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Faculty of Science and Technology

Common Core Science and Technology Departement

Fluid Mechanics Course

Intended for second-year engineering students in the core science and technology program (Semester 3)

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Preface

The course **Fluid Mechanics** is intended for second-year engineering students in the core science and technology program (Semester 3). It forms a fundamental component of the engineering curriculum, providing the theoretical and practical foundations for understanding the behavior of fluids—both liquids and gases—at rest and in motion.

Fluid Mechanics plays a central role in various branches of engineering, including mechanical, civil, chemical, and aerospace engineering. The principles covered in this course are essential for analyzing and designing systems involving fluid flow, such as pipelines, pumps, turbines, aircraft, and hydraulic structures. A solid grasp of these principles equips future engineers with the analytical and problem-solving tools needed to tackle real-world engineering challenges.

This course is organized into four comprehensive chapters, each addressing a major aspect of fluid behavior:

Chapter I: Fluid Statics introduces the basic definitions and properties of fluids, the concept of pressure, and the principles governing fluids at rest, including hydrostatic forces and buoyancy.

Chapter II: Fluid Kinematics focuses on describing fluid motion without reference to the forces that cause it, emphasizing flow visualization, the continuity equation, and different types of flow.

Chapter III: Dynamics of Ideal Incompressible Fluids develops the fundamental equations of motion, including Euler's and Bernoulli's equations, and illustrates their applications to ideal fluid flow problems.

Chapter IV: Real Incompressible Fluid Dynamics extends the discussion to real fluids, introducing the effects of viscosity, flow regimes characterized by the Reynolds number, head losses, and the concept of the boundary layer.

The objective of this course is to enable students to understand and apply the fundamental laws governing fluid behavior, develop physical intuition, and acquire the analytical skills necessary to model and solve engineering problems involving fluid systems.

It is hoped that these materials will serve as a clear and coherent introduction to Fluid Mechanics and provide a strong foundation for subsequent studies in areas such as mechanical engineering, hydraulics, and advanced fluid dynamics.

Contents

Chapter I Fluid statics

Fluid statics, a foundational branch of fluid mechanics, focuses on the behavior of fluids—liquids and gases—at rest. Unlike fluid dynamics, which examines flowing fluids, fluid statics analyzes the forces and pressures within and on surfaces submerged in static fluids, as well as the principles governing buoyancy and hydrostatic pressure.

I.1 Definition of a fluid

A fluid can be considered as a substance made up of a large number of very small material particles, free to move relative to one another. It is therefore a continuous, deformable, non-rigid material medium that can flow.

I.2 Physical properties of a fluid

I.2.1 Density

The density of a fluid is its mass per unit volume, represented by the letter ρ. Density has dimensions of M/L^3

$$\rho = \frac{M}{V} \qquad \text{I.1}$$

If the mass of 1 m^3 of liquid is 820 kg, its density is $\rho = 820$ kg/m^3.

Table I.1: Density of some Gases at Room Temperature and Pressure

Gas	ρ (kg/m^3)
Air	1.19
Carbon dioxide	1.82
Hydrogen	0.082 6
Oxygen	1.31

Table I.2: Density of Dry Air at Atmospheric Pressure

Temperature °C	ρ (kg/m^3)
23.15	1.413
16.85	1.218
46.85	1.103
146.85	0.840

I.2.2 Specific weight

The specific weight is weight per unit volume with dimension F/L^3 (N/m^3). Specific weight is related to density by:

$$\gamma = \frac{mg}{V} = \rho g \qquad \text{I.2}$$

I.2.3 Specific gravity

The specific gravity of a substance is the ratio of its density to the density of water ρ_w at 4° C:

$$Sg = \frac{\rho}{\rho_w} \qquad \text{I.3}$$

It is a **dimensionless quantity**, meaning it has no units.

Table I.3: Specific Gravity of some Common Liquids at 1.0 atm Pressure, (25° C)

Name	Specific gravity
Acetone	0.787
Water	1
Fluorine refrigerant R-22	1.197
Mercury	13.6

Table I.4: Specific Gravity of Water at Atmospheric Pressure

Temperature °C	*Sg*
0	0.9999
4	1.0000
12	0.9995
18	0.9986
100	0.9584

Table I.5: Specific Gravity of some Solids at Ordinary Atmospheric Temperature

Substance	Specific gravity
Balsa wood	0.11–0.14
Cardboard	0.69
Ice	0.917
Marble (رخام)	2.6–2.84
Emery (صنفرة)	4

I.2.4 Viscosity

Viscosity describes how much a fluid resists moving when a thin layer slides over another; this resistance is felt only when a shear force acts on the fluid. Different fluids deform at different speeds — low-viscosity fluids like water or gasoline flow easily, while high-viscosity fluids like tar or syrup flow much more slowly.

I.3 Fluid classification

I.3.1 Compressible and incompressible fluids

Compressible fluid: A fluid is said to be compressible when the volume occupied by a given mass varies as a function of external pressure. Gases are compressible fluids. For example, air, hydrogen and methane in their gaseous state are considered compressible fluids.

Incompressible fluid: A fluid is said to be incompressible when the volume occupied by a given mass does not vary as a function of external pressure. Liquids can be considered incompressible fluids (water, oil, etc.).

I.3.2 Ideal and real fluids (viscous fluid)

Ideal fluid: An ideal fluid is one in which cohesive forces are zero.
Real fluid (viscous fluid): In a real fluid, the tangential forces of internal friction opposing the relative sliding of the fluid layers are taken into account. This phenomenon of viscous friction occurs as the fluid moves.
Real fluids (viscous fluids) include both Newtonian and Non-Newtonian fluids, which are distinguished by how their viscosity responds to applied forces and flow conditions. The concepts of Newtonian and non-Newtonian fluids will be discussed in section IV.1

I.4. General principles and theorems

I.4.1. Concept of pressure and pressure scale

Definition of pressure
Pressure is defined as the force acting normal to an area divided by this area. If we assume the fluid to be a continuum, then at a point within the fluid the area can approach zero, Fig. I.1a, and so the pressure becomes

$$p = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} = \frac{dF}{dA} \qquad \text{I.4}$$

If the surface has a finite area and the pressure is *uniformly distributed* over this area, Fig. I.1*b*, then the *average pressure* is

$$p_{avg} = \frac{F}{A} \qquad \text{I.5}$$

From the definition, pressure has the dimension of FL^{-2} and in SI units is expressed as N/m^2, defined as a pascal, abbreviated as Pa, and pressures are commonly specified in pascals.

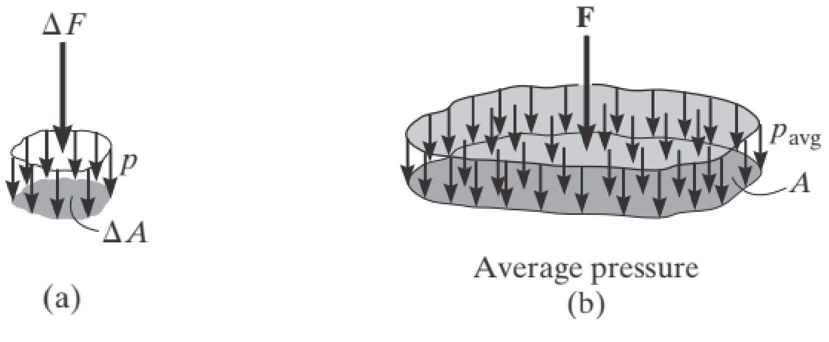


Figure I.1

Atmospheric pressure
Atmospheric pressure p_{atm} also known as barometric pressure is the normal force per unit area that the air exerts on a surface due to the weight of the column of air above it and the molecules' impacts. Its standard mean value at sea level is 1 atmosphere, defined as 101,3 pascals.

Absolute pressure and gage pressure

The pressure at a point within a fluid mass will be designated as either an ***absolute*** **pressure** p_{abs} or a ***gage*** **pressure** p_{gage}.

If a fluid such as air were removed from its container, a vacuum would exist and the pressure within the container would be zero. This is commonly referred to as ***zero absolute pressure***.

Any pressure that is measured above this value is referred to as the ***absolute pressure***, p_{abs}. For example, ***standard atmospheric pressure*** is the absolute pressure that is measured at sea level and at a temperature of 15°C. Its value is p_{atm} = 101.3 kPa.

The **gage pressure** is measured relative to the standard atmospheric pressure. Thus, a gage pressure of zero corresponds to a pressure that is equal to the standard atmospheric pressure.

Absolute pressures are always positive, but gage pressures can be either positive or negative depending on whether the pressure is above atmospheric pressure (a positive value) or below atmospheric pressure (a negative value). A negative gage pressure is also referred to as a *suction* or ***vacuum*** **pressure.**

Absolute pressure is measured relative to a perfect vacuum absolute zero pressure, whereas gage pressure is measured relative to the local atmospheric pressure. Thus, a gage pressure of zero corresponds to a pressure that is equal to the local atmospheric pressure. Absolute pressures are always positive, but gage pressures can be either positive or negative depending on whether the pressure is above atmospheric pressure (a positive value) or below atmospheric pressure (a negative value). A negative gage pressure is also referred to as a *suction* or ***vacuum*** **pressure.**

The absolute pressure and the gage pressure are related by:

$$p_{abs} = p_{gage} + p_{atm} \qquad \text{I.6}$$

For example, 151,3 pascals (abs) could be expressed as - 50 pascals (gage), if the local atmospheric pressure is 101,3 pascals, or alternatively 50 pascals suction or 50 pascals vacuum.

The concept of gage and absolute pressure is illustrated graphically in Fig. I.2 for two typical pressures located at points 1 and 2

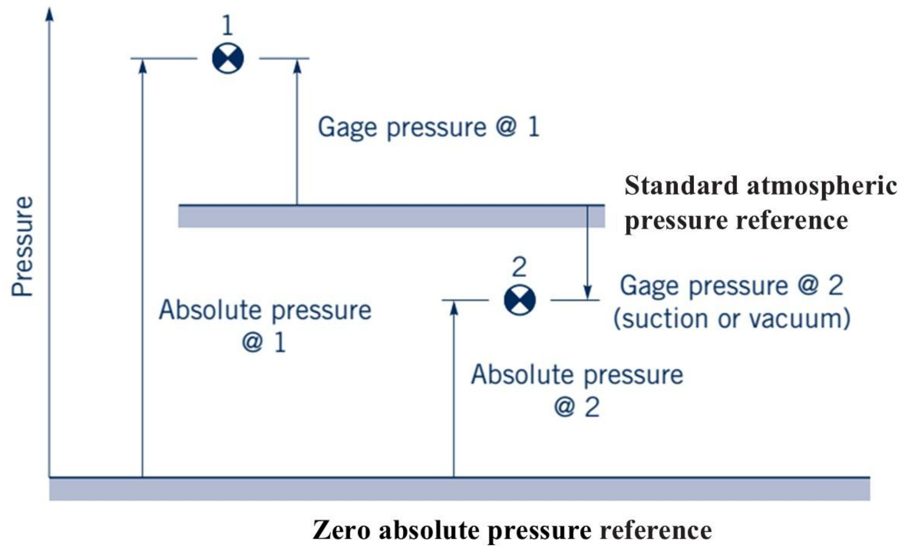


Figure I.2 Graphical representation of gage and absolute pressure (Pressure Scale).

I.4.2 Pressure forces at a point in a fluid

In a fluid, forces are classified as either **body forces** or **surface forces** (Figure I.3). These forces determine how the fluid behaves, whether it is static or moving. Body forces act throughout the entire volume of the fluid, while surface forces are applied to the fluid's boundaries.

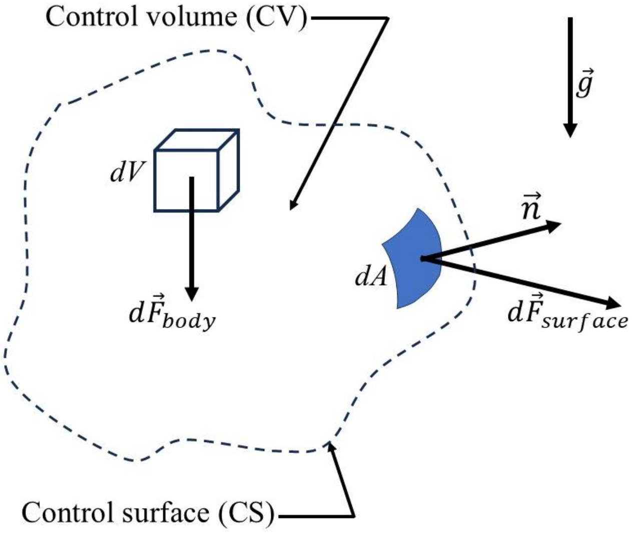


Figure I.3 Forces acting on a control volume

Body Forces

Body forces act uniformly across the fluid's volume, such as **gravitational force** (that is considered here) which pulls the fluid downward. These forces are proportional to the fluid's density and affect every part of the fluid equally. Other examples include electromagnetic forces or centrifugal forces in rotating fluids.

$$F_V = m.f = \rho.f.dv \qquad \text{I.7}$$

where:

F_V is the **volume force** or **total body force** acting on the fluid element,
m is the **mass** of the fluid element,
f is the **body force per unit mass** (gravitational acceleration),
ρ is the **density** of the fluid,
dv is the **differential volume** of the fluid element.

Surface Forces

Surface forces act on the boundaries of a fluid element. The most common surface force is **pressure**, which acts perpendicular to the fluid's surface. In moving fluids, surface forces also include **viscous stresses**, which arise from friction between fluid layers.

$$F_S = P.dS \qquad \text{I.8}$$

F_S represents the total force acting on a surface due to the pressure exerted by the fluid.
P is the **pressure** at a specific point within the fluid (measured in pascals, N/m²).
dS is a **differential element of surface area** (in m²) on which the pressure is acting.

I.4.3 Fundamental principle of fluid statics

Imagine a small cubic fluid element with side lengths *dx*, *dy*, and *dz*. The fluid is subjected to:

- **Pressure forces** acting on its surfaces.
- **Body forces** (e.g., gravitational force) acting throughout its volume.

We aim to find the net force per unit volume acting on this element and relate it to the pressure gradient.

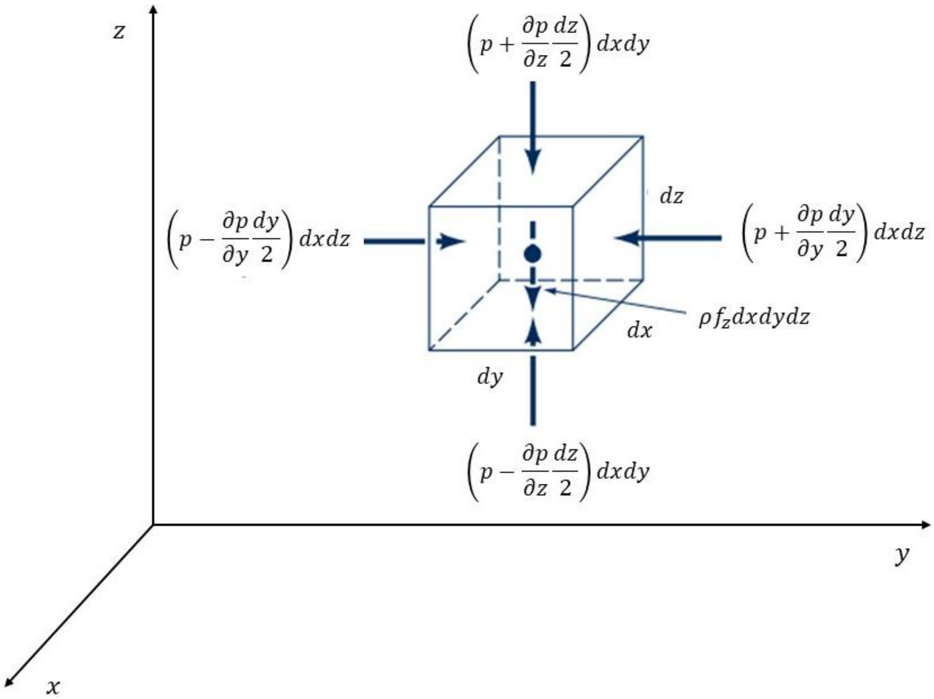


Figure I.4 Surface and body forces acting on small fluid element

Pressure Forces

Pressure acts perpendicular to the surfaces of the fluid element, and we consider the pressure forces along the *x*, *y*, and *z* directions.

- At the left face: the pressure force is $\left(p-\frac{\partial p}{\partial y}\frac{dy}{2}\right)dxdz$.
- At the right face: the pressure force is $\left(p+\frac{\partial p}{\partial y}\frac{dy}{2}\right)dxdz$.

The net pressure force along the *y*-direction is the difference between the forces on the two faces:

$$\text{Pressure force along } y = \left(p-\frac{\partial p}{\partial y}\frac{dy}{2}\right)dxdz-\left(p+\frac{\partial p}{\partial y}\frac{dy}{2}\right)dxdz=-\frac{\partial p}{\partial y}dxdydz$$

Similarly, we can compute the net pressure forces in the *z* and *x* directions:

$$\text{Pressure force along } z = -\frac{\partial p}{\partial z}dxdydz$$

$$\text{Pressure force along } x = -\frac{\partial p}{\partial x}dxdydz$$

Body Forces

Body forces (gravity) act uniformly throughout the volume of the element. Let the body force per unit mass be denoted by *fx*, *fy*, and *fz* in the *x*-, *y*-, and *z*-directions, respectively.

The total body force acting on the fluid element is given by:

Body force along $x = \rho f_x dxdydz$
Body force along $y = \rho f_y dxdydz$
Body force along $z = \rho f_z dxdydz$

For the fluid element to be in equilibrium, the sum of the forces in each direction must be zero. Using **Newton's second law**, the net force per unit volume in the *x*-, *y*-, and *z*-directions must equal zero.

In the *x*-direction: $-\frac{\partial p}{\partial x}dxdydz + \rho f_x dxdydz = 0$ and $f_x = \frac{1}{\rho}\frac{\partial p}{\partial x}$

Similarly, we get:

In the *y*-direction: $f_y = \frac{1}{\rho}\frac{\partial p}{\partial y}$

In the *z*-direction: $f_z = \frac{1}{\rho}\frac{\partial p}{\partial z}$

Now, multiplying the forces *fx*, *fy*, and *fz* by the differentials *dx*, *dy*, and *dz* respectively, we get:

$$f_x dx + f_y dy + f_z dz = \frac{1}{\rho}\left(\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz\right) \quad \text{I.9}$$

From the total derivative of the pressure *p*, we know:

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz$$

Substituting this into the previous expression, we get:

$$\frac{1}{\rho}dp = f_x dx + f_y dy + f_z dz \quad \text{I.10}$$

This is a **fundamental equation of fluid statics**, which defines the mode of variation of pressure with the coordinates *x*, *y* and *z*, the solution of many practical problems in fluid static depends on this equation.

