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On the study of the eigenvalues, the numerical range and the numerical radius of linear operator in Hilbert space

Presented by: AMMAR Aicha

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The jury :

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Dedication

I would like to dedicate this modest work to my beloved parents, my brothers and my sister.

Acknowledgement

First of all, I thank **Allah** for guiding and inspiring my patience and endurance throughout this thesis.

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Abstract

Our main goal in this thesis is to refine some well-known numerical radius inequalities of operators on a Hilbert space. We provide some new bounds of the numerical radius for one operator and for the off-diagonal parts of 2×2 operator matrices. Also, we establish several upper and lower bounds for the Euclidean operator radius of two linear operators in complex Hilbert space. We apply these results to reobtain some well known inequalities for the classical numerical radius. Finally, we give some bounds for the weighted numerical radius of one operator as well as for 2×2 operator matrices. We reobtain some well known inequalities for the classical numerical radius. New characterization for the weighted numerical radius is also given.

Key words: Numerical radius, Euclidean operator radius, weighted numerical radius , inequality.

Contents

Notations

- $\mathbb{N}$: The set of natural numbers.
- $\mathbb{R}$: The set of real numbers.
- $\mathbb{R}_+$: The set of non negative real numbers.
- $\mathbb{C}$: The set of complex numbers.
- $\mathbb{K}$: $\mathbb{R}$ or $\mathbb{C}$.
- E: Vector space .
- $\mathcal{H}$: Complex Hilbert space .
- $\langle .,. \rangle$: The inner product of $\mathcal{H}$.
- $\mathcal{B}(\mathcal{H})$: The set of all bounded linear operators on $\mathcal{H}$.
- T : A bounded linear operator defined on $\mathcal{H}$ ($T \in \mathcal{B}(\mathcal{H})$).
- $\|T\|$: The norm of T .
- $|T|$: The absolute value of T .
- I: Identity operator of T .
- T^*: The adjoint of T .
- $\Re(T)$: The real part of T .
- $\Im(T)$: The imaginary part of T.
- $\mathcal{R}(T)$: The range of T.
- $\sigma(T)$: The spectrum of T .

- $r(T)$: The spectral radius of T .
- $W(T)$: The numerical range of T.
- $w(T)$: The numerical radius of T.
- $c(T)$: The Crawford number of T.
- $\oplus$:The sign of direct sum.
- $w_e(B, C)$:The Euclidean operator radius of B, C.
- $\mathcal{K}(\mathcal{H})$:The set of all compact linear operators on a complex Hilbert space H. .

Introduction

The study of operator theory is an interesting topic, which becomes popular. Operator Theory is a crucial part of modern mathematics. It belongs to a larger domain which is functional analysis and it plays a pivotal role in many areas of pure and applied mathematics, as well as in the theoretical foundations of quantum mechanics, signal processing, and control theory. At its core, operator theory deals with the study of linear operators, which act on vector spaces.

One area of significant interest within operator theory is the study of the numerical range, also known as the field of values, and the numerical radius of operators, which offer valuable insights into the behavior, structure, and spectral properties of operators on Hilbert and Banach spaces. The numerical range of an operator A is a set of complex numbers defined by

$$W(A) = \{\langle Ax, x\rangle,\ x \in \mathcal{H},\ \|x\| = 1\},$$

where $\mathcal{H}$ is the Hilbert space. The numerical range plays a very significant role and have been studied extensively due to their enormous applications in engineering, quantum computing, quantum mechanics, numerical analysis, differential equations,.....

One of the central results in operator theory, *the numerical range theorem*, establishes that the numerical range of any bounded operator is always a convex set. The most important object related to the numerical range is the numerical radius $w(A)$ of a bounded operator A acting on a Hilbert space $\mathcal{H}$ which is the largest absolute value of the numbers in the numerical range i.e., $w(A) = \sup_{\lambda \in W(A)} |\lambda|$.

Although the numerical radius is always less than or equal to the operator norm, the relationship between the two quantities is generally nontrivial. The study of inequalities involving the numerical radius is thus a major focus in this thesis. Several inequalities involving the numerical radius of one operator and the numerical radii of operator matrices have been established by many researchers such as Kittaneh, Abu-Omar, Hirzallah, Kallol, Paul, Sal Moslehian, Dragomir, and others, see [18, 21, 24]. These inequalities serve not only as generalizations of classical results but also as powerful tools in deriving bounds for operator functions, sums, products, and commutators.

In this thesis, we give some recent results for the numerical radius, the Euclidean operator norm and the weighted numerical radius.

This thesis is divided into four chapters.

- In the first chapter, we present some basic definitions, properties, theorems and fundamental results for operators in Hilbert space that are useful throughout this thesis.
- In the second chapter, we provide some new bounds of the numerical radius for one operator and for the off-diagonal parts of 2×2 operator matrices. A refinement of the triangle inequality for the operator norm is also given.
- In the third chapter, we establish several upper and lower bounds for the Euclidean operator radius of two linear operators in complex Hilbert space. We apply these results to reobtain some well known inequalities for the classical numerical radius. Also, we give some inequalities for the Euclidean operator radius of the sums and the products of two operators.
- In the fourth chapter, we give some bounds for the weighted numerical radius of one operator as well as for 2×2 operator matrices. We reobtain some well known inequalities for the classical numerical radius. New characterization for the weighted numerical radius is also given.

Contributions

This work led to the production of three papers:

1) Aicha Ammar, Abdelkader Frakis and Fuad Kittaneh, *Numerical radius inequalities for the off-diagonal parts of* 2×2 *operator matrices*, Quaestiones Mathematicae, 46:11 (2023), 2277-2286.

2)Aicha Ammar, Abdelkader Frakis and Fuad Kittaneh , *New bounds for the Euclidean operator radius of two Hilbert space operators with applications*, Boletín de la sociedad matemática mexicana. (2024) 30:45 https://doi.org/10.1007/s40590-024-00621-8.

3) Aicha Ammar, Abdelkader Frakis and Fuad Kittaneh, *Weighted numerical radius inequalities for operators and* 2×2 *operator matrices*, KYUNGPOOK Math. J. 65(2025), 63-75. https://doi.org/10.5666/KMJ.2025.65.1.63.

Chapter 1

Basic concepts

In this chapter we collect some of basic properties of Hilbert space and operator theory, that will be used throughout this thesis. Most informations in this chapter can be found in almost every book on operator theory and matrix analysis, see for example [11, 17, 25, 31].

1.1 Inner product and Hilbert space

Definition 1.1.1. *Let E be a vector space over a field $\mathbb{K}$ ($\mathbb{C}$ or $\mathbb{R}$). A map $\|\cdot\| : E \to \mathbb{R}_+$ is called a norm on E if*

1. $\|x\| \geq 0$ *for all* $x \in E$ *and* $\|x\| = 0$ *if and only if* $x = 0$.
2. $\|\alpha x\| = |\alpha|\|x\|$ *for all* $x \in E$ *and* $\alpha \in \mathbb{K}$.
3. $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in E$ *(triangle inequality).*

$(E, \|\cdot\|)$ *is called a normed space.*

Definition 1.1.2. *Let $\mathcal{H}$ be a vector complex space. An inner product on $\mathcal{H}$ is a map $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ such that for all $x, y, z \in E$ and $\alpha \in \mathbb{C}$.*

1. $\langle x, x \rangle \geq 0$.
2. $\langle x, x \rangle = 0$ *if and only if* $x = 0$.
3. $\langle \alpha x + y, z \rangle = \alpha \langle x, z \rangle + \langle y, z \rangle$.
4. $\langle x, y \rangle = \overline{\langle y, x \rangle}$.

A complex vector space $\mathcal{H}$ together with an inner product $\langle \cdot, \cdot \rangle$ is called ***Inner product space.***

Theorem 1.1.1. *Any Inner product space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ is a normed space where $\|x\| = \sqrt{\langle x, x \rangle}$, which is called the associated norm of the inner product.*

Definition 1.1.3. *A Hilbert space is an Inner product space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ such that the associated norm is complete.*

Theorem 1.1.2. *Let $x, y \in \mathcal{H}$. Then*

$$|\langle x, y \rangle| \leq \|x\| \|y\|.$$

Theorem 1.1.3. *Let $x, y \in \mathcal{H}$. Then*

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

1.2 Bounded linear operators

Definition 1.2.1. *Let $\mathcal{H}$ be a Hilbert space. An operator $T : \mathcal{H} \longrightarrow \mathcal{H}$ is a*

1) ***Linear operator*** *if*

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) \text{ for all } x, y \in \mathcal{H} \text{ and } \alpha, \beta \in \mathbb{C}.$$

2) ***Bounded linear operator*** *if there exists $m > 0$ such that*

$$\|Tx\| \leq m\|x\| \quad \text{for all} \quad x \in \mathcal{H}.$$

Notation: Let $\mathcal{B}(\mathcal{H})$ be the set of all bounded linear operators on a complex Hilbert space $\mathcal{H}$.

Definition 1.2.2. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

1. The range of T is the set

$$\mathcal{R}(T) = \{Tx : x \in \mathcal{H}\}.$$

2. The kernel of T is the set

$$N(T) = \{x \in \mathcal{H} : Tx = 0\}.$$

Proposition 1.2.1. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

1. *$\mathcal{R}(T)$ is a linear subspace of $\mathcal{H}$.*

2. *$N(T)$ is a closed linear subspace of $\mathcal{H}$.*

Proposition 1.2.2 (Generalized Polarization identity)**.** *Let $T \in \mathcal{B}(\mathcal{H})$ and let $x, y \in \mathcal{H}$. Then*

$$\langle Tx, y\rangle = \frac{1}{4}\{\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle + i\langle T(x+iy), x+iy\rangle - i\langle T(x-iy), x-iy\rangle\}.$$

1.2.1 Adjoint of operator

Definition 1.2.3. *Let $T \in \mathcal{B}(\mathcal{H})$. Then there exists a unique $T^* \in \mathcal{B}(\mathcal{H})$ such that*

$$\langle Tx, y\rangle = \langle x, T^*y\rangle$$

for all $x, y \in \mathcal{H}$. The operator T^* is called the adjoint of T.

Proprieties 1.2.1. *Let $S, T \in \mathcal{B}(\mathcal{H})$. Then*

1. *$(T+S)^* = T^* + S^*$.*

2. *$(\alpha T)^* = \overline{\alpha} T^*$ for all $\alpha \in \mathbb{C}$.*

3. *$(T^*)^* = T$.*

4. *$(ST)^* = T^*S^*$.*

Definition 1.2.4. *Let $T \in \mathcal{B}(\mathcal{H})$. Then T is said to be*

- ***Self adjoint operator*** *if $T^* = T$.*
- ***Normal operator*** *if $TT^* = T^*T$.*
- ***Unitary operator*** *if $TT^* = T^*T = I$.*

1.3 Positive operator

Definition 1.3.1. *An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be positive semidefinite, written as $T \geq 0$, if T is self adjoint operator and $\langle Tx, x\rangle \geq 0$, for all $x \in \mathcal{H}$.*

T is further called positive definite, written as $T > 0$, if $\langle Tx, x\rangle > 0$ for all $x \in \mathcal{H}, x \neq 0$.

Theorem 1.3.1. *[11] Let $T \in M_n(\mathbb{C})$ be a Hermitian matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then*

- *T is positive definite if and only if $\lambda_k > 0$, for all $k = 1, 2, \ldots, n$.*
- *T is positive semidefinite if and only if $\lambda_k \geq 0$, for all $k = 1, 2, \ldots, n$.*
- *T is indefinite if and only if there are integers $j, k, j \neq k$, with $\lambda_j > 0$ and $\lambda_k < 0$.*

Theorem 1.3.2. *Let $T \in \mathcal{B}(\mathcal{H})$ be a positive operator and let $k \in \mathbb{N}^*$. Then there exists a unique positive operator $B \in \mathcal{B}(\mathcal{H})$ such that $T = B^k$, written as*

$$B = T^{\frac{1}{k}}.$$

Definition 1.3.2. *The absolute value of the operator $T \in \mathcal{B}(\mathcal{H})$ is defined as the square root of the positive operator T^*T and noted by $|T|$. That is,*

$$|T| = (T^*T)^{\frac{1}{2}}.$$

Theorem 1.3.3. *[29] Let $T \in \mathcal{B}(\mathcal{H})$ be self-adjoint, and let $x \in \mathcal{H}$ be any vector. Then*

$$|\langle Tx, x\rangle| \leq \langle |T|x, x\rangle.$$

1.4 Operator norm

Definition 1.4.1. *A map $N : \mathcal{B}(\mathcal{H}) \longrightarrow \mathbb{R}^+$ is called a norm if for all $T, S \in \mathcal{B}(\mathcal{H})$ and $\alpha \in \mathbb{C}$, it satisfies the following axioms:*

1. $N(T) \geq 0$.
2. $N(T) = 0$ *if and only if* $T = 0$.
3. $N(\alpha T) = |\alpha| N(T)$.
4. $N(T + S) \leq N(T) + N(S)$.
5. $N(TS) \leq N(T)N(S)$.

1.4.1 Usual operator norm

Definition 1.4.2. *Let $T \in \mathcal{B}(\mathcal{H})$. The usual operator norm is defined by*

$$\|T\| = \sup_{\|x\|=1} \|Tx\|.$$

An equivalent definition of the operator norm is

$$\|T\| = \sup_{\|x\|=\|y\|=1} |\langle Tx, y\rangle|.$$

Proposition 1.4.1. *[17] Let $T, S \in \mathcal{B}(\mathcal{H})$ and $n \in \mathbb{N}$. Then*

- $\|Tx\| \leq \|T\|\|x\|$ *for all $x \in \mathcal{H}$.*
- $\|TS\| \leq \|T\|\|S\|$.
- $\|T^n\| \leq \|T\|^n$.

Proposition 1.4.2. *[17] Let $T \in \mathcal{B}(\mathcal{H})$.Then*

- $\|T^*\| = \|T\|$.
- $\|T^*T\| = \|TT^*\| = \|T\|^2$.
- $\|\,|T|\,\| = \|\,|T^*|\,\| = \|T\|$.

Theorem 1.4.1. *Let $T \in \mathcal{B}(\mathcal{H})$ be a normal operator and $n \in \mathbb{N}$. Then*

$$\|T^n\| = \|T\|^n.$$

1.4.2 Schatten p-norm

Let $T \in \mathcal{K}(\mathcal{H})$ and $1 \leq p \leq \infty$. The Schatten p-norm is defined as

$$\|T\|_p = (tr|T|^p)^{\frac{1}{p}}.$$

It should be mentioned here that for $p = \infty$ and $p = 2$, the Schatten p-norm are the usual operator norm $\|T\| = \sup_{\|x\|=1} \|Tx\|$ and the Hilbert-Schmidt norm $\|T\|_2 = (tr|T|^2)^{\frac{1}{2}}$, respectively.

Next, we present lemmas which are needed throughout this thesis.

