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Abstract

On Fractional Differential Equations and Inclusions

This thesis investigates the global existence of solutions to fractional diffusion equations involving Caputo fractional derivatives, incorporating nonlinear memory effects and small initial data. We analyze the influence of both the nonlinearity parameter and the fractional derivative order on the admissible range of exponents and the corresponding solution estimates. Our analytical approach combines the Banach fixed-point theorem with established results for the associated linear fractional differential equations.

Key words: Global in time existence. Small data, Fractional deffusion equation. Critical exponents. Fractional differential equation.

ملخص

المعادلات والتضمينات التفاضلية الكسرية

تبحث هذه الأطروحة في الوجود العام لحلول معادلات الانتشار الكسورية ذات المشتقات الكسورية من نوع كابوتو، مع تضمين تأثيرات الذاكرة غير الخطية والبيانات الأولية الصغيرة. نقوم بتحليل لكيفية تأثير معامل اللاخطية وترتيب المشتق الكسوري بشكل مشترك على نطاق الأس المسموح به وتقديرات الحلول. يعتمد نهجنا التحليلي على الجمع بين نظرية نقطة التثبيت لباناخ والنتائج المعتمدة لمعادلات التفاضل الكسورية الخطية المقابلة.

الكلمات المفتاحية: الوجود العام. معادلة الانتشار الكسري. المعادلات التفاضلية الكسرية. الأسس الحرجة. بيانات أولية صغيرة.

Contents

Notation

Symbol	Description
$\mathbf{R}$	Set of real numbers.
$\mathbf{R}^+$	Set of positive real numbers.
$\mathbf{N}$	Set of natural numbers $\{1, 2, 3, \dots\}$.
$\mathbf{N}_0$	Set of non-negative integers $\{0, 1, 2, 3, \dots\}$.
$\lvert x \rvert$	Absolute value of x.
$\lVert x \rVert$	Norm of x.
$\lceil \beta \rceil$	Ceiling function (smallest integer $\geq \beta$).
$\lesssim$	Inequality/approximation: $a \lesssim b$ means $a \leq C \cdot b$ for some constant $C > 0$.
$\mathfrak{Re}(z)$	Real part of complex number z.
$\mathfrak{Im}(z)$	Imaginary part of complex number z.
$\Gamma(\cdot)$	Euler gamma function.
$\mathcal{E}_\beta(w)$	Standard Mittag-Leffler function.
$\mathcal{E}_{\beta,\gamma}(w)$	Mittag-Leffler function in two arguments α and β.
∂_x	Partial derivative.
$\Delta\phi$	Laplacian operator of ϕ.
$\mathfrak{L}\{\phi\}$	Laplace transform of ϕ.
$\mathfrak{F}\{\phi\}$	Fourier transform of ϕ.
$\mathfrak{T}_{b+}^{\beta}$	Riemann-Liouville fractional integral of order β.
$\mathfrak{D}_{b+}^{\beta}$	Riemann-Liouville fractional derivative of order β.
${}^{\mathcal{C}}\mathfrak{D}_{b+}^{\beta}$	Caputo fractional derivative of order β.

Introduction

In this thesis, we focus on studying Fractional Diffusion Equation with nonlinear memory. The model under consideration is as follows

$$\begin{cases} {}^{C}\mathfrak{D}_{\tau}^{\beta+1}\varphi - \Delta\varphi = \int_0^{\tau}(\tau-\theta)^{-\sigma}\left|\varphi(\theta,.)\right|^{\mu}\mathrm{d}\theta, \\ \varphi(0,\zeta) = \varphi_0(\zeta), \\ \varphi_{\tau}(0,\zeta) = \varphi_1(\zeta), \end{cases} \tag{1}$$

where $0 < \sigma, \beta < 1$, $\tau \in [0,\infty), \zeta \in \mathbf{R}^{\mathrm{m}}$, $\mu > 1$ and ${}^{C}\mathfrak{D}_{\tau}^{\beta+1}\varphi$ denotes the Caputo fractional derivative of order $\beta+1$ with respect to time τ. We prove the global existence of solutions to the Cauchy problem (1) for small initial data. Additionally, we illustrate the influence of the nonlinearity parameter σ together with the fractional derivative order β on the range of the exponent μ and the estimation of the solutions. This is achieved by applying the fixed point theorem.

What does means diffusion?

Diffusion is a physical phenomenon that describes the movement of particles (such as atoms, molecules, or ions) from a region of higher concentration to a region of lower concentration.This process is driven by the random movement of particles and gradually distributes them evenly throughout a medium over time.

Diffusion is described using mathematical equations called partial differential equations (PDEs). In these equations, $\varphi(\tau,\zeta)$ represents the concentration of the substance at a certain location ζ and time τ. The variables that change are time τ and one or more space coordinates ζ.

The behavior of diffusion is strongly influenced by factors such as the system, initial conditions (the concentration distribution at $\tau = 0$).boundary conditions (constraints on the concentration at the edges of the domain), the geometry of the domain, and any external influences or disturbances.

Diffusion occurs in a wide range of scientific and engineering disciplines, including chemistry (e.g., diffusion of gases or solutes in liquids), biology (e.g., nutrient transport in cells), materials science (e.g., diffusion of atoms in solids), environmental science (e.g., dispersion of pollutants), and heat transfer (e.g., thermal diffusion).

The diffusion process is typically described by solutions to either linear or nonlinear PDEs, such as Fick's second law of diffusion, also known as classical diffusion equation

$$\frac{\partial \varphi(\tau,\zeta)}{\partial \tau} = \mathfrak{K}\frac{\partial^2 \varphi(\tau,\zeta)}{\partial \zeta^2},$$

where $\mathfrak{K}$ is the diffusion coefficient, which describes how fast particles spread.

In higher dimensions, the equation generalizes to

$$\frac{\partial \varphi(\tau,\zeta)}{\partial \tau} = \mathfrak{K}\Delta \varphi(\tau,\zeta).$$

While the specific methods for solving diffusion equations vary depending on the application, the focus here is not on the solution techniques but rather on understanding the fundamental nature of diffusion.

Fractional diffusion equations (FDEs) are generalized forms of classical diffusion equations. These equations are abstract partial differential equations that involve fractional derivatives in time and, in some cases, in space. They are particularly useful for modeling anomalous diffusion, where the spread of particles deviates from the standard Fickian behavior described by classical diffusion. In FDEs, the standard time derivative is replaced with a fractional derivative. We refer to [18, 21, 2, 17, 12] to illustrate various applications of fractional derivative theory in modeling complex systems, including anomalous diffusion, viscoelastic materials, and memory-dependent processes.

Let us present some research relevant to our framework. In [6], D'Abbicco et al. study a semilinear fractional diffusive equation of the type

$$\begin{cases} {}^{\mathcal{C}}\mathfrak{D}_{\tau}^{\beta+1}\varphi - \Delta\varphi = |\varphi|^{\mu}, \\ \varphi(0,\zeta) = \varphi_0(\zeta), \\ \varphi_\tau(0,\zeta) = \varphi_1(\zeta), \end{cases} \tag{2}$$

where τ, ζ, β, μ and ${}^{C}\mathfrak{D}^{\beta+1}_{\tau}\varphi$ are as in problem (1).

The authors show that there exist two critical exponents for the global solutions of the Cauchy problem (2). The first critical exponent defined as

$$\bar{\mu} := 1 + \frac{1+\beta}{\frac{\mathfrak{m}}{2}(1+\beta)-1}.$$

They proved that global solutions to (2) exist if $\mu > \bar{\mu}$ and φ_0, φ_1 are sufficiently small in $\mathcal{L}^1 \cap \mathcal{L}^{\mu}$. The second critical exponent defined as

$$\tilde{\mu} := 1 + \frac{2\beta+2}{(\mathfrak{m}-2)(1+\beta)+2}.$$

This exponent appears when φ_1 is assumed to be zero. In this cas, it is shown that global solutions to (2) exist if $\mu > \tilde{\mu}$ and φ_0 is sufficiently small in $\mathcal{L}^1 \cap \mathcal{L}^{\mu}$.

The Cauchy problem described in (2), acts as an interpolation between two basic equations of mathematical physics, the semilinear wave equation which is obtained when $\beta = 1$ and the semilinear heat equation, which is obtained when $\beta = 0$.

In the case when $\beta = 1$, the corresponding semilinear wave equation

$$\begin{cases} \partial_{\tau\tau}\varphi - \Delta\varphi = |\varphi|^{\mu}, \\ \varphi(0,\zeta) = \varphi_0(\zeta), \\ \varphi_{\tau}(0,\zeta) = \varphi_1(\zeta), \end{cases} \tag{3}$$

has been extensively studied by numerous researchers. See the first work by Kato in [14], which demonstrated that for $1 < \mu < 1 + \frac{2}{\mathfrak{m}-1}$ (and for $\mu > 1$ if $\mathfrak{m} = 1$), there are no global generalized solutions to the Cauchy problem (3). It was conjectured in [24] and [25] that the critical exponent for (3), denoted as $\mu_{\mathfrak{c}}$, is given by the positive root of the quadratic equation

$$(\mathfrak{m}-1)\mu^2 - (\mathfrak{m}+1)\mu - 2 = 0. \tag{4}$$

Subsequently, John [13] proved this result for the case when $\mathfrak{m} = 3$.

Additionally, many authors in [9, 16, 26, 27] derived the global existence of solutions to (3) with small data in different spatial dimensions when $\mu > \mu_{\mathfrak{c}}$. On the other hand, for an appropriate choice of initial data, the author in [23], established that the solution to (3) blow-up in finite time for $1 < \mu < \mu_{\mathfrak{c}}$.

When $\beta = 1$, the semilinear fractional diffusive equation (2) reduces to the semilinear heat equation

$$\begin{cases} \partial_\tau \varphi - \Delta\varphi = |\varphi|^\mu, \\ \varphi(0,\zeta) = \varphi_0(\zeta), \end{cases} \tag{5}$$

In [8], Fujita constructed the critical exponent for the Cauchy problem (5) as $\mu_0 = 1 + \frac{2}{\mathfrak{m}}$. He proved the global existence of solutions to (5) for small initial data when $\mu > \mu_0$ and also, derived finite time blow-up of solutions for the case when $1 < \mu < \mu_0$.

In the case when nonlinear memory terms are present, meaning that the right-hand side of equation (5) is replaced by $\int_0^\tau (\tau-\theta)^{-\sigma} |\varphi(\theta,.)|^\mu \, d\theta$, it was shown in [4] that the critical exponent for (5) becomes

$$\tilde{\mu}_0 = max\left\{\bar{\mu}(\mathfrak{m},\sigma), \frac{1}{\sigma}\right\}, \tag{6}$$

where

$$\bar{\mu}(\mathfrak{m},\sigma) = 1 + \frac{4-2\sigma}{\mathfrak{m}-2+2\sigma}.$$

Additionally, it was established that the global solutions with small initial data exist if $\mu > \tilde{\mu}_0$, while the blow-up behavior of solutions in finite time occurs for $1 < \mu < \tilde{\mu}_0$

Numerous researchers have studied the influence of the nonlinear memory term on the global existence of solutions with small initial data for problems involving wave equations, damped wave equations, and fractional damped equations, see [11, 7] and [20].

In 2013, M. D'Abbicco [5] considered the following Cauchy problem

$$\varphi_{\tau\tau} - \Delta\varphi + \varphi_\tau = \int_0^\tau (\tau-\theta)^{-\sigma} |\varphi(\theta,.)|^\mu \, \mathrm{d}\theta, \tag{7}$$

$$\varphi(0,\zeta) = \varphi_0(\zeta), \;\; \varphi_\tau(0,\zeta) = \varphi_1(\zeta), \tag{8}$$

where, $1 \leq \mathfrak{m} \leq 5$, $\sigma \in (0,1)$ and $\mu > 1$.

The author established that The critical exponent for (7)-(8) is given by $\tilde{\mu}_0$, as defined in (6) for the heat equation with a nonlinear memory. Additionally, the author derived significant results regarding the global existence of solutions for $\mu > \tilde{\mu}_0$ and the no global existence of weak solutions if $1 < \mu < \tilde{\mu}_0$, in space dimension $1 \leq \mathfrak{m} \leq 5$.

The present thesis is structured into four chapters. Chapter 1 introduces essential notations and fundamental concepts related to various topics in analysis, including functional spaces, special functions (such as the gamma function and Mittag-Leffler function), and key results on the properties of Riemann-Liouville and Caputo fractional derivatives, as well as Laplace and Fourier transforms. The chapter concludes with a presentation of the Banach fixed-point theorem.

Chapter 2 focuses on fractional differential equations involving the Caputo fractional derivative. In Section 1, we prove the existence and uniqueness of solutions for the following nonlinear fractional differential equation of the form:

$$\left({}^{C}\mathfrak{D}_{0+}^{\beta}\varphi\right)(\tau) = \Phi[\tau, \varphi(\tau)], \tag{9}$$

under the condition

$$\varphi^{(\mathfrak{j})}(0) = \mathfrak{c}_{\mathfrak{j}}. \tag{10}$$

where $0 \leq \tau \leq T, \beta > 0$, and $\mathfrak{c}_{\mathfrak{j}} \in \mathbf{R}$ for $\mathfrak{j} = 0, 1, \cdots, \mathfrak{m} - 1$ with $\mathfrak{m} = \lceil \beta \rceil$. In Section 2, we derive the explicit solution to the following linear fractional differential equation

$$\left({}^{C}\mathfrak{D}_{0+}^{\beta}\varphi\right)(\tau) - \delta\varphi(\tau) = \Phi(\tau),$$

$$\varphi^{(\mathfrak{j})}(0) = \mathfrak{c}_{\mathfrak{j}},$$

by reducing the problem to an equivalent Volterra integral equation. In Section 3, an alternative method is presented to obtain the explicit solution, utilizing the Laplace transform method.

In Chapter 3, we study the Cauchy problem associated with the following fractional diffusion equation:

$$\begin{cases} {}^{C}\mathfrak{D}_{\tau}^{\beta+1}\varphi - \Delta\varphi = \Phi(\tau, \zeta), \\ \varphi(0, \zeta) = \varphi_0(\zeta), \\ \varphi_\tau(0, \zeta) = \varphi_1(\zeta), \end{cases} \tag{11}$$

First, we derive the explicit solution to the problem (11), utilizing the results established in Chapter 2. Subsequently, we establish $\mathcal{L}^{\upsilon} - \mathcal{L}^{\rho}$ estimates for the solution, which play a fundamental role in our main result.

In Chapter 4, we present our global existence results for the Cauchy problem (11) with

$$\Phi(\tau, \cdot) = \int_0^\tau (\tau - \theta)^{-\sigma} |\varphi(\theta, \cdot)|^\mu \, \mathrm{d}\theta,$$

and small initial data. Our approach relies on the Banach fixed-point theorem applied in a suitable subspace $\mathscr{C}\left([0,\infty), \mathcal{L}^1 \cap \mathcal{L}^\mu\right)$, endowed with a carefully chosen norm that enables us to derive the desired estimates for the solution.

Chapter 1

PRELIMINARIES

In this chapter, we present some useful notations and fundamental concepts related to various topics in analysis, such as functional spaces, special functions, fractional calculus, and the Laplace and Fourier transforms, which will be useful throughout our work (see [15, 22, 16]).

1.1 Notations and defnitions

Let $[b, d]$ be an interval of $\mathbf{R}$ and $1 \leq \vartheta \leq \infty$. We denote $\mathcal{L}^{\vartheta}([b, d], \mathbf{R})$ as the Banach space of all measurable functions from $[b, d]$ to $\mathbf{R}$, equipped with the norm:

$$\|\Phi\|_{\mathcal{L}^{\vartheta}} = \left(\int_b^d |\Phi(\theta)|^{\vartheta} \mathrm{d}\theta \right)^{1/\vartheta} \qquad \text{for } 1 \leq \vartheta < \infty,$$

and

$$\|\Phi\|_{\mathcal{L}^{\infty}} = \mathfrak{inf}\{\Lambda \geq 0 : |\Phi(\theta)| \leq \Lambda \text{ a.e. on } [b, d]\}.$$

We denote $\mathscr{C}([b, d], \mathbf{R})$ as the Banach space of all continuous functions from $[b, d]$ to $\mathbf{R}$, endowed with the norm:

$$\|\Phi\|_{\mathscr{C}} = \max_{b \leq \theta \leq d} |\Phi(\theta)|.$$

While $\mathscr{C}^{\mathfrak{m}}([b, d], \mathbf{R})$ with $\mathfrak{m} \in \mathbf{N}_0$, denotes the space of functions $\Phi : [b, d] \to \mathbf{R}$ that are $\mathfrak{m}$ times continuously differentiable on $[b, d]$ endowed with the norm

$$\|\Phi\|_{\mathscr{C}^{\mathfrak{m}}} = \sum_{\mathfrak{i}=0}^{\mathfrak{m}} \max_{b\le\theta\le d} \left|\Phi^{(\mathfrak{i})}(\theta)\right|, \quad \mathfrak{m}\in\mathbf{N}_0.$$

Let $[b,d]$ be a finite interval of $\mathbf{R}$ and $\delta\in\mathbf{R}$ with $0\le\delta<1$. We define the weighted space $\mathscr{C}_\delta([b,d],\mathbf{R})$ as the set of functions defined on $(b,d]$ such that $(\theta-b)^\delta\Phi(\theta)\in\mathscr{C}([b,d],\mathbf{R})$, equipped with the norm:

$$\|\Phi\|_{\mathscr{C}_\delta} = \left\|(\theta-b)^\delta\Phi(\theta)\right\|_{\mathscr{C}}, \quad \mathscr{C}_0[b,d]=\mathscr{C}[b,d].$$

For $\mathfrak{m}\in\mathbf{N}$, we introduce $\mathscr{C}_\delta^{\mathfrak{m}}([b,d],\mathbf{R})$ as the Banach space of functions Φ, such that $\Phi\in\mathscr{C}^{\mathfrak{m}-1}([b,d],\mathbf{R})$ and $\Phi^{(\mathfrak{m})}\in\mathscr{C}_\delta([b,d],\mathbf{R})$, equipped with the norm

$$\|\Phi\|_{\mathscr{C}_\delta^{\mathfrak{m}}} = \sum_{\mathfrak{i}=0}^{\mathfrak{m}-1} \left\|\Phi^{(\mathfrak{i})}\right\|_{\mathscr{C}} + \left\|\Phi^{(\mathfrak{m})}\right\|_{\mathscr{C}_\delta}, \quad \mathscr{C}_\delta^0[b,d]=\mathscr{C}_\delta[b,d].$$

Let $[b,d]$ be a finite interval of $\mathbf{R}$. We denote $\mathscr{AC}([b,d],\mathbf{R})$ as the space of absolutely continuous functions on the interval $[b,d]$, which coincides with the space of primitives of Lebesgue summable functions:

$$\Phi(\theta)\in\mathscr{AC}([b,d],\mathbf{R}) \Leftrightarrow \exists\Psi(\sigma)\in\mathcal{L}^1([b,d],\mathbf{R}) \text{ such that : } \Phi(\theta)=k+\int_b^\theta \Psi(\sigma)\mathrm{d}\sigma. \tag{1.1}$$

For $\mathfrak{m}\in\mathbf{N}$. We define the space of functions $\mathscr{AC}^{\mathfrak{m}}([b,d],\mathbf{R})$ as follows:
$\mathscr{AC}^{\mathfrak{m}}([b,d],\mathbf{R}) = \left\{\Phi:[b,d]\to\mathbf{R} \mid \Phi\in\mathscr{C}^{\mathfrak{m}-1}([b,d],\mathbf{R}) \text{ and } \Phi^{(\mathfrak{m}-1)}\in\mathscr{AC}([b,d],\mathbf{R})\right\}.$
In particular, $\mathscr{AC}^1([b,d],\mathbf{R})=\mathscr{AC}([b,d],\mathbf{R})$.

Lemma 1.1.1. *[22] Let $\Phi:[b,d]\to\mathbf{R}$. Then $\Phi\in\mathscr{AC}^{\mathfrak{m}}([b,d],\mathbf{R})$ if and only if*

$$\Phi(\theta)=\sum_{\mathfrak{i}=0}^{\mathfrak{m}-1}\kappa_{\mathfrak{i}}(\theta-b)^{\mathfrak{i}}+\left(\mathfrak{T}_{b+}^{\mathfrak{m}}\Psi\right)(\theta), \tag{1.2}$$

where $\kappa_{\mathfrak{i}}$ (for $\mathfrak{i}=0,1,\cdots,\mathfrak{m}-1$) are arbitrary constants, $\Psi(\sigma)\in\mathcal{L}^1([b,d],\mathbf{R})$ and

$$\left(\mathfrak{T}_{b+}^{\mathfrak{m}}\Psi\right)(\theta)=\frac{1}{(\mathfrak{m}-1)!}\int_b^\theta(\theta-\sigma)^{\mathfrak{m}-1}\Psi(\sigma)\mathrm{d}\sigma.$$

From (1.2), we have

$$\kappa_{\mathfrak{i}} = \frac{\Phi^{(\mathfrak{i})}(b)}{\mathfrak{i}!} \qquad \textit{and } \Psi(\sigma) = \Phi^{(\mathfrak{m})}(\sigma).$$

Lemma 1.1.2. *[15] Let $\mathfrak{m} \in \mathbf{N}_0$, $\delta \in \mathbf{R}$ such that $0 \leq \delta < 1$ and $\Phi : [b, d] \to \mathbf{R}$. Then $\Phi \in \mathscr{C}_\delta^{\mathfrak{m}}([b, d], \mathbf{R})$ if and only if*

$$\Phi(\theta) = \sum_{\mathfrak{i}=0}^{\mathfrak{m}-1} \kappa_{\mathfrak{i}} (\theta - b)^{\mathfrak{i}} + \frac{1}{(\mathfrak{m} - 1)!} \int_b^\theta (\theta - \sigma)^{\mathfrak{m}-1} \Psi(\sigma) \mathrm{d}\sigma, \tag{1.3}$$

where $\kappa_{\mathfrak{i}}$ (for $\mathfrak{i} = 0, 1, \cdots, \mathfrak{m}-1$) are arbitrary constants, $\Psi(\sigma) \in \mathscr{C}_\delta([b, d], \mathbf{R})$. In addition,

$$k_{\mathfrak{i}} = \frac{\Phi^{(\mathfrak{i})}(b)}{\mathfrak{i}!} \qquad \textit{and } \Psi(\sigma) = \Phi^{(\mathfrak{m})}(\sigma). \tag{1.4}$$

We conclude this section by presenting an essential result in functional analysis: Young's theorem. To begin, we first define the convolution, which is a fundamental operation in mathematics, particularly in Fourier analysis and functional analysis.
For two functions Φ and Ψ the convolution operator of Φ and Ψ, denoted by $\Phi * \Psi$, is defined as:

$$\Phi * \Psi := (\Phi * \Psi)(\xi) = \int_{\mathbf{R}^n} \Phi(\xi - \eta) \Psi(\eta) \mathrm{d}\eta. \tag{1.5}$$

Theorem 1.1.3. *Let $\Phi \in \mathcal{L}^\mu(\mathbf{R}^{\mathfrak{m}})$ and $\Psi \in \mathcal{L}^\rho(\mathbf{R}^{\mathfrak{m}})$. If $\mu, \rho, \vartheta \geq 1$ such that*

$$1 + \frac{1}{\rho} = \frac{1}{\mu} + \frac{1}{\vartheta},$$

then

$$\|\Phi * \Psi\|_{\mathcal{L}^\rho} \leq \|\Psi\|_{\mathcal{L}^\vartheta} \|\Phi\|_{\mathcal{L}^\mu}.$$

1.2 Special Functions

In this section, we outline the definitions and properties of the Euler gamma and Mittag-Leffler functions, which are crucial to the study of fractional calculus.

1.2.1 Gamma Function

The Gamma function, denoted by $\Gamma(w)$ is a crucial mathematical function that generalizes the factorial concept to complex and real number arguments.

For $w \in \mathbf{C}$ with $\mathfrak{Re}(w) > 0$, the Gamma function is defined by the integral:

$$\Gamma(w) = \int_0^\infty \mathfrak{e}^{-\theta}\theta^{w-1}\mathrm{d}\theta. \tag{1.6}$$

This integral is convergent for all complex $w \in \mathbf{C}$ with $\mathfrak{Re}(w) > 0$.

The Gamma function satisfies the property:

$$\Gamma(w+1) = w\Gamma(w) \quad \text{with } \mathfrak{Re}(w) > 0. \tag{1.7}$$

In particular

$$\Gamma(\mathfrak{m}+1) = \mathfrak{m}! \quad \text{where } \mathfrak{m} \in \mathbf{N}_0,$$

with (as usual) $0! = 1$.