Pressure Variation in a Fluid at Rest

For a fluid at rest $f_x = 0\ and\ \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$, $f_y = 0\ and\ \frac{1}{\rho}\frac{\partial p}{\partial y} = 0$, $f_z = -g\ and\ \frac{1}{\rho}\frac{\partial p}{\partial z} \neq 0$ and Eq. I.9 reduces to

$$0 + 0 - \mathrm{g}dz = \frac{1}{\rho}\left(0 + 0 + \frac{\partial p}{\partial z}dz\right)$$

$$\frac{\partial p}{\partial z} = -\rho g \quad \text{I.11}$$

Since *p* depends only on *z*, Eq. I.11 can be written as the ordinary differential equation

$$\frac{dp}{dz} = -\rho g \quad \text{I.12}$$

Rearranging:

$$dp = -\rho g dz \quad \text{I.13}$$

Therefore, pressure does vary in a static fluid in the z-direction—it increases with depth, as shown by Equation I.13. Integrating both sides yields (Figure I.5)

$$\int_{P_1}^{P_2} dp = -\int_{Z_1}^{Z_2} \rho g dz$$

ρ is constant because we consider incompressible fluid.

$$\int_{P_1}^{P_2} dp = -\rho g \int_{Z_1}^{Z_2} dz$$

to yield

$$P_2 - P_1 = -\rho g (Z_2 - Z_1)$$

Or

$$P_1 - P_2 = \rho g (Z_2 - Z_1) \qquad \text{I.14}$$

Where P_1 and P_2 are pressures at the vertical elevations Z_1 and Z_2 as is illustrated in Fig. I.4. Equation I.14 can be written in the compact form

$$P_1 - P_2 = \rho g h \qquad \text{I.15}$$

Or

$$P_1 = P_2 + \gamma h \qquad \text{I.16}$$

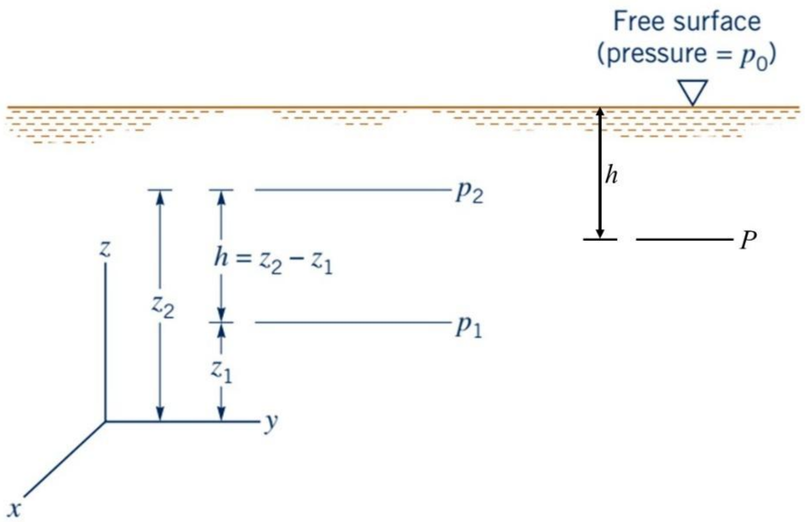


Figure I.5 Notation for pressure variation in a fluid at rest with a free surface (pressure scale).

where h is the distance, which is the depth of fluid measured downward from the location of p_2. This type of pressure distribution is commonly called a ***hydrostatic distribution,*** and Eq. I.16 shows that in an incompressible fluid at rest the pressure varies linearly with depth. The pressure must increase with depth to "hold up" the fluid above it.

It can also be observed from Eq. I.15 that the pressure difference between two points can be specified by the distance h since

$$h = \frac{P_1 - P_2}{\rho g} = \frac{P_1 - P_2}{\gamma} \qquad \text{I.17}$$

In this case h is called the ***pressure head*** and is interpreted as the height of a column of fluid of specific weight required to give a pressure difference.
When one works with liquids there is often a free surface, as is illustrated in Fig. I.4, and it is convenient to use this surface as a reference plane. The reference pressure would correspond to the pressure acting on the free surface (which would frequently be atmospheric pressure), it follows that the pressure p at any depth h below the free surface is given by the equation:

$$P = P_0 + \gamma h \qquad \text{I.18}$$

As is demonstrated by Eq. I.16 or I.18, the pressure in a homogeneous, incompressible fluid at rest depends on the depth of the fluid relative to some reference plane, and it is *not* influenced by the *size* or *shape* of the tank or container in which the fluid is held. Thus, in Fig. I.6 the pressure is the same at all points along the line AB, even though the containers have very irregular shapes. The actual value of the pressure along AB depends only on the depth, h, the surface pressure p_0, and the specific weight γ, of the liquid in the container.

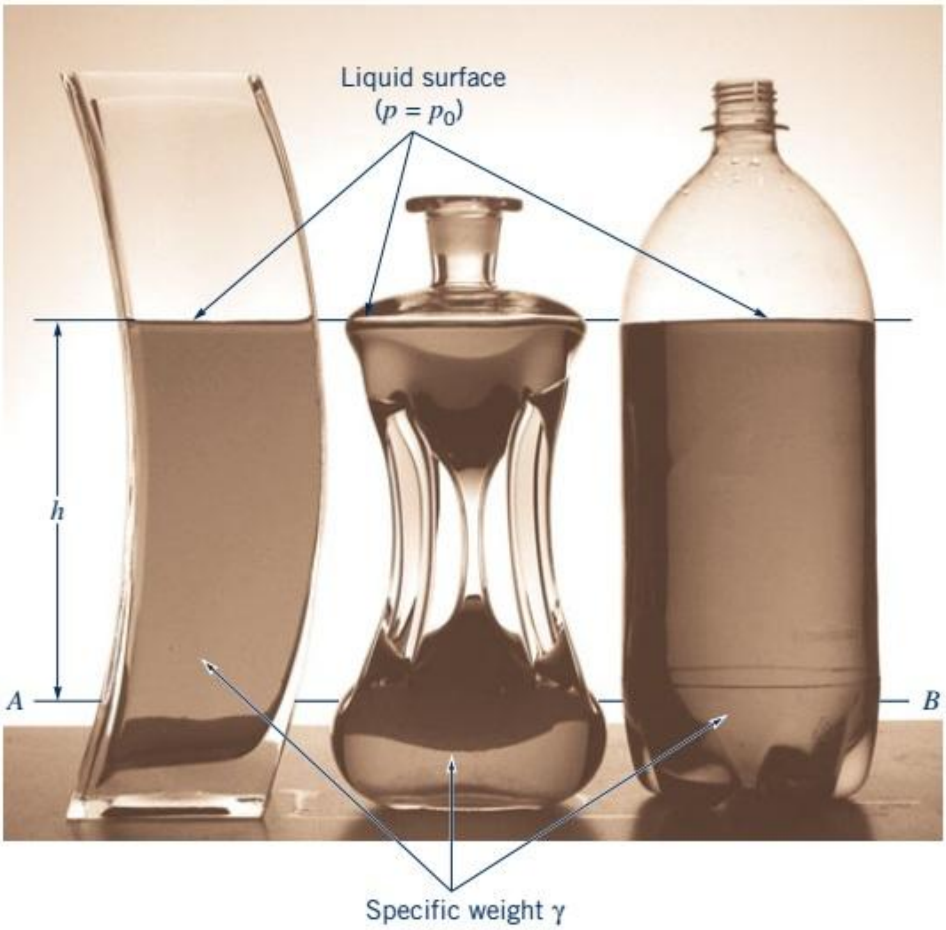


Figure I.6 Fluid pressure in containers of arbitrary shape.

I.5. Hydrostatic thrust

I.5.1 Definition

Hydrostatic thrust is "the resultant force produced by the distribution of hydrostatic pressure over a given surface, acting normal to that surface and derived from the weight of the fluid column above or adjacent to it". This concept is crucial whenever a static fluid exerts pressure, such as in underwater engineering and fluid mechanics studies.

I.5.2 Pressure distribution

When a surface is submerged in a fluid, forces develop on the surface due to the fluid. The determination of these forces is important in the design of storage tanks, ships, dams, and other hydraulic structures. For fluids at rest, we know that the force must be *perpendicular* to the surface since there are no shearing stresses present. We also know that the pressure will vary linearly with depth as shown in Fig. I.7 if the fluid is incompressible. For a horizontal surface, such as the bottom of a liquid filled tank (Fig. I.7*a*), the magnitude of the resultant force is simply $F_R = pA$, where p is the uniform pressure on the bottom and A is the area of the bottom. For the open tank shown $p = \gamma h$. Note that if atmospheric pressure acts on both sides of the bottom, as is illustrated, the *resultant* force on the bottom is simply due to the liquid in the tank. Since the pressure is constant and uniformly distributed over the bottom, the resultant force acts through the centroid of the area as shown in Fig. I.7*a*. As shown in Fig. I.7*b*, the pressure on the ends of the tank is not uniformly distributed. Determination of the resultant force for situations such as this is presented in sections I.5.3 and I.5.4.

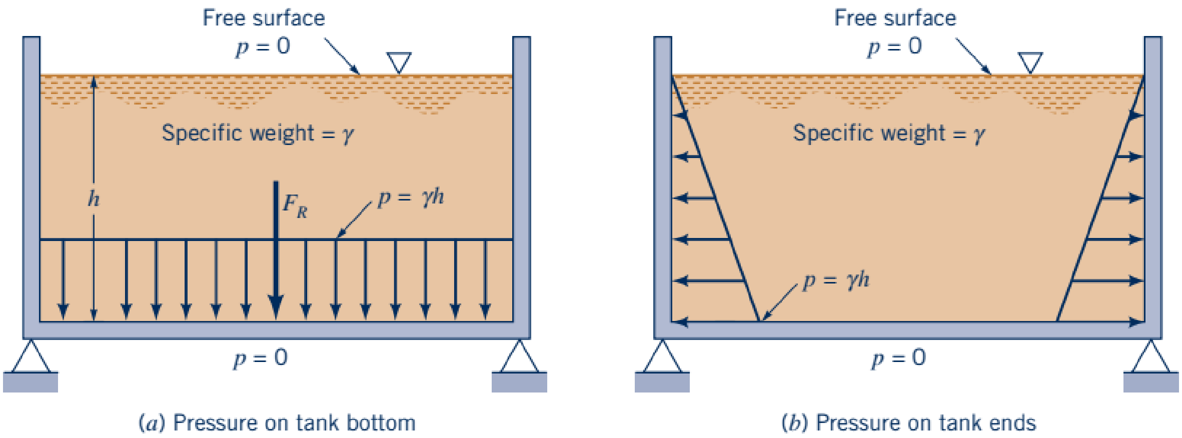


Figure I.7 (*a*) Pressure distribution and resultant hydrostatic force on the bottom of an open tank. (*b*) Pressure distribution on the ends of an open tank.

I.5.3 Hydrostatic force on a plane surface

Hydrostatic force magnitude and direction

For the more general case in which a submerged plane surface is inclined, as is illustrated in Fig. I.8, the determination of the resultant force acting on the surface is more involved. For the present we will assume that the fluid surface is open to the atmosphere. Let the plane in which the surface lies intersect the free surface at 0 and make an angle θ with this surface as in Fig. I.8. The x–y coordinate system is defined so that 0 is the origin and $y = 0$ (i.e., the x axis) is directed along the surface as shown. The area can have an arbitrary shape as shown. We wish to determine the magnitude and direction of the resultant force acting on one side of this area due to the liquid in contact with the area.

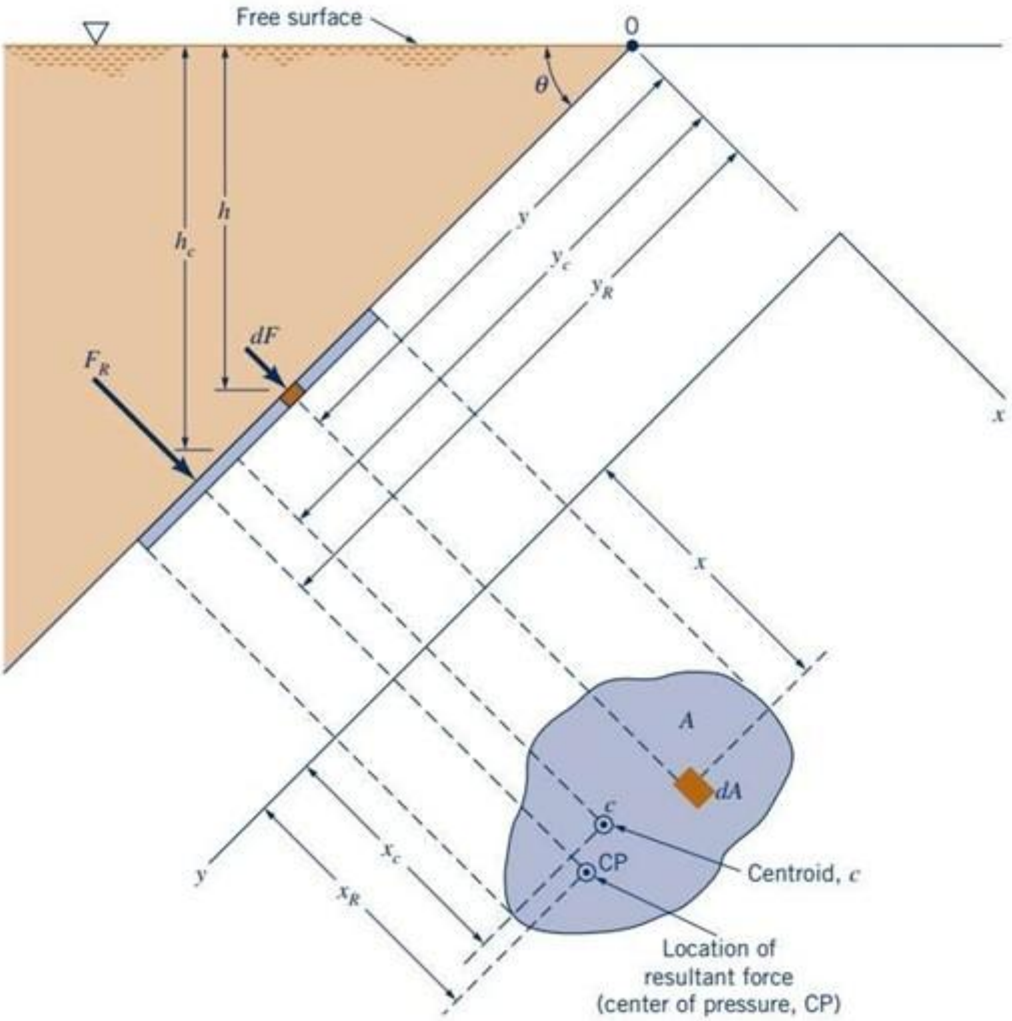


Figure I.8 Notation for hydrostatic force on an inclined plane surface of arbitrary shape.

At any given depth, *h*, the force acting on *dA* (the differential area of Fig. I.8) is:

$$dF = pdA = \gamma hdA$$

This force is perpendicular to the surface. Thus, the magnitude of the resultant force can be found by summing these differential forces over the entire surface. In equation form:

$$F_R = \int_A \gamma hdA = \int_A \gamma ysin\theta dA$$

Where $h = ysin\theta$. For constant γ and θ

$$F_R = \gamma sin\theta \int_A ydA \qquad \text{I.19}$$

The integral appearing in Eq. I.19 is the *first moment of the area* with respect to the *x* axis, so we can write

$$\int_A ydA = y_c A$$

Where y_c is the y coordinate of the centroid of area A measured from the x axis which passes through 0.
Equation I.19 can thus be written as

$$F_R = \gamma sin\theta y_c A$$

$$F_R = \gamma h_c A \qquad \text{I.20}$$

Since all the differential forces that were summed to obtain F_R are perpendicular to the surface, the resultant F_R must also be perpendicular to the surface.

Hydrostatic Force location

The location of hydrostatic force on a submerged surface refers to the specific point, known as the center of pressure, where the resultant force due to hydrostatic pressure acts upon the surface. This point is always found below the centroid of the submerged surface because the pressure increases with depth.

The y coordinate, y_R, of the resultant force can be determined by summation of moments around the x axis. That is, the moment of the resultant force must equal the moment of the distributed pressure force, or

$$F_R y_R = \int_A y dF = \int_A y\,(\gamma\, y sin\theta\, dA) = \int_A \gamma sin\theta\, y^2\, dA$$

where $dF = \gamma h dA = \gamma\, y sin\theta\, dA$
and, therefore, since $F_R = \gamma sin\theta y_c A$

$$\gamma sin\theta y_c A\, y_R = \int_A \gamma sin\theta\, y^2\, dA$$

$$y_R = \frac{\int_A y^2\, dA}{y_c A}$$

The integral in the numerator is the *second moment of the area (moment of inertia),* with respect to an axis formed by the intersection of the plane containing the surface and the free surface (x axis) $I_x = \int_A y^2\, dA$. Thus, we can write

$$y_R = \frac{I_x}{y_c A}$$

One can now be made of the parallel axis theorem to express as: $I_x = I_{xc} + A y_c^2$
Where I_{xc} is the second moment of the area with respect to an axis passing through its *centroid* and parallel to the x axis. Thus,

$$y_R = y_c + \frac{I_{xc}}{y_c A} \qquad \text{I.21}$$

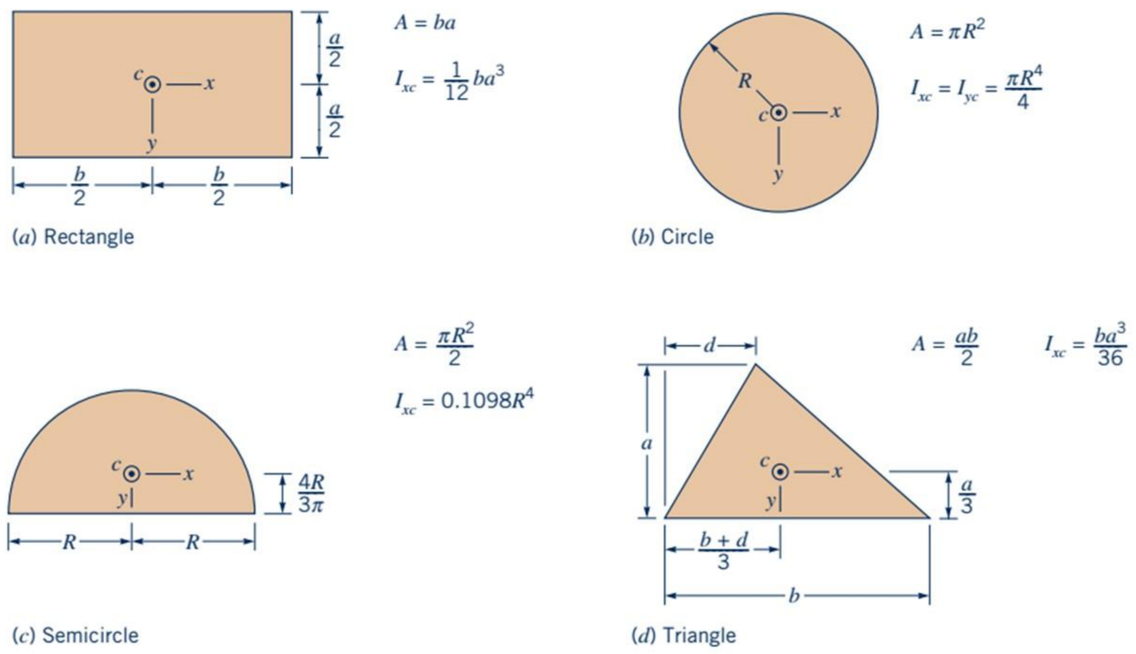


Figure I.9 Geometric properties of some common shapes.