Lemma 1.4.1. *[15](Buzano's inequality) Let $\mathcal{H}$ be a Hilbert space and $x, y, e \in \mathcal{H}$ with $\|e\| = 1$. Then*

$$|\langle x, e\rangle\langle e, y\rangle| \leq \frac{1}{2}(\|x\|\,\|y\| + |\langle x, y\rangle|).$$

Lemma 1.4.2. *[29](Mixed Schwarz inequality) Let $A \in \mathcal{B}(\mathcal{H})$ and $0 \le \alpha \le 1$. Then*

$$|\langle Ax, y\rangle|^2 \le \langle |A|^{2\alpha}x, x\rangle\langle |A^*|^{2(1-\alpha)}y, y\rangle$$

for all $x, y \in \mathcal{H}$.

Lemma 1.4.3. *[41](McCarthy's inequality) Let $T \in \mathcal{B}(\mathcal{H})$ be a positive semidefinite operator and let $x \in \mathcal{H}$ be any unit vector. Then*

a. $\langle Tx, x\rangle^r \le \langle T^r x, x\rangle$ *for* $r \ge 1$.

b. $\langle T^r x, x\rangle \le \langle Tx, x\rangle^r$ *for* $0 < r \le 1$.

1.5 The Cartesian decomposition

Theorem 1.5.1. *Let $T \in \mathcal{B}(\mathcal{H})$. Then there exist self-adjoint operators A and B such that $T = A + iB$ where $A = \Re(T) = \frac{T+T^*}{2}$ and $B = \Im(T) = \frac{1}{2i}(T - T^*)$. This decomposition is called the Cartesian decomposition of T. The operators A and B are called the real and imaginary parts of T, respectively.*

1.6 The spectral radius

Definition 1.6.1. *Let $T \in \mathcal{B}(\mathcal{H})$. The spectrum of T is defined as follows*

$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ *is not invertible*$\}$.

Definition 1.6.2. *Let $T \in \mathcal{B}(\mathcal{H})$. The spectral radius of T is defined as*

$$r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}.$$

Proprieties 1.6.1. *Let $T, S \in \mathcal{B}(\mathcal{H})$, $\alpha \in \mathbb{C}$ and let n be a positive integer.*

- $r(\alpha T) = |\alpha| r(T)$.
- $r(T) \le \|T\|$.
- $r(T^*) = r(T)$.
- $r(TS) = r(ST)$.

Theorem 1.6.1. *(**Spectral radius formula**):*

Let $T \in \mathcal{B}(\mathcal{H})$. Then

$$r(T) = \lim_{n\to+\infty} \|T^n\|^{\frac{1}{n}}.$$

Corrollary 1.6.1. *Let $T \in \mathcal{B}(\mathcal{H})$ be normal. Then*

$$r(T) = \|T\|.$$

Theorem 1.6.2. *Let $T, S \in \mathcal{B}(\mathcal{H})$ be such that $TS = ST$. Then*

$$r(T+S) \leq r(T) + r(S)$$

and

$$r(TS) \leq r(T)r(S).$$

Remark 1.6.1. *The spectral radius is not a norm. To see this, consider $T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, and noting that $r(T) = 0$.*

1.7 Numerical range

Definition 1.7.1. *Let $T \in \mathcal{B}(\mathcal{H})$. The numerical range of T is the subset of the complex plane defined by*

$$W(T) = \{\langle Tx, x\rangle : x \in \mathcal{H}; \|x\| = 1\}.$$

Proprieties 1.7.1. *[23] Let $T, S \in \mathcal{B}(\mathcal{H})$ and $\alpha, \beta \in \mathbb{C}$. Then*

1. *$W(\alpha I + \beta T) = \alpha + \beta W(T)$.*
2. *$W(T+S) \subseteq W(T) + W(S)$.*
3. *$W(T^*) = \{\overline{z} : z \in W(T)\}$.*
4. *$W(U^*TU) = W(T)$ for any unitary operator U.*

Example 1.7.1. *Let T be an 2×2 matrix. Then the numerical range of T is either an ellipse (circle), a straight line segment, or a single point. More specifically, by Schur's theorem, if one reduces Tunitarily to upper triangular form,*

$$T = U^* \begin{bmatrix} \lambda_1 & m \\ 0 & \lambda_2 \end{bmatrix} U,\ U \text{unitary}.$$

Then

(a) *T is not normal if and only if $m \neq 0$.*

- $\lambda_1 \neq \lambda_2$. *$W(T)$ is the interior and boundary of an ellipse with foci at λ_1, λ_2, length of minor axis is $|m|$. Length of major axis $(|m|^2 + |\lambda_1 - \lambda_2|^2)^{\frac{1}{2}}$.*
- $\lambda_1 = \lambda_2$. *$W(T)$ is the disk with center at λ_1 and radius $\frac{|m|}{2}$.*

(b) *T is normal $(m = 0)$.*

- $\lambda_1 \neq \lambda_2$. *$W(T)$ is the line segment joining λ_1 and λ_2.*
- $\lambda_1 = \lambda_2$. *$W(T)$ is the single point λ_1.*

Theorem 1.7.1. *[43][Toeplitz-Hausdorff]*
The numerical range of any operator T is a convex set.

In the following theorem, we cite a very important property of the numerical range of an operator.

Theorem 1.7.2. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$\sigma(T) \subseteq \overline{W(T)}.$$

Proof . *See [23].* ■

Theorem 1.7.3. *Let $T \in \mathcal{B}(\mathcal{H})$. Then T is self-adjoint if and only if $W(T)$ is real.*

Proof . *See [23].* ■

Theorem 1.7.4. *Let $T \in \mathcal{B}(\mathcal{H})$ be self- adjoint and $W(T)$ is $[m, M]$. Then $\|T\| = \sup\{|m|, |M|\}$.*

Proof . *See [23].* ■

1.8 Crawford number

The Crawford number of the operator $T \in \mathcal{B}(\mathcal{H})$ is defined by

$$c(T) = \inf \{|\langle Tx, x\rangle| : x \in \mathcal{H}, \|x\| = 1\}.$$

1.9 Numerical radius

Definition 1.9.1. *Let $T \in \mathcal{B}(\mathcal{H})$. The numerical radius of T is defined by*

$$w(T) = \sup\{|\lambda|, \lambda \in W(T)\} \text{ or } w(T) = \sup\{|\langle Tx, x\rangle| : x \in \mathcal{H}, \|x\| = 1\}.$$

Lemma 1.9.1. *Let $T \in \mathcal{B}(\mathcal{H})$ and $x \in \mathcal{H}$, we have*

$$|\langle Tx, x\rangle| \leq w(T)\|x\|^2.$$

Proof . *Let $x, y \in \mathcal{H}$ with $\|y\| = 1$. Then*

$$|\langle Ty, y\rangle| \leq \sup_{\|y\|=1} |\langle Ty, y\rangle| = w(T).$$

We put $y = \dfrac{x}{\|x\|}$, it follows that

$$\left|\left\langle T\frac{x}{\|x\|}, \frac{x}{\|x\|}\right\rangle\right| \leq w(T).$$

Hence,

$$|\langle Tx, x\rangle| \leq w(T)\|x\|^2.$$

■

The following theorem is a characterization of the numerical radius.

Theorem 1.9.1. *[44] Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$w(T) = \sup_{\theta \in \mathbb{R}} \|\Re(e^{i\theta}T)\|.$$

Also, the numerical radius is defined as $w(T) = \sup_{\theta \in \mathbb{R}} \|\Im(e^{i\theta}T)\|$.

Proof . *Let $x \in \mathcal{H}$ be any unit vector. Then*

$$\begin{aligned}
\sup_{\theta \in \mathbb{R}} \left\|\Re(e^{i\theta}T)\right\| &= \sup_{\theta \in \mathbb{R}} w(\Re(e^{i\theta}T)) \\
&= \sup_{\theta \in \mathbb{R}} \sup_{\|x\|=1} |\langle \Re(e^{i\theta}T)x, x\rangle| \\
&= \sup_{\|x\|=1} |\langle Tx, x\rangle| \\
&= w(T).
\end{aligned}$$

By replacing T by iT, we get $w(T) = w(iT) = \sup_{\theta\in\mathbb{R}} \left\|\Re(e^{i\theta}(iT))\right\| = \sup_{\theta\in\mathbb{R}} \left\|\Im(e^{i\theta}T)\right\|.$ ∎

The following inequality is known as the power inequality of the numerical radius.

Theorem 1.9.2. *Let $T \in \mathcal{B}(\mathcal{H})$ and $n \in \mathbb{N}$. Then*

$$w(T^n) \leq w^n(T).$$

Proof . *See [23].* ∎

1.10 Somes elementary inequalities

Theorem 1.10.1 (Young inequality)**.** *Let $a, b \in \mathbb{C}$ and $p, q > 1$ with $\frac{1}{P} + \frac{1}{q} = 1$. Then*

$$|ab| \leq \frac{1}{p}|a|^p + \frac{1}{q}|b|^q.$$

Theorem 1.10.2 (Arithmetic-geometric mean inequality)**.** *Let $a, b \geq 0$. Then*

$$\sqrt{ab} \leq \frac{a+b}{2}$$

or

$$a.b \leq \frac{a^2+b^2}{2}.$$

Theorem 1.10.3 (Jensen's inequality)**.** *[7] For $p \geq 2$ and for every finite positive sequence of real numbers $a_1, a_2, ..., a_n$, we have*

$$\Big(\frac{1}{n}\sum_{k=1}^{n} a_k\Big)^p \leq \frac{1}{n}\sum_{k=1}^{n} a_k^p - \frac{1}{n}\sum_{k=1}^{n} \big|a_k - \frac{1}{n}\sum_{j=1}^{n} a_j\big|^p.$$

Chapter 2

Numerical radius inequalities

In this chapter, we present several upper and lower bounds for the numerical radius of one operator and for the off-diagonal parts of 2×2 operator matrices. A refinement of the triangle inequality for the operator norm is also given.

2.1 Numerical radius inequalities of one operator

Theorem 2.1.1. *[23] Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$\frac{\|T\|}{2} \leq w(T) \leq \|T\|. \tag{2.1}$$

Proof . *Let $x \in \mathcal{H}$ be any unit vector. By the Cauchy-Schwarz inequality, we have*

$$|\langle Tx, x\rangle| \leq \|Tx\| \leq \|T\|.$$

By taking the supremum in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$, we obtain the second inequality in (2.1).

To prove the first inequality, let $x, y \in \mathcal{H}$ be any unit vectors. Using the generalized polarization identity gives

$$4\langle Tx, y\rangle = \langle T(x+y), (x+y)\rangle - \langle T(x-y), (x-y)\rangle + i\langle T(x+iy), (x+iy)\rangle - i\langle T(x-iy), (x-iy)\rangle.$$

Thus,

$$\begin{aligned} 4|\langle Tx, y\rangle| &\leq w(T)\left(\|x+y\|^2 + \|x-y\|^2 + \|x+iy\|^2 + \|x-iy\|^2\right) \\ &= 4w(T)\left(\|x\|^2 + \|y\|^2\right). \end{aligned}$$

By taking the supremum on both sides in the above inequality over $x, y \in \mathcal{H}$ with $\|x\| = \|y\| = 1$, we obtain

$$\frac{1}{2}\|T\| \leq w(T),$$

as required. ■

Remark 2.1.1. *The numerical radius of T defines a norm on $\mathcal{B}(\mathcal{H})$.*

Theorem 2.1.2. *Let $T \in \mathcal{B}(\mathcal{H})$. Then $r(T) \leq w(T) \leq \|T\|$.*

Kittaneh improved the inequalities in (2.1) *as follows.*

Theorem 2.1.3. *([28]) Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$\frac{1}{4}\|T^*T + TT^*\| \leq w^2(T) \leq \frac{1}{2}\|T^*T + TT^*\|. \tag{2.2}$$

Proof . *Let $x \in \mathcal{H}$ be any unit vector.It follows from the convexity of the function $f(t) = t^2$ that*

$$\begin{aligned} |\langle Tx, x\rangle|^2 &= \langle \Re(T)x, x\rangle^2 + \langle \Im(T)x, x\rangle^2 \\ &\geq \frac{1}{2}\Big(|\langle \Re(T)x, x\rangle| + |\langle \Im(T)x, x\rangle| \Big)^2 \\ &\geq \frac{1}{2} \left|\langle (\Re(T) \pm \Im(T))x, x\rangle\right|^2 . \end{aligned}$$

By taking the supremum on both sides in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$, we obtain

$$w^2(T) \geq \frac{1}{2}\|\Re(T) \pm \Im(T)\|^2 = \frac{1}{2}\|(\Re(T) \pm \Im(T))^2\|.$$

Hence,

$$\begin{aligned} 2w^2(T) &\geq \frac{1}{2}\left(\left\|(\Re(T) + \Im(T))^2\right\| + \left\|(\Re(T) - \Im(T))^2\right\|\right) \\ &\geq \frac{1}{2}\left\|(\Re(T) + \Im(T))^2 + (\Re(T) - \Im(T))^2\right\| \\ &= \left\|\Re(T)^2 + \Im(T)^2\right\| \\ &= \frac{1}{2}\|T^*T + TT^*\|. \end{aligned}$$

Therefore,

$$w^2(T) \geq \frac{1}{4}\|T^*T + TT^*\|,$$

which proves the first inequality in (2.2).

Next, let $x \in \mathcal{H}$ be any unit vector. It follows from the Cauchy-Schwarz inequality that

$$\begin{aligned}|\langle Tx, x\rangle|^2 &= \langle \Re(T)x, x\rangle^2 + \langle \Im(T)x, x\rangle^2 \\ &\leq \|\Re(T)x\|^2 + \|\Im(T)x\|^2 \\ &= \langle \Re(T)^2 x, x\rangle + \langle \Im(T)^2 x, x\rangle \\ &= \langle (\Re(T)^2 + \Im(T)^2)x, x\rangle.\end{aligned}$$

By taking the supremum on both sides in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$, we obtain

$$\begin{aligned}w^2(T) &\leq \|\Re(T)^2 + \Im(T)^2\| \\ &= \frac{1}{2}\|T^*T + TT^*\|,\end{aligned}$$

which proves the second inequality in (2.2). ■

Remark 2.1.2. *The inequalities (2.2) improves (2.1). Indeed*

$$w^2(T) \leq \frac{1}{2}\|T^*T + TT^*\| \leq \frac{1}{2}(\|T^*T\| + \|TT^*\|) = \frac{1}{2}(\|T\|^2 + \|T\|^2) = \|T\|^2.$$

On the other hand, we have

$$\|Tx\|^2 = \langle T^*Tx, x\rangle \leq \langle T^*Tx, x\rangle + \langle TT^*x, x\rangle = \langle (T^*T + TT^*)x, x\rangle.$$

So

$$\frac{1}{4}\|T\|^2 \leq \frac{1}{4}\|T^*T + TT^*\| \leq w^2(T).$$

In [12], Bhunia and Kallol gave the following theorem.

Theorem 2.1.4. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$w(T) \leq \inf_{\phi \in \mathbb{R}} \sqrt{\|H_\phi\|^2 + \|H_{\phi+\frac{\pi}{2}}\|^2},$$

where $H_\phi = \Re(e^{i\phi}T)$.

