time – Arrival Time.
- Service End Time = Service Start Time + Service Duration.

Results:

- Number of customers served: 5 customers
- Average waiting time: (0+0+0+1+3)/5 = 0.8 minutes
- Employee busy time: 6 + 5 + 7 + 8 + 5 = 31 minutes
- Employee utilization rate: 31 / 32 = 96.8%

Application According to Queueing Theory (M/M/1):

We apply the classical M/M/1 model (random arrival, random service, single server):

- λ (arrival rate) = 1 customer / 6 minutes = 0.1667 customer/min

- μ (service rate) = 1 customer / average service time = 1 / 6.2 ≈ 0.1613 customer/min

Theoretical Indicators:

- ρ (utilization rate) = λ / μ = 0.1667 / 0.1613 ≈ 1.033 → indicates system overload!
- Lq (average number of customers waiting) = $(\rho^2) / (1 - \rho)$ → undefined because $\rho > 1$

Conclusion: The system is unstable because the arrival rate exceeds the service rate, meaning the number of waiting customers will increase without bound.

What Does This Mean Managerially?

- The arrival rate must be reduced (e.g., by booking appointments in advance)
- Service time must be reduced (e.g., by using digital tools)
- Or an additional employee must be added to transform the system into M/M/2

Analysis:

- The employee is heavily utilized (>95%), which may cause fatigue or stress in the long run.
- The average waiting time is low, indicating operational efficiency, but the theoretical model warns of the risk of queue buildup over time.

Example 2: Problem Statement — Passenger Check-in Point at Algiers International Airport

Algiers International Airport facilitates passenger departures through a check-in point that includes document verification and travel procedures. The system consists of:

- Two employees dedicated to domestic flight services.
- Three employees dedicated to international flight services.

This check-in point operates from 6:00 AM to 2:00 PM.

Assumptions (Hypothetical):

Table. 12.3

Service Type	Number of Employees	Arrival Rate (Passenger/Minute)	Service Time Distribution (Minutes)
Domestic	2	Every 3 minutes ($\lambda = 1/3$)	Uniform between 2 and 4
International	3	Every 2 minutes ($\lambda = 1/2$)	Uniform between 3 and 5

Source: Author's own work

- Queue type: FIFO (First-In-First-Out).
- Objective: To calculate performance indicators:
  - Average waiting time.
  - Staff utilization rate.
  - Number of unserved passengers.

Requirements:

- Simulate passenger arrivals and services from 6:00 AM to 6:20 AM.
- Display the Event Calendar.
- Calculate the average waiting time, the number of served and unserved passengers.
- Calculate employee utilization.

Detailed Solution:

Time: 6:00 AM — Simulation Start

The first entities are created based on exponential distribution:

Table. 12.4

Entity	Flight Type	Arrival Time	Expected Service Time
1	International	6:02	4 minutes
2	Domestic	6:03	3 minutes
3	International	6:04	3.5 minutes
4	Domestic	6:06	3.5 minutes
5	International	6:06:30	4.5 minutes

Source: Author's own work

Event Sequence and State Table

- Event 1: Arrival of Entity 1 (International)
  - Time: 6:02
  - An international employee is available → enters service directly.
  - Service ends: 6:06
- Event 2: Arrival of Entity 2 (Domestic)
  - Time: 6:03
  - Domestic employee available → starts service.
  - Service ends: 6:06
- Event 3: Arrival of Entity 3 (International)
  - Time: 6:04
  - Another international employee available → enters service.
  - Service ends: 6:07:30
- Event 4: Arrival of Entity 4 (Domestic)
  - Time: 6:06
  - All employees are busy → joins the queue.
- Event 5: End of Service for Entity 1 (International)
  - Time: 6:06

- Employee becomes available for a new entity.

- Event 6: Arrival of Entity 5 (International)
  - Time: 6:06:30
  - Enters service immediately with the available employee.
  - Service duration: 4.5 minutes → ends at 6:11

(The simulation continues until 6:20 using the same method.)

Summarized in the following table:

Table. 12.5

Event	Time	Entity	Flight Type	Action	Service End Time
1	6:02	1	International	Enters service directly	6:06
2	6:03	2	Domestic	Available employee → starts service	6:06
3	6:04	3	International	Another available employee → service	6:07:30
4	6:06	4	Domestic	All employees busy → enters queue	—
5	6:06	1	International	End of service → employee available	—
6	6:06:30	5	International	Enters service with available employee	6:11

Source: Author's own work

Cumulative Results until 6:20 AM:

Passenger Summary:

Table. 12.6

Flight Type	Number of Arrivals	Number Served	Number Not Served
Domestic	7	6	1
International	9	7	2

Source: Author's own work

Average Waiting Time (Using Queueing Theory Formula:)

$Wq = Lq / \lambda$

Table. 12.7

Flight Type	Queue Length (Lq)	λ (Passenger/Minute)	Wq (Minutes)
Domestic	1.5	1/3	4.5
International	2.1	1/2	4.2

Source: Author's own work

Employee Utilization Rate:

Table. 12.8

Flight Type	Total Service Time (Minutes)	Available Time	Utilization Rate
Domestic	36	40	0.9
International	63	90	0.7

Source: Author’s own work

Analysis

This practical example demonstrates how Discrete Event Simulation can serve as an effective analytical tool for understanding the operational performance of airport systems. The results reveal that the service system experiences mild congestion during early morning hours.

Suggested improvements:

- Add one more domestic service agent to reduce waiting times.
- Rotate staff depending on workload distribution.
- Optionally integrate both queues into a unified queueing system.

3. The Monte Carlo Method

The Monte Carlo method is one of the probabilistic modeling techniques that relies on random repetition to assess the likelihood of different outcomes within a given system. This approach involves generating multiple random inputs based on appropriate probability distributions, which are then used in mathematical models to simulate the system's behavior. The simulation is repeated many times in order to build a statistically accurate representation of the possible outputs, thereby aiding in risk assessment and supporting decision-making under uncertainty. (Software Solutions Studio, 2021)

Example 3: Real Estate Company and Net Cash Flow Analysis Using Monte Carlo Simulation
A company specialized in selling ready-to-move residential units. In the upcoming month, the company anticipates cash inflows from customer bookings and cash outflows related to construction costs, labor wages, and marketing. Given the uncertainties in the real estate market, management seeks to estimate the probability of achieving a monthly profit or loss using Monte Carlo Simulation.
Data Inputs:
Expected Cash Inflows (In-Flow):

Table. 12.9

Inflow (DZD)	Probability
800,000	0.30
1,000,000	0.50
1,200,000	0.20

Source: Author’s own work

Expected Cash Outflows (Out-Flow):

Table. 12.10

Outflow (DZD)	Probability
700,000	0.40
900,000	0.35
1,100,000	0.25

Source: Author's own work

Requirements:

- Perform 10 Monte Carlo simulation trials to estimate the Net Cash Flow (NCF) for the month.
- Build a table showing:
  - ➢ Random numbers for inflow and outflow.
  - ➢ Corresponding values.
  - ➢ NCF = Inflow – Outflow.
- Construct the frequency and estimated probability distribution of NCF.
- Analyze the results to support financial decision-making.

Solution:

1. Convert Probabilities to Cumulative Intervals (0–99):

Inflows:

Table. 12.11

Inflow (DZD)	Probability	Random Number Interval
800,000	0.30	00 – 29
1,000,000	0.50	30 – 79
1,200,000	0.20	80 – 99

Source: Author's own work

Outflows:

Table. 12.12

Outflow (DZD)	Probability	Random Number Interval
700,000	0.40	00 – 39
900,000	0.35	40 – 74
1,100,000	0.25	75 – 99

Source: Author's own work

2. Generate 10 Pairs of Random Numbers*:*

Table. 12.13

Trial	Random In	Inflow Value	Random Out	Outflow Value	NCF (DZD)
1	12	800,000	37	700,000	100,000
2	78	1,000,000	83	1,100,000	-100,000
3	45	1,000,000	49	900,000	100,000
4	87	1,200,000	12	700,000	500,000
5	32	1,000,000	92	1,100,000	-100,000
6	09	800,000	61	900,000	-100,000
7	66	1,000,000	39	700,000	300,000

Trial	Random In	Inflow Value	Random Out	Outflow Value	NCF (DZD)
8	82	1,200,000	77	1,100,000	100,000
9	29	800,000	18	700,000	100,000
10	34	1,000,000	70	900,000	100,000

Source: Author's own work

3. Net Cash Flow Distribution:

Table. 12.14

NCF (DZD)	Frequency	Estimated Probability
-100,000	3	0.30
100,000	5	0.50
300,000	1	0.10
500,000	1	0.10
Total	10	1.00

Source: Author's own work

4. Managerial Analysis:

- 50% of the simulations resulted in a moderate profit (+100,000 DZD).
- There is a 30% chance of incurring a minor loss (-100,000 DZD).
- The probability of high profit (+300,000 or +500,000 DZD) is only 20%.
- While profit is more likely than loss, the financial manager should prepare for risk by:
    - Reducing fixed costs.
    - Diversifying revenue sources.
    - Enhancing inventory and marketing efficiency.

Note:
Although this example uses only 10 simulations for illustration purposes, real-world applications require a significantly larger number of iterations to ensure reliable outcomes. Increasing the number of trials enhances the accuracy of estimated probabilities and strengthens confidence in financial decision-making.
In summary, a higher number of simulations yields more robust and dependable results—particularly in volatile sectors such as real estate. It is generally recommended that managers perform at least 250 to 1,000 simulation runs to support sound strategic planning

Example 4: Investment in an Industrial Machine

An industrial company in Algiers is considering investing in a new industrial machine to expand its production capacity. The company has decided to purchase the new machine, which costs 12,000,000 DZD. This machine will be used to produce a specific product, generating annual cash inflows. However, the company faces challenges in estimating the expected returns over the long term due to fluctuations in market prices, maintenance costs, and the number of units sold. Therefore, the company decided to use Monte Carlo simulation to analyze the expected returns from this investment.
Given Data:

Table. 12.15

P	CF_1 (DZD)	P	CF_2 (DZD)	P	CF_3 (DZD)
0.25	2,000,000	0.20	2,500,000	0.15	4,000,000
0.30	3,000,000	0.30	3,000,000	0.20	4,500,000
0.35	4,000,000	0.20	4,000,000	0.45	5,000,000
0.10	4,500,000	0.30	4,500,000	0.20	5,500,000
1.00		1.00		1.00	

Source: Author's own work

Requirements:

- Calculate the expected returns for the machine in each of the three years using the simulation.
- Calculate the Net Present Value (NPV) of the project using a 10% discount rate.
- Perform Monte Carlo simulation to determine whether to accept or reject the investment.
- Calculate the mean and standard deviation of the Net Present Value after running the simulation.
- Determine the probability distribution for the Net Present Value after running the simulation.

Solution:

Step 1: Assigning Cash Flow Values for Year 1 (CF_1):

Table. 12.16

$CF_1 \times 10^3$ DZD	P	Cumulative Probability	Random Numbers
2.0	0.25	0.25	0 – 25
3.0	0.30	0.55	26 – 55
4.0	0.35	0.90	56 – 90
4.5	0.10	1.00	91 – 100
Total	1.00		

Source: Author's own work

Step 2: Assigning Cash Flow Values for Year 2 (CF_2):

Table. 12.17

$CF_2 \times 10^3$ DZD	P	Cumulative Probability	Random Numbers
2.5	0.20	0.20	0 – 20
3.0	0.30	0.50	21 – 50
4.0	0.20	0.70	51 – 70
4.5	0.30	1.00	71 – 100
Total	1.00		

Source: Author's own work

Step 3: Assigning Cash Flow Values for Year 3 (CF_3):

Table. 12.18

$CF_3 \times 10^3$ DZD	P	Cumulative Probability	Random Numbers
4.0	0.15	0.15	0 – 15
4.5	0.20	0.35	16 – 35
5.0	0.45	0.80	36 – 80
5.5	0.20	1.00	81 – 100
Total	1.00		

Source: Author's own work

Step 4: Calculating the Net Present Value (NPV):

We calculate the NPV using the following formula:

$$\mathrm{NPV} = \sum \frac{\mathrm{CF_n}}{(1+\mathrm{K})^{\mathrm{n}}}$$

Where:

- CF_n is the cash flow in year n.
- K is the discount rate (10%).
- n is the year number.