1.2.2 Mittag-Leffler Function

The one-parameter Mittag-Leffler function, denoted as $\mathcal{E}_\beta(.)$ for a parameter β, is a complex function that generalizes the exponential function and plays a crucial role in fractional calculus and related areas, It is defined by:

$$\mathcal{E}_\beta(w) = \sum_{\mathrm{j}=0}^{\infty} \frac{w^{\mathfrak{j}}}{\Gamma(\beta\mathfrak{j}+1)},$$

where $w \in \mathbf{C}$ and $\mathfrak{Re}(\beta) > 0$.

The two-parameter Mittag-Leffler function $\mathcal{E}_{\beta,\gamma}(\cdot)$ is an extension of the one-parameter Mittag-Leffler function and is defined by

$$\mathcal{E}_{\beta,\gamma}(w) = \sum_{\mathrm{j}=0}^{\infty} \frac{w^{\mathfrak{j}}}{\Gamma(\beta\mathfrak{j}+\gamma)}, \tag{1.8}$$

where w, $\gamma \in \mathbf{C}$ and $\mathfrak{Re}(\beta) > 0$.

When $\gamma = 1$, the function reduces to the one-parameter Mittag-Leffler function: $\mathcal{E}_{\beta,1}(w) = \mathcal{E}_\beta(w)$.

For specific values of $\mathcal{E}_{\beta,\gamma}(w)$, we provide the following table:

β	γ	$\mathcal{E}_{\beta,\gamma}(w)$
0	1	$\sum_{\mathrm{j}=0}^{\infty} \frac{w^{\mathrm{j}}}{\Gamma(1)} = \sum_{\mathrm{j}=0}^{\infty} w^{\mathrm{j}}$
1	1	$\sum_{\mathrm{j}=0}^{\infty} \frac{w^{\mathrm{j}}}{\Gamma(\mathrm{j}+1)} = \sum_{\mathrm{j}=0}^{\infty} \frac{w^{\mathrm{j}}}{\mathrm{j}!} = e^{w}$
2	1	$cosh(\sqrt{w})$
2	2	$\frac{sinh(\sqrt{w})}{\sqrt{w}}$

Table 1.1

1.3 Fractional Integral and Derivative

In this section, we will introduce some properties and definitions of the most common fractional integrals and fractional derivatives that will be used in this work.

1.3.1 Riemann-Liouville Fractional Integrals and Fractional Derivatives

Let $[b, d]$ be a finite interval on $\mathbf{R}$.

Definition 1.3.1. *[16] Let $\beta \in \mathbf{R}_{+}^{*}$, The left-sided fractional integral of Riemann-Liouville, denoted as $\mathfrak{T}_{b+}^{\beta}\varphi$ for a function $\varphi(\theta)$, is defined by the expression:*

$$\left(\mathfrak{T}_{b+}^{\beta}\varphi\right)(\tau) := \frac{1}{\Gamma(\beta)} \int_{b}^{\tau} \frac{\varphi(\theta)\mathrm{d}\theta}{(\tau-\theta)^{1-\beta}} \quad \text{with } \tau > b. \tag{1.9}$$

If $\beta = \mathfrak{m} \in \mathbf{N}$, the left-sided fractional integral of Riemann-Liouville coincides with the standard integral of order $\mathfrak{m}$, given by

$$\left(\mathfrak{T}_{b+}^{\mathfrak{m}}\varphi\right)(\tau) = \int_b^\tau \mathrm{d}\theta_1 \int_b^{\theta_1} \mathrm{d}\theta_2 \cdots \int_b^{\theta_{\mathfrak{m}-1}} \varphi(\theta_n)\,\mathrm{d}\theta_{\mathfrak{m}}$$
$$= \frac{1}{(\mathfrak{m}-1)!}\int_b^\tau (\tau-\theta)^{\mathfrak{m}-1}\varphi(\theta)\mathrm{d}\theta. \quad (1.10)$$

Example 1.3.1. *[16] Let $\beta \in \mathbf{R}_+^*$, $\gamma \in \mathbf{R}$ with $\gamma > -1$. Consider the function $\varphi(\theta) = (\theta - b)^\gamma$. The Riemann-Liouville fractional integral of $\varphi(\theta)$ is given by:*

$$\left(\mathfrak{T}_{b+}^{\beta}\varphi\right)(\tau) = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\beta)}(\tau-b)^{\beta+\gamma}. \quad (1.11)$$

Definition 1.3.2. *[16] Let $\beta \in \mathbf{R}_+$, the Riemann-Liouville fractional derivative, denoted as $\mathfrak{O}_{b+}^{\beta}\varphi$ for a function $\varphi(\theta)$, is defined by*

$$\left(\mathfrak{O}_{b+}^{\beta}\varphi\right)(\tau) = \frac{1}{\Gamma(\mathfrak{m}-\beta)}\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^{\mathfrak{m}}\int_b^\tau \frac{\varphi(\theta)\mathrm{d}\theta}{(\tau-\theta)^{\beta-\mathfrak{m}+1}} \qquad \text{with } \tau > b, \quad (1.12)$$

where $\mathfrak{m} = \lceil \beta \rceil$.

when $\beta = \mathfrak{m}$ with $\mathfrak{m} \in \mathbf{N}_0$, the following holds:

$$\left(\mathfrak{O}_{b+}^{\mathfrak{m}}\varphi\right)(\tau) = \varphi^{(\mathfrak{m})}(\tau),$$

and

$$\left(\mathfrak{O}_{b+}^{0}\varphi\right)(\tau) = \varphi(\tau),$$

where $\varphi^{(\mathfrak{m})}(\tau)$ represents the $\mathfrak{m}$-th order derivative of $\varphi(\tau)$.

Example 1.3.2. *[16] Let $\beta \in \mathbf{R}_+$, $\gamma \in \mathbf{R}$ with $\gamma > -1$. Consider the function $\varphi(\theta) = (\theta - b)^\gamma$.Then, the Riemann-Liouville fractional derivative of $\varphi(\theta)$ is given by:*

$$\left(\mathfrak{O}_{b+}^{\beta}\varphi\right)(\tau) = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\beta)}(\tau-b)^{\gamma-\beta}. \quad (1.13)$$

In the special case where $\gamma = 0$, this simplifies to:

$$\left(\mathfrak{O}_{b+}^{\beta}1\right)(\tau) = \frac{(\tau-b)^{-\beta}}{\Gamma(1-\beta)}. \quad (1.14)$$

It is important to note that, in general, the Riemann-Liouville fractional derivative of a constant is not zero.

Corollary 1.3.1. *[16] Let $\beta \in \mathbf{R}_+^*$ and $\mathfrak{m} = \lceil \beta \rceil$. Then $\left(\mathfrak{D}_{b+}^{\beta}\varphi\right)(\tau) = 0$ if and only if*

$$\varphi(\tau) = \sum_{\mathfrak{i}=1}^{\mathfrak{m}} k_{\mathfrak{i}}(\tau - b)^{\beta-\mathfrak{i}},$$

where $k_{\mathfrak{i}} \in \mathbf{R}$ for $\mathfrak{i} = 1, \cdots, \mathfrak{m}$ are arbitrary constants.

The following result delineates the necessary conditions for the existence of the fractional derivative $\mathfrak{D}_{b+}^{\beta}$ within the space $\mathscr{AC}^{\mathfrak{m}}([b,d],\mathbf{R})$.

Lemma 1.3.2. *[22] Let $\beta \in \mathbf{R}_+$ and $\mathfrak{m} = \lceil \beta \rceil$. If $\varphi(\tau) \in \mathscr{AC}^{\mathfrak{m}}([b,d],\mathbf{R})$, then the fractional derivative $\mathfrak{D}_{b+}^{\beta}\varphi$ exists almost everywhere on $[b,d]$ and it can be expressed in the following form:*

$$\left(\mathfrak{D}_{b+}^{\beta}\varphi\right)(\tau) = \sum_{\mathfrak{j}=0}^{\mathfrak{m}-1} \frac{\varphi^{(\mathfrak{j})}(b)}{\Gamma(1+\mathfrak{j}-\beta)}(\tau-b)^{\mathfrak{j}-\beta} + \frac{1}{\Gamma(\mathfrak{m}-\beta)} \int_b^{\tau} \frac{\varphi^{(\mathfrak{m})}(\theta)}{(\tau-\theta)^{\beta-\mathfrak{m}+1}} \mathrm{d}\theta. \tag{1.15}$$

Lemma 1.3.3. *[22] Let $\varphi(\tau) \in \mathcal{L}^{\upsilon}([b,d],\mathbf{R})$ such that $1 \leq \upsilon \leq \infty$. If $\beta, \gamma \in \mathbf{R}_+^*$, then the following identity holds almost everywhere on $[b,d]$:*

$$\left(\mathfrak{T}_{b+}^{\beta}\mathfrak{T}_{b+}^{\gamma}\varphi\right)(\tau) = \left(\mathfrak{T}_{b+}^{\beta+\gamma}\varphi\right)(\tau). \tag{1.16}$$

Moreover, if $\beta + \gamma > 1$, equation (1.16) is valid at every point in $[b,d]$.

The following statement demonstrates that the Riemann-Liouville fractional differentiation operator acts as the left inverse of the Riemann-Liouville fractional integration operator.

Lemma 1.3.4. *[22] Let $\varphi(\tau) \in \mathcal{L}^{\upsilon}([b,d],\mathbf{R})$ such that $1 \leq \upsilon \leq \infty$. If $\beta \in \mathbf{R}_+^*$, then the following identity holds almost everywhere on $[b,d]$:*

$$\left(\mathfrak{D}_{b+}^{\beta}\mathfrak{T}_{b+}^{\beta}\varphi\right)(\tau) = \varphi(\tau). \tag{1.17}$$

The property below establishes the relationship between the Riemann-Liouville fractional integration operator and the Riemann-Liouville fractional differentiation operator.

Property 1.3.5. *[16] Let $\varphi(\tau) \in \mathcal{L}^{\upsilon}([b,d],\mathbf{R})$ such that $1 \leq \upsilon \leq \infty$. If $\beta, \gamma \in \mathbf{R}_{+}^{*}$ with $\gamma < \beta$, then the following equality holds almost everywhere on $[b,d]$:*

$$\left(\mathfrak{D}_{b+}^{\gamma}\mathfrak{T}_{b+}^{\beta}\varphi\right)(\tau) = \mathfrak{T}_{b+}^{\beta-\gamma}\varphi(\tau). \tag{1.18}$$

If $\gamma = \mathfrak{j} \in \mathbf{N}$ and $\beta > \mathfrak{j}$, then

$$\left(\mathfrak{D}^{\mathfrak{j}}\mathfrak{T}_{b+}^{\beta}\varphi\right)(\tau) = \mathfrak{T}_{b+}^{\beta-\mathfrak{j}}\varphi(\tau). \tag{1.19}$$

In what follows, we will use the notation $\varphi_{\mathfrak{m}-\beta}(\tau) = \left(\mathfrak{T}_{b+}^{\mathfrak{m}-\beta}\varphi\right)(\tau)$ to denote the fractional integral of order $\mathfrak{m}-\beta$.

We now define the space of functions $\mathfrak{T}_{b+}^{\beta}(\mathcal{L}^{\upsilon})$, for $\beta > 0$ and $1 \leq \upsilon \leq \infty$, as follows

$$\mathfrak{T}_{b+}^{\beta}[\mathcal{L}^{\upsilon}] := \left\{\Phi : \Phi = \mathfrak{T}_{b+}^{\beta}\varphi, \quad \varphi \in \mathcal{L}^{\upsilon}([b,d],\mathbf{R})\right\}.$$

This space will be used in the following result, which establishes the composition of the operator $\mathfrak{T}_{b+}^{\beta}$ and the operator $\mathfrak{D}_{b+}^{\beta}$.

Lemma 1.3.6. *[16] Let $\beta \in \mathbf{R}_{+}^{*}$ and $\mathfrak{m} = \lceil \beta \rceil$.*

(a) If $1 \leq \upsilon \leq \infty$ and $\varphi(\tau) \in \mathfrak{T}_{b+}^{\beta}[\mathcal{L}^{\upsilon}]$, then

$$\left(\mathfrak{T}_{b+}^{\beta}\mathfrak{D}_{b+}^{\beta}\varphi\right)(\tau) = \varphi(\tau). \tag{1.20}$$

(b) If $\varphi(\tau) \in \mathcal{L}^{1}([b,d],\mathbf{R})$ and $\varphi_{\mathfrak{m}-\beta}(\tau) \in \mathscr{AC}^{\mathfrak{m}}([b,d],\mathbf{R})$, then the following equality holds almost everywhere on $[b,d]$:

$$\left(\mathfrak{T}_{b+}^{\beta}\mathfrak{D}_{b+}^{\beta}\varphi\right)(\tau) = \varphi(\tau) - \sum_{j=1}^{\mathfrak{m}} \frac{\varphi_{\mathfrak{m}-\beta}^{(\mathfrak{m}-j)}(b)}{\Gamma(\beta-j+1)}(\tau-b)^{\beta-j}. \tag{1.21}$$

If $0 < \beta < 1$, then

$$\left(\mathfrak{T}_{b+}^{\beta}\mathfrak{D}_{b+}^{\beta}\varphi\right)(\tau) = \varphi(\tau) - \frac{\varphi_{1-\beta}(b)}{\Gamma(\beta)}(\tau-b)^{\beta-1}. \tag{1.22}$$

If $\beta = \mathfrak{m} \in \mathbf{N}$, then

$$\left(\mathfrak{T}_{b+}^{\mathfrak{m}}\mathfrak{D}_{b+}^{\mathfrak{m}}\varphi\right)(\tau) = \varphi(\tau) - \sum_{k=0}^{\mathfrak{m}-1} \frac{\varphi^{(k)}(b)}{k!}(\tau-b)^{k}. \tag{1.23}$$

The following lemma establishes the existence of the Riemann-Liouville fractional integral in the space of functions $\mathscr{C}_\delta[b,d]$ as well as the the existence of the Riemann-Liouville fractional derivatives in the space of functions $\mathscr{C}_\delta^{\mathfrak{m}}[b,d]$.

Lemma 1.3.7. *[16] Let $\beta \in \mathbf{R}_+^*$ and $\delta \in \mathbf{R}$.*

(a) If $\delta \leq \beta$ with $0 \leq \delta < 1$, then the operator $\mathfrak{T}_{b+}^{\beta}$ is bounded from $\mathscr{C}_\delta[b,d]$ into $\mathscr{C}[b,d]$:

$$\left\|\mathfrak{T}_{b+}^{\beta}\Phi\right\|_{\mathscr{C}} \leq \mathfrak{c}_2\|\Phi\|_{\mathscr{C}_\delta},$$

with $\mathfrak{c}_2 = (d-b)^{\beta-\gamma}\frac{\Gamma(\beta)\Gamma(1-\delta)}{\Gamma(\beta)\Gamma(1+\beta-\delta)}$.

(b) If $\beta \in \mathbf{R}_+$ and $\varphi(\tau) \in \mathscr{C}_\delta^{\mathfrak{m}}([b,d],\mathbf{R})$ where $\mathfrak{m} = \lceil\beta\rceil$, then $\mathfrak{D}_{b+}^{\beta}\varphi$ exist on $(b,d]$ and can be expressed in the form given by (1.15).

The following lemma presents some results from fractional calculus that will be needed later in this thesis.

Lemma 1.3.8. *[15] Let $\beta \in \mathbf{R}_+^*$, $\mathfrak{m} = \lceil\beta\rceil$ and let $\delta \in \mathbf{R}$ with $0 \leq \delta < 1$.*

(a) If $\Phi(\tau) \in \mathscr{C}_\delta([b,d],\mathbf{R})$ and $\Phi_{\mathfrak{m}-\beta}(\tau) \in \mathscr{C}_\delta^{\mathfrak{m}}([b,d],\mathbf{R})$, then the relation (1.21) is satisfied at every point $\tau \in (b,d]$. In particular, if $0 < \beta < 1$ and $\Phi_{1-\beta}(\tau) \in \mathscr{C}_\delta^{1}([b,d],\mathbf{R})$, then the relation (1.22) holds.

(b) If $\Phi(\tau) \in \mathscr{C}([b,d],\mathbf{R})$ and $\Phi_{\mathfrak{m}-\beta}(\tau) \in \mathscr{C}^{\mathfrak{m}}([b,d],\mathbf{R})$, then (1.21) is satisfied at every point $\tau \in [b,d]$. In particular, when $\Phi(\tau) \in \mathscr{C}^{\mathfrak{m}}([b,d],\mathbf{R})$, then the equality (1.23) holds at every point $\tau \in [b,d]$.

1.3.2 Caputo Fractional Derivative

Definition 1.3.3. *[16] Let $\beta \in \mathbf{R}_+$. The left-sided Caputo fractional derivative, denoted as ${}^{C}\mathfrak{D}_{b+}^{\beta}\varphi$, for a function $\varphi(\theta)$, is defined in terms of the Riemann-Liouville fractional derivative by the following expression:*

$$\left({}^{C}\mathfrak{D}_{b+}^{\beta}\varphi\right)(\tau) := \left(\mathfrak{D}_{b+}^{\beta}\left[\varphi(\theta) - \sum_{\mathrm{j}=0}^{\mathfrak{m}-1}\frac{\varphi^{(\mathrm{j})}(b)}{\mathrm{j}!}(\theta-b)^{\mathrm{j}}\right]\right)(\tau), \tag{1.24}$$

where

$$\mathfrak{m} = \lceil\beta\rceil \text{ if } \beta \notin \mathbf{N}_0, \qquad \text{and } \mathfrak{m} = \beta \text{ if } \beta \in \mathbf{N}_0. \tag{1.25}$$

If $\beta \notin \mathbf{N}_0$ and both fractional derivatives $\left({}^{C}\mathfrak{D}_{b+}^{\beta}\varphi\right)(\tau)$ and $\left(\mathfrak{D}_{b+}^{\beta}\varphi\right)(\tau)$ of order $\beta \in \mathbf{R}_+$ exist for a suitable function $\varphi(\tau)$, then according to (1.13), they are related by the following expression:

$$\left({}^{C}\mathfrak{D}_{b+}^{\beta}\varphi\right)(\tau) = \left(\mathfrak{D}_{b+}^{\beta}\varphi\right)(\tau) - \sum_{\mathfrak{j}=0}^{\mathfrak{m}-1} \frac{\varphi^{(\mathfrak{j})}(b)}{\Gamma(\mathfrak{j}-\beta+1)}(\tau-b)^{\mathfrak{j}-\beta} \quad \text{where } \mathfrak{m} = \lceil \beta \rceil. \tag{1.26}$$

Also, we have

$$\left({}^{C}\mathfrak{D}_{b+}^{\beta}\varphi\right)(\tau) = \left(\mathfrak{D}_{b+}^{\beta}\varphi\right)(\tau),$$

under the condition

$$\varphi(b) = \varphi'(b) = \cdots = \varphi^{(\mathfrak{m}-1)}(b) = 0 \text{ where } \mathfrak{m} = \lceil \beta \rceil.$$

If $\beta = \mathfrak{m} \in \mathbf{N}_0$ and the ordinary derivative $\varphi^{(\mathfrak{m})}(x)$ exists, then we have

$$\left({}^{C}\mathfrak{D}_{b+}^{\mathfrak{m}}\varphi\right)(\tau) = \varphi^{(\mathfrak{m})}(\tau) \quad \text{where } \mathfrak{m} \in \mathbf{N}. \tag{1.27}$$

The Caputo fractional derivative $\left({}^{C}\mathfrak{D}_{b+}^{\beta}\varphi\right)(\tau)$ shares properties analogous to those of the Riemann-Liouville fractional derivative $\left(\mathfrak{D}_{b+}^{\beta}\varphi\right)(\tau)$ as described in (1.13), but it differs from the properties outlined in (1.14).

Example 1.3.3. *[16] Let $\beta \in \mathbf{R}_+^*$, $\gamma \in \mathbf{R}$ with $\gamma > -1$ and let $\mathfrak{m}$ be defined as in (1.25).Consider the function $\varphi(\theta) = (\theta - b)^{\gamma}$. Then*

(1) For $\gamma > \mathfrak{m} - 1$ the Caputo fractional derivative of $\varphi(\theta)$ is given by:

$$\left({}^{C}\mathfrak{D}_{b+}^{\beta}\varphi\right)(\tau) = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\beta)}(\tau-b)^{\gamma-1}.$$

(2) When $\gamma = \mathfrak{j}$ where $\mathfrak{j} = 0, 1, \cdots, \mathfrak{m}-1$, the Caputo fractional derivative vanishes:

$$\left({}^{C}\mathfrak{D}_{b+}^{\beta}\varphi\right)(\tau) = 0.$$

In particular, for the constant function $\varphi(\theta) = 1$ (corresponding to $\gamma = 0$), the Caputo fractional derivative is:

$$\left({}^{\mathcal{C}}\mathfrak{D}_{b+}^{\beta}1\right)(\tau) = 0.$$

From the definition of the Caputo fractional derivative, it follows that if the Riemann-Liouville fractional derivative in (1.24) exists, then the Caputo fractional derivative $\left({}^{\mathcal{C}}\mathfrak{D}_{b+}^{\beta}\varphi\right)(\tau)$ is well-defined. Specifically, it is defined for functions in the space $\mathscr{A}\mathscr{C}^{\mathfrak{m}}([b,d],\mathbf{R})$.

Theorem 1.3.9. *[16] Let $\beta \in \mathbf{R}_{+}$ and $\mathfrak{m}$ be defined as in (1.25). If $\varphi(\tau) \in \mathscr{A}\mathscr{C}^{\mathfrak{m}}([b,d],\mathbf{R})$, then $\left({}^{\mathcal{C}}\mathfrak{D}_{b+}^{\beta}\varphi\right)(\tau)$ exists almost everywhere on $[b,d]$.*

(a) If $\beta \notin \mathbf{N}_0$, the Caputo fractional derivative $\left({}^{\mathcal{C}}\mathfrak{D}_{b+}^{\beta}\varphi\right)(\tau)$ is expressed as follows:

$$\left({}^{\mathcal{C}}\mathfrak{D}_{b+}^{\beta}\varphi\right)(\tau) = \frac{1}{\Gamma(\mathfrak{m}-\beta)}\int_{b}^{\tau}\frac{\varphi^{(\mathfrak{m})}(\theta)\mathrm{d}\theta}{(\tau-\theta)^{\beta-\mathfrak{m}+1}},$$

where $\mathfrak{m} = \lceil \beta \rceil$.

(b) If $\beta = \mathfrak{m} \in \mathbf{N}_0$, then

$$\left({}^{\mathcal{C}}\mathfrak{D}_{b+}^{\beta}\varphi\right)(\tau) = \varphi^{(\mathfrak{m})}(\tau).$$

In particular, if $\beta = 0$, then

$$\left({}^{\mathcal{C}}\mathfrak{D}_{b+}^{0}\varphi\right)(\tau) = \varphi(\tau).$$

Lemma 1.3.10. *[16] Let $\beta \in \mathbf{R}_{+}^{*}$ and $\mathfrak{m}$ be defined as in (1.25). If $\varphi(\tau) \in \mathscr{C}^{\mathfrak{m}}([b,d],\mathbf{R})$ or $\varphi(\tau) \in \mathscr{A}\mathscr{C}^{\mathfrak{m}}([b,d],\mathbf{R})$, then*

$$\left(\mathfrak{T}_{b+}^{\beta}\,{}^{\mathcal{C}}\mathfrak{D}_{b+}^{\beta}\varphi\right)(\tau) = \varphi(\tau) - \sum_{\mathrm{j}=0}^{\mathfrak{m}-1}\frac{\varphi^{(\mathrm{j})}(b)}{\mathrm{j}!}(\tau-b)^{\mathrm{j}}.$$

In particular, when $0 < \beta \leq 1$, then

$$\left(\mathfrak{T}_{b+}^{\beta}\,{}^{\mathcal{C}}\mathfrak{D}_{b+}^{\beta}\varphi\right)(\tau) = \varphi(\tau) - \varphi(b).$$

1.4 Fourier and Laplace Transforms

In this section, we introduce the definitions and fundamental properties of the multidimensional Fourier transform and the one-dimensional Laplace transform, both of which play a crucial role in applications to partial differential equations (PDEs) and fractional differential equations (FDEs).