Proof . *We have,* $H_\theta = \Re(e^{i\theta}T) = \cos\theta\Re(T) - \sin\theta\Im(T)$*. Then for* $[0, 2\pi]$*, we get*

$$\begin{aligned}
H_{\theta+\phi} &= \cos(\theta+\phi)\Re(T) - \sin(\theta+\phi)\Im(T) \\
&= \cos\theta\big[\cos\phi\Re(T) - \sin\phi\Im(T)\big] - \sin\theta\big[\sin\phi\Re(T) + \cos\phi\Im(T)\big] \\
&= \cos\theta\cos\phi\Re(T) - \sin\phi\Im(T) - \sin\theta\Big[-\cos(\phi+\frac{\pi}{2})\Re(T) + \sin(\phi+\frac{\pi}{2})\Im(T)\Big] \\
&= \cos\theta\Re(e^{i\theta}T) + \sin\theta\Re(e^{i(\theta+\frac{\pi}{2})}T) \\
&= H_\theta\cos\theta + H_{\phi+\frac{\pi}{2}}\sin\theta.
\end{aligned}$$

Thus,

$$\|H_{\theta+\phi}\| \le \|H_\theta\cos\theta\| + \|H_{\phi+\frac{\pi}{2}}\sin\theta\|.$$

Taking the supremum in the above inequality over $\theta \in \mathbb{R}$*, we get*

$$w(T) \le \sqrt{\|H_\phi\|^2 + \|H_{\phi+\frac{\pi}{2}}\|^2}.$$

Hence,

$$w(T) \le \inf_{\phi\in\mathbb{R}} \sqrt{\|H_\phi\|^2 + \|H_{\phi+\frac{\pi}{2}}\|^2}.$$

■

Remark 2.1.3. *Noting that for* $\phi = 0$*,* $\|H_\phi\| = \|\Re(T)\|$ *and* $\|H_{\phi+\frac{\pi}{2}}\| = \|\Im(T)\|$*, it follows from Theorem (2.1.4) that*

$$w(T) \le \sqrt{\|\Re(T)\|^2 + \|\Im(T)\|^2}. \tag{2.3}$$

The following lemma plays an essential role in the proof of the next results.

Lemma 2.1.1. *Let* $a, b \ge 0$*,* $0 \le \alpha \le 1$ *and* $p, q > 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$*. Then*

(1) $a^\alpha b^{1-\alpha} \le \alpha a + (1-\alpha)b \le \big[\alpha a^r + (1-\alpha)b^r\big]^{\frac{1}{r}}$ *for* $r \ge 1$*.*

(2) $ab \le \frac{a^p}{p} + \frac{b^q}{q} \le \left(\frac{a^{pr}}{p} + \frac{b^{qr}}{q}\right)^{\frac{1}{r}}$ *for* $r \ge 1$*.*

Theorem 2.1.5. *[21] Let* $T \in \mathcal{B}(\mathcal{H})$*,* $0 < \alpha < 1$*, and* $r \ge 1$*. Then*

$$w^r(T) \le \frac{1}{2}\left\| |T|^{2\alpha r} + |T^*|^{2(1-\alpha)r} \right\|.$$

Proof . *Let $x \in \mathcal{H}$ be any unit vector. Then*

$$\begin{aligned}|\langle Tx, x\rangle| &\le \langle |T|^{2\alpha}x, x\rangle^{\frac{1}{2}} \langle |T^*|^{2(1-\alpha)}x, x\rangle^{\frac{1}{2}} \text{ (by Lemma 1.4.2)}\\ &\le 2^{-\frac{1}{r}} \left(\langle |T|^{2\alpha}x, x\rangle^r + \langle |T^*|^{2(1-\alpha)}x, x\rangle^r\right)^{\frac{1}{r}} \text{ (by Lemma 1.4.3 (a))}\\ &\le 2^{-\frac{1}{r}} \left(\langle |T|^{2\alpha r}x, x\rangle + \langle |T^*|^{2(1-\alpha)r}x, x\rangle\right)^{\frac{1}{r}} \text{ (by Lemma 2.1.1(1))}\end{aligned}$$

Thus,

$$|\langle Tx, x\rangle|^r \le \frac{1}{2}\left(\langle |T|^{2\alpha r}x, x\rangle + \langle |T^*|^{2(1-\alpha)r}x, x\rangle\right).$$

By taking the supremum on both sides in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$, we get the desired result. ■

In the following theorem, we give another upper bound for the numerical radius. It can be found in [21].

Theorem 2.1.6. *Let $T \in \mathcal{B}(\mathcal{H})$, $0 < \alpha < 1$, and $r \ge 1$. Then*

$$w^{2r}(T) \le \left\|\alpha|T|^{2r} + (1-\alpha)|T^*|^{2r}\right\|.$$

Corrollary 2.1.1. *[21] Let $T \in \mathcal{B}(\mathcal{H})$ and $r \ge 1$. Then*

$$w^{2r}(T) \le \frac{1}{2}\left\||T|^{2r} + |T^*|^{2r}\right\|. \tag{2.4}$$

Recently, Safshekan and Farokhinia [36] gave the following theorem

Theorem 2.1.7. *Let $T \in \mathcal{B}(\mathcal{H})$ and $r \ge 1$. Then*

$$w^{2r}(T) \le \frac{1}{4}\left\||T|^2 + |T^*|^2\right\|^r + \frac{1}{4}\left\||T|^2 - |T^*|^2\right\|^r + \frac{1}{2}w^r\left(T^2\right). \tag{2.5}$$

Also, Najafabadi and Moradi [34] gave a lower bound for the numerical radius.

Theorem 2.1.8. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$\frac{1}{2}\sqrt{2w^4(T) + \frac{1}{8}\left\|(T+T^*)^2(T-T^*)^2\right\|} \le w^2(T). \tag{2.6}$$

To prove the next result, we need the following lemma.

Lemma 2.1.2. *[29] Let $T \in \mathcal{B}(\mathcal{H})$. If f and g are two non-negative continuous functions on*

$[0,\infty)$ satisfying $f(t)g(t)=t$ for all $t\in[0,\infty)$, then

$$|\langle Tx,y\rangle|\leq\|f(|T|)x\|\|g(|T^*|)y\|$$

for all $x,y\in\mathcal{H}$.

Theorem 2.1.9. *[37] Let $T\in\mathcal{B}(\mathcal{H})$ and f and g be two non-negative continuous functions on $[0,\infty)$ satisfying $f(t)g(t)=t$, $t\geq 0$. Then*

$$w^{2r}(T)\leq\frac{1}{2}\Big(\|T\|^{2r}+\Big\|\frac{1}{p}f^{pr}(|A^2|)+\frac{1}{q}g^{qr}(|(A^2)^*|)\Big\|\Big) \tag{2.7}$$

for all $r\geq 1$, $p\geq q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $qr\geq 2$.

Proof . *Let $x\in\mathcal{H}$ be any unit vector. Then*

$$\begin{aligned}
|\langle T^2x,x\rangle|^r &\leq \|f(|T^2|)x\|^r\|g(|(T^2)^*|)x\|^r \text{ (by Lemma (2.1.2))}\\
&= \langle f^2(|T^2|)x,x\rangle^{\frac{r}{2}}\langle g^2(|(T^2)^*|)x,x\rangle^{\frac{r}{2}}\\
&\leq \frac{1}{p}\langle f^2(|T^2|)x,x\rangle^{\frac{pr}{2}}+\frac{1}{q}\langle g^2(|(T^2)^*|)x,x\rangle^{\frac{qr}{2}} \text{ (by Lemma 2.1.1 (2))}\\
&\leq \frac{1}{p}\langle f^{pr}(|T^2|)x,x\rangle+\frac{1}{q}\langle g^{qr}(|(T^2)^*|)x,x\rangle \text{ (by Lemma 1.4.3 (a))}\\
&= \Big\langle\Big(\frac{1}{p}f^{pr}(|T^2|)+\frac{1}{q}g^{qr}(|(T^2)^*|)\Big)x,x\Big\rangle.
\end{aligned}$$

By using Lemma 1.4.1 and Lemma 2.1.1 (1), we deduce that

$$|\langle Tx,x\rangle|^{2r}\leq\Big(\|Tx\|^r\|T^*x\|^r+\Big\langle\big(\frac{1}{p}f^{pr}(|T^2|)+\frac{1}{q}g^{qr}(|(T^2)^*|)x,x\Big\rangle\Big).$$

By taking the supremum on both sides in the above inequality over $x\in\mathcal{H}$ with $\|x\|=1$, we get the inequality (2.7).

■

Corrollary 2.1.2. *Let $T\in\mathcal{B}(\mathcal{H})$, $r\geq 1$ and $0\leq\alpha\leq 1$. Then*

$$w^{2r}(T)\leq\frac{1}{2}\|T\|^{2r}+\frac{1}{4}\big\||T|^{4r\alpha}+|T^*|^{4r(1-\alpha)}\big\|. \tag{2.8}$$

Proof . *By taking $f(t)=t^\alpha$, $g(t)=t^{1-\alpha}$ and $p=q=2$ in inequality 2.7, we deduce the desired result.* ■

Moradi and Sababheh [32] gave the following theorem.

Theorem 2.1.10. *Let $T, S \in \mathcal{B}(\mathcal{H})$. Then*

$$w(T+S) \leq \frac{1}{\sqrt{2}} w\big(|T| + |S| + i(|T^*| + |S^*|)\big). \tag{2.9}$$

Proof . *Let $x \in \mathcal{H}$, using Lemma 1.4.2 gives*

$$\begin{aligned}
|\langle (T+S)x, x\rangle| &\leq |\langle Tx, x\rangle| + |\langle Sx, x\rangle| \\
&\leq \langle |T|x, x\rangle^{\frac{1}{2}} \langle |T^*|x, x\rangle^{\frac{1}{2}} + \langle |S|x, x\rangle^{\frac{1}{2}} \langle |S^*|x, x\rangle^{\frac{1}{2}} \\
&\leq \frac{1}{2} \langle |T|x, x\rangle + \langle |T^*|x, x\rangle + \langle |S^*|x, x\rangle + \langle |S|x, x\rangle \\
&\qquad \text{(by Arithmetic-geometric mean inequality)} \\
&\leq \frac{1}{2} \left| \langle (|T| + |S|)x, x\rangle + \langle (|T^*| + |S^*|)x, x\rangle \right| \\
&\leq \frac{1}{\sqrt{2}} \langle (|T| + |S| + i(|T^*| + |S^*|))x, x\rangle \\
&\text{(by the scalar inequality } |a+b| \leq \sqrt{2}|a+ib| \text{ where } a, b \in \mathbb{R}).
\end{aligned}$$

By taking the supremum on both sides in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$, we get the desired result. ∎

Taking $S = 0$ in Theorem 2.1.10, we get the following corollary.

Corrollary 2.1.3. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$w^2(T) \leq \frac{1}{2} w^2\big(|T| + i|T^*|\big). \tag{2.10}$$

2.1.1 Numerical radius inequalities for the off-diagonal parts of 2×2 operator matrices

In this section, we give some upper and lower bounds of the numerical radius for the off-diagonal parts 2×2 operator matrices .

Let $A, B, C, D \in \mathcal{B}(\mathcal{H})$. Then the operator matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ can be considered as an operator on $\mathcal{H} \oplus \mathcal{H}$, and is defined for $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathcal{H} \oplus \mathcal{H}$ by $\begin{bmatrix} A & B \\ C & D \end{bmatrix} x = \begin{bmatrix} Ax_1 + Bx_2 \\ Cx_1 + Dx_2 \end{bmatrix}$.

Lemma 2.1.3. *[22] Let $A, B, C, D \in B(\mathcal{H})$. Then*

1. $w\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right) = \max\{w(A), w(D)\}$.

2. $w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) = w\left(\begin{bmatrix} 0 & C \\ B & 0 \end{bmatrix}\right)$.

3. $w\left(\begin{bmatrix} 0 & B \\ -B & 0 \end{bmatrix}\right) = w\left(\begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix}\right) = w(B)$.

4. $w\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right) = \max\{w(A+B), w(A-B)\}$.

   In particular,

$$w\left(\begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix}\right) = w(B).$$

Bhunia, Bag and Paul [13] have obtained the following results.

Theorem 2.1.11. *Let $B, C \in \mathcal{B}(\mathcal{H})$. Then*

$$w^2\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \geq \frac{1}{4}\max\left\{\left\||C|^2 + |B^*|^2\right\|, \left\||B|^2 + |C^*|^2\right\|\right\}. \tag{2.11}$$

$$w^2\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \leq \frac{1}{2}\max\left\{\left\||C|^2 + |B^*|^2\right\|, \left\||B|^2 + |C^*|^2\right\|\right\}.$$

Proof . *Let $T = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$ and $H_\theta = \Re(e^{i\theta}T)$, $K_\theta = \Im(e^{i\theta}T)$. A simple computation gives*

$$H_\theta^2 + K_\theta^2 = \frac{1}{2}\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix},$$

where $X = |B|^2 + |C^*|^2$ and $Y = |C|^2 + |B^*|^2$. Therefore, using Theorem 1.9.1, we get

$$\frac{1}{2}\left\|\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}\right\| = \|H_\theta^2 + K_\theta^2\| \leq \|H_\theta\|^2 + \|K_\theta\|^2 \leq 2w^2(T).$$

Hence,

$$\frac{1}{2}\max\{\|X\|, \|Y\|\} \leq 2w^2(T).$$

This completes the proof of the first inequality in Theorem 2.1.11. Again, we have

$$H_\theta^2 - \frac{1}{2}\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} = -K_\theta^2 \leq 0.$$

Therefore,

$$H_\theta^2 \leq \frac{1}{2}\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}$$

and so,

$$\|H_\theta^2\| \leq \frac{1}{2}\left\|\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}\right\| = \frac{1}{2}\max\{\|X\|, \|Y\|\}.$$

Taking the supremum over $\theta \in \mathbb{R}$ and using Theorem 1.9.1, we get

$$w^2(T) \leq \frac{1}{2}\max\{\|X\|, \|Y\|\}.$$

This completes the proof of the second inequality in Theorem 2.1.11.

■

Bani-Domi and Kittaneh [8] have proved the following result.

Theorem 2.1.12. *Let $A, D, B, C \in \mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned} \left\|\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right\|^2 \quad &\leq \quad \max\{\|A\|^2, \|D\|^2\} + \max\{\|B\|^2, \|C\|^2\} + w\left(\begin{bmatrix} 0 & C^*D \\ B^*A & 0 \end{bmatrix}\right) \\ &\quad + \max\{\|A\|, \|D\|\}\max\{\|B\|, \|C\|\}. \end{aligned}$$

Corrollary 2.1.4. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then*

$$\left\|\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right\|^2 = \max\{\|A+B\|^2, \|A-B\|^2\} \leq \|A\|^2 + \|B\|^2 + \|A\|\|B\| + w(B^*A). \tag{2.12}$$

In [10], Bhatia and Kittaneh gave the following arithmetic-geometric mean inequality for positive operators.