The simulation results are as follows:

Table. 12.19

S	The first year				The second year				The third year				A + B + C = NPV
				A				B				C	
	R N	CF	$(1.10)^{-1}$	CF × $(1.10)^{-1}$	R N	CF	$(1.10)^{-2}$	$CF_2 \times (1.10)^{-2}$	R N	CF	$(1.10)^{-3}$	$CF_3 \times (1.10)^{-3}$	
1	2	2	0.9091	1.8182	3	3	0.8264	2.4792	4	4	0.7513	3.0052	7.3026
2	1	3	0.9091	2.7273	4	4	0.8264	3.3056	5	5	0.7513	3.7565	9.7894
3	3	2	0.9091	1.8182	5	5	0.8264	4.1320	6	4.5	0.7513	3.3791	9.3292
4	2	3	0.9091	2.7273	3	3.5	0.8264	2.8944	7	5.5	0.7513	4.1337	9.7554
5	1	2	0.9091	1.8182	6	3	0.8264	2.4792	8	5	0.7513	3.7565	8.0540
													44.230

S: Simulation, R N: Random Number

Source: Author's own work

Step 5: Calculating the Mean and Standard Deviation:

- Mean (E(NPV)) $= \frac{44.2306}{5} = 8.84612\mathrm{DZD}$
- To calculate the Standard Deviation of the given Net Present Value (NPV) values, we proceed as follows:

- Calculate the Mean (Average):

- Construct the following table:

Table. 12.20

NPV (X)	X − Mean	(X − Mean)²
7.3026	-1.5435	2.3814
9.7894	0.9433	0.8898
9.3292	0.4831	0.2334
9.7554	0.9093	0.8270
8.0540	-0.7921	0.6274
Total		4.9589

Source: Author's own work

- Calculate the Variance:

$$\text{Variance} = \frac{\sum(X - \text{Mean})^2}{n} = \frac{4.9589}{5} = 0.9918$$

- Calculate the Standard Deviation:

$$\text{SD} = \sqrt{0.9918} \approx 0.9959$$

The standard deviation is approximately 0.9959 DZD

- Minimum NPV (MIN(NPV)) = 7.3026 DZD
- Maximum NPV (MAX(NPV)) = 9.7894 DZD

Step 6: Probability Distribution for NPV:

Table. 12.21

Value Range	Probability	Cumulative Probability
7 – 8	0.40	0.40
8 – 9	0.50	0.90
9 – 10	0.10	1.00

Source: Author's own work

Thus, we conclude that:

- Project Acceptance: Since the Net Present Value (NPV) is positive in most simulations, the project can be accepted, especially considering the expected profit over the three years.
- Possible Variability: Although the probability distribution shows good chances of profit, there are potential risks in some cases that may lead to lower-than-expected outcomes.

✓ Risk Assessment: With a low standard deviation, we can conclude that the project has relative stability in expected returns, which reduce.

Conclusion

Simulation is an indispensable tool for understanding and improving the performance of complex systems, especially in management environments characterized by uncertainty and rapid change. Whether deterministic or stochastic, static or dynamic, simulation offers a systematic way to test scenarios and analyze alternatives without the cost or risk of real-world experimentation. This lecture has highlighted the broad potential of simulation, particularly through Discrete Event Simulation and the Monte Carlo Method—two of the most widely used models for analyzing operations and supporting informed decision-making. As such, simulation represents a powerful tool that combines analytical precision with practical flexibility to support modern management.

Lecture 13: statistical methods in quality control

Introduction

In light of the rapid advancements in the fields of production and services, ensuring product and process quality has become a strategic necessity rather than an optional enhancement. Statistical Quality Control (SQC) has emerged as a scientifically grounded method that enables organizations to monitor their operational performance and ensure compliance with technical standards through data analysis and graphical tools.

The significance of SQC lies in its ability to differentiate between natural, random variations inherent in any production system and abnormal variations that may indicate a malfunction or systemic issue. As such, the application of statistical control not only enhances product quality but also supports proactive prevention and early intervention before defects escalate.

Lecture objectives

This lecture aims to enable participants to:

- Understand the core concepts of statistical quality control and its role in improving operational performance.
- Distinguish between full inspection and sampling inspection, along with the criteria for choosing each.
- Classify the types of control according to production stages (preventive, in-process, post-process).
- Differentiate between attribute control charts and variable control charts and recognize the appropriate context for their application.
- Learn how to calculate central lines and control limits (upper and lower) for different chart types.
- Analyze and interpret control charts to determine whether a process is statistically under control or requires corrective action.
- Utilize control chart results in data-driven decision-making to enhance quality and achieve continuous improvement.

1. Background

Statistical process control (SPC) methods have been utilized in industrial statistics for over eighty years, attributed to pioneers such as Shewhart, Romig, Wald, and Deming. SPC approaches are employed, among other purposes, to identify when a stable process, characterized by a constant mean and consistent variance, deviates from stability. SPC is utilized to evaluate the quality of a product that we are either manufacturing ourselves or procuring from a supplier. The initial aim pertains to "quality control" (QC) processes, whereas the subsequent objective relates to "acceptance sampling" methods. (Romeu)

Quality control approaches are predicated on the premise that performance metrics from "stable" processes adhere to a specific statistical distribution characterized by a constant mean and variance. Consequently, we regularly extract independent and random samples from these probability measures to confirm their stability. An instance of a measure of central tendency is the sample mean ($\bar{x}$). Instances of measures of variation or dispersion encompass the sample standard deviation (s) and the sample range (r). In stable and normally distributed processes, process measurements reside inside clearly defined areas of the quality control charts. The systematic and periodic evaluation of such PM, within the framework of QC charts, constitutes the core of QC. This START sheet offers a summary of many QC approaches. A forthcoming one will similarly address acceptability sampling. (Romeu)

Based on the above, it becomes clear that employing statistical methods to monitor processes represents an effective scientific approach to quality control and achieving operational efficiency. Utilizing data analysis tools is a fundamental means of detecting deviations and distinguishing between random variation and variation caused by specific factors. This enables organizations to improve their outputs and prevent potential issues before they escalate. In this context, control charts have emerged as one of the most prominent applied statistical tools, combining the power of quantitative analysis with the clarity of visual representation, making them a foundational component in modern quality systems.

2. General Concepts and Types of Statistical Quality Control

2.1. Basic concepts

Statistical quality control represents a methodological foundation for ensuring continuous conformity of products with quality standards and technical specifications. Two principal approaches are commonly employed: (Kedada & Elfayad, 2010, p. 393)

- Complete Inspection: This involves examining every unit in the production population without exception, ensuring full detection of defects. However, it is often costly, especially in mass production, and may result in product damage during the inspection process.
- Sampling Inspection: This method relies on selecting a limited number of items (a sample) from the production population, chosen randomly and representatively, with the aim of inferring the quality of the entire lot. Sampling is widely used in industrial applications due to its advantages in saving time, effort, and cost, while maintaining an acceptable level of accuracy.

In addition to method selection, statistical quality control is divided into two main types based on the stage of application in the production process: (Kedada & Elfayad, 2010, pp. 394-395)

- End-of-Production Inspection (Final Control): Conducted after production is completed to verify that the finished products meet specifications. Based on the sample results, the entire lot is either accepted or rejected.
- In-Process Control: Primarily applied in continuous or large-scale production lines. Samples are taken periodically during the production stages to monitor process stability and detect deviations early before defects accumulate.

Understanding these concepts and distinguishing between the types of control is essential for establishing an effective quality system that improves performance, reduces waste, and ensures customer satisfaction.

2.2. Determinants for Applying Sampling Methods: Considerations, Errors, and Implementation

The effectiveness of sampling in statistical quality control depends on several technical and methodological factors that collectively influence the sample size, the reliability of the method, and the accuracy of decisions based on it.

2.2.1. Determinants of Sample Size

Several key factors influence the determination of an appropriate sample size: (Kedada & Elfayad, 2010, p. 394)

- Inspection Cost: Larger sample sizes increase the cost of inspection operations.
- Potential Error Costs: When the cost of accepting defective units or rejecting good ones is high, a larger sample is recommended to reduce the likelihood of errors.
- Process Stability: In stable industrial environments with low variability, smaller samples can be used without significantly affecting inspection accuracy.

2.2.2. Statistical Errors in Sampling

Sampling inherently involves the risk of two types of statistical errors, each of which affects decision quality: (Kedada & Elfayad, 2010, p. 394)

- Type I Error (α): Occurs when conforming units are incorrectly rejected, resulting in economic losses due to unnecessary waste or rework.
- Type II Error (β): Occurs when non-conforming units are mistakenly accepted. This is considered more serious, as it can degrade product quality and harm customer satisfaction.

2.2.3. When Sampling is Preferred

Sampling methods are most suitable and efficient in the following industrial scenarios: (Kedada & Elfayad, 2010, p. 395)

- When complete inspection is too costly.
- When the inspection process is destructive to the unit.
- When the cost impact of a few defective units is relatively low.
- In cases of limited human or technical resources.
- When quick decision-making is required without significantly compromising overall quality.

2.2.4. Steps for Implementing Sampling Inspection

The inspection process via sampling follows a series of practical steps, depending on the nature of the supply: (Kedada & Elfayad, 2010, pp. 395-397)

- In the case of separate lots:
  - ✓ Determine the number of units in the lot (N).
  - ✓ Randomly select a sample (n) from the lot.
  - ✓ Inspect the sample to identify the number of conforming units (C) and non-conforming units (D).
  - ✓ Make the accept/reject decision based on the results and the established acceptance criteria.
- In continuous production:
  - ✓ Divide the output into sequential time-based sub-lots.
  - ✓ Draw two successive samples (n_1 and n_2) at different stages of production.
  - ✓ Analyze the results of both samples to determine lot conformity and make the final decision regarding acceptance or rejection.

2.3. Evaluating Inspector Accuracy Using the True Defect Rate Formula

To assess the accuracy of inspectors, the following equation may be applied: (Kedada & Elfayad, 2010, p. 397)

$$\text{True Defect Rate} = \frac{D - K}{D - K + B}$$

Where:

- D = Number of units the inspector classified as defective
- K = Number of good units mistakenly rejected (Type I Error)
- B = Number of defective units the inspector failed to identify (Type II Error)

Example 1: (Kedada & Elfayad, 2010, p. 397)

- The inspector reported 50 units as conforming.
- Upon review, only 8 were actually conforming.
- The number of mistakenly rejected good units was 10.

Thus, the calculation becomes:

$$\frac{52 - 42}{(10 + 8 - 50) + (8 - 50)} = \frac{80.8}{281} \approx 0.288$$

Statistical methods constitute a powerful tool for quality control, enabling scientifically grounded decisions on the acceptance or rejection of products. A solid understanding of sampling principles, error types, and acceptance criteria is essential to maintaining product quality and controlling costs. Achieving a balance between inspection accuracy and efficiency requires the intelligent, systematic application of statistical control techniques—ultimately enhancing trust in both the production process and the final product.

3. Quality control chart

In the following sections, the concept of quality control will be addressed as the general framework within which control charts operate. This includes an overview of its objectives and core functions, along with a classification of its types—namely, preventive control, in-process control, and post-process control. This will provide a broader understanding of the operational context of these tools and their vital role in ensuring that processes comply with performance and quality standards.

3.1. Definition of control chart

A control chart is a statistical method used to identify assignable causes of variance. A control chart typically comprises three horizontal lines: (1) a center line (CL) representing the intended standard, (2) an upper control limit (UCL) situated above the CL, and (3) a lower control limit (LCL) positioned below the CL. The CL is consistently a horizontal line, whereas the UCL and LCL may or may not be horizontal. Nevertheless, the distance between the Central Line (CL) and the Upper Control Limit (UCL) is equivalent to that between the CL and the Lower Control Limit (LCL). (Sinha & Vatsa, 2022, p. 302)

Control charts are analytical graphical tools used to monitor the quality and stability of production processes over time. These charts track the behavior of a specific variable across different time intervals, where the horizontal axis typically represents time periods or production units, and the vertical axis displays the measured values of the quality characteristic under observation. (Elbaladawi & Elhamidi, 2008, p. 243)

A Control Chart is a statistical instrument employed to oversee and regulate a process by illustrating data over time and detecting deviations that may signify problems. It graphs data points with a centerline (usually indicating the average or mean of the process) and upper and lower control limits that delineate the parameters of permissible fluctuation. Control charts assist in ascertaining if a process is statistically controlled or if notable variances necessitate further examination. (Kumar P. , 2024)

Based on the previous definitions, it can be concluded that control charts are among the most important statistical tools that enable organizations to systematically monitor production or service processes over time. Their significance lies in their ability to distinguish between natural random variations and out-of-control changes that may indicate a malfunction or deviation in the process. These charts visually represent performance data using three main components: a central line representing the process mean, and upper and lower control limits that define the acceptable range of variation. Through this representation, decision-makers can assess the stability and quality of the process and intervene when necessary to correct deviations before they escalate into significant issues. Thus, control charts not only support quality control but also serve as an effective tool for performance improvement and ongoing process regulation.