Fourier Transform

Let $\Phi(\theta)$ be a function of $\theta = (\theta_1, \theta_2, \cdots, \theta_{\mathfrak{m}}) \in \mathbf{R}^{\mathfrak{m}}$. The Fourier transform of $\Phi(\theta)$ is defined as:

$$(\mathfrak{F}\Phi)(\zeta) = \mathfrak{F}[\Phi(\theta)](\zeta) = \hat{\Phi}(\zeta) := \int_{\mathbb{R}^{\mathfrak{m}}} e^{i\zeta\cdot\theta}\Phi(\theta)d\theta \quad \text{for } \zeta \in \mathbf{R}^{\mathfrak{m}}. \tag{1.28}$$

The inverse Fourier transform is defined as:

$$\left(\mathfrak{F}^{-1}\Psi\right)(\zeta) = \mathfrak{F}^{-1}[\Psi(\theta)](\zeta) = \frac{1}{(2\pi)^{\mathfrak{m}}}\hat{\Psi}(-\zeta) := \frac{1}{(2\pi)^{\mathfrak{m}}}\int_{\mathbf{R}^{\mathfrak{m}}} e^{-i\zeta\cdot\theta}\Psi(\theta)d\theta \quad \text{for } \zeta \in \mathbf{R}^{\mathfrak{m}}. \tag{1.29}$$

The Fourier transform and its inverse are well-defined for functions $\Phi \in \mathcal{L}^1(\mathbf{R}^{\mathfrak{m}}, \mathbf{R})$(respectively, $\Phi \in \mathcal{L}^2(\mathbf{R}^{\mathfrak{m}}, \mathbf{R})$) which are functions that are absolutely integrable (respectively, square-integrable).

We present some basic properties of the Fourier transform:

Linearity:

$$\mathfrak{F}[\alpha\Phi(\theta) + \gamma\Psi(\theta)](\sigma) = \alpha\mathfrak{F}[\Phi(\theta)](\sigma) + \gamma\mathfrak{F}[\Psi(\theta)](\sigma), \quad \alpha, \gamma \in \mathbf{R}.$$

Differentiation Property:

$$\mathfrak{F}\left[\mathcal{D}^{\kappa}\Phi(\theta)\right](\zeta) = (-\mathfrak{i}\zeta)^{\kappa}(\mathfrak{F}\Phi)(\zeta) \quad \text{with } \zeta \in \mathbf{R}^{\mathfrak{m}}, \quad \kappa = (\kappa_1, \cdots, \kappa_{\mathfrak{m}}) \in \mathbf{N}^{\mathfrak{m}}. \tag{1.30}$$

for $\Phi(\theta) \in \mathscr{C}^{\kappa}(\mathbf{R}^{\mathfrak{m}}, \mathbf{R})$ such that $\mathcal{D}^{\mathfrak{j}}\Phi(\theta) \in \mathcal{L}^1(\mathbf{R}^{\mathfrak{m}}, \mathbf{R})$ for all $\mathfrak{j} \leq \kappa$, Where, $\mathcal{D}^{\kappa}$ denotes the partial derivative operator.

Laplace Operator Property:

$$\mathfrak{F}[\Delta\Phi(\theta)](\zeta) = -|\zeta|^2(\mathfrak{F}\Phi)(\zeta). \tag{1.31}$$

Here, Δ denotes the Laplacian operator, defined as

$$\Delta = \frac{\partial^2}{\partial \zeta_1^2} + \cdots + \frac{\partial^2}{\partial \zeta_{\mathfrak{m}}^2}.$$

The following theorem, known as the Fourier convolution theorem, provides the Fourier transform of the convolution operator:

Theorem 1.4.1. *Let Φ and Ψ be two functions in $\mathcal{L}_1(\mathbf{R}^{\mathfrak{m}}, \mathbf{R})$ (or $\mathcal{L}_2(\mathbf{R}^{\mathfrak{m}}, \mathbf{R})$). Then The Fourier transform of their convolution is given by*

$$(\mathfrak{F}(\Phi * \Psi))(\zeta) = (\mathfrak{F}\Phi)(\zeta) \cdot (\mathfrak{F}\Psi)(\zeta).$$

Equivalently, the convolution itself can be expressed as:

$$(\Phi * \Psi)(\zeta) = (\mathfrak{F}^{-1}(\mathfrak{F}\Phi) \cdot (\mathfrak{F}\Psi))(\zeta). \tag{1.32}$$

The Laplace Transform

Let be $\Phi(\theta)$ a function of $\theta \in \mathbf{R}^+$, The Laplace transform of $\Phi(\theta)$ is defined as:

$$(\mathfrak{L}\Phi)(\sigma) = \mathfrak{L}[\Phi(\theta)](\sigma) = \tilde{\Phi}(\sigma) := \int_0^\infty e^{-\sigma\theta}\Phi(\theta)d\theta \quad \text{where } \sigma \in \mathbf{C}. \tag{1.33}$$

The inverse Laplace transform for $\zeta \in \mathbf{R}^+$ is defined as follows:

$$\left(\mathfrak{L}^{-1}\Psi\right)(\zeta) = \mathfrak{L}^{-1}[\Psi(\sigma)](\zeta) := \frac{1}{2\pi i}\int_{\delta - i\infty}^{\delta + i\infty} e^{\sigma\zeta}\Psi(\sigma)d\sigma \quad \text{where } \delta = \mathfrak{Re}(\sigma).$$

The Laplace transform and its inverse are inverse operations, meaning:

$$\mathfrak{L}^{-1}\mathfrak{L}\Phi = \Phi,$$

and

$$\mathfrak{L}\mathfrak{L}^{-1}\Psi = \Psi.$$

Example 1.4.1. *[16] Let $\Phi(\theta) = \theta^\alpha$ with $\alpha > -1$. The Laplace transform of Φ is given by:*

$$\mathfrak{L}[\theta^\alpha](\sigma) = \frac{\Gamma(\alpha+1)}{\sigma^{\alpha+1}} \quad (\sigma > 0). \tag{1.34}$$

Indeed, from the definition of the Laplace transform:

$$(\mathfrak{L}\Phi)(\sigma) = \int_0^\infty e^{-\theta\sigma}\theta^\alpha d\theta,$$

by setting $\zeta = \theta\sigma$, we find:

$$\begin{aligned}(\mathfrak{L}\Phi)(\sigma) &= \int_0^\infty e^{-\zeta}\left(\frac{\zeta}{\sigma}\right)^\alpha \frac{d\zeta}{\sigma} \\ &= \frac{1}{\sigma^{\alpha+1}}\int_0^\infty e^{-\zeta}\zeta^\alpha d\zeta.\end{aligned}$$

The integral $\int_0^\infty e^{-\zeta}\zeta^\alpha d\zeta$ is the definition of the Gamma function $\Gamma(\alpha+1)$. Therefore, we have:

$$(\mathfrak{L}\Phi)(\sigma) = \frac{\Gamma(\alpha+1)}{\sigma^{\alpha+1}}.$$

From this result, we can also conclude the inverse Laplace transform of the function $\Psi(\sigma) = \frac{1}{\sigma^{\alpha+1}}$, which is given by:

$$\mathfrak{L}^{-1}\left[\frac{1}{\sigma^{\alpha+1}}\right](\theta) = \frac{\theta^\alpha}{\Gamma(\alpha+1)}.$$

Example 1.4.2. *[16] Let $\theta, \delta, \gamma \in \mathbf{R}_+^*$ and $\mu \in \mathbf{R}$. Consider the elementary function $\theta^{\gamma-1}\mathcal{E}_{\delta,\gamma}\left(\mu\theta^\delta\right)$, where $\mathcal{E}_{\delta,\gamma}(\cdot)$ is the two-parameter Mittag-Leffler function. The Laplace transform of this function is given by:*

$$\mathfrak{L}\left[\theta^{\gamma-1}\mathcal{E}_{\delta,\gamma}\left(\mu\theta^\delta\right)\right](\sigma) = \frac{\sigma^{\delta-\gamma}}{\sigma^\delta - \mu} \quad \text{with } \sigma > 0 \text{ and } \left|\mu\sigma^{-\delta}\right| < 1. \tag{1.35}$$

.

Indeed, from the definition of the Laplace transform:

$$\begin{aligned}
\mathfrak{L}\left[\theta^{\gamma-1}\mathcal{E}_{\delta,\gamma}\left(\mu\theta^{\delta}\right)\right](\sigma) &= \int_0^{\infty} e^{-\theta\sigma}\theta^{\gamma-1}\mathcal{E}_{\delta,\gamma}\left(\mu\theta^{\delta}\right)d\theta \\
&= \int_0^{\infty} e^{-\theta\sigma}\theta^{\gamma-1}\sum_{\kappa=0}^{\infty}\frac{(\mu\theta^{\delta})^{\kappa}}{\Gamma(\delta\kappa+\gamma)}d\theta \\
&= \sum_{\kappa=0}^{\infty}\frac{(\mu)^{\kappa}}{\Gamma(\delta\kappa+\gamma)}\int_0^{\infty} e^{-\theta\sigma}\theta^{\delta\kappa+\gamma-1}d\theta \\
&= \sum_{\kappa=0}^{\infty}\frac{(\mu)^{\kappa}}{\Gamma(\delta\kappa+\gamma)}\mathfrak{L}\left[\theta^{\delta\kappa+\gamma-1}\right](\sigma).
\end{aligned}$$

Going back to (1.34), one has

$$\begin{aligned}
\mathfrak{L}\left[\theta^{\gamma-1}\mathcal{E}_{\delta,\gamma}\left(\mu\theta^{\delta}\right)\right](\sigma) &= \sum_{\kappa=0}^{\infty}\frac{(\mu)^{\kappa}}{\Gamma(\beta\kappa+\gamma)}\cdot\frac{\Gamma(\delta\kappa+\gamma)}{\sigma^{\delta\kappa+\gamma}} \\
&= \sum_{\kappa=0}^{\infty}\frac{(\mu)^{\kappa}}{\sigma^{\delta\kappa+\gamma}} = \sum_{\kappa=0}^{\infty}\frac{1}{\sigma^{\gamma}}\cdot\left(\frac{\mu}{\sigma^{\delta}}\right)^{\kappa}.
\end{aligned}$$

This is a geometric series with ratio $\dfrac{\mu}{\sigma^{\delta}}$*. Summing the series under the condition* $|\mu\sigma^{-\delta}| < 1$*, we get:*

$$\mathfrak{L}\left[\theta^{\gamma-1}\mathcal{E}_{\delta,\gamma}\left(\mu\theta^{\delta}\right)\right](\sigma) = \frac{1}{\sigma^{\gamma}}\cdot\frac{1}{1-\dfrac{\mu}{\sigma^{\delta}}} = \frac{\sigma^{\delta-\gamma}}{\sigma^{\delta}-\mu}.$$

This result is a fundamental tool in the analysis of fractional differential equations with memory effects, as it connects the Mittag-Leffler function with the Laplace transform

We present some basic properties of the Laplace transform:
Linearity:

$$\mathfrak{L}[\alpha\Phi(\theta)+\gamma\Psi(\theta)](\sigma) = \alpha\mathfrak{L}[\Phi(\theta)](\sigma)+\gamma\mathfrak{L}[\Psi(\theta)](\sigma), \quad \alpha,\gamma\in\mathbf{R}.$$

Differentiation:

The Laplace transform of the κ-th order derivative of $\Phi(\theta)$, is given by:

$$\mathfrak{L}\left[\Phi^{(\kappa)}(\theta)\right](\sigma) = \sigma^{\kappa}(\mathfrak{L}\Phi)(\sigma) - \sum_{j=0}^{\kappa-1} \sigma^{\kappa-j-1}\Phi^{(\kappa)}(0) \quad \text{for } \kappa \in \mathbf{N}.$$

The Laplace transform of the convolution between $\Psi \in \mathcal{L}_1(\mathbf{R},\mathbf{R})$ (or $\mathcal{L}_2(\mathbf{R},\mathbf{R})$) and $\Phi \in \mathcal{L}_1(\mathbf{R},\mathbf{R})$ (or $\mathcal{L}_2(\mathbf{R},\mathbf{R})$) is given by

$$\mathfrak{L}\left[(\Phi * \Psi)(\theta)\right](\sigma) = \mathfrak{L}\left[\Phi(\theta)\right](\sigma) \cdot \mathfrak{L}\left[\Psi(\theta)\right](\sigma). \tag{1.36}$$

Laplace Transform of the Fractional Integrals and Derivatives

Let us now present some fundamental results that are essential for solving fractional differential equations using the Laplace transform. Specifically, we will focus on the Laplace transforms of fractional integrals and fractional derivatives under specific conditions. These conditions will be explicitly stated for both cases, ensuring the applicability of the Laplace transform to fractional-order operators.

First, let's recall the definition of the exponential order of a function. For $\theta > 0$, a function $\Phi(\theta)$ is said to be of exponential order $\gamma > 0$ if there exist constants $\lambda, d \in \mathbf{R}_+^*$ such that

$$|\Phi(\theta)| \leq \lambda \mathfrak{e}^{\gamma\theta} \quad \text{for all } \theta > d. \tag{1.37}$$

Lemma 1.4.2. *[16]*

(a) Let $\beta > 0$. If $\Phi(\theta) \in \mathcal{L}^1([0,d],\mathbf{R})$ for any $d > 0$ and of exponential order $\gamma > 0$, then

$$\left(\mathfrak{L}\mathfrak{T}_{0+}^{\beta}\Phi\right)(\sigma) = \sigma^{-\beta}(\mathfrak{L}\Phi)(\sigma),$$

for $\Re(\sigma) > \gamma$.

(b) Let $\beta > 0$, $\mathfrak{m} = \lceil \beta \rceil$. If $\Phi(\theta) \in \mathscr{A}\mathscr{C}^{\mathfrak{m}}([0,d],\mathbf{R})$ for any $d > 0$, and there exist the finite limits $\lim_{\theta\to 0+}[\mathcal{D}^{\kappa}\mathfrak{T}_{0+}^{\mathfrak{m}-\beta}\Phi(\theta)]$ and $\lim_{\theta\to\infty}[\mathcal{D}^{\kappa}\mathfrak{T}_{0+}^{\mathfrak{m}-\beta}\Phi(\theta)] = 0$ $\quad (\mathcal{D} = d/d\theta; \kappa = 0, 1, \ldots, \mathfrak{m}-1)$ then

$$\left(\mathfrak{L}\mathfrak{D}_{0+}^{\beta}\Phi\right)(\sigma) = \sigma^{\beta}(\mathfrak{L}\Phi)(\sigma) - \sum_{\kappa=0}^{\mathfrak{m}-1} \sigma^{\mathfrak{m}-\kappa-1}\mathcal{D}^{\kappa}\left(\mathfrak{T}_{0+}^{\mathfrak{m}-\beta}\Phi\right)(0+).$$

Lemma 1.4.3. *[16] Let $\beta > 0$ and $\mathfrak{m} = \lceil \beta \rceil$. If $\Phi(\theta) \in \mathscr{C}^{\mathfrak{m}}(\mathbf{R}^{+}, \mathbf{R})$ and also $\Phi^{(\mathfrak{m})}(\zeta) \in \mathcal{L}^{1}([0,d], \mathbf{R})$ for any $d > 0$ and of exponential order $\gamma > 0$, then the following relation holds:*

$$\left(\mathfrak{L}^{\mathcal{C}}\mathfrak{D}_{0+}^{\beta}\Phi\right)(\sigma) = \sigma^{\beta}(\mathfrak{L}\Phi)(\sigma) - \sum_{\kappa=0}^{\mathfrak{m}-1} \sigma^{\beta-\kappa-1}\Phi^{(\kappa)}(0). \tag{1.38}$$

1.5 Fixed Point Theorems

In this section, we present Banach's Fixed-Point Theorem, a highly useful tool in mathematics, particularly for solving differential equations and fractional differential equations. This theorem provides sufficient conditions under which a given function or mapping admits a unique fixed point. By reformulating a problem as a fixed-point problem, we can guarantee not only the existence but also the uniqueness of the solution. Moreover, this fixed point corresponds directly to the solution of the original problem.

Definition 1.5.1. *Let $(\Lambda, \mathbf{d})$ be a metric space and $\Upsilon : \Lambda \to \Lambda$ be a mapping. An element $\zeta \in \Lambda$ is called a fixed point of Υ if*

$$\Upsilon\zeta = \zeta.$$

Definition 1.5.2. *A mapping $\Upsilon : \Lambda \to \Lambda$ on a metric space $(\Lambda, \mathbf{d})$ is called a contraction if there exists $\mathfrak{K} \in [0, 1)$ such that*

$$\mathbf{d}(\Upsilon(\zeta), \Upsilon(\xi)) \leq \mathfrak{K}\,\mathbf{d}(\zeta, \xi) \quad \textit{for all } \zeta, \xi \in \Lambda.$$

Now we present the classical Banach fixed point theorem in a complete metric space

Theorem 1.5.1. *Let $(\Lambda, \mathbf{d})$ be a complete metric space and let $\Upsilon : \Lambda \to \Lambda$ be a contraction on Λ. Then Υ has a unique fixed point $\zeta^{*} \in \Lambda$.*

Chapter 2

FRACTIONAL DIFFERENTIAL EQUATIONS

In this chapter, we establish the existence and uniqueness of solutions for nonlinear fractional differential equations involving Caputo fractional derivatives. We also derive explicit solutions for linear fractional differential equations with Caputo derivatives through two complementary methods: reduction to Volterra integral equations and application of the Laplace transform.

Definition 2.0.1. *[19] An equation of the form*

$$\sum_{\mathfrak{j}=0}^{\mathfrak{m}} \mu_{\mathfrak{j}} \cdot \left({}^{C}\mathfrak{D}_{0+}^{\beta_{\mathfrak{j}}}\varphi\right)(\tau) = \Phi(\tau), \qquad \textit{for each } \tau > 0, \tag{2.1}$$

where ${}^{C}\mathfrak{D}_{0+}^{\beta_{\mathfrak{j}}}$ *denotes the Caputo fractional derivative of order* $\beta_{\mathfrak{j}}$ *with* $\beta_{\mathfrak{m}} > \beta_{\mathfrak{m}-1} > \cdots > \beta_0 \geq 0$, $\Phi : [0, \infty) \longrightarrow \mathbf{R}$ *is a given function, and* $\mu_{\mathfrak{j}} \in \mathbf{R}$ *for all* $\mathfrak{j} = 0, ..., \mathfrak{m}$ *with* $\mu_{\mathfrak{m}} = 1$, *is called a linear fractional differential equation.*

An equation that cannot be expressed in the form (2.1) is called a nonlinear equation.

Remark 2.0.1. *If* $\Phi(\tau) \equiv 0$, *the equation (2.1) reduces to the homogeneous case:*

$$\sum_{\mathfrak{j}=0}^{\mathfrak{m}} \mu_{\mathfrak{j}} \cdot \left({}^{C}\mathfrak{D}_{0+}^{\beta_{\mathfrak{j}}}\varphi\right)(\tau) = 0, \qquad \textit{for } \tau > 0.$$

2.1 The Existence and Uniqueness Theorem for Fractional Differential Equations

In this section, we investigate the existence and uniqueness of solutions for the following nonlinear fractional differential equation:

$$\left({}^{\mathcal{C}}\mathfrak{D}^{\beta}_{0+}\varphi\right)(\tau) = \Phi[\tau, \varphi(\tau)], \tag{2.2}$$

under the initial conditions:

$$\varphi^{(\mathfrak{j})}(0) = \mathfrak{c}_{\mathfrak{j}}, \tag{2.3}$$

where $0 \leq \tau \leq T, \beta > 0$, ${}^{\mathcal{C}}\mathfrak{D}^{\beta}_{0+}$ denotes the Caputo fractional derivative of order β, and $\mathfrak{c}_{\mathfrak{j}} \in \mathbf{R}$ for $\mathfrak{j} = 0, 1, \cdots, \mathfrak{m} - 1$ with $\mathfrak{m} = \lceil \beta \rceil$.

In particular, if $\beta = \mathfrak{m} \in \mathbf{N}$ the Cauchy problem described by (2.2)-(2.3) reduces, based on (1.27), to the usual Cauchy problem for an ordinary differential equation (ODE) of order $\mathfrak{m} \in \mathbf{N}$.

More precisely, the problem takes the form:

$$\varphi^{(\mathfrak{m})}(\tau) = \Phi[\tau, \varphi(\tau)], \tag{2.4}$$

$$\varphi^{(\mathfrak{j})}(0) = \mathfrak{c}_{\mathfrak{j}}, \tag{2.5}$$

where $0 \leq \tau \leq T$, and $\mathfrak{c}_{\mathfrak{j}} \in \mathbf{R}$ for $\mathfrak{j} = 0, 1, \cdots, \mathfrak{m} - 1$.

This Cauchy problem can be equivalently transformed into the Volterra integral equation of the second kind:

$$\varphi(\tau) = \sum_{i=0}^{\mathfrak{m}-1} \frac{\mathfrak{c}_{\mathfrak{i}}}{\mathfrak{i}!}\tau^{\mathfrak{i}} + \frac{1}{(\mathfrak{m}-1)!}\int_0^{\tau} (\tau - \theta)^{\mathfrak{m}-1}\Phi[\theta, \varphi(\theta)]\mathrm{d}\theta, \qquad \text{for } 0 \leq \tau \leq T. \tag{2.6}$$

Consequently, the problem (2.2)–(2.3) can be equivalently transformed into the Volterra integral equation:

$$\varphi(\tau) = \sum_{\mathfrak{i}=0}^{\mathfrak{m}-1} \frac{\mathfrak{c}_{\mathfrak{i}}}{\mathfrak{i}!}\tau^{\mathfrak{i}} + \frac{1}{\Gamma(\beta)}\int_0^{\tau} \frac{\Phi[\theta, \varphi(\theta)]\mathrm{d}\theta}{(\tau - \theta)^{1-\beta}} \qquad \text{for } 0 \leq \tau \leq T. \tag{2.7}$$

First, we define the space of functions $\mathscr{C}^{\beta,\mathfrak{p}}_{\delta}[0, T]$ for which we establish the conditions ensuring the existence of a unique solution $\varphi(\tau)$ to the problem

(2.2)-(2.3). For $\beta > 0$, $\delta \in \mathbf{R}$ with $0 \leq \delta < 1$ and $\mathfrak{p} \in \mathbf{N}$, the space is defined as follows:

$$\mathscr{C}_{\delta}^{\beta,\mathfrak{p}}[0,T] = \left\{ \varphi(\tau) \in \mathscr{C}^{\mathfrak{p}}[0,T] : {}^{\mathcal{C}}\mathfrak{D}_{0+}^{\beta}\varphi \in \mathscr{C}_{\delta}[0,T] \right\}, \tag{2.8}$$

$$\mathscr{C}_{0}^{\beta,\mathfrak{p}}[0,T] = \mathscr{C}^{\beta,\mathfrak{p}}[0,T], \quad \mathscr{C}_{\delta}^{\mathfrak{p},\mathfrak{p}}[0,T] = \mathscr{C}_{\delta}^{\mathfrak{p}}[0,T].$$

The following Theorem establishes an equivalence between the problem (2.2)–(2.3) and the integral equation (2.7) in the space of continuously differentiable functions. Specifically, it shows that solving the fractional differential equation (2.2) with the initial conditions (2.3) is equivalent to solving the integral equation (2.7) in the space $\mathscr{C}^{\mathfrak{p}}[0,T]$.

Prior to presenting and proving the results, consider the following assumptions:

(a) $\beta > 0$ and $\mathfrak{m} = \lceil \beta \rceil$, $0 \leq \delta < 1$ with $\delta \leq \beta$.

(b) U be an open set in $\mathbb{R}$, $\Phi : (0,T] \times U \to \mathbf{R}$ be a function such that, for any $\varphi \in U, \Phi[\tau, \varphi] \in \mathscr{C}_{\delta}[0,T]$.

(c) $\mathfrak{p} = \mathfrak{m}$ for $\beta \in \mathbf{N}$ and $\mathfrak{p} = \mathfrak{m} - 1$ for $\beta \notin \mathbf{N}$.