I.5.4 Hydrostatic Force on a Curved Surface

Another important area of interest involves determining forces on submerged curved surfaces. The hull of a floating ship is a curved surface in contact with liquid, as is the wall or sides of a drinking glass or funnel or culvert. To develop equations for these cases, consider the configuration illustrated in Figure I.10a. A curved surface is shown in profile and projected frontal views. Let us examine the element of area dA. The force acting is $p\ dA$ (Figure I.10b). It is convenient to resolve this force into horizontal and vertical components, dR_h and dR_y, respectively.

The horizontal component magnitude

We write the horizontal component of this force directly as:

$$dR_h = pdA\ cos\theta$$

where $dA\cos\theta$ is the vertical projection of dA. Integrating this expression gives a result similar to that for a submerged plane:

$$R_h = \gamma h_c A_v \qquad \text{I.22}$$

where: γh_c = pressure at the centroid of the surface

A_v = its area projected onto a vertical plane

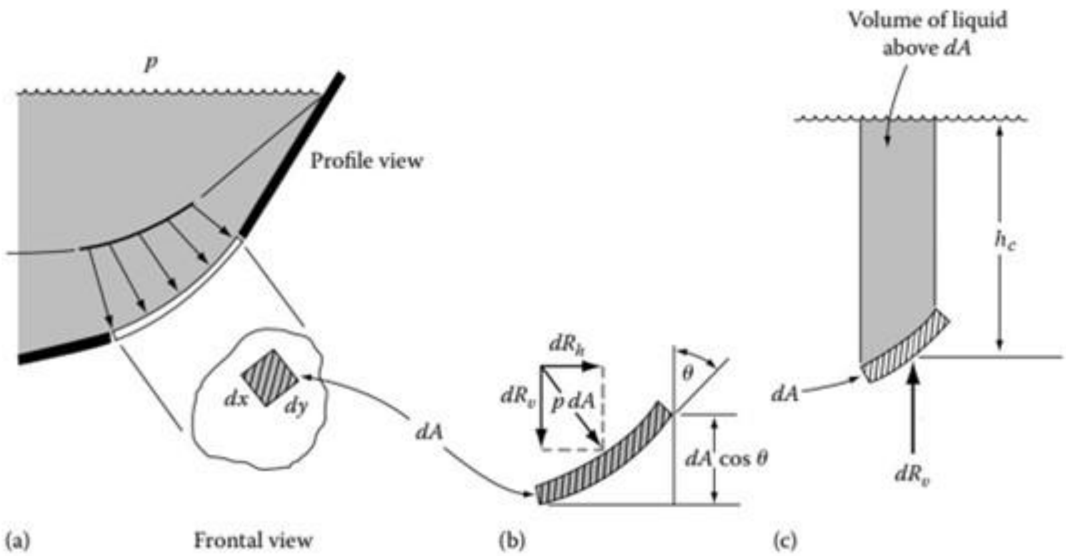


Figure I.10 A submerged, curved surface.

The horizontal component location

The line of action, or location, of the horizontal force R_h is found by summing moments as was done for the plane surfaces in Section I.5.3. The result is the same equation as was derived there except that we are now working with the vertical projection of the area—namely, A_v:

$$y_R = y_c + \frac{I_{xc}}{y_c A_v}$$

Here h_c is the distance from the free surface to the centroid of the area A_v. The second moment of inertia I_{xxc} also applies to the vertical projected area A_v.

The vertical component magnitude

Next consider the vertical component of force, which is given by

$$dR_v = pdA\ sin\theta$$

where $dA\ sin\theta$ is the horizontal projection of dA. Combining this result with the hydrostatic equation, we obtain

$$dR_v = \gamma h_c dA\ sin\theta \qquad \text{I.23}$$

where again h_c, as shown in Figure I.10c, is the vertical distance from the liquid surface to the centroid of dA. The quantity $h_c dA\ sin\theta$ is the volume of liquid above dA. Equation I.23 thus becomes

$$dR_v = \gamma dV$$

and, after integration, yields

$$R_v = \gamma V \qquad \text{I.24}$$

Therefore, the vertical component of force acting on a submerged curved surface equals the weight of the liquid above it.

I.6 Relative equilibrium

I.6.1 Constant translational acceleration of a liquid

In this section we will discuss both horizontal and vertical constant accelerated motion of a container of liquid, and we will study how the pressure varies within the liquid for these two motions.

Constant horizontal acceleration

Here we consider a differential element that has a length x and cross-sectional area A, Fig. I.11. The only horizontal forces acting on it are caused by the pressure of the adjacent liquid on each of its ends.

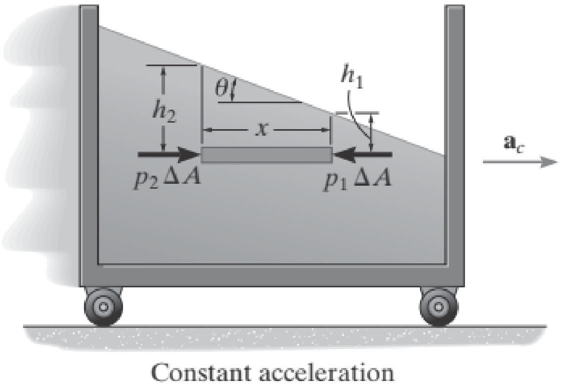


Figure I.11

Since the mass of the element is $\Delta m = \Delta W/g = \gamma(x\Delta A)/g$, ($\Delta W$ is the weight of the contained liquid) the equation of motion becomes

$$\xrightarrow{+} \Sigma F_x = ma_x; \qquad p_2\Delta A - p_1\Delta A = \frac{\gamma(x\Delta A)}{g}a_c$$

$$p_2 - p_1 = \frac{\gamma x}{g}a_c \qquad \text{I.25}$$

Using $p_1 = \gamma h_1$ and $p_2 = \gamma h_2$, we can also write this expression as

$$\frac{h_2 - h_1}{x} = \frac{a_c}{g} \qquad \text{I.26}$$

As noted in Fig. I.11, the term on the left of Eq. I.26 represents the *slope* of the liquid's free surface. Since this is equal to tan θ, then

$$tan\ \theta = \frac{a_c}{g}$$

Constant Vertical Acceleration

The forces acting on the vertical element of depth h and cross section ΔA, Fig. I.12, consist of the element's weight $\Delta W = \gamma\Delta V = \gamma(h\Delta A)$ and the pressure force on its bottom. Since the mass of the element is $\Delta m = \Delta W/g = \gamma\,(h\Delta A)/g$, application of the equation of motion yields

$$+\uparrow \Sigma F_y = ma_y; \qquad p\Delta A - \gamma(h\Delta A) = \frac{\gamma(h\Delta A)}{g}a_c$$

$$p = \gamma h\left(1 + \frac{a_c}{g}\right) \qquad \text{I.27}$$

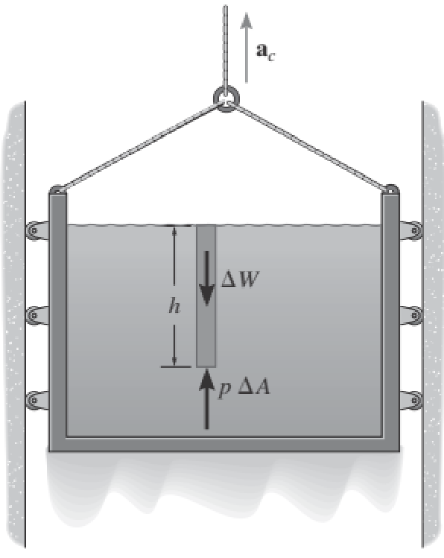


Figure I.12

Thus, the pressure within the liquid will *increase* by $\gamma h(a_c/g)$ when the container is accelerated *upward*.

I.6.2 Steady Rotation of a Liquid

If a liquid is placed into a cylindrical container that rotates at a constant angular velocity ω, Fig. I.13, the shear stress developed within the liquid will begin to cause the liquid to rotate with the container. Eventually, no relative motion within the liquid will occur, and the system will then rotate as a solid body. When this happens, the velocity of each fluid particle will depend on its distance from the axis of rotation. Those particles that are closer to the axis will move slower than those farther away. This motion will cause the liquid surface to form the shape of a *forced vortex*.

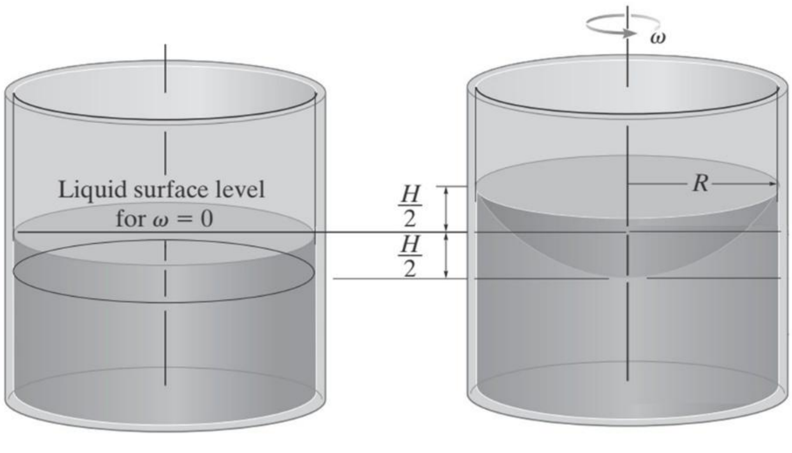


Figure I.13

The constant angular rotation ω of the cylinder–liquid system produces a pressure difference or gradient in the radial direction due to the *radial acceleration* of the liquid particles. This acceleration is the result of the *changing direction* of the *velocity* of each particle. If a particle is at a radial distance r from the axis of rotation, then from dynamics (or physics), its acceleration has a magnitude of $a_r = \omega^2 r$, and it acts toward the center of rotation. To study the radial pressure gradient, we will consider a ring element having a radius r, thickness r, and height h, Fig. I.14*b*. The pressures on the inner and outer sides of the ring are p and $p + (\partial p/\partial r)\, \Delta r$, respectively.

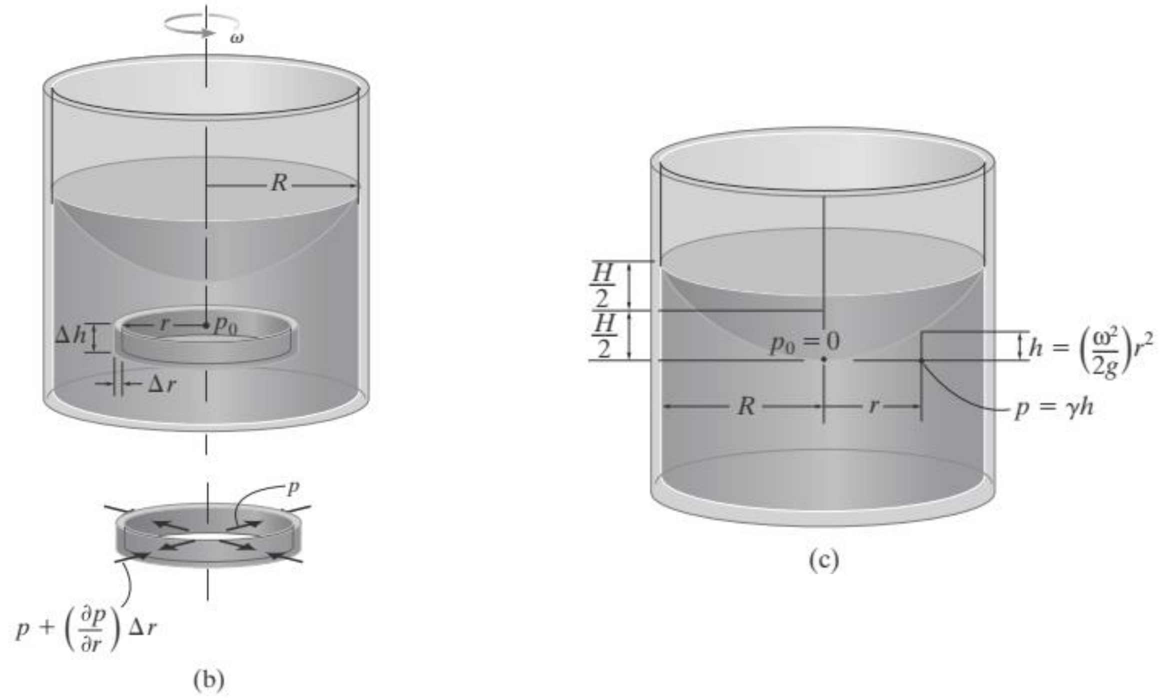


Fig. I.14

Since the mass of the ring is $\Delta m = \Delta W/g = \gamma\, \Delta V/g = \gamma(2\pi r)\, \Delta r \Delta h/g$, the equation of motion in the radial direction gives

$$\Sigma F_r = ma_r; -\left[p + \left(\frac{\partial p}{\partial r}\right)\Delta r\right](2\pi r\Delta h) + p(2\pi r\Delta h) = -\frac{\gamma(2\pi r)\Delta r\Delta h}{g}\omega^2 r$$

$$\frac{\partial p}{\partial r} = \left(\frac{\gamma\omega^2}{g}\right)r$$

Integrating, we obtain

$$p = \left(\frac{\gamma\omega^2}{2g}\right)r^2 + C$$

We can determine the constant of integration provided we know the pressure in the fluid at a specific point. Consider the point on the vertical axis at the free surface, where $r = 0$ and $p_0 = 0$, Fig. I.14*c*. Then $C = 0$, and so

$$p = \left(\frac{\gamma\omega^2}{2g}\right)r^2$$

The pressure increases with the square of the radius. Since $p = gh$, the equation of the free surface of the liquid, Fig. I.13*c,* becomes

$$h = \left(\frac{\omega^2}{2g}\right)r^2$$

This is the equation of a parabola.

I.7 Archimedes' principle

The Greek scientist Archimedes discovered the principle of buoyancy, which states that when a body is placed in a static fluid, it is buoyed up by a force that is equal to the weight of the fluid that is displaced by the body.

1.7.1. Submerged body

We consider the submerged body in Fig. I.15. Due to fluid pressure, the vertical resultant force *acting upward* on the bottom surface of the body, *ADC*, is equivalent to the weight of fluid contained above this surface, that is, within the volume *ADCyx*. Likewise, the resultant force due to pressure acting *downward* on the top surface of the body, *ABC*, is equivalent to the weight of fluid contained within the volume *ABCyx*. The *difference* in these forces acts upward, and is the ***buoyant force***. It is equivalent to the weight of an imaginary amount of fluid contained within the volume of the body, *ABCDA*. This force $\boldsymbol{F_b}$ acts through the ***center of buoyancy***, C_b, which is located at the centroid of the volume of liquid displaced by the body. If the density of the fluid is constant, then this force will remain constant, *regardless of how deep* the body is placed within the fluid.

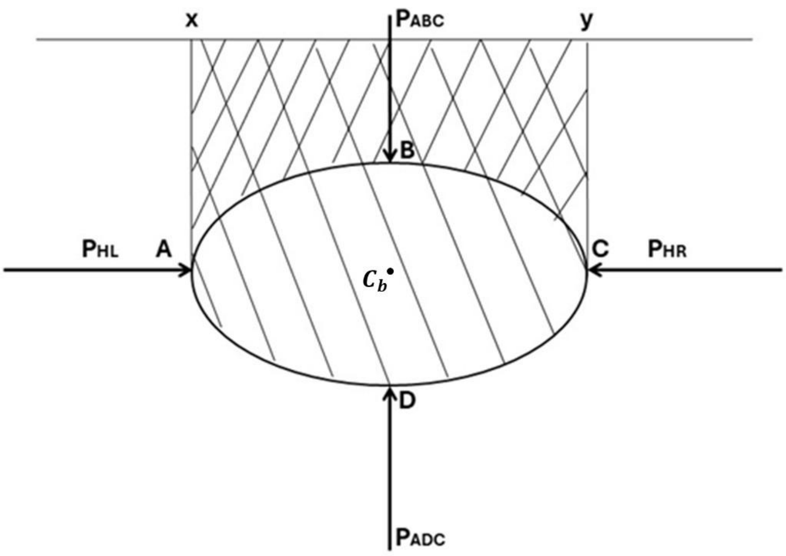


Figure I.15 Submerged body

From the diagram:

- Looking at the horizontal forces (P_{HL} and P_{HR} in the diagram):

The horizontal forces acting on the body ABCD are equal and opposite, so they cancel each other out.

- Looking at the vertical forces:

P_{ADC} = weight of volume $ADCyx$ = γ.(volume $ADCyx$)

P_{ABC} = weight of volume $ABCyx$ = γ.(volume $ABCyx$)

The net vertical force (buoyant force) $P_V = P_{ADC} - P_{ABC}$ = weight of the body's volume in liquid

$$P_V = \gamma . V$$

P_V is called Archimedes' thrust or ***buoyant force***.

This principle explains why objects float or sink in fluids, and it's fundamental to understanding ship design, hot air balloons, and many other applications in fluid mechanics.

1.7.2. Floating body

The same arguments of the previous section can also be applied to a floating body, as in Fig. I.16. Here the displaced amount of fluid is within the region *ABC*, the buoyant force is equal to the weight of fluid within this displaced volume, and the center of buoyancy C_b is at the centroid of this volume.

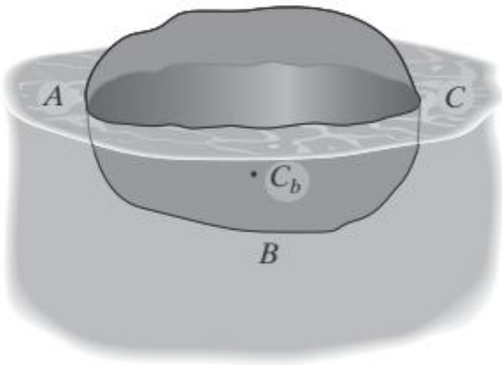


Figure I.16 Floating body

Chapter II Fluid Kinematics

Fluid Kinematics deals with the motion of fluids without necessarily considering the forces and moments which create the motion.

II.1 Description of fluid motion

II.1.1 Lagrangian Description

In the Lagrangian description, individual fluid particles are tracked from their starting positions as they move through space and time. Each particle's position, velocity, and acceleration are observed as they change over time. The particle's position vector *r*, varies with time, and its time derivative gives the particle's velocity.

$$V = V(t) = \frac{dr(t)}{dt} \qquad \text{II.1}$$

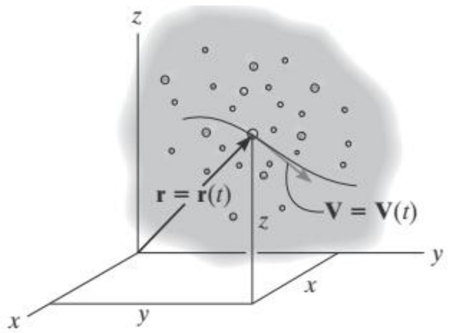


Figure II.1 Lagrangian description of motion follows a single fluid particle as it moves about within the system

II.1.2 Eulerian Description

In the Eulerian description, fluid velocity is measured at fixed points in space (x_0, y_0, z_0) within small surrounding volumes. To analyze the entire system, control volumes are placed at every point (*x, y, z*), allowing measurement of particle velocities across all points over time.