Lemma 2.1.4. *If $X, Y \in \mathcal{B}(\mathcal{H})$ are positive operators, then*

$$\|XY\| \leq \left\|\frac{X^2 + Y^2}{2}\right\|. \tag{2.13}$$

Theorem 2.1.13. *Let $B, C \in \mathcal{B}(\mathcal{H})$. Then*

$$\omega^2\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \leq \frac{1}{2}\max\left\{\|B\|^2, \|C\|^2\right\} + \frac{1}{2}\max\left\{\left\||B^*|^{2(1-\alpha)}|C|^{2\alpha}\right\|, \left\||C^*|^{2(1-\alpha)}|B|^{2\alpha}\right\|\right\},$$

for any $0 \leq \alpha \leq 1$.

Proof . *Let $T = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$ and $x \in \mathcal{H} \oplus \mathcal{H}$ with $\|x\| = 1$. Then, from Lemma 1.4.2, we get*

$$\begin{aligned} |\langle Tx, x\rangle|^2 &\leq \langle |T|^{2\alpha}x, x\rangle\langle |T^*|^{2(1-\alpha)}x, x\rangle \\ &= \left|\langle |T|^{2\alpha}x, x\rangle\langle x, |T^*|^{2(1-\alpha)}x\rangle\right|. \end{aligned}$$

Using Buzano's inequality in Lemma 1.4.1, we obtain

$$\begin{aligned} |\langle Tx, x\rangle|^2 &\leq \frac{1}{2}\left(\left\||T|^{2\alpha}x\right\| \left\||T^*|^{2(1-\alpha)}x\right\| + \left|\langle |T|^{2\alpha}x, |T^*|^{2(1-\alpha)}x\rangle\right|\right) \\ &\leq \frac{1}{2}\left(\left\||T|^{2\alpha}\right\| \left\||T^*|^{2(1-\alpha)}\right\| \|x\|^2 + \left\||T^*|^{2(1-\alpha)}|T|^{2\alpha}\right\| \|x\|^2\right). \end{aligned}$$

Taking the supremum over all $x \in \mathcal{H}, \|x\| = 1$, we deduce that

$$\begin{aligned} w^2(T) &\leq \frac{1}{2}\left(\left\||T|^{2\alpha}\right\| \left\||T^*|^{2(1-\alpha)}\right\| + \left\||T^*|^{2(1-\alpha)}|T|^{2\alpha}\right\|\right) \\ &\leq \frac{1}{2}\left(\left\||T|^2\right\|^{\alpha} \left\||T^*|^2\right\|^{1-\alpha} + \left\||T^*|^{2(1-\alpha)}|T|^{2\alpha}\right\|\right) \\ &= \frac{1}{2}\left(\left\||T|^2\right\| + \left\||T^*|^{2(1-\alpha)}|T|^{2\alpha}\right\|\right). \end{aligned}$$

On the other hand, we have $|T|^2 = \begin{bmatrix} |C|^2 & 0 \\ 0 & |B|^2 \end{bmatrix}$ and $|T^*|^2 = \begin{bmatrix} |B^*|^2 & 0 \\ 0 & |C^*|^2 \end{bmatrix}$. Hence,

$$w^2\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \leq \frac{1}{2}\max\left\{\|B\|^2, \|C\|^2\right\} + \frac{1}{2}\max\left\{\left\||B^*|^{2(1-\alpha)}|C|^{2\alpha}\right\|, \left\||C^*|^{2(1-\alpha)}|B|^{2\alpha}\right\|\right\}.$$

■

Corrollary 2.1.5. *Let $B, C \in \mathcal{B}(\mathcal{H})$. Then*

$$\omega^{2r}\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \leq \frac{1}{2}\max\left\{\|B\|^{2r}, \|C\|^{2r}\right\} + \frac{1}{2}\max\left\{\left\||B^*|^{2r(1-\alpha)}|C|^{2r\alpha}\right\|, \left\||C^*|^{2r(1-\alpha)}|B|^{2r\alpha}\right\|\right\},$$

for any $r \geq 1$ and $0 \leq \alpha \leq 1$.

Proof . *The result follows from Theorem 2.1.13, in view of the convexity of the function $f(t) = t^r$ on $[0, \infty)$, and the fact that $\|XY\|^r \leq \|X^r Y^r\|$ for all positive operators $X, Y \in \mathcal{B}(\mathcal{H})$ (see, e.g. [11, p. 256]).* ■

In particular, if we take $B = C$ in Corollary 2.1.5, then we get the following corollary.

Corrollary 2.1.6. *Let $B \in \mathcal{B}(\mathcal{H})$, $r \geq 1$ and $0 \leq \alpha \leq 1$. Then*

$$w^{2r}(B) \leq \frac{1}{2}\left(\|B\|^{2r} + \left\||B^*|^{2r(1-\alpha)}|B|^{2r\alpha}\right\|\right). \tag{2.14}$$

Remark 2.1.4. *Using the inequality (2.13), we get*

$$\left\||B^*|^{2r(1-\alpha)}|B|^{2r\alpha}\right\| \leq \left\|\frac{|B|^{4r\alpha} + |B^*|^{4r(1-\alpha)}}{2}\right\|.$$

Hence, the inequality (2.14) is a refinement of the inequality (2.8).

In order to prove the next theorem, we need the following lemma.

Lemma 2.1.5. *[19] Let $x, y, z \in \mathcal{H}$. Then*

$$|\langle y, x\rangle|^2 + |\langle x, z\rangle|^2 \leq \|x\|^2\left(\max\left\{\|y\|^2, \|z\|^2\right\} + |\langle y, z\rangle|\right). \tag{2.15}$$

Theorem 2.1.14. *Let $B, C \in \mathcal{B}(\mathcal{H})$ and $r \geq 1$. Then*

$$\begin{aligned} w^{2r}\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) &\leq \frac{1}{4}\max\left\{\left\||C|^2 + |B^*|^2\right\|^r, \left\||B|^2 + |C^*|^2\right\|^r\right\} \\ &+ \frac{1}{4}\max\left\{\left\||C|^2 - |B^*|^2\right\|^r, \left\||B|^2 - |C^*|^2\right\|^r\right\} \\ &+ \frac{1}{2}\max\left\{w^r(BC), w^r(CB)\right\}. \end{aligned}$$

Proof . *Let* $T = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$ *and* $x \in \mathcal{H} \oplus \mathcal{H}$ *with* $\|x\| = 1$. *Then*

$$\begin{aligned} |\langle Tx, x\rangle|^2 &= |\langle T^*x, x\rangle| \, |\langle Tx, x\rangle| \\ &= |\langle x, Tx\rangle| \, |\langle x, T^*x\rangle| \\ &\leq \frac{1}{2}\big(|\langle x, Tx\rangle|^2 + |\langle x, T^*x\rangle|^2 \big). \end{aligned}$$

Using the inequality (2.15), we have

$$\begin{aligned} |\langle Tx, x\rangle|^2 &\leq \frac{1}{2}\|x\|^2 \left(\max\left\{\|Tx\|^2, \|T^*x\|^2\right\} + |\langle Tx, T^*x\rangle|\right) \\ &= \frac{1}{4}\left(\|Tx\|^2 + \|T^*x\|^2 + \left|\|Tx\|^2 - \|T^*x\|^2\right|\right) + \frac{1}{2}|\langle Tx, T^*x\rangle| \\ &= \frac{1}{4}\left\{\left|\left\langle \left(|T^*|^2 + |T|^2\right)x, x\right\rangle\right| + \left|\left\langle \left(|T^*|^2 - |T|^2\right)x, x\right\rangle\right|\right\} + \frac{1}{2}|\langle Tx, T^*x\rangle| \\ &\leq \frac{1}{4}\left\| \begin{bmatrix} |C|^2 + |B^*|^2 & 0 \\ 0 & |B|^2 + |C^*|^2 \end{bmatrix} \right\| + \frac{1}{4}\left\| \begin{bmatrix} |C|^2 - |B^*|^2 & 0 \\ 0 & |B|^2 - |C^*|^2 \end{bmatrix} \right\| \\ &+ \frac{1}{2}w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}^2\right) \\ &= \frac{1}{4}\max\left\{\left\||C|^2 + |B^*|^2\right\|, \left\||B|^2 + |C^*|^2\right\|\right\} \\ &+ \frac{1}{4}\max\left\{\left\||C|^2 - |B^*|^2\right\|, \left\||B|^2 - |C^*|^2\right\|\right\} + \frac{1}{2}\max\{\omega(BC), \omega(CB)\}. \end{aligned}$$

Taking the supremum over $x \in \mathcal{H}, \|x\| = 1$, *we get*

$$\begin{aligned} \omega^2\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) &\leq \frac{1}{4}\max\left\{\left\||C|^2 + |B^*|^2\right\|, \left\||B|^2 + |C^*|^2\right\|\right\} \\ &+ \frac{1}{4}\max\left\{\left\||C|^2 - |B^*|^2\right\|, \left\||B|^2 - |C^*|^2\right\|\right\} \\ &+ \frac{1}{2}\max\{w(BC), w(CB)\}. \end{aligned}$$

Now, using the convexity of the function $f(t) = t^r$ *on* $[0, \infty)$, *we get the desired result.* ■

Remark 2.1.5. *If we take* $B = C$ *in Theorem 2.1.14, then we reobtain the inequality (2.5).*

An inequality related to Buzano's inequality is given in the following lemma.

Lemma 2.1.6. *If $x, y, e \in \mathcal{H}$ with $\|e\| = 1$ and $0 \leq \alpha \leq 1$, then*

$$|\langle x, e\rangle\langle e, y\rangle|^2 \leq \frac{3+\alpha}{4}\|x\|^2\|y\|^2 + \frac{1-\alpha}{4}|\langle x, y\rangle|^2.$$

Proof . *From Lemma 1.4.1, we have*

$$\begin{aligned}
|\langle x, e\rangle\langle e, y\rangle|^2 &\leq \frac{1}{4}(\|x\|\|y\| + |\langle x, y\rangle|)^2 \\
&= \frac{1}{4}(\|x\|^2\|y\|^2 + 2\|x\|\|y\||\langle x, y\rangle| + |\langle x, y\rangle|^2) \\
&= \frac{1}{4}(\|x\|^2\|y\|^2 + 2\|x\|\|y\||\langle x, y\rangle| + \alpha|\langle x, y\rangle|^2 + (1-\alpha)|\langle x, y\rangle|^2) \\
&\leq \frac{1}{4}(\|x\|^2\|y\|^2 + 2\|x\|^2\|y\|^2 + \alpha\|x\|^2\|y\|^2 + (1-\alpha)|\langle x, y\rangle|^2) \\
&= \frac{3+\alpha}{4}\|x\|^2\|y\|^2 + \frac{1-\alpha}{4}|\langle x, y\rangle|^2.
\end{aligned}$$

■

Now, we state our next result, which is based on Lemma 2.1.6.

Theorem 2.1.15. *Let $B, C \in \mathcal{B}(\mathcal{H})$ and $0 \leq \alpha \leq 1$, we have*

$$w^4\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \leq \frac{3+\alpha}{8}\max\{\||C|^4 + |B^*|^4\|, \||B|^4 + |C^*|^4\|\} + \frac{1-\alpha}{4}\max\{w^2(BC), w^2(CB)\}.$$

Proof . *Let* $T = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$ *and* $x \in \mathcal{H} \oplus \mathcal{H}$ *with* $\|x\| = 1$. *Then*

$$\begin{aligned}
|\langle Tx, x\rangle|^4 &= |\langle Tx, x\rangle\langle x, T^*x\rangle|^2 \\
&\leq \frac{3+\alpha}{4}\|Tx\|^2\|T^*x\|^2 + \frac{1-\alpha}{4}|\langle Tx, T^*x\rangle|^2 \qquad (\textit{by Lemma 2.1.6}) \\
&\leq \frac{3+\alpha}{8}(\|Tx\|^4 + \|T^*x\|^4) + \frac{1-\alpha}{4}|\langle T^2x, x\rangle|^2 \\
&\quad (\textit{by the arithmetic-geometric mean inequality}) \\
&\leq \frac{3+\alpha}{8}\langle(|T|^4 + |T^*|^4)x, x\rangle + \frac{1-\alpha}{4}|\langle T^2x, x\rangle|^2 \\
&= \frac{3+\alpha}{8}\left\langle \begin{bmatrix} |C|^4 + |B^*|^4 & 0 \\ 0 & |B|^4 + |C^*|^4 \end{bmatrix} x, x \right\rangle + \frac{1-\alpha}{4}\left|\left\langle \begin{bmatrix} BC & 0 \\ 0 & CB \end{bmatrix} x, x \right\rangle\right|^2 \\
&\leq \frac{3+\alpha}{8} w\left(\begin{bmatrix} |C|^4 + |B^*|^4 & 0 \\ 0 & |B|^4 + |C^*|^4 \end{bmatrix}\right) + \frac{1-\alpha}{4} w^2\left(\begin{bmatrix} BC & 0 \\ 0 & CB \end{bmatrix}\right) \\
&= \frac{3+\alpha}{8}\max\{\||C|^4 + |B^*|^4\|, \||B|^4 + |C^*|^4\|\} + \frac{1-\alpha}{4}\max\{w^2(BC), w^2(CB)\}.
\end{aligned}$$

Taking the supremum over $x \in \mathcal{H}, \|x\| = 1$, *we get*

$$\omega^4\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \leq \frac{3+\alpha}{8}\max\{\||C|^4 + |B^*|^4\|, \||B|^4 + |C^*|^4\|\} + \frac{1-\alpha}{4}\max\{w^2(BC), w^2(CB)\}.$$

■

In particular, taking $B = C$ *in Theorem 2.1.15, we obtain the following corollary.*

Corrollary 2.1.7. *If* $B \in \mathcal{B}(\mathcal{H})$, *then for* $0 \leq \alpha \leq 1$, *we have*

$$w^4(B) \leq \frac{3+\alpha}{8}\left\||B|^4 + |B^*|^4\right\| + \frac{1-\alpha}{4}w^2(B^2).$$

Remark 2.1.6. *For every $0 \leq \alpha \leq 1$, we have*

$$\begin{aligned} w^4(B) &\leq \frac{3+\alpha}{8}\left\| |B|^4 + |B^*|^4 \right\| + \frac{1-\alpha}{4} w^2(B^2) \\ &\leq \frac{3+\alpha}{8}\left\| |B|^4 + |B^*|^4 \right\| + \frac{1-\alpha}{4}\left\| B^2 \right\|^2 \\ &= \frac{3+\alpha}{8}\left\| |B|^4 + |B^*|^4 \right\| + \frac{1-\alpha}{4}\left\| |B||B^*| \right\|^2 \\ &\leq \frac{3+\alpha}{8}\left\| |B|^4 + |B^*|^4 \right\| + \frac{1-\alpha}{4}\left\| \frac{|B|^2 + |B^*|^2}{2} \right\|^2 \\ &\quad (\textit{by the inequality (2.13)}) \\ &= \frac{3+\alpha}{8}\left\| |B|^4 + |B^*|^4 \right\| + \frac{1-\alpha}{4}\left\| \left(\frac{|B|^2 + |B^*|^2}{2} \right)^2 \right\| \\ &\leq \frac{3+\alpha}{8}\left\| |B|^4 + |B^*|^4 \right\| + \frac{1-\alpha}{8}\left\| |B|^4 + |B^*|^4 \right\| \\ &\quad (\textit{by the operator convexity of the function } f(t) = t^2 \textit{ on } [0,\infty)) \\ &= \frac{1}{2}\left\| |B|^4 + |B^*|^4 \right\|. \end{aligned}$$

This means that Corollary 2.1.7 refines the inequality (2.4) for $r = 2$.