3.2. Goals of the Control Chart

The objectives of quality control charts are as follows: (Kumar P. , 2024)

1. Oversight of Process Stability Control charts function as a proficient instrument for monitoring operations throughout time to guarantee they stay inside established natural limitations. They assist in differentiating between inherent random fluctuations in the process and atypical deviations that may signify a malfunction or a particular reason for action. Timely identification of such abnormalities aids in preserving product quality and operational stability.
2. Recognizing Sources of Variation Control charts facilitate the detection of alterations impacting production or service operations, whether arising from random variations or systematic problems that may be rectified. Monitoring process behavior enables the identification of recurrent or developing issues that might affect efficiency and quality, so presenting genuine chances for enhancement.
3. Assessing Performance Efficiency Control charts serve as a criterion for evaluating the degree to which a process adheres to established quality standards and control limits. They provide the comparison of actual performance metrics against established upper and lower thresholds, aiding in the assessment of process consistency and the identification of trends or changes that may indicate quality deterioration or the necessity for procedure modifications.
4. Facilitating Decision-Making Control charts offer a visual, data-centric depiction of process dynamics, facilitating decision-making in operational contexts. Through the analysis of trends and deviations, managers and teams acquire clear insights that facilitate evidence-based choices concerning process adjustments or corrective measures.
5. Improving Quality Assurance Control charts, as a fundamental component of quality control systems, provide ongoing monitoring to guarantee adherence to defined standards. Through real-time performance monitoring, they reduce discrepancies and aid in sustaining superior quality in products or services.
6. Facilitating Ongoing Enhancement Control charts facilitate continuous improvement initiatives by monitoring performance subsequent to the adoption of any modifications or advancements. They assist teams in quantifying the effects of enhancements and assessing whether these modifications have successfully diminished variance and improved overall process performance—an essential component of comprehensive quality management and organizational excellence.

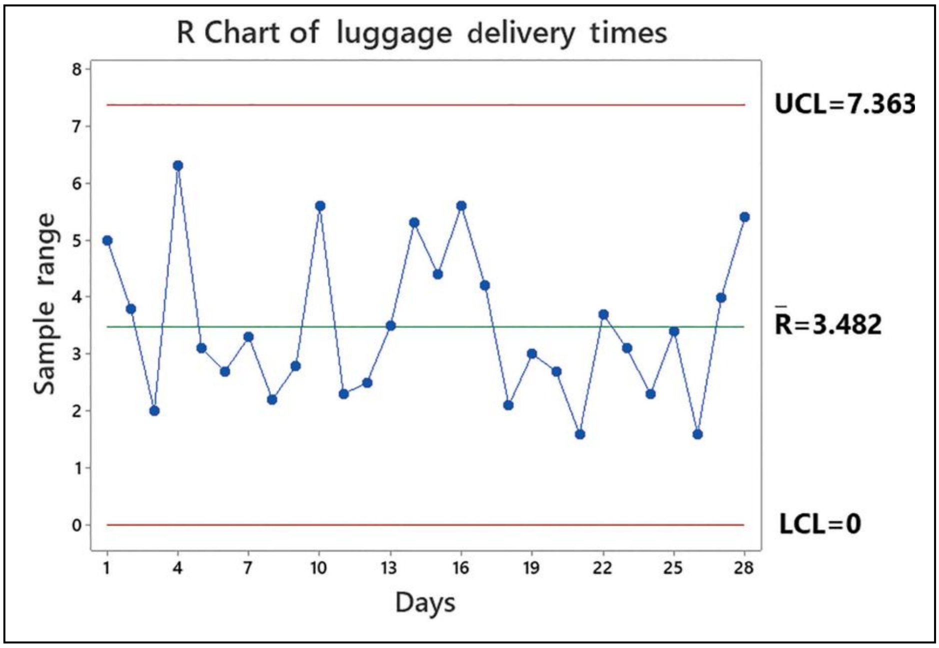
3.3. Components of Control Charts

Control charts are composed of the following components: (Kumar P. , 2024)

- Centerline (CL): Denotes the average or mean of the process data. It is often derived from previous data and used as a benchmark for comparison.
- Upper Control Limit (UCL): The maximum value within which process data points are anticipated to reside under standard operating conditions. It is positioned at a specific number of standard deviations above the mean.
- Lower Control Limit (LCL): The minimum value within which process data points are anticipated to reside under standard operating circumstances. It is positioned at a specific number of standard deviations under the centerline.
- Data Points: Singular measurements or values represented on the chart along a temporal continuum. This illustrates the process's performance in relation to the control limits.

The following example illustrates one type of control chart along with its components:

Fig. 13.1. Example of control chart



Source: (Sinha & Vatsa, 2022, p. 310)

3.4. Types of Control Charts

Depending on the nature of the measured data, control charts are categorized into two main types:

- Attribute control charts,
- Variable control charts,

3.4.1. Attribute control charts

3.4.1.1.P-Chart (Proportion Chart)

The P-Chart is used in quality control to monitor processes involving binary outcomes (e.g., defective/non-defective). It tracks the proportion of defects within production samples to determine whether the process remains within statistically acceptable limits. If the defect rate exceeds the upper control limit, this is considered a signal of a potential issue in the process. (Al-Baldawi & Al-Hamidi, 2008, p. 245)

In this case, the following formulas are used: (Al-Baldawi & Al-Hamidi, 2008, p. 245)

Calculating the Defect Proportion for Each Sample:

The defect proportion in each sample is calculated using the following formula:

$$P_i = \frac{r_i}{n_i}$$

Where:

- P_i: Proportion of defects in sample i
- r_i: Number of defective units in sample i
- n_i: Total number of units in sample i

Components of the P-Chart:

The chart is constructed based on a high confidence level of 99.73% (which corresponds to ±3 standard deviations from the mean). The main components are:

1. Central Line (C.L): Represents the average proportion of defects across all samples:

$$\bar{P} = \frac{\sum r_i}{\sum n_i} = \text{CL}$$

3. Upper Control Limit (U.C.L): Indicates the maximum acceptable defect proportion for a sample:

$$U.C.L = \bar{P} + 3\sqrt{\frac{\bar{P}(1-\bar{P})}{n}}$$

4. Lower Control Limit (L.C.L): Indicates the minimum acceptable defect proportion for a sample:

$$L.C.L = \bar{P} - 3\sqrt{\frac{\bar{P}(1-\bar{P})}{n}}$$

Chart Axes:

- Horizontal Axis (X-axis): Represents the sample number or its chronological order.
- Vertical Axis (Y-axis): Represents the defect proportion in each sample.

P-Charts are effective tools for detecting abnormal variations in a production process, allowing corrective actions to be taken before quality issues become widespread.

Example 2**:** In a factory specializing in the production of electric bulbs, a quality inspection was carried out by collecting samples at regular time intervals. Each sample consisted of 120 units. The number of defective units identified in each sample is shown in the following table:

Table. 13.1

Sample Number	1	2	3	4	5	6
Number of Defective Units	4	8	2	9	3	5

Source: Author's own work.

Required: Conduct statistical quality control for the light bulb production process in this factory.

The solution:

Statistical Results:

- Average Defect Proportion ($\bar{P}$):

$$\bar{P} = \frac{4+8+2+9+3+5}{6\times 120} = \frac{31}{720} \approx 0.0431 = CL$$

- Upper Control Limit (UCL):

$$UCL = 0.0431 + 3 \times 0.0186 \approx 0.0990$$

- Lower Control Limit (LCL):

$LCL = 0.0431 - 3 \times 0.0186 \approx -0.0127 \Rightarrow$ Adjusted to 0, as proportions cannot be negative.

Interpretation: (see Fig 13.2)

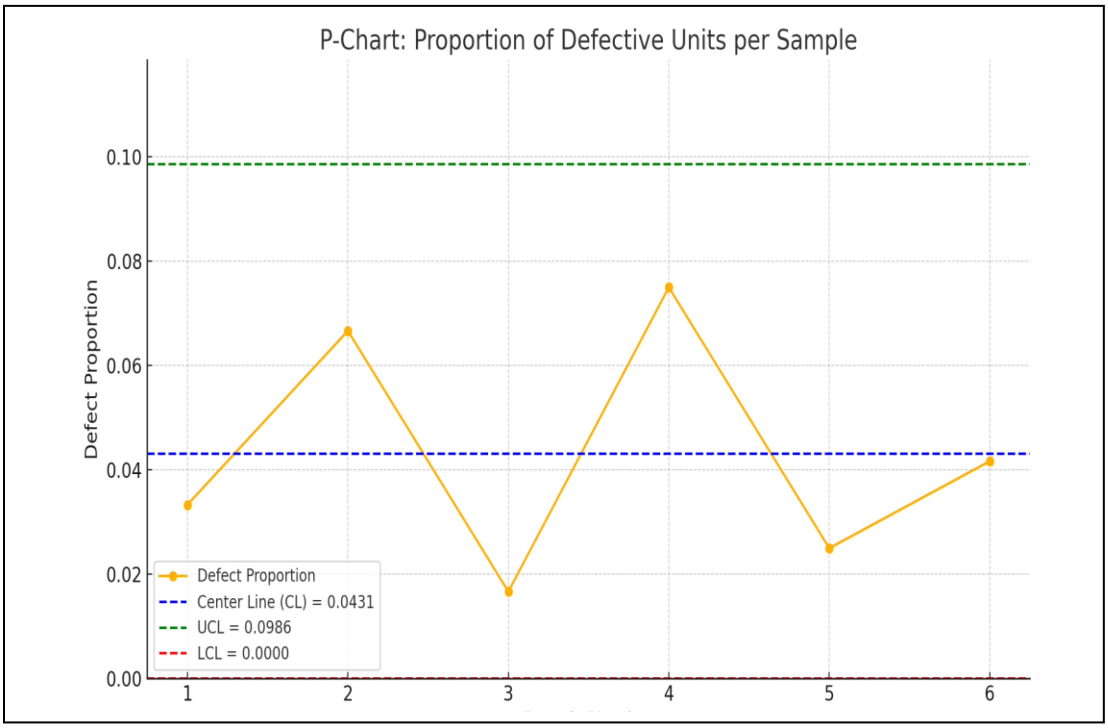
- All sample points fall within the control limits (between 0 and 0.0990).
- There are no signs of abnormal variation or out-of-control behavior.
- The production process can be considered statistically stable at this time.

Table. 13.2

Sample Number	1	2	3	4	5	6
Defect Proportion (P_i)	4/120= 0.0333	8/120= 0.0667	2/120= 0.0167	9/120= 0.0750	3/120= 0.0250	5/120= 0.0417

Source: Author's own work

Fig. 13.2



Source: Author's own work

3.4.1.2. Defective Unit Count Chart (Np-Chart)

In this chart type, it is assumed that the sample sizes are always equal. Thus, the number of defective units in each sample can be used to measure quality instead of the defect proportion. The components of the chart are as follows: (Al-Baldawi & Al-Hamidi, 2008, p. 247)

- Central Line (C.L):

$$C.L = NP$$

where NP is the average number of defective units, N is the sample size, and P is the overall defect proportion.

- Upper Control Limit (U.C.L):

$$U.C.L = NP + 3\sqrt{NP(1-P)}$$

- Lower Control Limit (L.C.L):

$$L.C.L = NP - 3\sqrt{NP(1-P)}$$

- The horizontal axis represents the number of defective units in each sample.
- The vertical axis represents the sample numbers (data points).