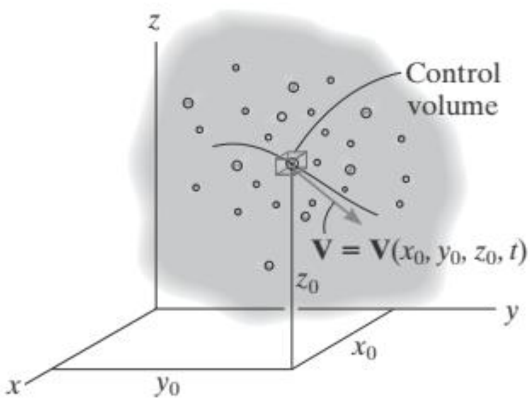


Figure II.2 Eulerian description of motion specifies a point or region within the system, and it measures the velocity of the particles that pass through this point or control volume

II.1.3 Graphical descriptions of fluid flow

Pathlines: The pathline for a fluid particle defines the "path" the particle travels over a period of time (Figure II.3a).

Streaklines: A streakline is defined as the position of all fluid particles that have all come from the same point of origin (Figure II.3b).

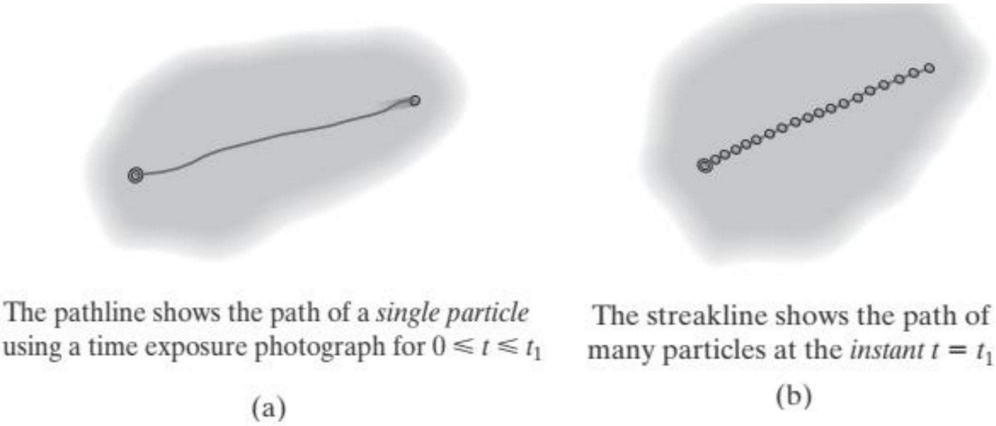
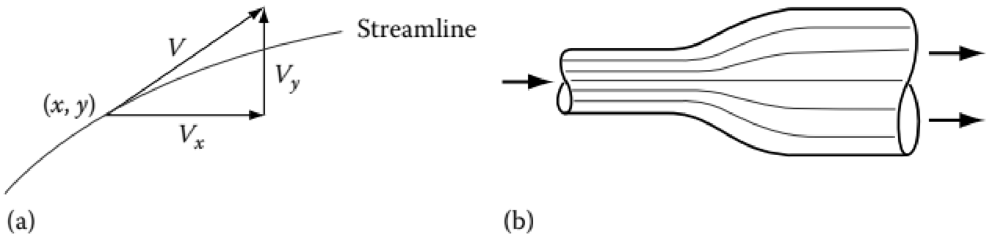


Figure II.3

Streamlines: A streamline is an instantaneous curve in a fluid flow field where the tangent at any point is parallel to the local velocity vector at that point. In other words, it's a curve that is always parallel to the direction of fluid motion at any given instant. There is no flow across a streamline.



(a) Velocity vector at a point on a streamline. (b) Streamlines in a diverging duct.

Figure II.4

As long as the flow is steady, the streamlines, pathlines, and streaklines will all coincide.

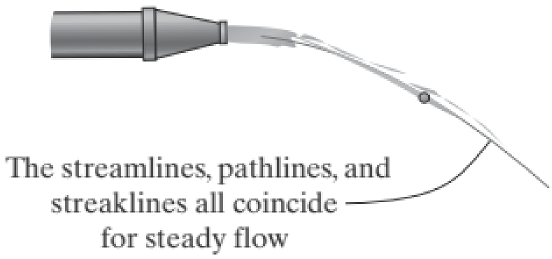


Figure II.5

Streamtubes: For some types of analysis, it is convenient to consider a *bundle of streamlines* that surround a region of flow, (Fig. II.6). Such a circumferential grouping is called a ***streamtube***. Here the fluid flows through the streamtube as if it were contained within a curved conduit.

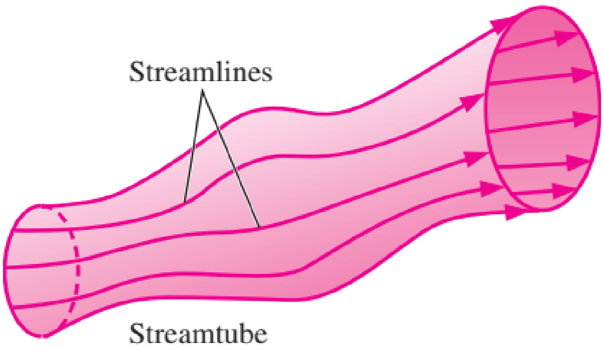


Figure II.6: A streamtube consists of a bundle of individual streamlines.

II.2 Continuity equation

II.2.1 The concept of flow rate

Volumetric Flow: The rate at which a *volume* of fluid flows through a cross-sectional area A is called the *volumetric flow*, or simply the *flow* or *discharge*. It can be determined provided we know the velocity profile for the flow across the area. For example, consider the flow of a viscous fluid through a pipe, such that its velocity profile has the axisymmetric shape shown in figure (Figure II.7).

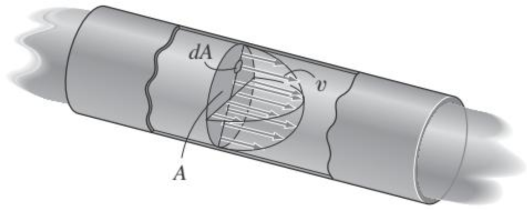


Figure II.7

If particles passing through the differential area dA have a velocity v, then during the time dt, a volume element of fluid of length vdt will pass through the area. Since this volume is $dV = (vdt)(dA)$, then the *volumetric flow* dQ through the area is determined by dividing the volume by dt, which gives $dQ = dV/dt = vdA$. If we integrate this over the entire cross-sectional area A, we have

$$Q = \int_A vdA \qquad \text{II.2}$$

Q (m^3/s)

When calculating Q, it is important to remember that the velocity must be *normal* to the cross-sectional area through which the fluid flows.

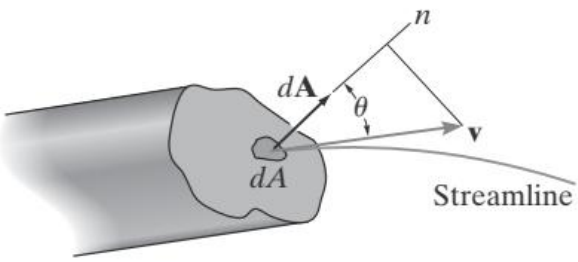


Figure II.8

If this is not the case, as in figure II.8, then we must consider the velocity's *normal component* $v\cos\theta$ for the calculation. By considering the area as a vector, $d\mathbf{A}$, where its normal is *positive outward*, we can use the dot product, $\mathbf{v}\cdot d\mathbf{A} = v\cos\theta\, dA$, to express the integral in the previous equation in a more general form, namely

$$Q = \int_A \mathbf{v}.d\mathbf{A} \qquad \text{II.3}$$

Mass Flow: Since the mass of the element in Fig. II.7 is $dm = \rho dV = \rho\,(vdt)dA$, the ***mass flow*** or ***mass discharge*** of the fluid through the entire cross section becomes

$$\dot{m} = \frac{dm}{dt} = \int_A \boldsymbol{\rho}\mathbf{v}.d\mathbf{A} \qquad \text{II.4}$$

II.2.2 Derivation of the continuity equation

- Fluid Property Description

Extensive Property: An extensive property is a property that depends on the amount of mass or volume in a system. It "extends" throughout the system. For example, momentum is an extensive property since it represents mass times velocity, $\boldsymbol{N} = mV$.

Intensive Property: Fluid properties that are independent of the system's mass are called intensive properties, $\boldsymbol{\eta}$ (eta). Examples include temperature and pressure.

<u>We can represent an extensive property $\boldsymbol{N}$ as an intensive property $\boldsymbol{\eta}$ simply by expressing it per unit mass, that is, $\boldsymbol{\eta} = N/m$.</u>

- Reynolds transport theorem*:* this theorem relates the time rate of change of *any* extensive property $\boldsymbol{N}$ of a system of fluid particles, defined from a Lagrangian description, to the changes of the same property from the viewpoint of the control volume, that is, as defined from a Eulerian description.

$$\left(\frac{DN}{Dt}\right)_{\text{syst}} = \frac{\partial}{\partial t}\int_{\text{cv}} \eta\rho d\forall + \int_{\text{cs}} \eta\rho\mathbf{V}\cdot d\mathbf{A} \qquad \text{II.5}$$

The first term on the right side is the *local change*, since it represents the time rate of change in the intensive property *within* the control volume. The second term on the right is the *convective change*, since it represents the *net flow* of the intensive property through the control surfaces.

- Continuity equation

Integral form

The conservation of mass states that within a region, apart from any nuclear process, matter can neither be created nor destroyed. From a Lagrangian point of view, the mass of all the particles in a system of particles must be *constant* over time, and so we require the change in the mass to be $(dm/dt)_{sys} = 0$. In order to develop a similar statement that relates to a control volume, we must use the Reynolds transport theorem, Eq. II.6. Here the extensive property $N = m$, and so the corresponding intensive property is mass per unit mass, or $\boldsymbol{\eta} = m/m = 1$. Therefore, the conservation of mass requires

$$\frac{\partial}{\partial t}\int_{\text{cv}} \rho d\forall + \int_{\text{cs}} \rho \mathbf{V} \cdot d\mathbf{A} = 0 \qquad \text{II.6}$$

This equation is often called the *continuity equation in integral form*. It states that the *local rate* of change of mass *within* the control volume, plus the *net convective rate* at which mass enters and exits the open control surfaces, must equal zero, Fig. II.9.

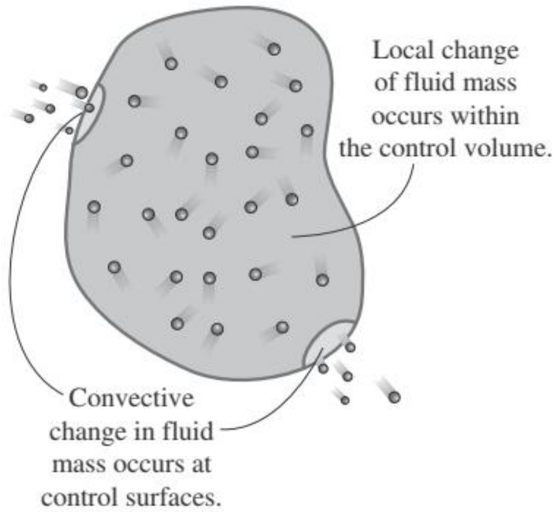


Figure II.9

Special Cases:

Provided we have a control volume with a fixed size that is completely filled with an incompressible fluid, then there will be no local change of the fluid mass within the control volume. In this case, the first term in Eq. II.6 is zero, and so the *net* mass flow into and out of the open control surfaces must be zero. In other words, "what flows in must flow out". Thus, for both steady and unsteady flow,

$$\int_{\text{cs}} \rho \, \mathbf{V} \cdot d\mathbf{A} = \Sigma \dot{m}_{\text{out}} - \Sigma \dot{m}_{\text{in}} = 0 \qquad \text{II.7}$$

Assuming the *average velocity* occurs through each control surface, then V will be constant, and integration yields,

$$\Sigma \rho \mathbf{V} \cdot \mathbf{A} = \Sigma \dot{m}_{\text{out}} - \Sigma \dot{m}_{\text{in}} = 0 \qquad \text{II.8}$$

Finally, if the *same fluid* is flowing at a *steady rate* into and out of the control volume, then the density can be factored out, and we have for incompressible steady flow,

$$\Sigma \mathbf{V} \cdot \mathbf{A} = \Sigma Q_{\text{out}} - \Sigma \text{Q}_{\text{in}} = 0 \qquad \text{II.9}$$

A conceptual application of this equation is shown in Fig. II.10. By our sign convention, notice that whenever fluid *exits* a control surface, V and A_{out} are *both* directed outward, and by the dot product this term is *positive*. If the fluid *enters* a control surface, V is directed inward and A_{in} is directed outward, and so their dot product will be *negative*.

$$\Sigma V \cdot A = 0$$
$$-V_A A_A - V_B A_B + V_C A_C = 0$$

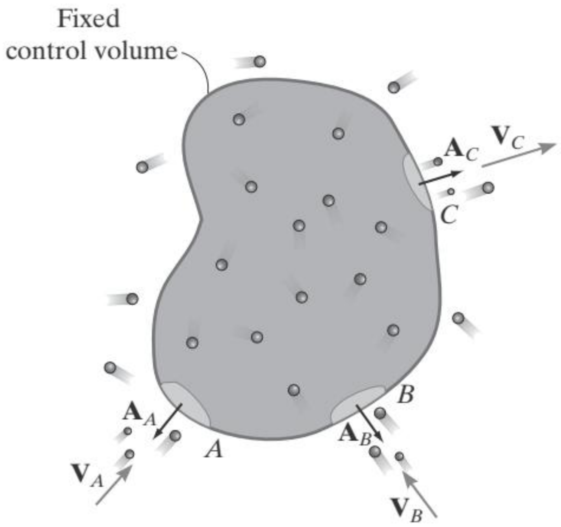


Figure II.10

Differential form

In this section we will derive the continuity equation (in differential form) for an element of fluid flowing through a fixed differential control volume that only has open control surfaces, Fig. II.11. We will assume three-dimensional flow, where the velocity field has components $u = u(x, y, z, t)$, $v = v(x, y, z, t)$, $w = w(x, y, z, t)$. Point (x, y, z) is at the center of the control volume, and at this point the density is defined by the scalar field $\rho = \rho\,(x, y, z, t)$.

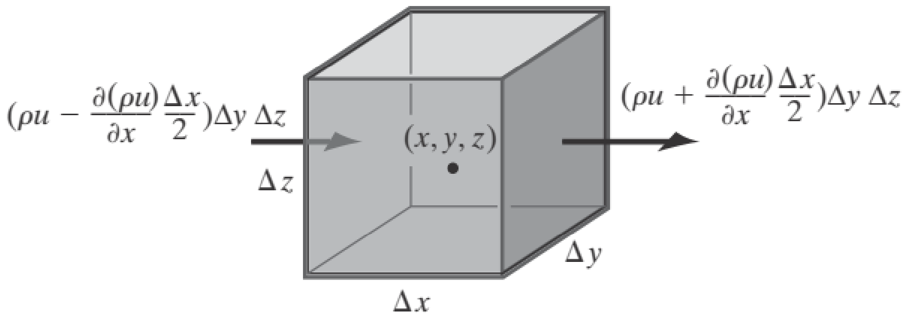


Fig. II.11

Within the control volume, local changes to the mass can occur due to the fluid's compressibility. Also, convective changes can occur from one control surface to another due to nonuniform flow. In Fig. II.11 these convective changes are considered only in the x direction, as noted by the partial derivatives at each control surface.

If we apply the continuity equation, Eq. II.6, to the control volume in the x direction, we have

$$\frac{\partial}{\partial t}\int_{\text{cv}} \rho dV + \int_{\text{cs}} \rho \mathbf{V}\cdot d\mathbf{A} = 0$$

$$\frac{\partial \rho}{\partial t}\Delta x \Delta y \Delta z + \left(\rho u + \frac{\partial(\rho u)}{\partial x}\frac{\Delta x}{2}\right)\Delta y \Delta z - \left(\rho u - \frac{\partial(\rho u)}{\partial x}\frac{\Delta x}{2}\right)\Delta y \Delta z = 0$$

Dividing by $\Delta x\ \Delta y\ \Delta z$, and simplifying, we get

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}=0 \qquad \text{II.10}$$

If we include the convective changes in the *y* and *z* directions, then the continuity equation becomes

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0 \qquad \text{II.11}$$

Finally, using the gradient operator $\nabla = \partial/\partial x\mathbf{i} + \partial/\partial y\mathbf{j} + \partial/\partial z\mathbf{k}$, and expressing the velocity as $\mathbf{V} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$, we can write the continuity equation for the differential element in vector form as

$$\frac{\partial\rho}{\partial t}+\nabla.\rho\boldsymbol{V}=0 \qquad \text{II.12}$$

Two-Dimensional Steady Flow of an Ideal Fluid.

Although we have developed the continuity equation in its most general form, often it has applications to two-dimensional steady-state flow of an ideal fluid. For this special case, the fluid is incompressible and so ρ is constant.

As a result Eq. II.11 then becomes

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad \text{II.13}$$

Or, from Eq. II.12, we can write

$$\nabla.\boldsymbol{V}=0 \qquad \text{II.14}$$

II.3 The Stream Function

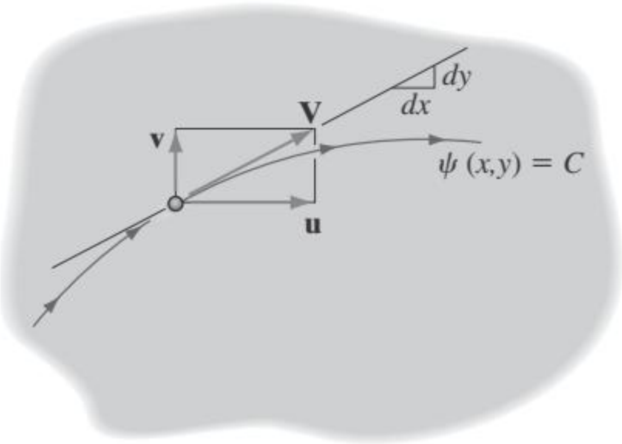
II.3.1 Definition

In two dimensions, one method for satisfying the equation of continuity is to replace the *two* unknown velocity components *u* and *v* by a *single unknown function*, thus reducing the number of unknowns, and thereby simplifying the analysis of an ideal fluid flow problem. In this section we will use the stream function as a means for doing this.