Theorem 2.1.1. *[16] Let the assumptions (a)-(c) hold. If $\varphi(\tau) \in \mathscr{C}^{\mathfrak{p}}[0,T]$, then $\varphi(\tau)$ is a solution of the Cauchy problem (2.2)-(2.3) if and only if $\varphi(\tau)$ is a solution of the Volterra integral equation (2.7).*

Proof. For $\beta = \mathfrak{m} \in \mathbf{N}$, let $\varphi(\tau) \in \mathscr{C}^{\mathfrak{m}}[0,T]$ be a solution of the Cauchy problem

$$\varphi^{(\mathfrak{m})}(\tau) = \Phi[\tau, \varphi(\tau)], \tag{2.9}$$

$$\varphi^{(\mathfrak{j})}(0) = \mathfrak{c}_{\mathfrak{j}}. \tag{2.10}$$

By applying the operator $\mathfrak{T}_{0+}^{\mathfrak{m}}$ to both sides of (2.9), we obtain

$$(\mathfrak{T}_{0+}^{\mathfrak{m}}\varphi^{(\mathfrak{m})})(\tau) = (\mathfrak{T}_{0+}^{\mathfrak{m}}\Phi[\theta, \varphi(\theta)])(\tau).$$

from (1.23) and (2.10), it follows that

$$\varphi(\tau) - \sum_{\mathfrak{i}=0}^{\mathfrak{m}-1} \frac{\mathfrak{c}_{\mathfrak{i}}}{\mathfrak{i}!}\tau^{\mathfrak{i}} = (\mathfrak{T}_{0+}^{\mathfrak{m}}\Phi[\theta, \varphi(\theta)])(\tau).$$

Hence

$$\varphi(\tau) = \sum_{\mathfrak{i}=0}^{\mathfrak{m}-1} \frac{\mathfrak{c}_\mathfrak{i}}{\mathfrak{i}!}\tau^\mathfrak{i} + \frac{1}{(\mathfrak{m}-1)!}\int_0^\tau (\tau-\theta)^{\mathfrak{m}-1}\Phi[\theta,\varphi(\theta)]\mathrm{d}\theta.$$

Conversely, if $\varphi(\tau) \in \mathscr{C}^\mathfrak{m}[0,b]$ satisfies the integral equation(2.6), then

$$\varphi(\tau) = \mathfrak{c}_0 + \sum_{i=1}^{\mathfrak{m}-1} \frac{\mathfrak{c}_\mathfrak{i}}{\mathfrak{i}!}\tau^\mathfrak{i} + \frac{1}{(\mathfrak{m}-1)!}\int_0^\tau (\tau-\theta)^{\mathfrak{m}-1}\Phi[\theta,\varphi(\theta)]\mathrm{d}\theta$$

By differentiating this equation once, we obtain

$$\varphi^{(1)}(\tau) = \mathfrak{c}_1 + \sum_{\mathfrak{i}=2}^{\mathfrak{m}-1} \frac{\mathfrak{c}_\mathfrak{i}}{(\mathfrak{i}-1)!}\tau^{\mathfrak{i}-1} + \frac{1}{(\mathfrak{m}-2)!}\int_0^\tau (\tau-\theta)^{\mathfrak{m}-2}\Phi[\theta,\varphi(\theta)]\mathrm{d}\theta.$$

Differentiating a second time yields

$$\varphi^{(2)}(\tau) = \mathfrak{c}_2 + \sum_{\mathfrak{i}=3}^{\mathfrak{m}-1} \frac{\mathfrak{c}_\mathfrak{i}}{(\mathfrak{i}-2)!}\tau^{\mathfrak{i}-2} + \frac{1}{(\mathfrak{m}-3)!}\int_0^\tau (\tau-\theta)^{\mathfrak{m}-3}\Phi[\theta,\varphi(\theta)]\mathrm{d}\theta.$$

Repeating this process, it follows that, for $\mathfrak{j} = 0, ..., \mathfrak{m}-2$

$$\varphi^{(\mathfrak{j})}(\tau) = \mathfrak{c}_\mathfrak{j} + \sum_{\mathfrak{i}=\mathfrak{j}+1}^{\mathfrak{m}-1} \frac{\mathfrak{c}_\mathfrak{i}}{(\mathfrak{i}-\mathfrak{j})!}\tau^{\mathfrak{i}-\mathfrak{j}} + \frac{1}{(\mathfrak{m}-\mathfrak{j}-1)!}\int_0^\tau (\tau-\theta)^{\mathfrak{m}-\mathfrak{j}-1}\Phi[\theta,\varphi(\theta)]\mathrm{d}\theta,$$

and

$$\varphi^{(\mathfrak{m}-1)}(\tau) = \mathfrak{c}_{\mathfrak{m}-1} + \int_0^\tau \Phi[\theta,\varphi(\theta)]\mathrm{d}\theta. \tag{2.11}$$

Taking the limit as $\tau \to 0$, we obtain

$$\varphi^{(\mathfrak{j})}(0) = \mathfrak{c}_\mathfrak{j} \quad \text{for } \mathfrak{j} = 0, 1, \cdots, \mathfrak{m}-1.$$

On the other hand, differentiating equation (2.11), we get

$$\varphi^{(\mathfrak{m})}(\tau) = \Phi[\tau,\varphi(\tau)].$$

The proof is complete for $\beta \in \mathbb{N}$.

For $\mathfrak{m}-1 < \beta < \mathfrak{m}$, let $\varphi(\tau) \in \mathscr{C}^{\mathfrak{m}-1}[0,T]$ be a solution of the Cauchy problem (2.2)-(2.3). From the definitions of the Caputo and Riemann-Liouville fractional derivatives, we have:

$$\left({}^{\mathcal{C}}\mathfrak{D}^{\beta}_{0+}\varphi\right)(\tau) = \left(\mathfrak{D}^{\beta}_{0+}\left[\varphi(\theta) - \sum_{\mathfrak{i}=0}^{\mathfrak{m}-1}\frac{\varphi^{(\mathfrak{i})}(0)}{\mathfrak{i}!}\theta^{\mathfrak{i}}\right]\right)(\tau) \tag{2.12}$$

$$= \left(\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^{\mathfrak{m}}\left(\mathfrak{T}^{\mathfrak{m}-\beta}_{0+}\left[\varphi(\theta) - \sum_{\mathfrak{i}=0}^{\mathfrak{m}-1}\frac{\varphi^{(\mathfrak{i})}(0)}{\mathfrak{i}!}\theta^{\mathfrak{i}}\right]\right)(\tau). \tag{2.13}$$

Under the condition $\Phi[\tau,\varphi] \in \mathscr{C}_{\delta}[0,T]$ as stated in Theorem 2.1.1, it follows that $\left({}^{\mathcal{C}}\mathfrak{D}^{\beta}_{0+}\varphi\right)(\tau) \in \mathscr{C}_{\delta}[0,T]$. Consequently, by Lemma 1.1.2, we obtain:

$$\left(\mathfrak{T}^{\mathfrak{m}-\beta}_{0+}\left[\varphi(\theta) - \sum_{\mathfrak{i}=0}^{\mathfrak{m}-1}\frac{\varphi^{(\mathfrak{i})}(0)}{\mathfrak{i}!}\theta^{\mathfrak{i}}\right]\right)(\tau) \in \mathscr{C}^{\mathfrak{m}}_{\delta}[0,T].$$

By applying the Riemann-Liouville fractional integral of order β to both sides of (2.12) and applying Lemma 1.3.8 for the term $\varphi(\theta) - \sum_{\mathfrak{i}=0}^{\mathfrak{m}-1}\frac{\varphi^{(\mathfrak{i})}(0)}{\mathfrak{i}!}\theta^{\mathfrak{i}}$, we get

$$\begin{aligned}\left(\mathfrak{T}^{\beta}_{0+}{}^{\mathcal{C}}\mathfrak{D}^{\beta}_{0+}\varphi\right)(\tau) &= \left(\mathfrak{T}^{\beta}_{0+}\mathfrak{D}^{\beta}_{0+}\left[\varphi(\theta) - \sum_{\mathfrak{i}=0}^{\mathfrak{m}-1}\frac{\varphi^{(\mathfrak{i})}(0)}{\mathfrak{i}!}\theta^{\mathfrak{i}}\right]\right)(\tau)\\ &= \varphi(\tau) - \sum_{\mathfrak{i}=0}^{\mathfrak{m}-1}\frac{\varphi^{(\mathfrak{i})}(0)}{\mathfrak{i}!}\tau^{\mathfrak{i}} - \sum_{k=1}^{\mathfrak{m}}\frac{\varphi^{(\mathfrak{m}-k)}_{n-\beta}(0+)}{\Gamma(\beta-k+1)}\tau^{\beta-k},\end{aligned} \tag{2.14}$$

where

$$\begin{aligned}\varphi_{\mathfrak{m}-\beta}(\tau) &= \left(\mathfrak{T}^{\mathfrak{m}-\beta}_{0+}\left[\varphi(\theta) - \sum_{\mathfrak{i}=0}^{\mathfrak{m}-1}\frac{\varphi^{(\mathfrak{i})}(0)}{\mathfrak{i}!}\theta^{\mathfrak{i}}\right]\right)(\tau)\\ &= \frac{1}{\Gamma(\mathfrak{m}-\beta)}\int_0^{\tau}(\tau-\theta)^{\mathfrak{m}-\beta-1}\left[\varphi(\theta) - \sum_{\mathfrak{i}=0}^{\mathfrak{m}-1}\frac{\varphi^{(\mathfrak{i})}(0)}{\mathfrak{i}!}\theta^{\mathfrak{i}}\right]\mathrm{d}\theta.\end{aligned} \tag{2.15}$$

Integrating equation (2.15) by parts, let $u(\theta) = \varphi(\theta) - \sum_{\mathfrak{i}=0}^{\mathfrak{m}-1}\frac{\varphi^{(\mathfrak{i})}(0)}{\mathfrak{i}!}\theta^{\mathfrak{i}}$ and $\mathrm{d}v(\theta) = (\tau-\theta)^{\mathfrak{m}-\beta-1}\mathrm{d}\theta$, which implies, $\mathrm{d}u(\theta) = \left[\varphi'(\theta) - \sum_{\mathfrak{i}=1}^{\mathfrak{m}-1}\frac{\varphi^{(\mathfrak{i})}(0)}{(\mathfrak{i}-1)!}\theta^{\mathfrak{i}-1}\right]\mathrm{d}\theta$ and $v(\theta) = -\frac{(\tau-\theta)^{\mathfrak{m}-\beta}}{\mathfrak{m}-\beta}$. Using the formula $\int u\,dv = uv - \int du\,v$, it follows

$$\varphi_{\mathfrak{m}-\beta}(\tau) = \frac{1}{\Gamma(\mathfrak{m}-\beta)}\left[-\frac{(\tau-\theta)^{\mathfrak{m}-\beta}}{\mathfrak{m}-\beta}\left(\varphi(\theta)-\sum_{\mathfrak{i}=0}^{\mathfrak{m}-1}\frac{\varphi^{(\mathfrak{i})}(0)}{\mathfrak{i}!}\theta^{\mathfrak{i}}\right)\Bigg|_0^{\tau}\right.$$
$$\left.+\int_0^{\tau}\frac{(\tau-\theta)^{\mathfrak{m}-\beta}}{\mathfrak{m}-\beta}\left(\varphi'(\theta)-\sum_{\mathfrak{i}=1}^{\mathfrak{m}-1}\frac{\varphi^{(\mathfrak{i})}(0)}{(\mathfrak{i}-1)!}\theta^{\mathfrak{i}-1}\right)\mathrm{d}\theta\right]$$
$$=\frac{1}{(\mathfrak{m}-\beta)\Gamma(\mathfrak{m}-\beta)}\int_0^{\tau}(\tau-\theta)^{\mathfrak{m}-\beta}\left(\varphi'(\theta)-\sum_{\mathfrak{i}=1}^{\mathfrak{m}-1}\frac{\varphi^{(\mathfrak{i})}(0)}{(\mathfrak{i}-1)!}\theta^{\mathfrak{i}-1}\right)\mathrm{d}\theta$$
$$=\frac{1}{\Gamma(\mathfrak{m}-\beta+1)}\int_0^{\tau}(\tau-\theta)^{\mathfrak{m}-\beta}\left(\varphi'(\theta)-\sum_{\mathfrak{i}=1}^{\mathfrak{m}-1}\frac{\varphi^{(\mathfrak{i})}(0)}{(\mathfrak{i}-1)!}\theta^{\mathfrak{i}-1}\right)\mathrm{d}\theta. \tag{2.16}$$

More precisely,

$$\varphi_{\mathfrak{m}-\beta}(\tau) = \left(\mathfrak{I}_{0+}^{\mathfrak{m}-\beta+1}\left[\varphi'(\theta)-\sum_{\mathfrak{i}=1}^{\mathfrak{m}-1}\frac{\varphi^{(\mathfrak{i})}(0)}{(\mathfrak{i}-1)!}\theta^{\mathfrak{i}-1}\right]\right)(\tau). \tag{2.17}$$

Differentiating equation (2.17), and applying (1.19) with $\mathfrak{j}=1$, we obtain

$$\varphi'_{\mathfrak{m}-\beta}(\tau) = \left(\mathfrak{I}_{0+}^{\mathfrak{m}-\beta}\left[\varphi'(\theta)-\sum_{\mathfrak{i}=1}^{\mathfrak{m}-1}\frac{\varphi^{(\mathfrak{i})}(0)}{(\mathfrak{i}-1)!}\theta^{\mathfrak{i}-1}\right]\right)(\tau).$$

By iterating this process $\mathfrak{m}-\mathfrak{j}$ times for $\mathfrak{j}=1,\cdots,\mathfrak{m}$, we obtain the following relation:

$$\varphi_{\mathfrak{m}-\beta}^{(\mathfrak{m}-\mathfrak{j})}(\tau) = \left(\mathfrak{I}_{0+}^{\mathfrak{m}-\beta}\left[\varphi^{(\mathfrak{m}-\mathfrak{j})}(\theta)-\sum_{\mathfrak{i}=\mathfrak{m}-\mathfrak{j}}^{\mathfrak{m}-1}\frac{\varphi^{(\mathfrak{i})}(0)}{(\mathfrak{i}-\mathfrak{m}+\mathfrak{j})!}\theta^{\mathfrak{i}-\mathfrak{m}+\mathfrak{j}}\right]\right)(\tau)$$
$$=\frac{1}{\Gamma(\mathfrak{m}-\beta)}\int_a^{\tau}(\tau-\theta)^{\mathfrak{m}-\beta-1}\left[\varphi^{(\mathfrak{m}-\mathfrak{j})}(\theta)-\sum_{\mathfrak{i}=\mathfrak{m}-\mathfrak{j}}^{\mathfrak{m}-1}\frac{\varphi^{(\mathfrak{i})}(0)}{(\mathfrak{i}-\mathfrak{m}+\mathfrak{j})!}\theta^{\mathfrak{i}-\mathfrak{m}+\mathfrak{j}}\right]\mathrm{d}\theta.$$

By setting $\theta=\sigma\tau$, which implies $\tau-\theta=\tau(1-\sigma)$ and $\mathrm{d}\theta=\tau\mathrm{d}\sigma$ we derive, for $\mathfrak{j}=1,\cdots,\mathfrak{m}$,

$$\begin{aligned}\varphi_{\mathfrak{m}-\beta}^{(\mathfrak{m}-\mathfrak{j})}(\tau) = \frac{\tau^{\mathfrak{m}-\beta}}{\Gamma(\mathfrak{m}-\beta)} \int_0^1 (1-\sigma)^{\mathfrak{m}-\beta-1} \Big[\varphi^{(\mathfrak{m}-\mathfrak{j})}(\sigma\tau) \\ - \sum_{\mathfrak{i}=\mathfrak{m}-\mathfrak{j}}^{\mathfrak{m}-1} \frac{\varphi^{(\mathfrak{i})}(0)}{(\mathfrak{i}-\mathfrak{m}+\mathfrak{j})!}(\sigma\tau)^{\mathfrak{i}-\mathfrak{m}+\mathfrak{j}}\Big] \mathrm{d}\sigma.\end{aligned} \tag{2.18}$$

Since $\beta < \mathfrak{m}$ and $\varphi^{(\mathfrak{m}-\mathfrak{j})}(\tau) \in \mathscr{C}[0,T]$ $(\mathfrak{j} = 1, \cdots, \mathfrak{m})$, then $\varphi_{\mathfrak{m}-\beta}^{(\mathfrak{m}-\mathfrak{j})}(0+) = 0$ for $\mathfrak{j} = 1, \cdots, \mathfrak{m}$, and therefore, (2.14) can be rewritten as

$$\left(\mathfrak{T}_{0+}^{\beta}\, {}^{\mathcal{C}}\mathfrak{D}_{0+}^{\beta}\varphi\right)(\tau) = \varphi(\tau) - \sum_{\mathfrak{i}=0}^{\mathfrak{m}-1} \frac{\varphi^{(\mathfrak{i})}(0)}{\mathfrak{i}!}\tau^{\mathfrak{i}}. \tag{2.19}$$

Since $\Phi[\tau,\varphi] \in \mathscr{C}_\delta[0,T]$ and $\gamma \le \beta$, it follows from Lemma 1.3.7(a) that, $\mathfrak{T}_{0+}^{\beta}\Phi[\theta,\varphi(\theta)] \in \mathscr{C}[0,T]$. By applying the Riemann-Liouville fractional integral of order β to both sides of (2.2), we obtain

$$\left(\mathfrak{T}_{0+}^{\beta}\, {}^{\mathcal{C}}\mathfrak{D}_{0+}^{\beta}\varphi\right)(\tau) = (\mathfrak{T}_{0+}^{\beta}\Phi[\theta,\varphi(\theta)])(\tau).$$

Using (2.19) along with the initial conditions (2.3), we have

$$\varphi(\tau) - \sum_{\mathfrak{i}=0}^{\mathfrak{m}-1} \frac{\mathfrak{c}_{\mathfrak{i}}}{\mathfrak{i}!}\tau^{\mathfrak{i}} = (\mathfrak{T}_{0+}^{\beta}\Phi[\theta,\varphi(\theta)])(\tau).$$

Hence

$$\varphi(\tau) = \sum_{\mathfrak{i}=0}^{\mathfrak{m}-1} \frac{\mathfrak{c}_{\mathfrak{i}}}{\mathfrak{i}!}\tau^{\mathfrak{i}} + \frac{1}{\Gamma(\beta)} \int_0^\tau \frac{\Phi[\theta,\varphi(\theta)]\mathrm{d}\theta}{(\tau-\theta)^{1-\beta}}.$$

This establishes the necessity.

Conversely, suppose $\varphi(\tau) \in \mathscr{C}^{\mathfrak{m}-1}[0,T]$ satisfies the Volterra integral equation (2.7). We will show that $\varphi(\tau)$ satisfies the initial conditions (2.3). Differentiating the equation (2.7) $\mathfrak{j}$-times, where $\mathfrak{j} = 1, \cdots, \mathfrak{m}-1$, and using the property(1.19), we obtain

$$\begin{aligned}\varphi^{(\mathfrak{j})}(\tau) &= \sum_{\mathfrak{i}=\mathfrak{j}}^{\mathfrak{m}-1} \frac{\mathfrak{c}_{\mathfrak{i}}}{(\mathfrak{i}-\mathfrak{j})!}\tau^{\mathfrak{i}-\mathfrak{j}} + (\mathfrak{T}_{0+}^{\beta-\mathfrak{j}}\Phi[\theta,\varphi(\theta)])(\tau) \\ &= \sum_{\mathfrak{i}=\mathfrak{j}}^{\mathfrak{m}-1} \frac{\mathfrak{c}_{\mathfrak{i}}}{(\mathfrak{i}-\mathfrak{j})!}\tau^{\mathfrak{i}-\mathfrak{j}} + \frac{1}{\Gamma(\beta-\mathfrak{j})} \int_0^\tau (\tau-\theta)^{\beta-\mathfrak{j}-1}\Phi[\theta,\varphi(\theta)]\mathrm{d}\theta.\end{aligned} \tag{2.20}$$

By setting $\theta = \sigma\tau$, which implies $\tau - \theta = \tau(1-\sigma)$ and $\mathrm{d}\theta = \tau \mathrm{d}\sigma$, we derive, for $\mathfrak{j} = 1, \cdots, \mathfrak{m}-1$,

$$\begin{aligned}\varphi^{(\mathfrak{j})}(\tau) = &\sum_{\mathfrak{i}=\mathfrak{j}}^{\mathfrak{m}-1} \frac{\mathfrak{c}_{\mathfrak{i}}}{(\mathfrak{i}-\mathfrak{j})!}\tau^{\mathfrak{i}-\mathfrak{j}} \\ &+ \frac{\tau^{\beta-\mathfrak{j}}}{\Gamma(\beta-\mathfrak{j})} \int_0^1 (1-\sigma)^{\beta-\mathfrak{j}-1}\Phi[\sigma\tau, \varphi(\sigma\tau)]\mathrm{d}\sigma.\end{aligned} \tag{2.21}$$

As a consequence of $\beta > \mathfrak{m}-1$ and the continuity of $\Phi[\tau, \varphi(\tau)]$, the integral in the expression (2.21) is well-defined and continuous. By taking the limit as $\tau \to 0+$, we obtain the initial conditions (2.3), for $\mathfrak{j} = 1, ..., \mathfrak{m}-1$.

For $\mathfrak{j} = 0$, Repeating the same process to the Volterra integral eqution (2.7), we derive $\varphi(0) = \mathfrak{c}_0$.

We now want to prove that $\varphi(\tau)$ satisfies the equation (2.2). From (2.7) together with (2.3) we have

$$\varphi(\tau) - \sum_{\mathfrak{i}=0}^{\mathfrak{m}-1} \frac{\varphi^{(\mathfrak{i})}(0)}{\mathfrak{i}!}\tau^{\mathfrak{i}} = \frac{1}{\Gamma(\beta)} \int_0^\tau \frac{\Phi[\theta, \varphi(\theta)]\mathrm{d}\theta}{(\tau-\theta)^{\theta-\beta}} \quad \text{for } 0 \le \tau \le T. \tag{2.22}$$

This equation can be rewritten in terms of the Riemann-Liouville fractional integral operator $\mathfrak{T}_{0+}^{\beta}$ as follow:

$$\varphi(\tau) - \sum_{\mathfrak{i}=0}^{\mathfrak{m}-1} \frac{\varphi^{(\mathfrak{i})}(0)}{\mathfrak{i}!}\tau^{\mathfrak{i}} = (\mathfrak{T}_{0+}^{\beta}\Phi[\theta, \varphi(\theta)])(\tau), \tag{2.23}$$

we apply the Riemann-Liouville fractional derivative operator $\mathfrak{D}_{0+}^{\beta}$ to both sides of (2.23). By taking (1.24) into account and using the property(1.17), we obtain

$$\left({}^{C}\mathfrak{D}_{0+}^{\beta}\varphi\right)(\tau) = \Phi[\tau, \varphi(\tau)]$$

The proof is complet for $\beta \notin \mathbf{N}$. □

Theorem 2.1.2. *[16] Let the assumptions (a)-(b) hold, and let Φ satisfies the Lipschitz condition with respect to the second variable, means that, there exists a constant $\lambda > 0$ such that*

$$|\Phi[\tau, \varphi_1] - \Phi[\tau, \varphi_2]| \le \lambda|\varphi_1 - \varphi_2| \quad \textit{for all } \tau \in (0, T], \textit{ and for all } \varphi_1, \varphi_2 \in U. \tag{2.24}$$

(i) *If* $\mathfrak{m}-1<\beta<\mathfrak{m},$ *where* $\mathfrak{m}\in\mathbf{N}$, *then the Cauchy problem (2.2)-(2.3) has a unique solution* $\varphi(\tau)$ *in the space* $\mathscr{C}_\delta^{\beta,\mathfrak{m}-1}[0,T]$.

(ii) *If* $\beta=\mathfrak{m}\in\mathbf{N}$, *then the Cauchy problem (2.4)-(2.5) has a unique solution* $\varphi(\tau)$ *in the space* $\mathscr{C}_\delta^{\mathfrak{m}}[0,T]$.

(iii) *In particular, if* $\delta=0$ *and* $\Phi[\tau,\varphi]\in\mathscr{C}[0,T]$, *the Cauchy problem (2.2)-(2.3) has a unique solution in the space* $\mathscr{C}^{\beta,\mathfrak{m}-1}[0,T]$, *while the Cauchy problem (2.4)-(2.5) has a unique solution in the space* $\mathscr{C}^{\mathfrak{m}}[0,T]$.