Practical Application Using Example 2

Referring back to Example 2 above

- Average Defect Proportion (P):

$$P = \frac{4+8+2+9+3+5}{6 \times 120} = \frac{31}{720} \approx 0.0431$$

- Central Line (C.L):

$$CL = N \cdot \mathrm{P} = 120 \cdot 0.0431 = 5.17$$

- Upper Control Limit (U.C.L):

$$UCL = 5.17 + 3 \cdot \sqrt{5.17 \cdot (1 - 0.0431)} \approx 11.84$$

- Lower Control Limit (L.C.L):

$$LCL = 5.17 - 3 \cdot \sqrt{5.17 \cdot (1 - 0.0431)} \approx -1.50$$

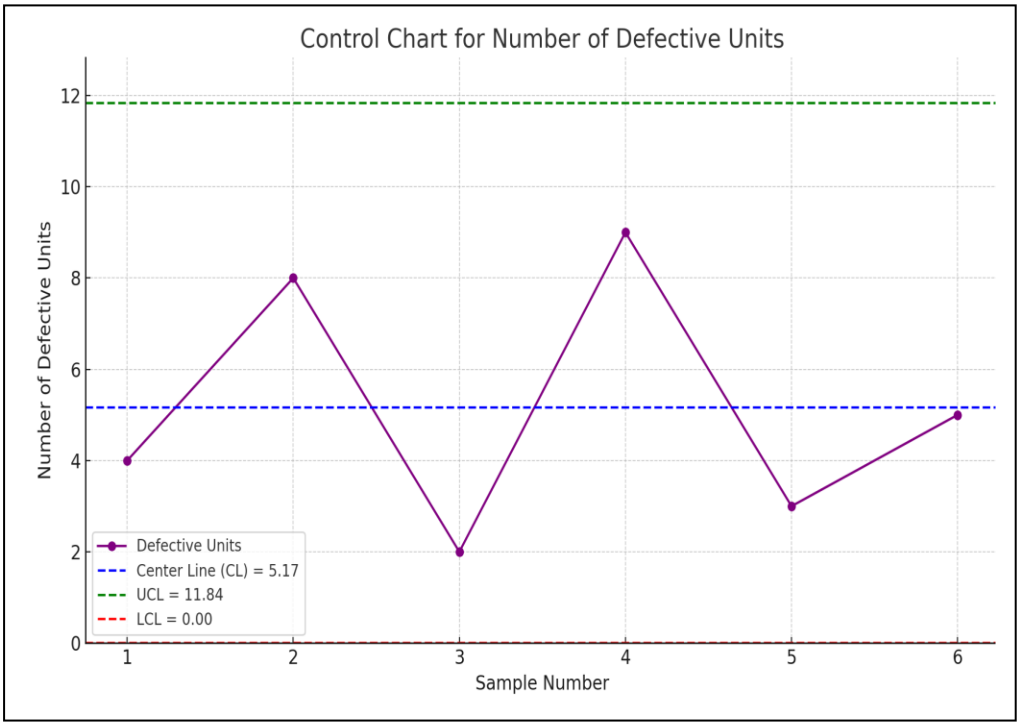
⇒ Adjusted to 0, since the count cannot be negative.

Results are:

- Central Line (CL): 5.17 defective units
- Upper Control Limit (UCL): ~11.84 defective units
- Lower Control Limit (LCL): Adjusted to 0

Based on Table 13.1 and the obtained results, the following chart is derived

Fig. 13. 3



Source: Author's own work

All observed defect counts (4, 8, 2, 9, 3, 5) fall within the control limits, indicating that the production process is statistically under control using the Np-Chart approach.

3.4.1.3. C-Chart for the Number of Defects per Unit

This type of control chart is specifically designed for monitoring the quality of individual units, and is commonly employed in the context of attribute (qualitative) quality control for finished products. The chart tracks the number of defects identified in several items across multiple samples. It operates under the assumption that the number of defects follows a Poisson distribution, making the C-chart a statistically appropriate tool for such analysis. (Al-Baldawi & Al-Hamidi, 2008, p. 248)

In this case, the following formulas are used: (Al-Baldawi & Al-Hamidi, 2008, pp. 248-249)

The C-chart consists of three essential components:

- Central Line (CL): Represents the average number of defects per unit.

$$CL = \bar{C}$$

- Upper Control Limit (UCL):

$$UCL = \bar{C} + 3\sqrt{\bar{C}}$$

- Lower Control Limit (LCL):

$$LCL = \bar{C} - 3\sqrt{\bar{C}}$$

The average number of defects $\bar{C}$ is computed using the formula:

$$\bar{C} = \frac{1}{K}\sum_{j=1}^{k} C_j$$

Where:

- C_{ij} is the number of defects observed in unit j during period i.
- K denotes the number of samples.
- n_K refers to the number of units in each sample.

Example 3:

In a production facility, a quality inspection was conducted on 7 samples, each consisting of 4 individual units. The number of defects found in each unit was recorded, as shown in the following table.

Table. 13.3

Sample	Unit 1	Unit 2	Unit 3	Unit 4	Total Defects (Cj)	Avg. Defects per Unit
1	2	3	2	4	11	2.75
2	5	4	6	7	22	5.50
3	1	2	2	1	6	1.50
4	0	1	2	1	4	1.00
5	2	1	0	1	4	1.00
6	4	3	5	4	16	4.00
7	6	5	7	8	26	6.50

Source: Author's own work

Required: Calculate the average number of defects, construct a C-Chart with the central line (CL), upper control limit (UCL), and lower control limit (LCL), and determine whether the process is statistically in control.

Solution

We begin by calculating the average number of defects per unit for each sample as follows:

Table. 13.4

Sample	Unit 1	Unit 2	Unit 3	Unit 4	Total Defects (Cj)	Avg. Defects per Unit
1	2	3	2	4	11	2.75
2	5	4	6	7	22	5.50
3	1	2	2	1	6	1.50
4	0	1	2	1	4	1.00
5	2	1	0	1	4	1.00
6	4	3	5	4	16	4.00
7	6	5	7	8	26	6.50

Source: Author's own work

Next, we compute the overall statistics as follows:

Average number of defects per sample (CL = $\bar{C}$):

$$\bar{C} = \frac{11 + 22 + 6 + 4 + 4 + 16 + 26}{7} = \frac{89}{7} \approx 12.71$$

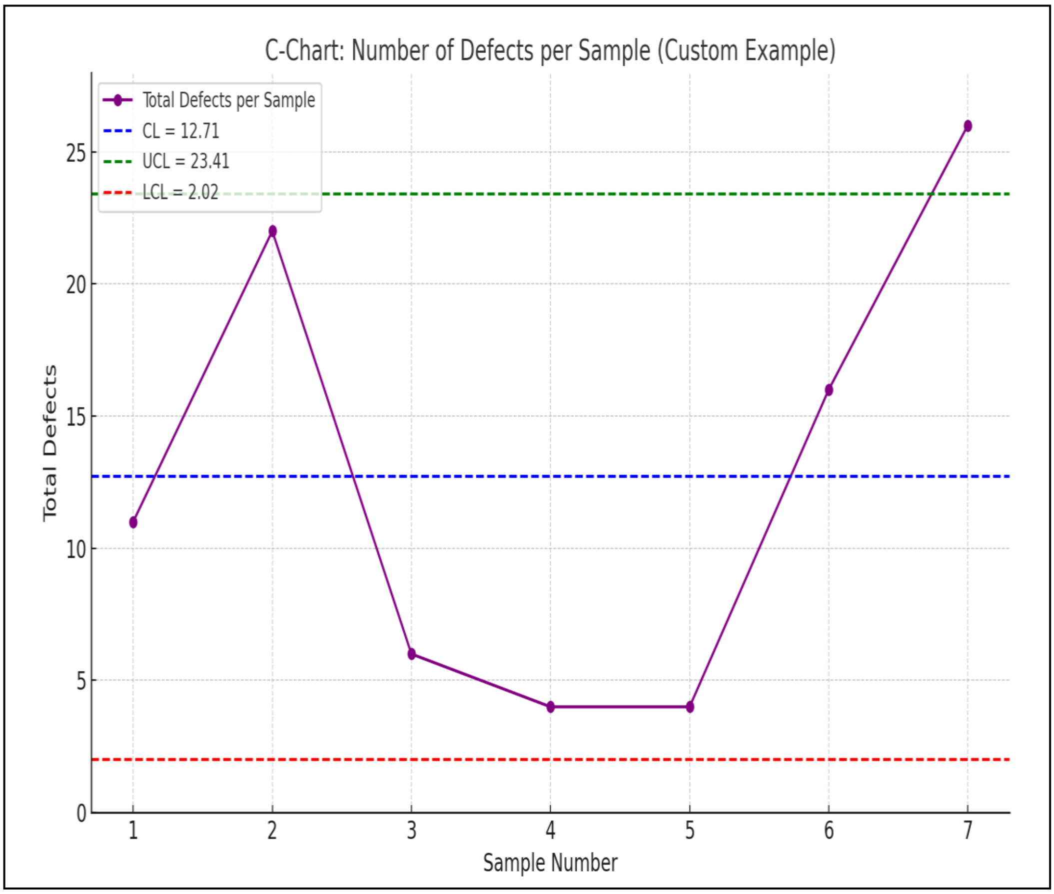
- Upper Control Limit (UCL):

$$UCL = \bar{C} + 3\sqrt{\bar{C}} \approx 12.71 + 3\sqrt{12.71} \approx 23.38$$

- Lower Control Limit (LCL):

$$LCL = \bar{C} - 3\sqrt{\bar{C}} \approx 12.71 - 3\sqrt{12.71} \approx 2.03$$

Fig. 13.3



Source: Author's own work

- The chart illustrates the position of each sample in relation to the control limits.
- Sample 7 recorded 26 defects, which exceeds the Upper Control Limit (UCL). This indicates an abnormal variation in the process.
- All other samples fall within the control limits, suggesting that the process is generally stable

3.4.2. Variable control charts

Control charts for variables are quantitative tools used in quality control for monitoring measurable characteristics such as diameter, weight, length, tensile strength, glossiness, and other physical or mechanical properties of a product. Unlike attribute charts, variable charts deal with continuous quantitative data. (Al-Baldawi & Al-Hamidi, 2008, p. 250)

3.4.2.1. Range Chart (R-Chart)

The R-chart is used to monitor the variability of measurement values within each sample. The range (R) is calculated as the difference between the maximum and minimum value within the sample: (Al-Baldawi & Al-Hamidi, 2008, p. 250)

$$R = X_{max} - x_{min}$$

Elements of the R-Chart: (Al-Baldawi & Al-Hamidi, 2008, p. 250)

- Central Line (CL: $\bar{R}$): Represents the arithmetic mean of the range values across all samples and is calculated using the formula:

$$\bar{R} = \frac{\sum_{i=1}^{K} R_i}{K}$$

Where: R_i: Range of the i-th sample
k: Total number of samples

- Upper Control Limit (UCL):

$$UCLL = D_4 \times \bar{R}$$

- Lower Control Limit (LCL):

$$LCL = D_3 \times \bar{R}$$

The constant values D_3 and D_4 are obtained from standard statistical tables (such as the Grant chart (see Table. 13.7)) based on the sample size.

Example 4

In a manufacturing process, a quality control engineer selects 6 samples, each containing 5 units, to measure a specific physical characteristic (e.g., weight, diameter). The recorded values for each unit in each sample are shown in the table below:

Table. 13.5

Sample	Unit 1	Unit 2	Unit 3	Unit 4	Unit 5
1	12	18	17	14	16
2	15	14	16	13	12
3	11	13	12	10	14
4	13	16	14	11	13
5	14	15	18	17	16
6	12	13	15	14	11

Source: Author's own work

Required:

- Calculate the range (R) for each sample.
- Compute the average range (R-bar),
- Determine:
  - Upper Control Limit (UCL)
  - Lower Control Limit (LCL)
- Draw the R-chart with:

- Control limits (UCL, LCL)
- Center line (CL)
- Sample ranges

- Conclude whether the process is statistically under control.