The ***stream function*** ψ (psi) is the equation that represents *all the equations of the streamlines*. In two dimensions, it is a function of *x* and *y*, and for the equation of *each streamline* it is equal to a *specific constant* $\psi(x, y) = C$.

II.3.2 Velocity Components

By definition, the velocity of a fluid particle is always tangent to the streamline along which it travels, Fig. II.12. As a result, we can relate the velocity components **u** and **v** to the slope of the tangent by proportion.



Velocity is tangent to streamline

Fig. II.12

As shown in the figure, $dy/dx = v/u$, or

$$udy - vdx = 0 \qquad \text{II.15}$$

Now, if we take the total derivative of the streamline equation $\psi(x, y) = C$, which describes the streamline in Fig. II.12, we have

$$d\psi = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy = 0 \qquad \text{II.16}$$

Comparing this with Eq. II.15, the two components of velocity can be related to ψ. We require

$$u = \frac{\partial\psi}{\partial y} \text{ and } v = -\frac{\partial\psi}{\partial x} \qquad \text{II.17}$$

Therefore, if we know the equation of any streamline, $\psi(x, y) = C$, we can obtain the velocity components of a particle that travels along it by using these equations. By obtaining the velocity components in this way, we can show that for steady flow the stream function *automatically satisfies the equation of continuity*. By direct substitution into Eq. II.13, we find

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0; \qquad \frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial\psi}{\partial x}\right) = 0$$

$$\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial^2\psi}{\partial y\partial x} = 0$$

II.4 Types of Fluid flow

II.4.1 Steady and Unsteady Flows

- Steady Flow: In steady flow, fluid properties (velocity, pressure, and density) at any given point in space remain constant over time. Mathematically, $\partial/\partial t = 0$ for any fluid property. Examples include water flowing at a constant rate through a pipe or air flowing steadily over an aircraft wing.

- **Unsteady Flow**: In unsteady flow, fluid properties at a given point in space change over time. Mathematically, $\partial/\partial t \neq 0$ for any fluid property. An example would be the flow of water through a pipe during the closing or opening of a valve.

II.4.2 Uniform Flow and non-uniform Flow

- **Uniform Flow** is a fluid flow in which characteristics and parameters remain unchanged with distance along the flow path. A steady flow through a long straight pipe of a constant diameter is an example of uniform flow.
In a steady uniform flow, an ideal fluid maintains the same velocity at all times and at each point (Fig. II.13), mathematically: $\frac{dv}{dt} = 0$ and $\frac{dv}{dx} = 0$.

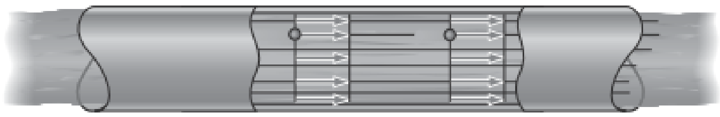


Fig. II.13: Steady uniform flow

In an unsteady uniform flow (Fig. II.14), the velocity of an ideal fluid is the same at all points at any given instant but changes with time, as is the case when a valve is slowly opened, mathematically: $\frac{dv}{dt} \neq 0$ and $\frac{dv}{dx} = 0$.

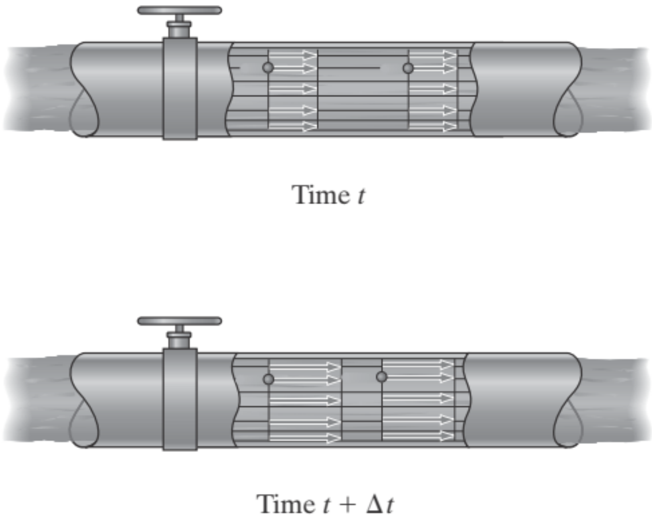


Fig. II.14: Unsteady uniform flow

- **Non-uniform Flow** is a flow in which characteristics and parameters vary and are different at different locations along the flow path. A steady flow through a pipe with a variable diameter exemplifies a non-uniform flow.
In a steady nonuniform flow (Fig. II.15), the velocity remains constant with time, but it is different from one location to the next, mathematically: $\frac{dv}{dt} = 0$ and $\frac{dv}{dx} \neq 0$.

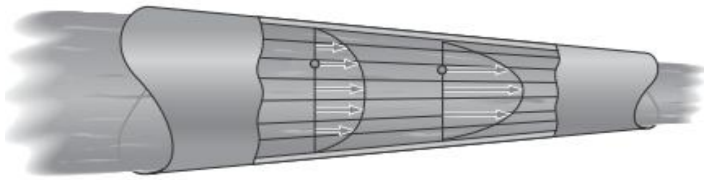


Fig. II.15: Steady nonuniform flow

In an unsteady nonuniform flow (Fig. II.16), the velocity is different at each point and changes at each time, such as when a valve is slowly opened in a pipe with a changing cross-section, mathematically: $\frac{dv}{dt} \neq 0$ and $\frac{dv}{dx} \neq 0$.

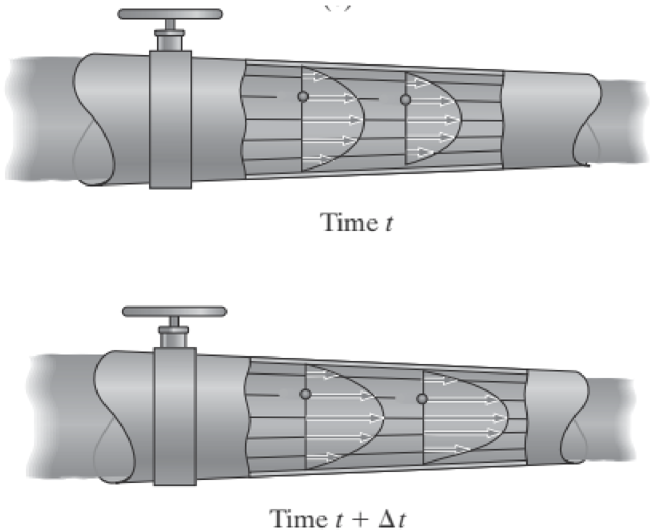


Fig. II.16: Unsteady nonuniform flow

II.4.3 Irrotational flow and Rotational flow

- Irrotational flow is flow in which fluid particles moving along the flow path do not undergo rotation. Mathematically: $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$ (irrotational two-dimensional flow).

Ideal fluids exhibit irrotational flow because no viscous friction forces act on ideal fluid elements, only pressure and gravitational forces. No rotation occurs in the ideal fluid because the entire element moves with the same velocity.

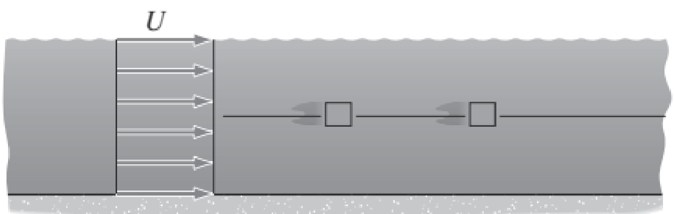


Fig. II.17: Irrotational flow (ideal fluid)

- Rotational flow is a fluid flow in which fluid particles moving along the flow path also rotate about their respective axes. *Mathematically:* $\frac{\partial u}{\partial y} \neq \frac{\partial v}{\partial x}$ (rotational two-dimensional flow).

In the figure bellow, the top and bottom surfaces of the element in the viscous fluid move at different velocities, and this will cause the vertical sides to rotate clockwise at the rate β. As a result, this produces rotational flow.

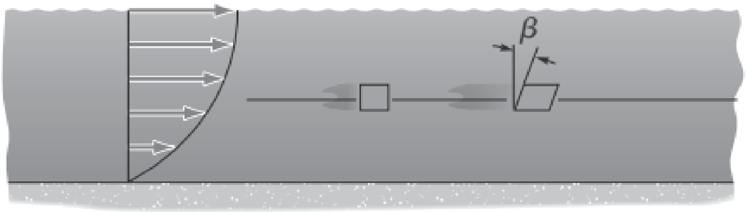


Fig. II.18: Rotational flow (viscous fluid)

Chapter III Dynamics of Ideal Incompressible Fluids

III.1 Forces acting on a fluid particle

Recall that the forces acting on a fluid element may be classified as **surface forces** and **body forces**; surface forces include both normal forces and tangential (shear) forces.
We shall consider the $\boldsymbol{x}$ component of the force acting on a differential element of mass $\boldsymbol{dm}$ and volume $\boldsymbol{dV = dx\ dy\ dz}$. Only those stresses that act in the $\boldsymbol{x}$ direction will give rise to surface forces in the $\boldsymbol{x}$ direction. If the stresses at the center of the differential element are taken to be $\boldsymbol{\sigma_{xx}}$, $\boldsymbol{\tau_{yx}}$, and $\boldsymbol{\tau_{zx}}$, then the stresses acting in the $\boldsymbol{x}$ direction on all faces of the element (obtained by a Taylor series expansion about the center of the element) are as shown in Fig. III.1.

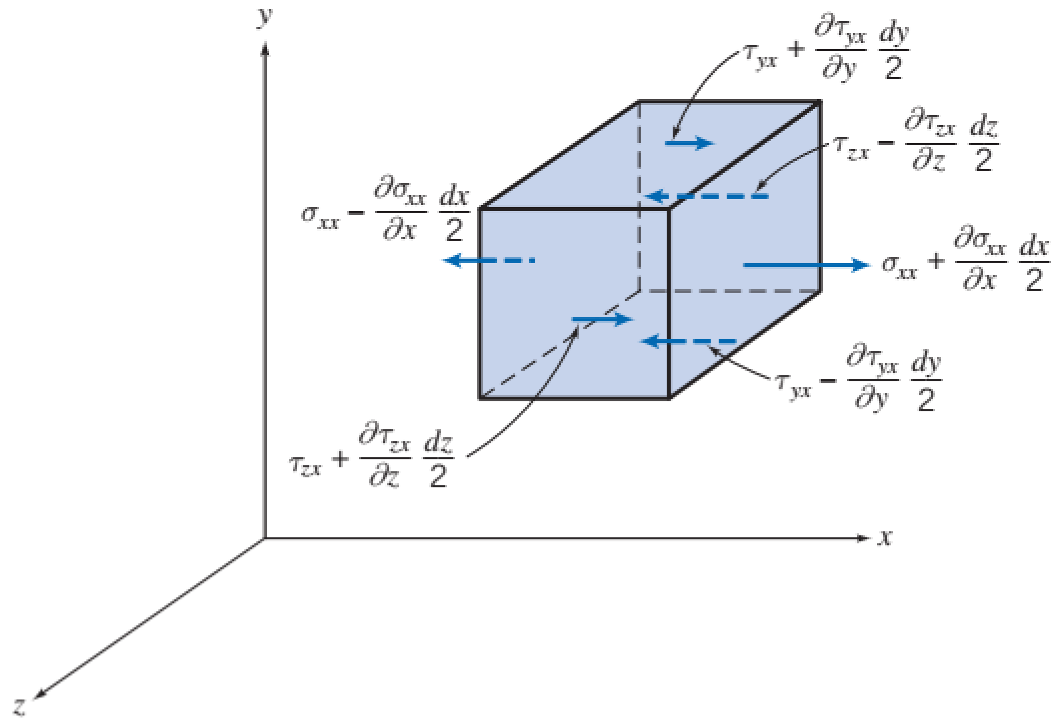


Fig. III.1 Stresses in the x direction on an element of fluid.

To obtain the net ***surface force*** in the $\boldsymbol{x}$ direction, $\boldsymbol{dF_{Sx}}$, we must sum the forces in the $\boldsymbol{x}$ direction. Thus,

$$dF_{Sx} = \left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}\frac{dx}{2}\right)dydz - \left(\sigma_{xx} - \frac{\partial \sigma_{xx}}{\partial x}\frac{dx}{2}\right)dydz + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\frac{dy}{2}\right)dxdz - \left(\tau_{yx} - \frac{\partial \tau_{yx}}{\partial y}\frac{dy}{2}\right)dxdz + \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\frac{dz}{2}\right)dxdy - \left(\tau_{zx} - \frac{\partial \tau_{zx}}{\partial z}\frac{dz}{2}\right)dxdy$$

On simplifying, we obtain

$$dF_{Sx} = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)dxdydz$$

Apart from these forces, there is also the ***body force*** due to the weight of the particle. If m is the particle's mass, this force is $dF_B = (dm)g = \rho g dx\ dy\ dz$.

The net force in the x direction, dF_x, is given by

$$dF_x = dF_{Sx} + dF_{Bx} = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho g_x\right) dxdydz \qquad \text{III.1}$$

We can derive similar expressions for the force components in the y and z directions:

$$dF_y = dF_{Sy} + dF_{By} = \left(\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{zy}}{\partial z} + \rho g_y\right) dxdydz \qquad \text{III.2}$$

$$dF_z = dF_{Sz} + dF_{Bz} = \left(\frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \rho g_z\right) dxdydz \qquad \text{III.3}$$

III.2 Equations of Motion

A dynamic equation describing fluid motion may be obtained by applying Newton's second law to a particle. Then, for an infinitesimal system of mass ***dm***, Newton's second law can be written

$$d\vec{F} = dm \left.\frac{\mathrm{d}\vec{V}}{dt}\right|_{system}$$

Provided the particle's velocity is expressed as a velocity field, **V** = **V**(***x***, ***y***, ***z***, ***t***), then the material derivative is used to determine the acceleration:

$$a = \frac{d\mathbf{V}}{dt} = \frac{\partial \mathbf{V}}{\partial t} + \frac{\partial \mathbf{V}}{\partial x}\frac{dx}{dt} + \frac{\partial \mathbf{V}}{\partial y}\frac{dy}{dt} + \frac{\partial \mathbf{V}}{\partial z}\frac{dz}{dt}$$

Thus

$$dF = dm\frac{d\mathbf{V}}{dt} = (\rho dx\, dy\, dz)\left[\frac{\partial \mathbf{V}}{\partial t} + u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z}\right]$$

When **V** = u**i** + v**j** + w**k**, the x, y, z components of this equation become

$$\begin{aligned}
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho g_x &= \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)\\
\frac{\partial \sigma_{yy}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho g_y &= \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)\\
\frac{\partial \sigma_{zz}}{\partial x} + \frac{\partial \tau_{xz}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \rho g_z &= \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)
\end{aligned} \qquad \text{III.4}$$

Or:

$$\begin{aligned}
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho g_x &= \rho\frac{du}{dt}\\
\frac{\partial \sigma_{yy}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho g_y &= \rho\frac{dv}{dt}\\
\frac{\partial \sigma_{zz}}{\partial x} + \frac{\partial \tau_{xz}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \rho g_z &= \rho\frac{dw}{dt}
\end{aligned} \qquad \text{III.5}$$

III.3 The Euler Equations

If we consider the fluid to be an ideal fluid, then the equations of motion will reduce to a simpler form. In particular, there will be no viscous shear stress on the particle (element), and the three normal stress components will represent the pressure. Since these normal stresses have all been defined in Fig. III.1 as positive outward, and as a convention, positive pressure produces a *compressive stress*, then $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p$. As a result, the general equations of motion for an ideal fluid particle become

$$-\frac{\partial p}{\partial x}+0+0+\rho g_x=\rho\frac{du}{dt}$$
$$-\frac{\partial p}{\partial x}+0+0+\rho g_y=\rho\frac{dv}{dt}$$
$$-\frac{\partial p}{\partial x}+0+0+\rho g_z=\rho\frac{dw}{dt}$$

$$-\frac{\partial p}{\partial x}+\rho g_x=\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)$$
$$-\frac{\partial p}{\partial y}+\rho g_y=\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right) \qquad \text{III.6}$$
$$-\frac{\partial p}{\partial z}+\rho g_z=\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)$$

These equations are called the ***Euler equations of motion***, expressed in *x*, *y*, *z* coordinates. Using the gradient operator, we can also write these equations in a more compact form, namely

$$-\nabla p+\rho g=\rho\left[\frac{\partial \boldsymbol{V}}{\partial t}+(\boldsymbol{V}.\nabla)\boldsymbol{V}\right] \qquad \text{III.7}$$

Two-Dimensional Steady Flow. In many cases we will have steady two-dimensional flow, and the *z* component of velocity $w = 0$. With $g_y = -g$, the Euler's equations become

$$-\frac{1}{\rho}\frac{\partial p}{\partial x}=u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}$$
$$-\frac{1}{\rho}\frac{\partial p}{\partial y}-g=u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y} \qquad \text{III.8}$$

III.4 The Bernoulli Equation and its Applications

III.4.1 The Bernoulli Equation

Assume we have irrotational two-dimensional flow so that: $\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x}$

If we substitute this condition into obtained equations for *Two-Dimensional Steady Flow*, we get

$$-\frac{1}{\rho}\frac{\partial p}{\partial x}=u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}$$
$$-\frac{1}{\rho}\frac{\partial p}{\partial y}-g=u\frac{\partial u}{\partial y}+v\frac{\partial v}{\partial y}$$

Since $\partial(u^2)/\partial x = 2u(\partial u/\partial x)$, $\partial(v^2)/\partial x = 2v(\partial v/\partial x)$, $\partial(u^2)/\partial y = 2u(\partial u/\partial y)$, and $\partial(v^2)/\partial y = 2v(\partial v/\partial y)$, the above equations become

$$-\frac{1}{\rho}\frac{\partial p}{\partial x}=\frac{1}{2}\frac{\partial(u^2+v^2)}{\partial x}$$
$$-\frac{1}{\rho}\frac{\partial p}{\partial y}-g=\frac{1}{2}\frac{\partial(u^2+v^2)}{\partial y}$$

Integrating with respect to *x* in the first equation, and with respect to *y* in the second equation, yields

$$-\frac{p}{\rho}+f(\mathrm{y})=\frac{1}{2}(u^2+v^2)=\frac{1}{2}V^2$$
$$-\frac{p}{\rho}-gy+h(x)=\frac{1}{2}(u^2+v^2)=\frac{1}{2}V^2$$

Here V is the fluid particle's velocity found from its components, $V^2 = u^2 + v^2$. Equating these two results, it is then necessary that $f(y) = -gy + h(x)$. The solution requires that $h(x)$ = Const., since x and y can vary independent of one another. As a result, the unknown function $f(y) = -gy$ + Const. Substituting this and $h(x)$ = Const. into the above two equations, we obtain in either case the Bernoulli equation, that is,

$$\frac{p}{\rho}+\frac{V^2}{2}+\mathrm{gy}=Const \qquad \text{III.9}$$

Or

$$\mathrm{y}+\frac{p}{\gamma}+\frac{V^2}{2g}=Const \qquad \text{III.10}$$

Thus, if the flow is *irrotational*, then the Bernoulli equation may be applied between *any two points* (x_1, y_1) and (x_2, y_2). Of course, as noted, we must *also* require the *fluid to be ideal* and the *flow to be steady*.