The following theorem yields an improvement of the triangle inequality for the operator norm.

Theorem 2.1.16. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then*

$$\|A+B\|^2 \leq \frac{1}{2}\left(\left\| |A|^2 + |B|^2 \right\| + \left\| |A^*|^2 + |B^*|^2 \right\| \right) + \|A\|\|B\| + w(B^*A). \tag{2.16}$$

Proof . *Let $x, y \in \mathcal{H}$ with $\|x\| = \|y\| = 1$. Then we have*

$$|\langle (A+B)x, y\rangle|^2 \leq |\langle Ax, y\rangle|^2 + |\langle Bx, y\rangle|^2 + 2|\langle Ax, y\rangle||\langle y, Bx\rangle|.$$

Using Buzano's inequality, we get

$$|\langle (A+B)x, y\rangle|^2 \leq |\langle Ax, y\rangle|^2 + |\langle Bx, y\rangle|^2 + \|Ax\|\|Bx\| + |\langle Ax, Bx\rangle|.$$

Now, using Lemma 1.4.2, for $\alpha = \dfrac{1}{2}$, *we have*

$$\begin{aligned}
|\langle (A+B)x, y\rangle|^2 &\le \langle |A|x,x\rangle\langle |A^*|y,y\rangle + \langle |B|x,x\rangle\langle |B^*|y,y\rangle + \|Ax\|\|Bx\| + |\langle B^*Ax, x\rangle| \\
&\le \langle (|A|^2+|B|^2)x,x\rangle^{\frac{1}{2}}(|A^*|^2+|B^*|^2)y,y\rangle^{\frac{1}{2}} + \|Ax\|\|Bx\| + |\langle B^*Ax, x\rangle| \\
&\quad (\textit{by the Cauchy-Schwarz inequality}) \\
&\le \frac{1}{2}\left[\left\langle \left(|A|^2+|B|^2\right)x,x\right\rangle + \left\langle \left(|A^*|^2+|B^*|^2\right)y,y\right\rangle\right] + \|Ax\|\|Bx\| \\
&+ |\langle B^*Ax, x\rangle| \qquad (\textit{by the arithmetic-geometric mean inequality}).
\end{aligned}$$

Taking the supremum over $x, y \in \mathcal{H}$, $\|x\| = \|y\| = 1$, *we deduce the desired inequality.* ■

Remark 2.1.7. *It is easy to see that*

$$\frac{1}{2}\left(\left\||A|^2+|B|^2\right\| + \left\||A^*|^2+|B^*|^2\right\|\right) + \|A\|\|B\| + w(B^*A) \le (\|A\|+\|B\|)^2 .$$

Thus, the inequality (2.16) is a refinement of the triangle inequality for the operator norm.

Corrollary 2.1.8. *Let* $A, B \in \mathcal{B}(\mathcal{H})$. *Then*

$$\left\|\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right\|^2 \le \frac{1}{2}\left(\left\||A|^2+|B|^2\right\| + \left\||A^*|^2+|B^*|^2\right\|\right) + \|A\|\|B\| + w(B^*A).$$

Remark 2.1.8. *The above inequality is a refinement of the inequality (2.12).*

Our last main result for the numerical radii of the off-diagonal parts of 2×2 *operator matrices can be stated as follows.*

Theorem 2.1.17. *Let* $B, C \in \mathcal{B}(\mathcal{H})$. *Then*

$$\frac{1}{2}\sqrt{2w^4\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) + \frac{1}{8}\|B+C^*\|^2\|B-C^*\|^2} \le w^2\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right). \qquad (2.17)$$

Proof . *Let* $T = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$. *Then* $w(T) \ge \|\Re(T)\|$ *and* $w(T) \ge \|\Im(T)\|$. *So,*

$$w(T) \ge \left\|\frac{B+C^*}{2}\right\| \quad \textit{and} \quad w(T) \ge \left\|\frac{B-C^*}{2i}\right\|.$$

Hence,

$$\begin{aligned} w^2(T) &= \frac{1}{2}\sqrt{2w^4(T)+2w^4(T)} \\ &\geq \frac{1}{2}\sqrt{2w^4(T)+2\|\Re(T)\|^2\|\Im(T)\|^2} \\ &= \frac{1}{2}\sqrt{2w^4\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)+\frac{1}{8}\left\|B+C^*\right\|^2\left\|B-C^*\right\|^2}. \end{aligned}$$

■

Remark 2.1.9. *It is well known that*

$$\frac{1}{4}\left\||T|^2+|T^*|^2\right\| = \frac{1}{2}\left\|\Re^2(T)+\Im^2(T)\right\| = \frac{1}{4}\left\|\begin{bmatrix} |C|^2+|B^*|^2 & 0 \\ 0 & |C^*|^2+|B|^2 \end{bmatrix}\right\|.$$

Now, taking $A=\Re^2(T)$ *and* $B=\Im^2(T)$ *in the inequality (2.16), we get*

$$\begin{aligned} \left\|\Re^2(T)+\Im^2(T)\right\| &\leq \sqrt{\|\Re^4(T)+\Im^4(T)\|+\|\Re^2(T)\|\|\Im^2(T)\|+w(\Im^2(T)\Re^2(T))} \\ &\leq \sqrt{2w^4(T)+\|\Re(T)\|^2\|\Im(T)\|^2+\|(\Im^2(T)\Re^2(T))\|} \\ &\leq \sqrt{2w^4(T)+2\|\Re(T)\|^2\|\Im(T)\|^2} \\ &= \sqrt{2w^4(T)+\frac{1}{8}\left\|B+C^*\right\|^2\left\|B-C^*\right\|^2}. \end{aligned}$$

This implies that

$$\frac{1}{4}\left\|\begin{bmatrix} |C|^2+|B^*|^2 & 0 \\ 0 & |C^*|^2+|B|^2 \end{bmatrix}\right\| \leq \frac{1}{2}\sqrt{2w^4\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)+\frac{1}{8}\left\|B+C^*\right\|^2\left\|B-C^*\right\|^2}.$$

Hence,

$$\frac{1}{4}\max\left\{\left\||C|^2+|B^*|^2\right\|,\left\||B|^2+|C^*|^2\right\|\right\} \leq \frac{1}{2}\sqrt{2w^4\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)+\frac{1}{8}\left\|B+C^*\right\|^2\left\|B-C^*\right\|^2}. \tag{2.18}$$

This proves that the inequality (2.17) is an improvement of the inequality (2.11).

Remark 2.1.10. *If we take $B = C$ in Theorem 2.1.17, then we get*

$$\frac{1}{2}\sqrt{2w^4(B) + \frac{1}{8}\left\|B + B^*\right\|^2 \left\|B - B^*\right\|^2} \leq w^2(B). \tag{2.19}$$

The inequality (2.19) is sharper than the inequality (2.6).
Also, if we take $B = C$ in the inequality (2.18), then from the inequality (2.19), we deduce that

$$\frac{1}{4}\left\||B|^2 + |B^*|^2\right\| \leq w^2(B).$$

This means that the inequality (2.19) is a refinement of the first inequality in (2.2).

Chapter 3

Some inequalities for the Euclidean operator radius

In this chapter, we present some new upper and lower bounds for the Euclidean operator radius of a pair of Hilbert space operators. Some of these bounds refine certain existing ones. As applications of these results, we provide some new bounds for the classical numerical radius.

3.1 Some inequalities for the Euclidean operator radius

In this section, we present some results related to Euclidean operator radius.

Definition 3.1.1. *Let $B, C \in \mathcal{B}(\mathcal{H})$. The Euclidean operator radius is defined by*

$$w_e(B,C) = \sup_{\|x\|=1} \left(|\langle Bx, x\rangle|^2 + |\langle Cx, x\rangle|^2\right)^{\frac{1}{2}}.$$

Lemma 3.1.1. *Let $B, C \in \mathcal{B}(\mathcal{H})$ be self-adjoint operators. Then*

$$w_e(B,C) = w(B + iC).$$

Proof . *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned} w_e^2(B,C) &= \sup_{\|x\|=1} \left(|\langle Bx, x\rangle|^2 + |\langle Cx, x\rangle|^2\right) \\ &= \sup_{\|x\|=1} |\langle (B+iC)x, x\rangle|^2 \\ &= w^2(B+iC). \end{aligned}$$

■

It is mentioned in [27] and [40] that $w_e(\cdot,\cdot) : \mathcal{B}^2(\mathcal{H}) \to [0,\infty)$ is a norm that satisfies the inequality

$$\frac{\sqrt{2}}{4}\|B^*B + C^*C\|^{\frac{1}{2}} \le w_e(B,C) \le \|B^*B + C^*C\|^{\frac{1}{2}}. \tag{3.1}$$

A sharp lower bound for the Euclidean operator radius can be stated as follows

Theorem 3.1.1. *[19] Let $B, C \in \mathcal{B}(\mathcal{H})$. Then*

$$\frac{\sqrt{2}}{2}\left(w(B^2+C^2)\right)^{\frac{1}{2}} \le w_e(B,C) \le \| B^*B + C^*C \|^{\frac{1}{2}} . \tag{3.2}$$

Proof . *Let $x \in \mathcal{H}$ be any unit vector. Then*

$$\begin{aligned} |\langle Bx,x\rangle|^2 + |\langle Cx,x\rangle|^2 &\ge \frac{1}{2}\big(|\langle Bx,x\rangle| + |\langle Cx,x\rangle|\big)^2 \\ &\ge \frac{1}{2}|\langle (B \pm C)x,x\rangle|^2 \end{aligned} \tag{3.3}$$

Taking the supremum over $x \in \mathcal{H}$ with $\|x\| = 1$ in (3.3), we deduce that

$$w_e^2(B,C) \ge \frac{1}{2}w^2(B \pm C). \tag{3.4}$$

Hence,

$$\begin{aligned} 2w_e^2(B,C) &\ge \frac{1}{2}\Big(w^2(B+C) + w^2(B-C)\Big) \\ &\ge \frac{1}{2}\Big(w\big((B+C)^2\big) + w\big((B-C)^2\big)\Big) \\ &\ge \frac{1}{2}\Big(w\big((B+C)^2 + (B-C)^2\big)\Big) \\ &= w(B^2+C^2), \end{aligned}$$

which prove the first inequality in (3.2).

To prove the second inequality in (3.2), let $x \in \mathcal{H}$ be any unit vector. Then

$$\begin{aligned}|\langle Bx,x\rangle|^2+|\langle Cx,x\rangle|^2 &\leq \|Bx\|^2\|x\|^2+\|Cx\|^2\|x\|^2\\ &\leq \|Bx\|^2+\|Cx\|^2\\ &\leq \langle Bx,Bx\rangle+\langle Cx,Cx\rangle\\ &\leq \langle B^*Bx,x\rangle+\langle C^*Cx,x\rangle\\ &\leq \langle (B^*B+C^*C)x,x\rangle.\end{aligned}$$

Taking the supremum in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$, we obtain

$$w_e^2(B,C) \leq \|B^*B+C^*C\|$$

■

Corrollary 3.1.1. *[19] Let $B, C \in \mathcal{B}(\mathcal{H})$ be self-adjoint. Then*

$$\frac{\sqrt{2}}{2}\|B^2+C^2\|^{\frac{1}{2}} \leq w_e(B,C) \leq \|B^2+C^2\|^{\frac{1}{2}}. \tag{3.5}$$

The following particular case reads as follows

Corrollary 3.1.2. *Let $A \in \mathcal{B}(\mathcal{H})$ and $\alpha, \beta \in \mathbb{C}$. Then*

$$\frac{1}{2}w\big(\alpha^2A^2+\beta^2(A^*)^2\big) \leq \big(|\alpha|^2+|\beta|^2\big)w^2(A) \quad \leq \||\alpha|^2A^*A+|\beta|^2AA^*\|. \tag{3.6}$$

Proof . *If we choose in Theorem (3.1.1), $B = \alpha A$ and $C = \beta A^*$, we obtain*

$$\begin{aligned}w_e^2(\alpha A,\beta A^*) &= \sup_{\|x\|=1}\big(|\langle \alpha Ax,x\rangle|^2+|\langle \beta A^*x,x\rangle|^2\big)\\ &= \sup_{\|x\|=1}(|\alpha|^2\ |\langle Ax,x\rangle|^2+|\beta|^2\ |\langle A^*x,x\rangle|^2)\\ &= \sup_{\|x=1}\big(|\alpha|^2+|\beta|^2)|\langle Ax,x\rangle|^2\big)\\ &= (|\alpha|^2+|\beta|^2)w^2(A)\end{aligned}$$

Hence,

$$w(B^2+C^2) = w\big(\alpha^2A^2++\beta^2(A^*)^2\big).$$

■

Remark 3.1.1. *If we choose (3.6) $\alpha = \beta \neq 0$, then we get the inequality*

$$\frac{1}{4}\|A^2 + (A^*)^2\| \leq w^2(A) \leq \frac{1}{2}\|A^*A + AA^*\| \tag{3.7}$$

for any operator $A \in \mathcal{B}(\mathcal{H})$.
If we choose in (3.6), $\alpha = 1\,, \beta = i\,,$ then

$$\frac{1}{4}w\big(A^2 - (A^*)^2\big) \leq w^2(A) \tag{3.8}$$

for any $A \in \mathcal{B}(\mathcal{H})$.