Remark 2.1.1. *The established results on existence and uniqueness of solutions for Caputo-type fractional differential equations can be naturally generalized to fractional differential inclusions. This generalization is achieved by substituting the single-valued right-hand side function* Φ *with a set-valued mapping* Ψ, *yielding:*

$$\left({}^C\mathfrak{D}_{0+}^{\beta}\varphi\right)(\tau)\in\Psi[\tau,\varphi(\tau)].$$

The existence of solutions can be guaranteed under standard regularity conditions on Ψ, *including upper semicontinuity and the requirement that* Ψ *takes nonempty, compact, and convex values. The proof typically relies on fixed-point theorems adapted for multivalued operators. However, Uniqueness requires additional conditions, particularly the Lipschitz continuity of* Ψ *when equipped with the Hausdorff metric. for more detail of these results, we refer to [1].*

2.2 Linear Fractional Differential Equations

2.2.1 Reduction to Volterra Integral Equation Method for Explicity Solving Linear Fractional Differential Equation

In this section, we derive the explicit solution to the linear fractional differential equation involving the Caputo fractional derivative of the form

$$\left({}^C\mathfrak{D}_{0+}^{\beta}\varphi\right)(\tau)-\mu\varphi(\tau)=\Phi(\tau), \tag{2.25}$$

$$\varphi^{(\mathrm{j})}(0)=\mathfrak{c}_{\mathrm{j}}, \tag{2.26}$$

where $0 \leq \tau \leq T$, $\beta \notin \mathbf{N}, \mathfrak{m}-1 < \beta < \mathfrak{m}$ with $\mathfrak{m} \in \mathbf{N}$, $\mu \in \mathbf{R}$, ${}^{C}\mathfrak{D}^{\beta}_{0+}$ denotes the Caputo fractional fractional derivative of order β, and $\mathfrak{c}_{\mathfrak{j}} \in \mathbf{R}$ for $\mathfrak{j} = 0, 1, \cdots, \mathfrak{m}-1$.
By reducing this problem to an equivalent Volterra integral equation and using the method of successive approximations, we derive its explicit solution. The results are obtained under the assumptions of the following theorem:

Theorem 2.2.1. *[16] Let $\beta > 0$ such that $\mathfrak{m}-1 < \beta < \mathfrak{m}$ where $\mathfrak{m} \in \mathbf{N}$. Moreover let $0 \leq \delta < 1$ with $\delta \leq \beta$. Assume that $\mu \in \mathbf{R}$.*

If $\Phi(\tau) \in \mathscr{C}_{\delta}[0, T]$, then there exists a unique solution $\varphi(\tau)$ to the Cauchy problem (2.25)-(2.26) in the space $\mathscr{C}^{\beta,\mathfrak{m}-1}_{\delta}[0, T]$. It is given by

$$\varphi(\tau) = \sum_{\mathfrak{i}=0}^{\mathfrak{m}-1} \mathfrak{c}_{\mathfrak{i}} \tau^{\mathfrak{i}} \mathcal{E}_{\beta,\mathfrak{i}+1}\left(\mu\tau^{\beta}\right) + \int_0^{\tau} (\tau-\theta)^{\beta-1} \mathcal{E}_{\beta,\beta}\left(\mu(\tau-\theta)^{\beta}\right) \Phi(\theta) \mathrm{d}\theta. \quad (2.27)$$

When $\delta = 0$ and $\Phi(\tau) \in \mathscr{C}[0, T]$, the solution $\varphi(\tau)$ in (2.27) belongs to $\mathscr{C}^{\beta,\mathfrak{m}-1}[0, T]$.

Remark 2.2.1. *If $\Phi(\tau) \equiv 0$ (the homogeneous case), there exists a unique solution $\varphi(\tau)$ to the Cauchy problem (2.25)-(2.26) in the space $\mathscr{C}^{\beta,\mathfrak{m}-1}_{\delta}[0, T]$. It is given by*

$$\varphi(\tau) = \sum_{\mathfrak{i}=0}^{\mathfrak{m}-1} \mathfrak{c}_{\mathfrak{i}} \tau^{\mathfrak{i}} \mathcal{E}_{\beta,\mathfrak{i}+1}\left(\mu\tau^{\beta}\right).$$

Example 2.2.1. *Consider the the Cauchy problem*

$$\left({}^{C}\mathfrak{D}^{\beta}_{0+}\varphi\right)(\tau) - \mu\varphi(\tau) = \Phi(\tau) \quad \text{for } 1 < \beta < 2, \quad (2.28)$$

$$\varphi(0) = \mathfrak{c} \in \mathbf{R}, \quad \varphi'(0) = \mathfrak{e} \in \mathbf{R}, \quad (2.29)$$

the solution to the Cauchy problem (2.28)-(2.29) is given by

$$\begin{aligned}\varphi(\tau) = {} & \mathfrak{c}\mathcal{E}_{\beta,1}\left(\mu\tau^{\beta}\right) + \mathfrak{e}\tau\mathcal{E}_{\beta,2}\left(\mu\tau^{\beta}\right) \\ & + \int_0^{\tau} (\tau-\theta)^{\beta-1} \mathcal{E}_{\beta,\beta}\left(\mu(\tau-\theta)^{\beta}\right) \Phi(\theta) \mathrm{d}\theta.\end{aligned}$$

In particular, if $\Phi(\tau) \equiv 0$, the solution to the Cauchy problem (2.28)-(2.29) has the form

$$\varphi(\tau) = \mathfrak{c}\mathcal{E}_{\beta,1}\left(\mu\tau^{\beta}\right) + \mathfrak{e}\tau\mathcal{E}_{\beta,2}\left(\mu\tau^{\beta}\right)$$

Proof of Theorem 2.2.1. Let $\beta > 0$ such that $\mathfrak{m} - 1 < \beta < \mathfrak{m}$ and $0 \le \delta < 1$ with $\delta \le \beta$, assume that $\Phi(\tau) \in \mathscr{C}_\delta[0,T]$. According to Theorem 2.1.1, it follows that the solution to the Cauchy problem (2.25)-(2.26) satisfies in $\mathscr{C}^{\mathfrak{m}-1}[0,T]$ the Volterra integral equation (2.7), we have

$$\varphi(\tau) = \sum_{\mathfrak{i}=0}^{\mathfrak{m}-1} \frac{\mathfrak{c}_\mathfrak{i}}{\mathfrak{i}!}\tau^\mathfrak{i} + \frac{1}{\Gamma(\beta)} \int_0^\tau \frac{(\Phi(\theta) + \mu\varphi(\theta))\mathrm{d}\theta}{(\tau-\theta)^{1-\beta}},$$

yields

$$\varphi(\tau) = \sum_{\mathfrak{i}=0}^{\mathfrak{m}-1} \frac{\mathfrak{c}_\mathfrak{i}}{\mathfrak{i}!}\tau^\mathfrak{i} + \frac{\mu}{\Gamma(\beta)} \int_0^\tau \frac{\varphi(\theta)\mathrm{d}\theta}{(\tau-\theta)^{1-\beta}} + \frac{1}{\Gamma(\beta)} \int_0^\tau \frac{\Phi(\theta)\mathrm{d}\theta}{(\tau-\theta)^{1-\beta}}, \tag{2.30}$$

now, we use the method of successive approximations to solve equation (2.30). First we set

$$\varphi_0(\tau) = \sum_{\mathfrak{i}=0}^{\mathfrak{m}-1} \frac{\mathfrak{c}_\mathfrak{i}}{\mathfrak{i}!}\tau^\mathfrak{i}, \tag{2.31}$$

and

$$\varphi_k(\tau) = \varphi_0(\tau) + \frac{\mu}{\Gamma(\beta)} \int_0^\tau \frac{\varphi_{k-1}(\theta)\mathrm{d}\theta}{(\tau-\theta)^{1-\beta}} + \frac{1}{\Gamma(\beta)} \int_0^\tau \frac{\Phi(\theta)\mathrm{d}\theta}{(\tau-\theta)^{1-\beta}} (k \in \mathbf{N}). \tag{2.32}$$

We can rewrite (2.32) as follows:

$$\varphi_k(\tau) = \varphi_0(\tau) + \mu\left(\mathfrak{T}_{0+}^\beta \varphi_{k-1}\right)(\tau) + \left(\mathfrak{T}_{0+}^\beta \Phi\right)(\tau). \tag{2.33}$$

Using (2.31) together with (1.11), we find, for $k = 1$

$$\begin{aligned}\varphi_1(\tau) &= \sum_{\mathfrak{i}=0}^{\mathfrak{m}-1} \frac{\mathfrak{c}_\mathfrak{i}}{\Gamma(\mathfrak{i}+1)}\tau^\mathfrak{i} + \mu \sum_{\mathfrak{i}=0}^{\mathfrak{m}-1} \frac{\mathfrak{c}_\mathfrak{i}}{\Gamma(\beta+\mathfrak{i}+1)}\tau^{\beta+\mathfrak{i}} + \left(\mathfrak{T}_{0+}^\beta \Phi\right)(\tau) \\ &= \sum_{\mathfrak{i}=0}^{\mathfrak{m}-1} \mathfrak{c}_\mathfrak{i} \sum_{\mathfrak{j}=0}^{1} \frac{\mu^\mathfrak{j}\tau^{\beta\mathfrak{j}+\mathfrak{i}}}{\Gamma(\beta\mathfrak{j}+\mathfrak{i}+1)} + \frac{1}{\Gamma(\beta)} \int_0^\tau (\tau-\theta)^{\beta-1}\Phi(\theta)\mathrm{d}\theta.\end{aligned}$$

And for $k = 2$, we get

$$\begin{aligned}\varphi_2(\tau) &= \sum_{\mathfrak{i}=0}^{\mathfrak{m}-1} \frac{\mathfrak{c}_\mathfrak{i}}{\Gamma(\mathfrak{i}+1)}\tau^\mathfrak{i} \\ &+ \mu \sum_{\mathfrak{i}=0}^{\mathfrak{m}-1} \mathfrak{c}_\mathfrak{i} \sum_{\mathfrak{j}=0}^{1} \frac{\mu^\mathfrak{j}}{\Gamma(\beta\mathfrak{j}+\mathfrak{i}+1)} \left(\mathfrak{T}_{0+}^\beta \theta^{\beta\mathfrak{j}+\mathfrak{i}}\right)(\tau) + \mu\left(\mathfrak{T}_{0+}^\beta \mathfrak{T}_{0+}^\beta \Phi\right)(\tau) + \left(\mathfrak{T}_{0+}^\beta \Phi\right)(\tau),\end{aligned}$$

which, together with (1.16), leads to

$$\varphi_2(\tau) = \sum_{\mathfrak{i}=0}^{\mathfrak{m}-1} \mathfrak{c}_\mathfrak{i} \sum_{\mathfrak{j}=0}^{2} \frac{\mu^\mathfrak{j} \tau^{\beta\mathfrak{j}+\mathfrak{i}}}{\Gamma(\beta\mathfrak{j}+\mathfrak{i}+1)} + \int_0^\tau \left[\sum_{\mathfrak{j}=1}^{2} \frac{\mu^{\mathfrak{j}-1}}{\Gamma(\beta\mathfrak{j})} (\tau-\theta)^{\beta\mathfrak{j}-1} \right] \Phi(\theta) \mathrm{d}\theta.$$

Repeating this process, we obtain, for $k \in \mathbf{N}$

$$\begin{aligned}\varphi_k(\tau) &= \sum_{\mathfrak{i}=0}^{\mathfrak{m}-1} \mathfrak{c}_\mathfrak{i} \sum_{\mathfrak{j}=0}^{k} \frac{\mu^\mathfrak{j} \tau^{\beta\mathfrak{j}+\mathfrak{i}}}{\Gamma(\beta\mathfrak{j}+\mathfrak{i}+1)} + \int_0^\tau \left[\sum_{\mathfrak{j}=1}^{k} \frac{\mu^{\mathfrak{j}-1}}{\Gamma(\beta\mathfrak{j})} (\tau-\theta)^{\beta\mathfrak{j}-1} \right] \Phi(\theta) \mathrm{d}\theta \\ &= \sum_{\mathfrak{i}=0}^{\mathfrak{m}-1} \mathfrak{c}_\mathfrak{i} \tau^\mathfrak{i} \sum_{\mathfrak{j}=0}^{k} \frac{(\mu\tau^\beta)^\mathfrak{j}}{\Gamma(\beta\mathfrak{j}+\mathfrak{i}+1)} + \int_0^\tau (\tau-\theta)^{\beta-1} \sum_{\mathfrak{j}=0}^{k} \frac{(\mu(\tau-\theta)^\beta)^\mathfrak{j}}{\Gamma(\beta\mathfrak{j}+\beta)} \Phi(\theta) \mathrm{d}\theta.\end{aligned}$$

Passing to the limit as $k \to \infty$, having in mind (1.8), we derive

$$\varphi(\tau) = \sum_{\mathfrak{i}=0}^{\mathfrak{m}-1} \mathfrak{c}_\mathfrak{i} \tau^\mathfrak{i} \mathcal{E}_{\beta,\mathfrak{i}+1}\left(\mu\tau^\beta\right) + \int_0^\tau (\tau-\theta)^{\beta-1} \mathcal{E}_{\beta,\beta}\left(\mu(\tau-\theta)^\beta\right) \Phi(\theta) \mathrm{d}\theta. \quad (2.34)$$

This provides an explicit solution to the Cauchy problem (2.25)-(2.26).

Furthermore, the function $\Phi[\tau,\varphi] = \mu\varphi + \Phi(\tau)$ satisfies the Lipschitz condition with respect to the second variable, As a result of Theorem 2.1.2, the Cauchy problem (2.25)-(2.26) has a unique solution in the space $\mathscr{C}_\delta^{\beta,\mathfrak{m}-1}[0,T]$. This concludes the proof. □

2.2.2 Laplace Transform Method for Explicity Solving Linear Fractional Differential Equation

In the present section, we construct the explicit solution to the linear problem (2.25)-(2.26) by using the Laplace transform method.

The general solution to the nonhomogeneous fractional differential equation (2.25) can be expressed as the sum of the general solution to the corresponding homogeneous equation and a particular solution to the nonhomogeneous equation.

Let us start by deriving the explicit solution to the corresponding homogeneous equation:

$$\left({}^{\mathcal{C}}\mathfrak{D}_{0+}^{\beta}\varphi\right)(\tau) - \mu\varphi(\tau) = 0, \quad (2.35)$$

where $\tau > 0$, $\mathfrak{m}-1 < \beta \leq \mathfrak{m}$ with $\mathfrak{m} \in \mathbf{N}$, and $\mu \in \mathbf{R}$.

Theorem 2.2.2. *[16] Let $\mathfrak{m}-1<\beta\leq\mathfrak{m}$ where $\mathfrak{m}\in\mathbf{N}$, and let $\mu\in\mathbf{R}$. The fundamental system of solutions to equation (2.35) is given by the functions*

$$\varphi_{\mathfrak{i}}(\tau)=\tau^{\mathfrak{i}}\mathcal{E}_{\beta,\mathfrak{i}+1}\left(\mu\tau^{\beta}\right)\qquad \textit{where } \mathfrak{i}=0,\cdots,\mathfrak{m}-1.$$

Proof. By applying the Laplace transform to both side of (2.35), we obtain

$$\left(\mathfrak{L}^{\mathcal{C}}\mathfrak{D}_{0+}^{\beta}\varphi\right)(\sigma)-\mu\left(\mathfrak{L}\varphi\right)(\sigma)=0.$$

Using (1.38), we get

$$\sigma^{\beta}(\mathfrak{L}\varphi)(\sigma)-\sum_{\mathfrak{i}=0}^{\mathfrak{m}-1}\sigma^{\beta-\mathfrak{i}-1}\varphi^{(\mathfrak{i})}(0)-\mu\left(\mathfrak{L}\varphi\right)(\sigma)=0,\tag{2.36}$$

yields

$$(\mathfrak{L}\varphi)(\sigma)[\sigma^{\beta}-\mu]=\sum_{\mathfrak{i}=0}^{\mathfrak{m}-1}\sigma^{\beta-\mathfrak{i}-1}\varphi^{(\mathfrak{i})}(0),$$

then

$$(\mathfrak{L}\varphi)(\sigma)=\sum_{\mathfrak{i}=0}^{\mathfrak{m}-1}\mathfrak{a}_{\mathfrak{i}}\frac{\sigma^{\beta-\mathfrak{i}-1}}{\sigma^{\beta}-\mu},\tag{2.37}$$

where

$$\mathfrak{a}_{\mathfrak{i}}=\varphi^{(\mathfrak{i})}(0)\text{ for }\mathfrak{i}=0,\cdots,\mathfrak{m}-1.\tag{2.38}$$

From (1.35) with $\gamma=\mathfrak{i}+1$ and $\delta=\beta$, it follows that

$$(\mathfrak{L}\varphi)(\sigma)=\sum_{\mathfrak{i}=0}^{\mathfrak{m}-1}\mathfrak{a}_{\mathfrak{i}}\mathfrak{L}\left[\tau^{\mathfrak{i}}\mathcal{E}_{\beta,\mathfrak{i}+1}\left(\mu\tau^{\beta}\right)\right](\sigma)\quad\left(\left|\mu\sigma^{-\beta}\right|<1\right).\tag{2.39}$$

Applying the inverse Laplace transform to both side of (2.39), we derive the following expression for the solution to the homogeneous equation (2.35)

$$\varphi(\tau)=\sum_{\mathfrak{i}=0}^{\mathfrak{m}-1}\mathfrak{a}_{\mathfrak{i}}\varphi_{\mathfrak{i}}(\tau),\quad \varphi_{\mathfrak{i}}(\tau)=\tau^{\mathfrak{i}}\mathcal{E}_{\beta,\mathfrak{i}+1}\left(\mu\tau^{\beta}\right).\tag{2.40}$$

Note that $\{\varphi_0(\tau),\cdots,\varphi_{\mathfrak{m}-1}(\tau)\}$ are solutions to the homogeneous equation (2.35), indeed, for all $\mathfrak{i}=0,\cdots,\mathfrak{m}-1$

$$\left({}^{\mathcal{C}}\mathfrak{D}_{0+}^{\beta}\left[\tau^{\mathfrak{i}}\mathcal{E}_{\beta,\mathfrak{i}+1}\left(\mu\tau^{\beta}\right)\right]\right)(\tau)-\mu\tau^{\mathfrak{i}}\mathcal{E}_{\beta,\mathfrak{i}+1}\left(\mu\tau^{\beta}\right)=0.$$

On the other hand, the $\mathfrak{j}$-th derivative of $\varphi_{\mathfrak{i}}(\tau)$ is given by

$$\varphi_{\mathfrak{i}}^{(\mathfrak{j})}(\tau) = \tau^{\mathfrak{i}-\mathfrak{j}}\mathcal{E}_{\beta,\mathfrak{i}-\mathfrak{j}+1}\left(\mu\tau^{\beta}\right). \tag{2.41}$$

Now, if the solutions $\{\varphi_0(\tau), \cdots, \varphi_{\mathfrak{m}-1}(\tau)\}$ are linearly independent then these solutions form the fundamental system of solutions of the homogeneous equation (2.35). To verify the linear independence of these functions, we use the Wronskian method.

From (2.41), we have

$$\varphi_{\mathfrak{j}}^{(\mathfrak{j})}(0) = 1 \text{ for } \mathfrak{j} = 0, \cdots, \mathfrak{m}-1, \tag{2.42}$$

$$\varphi_{\mathfrak{i}}^{(\mathfrak{j})}(0) = 0 \text{ for } \mathfrak{j}, \mathfrak{i} = 0, \cdots, \mathfrak{m}-1 \text{ and } \mathfrak{i} > \mathfrak{j}. \tag{2.43}$$

If $\mathfrak{i} < \mathfrak{j}$, using the definition (1.8), we can rewrite (2.41) as

$$\varphi_{\mathfrak{i}}^{(\mathfrak{j})}(\tau) = \mu\tau^{\beta+\mathfrak{i}-\mathfrak{j}}\mathcal{E}_{\beta,\beta+\mathfrak{i}-\mathfrak{j}+1}\left(\mu\tau^{\beta}\right),$$

since $\beta + \mathfrak{i} - \mathfrak{j} > 0$ for all $\mathfrak{j}, \mathfrak{i} = 0, \cdots, \mathfrak{m}-1$, we get

$$\varphi_{\mathfrak{i}}^{(\mathfrak{j})}(0) = 0 \text{ for } \mathfrak{j}, \mathfrak{i} = 0, \cdots, \mathfrak{m}-1 \text{ and } \mathfrak{i} < \mathfrak{j}. \tag{2.44}$$

Now, we define the Wronskian as follows:

$$\mathscr{W}(\tau) = \det\left(\varphi_{\mathfrak{i}}^{(\mathfrak{j})}(\tau)\right)_{\mathfrak{j},\mathfrak{i}=0}^{\mathfrak{m}-1}.$$

Combining (2.42)-(2.43) and (2.44), it follows that

$\mathscr{W}(0) = 1 \neq 0$. Consequently, the solutions $\{\varphi_0(\tau), \cdots, \varphi_{\mathfrak{m}-1}(\tau)\}$ are linearly independent and form the fundamental system of solutions to the homogeneous equation (2.35). □

Corollary 2.2.3. *Consider the fractional diferential equation of the form:*

$$\left({}^{\mathcal{C}}\mathfrak{D}_{0+}^{\beta}\varphi\right)(\tau) - \mu\varphi(\tau) = 0 \quad \textit{for each } \tau > 0, \textit{ and } 0 < \beta \leq 1.$$

The solution to this equation is given by

$$\varphi(\tau) = \mathcal{E}_{\beta}\left(\mu\tau^{\beta}\right).$$

For the following fractional diferential equation:

$$\left({}^{\mathcal{C}}\mathfrak{D}_{0+}^{\beta}\varphi\right)(\tau) - \mu\varphi(\tau) = 0 \quad \textit{for each } \tau > 0, \textit{ and } 1 < \beta \leq 2.$$

The fundamental system of solutions is formed by the following linearly independent functions:

$$\varphi_1(\tau) = \mathcal{E}_{\beta}\left(\mu\tau^{\beta}\right), \quad \varphi_2(\tau) = \tau\mathcal{E}_{\beta,2}\left(\mu\tau^{\beta}\right).$$

Now by using the same approach, we will determine the particular solutions to the corresponding nonhomogeneous equations:

$$\left({}^{C}\mathfrak{D}_{0+}^{\beta}\varphi\right)(\tau)-\mu\varphi(\tau)=\Phi(\tau)\quad \text{for } \tau>0 \text{ and } \quad \beta>0. \tag{2.45}$$

Theorem 2.2.4. *[16] Let $\beta>0$ and $\mu\in\mathbf{R}$, also let $\Phi:\mathbf{R}_{+}\to\mathbf{R}$ be a given function. Then the equation (2.45) is solvable, and the function*

$$\varphi(\tau)=\int_{0}^{\tau}(\tau-\theta)^{\beta-1}\mathcal{E}_{\beta,\beta}\left(\mu(\tau-\theta)^{\beta}\right)\Phi(\theta)\mathrm{d}\theta \tag{2.46}$$

is a particular solution for it, provided that the integral in (2.46) is convergent.