III.4.2 Applications of the Bernoulli Equation

a) Venturi meter. A *venturi meter* is a device that can be used to measure the average velocity or the flow of an incompressible fluid through a pipe, Fig. III.2.

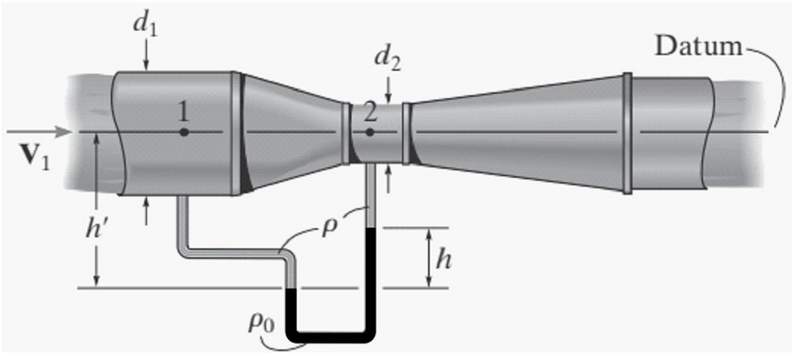


Fig. III.2 Venturi meter

$$\mathrm{y}_1+\frac{p_1}{\gamma}+\frac{V_1^2}{2g}=\mathrm{y}_2+\frac{p_2}{\gamma}+\frac{V_2^2}{2g}$$
$$0+\frac{p_1}{\rho g}+\frac{V_1^2}{2g}=0+\frac{p_2}{\rho g}+\frac{V_2^2}{2g}$$

In addition, the continuity equation can be applied at points 1 and 2. For steady flow we have

$$V_1\frac{\pi d_1^2}{4}=V_2\frac{\pi d_2^2}{4}$$

Combining these two results and solving for V_1, we get

The static pressure difference (p_2 - p_1) is often measured using a pressure transducer or a manometer. For example, if a manometer is used, as in Fig. III.2, and ρ is the density of the fluid in the pipe, and ρ_0 is the density for the fluid in the manometer, then applying the manometer rule, we have

$$p_1 + \rho g h' - \rho_0 g h - \rho g(h' - h) = p_2$$

$$p_2 - p_1 = (\rho - \rho_0) g h \qquad \text{III.11}$$

b) Flow from a Large Reservoir.

b.1 Expression of velocity

When water flows from a tank or reservoir through a drain, (figure bellow), the flow is *unsteady*. If we assume that water is an ideal fluid, then the Bernoulli equation can be applied between points *A* and *B*. Setting the gravitational datum at *B*, and using gage pressures, where $p_A = p_B = 0$, we have

$$y_A + \frac{p_A}{\gamma} + \frac{V_A^2}{2g} = y_B + \frac{p_B}{\gamma} + \frac{V_B^2}{2g}$$

$$h + 0 + 0 = 0 + 0 + \frac{V_B^2}{2}$$

$$V_B = \sqrt{2gh} \qquad \text{III.12}$$

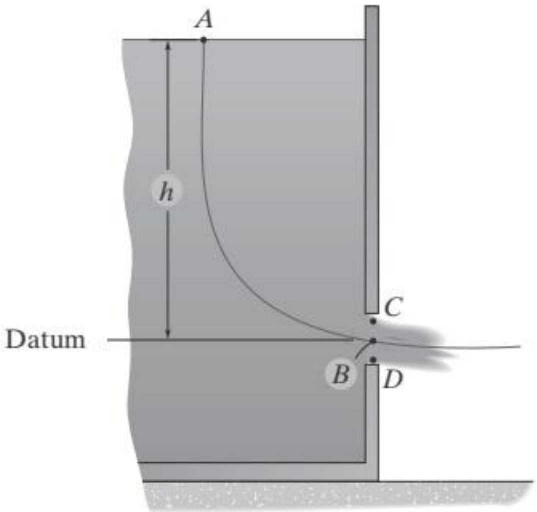


Fig. III.3 Flow from a Large Reservoir

This result is known as *Torricelli's law* since it was first formulated by Evangelista Torricelli in the 17th century.

b.2 Tank emptying time

The tank emptying process applies Bernoulli's equation to relate fluid velocity to height. It enables calculating the discharge rate and the time required for the tank to empty.

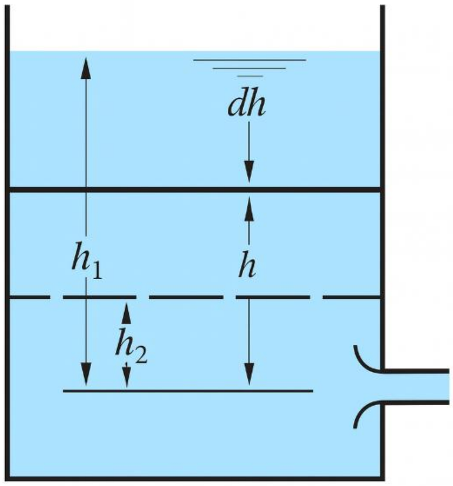


Fig. III.4 Tank emptying time

$$v = \sqrt{2gh}$$

The instantaneous flow rate is

$$Q = vA_o = \sqrt{2gh}A_o$$

$\boldsymbol{A_o}$ is the orifice area

In the time interval ***dt***, the small volume flow ***dV*** is written as ***Qdt***. In the same time interval, the head height decreases by ***dh*** and the discharged volume is equal to the tank surface area $\boldsymbol{A_T}$ multiplied by ***dh***. By equating these values, we obtain:

$$Qdt = \sqrt{2gh}A_o dt = -A_T dh$$

where the negative sign indicates that ***h*** decreases when ***t*** increases. Solving for ***t***, we obtain

$$t = \int_{t_1}^{t_2} dt = \frac{-A_T}{A_o\sqrt{2g}} \int_{h_1}^{h_2} h^{-1/2} dh$$

Or

$$t = t_2 - t_1 = \frac{2A_T}{A_o\sqrt{2g}}\left(h_1^{1/2} - h_2^{1/2}\right)$$

Equation of ***t*** can be rewritten by multiplying and dividing by. It results in

$$t = t_2 - t_1 = \frac{A_T(h_1 - h_2)}{\frac{1}{2}(A_o\sqrt{2gh_1} + A_o\sqrt{2gh_2})} \qquad \text{III.13}$$

c) Pitot tube.

c.1 Flow in an Open Channel.
One method of determining the velocity of a moving liquid in an *open channel*, such as a river, is to immerse a bent tube into the stream and observe the height h to which the liquid rises within the tube, Fig. III.5. Such a device is called a *stagnation tube*, or a *Pitot tube*, named after Henri Pitot who invented it in the early 18th century.

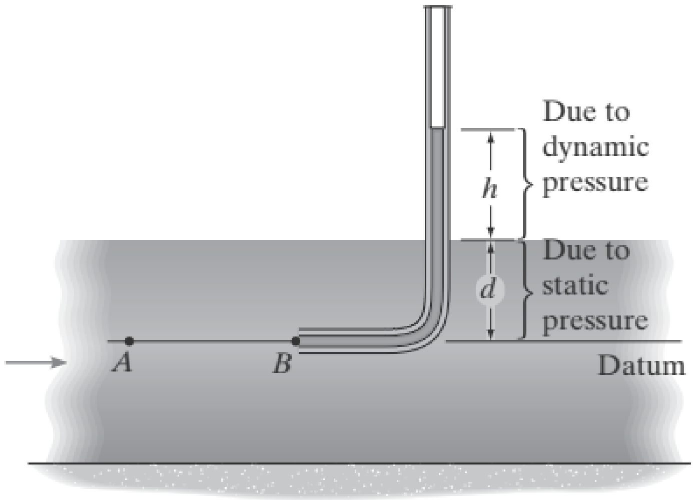


Fig. III.5 Flow in an Open Channel

To show how it works, consider the two points A and B located on the horizontal streamline. Point A is upstream within the fluid, where the velocity of flow is V_A and the pressure is $p_A = \rho gd$. Point B is at the opening of the tube. It is the stagnation point, since the velocity of flow has momentarily been reduced to zero due to its impact with the liquid within the tube. The liquid at this point produces *both* a *static pressure*, which causes the liquid in the tube to rise to a level d, and a *dynamic pressure*, which forces additional liquid farther up the vertical segment to a height h above the liquid surface. Thus the total pressure of the liquid at B is $p_B = \rho g(d + h)$. Applying the Bernoulli equation with the gravitational datum on the streamline, we have

$$y_A + \frac{p_A}{\gamma} + \frac{V_A^2}{2g} = y_B + \frac{p_B}{\gamma} + \frac{V_B^2}{2g}$$

$$0 + \frac{\gamma d}{\gamma} + \frac{V_A^2}{2g} = 0 + \frac{\gamma(d+h)}{\gamma} + 0$$

$$V_A = \sqrt{2gh} \qquad \text{III.14}$$

c.2 Flow in a Closed Conduit. Hence, by measuring h on the Pitot tube, the velocity of the flow can be determined. If the liquid is flowing in a *closed conduit* or pipe, Fig. III.6, then it will be necessary to use both a piezometer and a Pitot tube to determine the velocity of the flow. The ***piezometer*** measures the *static pressure* at A. This pressure is caused by the internal pressure in the pipe, ρgh, and the hydrostatic pressure ρgd, caused by the weight of the fluid. The total pressure at A is therefore $\rho g(h + d)$. The total pressure at the stagnation point B will be larger than this, due to the dynamic pressure $V_A^2/2$. If we apply the Bernoulli equation at points

A and B on the streamline, using the measurements h and $(l + h)$ from these two tubes, the velocity V_A can be obtained.

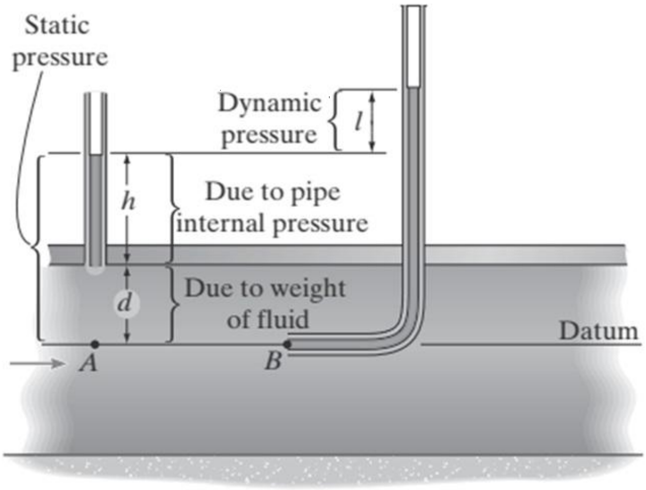


Fig. III.6 Flow in a Closed Conduit

$$y_A + \frac{p_A}{\gamma} + \frac{V_A^2}{2g} = y_B + \frac{p_B}{\gamma} + \frac{V_B^2}{2g}$$

$$0 + \frac{\gamma(h+d)}{\gamma} + \frac{V_A^2}{2g} = 0 + \frac{\gamma(h+d+l)}{\gamma} + 0$$

$$V_A = \sqrt{2gl} \qquad \text{III.15}$$

III.5 Momentum Equation

III.5.1 General Momentum Equation

The design of many hydraulic structures, such as floodgates and flow diversion blades, as well as pumps and turbines, depends upon the forces that a fluid flow exerts on them. In this section we will obtain these forces by using a linear momentum analysis, which is based on Newton's second law of motion, written in the form $\mathbf{F} = m\mathbf{a} = d(m\mathbf{V})/dt$. For application of this equation, it is important to measure the time rate of change in the momentum, $m\mathbf{V}$, from an *inertial* or nonaccelerating frame of reference, that is, a reference that either is fixed or moves with constant velocity.

Because of the fluid flow, a control volume approach works best for this type of analysis, and so we will apply the Reynolds transport theorem to determine the time derivative $d(m\mathbf{V})/dt$ before we apply Newton's second law. Linear momentum is an extensive property of a fluid, where

$\mathbf{N} = m\mathbf{V}$, and so $\eta = m\mathbf{V}/m = \mathbf{V}$. Therefore, Eq. II.5 becomes

$$\left(\frac{dN}{dt}\right)_{\text{syst}} = \frac{\partial}{\partial t}\int_{\text{cv}} \eta\rho\, d\forall + \int_{\text{cs}} \eta\rho\mathbf{V}\cdot d\mathbf{A}$$

$$\left(\frac{d(m\mathbf{V})}{dt}\right)_{\text{syst}} = \frac{\partial}{\partial t}\int_{\text{cv}} \mathbf{V}\rho\, d\forall + \int_{\text{cs}} \mathbf{V}\rho\mathbf{V}\cdot d\mathbf{A}$$

Now, substituting this result into Newton's second law of motion, we obtain our result, the *linear momentum equation*.

$$\mathbf{\Sigma F} = \frac{\partial}{\partial t}\int_{\mathrm{cv}} \mathbf{V}\rho \, d\forall + \int_{\mathrm{cs}} \mathbf{V}\rho\mathbf{V} \cdot d\mathbf{A} \qquad \text{III.16}$$

It is *very important* to realize how the velocity **V** is used in the last term of this equation. It stands alone as a *vector quantity* **V**, and as a result it has *components* along the *x*, *y*, *z* axes. But it is also involved in the dot product operation with d**A** in order to define the *mass flow through* an open control surface, that is, $\rho\mathbf{V} \cdot d\mathbf{A}$. This is a *scalar quantity*, and so it *does not have components*.

III.5.2 Steady-state momentum theorem

If the flow is *steady*, then no local change of momentum will occur within the control volume, and the first term on the right of Eq. III.11 will be equal to zero. Therefore

$$\mathbf{\Sigma F} = \int_{\mathrm{cs}} \mathbf{V}\rho\mathbf{V} \cdot d\mathbf{A} \qquad \text{III.17}$$

Furthermore, if we have an *ideal fluid*, then ρ is constant and viscous friction is zero. Thus the velocity will be uniformly distributed over the open control surfaces, and so integration of Eq. III.12 gives

$$\mathbf{\Sigma F} = \mathbf{\Sigma V}\rho\mathbf{V} \cdot \mathbf{A} \qquad \text{III.18}$$

The above equations are often used in engineering, to obtain the fluid forces acting on various types of surfaces that deflect or transport the flow.
If there is only one entrance and one exit, as in Fig. III.7, the momentum equation becomes

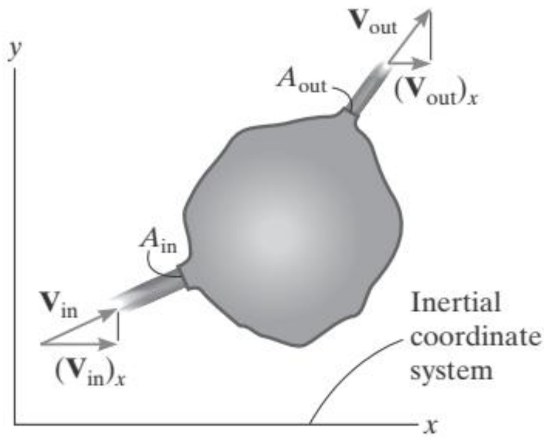
$$\mathbf{\Sigma F} = (V_{in})_x(-\rho V_{in}A_{in}) + (V_{out})_x(\rho V_{out}A_{out}) \qquad \text{III.19}$$

Here, $(V_{in})_x$ and $(V_{out})_x$ are the *x* components of $\mathbf{V}_{in}$ and $\mathbf{V}_{out}$. They both act in the $+x$ direction, Fig. III.7. When writing the expression for the dot products, we have followed our positive sign convention, that is, A_{in} and A_{out} are *both* positive out, but V_{in} is negative, since it is directed into the control volume. For this reason, $\rho V_{in}A_{in}$ is a negative quantity.
Using continuity,

$$\dot{m} = \rho V_{in}A_{in} = \rho V_{out}A_{out} \qquad \text{III.20}$$

the momentum equation takes the simplified form

$$\mathbf{\Sigma F} = \dot{m}((V_{out})_x - (V_{in})_x) \qquad \text{III.21}$$



Control Volume

Fig. III.7

III.5.3 Momentum Equation Applied to Stationary Deflector (Impact of jets on surfaces and reactions)

Let us first consider the stationary deflector, illustrated in Fig. III.8. Bernoulli's equation allows us to conclude that the magnitudes of the velocity vectors are equal (i.e., $V_2 = V_1$), since the pressure is assumed to be constant external to the fluid jet and elevation changes are negligible (see Eq. III.10). Assuming steady, uniform flow, the momentum equation takes the form of Eq. III.16, which for the *x*- and *y*-directions becomes

Momentum in *x*-direction:

$$-R_x = \dot{m}(V_{2x} - V_1) = \dot{m}(V_2 \cos\ \alpha - V_1) = \dot{m}V_1(\cos\ \alpha - 1)$$

Momentum in *y*-direction:

$$R_y = \dot{m}V_{2y} = \dot{m}V_2 \sin\ \alpha = \dot{m}V_1 \sin\ \alpha$$

For given jet conditions the reaction force components R_x and R_y can be calculated.

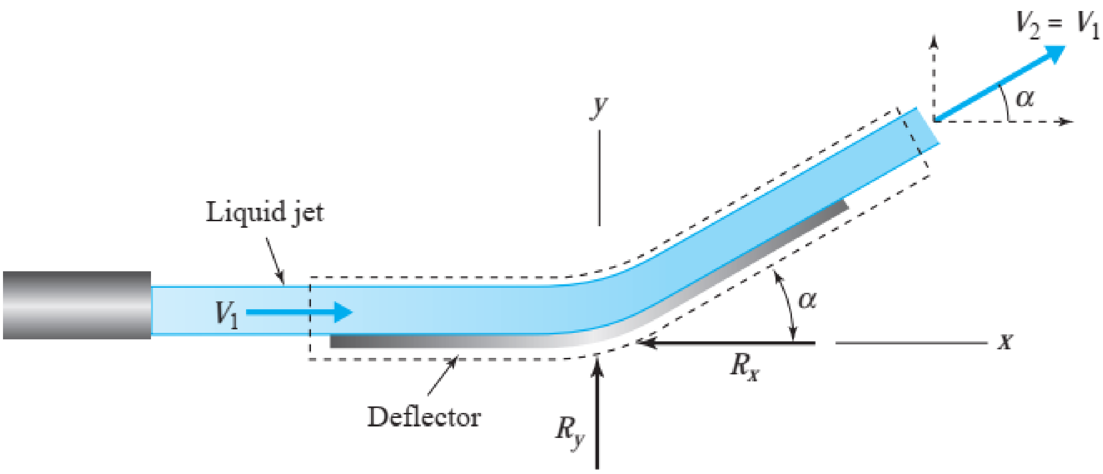


Fig. III.8 Stationary Deflector

Chapter IV: Real incompressible fluid dynamics

IV.1 Viscosity

IV.1.1 Dynamic viscosity

Consider a fluid-filled space formed by two horizontal parallel plates (Figure IV.1). The upper plate has an area A in contact with the fluid and is pulled to the right with a force F_1 at a velocity V_1. A velocity distribution like that illustrated in Figure IV.1 would result.