Theorem 3.1.2. *[19] Let $B, C \in \mathcal{B}(\mathcal{H})$. Then*

$$\frac{\sqrt{2}}{2}\max\Big\{w(B+C), w(B-C)\Big\} \leq w_e(B,C) \leq \frac{\sqrt{2}}{2}\Big(w^2(B+C) + w^2(B-C)\Big)^{\frac{1}{2}}. \tag{3.9}$$

Proof . *Let $x \in \mathcal{H}$ be unit vector. Then*

$$w_e^2(B,C) \geq \frac{1}{2}w^2(B+C) \quad and \quad w_e^2(B,C) \geq \frac{1}{2}w^2(B-C).$$

Thus,

$$w_e(B,C) \geq \frac{\sqrt{2}}{2}\max\{w(B+C), w(B-C)\}.$$

To prove the second inequality, let $x \in \mathcal{H}$ be any unit vector. Using the parallelogram identity gives

$$\begin{aligned} 2\big(|\langle Bx,x\rangle|^2 + |\langle Cx,x\rangle|^2\big) &= |\langle Bx,x\rangle - \langle Cx,x\rangle|^2 + |\langle Bx,x\rangle + \langle Cx,x\rangle|^2 \\ &\leq w^2(B-C) + w^2(B+C) \end{aligned}$$

∎

Proposition 3.1.1. *Let $B, C \in \mathcal{B}(\mathcal{H})$. Then*

$$w_e(B,C) \leq \big(w^2(C-B) + 2w(B)w(C)\big)^{\frac{1}{2}}. \tag{3.10}$$

Proof . *Let $x \in \mathcal{H}$ be any unit vector. Then*

$$|\langle Cx,x\rangle|^2 - 2Re\big(\langle Cx,x\rangle\overline{\langle Bx,x\rangle}\big) + |\langle Bx,x\rangle|^2 = |\langle Cx,x\rangle - \langle Bx,x\rangle|^2 \leq w^2(C-B).$$

Thus,

$$\begin{aligned}|\langle Cx,x\rangle|^2+|\langle Bx,x\rangle|^2 &\le w^2(C-B)+2Re\big(\langle Cx,x\rangle\overline{\langle Bx,x\rangle}\big)\\ &\le w^2(C-B)+2|\langle Cx,x\rangle||\langle Bx,x\rangle|.\end{aligned} \tag{3.11}$$

Taking the supremum in (3.11) over $\|x\|=1$, *we deduce the desired inequality (3.10). In particular, if* B *and* C *are self- adjoint operators, then*

$$w_e(B,C)\le(\|B-C\|^2+2\|B\|\|C\|)^{\frac{1}{2}}. \tag{3.12}$$

■

Now, if we apply the inequality (3.12) for $B=\dfrac{A+A^*}{2}$ *and* $C=\dfrac{A-A^*}{2i}$, *then*

$$w(A)\le\left(\Big\|\frac{(1+i)A+(1-i)A^*}{2}\Big\|^2+2\Big\|\frac{A+A^*}{2}\Big\|\Big\|\frac{A-A^*}{2}\Big\|\right)^{\frac{1}{2}}.$$

Proposition 3.1.2. *Let* $B,C\in\mathcal{B}(\mathcal{H})$. *Then*

$$w_e(B,C)\le\Big(2\min\{w^2(B),w^2(C)\}+w(B-C)w(B+C)^{\frac{1}{2}}. \tag{3.13}$$

Proof . *From the parallelogram identity,*

$$2\big(|\langle Bx,x\rangle|^2+|\langle Cx,x\rangle|^2\big)=|\langle Bx,x\rangle-\langle Cx,x\rangle|^2+|\langle Bx,x\rangle+\langle Cx,x\rangle|^2,$$

we get

$$2w_e^2(B,C)=w_e^2(B-C,B+C). \tag{3.14}$$

Now, if we apply Proposition (3.1.1) for $B-C,B+C$ *instead of* B *and* C, *we obtain*

$$w_e^2(B-C,B+C)\le 4w^2(C)+2w(B-C)-w(B+C). \tag{3.15}$$

Hence,

$$w_e^2(B,C)\le 2w^2(C)+w(B-C)w(B+C). \tag{3.16}$$

Now, if we exchange C *by* B *in the inequality (3.16), then we get*

$$w_e^2(B,C)\le 2w^2(B)+w(B-C)w(B+C).$$

Hence, the required result is obtained. ■

Theorem 3.1.3. *[19] Let $B, C \in \mathcal{B}(\mathcal{H})$. Then*

$$w_e^2(B,C) \le \max\{\|B\|^2, \|C\|^2\} + w(C^*B). \tag{3.17}$$

Corrollary 3.1.3. *Let $B, C \in \mathcal{B}(\mathcal{H})$. Then*

$$w_e^2(B,C) \le \frac{1}{2}\Big(\max\{\|B-C\|^2, \|B+C\|^2\} + w\big((B^*-C^*)(B+C)\big)\Big). \tag{3.18}$$

The constant $\frac{1}{2}$ in the inequality (3.18) is the best possible.

Proof . *By replacing $B+C$, $B-C$ instead of B, C, respectively, in the inequality (3.17) and using the identity $w_e^2(B+C, B-C) = 2w_e(B,C)^2$, yields*

$$w_e(B,C)^2 = \frac{1}{2}w_e^2(B+C, B-C) \le \frac{1}{2}\max\{\|B+C\|^2, \|B-C\|^2\} + w\big((B-C)^2(B+C)\big),$$

as required. ∎

Corollary 3.1.4. *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$w^2(A) \le \frac{1}{4}\Big(\max\{\|A+A^*|^2, \|A-A^*\|^2\} + w\big((A^*-A)(A+A^*)\big)\Big). \tag{3.19}$$

Proof . *If $A = B + iC$ is the Cartesian decomposition of A, then $w_e^2(B,C) = w^2(A)$.*

$$w(C^*B) = \frac{1}{4}w\big((A^*-A)(A+A^*)\big).$$

From the inequality (3.17), we obtain the desired inequality.

∎

Theorem 3.1.4. *[19] Let $B, C \in \mathcal{B}(\mathcal{H})$. Then*

$$w_e^2(B,C) \le \frac{1}{2}\big(\|B^*B + C^*C\| + \|B^*B - C^*C\|\big) + w(C^*B). \tag{3.20}$$

Corrollary 3.1.5. *Let $B, C \in \mathcal{B}(\mathcal{H})$. Then*

$$w_e^2(B,C) \le \frac{1}{2}\Big(\|B^*B + C^*C\| + \|B^*C + C^*B\| + w\big((B^*-C^*)(B+C)\big)\Big). \tag{3.21}$$

Corrollary 3.1.6. *Let $B, C \in \mathcal{B}(\mathcal{H})$ be self-adjoint. Then*

$$w_e^2(B,C) \le \frac{1}{2}\big(\|B^2 + C^2\| + \|B^2 - C^2\|\big) + w(CB). \tag{3.22}$$

Remark 3.1.2. *We observe that, if B and C are chosen to be the Cartesian decomposition of A, then we obtain from the inequality (3.22) the following inequality*

$$w^2(A) \leq \frac{1}{4}\Big(\|A^*A + AA^*\| + \|A^2 + (A^*)^2\| + w\big((A^* - A)(A + A^*)\big)\Big). \tag{3.23}$$

The constant $\frac{1}{4}$ is the best possible.
Now, if we choose in the inequality (3.20) *$B = A$ and $C = A^*$, we deduce that*

$$w^2(A) \leq \frac{1}{4}\big(\|A^*A + AA^*\|\|A^*A - AA^*\|\big) + \frac{1}{2}w(A^2). \tag{3.24}$$

Theorem 3.1.5. *[27] Let $B, C \in \mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned}&\frac{1}{2}w(B^2 + C^2) + \frac{1}{2}\max\big\{w(B), w(C)\big\}\big|w(B + C) - w(B - C)\big| \\ &\leq w_e^2(B, C) \leq \min\big\{w(|B| + i|C|)w(|B^*| + i|C^*|), w(|B| + i|C^*|)w(|B^*| + i|C|)\big\}.\end{aligned}$$

Proof . *Let $x \in \mathcal{H}$ be any unit vector. Then*

$$\begin{aligned}|\langle Bx, x\rangle|^2 + |\langle Cx, x\rangle|^2 &\geq \frac{1}{2}\big(|\langle Bx, x\rangle| + |\langle Cx, x\rangle|\big)^2 \\ &\geq \frac{1}{2}(|\langle Bx, x\rangle \pm \langle Cx, x\rangle|)^2 \\ &= \frac{1}{2}|\langle (B \pm C)x, x\rangle|^2.\end{aligned}$$

Taking the supremum in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$, we have

$$w_e^2(B, C) \geq \frac{1}{2}w^2(B \pm C). \tag{3.25}$$

Hence, it follows from the inequality (3.25) that

$$\begin{aligned}w_e^2(B, C) &\geq \frac{1}{2}\max\big\{w^2(B + C), w^2(B - C)\big\} \\ &= \frac{w^2(B + C) + w^2(B - C)}{4} + \frac{|w^2(B + C) - w^2(B - C)|}{4} \\ &\geq \frac{w\big((B + C)^2\big) + w\big((B - C)^2\big)}{4} + \big(w(B + C) + w(B - C)\big)\frac{\big|w(B + C) - w(B - C)\big|}{4} \\ &\geq \frac{w\big((B + C)^2 + w(B - C)^2\big)}{4} + \big(w(B + C) + (B - C)\big)\frac{\big|w(B + C) - w(B - C)\big|}{4}.\end{aligned} \tag{3.26}$$

Therefore,

$$w_e^2(B,C) \geq \frac{w(B^2+C^2)}{2} + \frac{w(B)}{2}\big|w(B+C) - w(B-C)\big|.$$

Interchanging B and C in the inequality (3.26), we obtain

$$w_e^2(B,C) \geq \frac{w(B^2+C^2)}{2} + \frac{w(C)}{2}\big|w(B+C) - w(B-C)\big|. \tag{3.27}$$

Hence, the required first inequality follows from the inequalities (3.26) *and* (3.27).

To prove the second inequality, let $x \in \mathcal{H}$ be any unit vector.

$$\begin{aligned}
|\langle Bx,x\rangle|^2 + |\langle Cx,x\rangle|^2 &\leq \langle |B|x,x\rangle\langle |B^*|x,x\rangle + \langle |C|x,x\rangle\langle |C^*|x,x\rangle \\
&\quad (\text{by Lemma } (1.4.2)) \\
&\leq \Big(\big(\langle |B|x,x\rangle^2 + \langle |C|x,x\rangle^2\big)\big(\langle |B^*|x,x\rangle^2 + \langle |C^*|x,x\rangle^2\big)\Big)^{\frac{1}{2}} \\
&= \big(|\langle |B|x,x\rangle + i\langle |C|x,x\rangle|^2 |\langle |B^*|x,x\rangle + i\langle |C^*|x,x\rangle|^2\big)^{\frac{1}{2}} \\
&= |\langle (|B|+i|C|)x,x\rangle||\langle (|B^*|+i|C^*|)x,x\rangle| \\
&\leq w(|B|+i|C|)w(|B^*|+i|C^*|).
\end{aligned}$$

Taking the supremum in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$, we obtain

$$w_e^2(B,C) \leq w(|B|+i|C|)w(|B^*|+i|C^*|). \tag{3.28}$$

Replacing C by C^* in the inequality (3.28), we get

$$w_e^2(B,C) \leq w(|B|+i|C^*|)w(|B^*|+i|C|). \tag{3.29}$$

Hence, the required second inequality follows from the inequalities in (3.28) and (3.29). ■

Remark 3.1.3. *The lower bound of $w_e(B,C)$ in Theorem 3.1.5 is stronger than the lower bound in (3.2). Also, it not difficult to verify that*

$$w^2(|B|+i|C|) \leq \|B^*B + C^*C\|.$$

Therefore,

$$w(|B|+i|C|)w(|B^*|+i|C^*|) \leq \|B^*B + C^*C\|^{\frac{1}{2}}\|BB^* + CC^*\|^{\frac{1}{2}}.$$

Similarly,

$$w(|B|+i|C^*|)w(|B^*|+i|C|)\leq \|B^*B+CC^*\|^{\frac{1}{2}}\|BB^*+C^*C\|^{\frac{1}{2}}.$$

Hence, it follows from the second inequality in Theorem 3.1.5 that

$$w_e^2(B,C)\leq \min\left\{\|B^*B+C^*C\|^{\frac{1}{2}}\|BB^*+CC^*\|^{\frac{1}{2}}, \|B^*B+CC^*\|^{\frac{1}{2}}\|BB^*+C^*C\|^{\frac{1}{2}}\right\}.$$

If $\|BB^*+CC^*\|\leq\|B^*B+C^*C\|$, *then the above bound of* $w_e(B,C)$ *is better than the upper bound in the inequality* (3.1).

The following corollary is an immediate consequence of Theorem 3.1.5

Corollary 3.1.7. *Let* $B,C\in\mathcal{B}(\mathcal{H})$ *be self- adjoint. Then*

$$\begin{aligned}\frac{1}{2}\left\|B^2+C^2\right\|+\frac{1}{2}\max\left\{\|B\|,\|C\|\right\}\big|\|B+C\|-\|B-C\|\big|&\\ \leq w_e^2(B,C)\leq w^2(|B|+i|C|).&\end{aligned}\tag{3.30}$$

Note that the second inequality in (3.30) is better than the one in (3.5). In particular, by considering $B=\Re(A)$ *and* $C=\Im(A)$ *in (3.30), we obtain the following new upper and lower bounds for the numerical radius of the operator* A *.*

Corollary 3.1.8. *Let* $A\in\mathcal{B}(\mathcal{H})$. *Then*

$$\begin{aligned}\frac{1}{4}\|A^*A+AA^*\|+\frac{1}{2}\max\{\|\Re(A)\|,\|\Im(A)\|\}\big|\|\Re(A)+\Im(A)\|-\|\Re(A)-\Im(A)\|\big|&\\ \leq w^2(A)\leq w^2\big(|\Re(A)|+i|\Im(A)|\big).&\end{aligned}$$

Remark 3.1.4. *We have* $w^2\big(|\Re(A)|+i|\Im(A)|\big)\leq\frac{1}{2}\|A^*A+AA^*\|$. *Therefore, the inequality in Corollary (3.1.8) is stronger than the second inequality in (2.2). Also,* $\frac{1}{2}\|A^*A+AA^*\|\leq \|\Re(A)\|^2+\|\Im(A)\|^2$. *So, the upper bound for* $w(A)$ *in Corollary 3.1.8 is stronger than upper bound in* (2.3).

If we consider $B=A$ *and* $C=A^*$ *in Theorem (3.1.5), we obtain the following corollary.*

Corollary 3.1.9. *Let* $A\in\mathcal{B}(\mathcal{H})$. *Then*

$$\frac{1}{2}\|\Re(A^2)\|+\frac{1}{2}w(A)\big|\|\Re(A)\|-\|\Im(A)\|\big|\leq w^2(A)\leq\frac{1}{2}w(|A|+i|A^*|)w(|A^*|+i|A|).$$

Remark 3.1.5. *Clearly, the first inequality in Corollary (3.1.9) is sharper than the inequality* $\frac{1}{2}\|\Re(A^2)\|\leq w^2(A)$, *which is given in (3.7). Observe that*

$\frac{1}{2}w(|A|+i|A^*|)w(|A^*|+i|A|) \leq \frac{1}{2}\|A^*A+AA^*\|$, *and so the second inequality in Corollary* (3.1.9) *is stronger than the second inequality in (2.1).*

For the rest of our result, we need the following lemma which can be found in [3].