Proof. Applying the Laplace transform to both sides of (2.45) and using (1.38), we get

$$(\mathfrak{L}\varphi)(\sigma)=\frac{(\mathfrak{L}\Phi)(\sigma)}{\sigma^{\beta}-\mu}, \tag{2.47}$$

then, by applying the inverse Laplace transform to both sides of (2.47), we derive a particular solution to the equation (2.46) of the form:

$$\varphi(\tau)=\left(\mathfrak{L}^{-1}\left[\frac{(\mathfrak{L}\Phi)(\sigma)}{\sigma^{\beta}-\mu}\right]\right)(\tau). \tag{2.48}$$

Using the Laplace convolution formula (1.36), which states:

$$\mathfrak{L}^{-1}\left[(\mathfrak{L}\Phi)(\sigma)(\mathfrak{L}\Psi)(\sigma)\right](\tau)=\int_{0}^{\tau}\Psi(\tau-\theta)\Phi(\theta)\mathrm{d}\theta,$$

by setting

$$(\mathfrak{L}\Psi)(\sigma)=\frac{1}{\sigma^{\beta}-\mu}.$$

From (1.35) with $\delta=\gamma=\beta$, it follows

$$\Psi(\tau)=\tau^{\beta-1}\mathcal{E}_{\beta,\beta}\left(\mu\tau^{\beta}\right),$$

thus

$$\varphi(\tau)=\int_{0}^{\tau}(\tau-\theta)^{\beta-1}\mathcal{E}_{\beta,\beta}\left(\mu(\tau-\theta)^{\beta}\right)\Phi(\theta)d\theta.$$

□

Example 2.2.2. *Consider the differential equation of second order*

$$\varphi^{(2)}(\tau) - \mu\varphi(\tau) = \Phi(\tau) \quad \text{for } \tau > 0 \text{ and } \mu \in \mathbf{R},$$

then, its particular solution is given by

$$\varphi(\tau) = \int_0^\tau (\tau-\theta)\mathcal{E}_{2,2}\left[\mu(\tau-\theta)^2\right]\Phi(\theta)\mathrm{d}\theta,$$

we derive that

$$\varphi(\tau) = \int_0^\tau (\tau-\theta)\frac{\sinh[\sqrt{\mu(\tau-\theta)^2}]}{\sqrt{\mu(\tau-\theta)^2}}\Phi(\theta)\mathrm{d}\theta,$$

if $\mu < 0$

$$\varphi(\tau) = \int_0^\tau \frac{\sinh\left[\sqrt{(-\mu)}(\tau-\theta)\right]}{\sqrt{(-\mu)}}\Phi(\theta)\mathrm{d}\theta,$$

if $\mu > 0$

$$\varphi(\tau) = \int_0^\tau \frac{\sinh[\sqrt{\mu}(\tau-\theta)]}{\sqrt{\mu}}\Phi(\theta)\mathrm{d}\theta$$

Theorem 2.2.5. *[16] Let* $\mathfrak{m}-1 < \beta \leq \mathfrak{m}$, *where* $\mathfrak{m} \in \mathbf{N}$, *and let* $\mu \in \mathbf{R}$. *Moreover, let* $\Phi : \mathbf{R}_+ \to \mathbf{R}$ *be a given function. Then, the fractional differential equation (2.45) is solvable, and the function*

$$\varphi(\tau) = \sum_{\mathfrak{i}=0}^{\mathfrak{m}-1} \mathfrak{e}_\mathfrak{i}\tau^\mathfrak{i}\mathcal{E}_{\beta,\mathfrak{i}+1}\left(\mu\tau^\beta\right) + \int_0^\tau (\tau-\theta)^{\beta-1}\mathcal{E}_{\beta,\beta}\left(\mu(\tau-\theta)^\beta\right)\Phi(\theta)\mathrm{d}\theta$$

is a general solution for it, where $\mathfrak{e}_\mathfrak{i} \in \mathbb{R}$ *for* $\mathfrak{i} = 0, \cdots, \mathfrak{m}-1$ *are arbitrary constants.*

When, $0 < \beta \leq 1$, *the general solutions to (2.45) is given by:*

$$\varphi(\tau) = \mathfrak{e}_0\mathcal{E}_{\beta,1}\left(\mu\tau^\beta\right) + \int_0^\tau (\tau-\theta)^{\beta-1}\mathcal{E}_{\beta,\beta}\left(\mu(\tau-t)^\beta\right)\Phi(\theta)\mathrm{d}\theta.$$

While, when $1 < \beta \leq 2$, *the general solutions to (2.45) is given by:*

$$\varphi(\tau) = \mathfrak{e}_1\mathcal{E}_{\beta,1}\left(\mu\tau^\beta\right) + \mathfrak{e}_2\tau\mathcal{E}_{\beta,2}\left(\mu\tau^\beta\right) + \int_0^\tau (\tau-\theta)^{\beta-1}\mathcal{E}_{\beta,\beta}\left(\mu(\tau-\theta)^\beta\right)\Phi(\theta)\mathrm{d}\theta,$$

where $\mathfrak{e}_1, \mathfrak{e}_2 \in \mathbf{R}$ *are arbitrary constants.*

Cauchy Problem for Fractional Differential Equation

The explicit solutions for the fractional differential equations derived in the previous section, can be directly applied to solve initial value problems for such equations. For instance, we consider the Cauchy problem

$$\left({}^{C}\mathfrak{D}_{0+}^{\beta}\varphi\right)(\tau) - \mu\varphi(\tau) = \Phi(\tau), \tag{2.49}$$

$$\varphi^{(\mathrm{j})}(0) = \mathfrak{c}_{\mathrm{j}}, \tag{2.50}$$

where $\tau > 0$, $\mathfrak{m} - 1 < \beta \leq \mathfrak{m}$ with $\mathfrak{m} \in \mathbf{N}$, $\mu \in \mathbf{R}$ and $\mathfrak{c}_{\mathrm{j}} \in \mathbf{R}$ for $\mathrm{j} = 0, \cdots, \mathfrak{m} - 1$.

The following result follows from Theorem 2.2.5.

Proposition 2.2.6. *[16] Let $\mathfrak{m} - 1 < \beta \leq \mathfrak{m}$ where $\mathfrak{m} \in \mathbf{N}$, and let $\mu \in \mathbf{R}$. Also let $\Phi : \mathbf{R}_{+} \to \mathbf{R}$ be a given function. Then, the Cauchy problem (2.49)-(2.50) is solvable, and the function*

$$\varphi(\tau) = \sum_{\mathrm{i}=0}^{\mathfrak{m}-1} \mathfrak{c}_{\mathrm{i}}\tau^{\mathrm{i}}\mathcal{E}_{\beta,\mathrm{i}+1}\left(\mu\tau^{\beta}\right) + \int_{0}^{\tau}(\tau-\theta)^{\beta-1}\mathcal{E}_{\beta,\beta}\left(\mu(\tau-\theta)^{\beta}\right)\Phi(\theta)\mathrm{d}\theta$$

is a solution for it.

Chapter 3

LINEAR FRACTIONAL DIFFUSION EQUATION

In this chapter, we study the linear fractional diffusive equation of the form:

$$\begin{cases} {}^{\mathcal{C}}\mathfrak{D}_{\tau}^{\beta+1}\varphi - \Delta\varphi = \Phi(\tau,\zeta), \\ \varphi(0,\zeta) = \varphi_0(\zeta), \\ \varphi_\tau(0,\zeta) = \varphi_1(\zeta), \end{cases} \tag{3.1}$$

where $\tau \geq 0, \zeta \in \mathbf{R}^{\mathfrak{m}}, 0 < \beta < 1$, and ${}^{\mathcal{C}}\mathfrak{D}_{\tau}^{\beta+1}$ denotes the Caputo fractional derivative of order $\beta + 1$ with respect to time τ, Δ is the Laplace operator in space.

The linear fractional diffusion equation described in (3.1), acts as an interpolation between two basic equations of mathematical physics, the linear wave equation which is obtained when $\beta = 1$ and the linear heat equation, which is obtained when $\beta = 0$.

3.1 Explicit Solution to the Linear Fractional Diffusion Equation

In this section, we derive the explicit solution to the linear problem (3.1), building on the results established in the previous chapter. Indeed,

consider the linear problem (3.1). By applying the Fourier transform with respect to the spatial variable ζ, denoted as $\mathfrak{F}_\zeta$ to both sides of (3.1) and using (1.31), we obtain

$$\begin{cases} {}^{C}\mathfrak{D}_{\tau}^{\beta+1}\hat{\varphi} + |\xi|^2\hat{\varphi} = \hat{\Phi}(\tau,\xi), \qquad \text{for } \tau > 0, \\ \hat{\varphi}(0,\xi) = \hat{\varphi}_0(\xi), \\ \hat{\varphi}_\tau(0,\xi) = \hat{\varphi}_1(\xi), \end{cases}$$

from Theorem 2.2.1, it follows

$$\begin{aligned} \hat{\varphi}(\tau,\xi) &= \hat{\varphi}_0(\xi)\mathcal{E}_{\beta+1,1}\left(-|\xi|^2\tau^{\beta+1}\right) + \hat{\varphi}_1(\xi)\tau\mathcal{E}_{\beta+1,2}\left(-|\xi|^2\tau^{\beta+1}\right) \\ &\quad + \int_0^\tau (\tau-\theta)^\beta \mathcal{E}_{\beta+1,\beta+1}\left(-|\xi|^2(\tau-\theta)^{\beta+1}\right)\hat{\Phi}(\theta,\xi)\mathrm{d}\theta, \end{aligned} \tag{3.2}$$

where $\mathcal{E}_{1+\beta,\eta}$ is the two-parameter Mittag-Leffler function defined in (1.8).

Applying the invers Fourier transform to both sides of (3.2), we obtain

$$\begin{aligned} \varphi(\tau,\zeta) &= \mathfrak{F}_\xi^{-1}[\hat{\varphi}_0(\xi)\mathcal{E}_{\beta+1,1}\left(-|\xi|^2\tau^{\beta+1}\right)](\zeta) + \mathfrak{F}_\xi^{-1}[\hat{\varphi}_1(\xi)\tau\mathcal{E}_{\beta+1,2}\left(-|\xi|^2\tau^{\beta+1}\right)](\zeta) \\ &\quad + \int_0^\tau (\tau-\theta)^\beta \mathfrak{F}^{-1}[\mathcal{E}_{\beta+1,\beta+1}\left(-|\xi|^2(\tau-\theta)^{\beta+1}\right)\hat{\Phi}(\theta,\xi)](\zeta)\mathrm{d}\theta. \end{aligned} \tag{3.3}$$

Defining $\mathcal{K}_{1+\beta,\eta}(\tau,\zeta) = \mathfrak{F}_\xi^{-1}\left(\mathcal{E}_{1+\beta,\eta}\left(-\tau^{1+\beta}|\xi|^2\right)\right)$ where $\eta = 1, 1+\beta, 2$, and Substituting in (3.3), we get

$$\begin{aligned} \varphi(\tau,\zeta) &= \mathfrak{F}_\xi^{-1}[\hat{\varphi}_0(\xi)\mathfrak{F}_\xi[\mathcal{K}_{1+\beta,1}(\tau,\zeta)](\xi)](\zeta) + \tau\mathfrak{F}_\xi^{-1}[\hat{\varphi}_1(\xi)\mathfrak{F}_\xi[\mathcal{K}_{1+\beta,2}(\tau,\zeta)](\xi)](\zeta) \\ &\quad + \int_0^\tau (\tau-\theta)^\beta \mathfrak{F}^{-1}[\mathfrak{F}_\xi[\mathcal{K}_{1+\beta,1+\beta}(\tau-\theta,\zeta)](\xi)]\hat{\Phi}(\theta,\xi)](\zeta)\mathrm{d}\theta, \end{aligned} \tag{3.4}$$

using (1.32), then the solution to the problem (3.1) can be expressed in the following form

$$\begin{aligned} \varphi(\tau,\cdot) &= \mathcal{K}_{1+\beta,1}(\tau,\cdot) *_{(\zeta)} \varphi_0 + \tau\mathcal{K}_{1+\beta,2}(\tau,\cdot) *_{(\zeta)} \varphi_1 \\ &\quad + \int_0^\tau (\tau-\theta)^\beta \mathcal{K}_{1+\beta,1+\beta}(\tau-\theta,\cdot) *_{(\zeta)} \Phi((\theta,\cdot))\mathrm{d}\theta, \end{aligned} \tag{3.5}$$

where

$$\mathcal{K}_{1+\beta,\eta}(\tau,\zeta) = \mathfrak{F}^{-1}\left(\mathcal{E}_{1+\beta,\eta}\left(-\tau^{1+\beta}|\xi|^2\right)\right).$$

3.2 $\mathcal{L}^{\upsilon} - \mathcal{L}^{\rho}$ Estimates for the Solution to the Linear Fractional Diffusion Equation

In order to derive $\mathcal{L}^{\upsilon} - \mathcal{L}^{\rho}$ estimates for the solution to the linear problem (3.1), where $1 \leq \upsilon \leq \rho \leq \infty$. we estimate $\mathcal{K}_{1+\beta,\eta}(t,\cdot)$ in $\mathcal{L}^{\mu}$ norms. Taking into account the scaling property (see [6]):

$$\left\| \mathfrak{F}^{-1}\left(\mathbb{M}\left(\tau^{\frac{1}{\kappa}} |\cdot|^2 \right)\right) \right\|_{\mathcal{L}^{\mu}} = \tau^{-\frac{\mathfrak{m}}{2\kappa}\left(1-\frac{1}{\mu}\right)} \left\| \mathfrak{F}^{-1}\left(\mathbb{M}\left(|\cdot|^2 \right)\right) \right\|_{\mathcal{L}^{\mu}}, \tag{3.6}$$

it suffices to estimate $\mathcal{K}_{1+\beta,\eta}(1,\cdot)$ in $\mathcal{L}^{\mu}$ norms, where $\eta = 1, \beta+1, 2$.

Proposition 3.2.1. *[6]*

1. *For all* $0 < \beta < 1$ *and for all* $1 \leq \mu \leq \infty$, *if*
$$1 - \frac{1}{\mu} < \frac{2}{\mathfrak{m}},$$
then
$$\mathcal{K}_{1+\beta,1}(1,\cdot) \in \mathcal{L}^{\mu}$$
2. *For all* $0 < \beta < 1$ *and for all* $1 \leq \mu \leq \infty$, *if*
$$1 - \frac{1}{\mu} < \frac{4}{\mathfrak{m}},$$
then
$$\mathcal{K}_{1+\beta,1+\beta}(1,\cdot) \in \mathcal{L}^{\mu}$$
3. *For all* $0 < \beta < 1$ *and for all* $1 \leq \mu \leq \infty$, *if*
$$1 - \frac{1}{\mu} < \frac{2}{\mathfrak{m}},$$
then
$$\mathcal{K}_{1+\beta,2}(1,\cdot) \in \mathcal{L}^{\mu}$$

In what follows, we will make use of the following lemma.

Lemma 3.2.2. *[6]*

Let $\mathfrak{p} < 1$ *and* $\mathfrak{q} \in \mathbf{R}$. *Then:*

$$\int_0^\tau (\tau-\theta)^{-\mathfrak{p}}(1+\theta)^{-\mathfrak{q}}\mathrm{d}\theta \lesssim \begin{cases} (1+\tau)^{-\mathfrak{p}}, & \text{if } \mathfrak{p} < 1 < \mathfrak{q}, \\ (1+\tau)^{-1}\log(1+\tau), & \text{if } \mathfrak{p} < 1 = \mathfrak{q}, \\ (1+\tau)^{1-\mathfrak{p}-\mathfrak{q}}, & \text{if } \mathfrak{p}, \mathfrak{q} < 1. \end{cases} \tag{3.7}$$

The following theorem establishes $\mathcal{L}^{\upsilon} - \mathcal{L}^{\rho}$ estimates for the solution to the linear problem (3.1), where $1 \leq \upsilon \leq \rho \leq \infty$.

Theorem 3.2.3. *[6] Let* $\mathfrak{m} \geq 1$ *and* $\rho \in [1,\infty]$. *Assume that* $\varphi_0 \in \mathcal{L}^{\upsilon_0}, \varphi_1 \in \mathcal{L}^{\upsilon_1}$, *and that* $\Phi(\tau,\cdot) \in \mathcal{L}^{\upsilon_2}$, *with* $\upsilon_i \in [1,\rho]$, *satisfying*

$$\frac{1}{\upsilon_i} - \frac{1}{\rho} < \frac{2}{\mathfrak{m}}, \tag{3.8}$$

for $i = 0,1,2$. *Assume that*

$$\|\Phi(\tau,\cdot)\|_{\mathcal{L}^{\upsilon_2}} \leq \mathfrak{K}(1+\tau)^{-\sigma}, \quad \forall \tau \geq 0, \tag{3.9}$$

for some $\mathfrak{K} > 0$ *and* $\sigma \in \mathbf{R}$. *Then the solution to (3.1) verifies the following estimate:*

$$\begin{aligned} \|\varphi(\tau,\cdot)\|_{\mathcal{L}^{\rho}} &\leq \mathfrak{c}\tau^{-\frac{\mathfrak{m}(1+\beta)}{2}\left(\frac{1}{\upsilon_0}-\frac{1}{\rho}\right)}\|\varphi_0\|_{\mathcal{L}^{\upsilon_0}} + \mathfrak{c}\tau^{1-\frac{\mathfrak{m}(1+\beta)}{2}\left(\frac{1}{\upsilon_1}-\frac{1}{\rho}\right)}\|\varphi_1\|_{\mathcal{L}^{\upsilon_1}} \\ &+ \begin{cases} \mathfrak{c}\mathfrak{K}(1+\tau)^{\beta-\frac{\mathfrak{m}(1+\beta)}{2}\left(\frac{1}{\upsilon_2}-\frac{1}{\rho}\right)} & \text{if } \sigma > 1, \\ \mathfrak{c}\mathfrak{K}(1+\tau)^{\beta-\frac{\mathfrak{m}(1+\beta)}{2}\left(\frac{1}{\upsilon_2}-\frac{1}{\rho}\right)}\log(1+\tau) & \text{if } \sigma = 1, \\ \mathfrak{c}\mathfrak{K}(1+\tau)^{1-\sigma+\beta-\frac{\mathfrak{m}(1+\beta)}{2}\left(\frac{1}{\upsilon_2}-\frac{1}{\rho}\right)} & \text{if } \sigma < 1, \end{cases} \end{aligned}$$

for any $\tau > 0$, *where* $\mathfrak{c}$ *does not depend on the data.*

Proof. Consider the linear problem (3.1), assume that $\varphi_0 \in \mathcal{L}^{\upsilon_0}$, $\varphi_1 \in \mathcal{L}^{\upsilon_1}$ and $\Phi(\tau,\cdot) \in \mathcal{L}^{\upsilon_2}$ satisfy (3.8). The solution to the problem can be expressed in the following form:

$$\begin{aligned} \varphi(\tau,\cdot) &= \mathcal{K}_{1+\beta,1}(\tau,\cdot) *_{(\zeta)} \varphi_0 + \tau\mathcal{K}_{1+\beta,2}(\tau,\cdot) *_{(\zeta)} \varphi_1 \\ &\quad + \int_0^\tau (\tau-\theta)^{\beta}\mathcal{K}_{1+\beta,1+\beta}(\tau-\theta,\cdot) *_{(\zeta)} \Phi((\theta,\cdot))\mathrm{d}\theta. \end{aligned}$$

For $\rho \in [1,\infty]$, the $\mathcal{L}^\rho$ norm of the solution satisfies:

$$\begin{aligned}\|\varphi(\tau,\cdot)\|_{\mathcal{L}^\rho} \leq \|\mathcal{K}_{1+\beta,1}(\tau,\cdot) *_{(\zeta)} \varphi_0\|_{\mathcal{L}^\rho} + \tau\|\mathcal{K}_{1+\beta,2}(\tau,\cdot) *_{(\zeta)} \varphi_1\|_{\mathcal{L}^\rho} \\ + \int_0^\tau (\tau-\theta)^\beta \|\mathcal{K}_{1+\beta,1+\beta}(\tau-\theta,\cdot) *_{(\zeta)} \Phi((\theta,\cdot))\|_{\mathcal{L}^\rho} \mathrm{d}\theta.\end{aligned} \tag{3.10}$$

By applying Young's convolution inequality, for $\rho, \mu \in [1,\infty]$ and $\upsilon_0 \in [1,\rho]$ such that $\dfrac{1}{\rho} + 1 = \dfrac{1}{\mu} + \dfrac{1}{\upsilon_0}$, we obtain

$$\|\mathcal{K}_{1+\beta,1}(\tau,\cdot) *_{(\zeta)} \varphi_0\|_{\mathcal{L}^\rho} \leq \|\mathcal{K}_{1+\beta,1}(\tau,\cdot)\|_{\mathcal{L}^\mu} \|\varphi_0\|_{\mathcal{L}^{\upsilon_0}}. \tag{3.11}$$

Using the scaling property (3.6), it follows that

$$\|\mathcal{K}_{1+\beta,1}(\tau,\cdot) *_{(\zeta)} \varphi_0\|_{\mathcal{L}^\rho} \leq \tau^{-\frac{\mathfrak{m}(1+\beta)}{2}\left(\frac{1}{\upsilon_0}-\frac{1}{\rho}\right)} \|\mathcal{K}_{1+\beta,1}(1,\cdot)\|_{\mathcal{L}^\mu} \|\varphi_0\|_{\mathcal{L}^{\upsilon_0}}. \tag{3.12}$$

From Proposition 3.2.1, under the condition (3.8), which is equivalent to

$$1 - \frac{1}{\mu} < \frac{2}{\mathfrak{m}},$$

we derive

$$\|\mathcal{K}_{1+\beta,1}(\tau,\cdot) *_{(\zeta)} \varphi_0\|_{\mathcal{L}^\rho} \leq \tau^{-\frac{\mathfrak{m}(1+\beta)}{2}\left(\frac{1}{\upsilon_0}-\frac{1}{\rho}\right)} \|\varphi_0\|_{\mathcal{L}^{\upsilon_0}}. \tag{3.13}$$

By the same process, we obtain

$$\|\mathcal{K}_{1+\beta,2}(\tau,\cdot) *_{(\zeta)} \varphi_1\|_{\mathcal{L}^\rho} \leq \tau^{-\frac{\mathfrak{m}(1+\beta)}{2}\left(\frac{1}{\upsilon_1}-\frac{1}{\rho}\right)} \|\varphi_1\|_{\mathcal{L}^{\upsilon_1}}. \tag{3.14}$$

Similarly, for the part related to the term Φ, we have

$$\|\mathcal{K}_{1+\beta,1+\beta}(\tau-\theta,\cdot) *_{(\zeta)} \Phi(\theta,\cdot)\|_{\mathcal{L}^\rho} \leq (\tau-\theta)^{-\frac{\mathfrak{m}(1+\beta)}{2}\left(\frac{1}{\upsilon_2}-\frac{1}{\rho}\right)} \|\Phi(\theta,\cdot)\|_{\mathcal{L}^{\upsilon_2}}. \tag{3.15}$$

Under the assumption (3.9), the estimation (3.15) yields

$$\|\mathcal{K}_{1+\beta,1+\beta}(\tau-\theta,\cdot) *_{(\zeta)} \Phi(\theta,\cdot)\|_{\mathcal{L}^\rho} \leq \mathfrak{K}(\tau-\theta)^{-\frac{\mathfrak{m}(1+\beta)}{2}\left(\frac{1}{\upsilon_2}-\frac{1}{\rho}\right)} (1+\theta)^{-\sigma}.$$

It follows that

$$\int_0^\tau (\tau-\theta)^\beta \|\mathcal{K}_{1+\beta,1+\beta}(\tau-\theta,\cdot) *_{(\zeta)} \Phi(\theta,\cdot)\|_{\mathcal{L}^\rho} \mathrm{d}\theta \leq \mathfrak{K} \int_0^\tau (\tau-\theta)^{\beta-\frac{\mathfrak{m}(1+\beta)}{2}\left(\frac{1}{\upsilon_2}-\frac{1}{\rho}\right)} (1+\theta)^{-\sigma} \mathrm{d}\theta$$

Applying Lemma (3.2.2) with $\mathfrak{p} = \frac{\mathfrak{m}(1+\beta)}{2}\left(\frac{1}{\upsilon_2} - \frac{1}{\rho}\right) - \beta$ and $\mathfrak{q} = \sigma$, we obtain

$$\begin{aligned} &\int_0^\tau (\tau-\theta)^\beta \|\mathcal{K}_{1+\beta,1+\beta}(\tau-\theta,\cdot) *_{(\zeta)} \Phi(\theta,\cdot)\|_{\mathcal{L}^\rho} \mathrm{d}\theta \\ &\lesssim \begin{cases} (1+\tau)^{\beta - \frac{\mathfrak{m}(1+\beta)}{2}\left(\frac{1}{\upsilon_2}-\frac{1}{\rho}\right)}, & \text{if } \sigma > 1, \\ (1+\tau)^{-1}\log(1+\tau), & \text{if } \sigma = 1, \\ (1+\tau)^{1+\beta-\frac{\mathfrak{m}(1+\beta)}{2}\left(\frac{1}{\upsilon_2}-\frac{1}{\rho}\right)-\sigma}, & \text{if } \sigma < 1. \end{cases} \end{aligned} \tag{3.16}$$

Combining (3.13), (3.14) with (3.16), we obtain the estimate of the solution.
□

Chapter 4

NONLINEAR MEMORY TERM FOR FRACTIONAL DIFFUSION EQUATION

This chapter focuses on the study of the Cauchy problem, represented by the following equation:

$$^{C}\mathfrak{D}_{\tau}^{\beta+1}\varphi-\Delta\varphi = \int_0^{\tau} (\tau-\theta)^{-\sigma} \left|\varphi(\theta,.)\right|^{\mu} \mathrm{d}\theta, \quad \varphi(0,\zeta) = \varphi_0(\zeta), \quad \varphi_{\tau}(0,\zeta) = \varphi_1(\zeta). \tag{4.1}$$

where $\tau \in [0,\infty), \zeta \in \mathbf{R}^{\mathfrak{m}}$, $\beta, \sigma \in (0,1)$ and $\mu > 1$. $^{C}\mathfrak{D}_{\tau}^{\beta+1}\varphi$ denotes the Caputo fractional derivative of order $\beta+1$ with respect to time τ and the integral $\int_0^{\tau}(\tau-\theta)^{-\sigma} \left|\varphi(\theta,.)\right|^{\mu} \mathrm{d}\theta$ represents the memory term. We prove the global existence of solutions to the Cauchy problem (4.1) for small initial data. Additionally, we illustrate the influence of the nonlinearity parameter σ and the fractional derivative order β on the range of the exponent μ and the estimation of the solutions. This is achieved by using Banach fixed point theorem.

4.1 Existence and Uniqueness Theorems

We now present our global existence results for the Cauchy problem (4.1). The first result is obtained under the condition that the second initial data

φ_1 is zero, while the second result is derived for the case where both initial data φ_0 and φ_1 are exist.