Solution:

Table. 13.6

Sample	Max Value	Min Value	Range (R)
1	18	12	6
2	16	12	4
3	14	10	4
4	16	11	5
5	18	14	4
6	15	11	4

Source: Author's own work

Step 1: Average Range (R-bar): $\bar{R} = \frac{6+4+4+5+4+4}{6} = \frac{27}{6} = 4.50$

Step 2: Control Limits:

Using constants:

- D3 = 0 (see Table. 13.7)
- D4 = 2.114 (see Table. 13.7)

$UCL = D_4 \times \bar{R} = 2.114 \times 4.50 = 9.51$

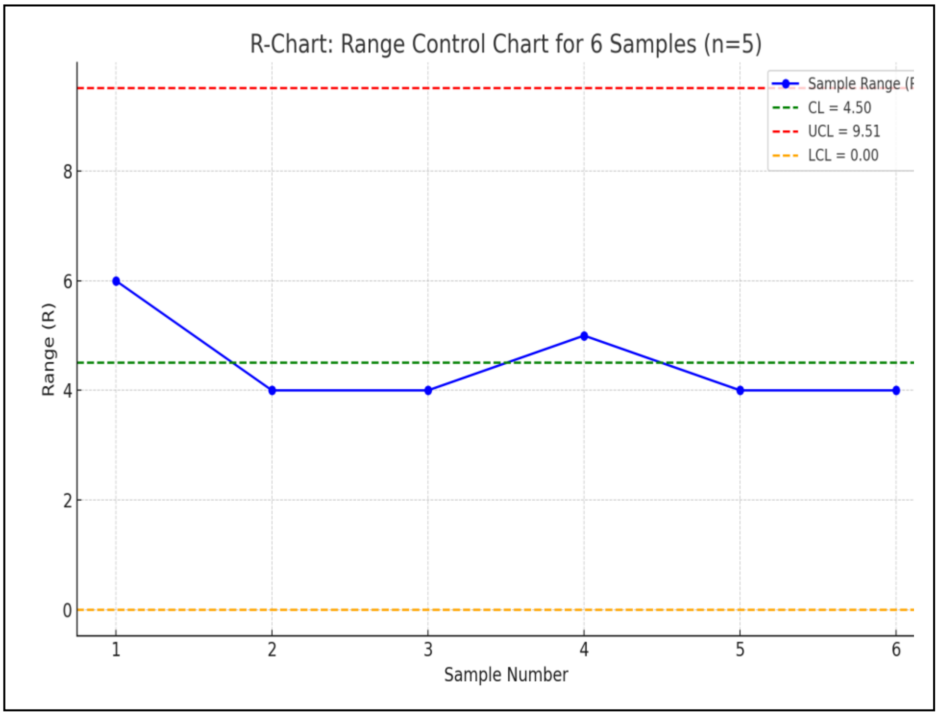
$LCL = D_3 \times \bar{R} = 0 \times 4.50 = 0$

Step 3: Interpretation: All sample ranges fall within the control limits [0, 9.51], indicating the process is statistically under control.

R-Chart:

- X-axis: Sample Number
- Y-axis: Range (R)
- Blue line: Sample Ranges
- Green dashed line: Center Line (CL = 4.50)
- Red dashed line: Upper Control Limit (UCL = 9.51)
- Orange dashed line: Lower Control Limit (LCL = 0)

Fig. 13.4



Source: Author's own work

Table. 13.7 Control Chart Constants

	X-bar Chart Constants		for sigma estimate	R Chart Constants		S Chart Constants	
Sample Size = m	A_2	A_3	d_2	D_3	D_4	B_3	B_4
2	1.880	2.659	1.128	0	3.267	0	3.267
3	1.023	1.954	1.693	0	2.574	0	2.568
4	0.729	1.628	2.059	0	2.282	0	2.266
5	0.577	1.427	2.326	0	2.114	0	2.089
6	0.483	1.287	2.534	0	2.004	0.030	1.970
7	0.419	1.182	2.704	0.076	1.924	0.118	1.882
8	0.373	1.099	2.847	0.136	1.864	0.185	1.815
9	0.337	1.032	2.970	0.184	1.816	0.239	1.761
10	0.308	0.975	3.078	0.223	1.777	0.284	1.716
11	0.285	0.927	3.173	0.256	1.744	0.321	1.679
12	0.266	0.886	3.258	0.283	1.717	0.354	1.646
13	0.249	0.850	3.336	0.307	1.693	0.382	1.618
14	0.235	0.817	3.407	0.328	1.672	0.406	1.594
15	0.223	0.789	3.472	0.347	1.653	0.428	1.572
16	0.212	0.763	3.532	0.363	1.637	0.448	1.552

	X-bar Chart Constants		for sigma estimate	R Chart Constants		S Chart Constants	
Sample Size = m	A_2	A_3	d_2	D_3	D_4	B_3	B_4
17	0.203	0.739	3.588	0.378	1.622	0.466	1.534
18	0.194	0.718	3.640	0.391	1.608	0.482	1.518
19	0.187	0.698	3.689	0.403	1.597	0.497	1.503
20	0.180	0.680	3.735	0.415	1.585	0.510	1.490
21	0.173	0.663	3.778	0.425	1.575	0.523	1.477
22	0.167	0.647	3.819	0.434	1.566	0.534	1.466
23	0.162	0.633	3.858	0.443	1.557	0.545	1.455
24	0.157	0.619	3.895	0.451	1.548	0.555	1.445
25	0.153	0.606	3.931	0.459	1.541	0.565	1.435

Control chart constants for X-bar, R, S, Individuals (called "X" or "I" charts), and MR (Moving Range) Charts.

NOTES:

- To construct the "X" and "MR" charts (these are companions), we compute the Moving Ranges as:
- R_2 = range of 1st and 2nd observations
- R_3 = range of 2nd and 3rd observations
- R_4 = range of 3rd and 4th observations
- etc.

Where the "average" moving range or "MR-bar" being the average of these ranges with the "sample size" for each of these ranges being n = 2 since each is based on consecutive observations… this should provide an estimated standard deviation (needed for the "I" chart) of:

$$\sigma = \frac{\text{MR-bar}}{d_2}$$

Where the value of d_2 is based on, as just stated, m = 2.

Similarly, the UCL and LCL for the MR chart will be:

$$\text{LCL} = D_3 \cdot \text{MR-bar}, \quad \text{UCL} = D_4 \cdot \text{MR-bar}$$

But since D_3 = 0 when n = 0 (or, more accurately, is "not applicable"), there will be no LCL for the MR chart, just a UCL.

Source: (Bessegato)

3.4.2.2. $\bar{X}$ Control Chart ($\bar{X}$-Chart)

The $\bar{X}$-Chart is a key statistical process control tool used to monitor the stability of the process mean over time. It is constructed by calculating the average value of each sample, and its control limits are determined based on the variability within each sample. (Al-Baldawi & Al-Hamidi, 2008, p. 252)

The chart includes the following components: (Al-Baldawi & Al-Hamidi, 2008, p. 252)

- Central Line (CL): This represents the average of all sample means and is computed as:

$$\bar{X} = \frac{1}{k}\sum_{j=1}^{k} \bar{X}_j$$

Where:

- $\bar{X}_j$: Mean of sample j
- k: Total number of samples

- Upper Control Limit (UCL):

$$UCL = \bar{X} + A_2 \cdot \bar{R}$$

- Lower Control Limit (LCL):

$$LCL = \bar{X} - A_2 \cdot \bar{R}$$

The constant A_2 is obtained from standard statistical tables (e.g., Grant's Table) and depends on the sample size.

- X-axis: Represents the sample numbers.
- Y-axis: Represents the sample means.

Practical Application Using Example 4

Referring back to previous Example 4

1. Sample Calculations

- Mean of each sample:

$$\bar{X}_j = \frac{X_1 + X_2 + X_3 + X_4 + X_5}{5}$$

- Range of each sample:

$$R_j = X_{\max} - X_{\min}$$

- Overall mean of the sample means:

$$\bar{\bar{X}} = \frac{\sum_{j=1}^{6} \bar{X}_j}{6}$$

- Average range:

$$\bar{R} = \frac{\sum_{j=1}^{6} R_j}{6}$$

- Control limits (for n = 5):

From the A2 constant table (see Table. 13.7):

$$A_2 = 0.577$$

- Upper Control Limit (UCL):

$$UCL = \bar{\bar{X}} + A_2 \cdot \bar{R}$$

- Lower Control Limit (LCL):

$$LCL = \bar{X} - A_2 \cdot \bar{R}$$

2. Calculation Table

Table. 13.8

Sample	Unit 1	Unit 2	Unit 3	Unit 4	Unit 5	Mean ($\bar{X}_k$)	Range (R_k)
1	12	18	17	14	16	15.40	6
2	15	14	16	13	12	14.00	4
3	11	13	12	10	14	12.00	4
4	13	16	14	11	13	13.40	5
5	14	15	18	17	16	16.00	4
6	12	13	15	14	11	13.00	4

Source: Author's own work

3. Final Calculations

$$\bar{\bar{X}} = \frac{15.4 + 14 + 12 + 13.4 + 16 + 13}{6} = \frac{83.8}{6} \approx 13.97$$

$$\bar{R} = \frac{6 + 4 + 4 + 5 + 4 + 4}{6} = \frac{27}{6} = 4.50$$

$$UCL = 13.97 + (0.577 \times 4.50) = 13.97 + 2.59 = 16.56$$

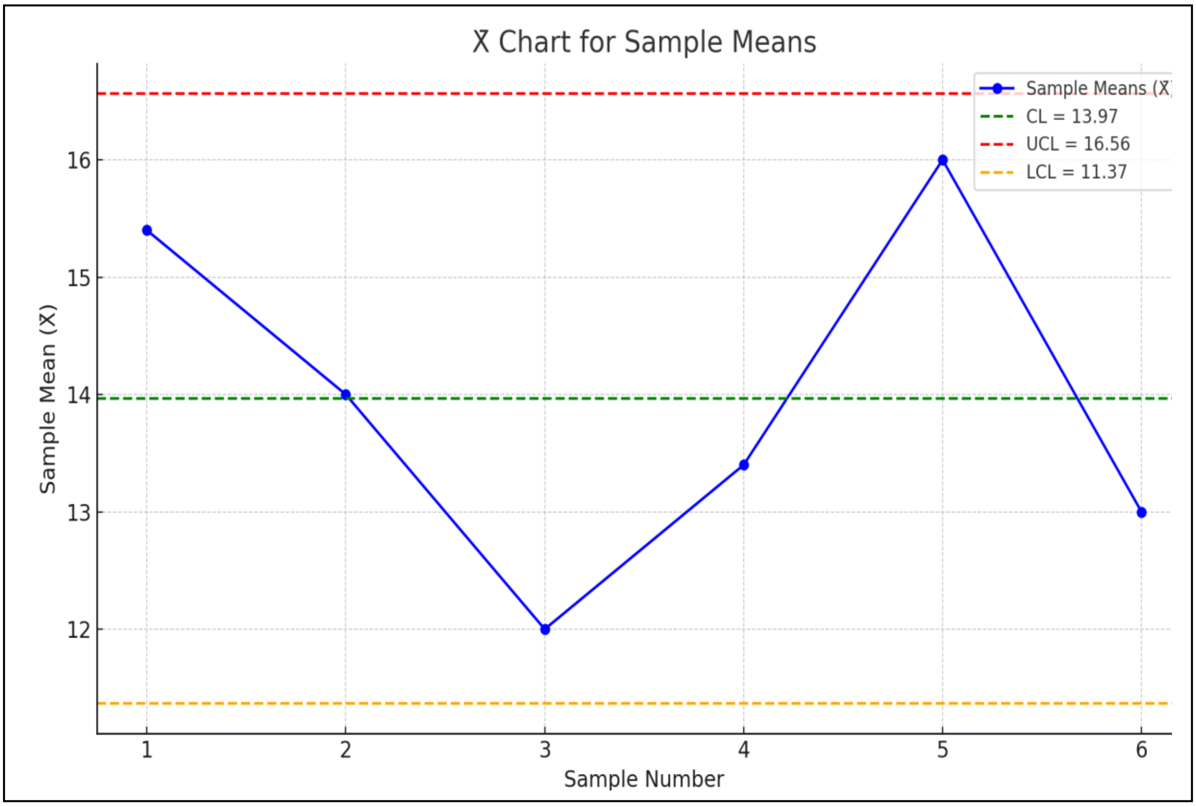
$$LCL = 13.97 - 2.59 = 11.38$$

4. $\bar{X}$ Control Chart (see Fig. 13.5)

- X-axis: Sample Number
- Y-axis: Sample Mean
- Green Line: Central Line (CL = 13.97)
- Red Line: Upper Control Limit (UCL = 16.56)
- Orange Line: Lower Control Limit (LCL = 11.38)

Interpretation: All sample means ($\bar{X}$) lie within the control limits, which indicates that the production process is statistically under control.