Lemma 3.1.2. *Let $x, y \in \mathcal{H}$. Then for any $t \geq 0$,*

$$|\langle x,y\rangle|^2 \leq \frac{1}{t+1}\|x\|\|y\||\langle x,y\rangle| + \frac{t}{1+t}\|x\|^2\|y\|^2.$$

Theorem 3.1.6. *Let $B, C \in \mathcal{B}(\mathcal{H})$ and let $t \geq 0$. Then*

$$w_e^2(B,C) \leq \frac{t}{1+t}\|B^*B+C^*C\| + \frac{1}{1+t}w_e(B,C)\|B^*B+C^*C\|^{\frac{1}{2}}. \tag{3.31}$$

Proof . *Let $x \in \mathcal{H}$ be any unit vector. Setting $e_2 = |\langle Bx,x\rangle|^2 + |\langle Cx,x\rangle|^2$, it follows that*

$$\begin{aligned}
e_2 &\leq \frac{1}{1+t}\big(\|Bx\||\langle Bx,x\rangle| + \|Cx\||\langle Cx,x\rangle|\big) + \frac{t}{1+t}\big(\|Bx\|^2+\|Cx\|^2\big) \\
&\quad (\text{by Lemma 3.1.2}) \\
&\leq \frac{1}{1+t}(|\langle Bx,x\rangle|^2 + |\langle Cx,x\rangle|^2)^{\frac{1}{2}}(\|Bx\|^2+\|Cx\|^2)^{\frac{1}{2}} + \frac{t}{1+t}(\|Bx\|^2+\|Cx\|^2) \\
&\quad (\text{by the Cauchy-Schwarz inequality}) \\
&\leq \frac{t}{1+t}|\langle (B^*B+C^*C)x,x\rangle| + \frac{1}{1+t}w_e(B,C)|\langle (B^*B+C^*C)x,x\rangle|^{\frac{1}{2}}.
\end{aligned}$$

Taking the supremum of both sides in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$, we get the required inequality. ■

Remark 3.1.6. *The inequality (3.31) is a refinement of the second inequality in (3.2). Indeed,*

$$\begin{aligned}
w_e^2(B,C) &\leq \frac{t}{1+t}\|B^*B+C^*C\| + \frac{1}{1+t}w_e(B,C)\|B^*B+C^*C\|^{\frac{1}{2}} \\
&\leq \frac{t}{1+t}\|B^*B+C^*C\| + \frac{1}{1+t}\|B^*B+C^*C\| \\
&\quad (\text{by the second inequality in (3.2)}) \\
&\leq \|B^*B+C^*C\|.
\end{aligned}$$

Next lower bound for $w_e(B,C)$ reads as follows.

Theorem 3.1.7. *[27] Let $B, C \in \mathcal{B}(\mathcal{H})$. Then*

$$\frac{1}{2}\max\left\{w^2(B+C)+c^2(B-C), w^2(B-C)+c^2(B+C)\right\} \leq w_e^2(B,C),$$

where $c(Y) = \inf_{\|x\|=1} |\langle Yx, x\rangle|$.

Remark 3.1.7. *Clearly, the bound in Theorem (3.1.7) is stronger than the first bound in* (3.8).

In the following theorem, we give a lower bound for $w_e(\cdot,\cdot)$.

Theorem 3.1.8. *[27] Let* $B, C \in \mathcal{B}(\mathcal{H})$. *Then*

$$\max\{w^2(B) + c^2(C), w^2(C) + c^2(B)\} \le w_e^2(B, C).$$

Proof . *Let* $x \in \mathcal{H}$ *be any unit vector. Then*

$$|\langle Bx, x\rangle + \langle Cx, x\rangle|^2 + |\langle Bx, x\rangle - \langle Cx, x\rangle|^2 = 2(|\langle Bx, x\rangle|^2 + |\langle Cx, x\rangle|^2).$$

Thus,

$$|\langle (B+C)x, x\rangle|^2 + |\langle (B-C)x, x\rangle|^2 = 2(|\langle Bx, x\rangle|^2 + |\langle Cx, x\rangle|^2).$$

Hence,

$$w_e^2(B+C, B-C) = 2w_e^2(B, C). \tag{3.32}$$

Now replacing B *by* $B+C$ *and* C *by* $B-C$ *in Theorem (3.1.7), we obtain that*

$$2\max\left\{w^2(B) + c^2(C), w^2(C) + c^2(B)\right\} \le w_e^2(B+C, B-C). \tag{3.33}$$

Using the identity (3.32) in (3.33) gives the required inequality. ■

Theorem 3.1.9. *[27] Let* $B, C \in \mathcal{B}(\mathcal{H})$. *Then*

$$w_e^2(B, C) \le \min\{w^2(B+C), w^2(B-C)\} + \frac{1}{2}\|C^*C + BB^*\| + w(BC). \tag{3.34}$$

Proof . *Let* $x \in \mathcal{H}$ *be any unit vector. Then*

$$\begin{aligned}|\langle Cx, x\rangle|^2 - 2Re[\langle Cx, x\rangle\overline{\langle Bx, x\rangle}] + |\langle Bx, x\rangle|^2 &= |\langle Cx, x\rangle - \langle Bx, x\rangle|^2\\ &= |\langle (C-B)x, x\rangle|^2\\ &\le w^2(C-B).\end{aligned}$$

Thus,

$$\begin{aligned}|\langle Cx,x\rangle|^2+|\langle Bx,x\rangle|^2 &\le w^2(C-B)+2Re[\langle Cx,x\rangle\overline{\langle Bx,x\rangle}]\\ &\le w^2(C-B)+2|\langle Cx,x\rangle\langle Bx,x\rangle|\\ &\le w^2(C-B)+\|Cx\|\|B^*x\|+|\langle Cx,B^*x\rangle|(\text{by Lemma (1.4.1)})\\ &\le w^2(C-B)+\frac{1}{2}(\|Cx\|^2+\|B^*x\|^2)+w(BC)\\ &\le w^2(C-B)+\frac{1}{2}\|C^*C+BB^*\|+w(BC).\end{aligned}$$

Taking the supremum of both sides in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$, we obtain

$$w_e^2(B,C) \le w^2(B-C)+\frac{1}{2}\|C^*C+BB^*\|+w(BC). \tag{3.35}$$

Replacing C by $-C$ in the above inequality, we get

$$w_e^2(B,C) \le w^2(B+C)+\frac{1}{2}\|C^*C+BB^*\|+w(BC). \tag{3.36}$$

Hence, the desired inequality follows directly from (3.35) and (3.36). ■

Remark 3.1.8. *If we take $B = C = A$ in (3.35), then we get the following upper bound (see [2])*

$$w^2(A) \le \frac{1}{4}\|A^*A+AA^*\|+\frac{1}{2}w(A^2).$$

Theorem 3.1.10. *[27] Let $B, C \in \mathcal{B}(\mathcal{H})$. Then*

$$\max_{0\le\alpha\le1} w(\alpha B^2+(1-\alpha)C^2) \le w_e^2(B,C). \tag{3.37}$$

Here we give an example in which we illustrate that the inequality (3.37) can be better than the inequality of Theorem (3.1.1). To see this, consider $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$,

where $B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ *and* $C^2 = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$.

Thus,

$$w\Big(\frac{B^2}{2}+\frac{C^2}{2}\Big) = 2 < 4 = \max_{0\le\alpha\le1} w(\alpha B^2+(1-\alpha)C^2).$$

Remark 3.1.9. *If we replace B by $\Re(A)$ and C by $\Im(A)$ in Theorem (3.1.10), we obtain*

$$\left\| \alpha(\Re(A))^2 + (1-\alpha)(\Im(A))^2 \right\| \leq w^2(A). \tag{3.38}$$

In particular, for $\alpha = \frac{1}{2}$, we get

$$\frac{1}{4}\|A^*A + AA^*\| \leq w^2(A).$$

Next, we give the following theorem.

Theorem 3.1.11. *[27] Let $B, C \in \mathcal{B}(\mathcal{H})$. Then*

$$w_e^2(B,C) \leq w^2(\sqrt{\alpha}B + \sqrt{1-\alpha}C) + w^2(\sqrt{1-\alpha}B + \sqrt{\alpha}C),$$

for all $\alpha \in [0,1]$.

Proof . *Let $x \in \mathcal{H}$ be any unit vector. Then*

$$\begin{aligned}
|\langle Bx,x\rangle|^2 + |\langle Cx,x\rangle|^2 &= |\langle \sqrt{\alpha}Bx,x\rangle + \langle \sqrt{1-\alpha}Cx,x\rangle|^2 + |\langle \sqrt{1-\alpha}Bx,x\rangle - \langle \sqrt{\alpha}Cx,x\rangle|^2 \\
&= |\langle (\sqrt{\alpha}B + \sqrt{1-\alpha}C)x,x\rangle|^2 + |\langle (\sqrt{1-\alpha}B - \sqrt{\alpha}C)x,x\rangle|^2 \\
&\leq w^2(\sqrt{\alpha}B + \sqrt{1-\alpha}C) + w^2(\sqrt{1-\alpha}B - \sqrt{\alpha}C).
\end{aligned}$$

Taking the supremum in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$, we obtain the desired inequality. ■

Remark 3.1.10. *In particular, if we take $\alpha = \frac{1}{2}$ in Theorem (3.1.11), then we obtain the following upper bound, see (Theorem (3.1.2)), for the Euclidean operator radius*

$$w_e^2(B,C) \leq \frac{1}{2}\big(w^2(B+C) + w^2(B-C)\big).$$

Theorem 3.1.12. *Let $B, C \in \mathcal{B}(\mathcal{H})$ and let $\alpha, \beta \in \mathbb{R}^*$. Then*

$$w_e^2(B,C) \leq \frac{1}{\alpha^2+\beta^2}\left(w^2(\alpha B + \beta C) + w^2(\beta B - \alpha C)\right).$$

Proof . *Let $x \in \mathcal{H}$ be any unit vector. Then*

$$\begin{aligned} |\langle Bx, x\rangle|^2 + |\langle Cx, x\rangle|^2 &= \frac{1}{\alpha^2+\beta^2}\left(|\langle(\alpha B+\beta C)x, x\rangle|^2 + |\langle(\beta B-\alpha C)x, x\rangle|^2\right) \\ &\le \frac{1}{\alpha^2+\beta^2}\left(w^2(\alpha B+\beta C) + w^2(\beta B-\alpha C)\right). \end{aligned}$$

Taking the supremum in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$, we obtain the desired result. ∎

Corrollary 3.1.10. *Let $B, C \in \mathcal{B}(\mathcal{H})$ be self-adjoint and let $\alpha, \beta \in \mathbb{R}^*$. Then*

$$w_e^2(B, C) \le \frac{1}{\alpha^2+\beta^2}\left(\|\alpha B+\beta C)\|^2 + \|\beta B-\alpha C\|^2\right). \tag{3.39}$$

Remark 3.1.11. *If we take $\alpha = \beta$ in the inequality (3.39), then we obtain the following inequality*

$$w_e^2(B, C) \le \frac{1}{2}\left(\|B+C\|^2 + \|B-C\|^2\right),$$

which was already given in [19].

Taking $B = \Re(A)$ and $C = \Im(A)$ in the inequality (3.39), gives the following corollary.

Corrollary 3.1.11. *Let $A \in \mathcal{B}(\mathcal{H})$ and let $\alpha, \beta \in \mathbb{R}^*$. Then*

$$w^2(A) \le \frac{1}{\alpha^2+\beta^2}\left(\|\alpha\Re(A)+\beta\Im(A)\|^2 + \|\beta\Re(A)-\alpha\Im(A)\|^2\right). \tag{3.40}$$

Remark 3.1.12. *1. If we take $\alpha = 1$ and $\beta = 0$ in the inequality (3.40), then we reobtain the inequality (2.3).*

The following result can be found in [19].

Proposition 3.1.1. *Let $B, C \in \mathcal{B}(\mathcal{H})$. Then*

$$w_e^2(B, C) \le w^2(C-B) + 2w(B)w(C).$$

In the following theorem we give an extension of Proposition 3.1.1. First, we provide an extension of the parallelogram identity. For $\alpha, \beta \in \mathbb{R}^*$ (the nonzero real numbers), we have

$$(\alpha^2+\beta^2)\left(|\langle Bx, x\rangle|^2 + |\langle Cx, x\rangle|^2\right) = |\langle(\alpha B-\beta C)x, x\rangle|^2 + |\langle(\beta B+\alpha C)x, x\rangle|^2.$$

Hence, we get

$$(\alpha^2+\beta^2)w_e^2(B, C) = w_e^2(\alpha B-\beta C, \beta B+\alpha C). \tag{3.41}$$

Theorem 3.1.13. *Let $B, C \in \mathcal{B}(\mathcal{H})$ and let $\alpha, \beta \in \mathbb{R}^*$. Then*

$$w_e^2(B,C) \leq \frac{1}{\alpha^2+\beta^2}\Big(w^2\big((\beta-\alpha)B-(\beta+\alpha)C\big)+2w(\alpha B+\beta C)w(\beta B-\alpha C)\Big).$$

Proof . *Apply Proposition* 3.1.1 *for $\alpha B+\beta C$ and $\beta B-\alpha C$ instead of B and C, respectively, we have*

$$w_e^2(\alpha B+\beta C, \beta B-\alpha C) \leq w^2\big((\beta-\alpha)B-(\alpha+\beta)C\big)+2w(\alpha B+\beta C)w(\beta B-\alpha C).$$

The desired result follows by using the identity (3.41). ■

Lemma 3.1.3. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be self-adjoint. Then*

$$w^2(A+iB) \leq \|A^2+B^2\|.$$

Theorem 3.1.14. *Let $B, C \in \mathcal{B}(\mathcal{H})$ and let $0 < p, q \leq 1$. Then*

$$w_e^2(B,C) \leq (p^2+q^2)^{\frac{1}{2}} w\big(|B|^{\frac{1}{p}}+i|C|^{\frac{1}{q}}\big)+\big((1-p)^2+(1-q)^2\big)^{\frac{1}{2}} w\big(|B^*|^{\frac{1}{1-p}}+i|C^*|^{\frac{1}{1-q}}\big).$$

Proof . *Let $x \in \mathcal{H}$ be any unit vector. Let e_2 be as described in the proof of Theorem 3.1.6, it follows that*

$$\begin{aligned}
e_2 &\leq \langle |B|x,x\rangle\langle |B^*|x,x\rangle+\langle |C|x,x\rangle\langle |C^*|x,x\rangle (\textit{by Lemma 1.4.2 with } \alpha=\frac{1}{2})\\
&\leq p\langle |B|x,x\rangle^{\frac{1}{p}}+(1-p)\langle |B^*|x,x\rangle^{\frac{1}{1-p}}+q\langle |C|x,x\rangle^{\frac{1}{q}}+(1-q)\langle |C^*|x,x\rangle^{\frac{1}{1-q}}\\
&\quad (\textit{by the Young's inequality})\\
&\leq \Big(\big((1-p)^2+(1-q)^2\big)\Big(\langle |B^*|x,x\rangle^{\frac{2}{1-p}}+\langle |C^*|x,x\rangle^{\frac{2}{1-q}}\Big)\Big)^{\frac{1}{2}}\\
&\quad +\Big(\big(p^2+q^2\big)\Big(\langle |B|x,x\rangle^{\frac{2}{p}}+\langle |C|x,x\rangle^{\frac{2}{q}}\Big)\Big)^{\frac{1}{2}}\\
&\quad (\textit{by the Cauchy-Schwarz inequality})\\
&\leq \sqrt{(1-p)^2+(1-q)^2}\,\Big|\langle (|B^*|^{\frac{1}{1-p}}+i|C^*|^{\frac{1}{1-q}})x,x\rangle\Big|\\
&\quad +\sqrt{p^2+q^2}\,\Big|\langle (|B|^{\frac{1}{p}}+i|C|^{\frac{1}{q}})x,x\rangle\Big| (\textit{by Lemma 1.4.3 (a)}).
\end{aligned}$$