Theorem 4.1.1. *[3] Let $\mathfrak{m} \geq 1$ and $\beta, \sigma \in (0,1)$ such that $\sigma > \frac{\beta+1}{2}$ if $\mathfrak{m} = 1$, we assume that $\varphi_1 = 0$, and the exponent μ satisfies*

$$\begin{cases} 1 + \dfrac{\beta - \sigma + 2}{\frac{\mathfrak{m}}{2}(1+\beta) - \beta - 1 + \sigma} < \mu & \text{if } \mathfrak{m} = 1, 2, \\ 1 + \dfrac{\beta - \sigma + 2}{\frac{\mathfrak{m}}{2}(1+\beta) - \beta - 1 + \sigma} < \mu < 1 + \dfrac{2}{\mathfrak{m} - 2} & \text{if } \mathfrak{m} \geq 3, \end{cases} \tag{4.2}$$

thus, there exists a small $\epsilon > 0$ such that for all $\varphi_0 \in \mathcal{L}^1 \cap \mathcal{L}^\mu$,

$$\|\varphi_0\|_{\mathcal{L}^1 \cap \mathcal{L}^\mu} \leq \epsilon.$$

Then, the Cauchy problem (4.1) admits a unique global (in time) solution in

$$\varphi \in \mathscr{C}\left([0,\infty), \mathcal{L}^1 \cap \mathcal{L}^\mu\right). \tag{4.3}$$

Moreover, the solution satisfies the following decay estimate:

$$\|\varphi(\tau,\cdot)\|_{\mathcal{L}^\rho} \lesssim (1+\tau)^{1-\sigma+\beta-\frac{\mathfrak{m}}{2}(1+\beta)\left(1-\frac{1}{\rho}\right)} \|\varphi_0\|_{\mathcal{L}^1 \cap \mathcal{L}^\mu}, \tag{4.4}$$

for all $\rho \in [1,\mu]$, and all $\tau \geq 0$.

Example 4.1.1. *the following table provides examples of the acceptable range for p, which depends on $\mathfrak{m}$, β, and σ as specified in the main assumption (4.2) of Theorem 4.1.1.*

$\mathfrak{m}$	σ	β	μ
2	$\frac{1}{3} < \sigma < \frac{2}{3}$	$3\sigma - 1$	$3 + \frac{1}{\sigma} < \mu < \infty$
3	$1 - \frac{\beta}{2}$	$0 < \beta < \frac{2}{3}$	$\frac{5}{3} + \beta < \mu < 3$
4	$\frac{7}{8}$	$\frac{2}{3}$	$\frac{104}{61} < \mu < 2$

Theorem 4.1.2. *[3] Let $\mathfrak{m} \geq 2$ and $\beta, \sigma \in (0,1)$ with $\sigma \geqslant \beta$, we assume that the exponent μ satisfies:*

$$\begin{cases} 1 + \dfrac{2}{\frac{\mathfrak{m}}{2}(1+\beta) - 1} < \mu & \text{if } \mathfrak{m} = 1, 2, \\ 1 + \dfrac{2}{\frac{\mathfrak{m}}{2}(1+\beta) - 1} < \mu < 1 + \dfrac{2}{\mathfrak{m} - 2} & \text{if } \mathfrak{m} \geq 3, \end{cases} \tag{4.5}$$

thus, there exists a small $\epsilon > 0$ such that for all $\varphi_0, \varphi_1 \in \mathcal{L}^1 \cap \mathcal{L}^\mu$,

$$\|\varphi_0\|_{\mathcal{L}^1 \cap \mathcal{L}^\mu} \leq \epsilon, \tag{4.6}$$

$$\|\varphi_1\|_{\mathcal{L}^1 \cap \mathcal{L}^\mu} \leq \epsilon, \tag{4.7}$$

Then, the Cauchy problem (4.1) admits a unique global (in time) solution in

$$\varphi \in \mathscr{C}\left([0, \infty), \mathcal{L}^1 \cap \mathcal{L}^\mu\right). \tag{4.8}$$

Moreover, the solution satisfies the following decay estimate:

$$\|\varphi(\tau, \cdot)\|_{\mathcal{L}^\rho} \lesssim (1+\tau)^{1 - \frac{\mathfrak{m}}{2}(1+\beta)\left(1 - \frac{1}{\rho}\right)} \left(\|\varphi_0\|_{\mathcal{L}^1 \cap \mathcal{L}^\mu} + \|\varphi_1\|_{\mathcal{L}^1 \cap \mathcal{L}^\mu}\right), \tag{4.9}$$

for all $\rho \in [1, \mu]$, and all $\tau \geq 0$.

Remark 4.1.1. *If we take $\mathfrak{m} = 4$, $\beta = \dfrac{2}{3}$, and $\sigma = \dfrac{7}{8}$ as in Example 4.1.1, and assume $\varphi_1 \neq 0$, then the admissible range for μ changes from $\dfrac{104}{61} < \mu < 2$ to $\dfrac{13}{7} < \mu < 2$.*

4.2 Approach and Proofs

In order to proceed with our proofs, we define the solution space as follows:

$$\Omega = \left\{\varphi \in \mathscr{C}\left([0, \infty), \mathcal{L}^1 \cap \mathcal{L}^\mu\right) : \|\varphi\|_\Omega < \infty\right\}.$$

From (3.5), a function $\varphi \in \Omega$ is a solution to the Cauchy problem (4.1) if and only if it satisfies the following equality:

$$\varphi(\tau, \zeta) = \varphi^{\mathfrak{lin}}(\tau, \zeta) + \varphi^{\mathfrak{nlin}}(\tau, \zeta),$$

where

$$\varphi^{\mathfrak{lin}}(\tau, \cdot) = \mathcal{K}_{1+\beta,1}(\tau, \cdot) *_{(\zeta)} \varphi_0 + \tau \mathcal{K}_{1+\beta,2}(\tau, \cdot) *_{(\zeta)} \varphi_1,$$

represents a solution to the Cauchy problem

$$ {}^{C}\mathfrak{D}_{\tau}^{\beta+1}\varphi - \Delta\varphi = 0, \quad \varphi(0,\zeta) = \varphi_0(\zeta), \quad \varphi_\tau(0,\zeta) = \varphi_1(\zeta), $$

and

$$ \varphi^{\mathfrak{nlin}}(\tau,.) := \int_0^\tau (\tau-\theta)^\beta \mathcal{K}_{1+\beta,1+\beta}(\tau-\theta,\cdot) *_{(\zeta)} \int_0^\theta (\theta-\tau)^{-\sigma}|\varphi(\tau,\cdot)|^\mu \,\mathrm{d}\tau\mathrm{d}\theta, $$

represents a solution to the Cauchy problem

$$ {}^{C}\mathfrak{D}_{\tau}^{\beta+1}\varphi - \Delta\varphi = \int_0^\tau (\tau-\theta)^{-\sigma}|\varphi(\tau,\cdot)|^\mu \,\mathrm{d}\theta, \quad \varphi(0,\zeta) = 0, \quad \varphi_\tau(0,\zeta) = 0, $$

and

$$ \mathcal{K}_{1+\beta,\eta}(\tau,\zeta) = \mathfrak{F}^{-1}(\mathcal{E}_{1+\beta,\eta}(-\tau^{1+\beta}|\xi|^2)), $$

where $\mathcal{E}_{1+\beta,\eta}$ is the two-parameter Mittag-Leffler function defined in (1.8).

We transform the Cauchy problem (4.1) into a fixed point problem. we introduce the operator $\mathcal{N}:\Omega\to\Omega$, defined as:

$$ \mathcal{N}\varphi := \mathcal{N}\varphi(\tau,\zeta) = \varphi^{\mathfrak{lin}}(\tau,\zeta) + \varphi^{\mathfrak{nlin}}(\tau,\zeta). $$

To proof our global existence results, We shall show that the operator $\mathcal{N}$ satisfies the assumptions of the Banach's fixed point theorem. The proof follows the scheme. First we define a suitable norm $\|\cdot\|_\Omega$ on the space Ω, which is chosen to reflect the desired decay rates for the solution to the Cauchy problem (4.1). then we prove the following inequalities

$$ \|\mathcal{N}\varphi\|_\Omega \lesssim \|\,(\varphi_0,\varphi_1)\,\|_{L^1\cap L^\mu} + \|\varphi\|_\Omega^\mu, \tag{4.10} $$

$$ \|\mathcal{N}\tilde{\varphi} - \mathcal{N}\hat{\varphi}\|_\Omega \lesssim \|\tilde{\varphi}-\hat{\varphi}\|_\Omega \left(\|\tilde{\varphi}\|_\Omega^{\mu-1} + \|\hat{\varphi}\|_\Omega^{\mu-1}\right). \tag{4.11} $$

Indeed. Let $\Omega_W = \{\varphi\in\Omega : \|\varphi\|_\Omega \le W\}$, where $W = 2\mathbf{C}\,\|\varphi_0\|_{L^1\cap L^\mu}$ is chosen such that the following condition holds:

$$ \mathbf{C}W^{\mu-1} < \frac{1}{2}. \tag{4.12} $$

By using (4.10) and (4.12), we have for each $\varphi\in\Omega_W$

$$ \begin{aligned} \|\mathcal{N}\varphi\|_\Omega &\le \mathbf{C}\,\|(\varphi_0,\varphi_1)\|_{\mathcal{L}^1\cap\mathcal{L}^\mu} + \mathbf{C}\|\varphi\|_\Omega^\mu \\ &\le \frac{W}{2} + \frac{W}{2} \\ &\le W. \end{aligned} $$

It follows that $\mathcal{N}$ maps Ω_W into itself.
Then we show that $\mathcal{N}$ is a contraction on Ω_W, let $\overline{\varphi}, \tilde{\varphi} \in \Omega_W$, using (4.11) and (4.12), it follows

$$\|\mathcal{N}\overline{\varphi} - \mathcal{N}\tilde{\varphi}\|_\Omega \leq 2\mathbf{C}W^{\mu-1}\|\overline{\varphi} - \tilde{\varphi}\|_\Omega,$$

with $2\mathbf{C}W^{\mu-1} < 1$. This will allow us to conclude that $\mathcal{N}$ is a contraction on Ω_W. By Banach's fixed point theorem, we deduce that $\mathcal{N}$ has a unique fixed point φ in Ω_W, which corresponds to the global solution of the Cauchy problem (4.1).

Proof of Theorem 4.1.1. Let

$$\Omega = \left\{\varphi \in \mathscr{C}\left([0,\infty), \mathcal{L}^1 \cap \mathcal{L}^\mu\right) : \|\varphi\|_\Omega < \infty\right\},$$

endowed with the norm

$$\|\varphi\|_\Omega = \sup_{\tau \geq 0}(1+\tau)^{-\beta+\sigma-1}\{\|\varphi(\tau,\cdot)\|_{\mathcal{L}^1} + (1+\tau)^{\frac{\mathfrak{m}}{2}(1+\beta)\left(1-\frac{1}{\mu}\right)}\|\varphi(\tau,\cdot)\|_{\mathcal{L}^\mu}\}. \tag{4.13}$$

Let $\varphi_0 \in \mathcal{L}^1 \cap \mathcal{L}^\mu$. By Theorem 3.2.3, taking $\rho = 1$ and $\vartheta_0 = 1$, and noting that condition (3.8) is satisfied, we get

$$\left\|\varphi^{\mathfrak{lin}}(\tau,\cdot)\right\|_{\mathcal{L}^1} \lesssim \|\varphi_0\|_{\mathcal{L}^1}. \tag{4.14}$$

For the case where $\rho = \mu$, we take $\vartheta_0 = \mu$ for $\tau \in [0,1]$ and $\vartheta_0 = 1$ for $\tau \geq 1$, We observe that condition (3.8) is satisfied for $\mathfrak{m} = 1, 2$, while for $\mathfrak{m} > 2$, an additional condition $\mu < 1 + \frac{2}{\mathfrak{m}-2}$ is generated. Then we obtain

$$\left\|\varphi^{\mathfrak{lin}}(\tau,\cdot)\right\|_{\mathcal{L}^\mu} \lesssim \begin{cases} \|\varphi_0\|_{\mathcal{L}^\mu} & \tau \in [0,1], \\ \tau^{-\frac{\mathfrak{m}}{2}(1+\beta)\left(1-\frac{1}{\mu}\right)}\|\varphi_0\|_{\mathcal{L}^1} & \tau \geq 1, \end{cases}$$

entails

$$\left\|\varphi^{\mathfrak{lin}}(\tau,\cdot)\right\|_{\mathcal{L}^\mu} \lesssim (1+\tau)^{-\frac{\mathfrak{m}}{2}(1+\beta)\left(1-\frac{1}{\mu}\right)}\left(\|\varphi_0\|_{\mathcal{L}^1} + \|\varphi_0\|_{\mathcal{L}^\mu}\right). \tag{4.15}$$

Now we are ready to estimate $\|\varphi^{\mathfrak{lin}}\|_\Omega$. From (4.13),

$$\|\varphi^{\mathfrak{lin}}\|_\Omega = \sup_{\tau \geq 0}(1+\tau)^{-\beta+\sigma-1}\{\|\varphi^{\mathfrak{lin}}(\tau,\cdot)\|_{\mathcal{L}^1} + (1+\tau)^{\frac{\mathfrak{m}}{2}(1+\beta)\left(1-\frac{1}{\mu}\right)}\|\varphi^{\mathfrak{lin}}(\tau,\cdot)\|_{\mathcal{L}^\mu}\}.$$

Using estimates (4.14) and (4.15), we obtain

$$\|\varphi^{\mathfrak{lin}}\|_\Omega \lesssim \sup_{\tau\geq 0}(1+\tau)^{-\beta+\sigma-1}\{\|\varphi_0\|_{\mathcal{L}^1} + (1+\tau)^{\frac{\mathfrak{m}}{2}(1+\beta)\left(1-\frac{1}{\mu}\right)}(1+\tau)^{-\frac{\mathfrak{m}}{2}(1+\beta)\left(1-\frac{1}{\mu}\right)}\left(\|\varphi_0\|_{\mathcal{L}^1} + \|\varphi_0\|_{\mathcal{L}^\mu}\right)\}$$
$$\lesssim \|\varphi_0\|_{\mathcal{L}^1} + \|\varphi_0\|_{\mathcal{L}^\mu}.$$

We conclude that

$$\|\varphi^{\mathfrak{lin}}\|_\Omega \lesssim \|\varphi_0\|_{\mathcal{L}^1\cap\mathcal{L}^\mu}. \tag{4.16}$$

We still have to estimate $\|\varphi^{\mathfrak{nlin}}\|_\Omega$. At this point, we set

$$\Phi(\tau,\zeta) = \int_0^\tau (\tau-\theta)^{-\sigma}|\varphi(\theta,\zeta)|^\mu\,\mathrm{d}\theta.$$

We have

$$\begin{aligned}\|\Phi(\tau,\cdot)\|_{\mathcal{L}^1} &= \left\|\int_0^\tau (\tau-\theta)^{-\sigma}|\varphi(\theta,\cdot)|^\mu\,\mathrm{d}\theta\right\|_{\mathcal{L}^1}\\ &\lesssim \int_0^\tau (\tau-\theta)^{-\sigma}\|\varphi(\theta,\cdot)\|^\mu_{\mathcal{L}^\mu}\,\mathrm{d}\theta.\end{aligned} \tag{4.17}$$

From 4.13, for any $\varphi\in\Omega$

$$\|\varphi\|_\Omega \geq (1+\theta)^{-\beta+\sigma-1}\{\|\varphi(\theta,\cdot)\|_{\mathcal{L}^1} + (1+\theta)^{\frac{\mathfrak{m}}{2}(1+\beta)\left(1-\frac{1}{\mu}\right)}\|\varphi(\theta,\cdot)\|_{\mathcal{L}^\mu}\},$$

it follows that

$$\|\varphi(\theta,\cdot)\|_{\mathcal{L}^\mu} \leq (1+\theta)^{\beta-\sigma+1-\frac{\mathfrak{m}}{2}(1+\beta)\left(1-\frac{1}{\mu}\right)}\|\varphi\|_\Omega,$$

then

$$\|\varphi(\theta,\cdot)\|^\mu_{\mathcal{L}^\mu} \leq (1+\theta)^{\mu\beta+\mu(1-\sigma)-\frac{\mathfrak{m}}{2}(1+\beta)(\mu-1)}\|\varphi\|^\mu_\Omega,$$

Combing this last estimate with (4.17), and setting $\lambda = -\mu\beta-\mu(1-\sigma)+\frac{\mathfrak{m}}{2}(1+\beta)(\mu-1)$, we obtain

$$\|\Phi(\tau,\cdot)\|_{\mathcal{L}^1} \lesssim \|\varphi\|^\mu_\Omega\int_0^\tau (\tau-\theta)^{-\sigma}(1+\theta)^{-\lambda}\,\mathrm{d}\theta.$$

Assume now that $\lambda > 1$, which is equivalent to condition (4.2), indeed

$$\begin{aligned}
\lambda &> 1 \\
\frac{\mathfrak{m}}{2}(1+\beta)(\mu-1) - p\beta - \mu(1-\sigma) &> 1 \\
\frac{\mathfrak{m}}{2}(1+\beta)\mu - \frac{\mathfrak{m}}{2}(1+\beta) - \mu\beta - \mu(1-\sigma) &> 1 \\
\mu\left(\frac{\mathfrak{m}}{2}(1+\beta) - \beta - (1-\sigma)\right) - \frac{\mathfrak{m}}{2}(1+\beta) &> 1 \\
\mu\left(\frac{\mathfrak{m}}{2}(1+\beta) - \beta - (1-\sigma)\right) &> 1 + \frac{\mathfrak{m}}{2}(1+\beta) \\
\mu &> \frac{1+\frac{\mathfrak{m}}{2}(1+\beta)}{\frac{\mathfrak{m}}{2}(1+\beta) - \beta - (1-\sigma)} \\
\mu &> \frac{1+\frac{\mathfrak{m}}{2}(1+\beta) - \beta - (1-\sigma) + \beta + (1-\sigma)}{\frac{\mathfrak{m}}{2}(1+\beta) - \beta - (1-\sigma)} \\
\mu &> 1 + \frac{(1+\beta)+(1-\sigma)}{\frac{\mathfrak{m}}{2}(1+\beta) - \beta - (1-\sigma)} \\
\mu &> 1 + \frac{\beta-\sigma+2}{\frac{\mathfrak{m}}{2}(1+\beta) - \beta - (1-\sigma)}.
\end{aligned}$$

By using Lemma 3.2.2, we get

$$\|\Phi(\tau,\cdot)\|_{\mathcal{L}^1} \lesssim \|\varphi\|_{\Omega}^{\mu}(1+\tau)^{-\sigma}.$$

According to Theorem 3.2.3, with $\mathfrak{K} = \mathfrak{c}\|\phi\|_{\Omega}^{\mu}$ and $\vartheta_2 = 1$, we obtain
For $\rho = 1$

$$\|\varphi^{\mathfrak{nlin}}(\tau,\cdot)\|_{\mathcal{L}^1} \lesssim (1+\tau)^{\beta-\sigma+1}\|\varphi\|_{\Omega}^{\mu}, \tag{4.18}$$

while for $\rho = \mu$,

$$\|\varphi^{\mathfrak{nlin}}(\tau,\cdot)\|_{\mathcal{L}^{\mu}} \lesssim (1+\tau)^{\beta-\sigma+1-\frac{\mathfrak{m}}{2}(1+\beta)\left(1-\frac{1}{\mu}\right)}\|\varphi\|_{\Omega}^{\mu}. \tag{4.19}$$

From (4.13), it follows that

$$\|\varphi^{\mathfrak{nlin}}\|_{\Omega} = \sup_{\tau\geq 0}(1+\tau)^{-\beta+\sigma-1}\{\|\varphi^{\mathfrak{nlin}}(\tau,\cdot)\|_{\mathcal{L}^1} + (1+\tau)^{\frac{\mathfrak{m}}{2}(1+\beta)\left(1-\frac{1}{\mu}\right)}\|\varphi^{\mathfrak{nlin}}(\tau,\cdot)\|_{\mathcal{L}^{\mu}}\}.$$

Using estimates (4.18) and (4.19), we obtain

$$\|\varphi^{\mathfrak{nlin}}\|_{\Omega} \lesssim \sup_{\tau\geq 0}(1+\tau)^{-\beta+\sigma-1}\{(1+\tau)^{\beta-\sigma+1}\|\varphi\|_{\Omega}^{\mu} + (1+\tau)^{\frac{\mathfrak{m}}{2}(1+\beta)\left(1-\frac{1}{\mu}\right)+\beta-\sigma+1-\frac{\mathfrak{m}}{2}(1+\beta)\left(1-\frac{1}{\mu}\right)}\|\varphi\|_{\Omega}^{\mu}\}.$$

We conclude that

$$\|\varphi^{\mathfrak{nlin}}\|_{\Omega} \lesssim \|\varphi\|_{\Omega}^{\mu}. \tag{4.20}$$

Now, we prove the inequality (4.10). We have

$$\begin{aligned}\|\mathcal{N}\varphi\|_{\Omega} &= \|\varphi^{\mathfrak{lin}} + \varphi^{\mathfrak{nlin}}\|_{\Omega} \\ &\leq \|\varphi^{\mathfrak{lin}}\|_{\Omega} + \|\varphi^{\mathfrak{nlin}}\|_{\Omega}.\end{aligned}$$

From (4.16) and (4.20), we derive

$$\|\mathcal{N}\varphi\|_{\Omega} \lesssim \|\varphi_0\|_{L^1\cap L^{\mu}} + \|\varphi\|_{\Omega}^{\mu}.$$

Let us now prove the inequality (4.11). Let $\overline{\varphi}, \tilde{\varphi} \in \Omega$,

$$\mathcal{N}\overline{\varphi} - \mathcal{N}\tilde{\varphi} = \int_0^{\tau} (\tau-\theta)^{\beta}\mathcal{K}_{1+\beta,1+\beta}(\tau-\theta,\cdot) *_{(\zeta)} \int_0^{\theta} (\theta-s)^{-\sigma}(|\overline{\varphi}(s,\cdot)|^{\mu} - |\tilde{\varphi}(s,\cdot)|^{\mu})\,\mathrm{d}s\mathrm{d}\theta.$$

Setting $\hat{\Phi}(\tau,\zeta) = \int_0^{\tau}(\tau-\theta)^{-\sigma}(|\overline{\varphi}(\theta,\zeta)|^{\mu} - |\tilde{\varphi}(\theta,\zeta)|^{\mu})\,\mathrm{d}\theta$, it follows that

$$\begin{aligned}\left\|\hat{\Phi}(\tau,\cdot)\right\|_{\mathcal{L}^1} &= \left\|\int_0^{\tau}(\tau-\theta)^{-\sigma}(|\overline{\varphi}(\theta,\cdot)|^{\mu} - |\tilde{\varphi}(\theta,\cdot)|^{\mu})\,\mathrm{d}\theta\right\|_{\mathcal{L}^1} \\ &\lesssim \int_0^{\tau}(\tau-\theta)^{-\sigma}\|(|\overline{\varphi}(\theta,\cdot)|^{\mu} - |\tilde{\varphi}(\theta,\cdot)|^{\mu})\|_{\mathcal{L}^1}\,\mathrm{d}\theta.\end{aligned}$$

By using Holder's inequality, we obtain

$$\left\|\hat{\Phi}(\tau,\cdot)\right\|_{\mathcal{L}^1} \lesssim \int_0^{\tau}(\tau-\theta)^{-\sigma}\|\overline{\varphi}(\theta,\cdot) - \tilde{\varphi}(\theta,\cdot)\|_{\mathcal{L}^{\mu}}(\|\overline{\varphi}(\theta,\cdot)\|_{\mathcal{L}^{\mu}}^{\mu-1} + \|\tilde{\varphi}(\theta,\cdot)\|_{\mathcal{L}^{\mu}}^{\mu-1})\,\mathrm{d}\theta. \tag{4.21}$$

From the definition of the norm $\|\cdot\|_{\Omega}$ in (4.13), we get

$$\|\overline{\varphi}(\theta,\cdot) - \tilde{\varphi}(\theta,\cdot)\|_{\mathcal{L}^{\mu}} \leq (1+\theta)^{\beta+1-\sigma-\frac{\mathfrak{m}}{2}(1+\beta)\left(1-\frac{1}{\mu}\right)}\|\overline{\varphi} - \tilde{\varphi}\|_{\Omega},$$

and

$$\|\overline{\varphi}(\theta,\cdot)\|_{\mathcal{L}^{\mu}}^{\mu-1} + \|\tilde{\varphi}(\theta,\cdot)\|_{\mathcal{L}^{\mu}}^{\mu-1} \leq (1+\theta)^{(\mu-1)\beta+(\mu-1)-(\mu-1)\sigma-\frac{\mathfrak{m}}{2}(1+\beta)(\mu-1)+\frac{\mathfrak{m}}{2}(1+\beta)\left(1-\frac{1}{\mu}\right)}(\|\overline{\varphi}\|_{\Omega}^{\mu-1} + \|\tilde{\varphi}\|_{\Omega}^{\mu-1}),$$

which implies

$$\begin{aligned}&\|\overline{\varphi}(\theta,\cdot) - \tilde{\varphi}(\theta,\cdot)\|_{\mathcal{L}^{\mu}}(\|\overline{\varphi}(\theta,\cdot)\|_{\mathcal{L}^{\mu}}^{\mu-1} + \|\tilde{\varphi}(\theta,\cdot)\|_{\mathcal{L}^{\mu}}^{\mu-1}) \\ &\leq (1+\theta)^{\mu\beta+\mu(1-\sigma)-\frac{\mathfrak{m}}{2}(1+\beta)(\mu-1)}\|\overline{\varphi} - \tilde{\varphi}\|_{\Omega}(\|\overline{\varphi}\|_{\Omega}^{\mu-1} + \|\tilde{\varphi}\|_{\Omega}^{\mu-1}).\end{aligned} \tag{4.22}$$