Fig. 13.5



Source: Author's own work

3.4.2.3. Standard Deviation Control Chart (σ-Chart)

The σ-chart is a statistical tool used to monitor the variability within a process by tracking the standard deviation of sample observations. The chart is constructed based on the following elements:

- Center Line (C.L): This represents the average standard deviation across all samples, calculated using the formula:

$$\bar{\sigma} = \frac{1}{K}\sum_{i=1}^{K}\sigma_i$$

Where:

- σ_i is the standard deviation for sample i, computed as:

$$\sigma_i = \sqrt{\frac{\sum_{i=1}^{K}(X_i - \bar{X})^2}{n}}$$

- n: number of observations per sample.
- K: total number of samples.

4. Control Limits:

- Upper Control Limit (UCL):

$$UCL = B_4 \times \bar{\sigma}$$

- Lower Control Limit (LCL):

$$LCL = B_3 \times \bar{\sigma}$$

The constants B_3 and B_4 are derived from statistical reference tables (e.g., Grant's Table (see Table. 13.7)) according to the sample size.

- The horizontal axis in the control chart represents the sample number.
- The vertical axis displays the standard deviation values for each sample.

Practical Application Using Example 4

Referring back to previous Example 4

1. Standard Deviation for a Single Sample (σ_i):

$$\boldsymbol{\sigma_i} = \sqrt{\frac{\sum_{i=1}^{K} (X_i - \bar{X})^2}{n}}$$

2. Average Standard Deviation for All Samples ($\bar{\sigma}$):

$$\bar{\sigma} = \frac{1}{K} \sum_{i=1}^{K} \sigma_i$$

Applied Example on Sample 1:

Sample 1 Data: 12, 18, 17, 14, 16 Number of units (n) = 5

Step 1: Compute the Sample Mean

$$\bar{X}_1 = \frac{12 + 18 + 17 + 14 + 16}{5} = \frac{77}{5} = 15.4$$

Step 2: Apply the Standard Deviation Formula

$$\sigma_1 = \sqrt{\frac{(12 - 15.4)^2 + (18 - 15.4)^2 + (17 - 15.4)^2 + (14 - 15.4)^2 + (16 - 15.4)^2}{5}}$$

$$\sigma_1 = \sqrt{\frac{(-3.4)^2 + (2.6)^2 + (1.6)^2 + (-1.4)^2 + (0.6)^2}{5}}$$

$$\sigma_1 = \sqrt{\frac{11.56 + 6.76 + 2.56 + 1.96 + 0.36}{5}} = \sqrt{\frac{23.2}{5}} = \sqrt{4.64} \approx 2.1541$$

In the same manner, the standard deviation for the remaining samples is calculated, resulting in the following table:

Table. 13.9 Summary Table for All Samples

Sample	Sample Mean ($\bar{X}$)	Standard Deviation (σ_i)
1	15.40	2.1541
2	14.00	1.4142
3	12.00	1.4142
4	13.40	1.7205
5	16.00	1.4142
6	13.00	1.2605

Source: Author's own work

Final Calculations

$\bar{\sigma}$ = (2.1541 + 1.4142 + 1.4142 + 1.7205 + 1.4142 + 1.2605) / 6 ≈ 1.5726=CL

Using constants for n = 5: B3 = 0, B4 = 2.089 (see Table 13.7)

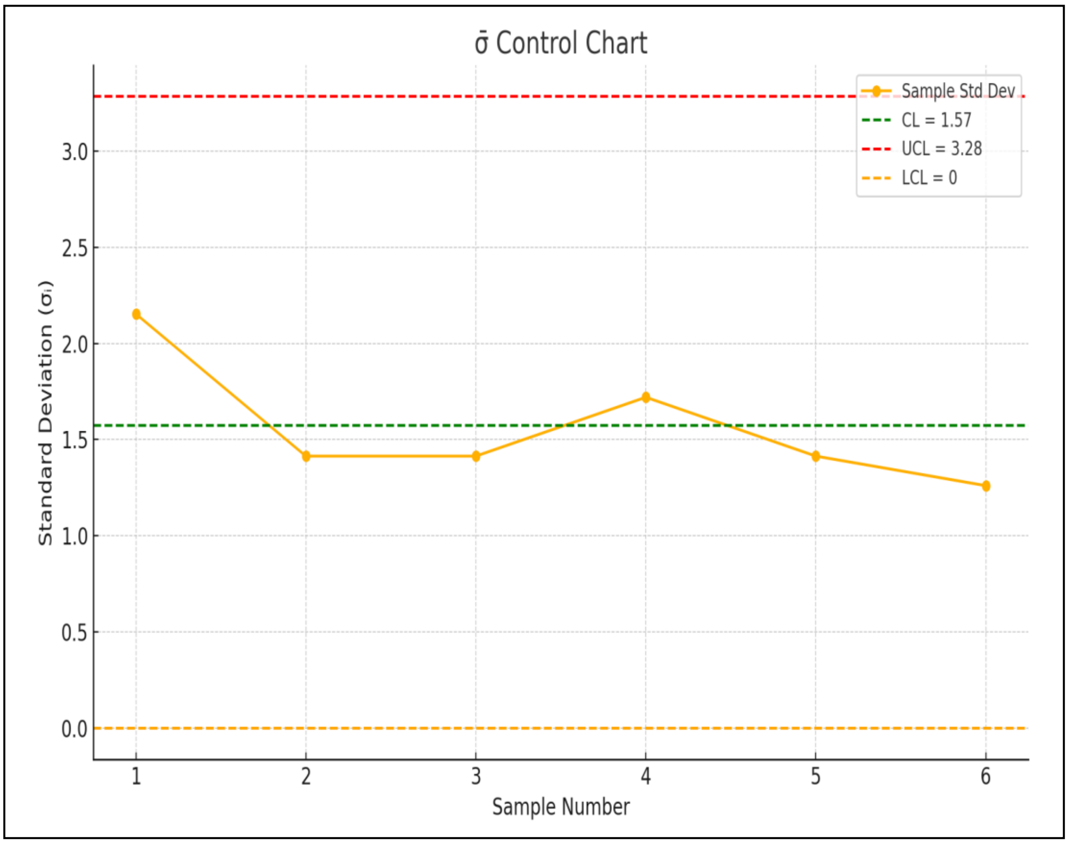
UCL = B4 × $\bar{\sigma}$ = 2.089 × 1.5726 ≈ 3.2852

LCL = B3 × $\bar{\sigma}$ = 0 × 1.5726 = 0

$\bar{\sigma}$ Control Chart (see Fig. 13.6)

- X-axis: Sample Number
- Y-axis: Sample Mean
- Green Line: Central Line (CL = 1.57)
- Red Line: Upper Control Limit (UCL = 3.28)
- Orange Line: Lower Control Limit (LCL = 0)

Fig. 13.6



Source: Author's own work

Interpretation: All sample means ($\bar{\sigma}$) lie within the control limits, which indicates that the production process is statistically under control.

Conclusion

From the discussion above, it is evident that Statistical Quality Control offers a structured methodology for organizations to transition from reactive troubleshooting to proactive monitoring and control. Through the scientific interpretation of control charts, deviations from normal process behavior can be detected early, allowing for timely adjustments to reduce waste and prevent defects.

Moreover, embracing a culture that relies on data-driven decision-making fosters a more efficient and reliable production environment. Therefore, mastering the principles and techniques of statistical control is a foundational step for anyone aiming to establish a robust and sustainable quality management system.

Lecture 14: Dynamic Programming

Introduction

The term linear programming dates back to 1957 when Richard published his findings from his research at RAND company.

Dynamic programming is used in decision problems that contain many sequential and consecutive stages, as it is a computational technique aimed at finding the optimal solution. the word dynamic does not mean that it has anything to do with time, that is, time, despite its importance, does not enter into it as one of the variables on which the optimal solution is based, but the meaning of dynamic means that the optimal solution includes many optimal partial solutions that relate to each of the stages covered by the issue where the optimal solution is presented for each partial problem that appears at each stage.

Issues in which decisions are made sequentially and sequentially involve three basic components: the stage, the options or alternatives (variables), and the state of the system at each stage

Dynamic programming is based on breaking down a problem into multiple stages, each stage containing some alternatives, which also results in a sequence of alternatives throughout the entire problem (Al-Baldawi & Al-Hamidi, 2008, p. 99)

1. Methods for solving sequential decision problems (with sequential stages)

There are two methods for solving decision issues qs following: (Al-Baldawi & Al-Hamidi, 2008, pp. 99-100)

1.1.The arithmetic method:

This method is based on identifying all the different paths within the network diagram, and then determining the lowest cost or the greatest possible return or profit by calculating the sum of all the values in each path from the first path to the last path and selecting the path according to the following two cases:

1.1.1.Minimal Cost (MIN): The path with the lowest total cost is selected.

1.1.2.In the case of maximising returns or profits (MAX): The path that achieves the highest possible profit or return is selected.

1.2.Linear programming method: It includes two types of algorithms, one forward and one backward, as follows:

1.2.1.The Forward solution algorithm:

The solution is to collect the costs or returns for each stage and each alternative with the best outcome resulting from the stage and the previous alternative (or the previous alternatives if there is more than one alternative). In the case of minimizing costs, we choose to combine the cost of the current stage and the current alternative with the lowest cost provided by one of the alternatives in The immediately preceding stage. In the case of returns, we choose the largest profit achieved by one of the alternatives in the previous stage and combine it with the current alternative. This is if these alternatives in the previous stage come directly before the current alternative in the current stage, that is, they are drawn in the network diagram with arrows starting from the alternatives in The previous stage to reach the current alternative in the current stage.

Provided that the value 0 is given to the first alternative in the first stage in the network diagram, whether the matter relates to costs or returns and profits.

This method is based on the following two laws:

MIN case:

$C^*_{s(x)}$=Min [C(N,X)+ $C^*_{S-1(N)}$]

MAX case:

$C^*_{s(x)}$=Max [C(N,X)+ $C^*_{S-1(N)}$]

Where

$C^*_{s(x)}$: Cost or return at position x and stage s.

C(N, X): The cost (or return) of moving from alternative n to alternative x.

1.2.2.The backward or inverse solution algorithm:

It is the opposite of the forward solution algorithm, that is, it reverses it in the direction, so that it starts from the back, that is, from the last alternative in the last stage within the network diagram, and with the same instructions as in the forward solution algorithm, with the last alternative in the last stage given the value 0.

This method is based on the following two laws:

MIN case:

$C^*_{s(x)}$=Min [C(N, X)+ $C^*_{S+1(N)}$]

MAX case:

$C^*_{s(x)}$=Max [C(N, X)+ $C^*_{S+1(N)}$]

Where

$C^*_{s(x)}$: Cost or return at position x and stage s.

C(N, X): The cost (or return) of moving from alternative n to alternative x.

2.Exemples and its solutions

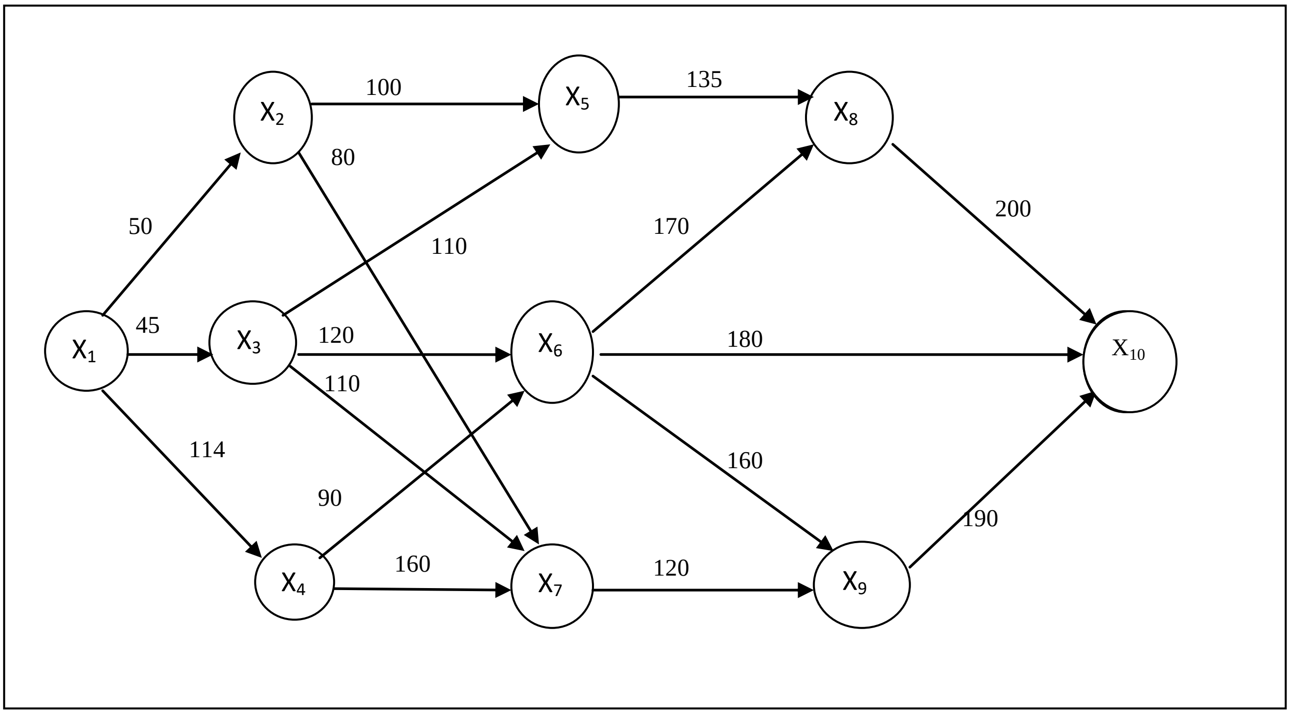
2.1.Example 1

Assuming that a company produces a specific product by passing through different production workshops, each workshop has specific production costs as shown in the following figure (Fig. 12-1).