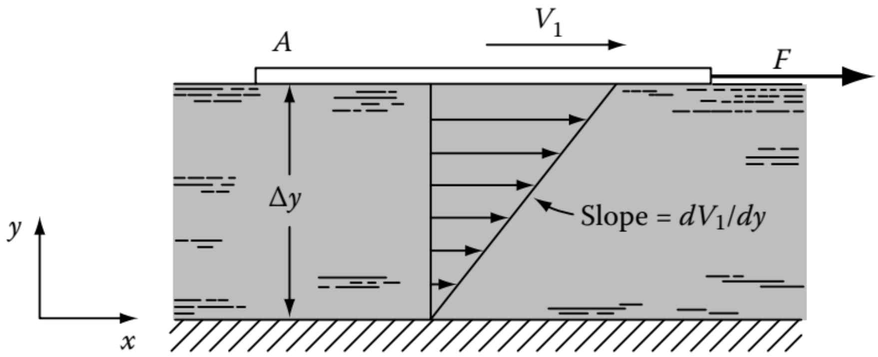


Figure IV.1 Shear stress applied to a fluid.

The fluid velocity at the moving plate is V_1 because the fluid adheres to that surface. This phenomenon is called the nonslip condition. At the bottom, the velocity is zero with respect to the boundary, owing again to the nonslip condition.

The slope of the velocity distribution is dV_1/dy. If this experiment is repeated with F_2 as the force, a different slope or strain rate results: dV_2/dy. In general, to each applied force there corresponds only one shear stress ($\tau = F/A$) and only one strain rate (dV/dy).

If data from a series of these experiments were plotted as τ versus dV/dy, Figure IV.1 would result for a fluid such as water.

The points lie on a straight line that passes through the origin. The slope of the resulting line in Figure IV.1 is the viscosity (μ) of the fluid because it is a measure of the fluid's resistance to shear. In other words, viscosity indicates how a fluid will react (dV/dy) under the action of an external shear stress (τ).

The plot of Figure 1.2 is a straight line that passes through the origin. This result is characteristic of a Newtonian fluid. Examples of Newtonian fluids are water, oil, and air.

Newtonian fluids follow Newton's law of viscosity and are represented by the equation:

$$\tau = \mu \frac{dV}{dy} \qquad \text{IV.1}$$

where:

τ = the applied shear stress in dimensions of F/L^2 (N/m^2)

μ is called the absolute or dynamic viscosity of the fluid in dimensions of $F.T/L^2$ ($N.s/m^2$)

dV/dy = the strain rate in dimensions of $1/T$ (1/s)

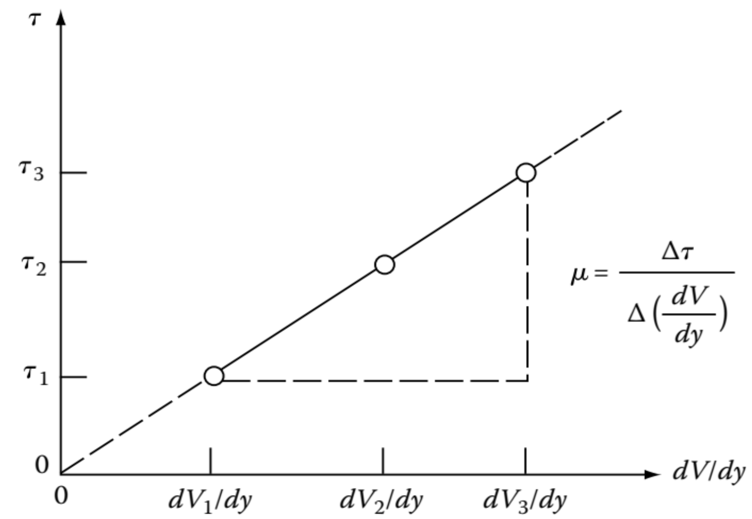


Figure IV.2 A plot of τ versus dV/dy (a rheological diagram) for Newtonian fluids.

If a fluid cannot be described by Equation IV.1, it is called a non-Newtonian fluid. A graph of τ versus dV/dy, called a rheological diagram, is shown in Figure IV.3 for several types of fluids.

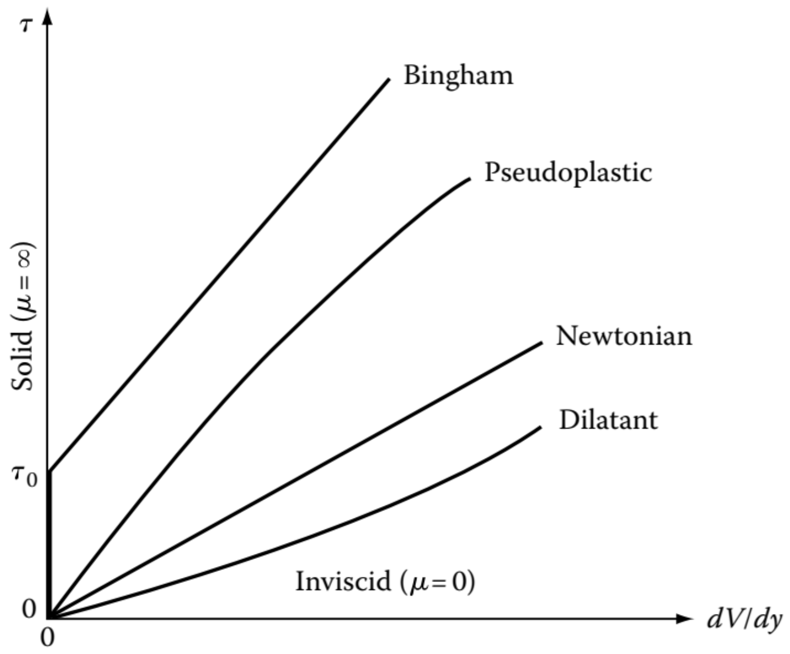


Figure IV.3 A rheological diagram for Newtonian and non-Newtonian fluids.

IV.1.2 Kinematic viscosity

The ratio of absolute viscosity to density is called the kinematic viscosity ν:

$$\nu = \frac{\mu}{\rho} \qquad \text{IV.2}$$

The dimensions of kinematic viscosity are L^2/T (m^2/s).

It is important to note that the viscosity and density of fluids both change with temperature.

IV.2 Stress–Deformation Relationships

For incompressible Newtonian fluids it is known that the stresses are linearly related to the rates of deformation and can be expressed in Cartesian coordinates as

For normal stresses:

$$\sigma_{xx} = -p + 2\mu \frac{\partial u}{\partial x} \qquad \text{IV.3}$$

$$\sigma_{yy} = -p + 2\mu \frac{\partial v}{\partial y} \qquad \text{IV.4}$$

$$\sigma_{zz} = -p + 2\mu \frac{\partial w}{\partial z} \qquad \text{IV.5}$$

For shearing stresses

$$\tau_{xy} = \tau_{yx} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \qquad \text{IV.6}$$

$$\tau_{yz} = \tau_{zy} = \mu \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \qquad \text{IV.7}$$

$$\tau_{zx} = \tau_{xz} = \mu \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \qquad \text{IV.8}$$

where p is the pressure, the negative of the average of the three normal stresses; that is, as indicated by the figure in IV.4,

$$-p = \frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$$

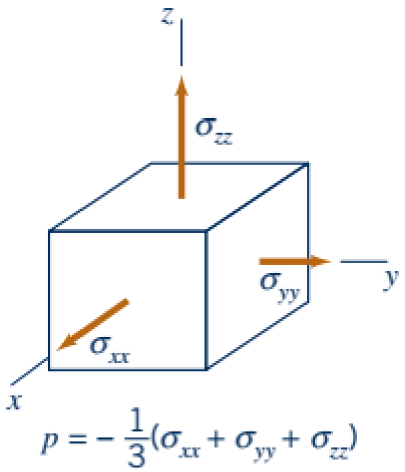


Figure: IV.4

For viscous fluids in motion the normal stresses are not necessarily the same in different directions, thus, the need to define the pressure as the average of the three normal stresses. For fluids at rest, or frictionless fluids, the normal stresses are equal in all directions.

In cylindrical polar coordinates the stresses for incompressible Newtonian fluids are expressed as (for normal stresses)

$$\sigma_{rr} = -p + 2\mu \frac{\partial v_r}{\partial r} \quad \text{IV.9}$$

$$\sigma_{\theta\theta} = -p + 2\mu \left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) \quad \text{IV.10}$$

$$\sigma_{zz} = -p + 2\mu \frac{\partial v_z}{\partial z} \quad \text{IV.11}$$

(for shearing stresses)

$$\tau_{r\theta} = \tau_{\theta r} = \mu \left[r \frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right] \quad \text{IV.12}$$

$$\tau_{\theta z} = \tau_{z\theta} = \mu \left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right) \quad \text{IV.13}$$

$$\tau_{zr} = \tau_{rz} = \mu \left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right) \quad \text{IV.14}$$

The double subscript has a meaning similar to that of stresses expressed in Cartesian coordinates— that is, the first subscript indicates the plane on which the stress acts and the second subscript the direction. Thus, for example, σ_{rr} refers to a stress acting on a plane perpendicular to the radial direction and in the radial direction (thus a normal stress). Similarly, $\tau_{r\theta}$ refers to a stress acting on a plane perpendicular to the radial direction but in the tangential (θ direction) and is therefore a shearing stress.

IV.3 The Navier–Stokes Equations

The stresses as defined in the preceding section can be substituted into the differential equations of motion (Eqs. III.4) and simplified by using the continuity equation for incompressible flow (Eq. II.11). For rectangular coordinates (Figure IV.5) the results are:

- *(x direction)*

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \quad \text{IV.15}$$

- *(y direction)*

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \rho g_y + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \quad \text{IV.16}$$

- *(z direction)*

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \rho g_z + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) \quad \text{IV.17}$$

where u, v, and w are the x, y, and z components of velocity as shown in the figure in the margin of the previous page.

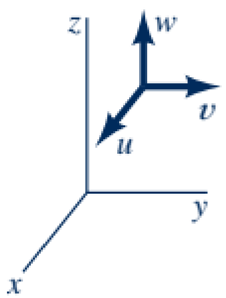


Figure: IV.5

These equations are commonly called the *Navier–Stokes equations*, when combined with the conservation of mass equation (Eq. II.11), provide a complete mathematical description of the flow of incompressible Newtonian fluids. We have four equations and four unknowns and therefore the problem is "well-posed" in mathematical terms.

In terms of cylindrical polar coordinates (Figure IV.6), the Navier–Stokes equations can be written as

(r direction)

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}\right)=-\frac{\partial p}{\partial r}+\rho g_r+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right)-\frac{v_r}{r^2}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial^2 v_r}{\partial z^2}\right] \quad \text{IV.18}$$

(θ direction)

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right)=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\rho g_\theta+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right)-\frac{v_\theta}{r^2}+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}+\frac{\partial^2 v_\theta}{\partial z^2}\right] \quad \text{IV.19}$$

(z direction)

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\rho g_z+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right] \quad \text{IV.20}$$

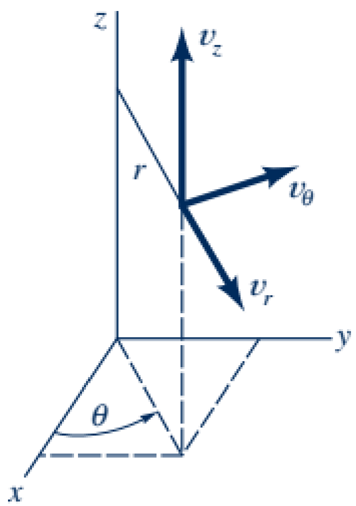


Figure: IV.6

IV.4 Steady, Laminar Flow in Circular Tubes (Poiseuille flow)

Consider the flow through a horizontal circular tube of radius R. Because of the cylindrical geometry it is convenient to use cylindrical coordinates. We assume that the flow is parallel to the walls so that $v_r = 0$ and $v_\theta = 0$.

For steady, incompressible flow, the differential form of the continuity equation in cylindrical coordinates is

$$\frac{1}{r}\frac{\partial(rv_r)}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial v_z}{\partial z}=0$$

From this equation $\partial v_z / \partial z = 0$. Also, for steady, axisymmetric flow, v_z is not a function of t or θ, so the velocity, v_z, is only a function of the radial position within the tube—that is, $v_z = v_z(r)$. Under these conditions the Navier–Stokes equations (Eqs. IV.18, IV.19 and IV.20) reduce to:

$$0 = -\rho g \sin\theta - \frac{\partial p}{\partial r} \qquad \text{IV.21}$$

$$0 = -\rho g \cos\theta - \frac{1}{r}\frac{\partial p}{\partial \theta} \qquad \text{IV.22}$$

$$0 = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)\right] \qquad \text{IV.23}$$

where we have used the relationships $g_r = -g\sin\theta$ and $g_\theta = -g\cos\theta$.
Equations IV.21 and IV.22 can be integrated to give:

$$p = -\rho g(r\sin\theta) + f_1(z)$$

or

$$p = -\rho g y + f_1(z) \qquad \text{IV.24}$$

Equation IV.24 indicates that the pressure is hydrostatically distributed at any particular cross section, and the z component of the pressure gradient, $\partial p / \partial z$, is not a function of r or θ.

The equation of motion in the z direction (Eq. IV.23) can be written in the form:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = \frac{1}{\mu}\frac{\partial p}{\partial z}$$

and integrated (using the fact that $\partial p / \partial z =$ constant) to give

$$r\frac{\partial v_z}{\partial r} = \frac{1}{2\mu}\left(\frac{\partial p}{\partial z}\right) r^2 + c_1$$

Integrating again we obtain

$$v_z = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right) r^2 + c_1 \ln r + c_2 \qquad \text{IV.25}$$

Since we wish v_z to be finite at the center of the tube ($r = 0$), it follows that $c_1 = 0$ [since ln $(0) = -\infty$]. At the wall ($r = R$) the velocity must be zero so that

$$c_2 = -\frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right) R^2$$

and the velocity distribution becomes

$$v_z = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)(r^2 - R^2) \qquad \text{IV.26}$$

Thus, at any cross section the velocity distribution is parabolic.

To obtain a relationship between the volume rate of flow, Q, passing through the tube and the pressure gradient, we consider the flow through the differential, washer-shaped ring of Fig. 6.34b. Since v_z is constant on this ring, the volume rate of flow through the differential area $dA = (2\pi r)dr$ is

$$dQ = v_z(2\pi r)dr$$

and therefore

$$Q = 2\pi \int_0^R v_z r dr \qquad \text{IV.27}$$

Equation IV.26 for v_z can be substituted into Eq. IV.27, and the resulting equation integrated to yield

$$Q = -\frac{\pi R^4}{8\mu}\left(\frac{\partial p}{\partial z}\right) \qquad \text{IV.28}$$

This relationship can be expressed in terms of the pressure drop, Δp, which occurs over a length, ℓ, along the tube, since

$$\frac{\Delta p}{\ell} = -\frac{\partial p}{\partial z}$$

and therefore

$$Q = \frac{\pi R^4 \Delta p}{8\mu\ell} \qquad \text{IV.29}$$

For a given pressure drop per unit length, the volume rate of flow is inversely proportional to the viscosity and proportional to the tube radius to the fourth power. A doubling of the tube radius produces a 16-fold increase in flow! Equation IV.29 is commonly called *Poiseuille's law*.

In terms of the mean velocity, V, where $V = Q/\pi R^2$, Eq. IV.29 becomes

$$V = \frac{R^2 \Delta p}{8\mu\ell} \qquad \text{IV.30}$$

The maximum velocity $v_{\max}$ occurs at the center of the tube, where from Eq. IV.26

$$v_{\max} = -\frac{R^2}{4\mu}\left(\frac{\partial p}{\partial z}\right) = \frac{R^2 \Delta p}{4\mu\ell} \qquad \text{IV.31}$$

so that

$$v_{\max} = 2V$$

The velocity distribution, as shown by the figure in the margin, can be written in terms of $v_{\max}$ as

$$\frac{v_z}{v_{\max}} = 1 - \left(\frac{r}{R}\right)^2 \qquad \text{IV.32}$$

IV.5 Fluid flow regimes - Reynolds number

The flow of a fluid in a pipe may be laminar flow or it may be turbulent flow. Osborne Reynolds (1842–1912), a British scientist and mathematician, was the first to distinguish the difference between these two classifications of flow by using a simple apparatus as shown by the figure (a), which is a sketch of Reynolds' dye experiment. Reynolds injected dye into a pipe in which water flowed due to gravity. If water runs through a pipe of diameter *D* with an average velocity *V*, the following characteristics are observed by injecting neutrally buoyant dye as shown. For "small enough flowrates" the dye streak (a streakline) will remain as a well-defined line as it flows along, with only slight blurring due to molecular diffusion of the dye into the surrounding water. For a somewhat larger "intermediate flowrate" the dye streak fluctuates in time and space, and intermittent bursts of irregular behavior appear along the streak. On the other hand, for "large enough flowrates" the dye streak almost immediately becomes blurred and spreads across the entire pipe in a random fashion. These three characteristics, denoted as ***laminar***, ***transitional***, and ***turbulent*** flow, respectively, are illustrated in Figure b.

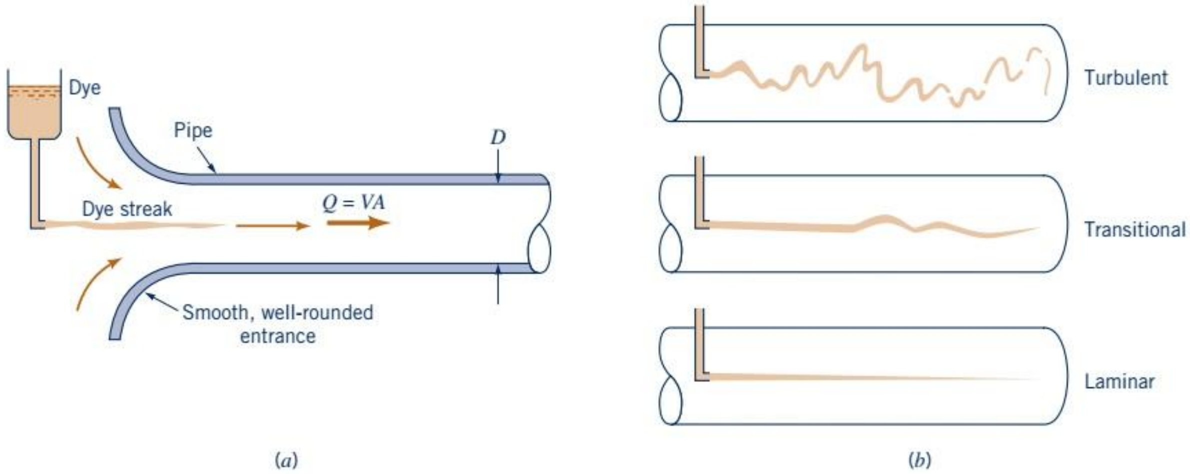


Figure IV.7 (a) Experiment to illustrate type of flow. (b) Typical dye streaks.