Combining this last estimate with (4.21), and setting $\lambda = -\mu\beta - \mu(1-\sigma) + \frac{\mathfrak{m}}{2}(1+\beta)(\mu-1)$, we obtain

$$\left\|\hat{\Phi}(\tau,\cdot)\right\|_{\mathcal{L}^1} \lesssim \|\overline{\varphi} - \tilde{\varphi}\|_\Omega(\|\overline{\varphi}\|_\Omega^{\mu-1} + \|\tilde{\varphi}\|_\Omega^{\mu-1}) \int_0^\tau (\tau-\theta)^{-\sigma}(1+\theta)^{-\lambda}\, \mathrm{d}\theta.$$

Applying Lemma 3.2.2, under the assumption $\lambda > 1$, we obtain

$$\left\|\hat{\Phi}(\tau,\cdot)\right\|_{\mathcal{L}^1} \lesssim \|\overline{\varphi} - \tilde{\varphi}\|_\Omega(\|\overline{\varphi}\|_\Omega^{\mu-1} + \|\tilde{\varphi}\|_\Omega^{\mu-1})(1+\tau)^{-\sigma}$$

According to Theorem 3.2.3, with $\mathfrak{K} = \mathfrak{c}\|\overline{\varphi} - \tilde{\varphi}\|_\Omega(\|\overline{\varphi}\|_\Omega^{\mu-1} + \|\tilde{\varphi}\|_\Omega^{\mu-1})$, and $\vartheta_2 = 1$, we get,
for $\rho = 1$

$$\|(\mathcal{N}\overline{\varphi} - \mathcal{N}\tilde{\varphi})(\tau,\cdot)\|_{\mathcal{L}^1} \lesssim (1+\tau)^{\beta-\sigma+1}\|\overline{\varphi} - \tilde{\varphi}\|_\Omega(\|\overline{\varphi}\|_\Omega^{\mu-1} + \|\tilde{\varphi}\|_\Omega^{\mu-1}), \quad (4.23)$$

while for $\rho = \mu$,

$$\|(\mathcal{N}\overline{\varphi} - \mathcal{N}\tilde{\varphi})(\tau,\cdot)\|_{\mathcal{L}^\mu} \lesssim (1+\tau)^{-\frac{\mathfrak{m}}{2}(1+\beta)\left(1-\frac{1}{\mu}\right)+\beta-\sigma+1}\|\overline{\varphi} - \tilde{\varphi}\|_\Omega(\|\overline{\varphi}\|_\Omega^{\mu-1} + \|\tilde{\varphi}\|_\Omega^{\mu-1}), \quad (4.24)$$

we have

$$\|\mathcal{N}\overline{\varphi} - \mathcal{N}\tilde{\varphi}\|_\Omega = \sup_{\tau\geq 0}(1+\tau)^{-\beta+\sigma-1}\{\|(\mathcal{N}\overline{\varphi} - \mathcal{N}\tilde{\varphi})(\tau,\cdot)\|_{\mathcal{L}^1} + (1+\tau)^{\frac{\mathfrak{m}}{2}(1+\beta)\left(1-\frac{1}{\mu}\right)}\|(\mathcal{N}\overline{\varphi} - \mathcal{N}\tilde{\varphi})(\tau,\cdot)\|_{\mathcal{L}^\mu}\}.$$

Using (4.23) and (4.23), we conclude

$$\|\mathcal{N}\overline{\varphi} - \mathcal{N}\tilde{\varphi}\|_\Omega \lesssim \|\overline{\varphi} - \tilde{\varphi}\|_\Omega(\|\overline{\varphi}\|_\Omega^{\mu-1} + \|\tilde{\varphi}\|_\Omega^{\mu-1}).$$

We now want to prove estimate (4.4) of the solution to the Cauchy problem (4.1). By using the definition of the norm $\|\cdot\|_\Omega$ in (4.13), we get

$$\|\varphi(\tau,\cdot)\|_{\mathcal{L}^1} \leq (1+\tau)^{1-\sigma+\beta}\|\varphi\|_\Omega, \quad (4.25)$$

and

$$\|\varphi(\tau,\cdot)\|_{\mathcal{L}^\mu} \leq (1+\tau)^{1-\sigma+\beta-\frac{\mathfrak{m}}{2}(1+\beta)\left(1-\frac{1}{\mu}\right)}\|\varphi\|_\Omega. \quad (4.26)$$

Let us recall that the solution to the Cauchy problem (4.1) satisfies

$$\|\varphi\|_\Omega \leq 2\mathbf{C}\,\|\varphi_0\|_{\mathcal{L}^1\cap\mathcal{L}^\mu}\,, \quad (4.27)$$

which follows from the fact that $\mathcal{N}$ has a unique fixed point φ in Ω_W. Combining (4.25) and (4.26) with (4.27), we conclude that

$$\|\varphi(\tau,\cdot)\|_{L^\rho} \lesssim (1+\tau)^{1-\sigma+\beta-\frac{\mathfrak{m}}{2}(1+\beta)\left(1-\frac{1}{\rho}\right)} \|\varphi_0\|_{L^1\cap L^\mu},$$

for all $\rho \in [1,\mu]$, and all $\tau \geq 0$.

□

Proof of Theorem 4.1.2. Let

$$\Omega = \left\{\varphi \in \mathscr{C}\left([0,\infty), \mathcal{L}^1 \cap \mathcal{L}^\mu\right) : \|\varphi\|_\Omega < \infty\right\},$$

endowed with the norm

$$\|\varphi\|_\Omega = \sup_{\tau\geq 0}(1+\tau)^{-1}\{\|\varphi(\tau,\cdot)\|_{\mathcal{L}^1} + (1+\tau)^{\frac{\mathfrak{m}}{2}(1+\beta)\left(1-\frac{1}{\mu}\right)}\|\varphi(\tau,\cdot)\|_{\mathcal{L}^\mu}\}. \quad (4.28)$$

Let $\varphi_0, \phi_1 \in \mathcal{L}^1 \cap \mathcal{L}^\mu$. By Theorem (3.2.3), taking $\rho = 1$ and $\vartheta_0 = 1$, and noting that condition (3.8) is satisfied, we get

$$\left\|\varphi^{\mathfrak{lin}}(\tau,\cdot)\right\|_{\mathcal{L}^1} \lesssim \|\varphi_0\|_{\mathcal{L}^1} + \tau\|\varphi_1\|_{\mathcal{L}^1},$$

thus,

$$\left\|\varphi^{\mathfrak{lin}}(\tau,\cdot)\right\|_{\mathcal{L}^1} \lesssim (1+\tau)\left(\|\varphi_0\|_{\mathcal{L}^1} + \|\varphi_1\|_{\mathcal{L}^1}\right). \quad (4.29)$$

For the case where $\rho = \mu$, we take $\vartheta_0 = \vartheta_1 = \mu$ for $\tau \in [0,1]$ and $\vartheta_0 = \vartheta_1 = 1$ for $\tau \geq 1$, We observe that condition (3.8) is satisfied for $\mathfrak{m} = 1, 2$, while for $\mathfrak{m} > 2$, an additional condition $\mu < 1 + \frac{2}{\mathfrak{m}-2}$ is generated. Then we get

$$\left\|\varphi^{\mathfrak{lin}}(\tau,\cdot)\right\|_{\mathcal{L}^\mu} \lesssim \begin{cases} \|\varphi_0\|_{\mathcal{L}^\mu} + \tau\|\varphi_1\|_{\mathcal{L}^\mu} & \tau \in [0,1], \\ \tau^{-\frac{\mathfrak{m}}{2}(1+\beta)\left(1-\frac{1}{\mu}\right)}\left(\|\varphi_0\|_{\mathcal{L}^1} + \tau\|\varphi_1\|_{\mathcal{L}^1}\right) & \tau \geq 1, \end{cases}$$

entails

$$\left\|\varphi^{\mathfrak{lin}}(\tau,\cdot)\right\|_{\mathcal{L}^\mu} \lesssim (1+\tau)^{1-\frac{\mathfrak{m}}{2}(1+\beta)\left(1-\frac{1}{\mu}\right)}\left(\|\varphi_0\|_{\mathcal{L}^1} + \|\varphi_1\|_{\mathcal{L}^1} + \|\varphi_0\|_{\mathcal{L}^\mu} + \|\varphi_1\|_{\mathcal{L}^\mu}\right), \quad \text{for } \tau \geq 0. \quad (4.30)$$

Now we are ready to estimate $\|\varphi^{\mathfrak{lin}}\|_\Omega$. From (4.28)

$$\|\varphi\|_\Omega = \sup_{\tau\geq 0}(1+\tau)^{-1}\{\|\varphi(\tau,\cdot)\|_{\mathcal{L}^1} + (1+\tau)^{\frac{\mathfrak{m}}{2}(1+\beta)\left(1-\frac{1}{\mu}\right)}\|\varphi(\tau,\cdot)\|_{\mathcal{L}^\mu}\}.$$

Using estimates (4.29) and (4.30), we obtain

$$\|\varphi^{\mathfrak{lin}}\|_\Omega \lesssim \sup_{\tau\geq 0}(1+\tau)^{-1}\{(1+\tau)\left(\|\varphi_0\|_{\mathcal{L}^1}+\|\varphi_1\|_{\mathcal{L}^1}\right)$$
$$+(1+\tau)^{\frac{\mathfrak{m}}{2}(1+\beta)\left(1-\frac{1}{\mu}\right)}(1+\tau)^{1-\frac{\mathfrak{m}}{2}(1+\beta)\left(1-\frac{1}{\mu}\right)}\left(\|\varphi_0\|_{\mathcal{L}^1}+\|\varphi_1\|_{\mathcal{L}^1}+\|\varphi_0\|_{\mathcal{L}^\mu}+\|\varphi_1\|_{\mathcal{L}^\mu}\right)\}.$$

We conclude that

$$\|\varphi^{\mathfrak{lin}}\|_\Omega \lesssim \|\varphi_0,\varphi_1\|_{\mathcal{L}^1\cap\mathcal{L}^\mu}\,. \tag{4.31}$$

We still have to estimate $\|\varphi^{\mathfrak{nlin}}\|_\Omega$. At this point, setting

$$\Phi(\tau,\zeta)=\int_0^\tau(\tau-\theta)^{-\sigma}|\varphi(\theta,\zeta)|^\mu\,d\theta,$$

we have

$$\|\Phi(\tau,\cdot)\|_{\mathcal{L}^1}\lesssim\int_0^\tau(\tau-\theta)^{-\sigma}\|\varphi(\theta,\cdot)\|^\mu_{\mathcal{L}^\mu}\,d\theta. \tag{4.32}$$

From (4.28), for any $\varphi\in\Omega$

$$\|\varphi\|_\Omega\geq(1+\theta)^{-1}\{\|\varphi(\theta,\cdot)\|_{\mathcal{L}^1}+(1+\theta)^{\frac{\mathfrak{m}}{2}(1+\beta)\left(1-\frac{1}{\mu}\right)}\|\varphi(\theta,\cdot)\|_{\mathcal{L}^\mu}\},$$

it follows that

$$\|\varphi(\theta,\cdot)\|_{\mathcal{L}^\mu}\leq(1+\theta)^{1-\frac{\mathfrak{m}}{2}(1+\beta)\left(1-\frac{1}{\mu}\right)}\|\varphi\|_\Omega,$$

thus

$$\|\varphi(\theta,\cdot)\|^\mu_{\mathcal{L}^\mu}\leq(1+\theta)^{\mu-\frac{\mathfrak{m}}{2}(1+\beta)(\mu-1)}\|\varphi\|^\mu_\Omega.$$

Combing this last estimate with (4.32), and setting $\lambda=-\mu+\frac{\mathfrak{m}}{2}(1+\beta)(\mu-1)$, we obtain

$$\|\Phi(\tau,\cdot)\|_{\mathcal{L}^1}\lesssim\|\varphi\|^\mu_\Omega\int_0^\tau(\tau-\theta)^{-\sigma}(1+\theta)^{-\lambda}\,\mathrm{d}\theta.$$

Assume now that $\lambda > 1$, which is equivalent to condition (4.5), indeed

$$\begin{aligned}
\lambda &> 1\\
\frac{\mathfrak{m}}{2}(1+\beta)(\mu-1)-\mu &> 1\\
\frac{\mathfrak{m}}{2}(1+\beta)\mu-\frac{\mathfrak{m}}{2}(1+\beta)-\mu &> 1\\
\mu\left(\frac{\mathfrak{m}}{2}(1+\beta)-1\right)-\frac{\mathfrak{m}}{2}(1+\beta) &> 1\\
\mu\left(\frac{\mathfrak{m}}{2}(1+\beta)-1\right) &> 1+\frac{\mathfrak{m}}{2}(1+\beta)\\
\mu &> \frac{1+\frac{\mathfrak{m}}{2}(1+\beta)}{\frac{\mathfrak{m}}{2}(1+\beta)-1}\\
\mu &> \frac{1+\frac{\mathfrak{m}}{2}(1+\beta)-1+1}{\frac{\mathfrak{m}}{2}(1+\beta)-1}\\
\mu &> 1+\frac{2}{\frac{\mathfrak{m}}{2}(1+\beta)-1}.
\end{aligned}$$

By using Lemma 3.2.2, we get

$$\|\Phi(\tau,\cdot)\|_{\mathcal{L}^1} \lesssim \|\varphi\|_{\Omega}^{\mu}(1+\tau)^{-\sigma}.$$

According to Theorem 3.2.3, with $\mathfrak{K}=\mathfrak{c}\|\varphi\|_{\Omega}^{\mu}$ and $\vartheta_2=1$, we obtain
For $\rho=1$

$$\|\varphi^{\mathfrak{nlin}}(\tau,\cdot)\|_{\mathcal{L}^1} \lesssim (1+\tau)^{\beta-\sigma+1}\|\varphi\|_{\Omega}^{\mu}, \tag{4.33}$$

while for $\rho=\mu$,

$$\|\varphi^{\mathfrak{nlin}}(\tau,\cdot)\|_{\mathcal{L}^\mu} \lesssim (1+\tau)^{\beta-\sigma+1-\frac{\mathfrak{m}}{2}(1+\beta)\left(1-\frac{1}{\mu}\right)}\|\varphi\|_{\Omega}^{\mu}. \tag{4.34}$$

From (4.28), it follows that

$$\|\varphi^{\mathfrak{nlin}}\|_{\Omega}=\sup_{\tau\geq 0}(1+\tau)^{-1}\{\|\phi^{\mathfrak{nlin}}(\tau,\cdot)\|_{\mathcal{L}^1}+(1+\tau)^{\frac{\mathfrak{m}}{2}(1+\beta)\left(1-\frac{1}{\mu}\right)}\|\varphi^{\mathfrak{nlin}}(\tau,\cdot)\|_{\mathcal{L}^\mu}\}.$$

Using estimates (4.33) and (4.34), taking in consideration $\beta \leq \sigma$, we obtain

$$\|\varphi^{\mathfrak{nlin}}\|_{\Omega} \lesssim \sup_{\tau\geq 0}(1+\tau)^{-1}\{(1+\tau)\|\varphi\|_{\Omega}^{\mu}+(1+\tau)^{\frac{\mathfrak{m}}{2}(1+\beta)\left(1-\frac{1}{\mu}\right)+1-\frac{\mathfrak{m}}{2}(1+\beta)\left(1-\frac{1}{\mu}\right)}\|\varphi\|_{\Omega}^{\mu}\},$$

then

$$\|\varphi^{\mathfrak{nlin}}\|_{\Omega} \lesssim \|\varphi\|_{\Omega}^{\mu}. \tag{4.35}$$

From (4.31) and (4.35) we derive

$$\|\mathcal{N}\varphi\|_\Omega \lesssim \|\,(\varphi_0,\varphi_1)\,\|_{\mathcal{L}^1\cap\mathcal{L}^\mu} + \|\varphi\|_\Omega^\mu.$$

Let us now prove the inequality (4.11). Let $\overline{\varphi},\tilde{\varphi}\in\Omega$,

$$\mathcal{N}\overline{\varphi}-\mathcal{N}\tilde{\varphi} = \int_0^\tau (\tau-\theta)^\beta \mathcal{K}_{1+\beta,1+\beta}(\tau-\theta,\cdot)*_{(\zeta)}\int_0^\theta (\theta-s)^{-\sigma}(|\overline{\varphi}(s,\cdot)|^\mu-|\tilde{\varphi}(s,\cdot)|^\mu)\,\mathrm{d}s\mathrm{d}\theta.$$

We set $\hat{\Phi}(\tau,\zeta) = \int_0^\tau(\tau-\theta)^{-\sigma}(|\overline{\varphi}(\theta,\zeta)|^\mu - |\tilde{\varphi}(\theta,\zeta)|^\mu)\,\mathrm{d}\theta$, using Holder's inequality, we obtain

$$\left\|\hat{\Phi}(\tau,\cdot)\right\|_{\mathcal{L}^1} \lesssim \int_0^\tau (\tau-\theta)^{-\sigma}\|\overline{\varphi}(\theta,\cdot)-\tilde{\varphi}(\theta,\cdot)\|_{\mathcal{L}^\mu}(\|\overline{\varphi}(\theta,\cdot)\|_{\mathcal{L}^\mu}^{\mu-1}+\|\tilde{\varphi}(\theta,\cdot)\|_{\mathcal{L}^\mu}^{\mu-1})\,\mathrm{d}\theta. \tag{4.36}$$

From the definition of the norm $\|\cdot\|_\Omega$ in (4.28), we get

$$\|\overline{\varphi}(\theta,\cdot)-\tilde{\varphi}(\theta,\cdot)\|_{\mathcal{L}^\mu} \le (1+\theta)^{1-\frac{\mathfrak{m}}{2}(1+\beta)\left(1-\frac{1}{\mu}\right)}\|\overline{\varphi}-\tilde{\varphi}\|_\Omega,$$

and

$$\|\overline{\varphi}(\theta,\cdot)\|_{\mathcal{L}^\mu}^{\mu-1}+\|\tilde{\varphi}(\theta,\cdot)\|_{\mathcal{L}^\mu}^{\mu-1} \le (1+\theta)^{\mu-1-\frac{\mathfrak{m}}{2}(1+\beta)(\mu-1)+\frac{\mathfrak{m}}{2}(1+\beta)\left(1-\frac{1}{\mu}\right)}(\|\overline{\varphi}\|_\Omega^{\mu-1}+\|\tilde{\varphi}\|_\Omega^{\mu-1}),$$

which implies

$$\begin{aligned}&\|\overline{\varphi}(\theta,\cdot)-\tilde{\varphi}(\theta,\cdot)\|_{\mathcal{L}^\mu}(\|\overline{\varphi}(\theta,\cdot)\|_{\mathcal{L}^\mu}^{\mu-1}+\|\tilde{\varphi}(\theta,\cdot)\|_{\mathcal{L}^\mu}^{\mu-1})\\ &\le (1+\theta)^{\mu-\frac{\mathfrak{m}}{2}(1+\beta)(\mu-1)}\|\overline{\varphi}-\tilde{\varphi}\|_\Omega(\|\overline{\varphi}\|_\Omega^{\mu-1}+\|\tilde{\varphi}\|_\Omega^{\mu-1}).\end{aligned} \tag{4.37}$$

Combing this last estimate with (4.36), and setting $\lambda = -\mu+\frac{\mathfrak{m}}{2}(1+\beta)(\mu-1)$, we get

$$\left\|\hat{\Phi}(\tau,\cdot)\right\|_{\mathcal{L}^1} \lesssim \|\overline{\varphi}-\tilde{\varphi}\|_\Omega(\|\overline{\varphi}\|_\Omega^{\mu-1}+\|\tilde{\varphi}\|_\Omega^{\mu-1})\int_0^\tau(\tau-\theta)^{-\sigma}(1+\theta)^{-\lambda}\,\mathrm{d}\theta.$$

Applying Lemma 3.2.2, under the assumption $\lambda > 1$, we obtain

$$\left\|\hat{\Phi}(\tau,\cdot)\right\|_{\mathcal{L}^1} \lesssim \|\overline{\varphi}-\tilde{\varphi}\|_\Omega(\|\overline{\varphi}\|_\Omega^{\mu-1}+\|\tilde{\varphi}\|_\Omega^{\mu-1})(1+\tau)^{-\sigma}.$$

According to Theorem 3.2.3, with $\mathfrak{K} = \mathfrak{c}\|\overline{\varphi}-\tilde{\varphi}\|_\Omega(\|\overline{\varphi}\|_\Omega^{\mu-1}+\|\tilde{\varphi}\|_\Omega^{\mu-1})$, and $\vartheta_2 = 1$, we obtain
For $\rho = 1$

$$\|(\mathcal{N}\overline{\varphi}-\mathcal{N}\tilde{\varphi})(\tau,\cdot)\|_{\mathcal{L}^1} \lesssim (1+\tau)^{\beta-\sigma+1}\|\overline{\varphi}-\tilde{\varphi}\|_\Omega(\|\overline{\varphi}\|_\Omega^{\mu-1}+\|\tilde{\varphi}\|_\Omega^{\mu-1}), \tag{4.38}$$

while for $\rho = \mu$,

$$\|(\mathcal{N}\overline{\varphi}-\mathcal{N}\tilde{\varphi})(\tau,\cdot)\|_{\mathcal{L}^\mu} \lesssim (1+\tau)^{-\frac{\mathfrak{m}}{2}(1+\beta)\left(1-\frac{1}{\mu}\right)+\beta-\sigma+1}\|\overline{\varphi}-\tilde{\varphi}\|_\Omega(\|\overline{\varphi}\|_\Omega^{\mu-1}+\|\tilde{\varphi}\|_\Omega^{\mu-1}). \tag{4.39}$$

Using estimates (4.38) and (4.39), taking in consideration $\beta \leq \sigma$, we obtain

$$\begin{aligned}\|\mathcal{N}\overline{\varphi} - \mathcal{N}\tilde{\varphi}\|_\Omega \lesssim \sup_{\tau\geq 0}(1+\tau)^{-1}\{&(1+\tau)\|\overline{\varphi}-\tilde{\varphi}\|_\Omega(\|\overline{\varphi}\|_\Omega^{\mu-1}+\|\tilde{\varphi}\|_\Omega^{\mu-1}) \\ &+ (1+\tau)^{\frac{\mathfrak{m}}{2}(1+\beta)\left(1-\frac{1}{\mu}\right)+1-\frac{\mathfrak{m}}{2}(1+\beta)\left(1-\frac{1}{\mu}\right)}\|\overline{\varphi}-\tilde{\varphi}\|_\Omega(\|\overline{\varphi}\|_\Omega^{\mu-1}+\|\tilde{\varphi}\|_\Omega^{\mu-1})\},\end{aligned}$$

then

$$\|\mathcal{N}\overline{\varphi} - \mathcal{N}\tilde{\varphi}\|_\Omega \lesssim \|\overline{\varphi}-\tilde{\varphi}\|_\Omega(\|\overline{\varphi}\|_\Omega^{\mu-1}+\|\tilde{\varphi}\|_\Omega^{\mu-1}). \tag{4.40}$$

Estimation (4.9) is obtained by following the same steps used to prove estimation (4.4).

□

Conclusion

In this thesis, we investigated the global existence of solutions to fractional diffusion equations with nonlinear memory and established estimates for these global solutions. Our approach relies on the Banach fixed-point theorem and previous results for linear fractional differential equations.

In Chapter 1, we presented the necessary analytical tools, including notations, functional spaces, and fundamental concepts in fractional calculus, Laplace and Fourier transforms, which form the foundation for our work.

In Chapter 2, we formulated the problem as a Volterra integral equation to prove existence and uniqueness of solutions for the associated linear fractional differential equations and obtain their explicit representation.

In Chapter 3, we reduced the fractional diffusion equations (without the nonlinear memory terms) to the linear fractional differential equations investigated in Chapter 2. This reduction enables us to derive explicit solution formulas and establish Linear estimates for the solutions.

Finally, in Chapter 4, we introduced an appropriate solution space and reformulate our problem in terms of a nonlinear operator. Then, we demonstrated that this operator admits a unique fixed point, which yields both the global existence of solutions and the desired a priori estimates.

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