Questions:

1. Find the optimal solution by using *The arithmetic method.*
2. Find the optimal solution by using *The Forward solution algorithm*
3. Find the optimal solution by using *The backward or inverse solution algorithm*

Fig. 12-1



Source: Author's own work

The solution

Firstly, we begin by dividing the network diagram into several stages by drawing a vertical dashed line, as in the following Fig. 12-2.

Secondly, one of the following methods will be used for calculation and choosing the best path which achieves the lowest cost.

1.Finding the optimal solution by using *The arithmetic method.*

1) the cost of a path (x_1-x_2-x_5-x_8-x_{10}) is 50+100+135+200= 485.

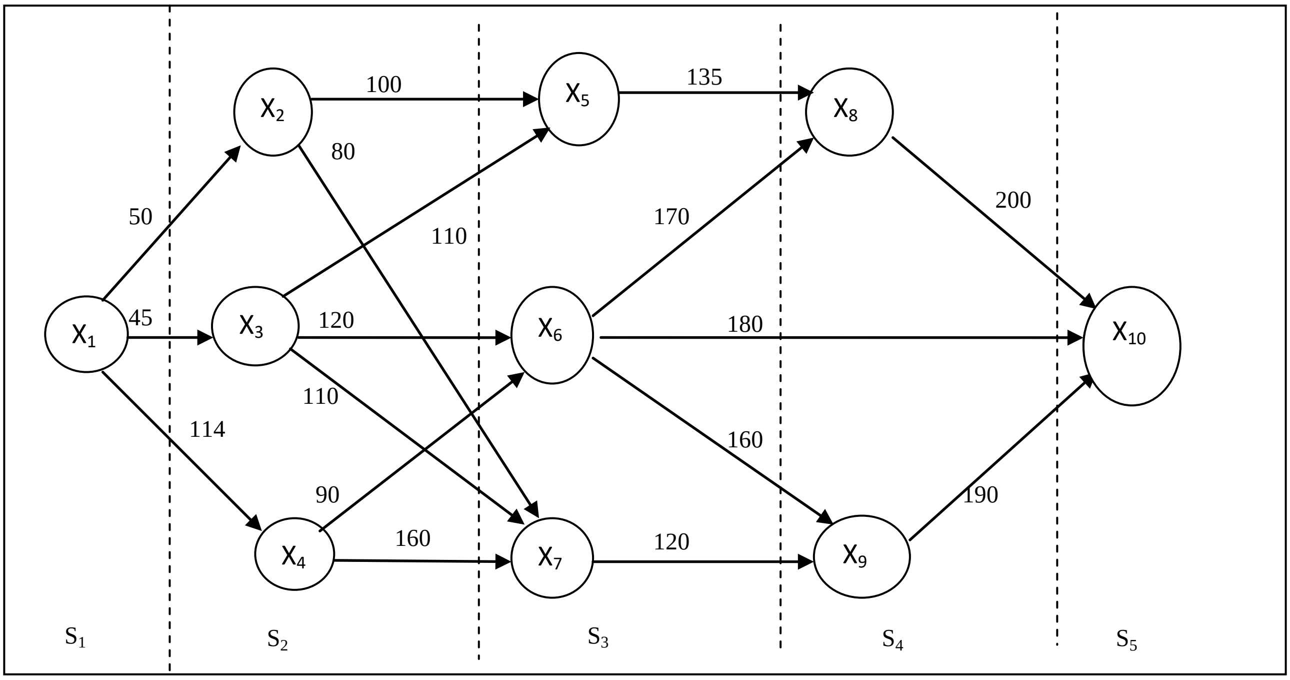
2) the cost of a path (x_1-x_2-x_7-x_9-x_{10}) is 50+80+120+190= 440.

3) the cost of a path (x_1-x_3-x_5-x_8-x_{10}) is 45+110+135+200= 490.

4) the cost of a path (x_1-x_3-x_6-x_8-x_{10}) is 45+120+170+200= 535.

5) the cost of a path (x_1-x_3-x_6 -x_{10}) is: 45+120+180= 345.

Fig. 12-2



Source: Author's own work

6) the cost of a path (x_1-x_3-x_6-x_9-x_{10}) is:45+120+160+190= 515.

7) the cost of a path (x_1-x_3-x_7-x_9-x_{10}) is 45+110+120+190=465.

8) the cost of a path (x_1-x_4-x_6-x_8-x_{10}) is 114+90+170+200= 574.

9) the cost of a path (x_1-x_4-x_6- x_{10}) is 114+90+180= 384.

10) the cost of a path (x_1-x_4-x_7-x_9-x_{10}) is: 114+160+120+190= 584.

Based on previous calculations the lowest cost is: 345 monetary units which is calculated from path number 5, thus the best path is the fifth path.

1. Finding the optimal solution by using *The Forward solution algorithm*

We use the following rule:

$C^*_{s(x)}$=Min [C(N,X)+ $C^*_{S-1(N)}$]

Assuming that $C^*_{1(1)}$=0

$C^*_{2\,(2)}$=Min [C(1,2)+ $C^*_{1(1)}$]= Min [50+0]= 50.

$C^*_{2\,(3)}$=Min [C(1,3)+ $C^*_{1(1)}$]= Min [45+0]= 45.

$C^*_{2\,(4)}$=Min [C(1,4)+ $C^*_{1(1)}$]= Min [114+0]= 114.

$C^*_{3\,(5)}$=Min [C(2,5)+ $C^*_{2(2)}$, C(3,5)+ $C^*_{2(3)}$]= Min [100+50, 110+45]= Min [150, 155]= 150.

$C^*_{3\,(6)}$=Min [C(3,6)+ $C^*_{2(3)}$, C(4,6)+ $C^*_{2(4)}$]= Min [120+45, 90+114]= Min [165, 204]= 165.

$C^*_{3\ (7)}$=Min [C(2,7)+ $C^*_{2(2)}$, C(3,7)+ $C^*_{2(3)}$,C(4,7)+ $C^*_{2(4)}$]= Min [80+50, 110+45, 160+114]= Min [130, 155, 274]= 130.

$C^*_{4\ (8)}$=Min [C(5,8)+ $C^*_{3(5)}$, C(6,8)+ $C^*_{3(6)}$]= Min [135+150, 170+165]= Min [285, 335]= 285.

$C^*_{4\ (9)}$=Min [C(6,9)+ $C^*_{3(6)}$, C(7,9)+ $C^*_{3(7)}$]= Min [160+165, 120+130]= Min [325, 250]= 250.

$C^*_{5\ (10)}$=Min [C(8,10)+ $C^*_{4(8)}$, C(6,10)+ $C^*_{3(6)}$,C(9,10)+ $C^*_{4(9)}$]= Min [200+285, 180+165, 190+250]= Min [485, 345, 440]= 345.

Based on previous calculations the lowest cost is 345 monetary units which is calculated from the shaded path during calculations. It is (x_1-x_3-x_6 -x_{10}).

2. Finding the optimal solution by using *The backward or inverse solution algorithm*

We use the following rule:

$C^*_{s(x)}$=Min [C(N,X)+ $C^*_{S+1(N)}$]

Assuming that $C^*_{5(10)}$=0

$C^*_{4\ (8)}$=Min [C(8,10)+ $C^*_{5(10)}$]= Min [200+0]= 200.

$C_{4\ (9)}$=Min [C(9,10)+ $C^*_{5(10)}$]= Min [190+0]= 190.

$C^*_{3\ (5)}$=Min [C(5,8)+ $C^*_{4(8)}$]= Min [135+200]= 335.

$C^*_{3(6)}$=Min [C(6,8)+ $C^*_{4(8)}$, C(6,10)+ $C^*_{5(10)}$,C(9,10)+ $C^*_{5(10)}$]= Min [170+200, 180+0, 160+190]= Min [370, 180, 350]= 180.

$C^*_{3\ (7)}$=Min [C(7,9)+ $C^*_{4(9)}$]= Min [120+190]= 310.

$C^*_{2(2)}$=Min [C(2,5)+ $C^*_{3(5)}$, C(2,7)+ $C^*_{3(7)}$]= Min [100+335, 80+310]= Min [435, 390]= 390.

$C^*_{2(3)}$=Min [C(3,5)+ $C^*_{3(5)}$, C(3,6)+$C^*_{3(6)}$,C(3,7)+ $C^*_{3(7)}$]= Min [110+335, 120+180, 110+310]= Min [445, 300, 420]= 300.

$C^*_{2(4)}$=Min [C(4,6)+ $C^*_{3(6)}$, C(4,7)+ $C^*_{3(7)}$]= Min [90+180, 160+310]= Min [270, 470]= 270.

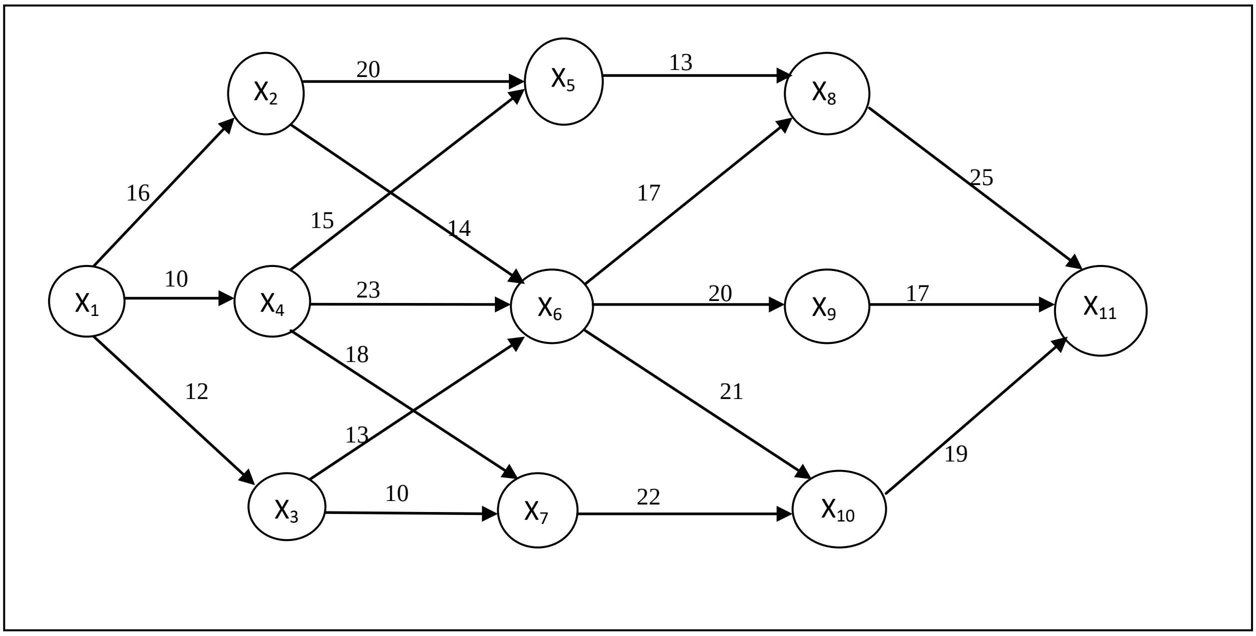
$C^*_{1(1)}$=Min [C(1,2)+ $C^*_{2(2)}$, C(1,3)+$C^*_{2(3)}$,C(1,4)+ $C^*_{2(4)}$]= Min [50+390, 45+300, 114+270]= Min [440, 345, 384]= 345.