Reynolds showed that the parameter used to determine whether flow is laminar or turbulent is a dimensionless number called the Reynolds number given by the following expression:

$$Re = \frac{uD}{\nu} \qquad \text{IV.33}$$

u: Average flow velocity through the section under consideration in (m/s)
D: Pipe diameter or width of the fluid stream in (m).
ν: Kinematic viscosity of the fluid (m^2/s).
The different flow regimes can be classified according to Reynolds number (as an indication) as follows:
If $Re < 2000$, flow is laminar
If $Re > 2000$, flow is turbulent:
- Smooth turbulent if $2000 < Re < 100000$
- Rough turbulent if $Re > 100000$

IV.6 Head losses

Total head loss is the sum of linear (distributed) and singular (concentrated) head losses.

IV.6.1 Linear head losses (Major losses or frictional losses)

Linear pressure losses are pressure losses evenly distributed along a pipe.

a. Concept of pipe roughness

Unlike a smooth surface, a rough surface implies a surface condition whose irregularities have a direct effect on friction forces. A rough surface can be considered as being made up of a series of elementary protuberances characterized by a height, noted k, and called Roughness. In order to compare roughness with respect to pipe diameter, the ratio known as relative roughness is introduced: $\varepsilon = \frac{k}{D}$

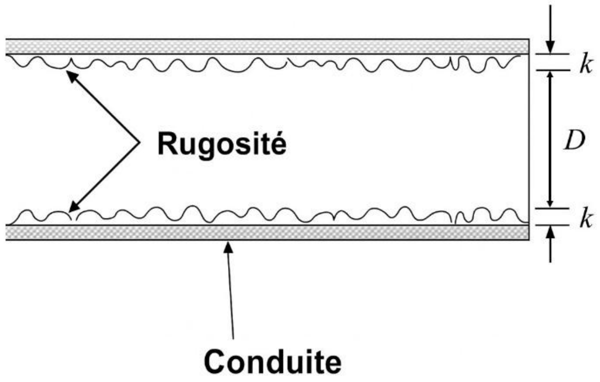


Figure IV.8

b. Expression of linear head loss

Linear head loss J_l is calculated using the Darcy-Weisbach formula:

$$J_l = f\frac{l}{D}\frac{v^2}{2g} \qquad \text{IV.34}$$

l: Pipe length (m)
D: Diameter of flow cross-section (m)
v : Flow velocity (m/s)
f: Coefficient of friction (unitless)

c. Expressions of the coefficient of friction *f*

In laminar flow, the coefficient of friction depends solely on the Reynolds number Re, according to the formula:

$$f = \frac{64}{Re} \qquad \text{IV.35}$$

In turbulent conditions, the Colebrook-White and Blasius formulas can be used:
Colebrook-White for hydraulically rough flow:

$$\frac{1}{\sqrt{f}} = 2log\left(\frac{3.71D}{k}\right) \qquad \text{IV.36}$$

Colebrook-White for hydraulically smooth flow:

$$\frac{1}{\sqrt{f}} = 2log\left(\frac{Re\sqrt{f}}{2.51}\right) \qquad \text{IV.37}$$

Blasius for Re < 10^5

$$f = \frac{0.316}{Re^{0.25}} \qquad \text{IV.38}$$

d- Moody diagram

Nikuradse's work on head loss in pipes has led to the development of a graph (Moody Diagram) (Figure IV.9) for determining the coefficient *f* as a function of *Re* for different types of flow and relative roughnesses *k/D* ranging from 1/30 to 1/1014:

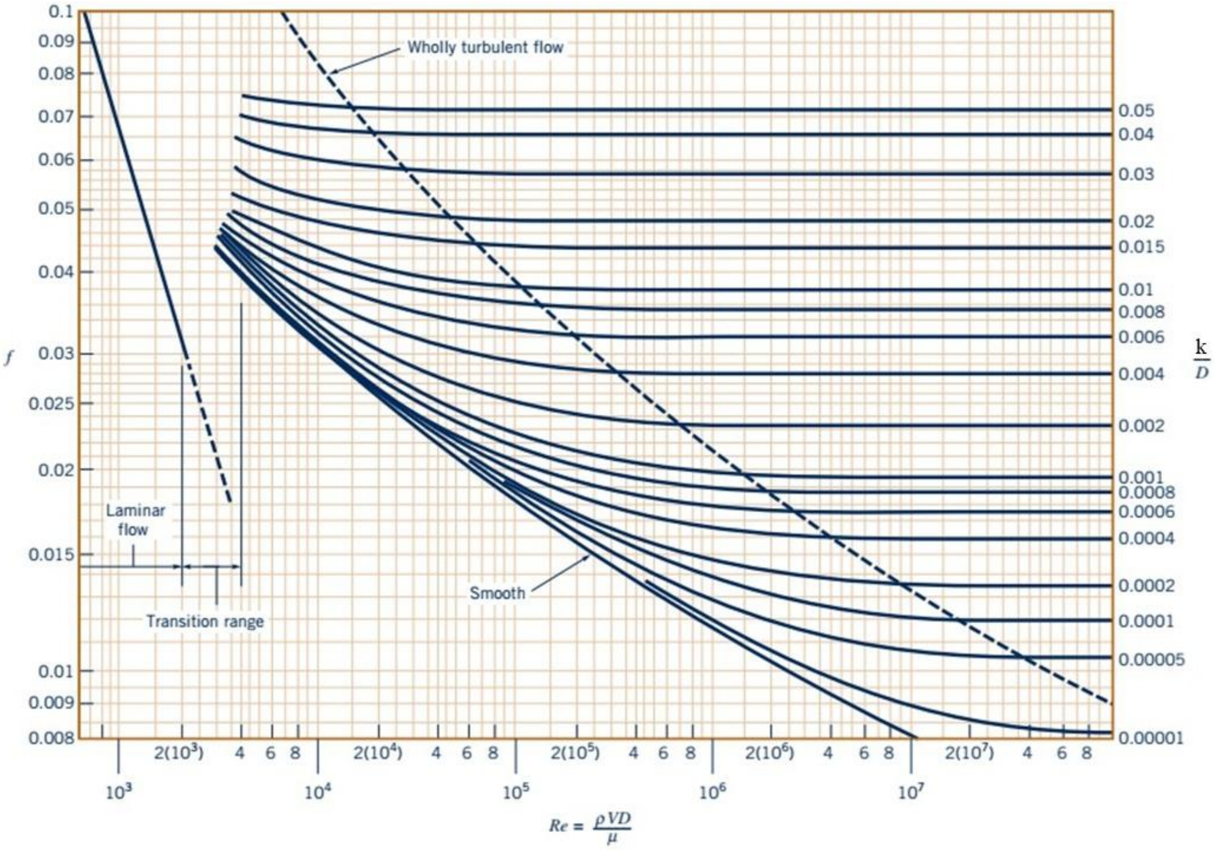


Figure IV.9: Moody diagram

IV.6.2 Minor head losses (Local losses)

Minor losses are the result of turbulent mixing of the fluid within the connection as the fluid passes through it. Minor losses occur when the flow geometry changes — for example, at bends, elbows, valves, expansions, contractions, and other fittings. These losses are localized at specific points in the system.

They are caused by several factors:

- **Changes in flow direction:** Bends and elbows create turbulence and result in pressure drops.

- **Changes in flow area:** Contractions and expansions lead to flow separation and energy dissipation.
- **Obstructions in the flow path:** Valves, fittings, and other components interfere with the flow, increasing resistance.

Minor losses are calculated using the following equation:

$$j_s = k\,\frac{v^2}{2g} \qquad \text{IV.39}$$

k is called the *resistance* or *loss coefficient*, which is determined from experiment. It is dependent on the type and geometry of the fitting.

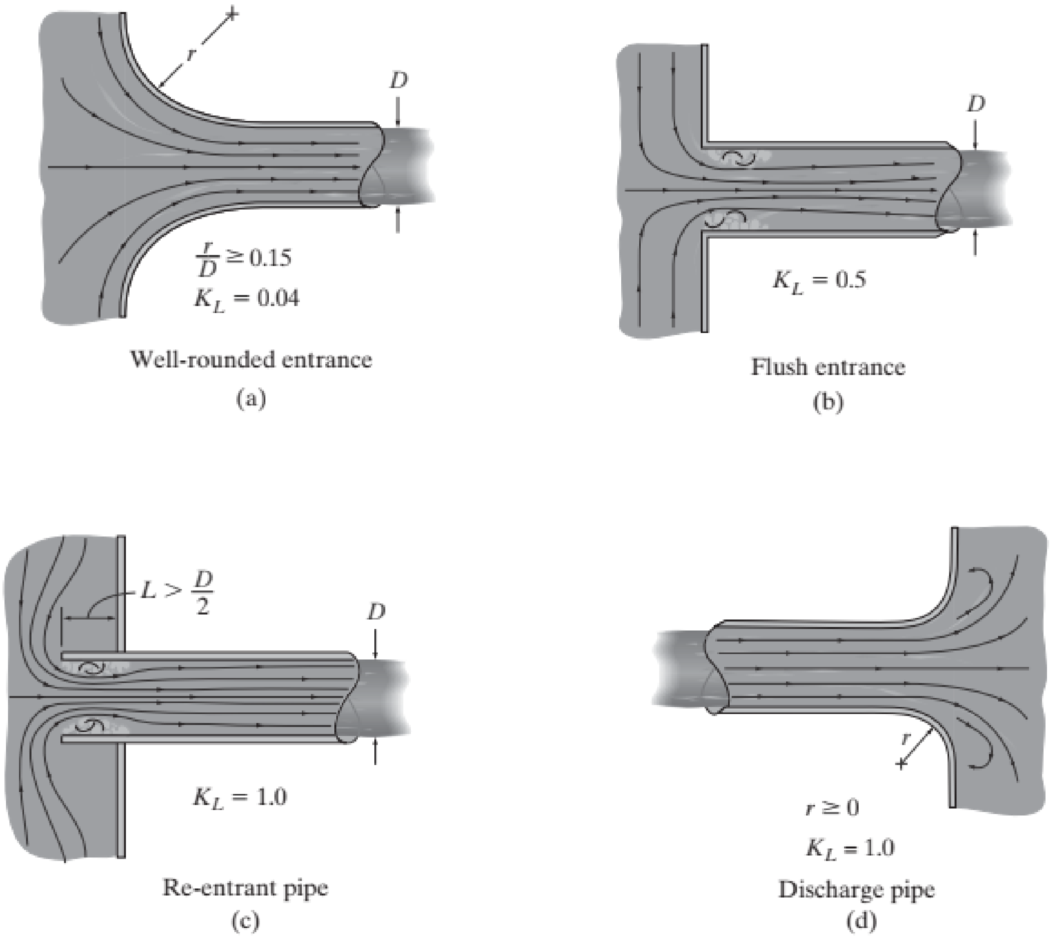
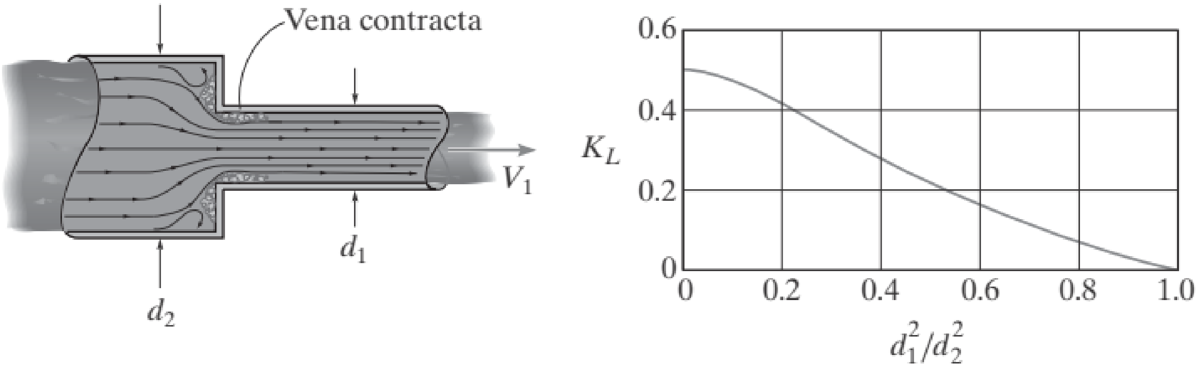
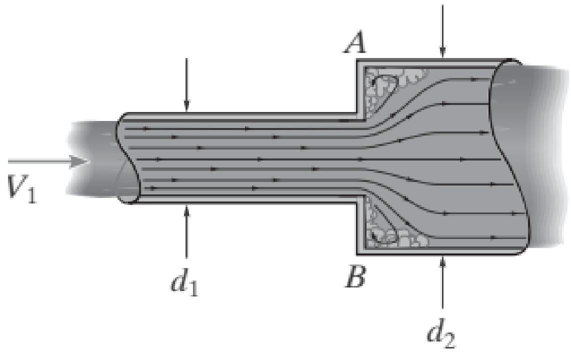


Figure IV.10: Inlet and exit transitions

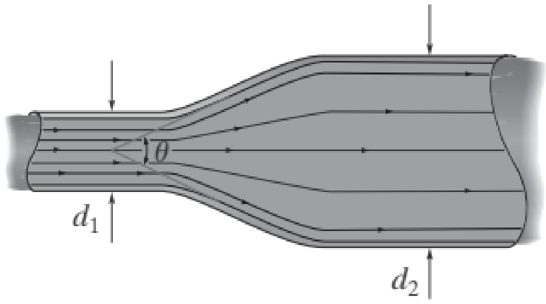


Sudden contraction



$$K_L = \left(1 - \frac{d_1^2}{d_2^2}\right)^2$$

Sudden expansion



θ ($\frac{d_2}{d_1} = 4$)	K_L
10°	0.13
20°	0.40
30°	0.80

Conical diffuser

Figure IV.11: Expansion and Contraction.

IV.7 Generalization of Bernoulli's theorem to real fluids

IV.7.1 Bernoulli's equation with head losses

The final form of Bernoulli's equation for a real liquid net is as follows:

$$z_A + \frac{p_A}{\gamma} + \frac{v_A^2}{2g} = z_B + \frac{p_B}{\gamma} + \frac{v_B^2}{2g} + J \qquad \text{IV.40}$$

$J = J_l + J_s$ with:

J: Total head losses.
J_l: Linear head losses.
J_s: Singular head losses.

IV.7.2 Bernoulli's equation with energy production

If the term W_{AB} denotes the mechanical work exchanged between the fluid and any machines present between points A and B, the Bernoulli equation can be expressed in the following general form:

$$z_A + \frac{p_A}{\gamma} + \frac{v_A^2}{2g} = z_B + \frac{p_B}{\gamma} + \frac{v_B^2}{2g} + \sum\left(\frac{J_l + J_s}{J}\right) + W_{AB} \qquad \text{IV.41}$$

If the machine supplies energy to the fluid (pump), then: $W_{AB} > 0$
If the machine receives energy from the fluid (turbine), then: $W_{AB} < 0$
If there is no machine between points 1 and 2, then: $W_{AB} = 0$

IV.8 Concept of boundary layer

The boundary layer is the thin region of fluid near a solid surface where the effects of viscosity are significant. Within this layer, the fluid velocity changes from zero at the surface (due to the no-slip condition) to the free-stream velocity away from the surface. It plays a key role in friction, drag, and heat transfer in fluid flow.
When a fluid moves over a flat surface, the layer of fluid particles next to the surface has zero velocity. As we move further away from the surface, each layer moves faster, eventually reaching the free-stream velocity U. This phenomenon is due to the shear stress between fluid layers. In the case of a Newtonian fluid, this shear stress is directly proportional to the velocity gradient, described by $\tau = \mu\left(\frac{du}{dy}\right)$.
The velocity gradient and corresponding shear stress are greatest at the surface itself and decrease gradually with distance from the surface, eventually becoming negligible far away. This means that, away from the surface, the flow becomes uniform, with little or no shear between fluid layers and almost no sliding. In 1904, Ludwig Prandtl identified this distinct behavior and named the variable velocity region near the surface the boundary layer.

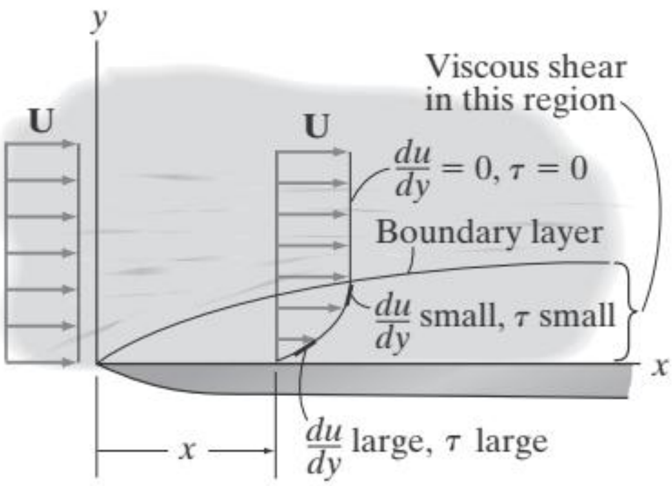


Fig. IV.12: Shear is proportional to the velocity gradient

References

[1] Hibbeler, R. C. (2017). *Fluid mechanics in SI units*. Pearson Education India.

[2] ZEYTOUNIAN R. K. (1991), « Mécanique des fluides fondamentale », Springer, Verlag Berlin Heidelberg.

[3] Potter, M. C., & Wiggert, D. C. (2008). *Shaums outline of fluid mechanics*. McGraw-HilI.

[4] RANALD V. GILES (1986), « Mécanique des fluides et hydraulique: cours et problèmes », Série Schaum, Paris.

[5] JOHN K. VENNARD (1962), « Elementary fluid mechanics », Library of congress catalog, United States of America.

[6] Munson, B. R., Young, D. F., & Okiishi, T. H. (1995). Fundamentals of fluid mechanics. *Oceanographic Literature Review*, *10*(42), 831.

[7] Janna, W. S. (2009). *Introduction to fluid mechanics*. CRC press.

[8] Potter, M. C., Wiggert, D. C., & Ramadan, B. H. (2016). *Mechanics of fluids*. Cengage Learning.

[9] CHASSAING P. (2010), « Mécanique des fluides », Cépaduèces-éditions, Toulouse.

[10] Pritchard, P. J., & Mitchell, J. W. (2016). *Fox and McDonald's introduction to fluid mechanics*. John Wiley & Sons.

[11] Rajput, R. K. (2004). *A textbook of fluid mechanics and hydraulic machines*. S. Chand Publishing.

[12] LUMBROSO H. (1994), « Problèmes résolus de mécanique des fluides », Dunod, Paris.

[13] DARROZES J. S., FRANCOIS C. (1982), « Mécanique des fluides incompressibles », Springer, Verlag Berlin Heidelberg.

[14] Amiroudine, S., & Battaglia, J. L. (2011). *Mécanique des fluides*. Dunod.

[15] Morrison, F. A. (2013). *An introduction to fluid mechanics*. Cambridge University Press.