Based on previous calculations the lowest cost is 345 monetary units which is calculated from the shaded path during calculations. It is (x_1-x_3-x_6 -x_{10}).

2.2.Example 2

Air Algerie intends to establish an airline linking the capital to Medina, and given the possibility of transferring passengers to several airports during its journey before reaching Medina airport, so that it achieves additional profits from transferring passengers between each airport, It has identified profit-making paths (in thousands of monetary units) as follows (Fig. 12-3).

Fig. 12-3



Source: Author's own work

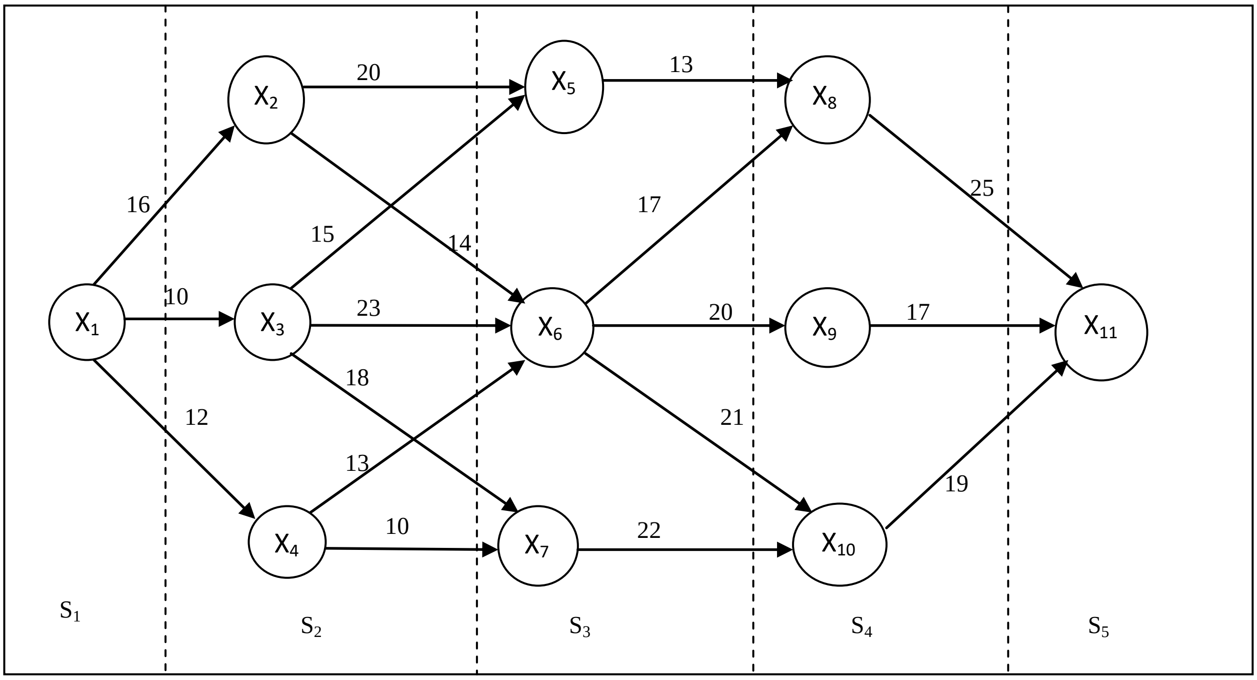
Questions:

1. Find the optimal solution by using *The arithmetic method.*
2. Find the optimal solution by using *The Forward solution algorithm*
3. Find the optimal solution by using *The backward or inverse solution algorithm*

The solution

Firstly, we begin by dividing the network diagram into several stages by drawing a vertical dashed line, as in the following Fig. 12-4.

Fig. 12-4



Source: Author's own work

Secondly, one of the following methods will be used for calculation and choosing the best path which achieves the highest profit.

1. Finding the optimal solution by using *The arithmetic method.*

1) the profit of a path (x_1-x_2-x_5-x_8-x_{11}) is 16+20+13+25=74.

2) the profit of a path (x_1-x_2-x_6-x_8-x_{11}) is 16+14+17+25=72.

3) the profit of a path (x_1-x_2-x_6-x_9-x_{11}) is 16+14+20+17=67.

4)the profit of a path (x_1-x_2-x_6-x_{10}-x_{11}) is 16+14+21+19=70.

5) the profit of a path (x_1-x_3-x_5-x_8-x_{11}) is10+15+13+25=63.

6)the profit of a path (x_1-x_3-x_6-x_8-x_{11}) is 10+23+17+25=75.

7) the profit of a path (x_1-x_3-x_6-x_9-x_{11}) is 10+23+20+17=70.

8) the profit of a path (x_1-x_3-x_6-x_{10}-x_{11}) is 10+23+21+19=73.

9) the profit of a path (x_1-x_3-x_7 - x_{10}-x_{11}) is 10+18+22+19=69.

10) the profit of a path (x_1-x_3-x_6-x_8-x_{11}) is 12+13+17+25=67.

11) the profit of a path (x_1-x_4-x_6-x_9-x_{11}) is 12+13+20+17=62.

12) the profit of a path (x_1-x_4-x_6-x_{10}-x_{11}) is 12+13+21+19=65.

13) the profit of a path (x_1-x_4-x_7- x_{10}- x_{11}) is 12+10+22+19=63.

Based on previous calculations the highest profit is: 75 thousand monetary units which is calculated from path number 6, thus the best path is the sixth path.

1. Finding the optimal solution by using *The Forward solution algorithm*

We use the following rule:

$C^*_{s(x)}$=Max [C(N,X)+ $C^*_{S-1(N)}$]

Assuming that $C^*_{1(1)}$=0

$C^*_{2\,(2)}$=Max [C(1,2)+ $C^*_{1(1)}$]= Max [16+0]= 16.

$C^*_{2\,(3)}$=Max [C(1,3)+ $C^*_{1(1)}$]= Max [10+0]= 10.

$C^*_{2\,(4)}$=Max [C(1,4)+ $C^*_{1(1)}$]= Max [12+0]= 12.

$C^*_{3\,(5)}$=Max [C(2,5)+ $C^*_{2(2)}$, C(3,5)+ $C^*_{2(3)}$]= Max [20+16, 15+10]= Max [36, 25]= 36.

$C^*_{3\,(6)}$=Max [C(2,6)+ $C^*_{2(2)}$, C(3,6)+ $C^*_{2(3)}$, C(4,6)+ $C^*_{2(4)}$]= Max [14+16, 23+10, 13+12]= Max [30, 33, 25]= 33.

$C^*_{3\,(7)}$=Max [C(3,7)+ $C^*_{2(3)}$, C(4,7)+ $C^*_{2(4)}$]= Max [18+10, 10+12]= Max [28, 22]= 28.

$C^*_{4\ (8)}$=Max [C(5,8)+ $C^*_{3(5)}$, C(6,8)+ $C^*_{3(6)}$]= Max [13+36, 17+33]= Max [49, 50]= 50.

$C^*_{4\ (9)}$=Max [C(6,9)+ $C^*_{3(6)}$]= Max [20+33]= 53.

$C^*_{4\ (10)}$=Max [C(6,10)+ $C^*_{3(6)}$, C(7,10)+ $C^*_{3(7)}$]= Max [21+33, 22+28]= Max [54, 50]= 54.

$C^*_{5\ (11)}$=Max [C(8,11)+ $C^*_{4(8)}$, C(9,11)+ $C^*_{4(9)}$, C(10,11)+ $C^*_{4(10)}$]= Max [25+50, 17+53, 19+54]= Max [75, 70, 73]= 75.

Based on previous calculations the highest profit is 75 thousand monetary units

which are calculated from the shaded path during calculations. It is (x_1-x_3-x_6-x_8-x_{11}).

2. Finding the optimal solution by using *The backward or inverse solution algorithm*

We use the following rule:

$C^*_{s(x)}$=Max [C(N,X)+ $C^*_{S+1(N)}$]

Assuming that $C^*_{5(11)}$=0

$C^*_{4\ (8)}$=Max [C(8,11)+ $C^*_{5(11)}$]= Max [25+0]= 25.

$C^*_{4\ (9)}$=Max [C(9,11)+ $C^*_{5(11)}$]= Max [17+0]= 17.

$C^*_{4\ (10)}$=Max [C(10,11)+ $C^*_{5(11)}$]= Max [19+0]= 19.

$C^*_{3\ (5)}$=Max [C(5,8)+ $C^*_{4(8)}$]= Max [13+25]= 38.

$C^*_{3\ (6)}$=Max [C(6,8)+ $C^*_{4(8)}$, C(6,9)+ $C^*_{4(9)}$, C(6,10)+ $C^*_{4(10)}$]= Max [17+25, 20+17, 21+19]= Max [42, 37, 40]= 42.

$C^*_{3\ (7)}$=Max [C(7,10)+ $C^*_{4(10)}$]= Max [22+19]=41.

$C^*_{2\ (2)}$=Max [C(2,5)+ $C^*_{3(5)}$, C(2,6)+ $C^*_{3(6)}$]= Max [20+38, 14+42]= Max [58, 56]= 58.

$C^*_{2\ (3)}$=Max [C(3,5)+ $C^*_{3(5)}$, C(3,6)+ $C^*_{3(6)}$, C(3,7)+ $C^*_{3(7)}$]= Max [15+38, 23+42, 18+41]= Max [53, 65, 59]= 65.

$C^*_{2\ (4)}$=Max [C(4,6)+ $C^*_{3(6)}$, C(4,7)+ $C^*_{3(7)}$]= Max [13+42, 10+41]= Max [55, 51]= 55.

$C^*_{1\ (1)}$=Max [C(1,2)+ $C^*_{2(2)}$, C(1,3)+ $C^*_{2(3)}$, C(1,4)+ $C^*_{2(4)}$]= Max [16+58, 10+65, 12+55]= Max [74, 75, 67]= 75.

Based on previous calculations the highest profit is 75 thousand monetary units

which are calculated from the shaded path during calculations. It is (x_1-x_3-x_6-x_8-x_{11}).

Conclusion

In this lecture, we explored dynamic programming as an effective analytical tool used to solve decision-making problems that proceed through sequential stages, where each partial decision must be made based on the outcomes of previous or subsequent decisions in a logical and systematic sequence.

We clarified that the term "dynamic" does not refer to time, as might be assumed from the name, but rather indicates that the optimal solution to the overall problem is composed of a series of optimal partial solutions at each stage.

We covered two main methods for solving such problems:

- The arithmetic method, which involves calculating all possible paths and selecting the one that results in the lowest total cost or the highest possible return.
- The dynamic linear programming method, which includes both the forward solution algorithm and the backward (or inverse) solution algorithm, where the optimal value is determined step by step either from the first to the last stage or in reverse, depending on the objective (minimizing cost or maximizing profit).

To enhance practical understanding, we presented two applied examples:

- The first focused on cost minimization, where all three methods were applied to find the least-cost path through the stages of production.
- The second addressed profit maximization, using the same structured approach to determine the path yielding the highest possible return.

This lecture demonstrates that dynamic programming is a flexible and efficient approach for solving complex problems that involve sequential decisions, and it serves as a vital tool in fields such as planning, production, finance, and engineering.

Conclusion of The book

In conclusion, this book underscores the foundational role of quantitative methods in scientific management thinking, providing students and researchers with effective analytical tools to understand real-world problems and make decisions grounded in logic and data. Through this work, we have aimed to deliver a comprehensive resource that blends theoretical foundations with practical applications—contributing to the development of students' analytical skills and their ability to employ mathematical and statistical models across various management contexts.

We hope that these lectures serve as a stimulating academic reference, supporting students throughout their academic journey and equipping them with the quantitative reasoning necessary to address the complex challenges of today's business environment. Moreover, we aspire for this book to open new horizons for further research and practical applications in modern management disciplines, aligned with the principles of quality and effectiveness in decision-making.

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