Taking the supremum of both sides in the above inequality over $x \in \mathcal{H}$ with $\|x\|=1$, we get the desired inequality. ■

Theorem 3.1.15. *Let $B, C \in \mathcal{B}(\mathcal{H})$. Then*

$$\sup_{\alpha,\beta\in\mathbb{R}^*} \frac{1}{\alpha^2+\beta^2}\Big(w\big(\alpha^2B^2+\beta^2C^2\big)+\frac{1}{2}\big|w^2\big(\alpha B+\beta C\big)-w^2(\alpha B-\beta C)\big|\Big) \leq w_e^2(B,C).$$

Proof . *Let $x \in \mathcal{H}$ be any unit vector. By the Cauchy-Schwarz inequality, we have*

$$\left(|\alpha||\langle Bx,x\rangle|+|\beta||\langle Cx,x\rangle|\right)^2 \leq \left(|\langle Bx,x\rangle|^2+|\langle Cx,x\rangle|^2\right)(\alpha^2+\beta^2).$$

Hence,

$$\begin{aligned}
|\langle Bx,x\rangle|^2+|\langle Cx,x\rangle|^2 &\geq \frac{1}{\alpha^2+\beta^2}(|\alpha||\langle Bx,x\rangle|+|\beta||\langle Cx,x\rangle|)^2\\
&= \frac{1}{\alpha^2+\beta^2}(|\langle \alpha Bx,x\rangle|+|\langle \beta Cx,x\rangle|)^2\\
&\geq \frac{1}{\alpha^2+\beta^2}|\langle \alpha Bx,x\rangle \pm \langle \beta Cx,x\rangle|^2\\
&= \frac{1}{\alpha^2+\beta^2}|\langle (\alpha B\pm\beta C)x,x\rangle|^2.
\end{aligned}$$

Taking the supremum of both sides in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$, we get $w_e^2(B,C) \geq \frac{1}{\alpha^2+\beta^2}w^2(\alpha B \pm \beta C)$. Thus,

$$\begin{aligned}
w_e^2(B,C) &\geq \frac{1}{\alpha^2+\beta^2}\max\left\{w^2(\alpha B+\beta C), w^2(\alpha B-\beta C)\right\}\\
&= \frac{1}{2(\alpha^2+\beta^2)}\Big(w^2(\alpha B+\beta C)+w^2(\alpha B-\beta C)\\
&\qquad +\big|w^2(\alpha B+\beta C)-w^2(\alpha B-\beta C)\big|\Big)\\
&\geq \frac{1}{2(\alpha^2+\beta^2)}\Big(w\big((\alpha B+\beta C)^2+(\alpha B-\beta C)^2\big)\\
&\qquad +\big|w^2(\alpha B+\beta C)-w^2(\alpha B-\beta C)\big|\Big)\\
&\geq \frac{1}{2(\alpha^2+\beta^2)}\Big(w(2\alpha^2B^2+2\beta^2C^2)\\
&\qquad +\big|w^2(\alpha B+\beta C)-w^2(\alpha B-\beta C)\big|\Big).
\end{aligned}$$

Therefore, the desired inequality is obtained. ■

Remark 3.1.13. *If we take $\alpha = \beta$ in Theorem 3.1.15, then we get*

$$\frac{1}{2}w\big(B^2+C^2\big)+\frac{1}{4}\big|w^2\big(B+C\big)-w^2(B-C)\big| \leq w_e^2(B,C). \tag{3.42}$$

Clearly, the inequality (3.42) is a refinement of the first inequality in (3.2).

Putting $B = \Re(A)$ and $C = \Im(A)$ in Theorem 3.1.15, we obtain the following corollary.

Corrollary 3.1.12. *Let $A \in \mathcal{B}(\mathcal{H})$ and let $\alpha, \beta \in \mathbb{R}^*$. Then*

$$w^2(A) \geq \frac{1}{\alpha^2+\beta^2}\Big(w\big(\alpha^2\Re^2(A)+\beta^2\Im^2(A)\big) + \frac{1}{2}\big|w^2\big(\alpha\Re(A)+\beta\Im(A)\big) - w^2\big(\alpha\Re(A)-\beta\Im(A)\big)\big|\Big).$$

Remark 3.1.14. *If we take $\alpha = \beta$ in Corollary 3.1.12, we get*

$$\frac{1}{4}\|A^*A+AA^*\| + \frac{1}{4}\big|\|\Re(A)+\Im(A)\|^2 - \|\Re(A)-\Im(A)\|^2\big| \leq w^2(A). \tag{3.43}$$

The inequality (3.43) is an improvement of the first inequality in (2.2).

Theorem 3.1.16. *Let $B, C \in \mathcal{B}(\mathcal{H})$. Then*

$$\frac{1}{2}\max\big\{\|\Re(B+C)\|^2 + c^2(\Im(B+C)), \|\Im(B+C)\|^2 + c^2(\Re(B+C))\big\} \leq w_e^2(B,C),$$

where $c(Y) = \inf_{\|x\|=1}|\langle Yx, x\rangle|$.

Proof . *Let $x \in \mathcal{H}$ be any unit vector. Let e_2 be as described in the proof of Theorem 3.1.6, it follows that*

$$\begin{aligned} e_2 &= \langle\Re(B)x,x\rangle^2 + \langle\Im(B)x,x\rangle^2 + \langle\Re(C)x,x\rangle^2 + \langle\Im(C)x,x\rangle^2 \\ &\geq \frac{1}{2}\left(|\langle\Re(B+C)x,x\rangle|^2 + |\langle\Im(B+C)x,x\rangle|^2\right) \\ &\geq \frac{1}{2}\left(|\langle\Re(B+C)x,x\rangle|^2 + c^2\big(\Im(B+C)\big)\right). \end{aligned}$$

Taking the supremum of both sides in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$, we obtain

$$w_e^2(B,C) \geq \frac{1}{2}\|\Re(B+C)\|^2 + \frac{1}{2}c^2\big(\Im(B+C)\big).$$

Similarly, we can prove that

$$w_e^2(B,C) \geq \frac{1}{2}\|\Im(B+C)\|^2 + \frac{1}{2}c^2\big(\Re(B+C)\big).$$

Therefore, the desired inequality is obtained. ∎

Remark 3.1.15. *If we choose $B = C = A$ in Theorem* 3.1.16*, then we get*

$$\max\left\{\|\Re(A)\|^2 + c^2(\Im(A)), \|\Im(A)\|^2 + c^2(\Re(A))\right\} \leq w^2(A),$$

which was given in [12].

Finally, we present the following inequality involving non-negative continous functions.

Theorem 3.1.17. *[27] Let $B, C \in \mathcal{B}(\mathcal{H})$ and let f, g be two non-negative continous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then*

$$\frac{1}{2}\|B + C\|^2 \leq w_e\big(f^2(|B|), f^2(|C|)\big)w_e\big(g^2(|B^*|), g(|C^*|)\big).$$

In particular,

$$\frac{1}{2}\|B + C\|^2 \leq w_e(|B|, |C|)w_e(|B^*|, |C^*|).$$

Proof . *Let $x, y \in \mathcal{H}$ be two unit vectors. Then*

$$\begin{aligned}
|\langle(B + C)x, y\rangle|^2 &= |\langle Bx, y\rangle + \langle Cx, y\rangle|^2 \\
&\leq 2(|\langle Bx, y\rangle|^2 + |\langle Cx, y\rangle|^2) \\
&\leq 2(\|f(|B|)x\|^2\|g(|B^*|)y\|^2 + \|f(|C|)x\|^2\|g(|C^*|)y\|^2) \\
&\quad (\textit{using Lemma (2.1.2)}) \\
&= 2(\langle f^2(|B|)x, x\rangle\langle g^2(|B^*|)y, y\rangle + \langle f^2(|C|)x, x\rangle\langle g^2(|C^*|)y, y\rangle) \\
&\leq 2(\langle f^2(|B|)x, x\rangle^2 + \langle f^2(|C|)x, x\rangle^2)^{\frac{1}{2}}(\langle g^2(|B^*|)x, x\rangle^2 + \langle g^2(|C^*|)x, x\rangle^2)^{\frac{1}{2}} \\
&\leq 2w_e\big(f^2(|B|), f^2(|C|)\big)w_e\big(g^2(|B^*|), g^2(|C^*|)\big).
\end{aligned}$$

Taking the supremum in the above inequality over $x, y \in \mathcal{H}$ with $\|x\| = \|y\| = 1$, we obtain

$$\frac{1}{2}\|B + C\|^2 \leq w_e(f^2(|B|), f^2(|C|))w_e(g^2(|B^*|), g^2(|C^*|)).$$

In particular, if we take $f(t) = g(t) = t^{\frac{1}{2}}$, then

$$\frac{1}{2}\|B + C\|^2 \leq w_e(|B|, |C|)w_e(|B^*|, |C^*|),$$

as required.

■

3.2 Power inequalities of the Euclidean operator radius

Theorem 3.2.1. *[27] Let $B, C \in \mathcal{B}(\mathcal{H})$. Then*

$$w_e^{2r}(B,C) \le \frac{1}{2}w^{2r}(B+C) + \frac{1}{2}w^{2r}(B-C) - 2^r \inf_{\|x\|=1} |Re(\langle Bx, x\rangle \overline{\langle Cx, x\rangle})| \text{ for } r \ge 2.$$

Proof . *Let $x \in \mathcal{H}$ be any unit vector and let e_2 be as described in the proof of Theorem 3.1.6. Then*

$$\begin{aligned}
e_2^r &= \left(\frac{1}{2}|\langle Bx, x\rangle + \langle Cx, x\rangle|^2 + \frac{1}{2}|\langle Bx, x\rangle - \langle Cx, x\rangle|^2\right)^r \\
&= \left(\frac{1}{2}|\langle (B+C)x, x\rangle|^2 + \frac{1}{2}|\langle (B-C)x, x\rangle|^2\right)^r \\
&\le \frac{1}{2}|\langle (B+C)x, x\rangle|^{2r} + \frac{1}{2}|\langle (B-C)x, x\rangle|^{2r} - \frac{1}{2}\left|\frac{1}{2}|\langle (B+C)x, x\rangle|^2 - \frac{1}{2}|\langle (B-C)x, x\rangle|^2\right|^r \\
&-\frac{1}{2}\left|\frac{1}{2}|\langle (B-C)x, x\rangle|^2 - \frac{1}{2}|\langle (B+C)x, x\rangle|^2\right|^r \quad (\textit{using Jensen's inequality }) \\
&= \frac{1}{2}|\langle (B+C)x, x\rangle|^{2r} + \frac{1}{2}|\langle (B-C)x, x\rangle|^{2r} - \frac{1}{2^r}\left||\langle (B+C)x, x\rangle|^2 - |\langle (B-C)x, x\rangle|^2\right|^r \\
&= \frac{1}{2}|\langle (B+C)x, x\rangle|^{2r} + \frac{1}{2}|\langle (B-C)x, x\rangle|^{2r} - \frac{2^{2r}}{2^r}|Re(\langle Bx, x\rangle \overline{\langle Cx, x\rangle})| \\
&\le \frac{1}{2}w^{2r}(B+C) + \frac{1}{2}w^{2r}(B-C) - 2^r \inf_{\|x\|=1} |Re(\langle Bx, x\rangle \overline{\langle Cx, x\rangle})|.
\end{aligned}$$

Taking the supremum of both sides in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$, we get the desired result. ■

3.3 Characterization of the Euclidean operator radius

The following lemma can be found in [24].

Lemma 3.3.1. *Let $B, C \in \mathcal{B}(\mathcal{H})$. Then*

$$w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) = \frac{1}{2}\sup_{\theta \in \mathbb{R}} \|e^{i\theta}B + e^{-i\theta}C^*\|.$$

Proof . *We have*

$$
\begin{aligned}
w\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) &= \sup_{\theta\in\mathbb{R}} \left\| \Re\left(e^{i\theta}\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right)\right\| \\
&= \frac{1}{2}\sup_{\theta\in\mathbb{R}} \left\| \begin{bmatrix} 0 & e^{i\theta}A + e^{-i\theta}B^* \\ e^{-i\theta}A^* + e^{i\theta}B & 0 \end{bmatrix}\right\| \\
&= \frac{1}{2}\sup_{\theta\in\mathbb{R}} \left\| e^{i\theta}A + e^{-i\theta}B^*\right\|.
\end{aligned}
$$

■

Our first main result can be stated as follows. In the sequel, μ, ν are assumed to be positive real numbers with $\mu^2 + \nu^2 = 1$.

Theorem 3.3.1. *Let $B, C \in \mathcal{B}(\mathcal{H})$. Then*

$$w_e^2(B,C) = \sup_{\mu^2+\nu^2=1} \sup_{\theta\in\mathbb{R}} w^2\big(\mu e^{i\theta}B + \nu e^{-i\theta}C\big).$$

Proof . *Let $x \in \mathcal{H}$ be any unit vector. Using the two identities $\sup_{|z|=1}\Re(za) = |a|$ and $\sup_{\theta\in\mathbb{R}}\left|e^{i\theta}a + e^{-i\theta}\overline{b}\right| = |a| + |b|$, where $z = x + iy$, $x, y \in \mathbb{R}$ and $a, b \in \mathbb{C}$, it follows that*

$$
\begin{aligned}
|\langle Bx,x\rangle|^2 + |\langle Cx,x\rangle|^2 &= \sup_{\mu^2+\nu^2=1} (\mu|\langle Bx,x\rangle| + \nu|\langle Cx,x\rangle|)^2 \\
&= \sup_{\mu^2+\nu^2=1} \sup_{\theta\in\mathbb{R}} |e^{i\theta}\langle \mu Bx,x\rangle + e^{-i\theta}\langle \nu Cx,x\rangle|^2 \\
&= \sup_{\mu^2+\nu^2=1} \sup_{\theta\in\mathbb{R}} |\langle (e^{i\theta}\mu B + e^{-i\theta}\nu C)x,x\rangle|^2.
\end{aligned}
$$

Taking the supremum of both sides in the above equality over $x \in \mathcal{H}$ with $\|x\| = 1$, we obtain the desired equality. ■

If we apply Theorem 3.3.1 for $B - C$ and $B + C$ instead of B and C, respectively, and we use the identity (3.14), then we obtain the following equality.

$$w_e^2(B,C) = \sup_{\mu^2+\nu^2=1} \sup_{\theta\in\mathbb{R}} \frac{1}{2} w^2\big(\mu e^{i\theta}(B - C) + \nu e^{-i\theta}(B + C)\big). \tag{3.44}$$

Corrollary 3.3.1. *Let $B, C \in \mathcal{B}(\mathcal{H})$. Then*

$$\sup_{\mu^2+\nu^2=1} w\left(\begin{bmatrix} 0 & \mu B \\ \nu C & 0 \end{bmatrix}\right) \le w_e(B,C) \le \sup_{\mu^2+\nu^2=1} 2w\left(\begin{bmatrix} 0 & \mu B \\ \nu C & 0 \end{bmatrix}\right).$$

Proof . *We have*

$$\begin{aligned} w_e(B,C) &= w_e(B,C^*) \\ &= \sup_{\mu^2+\nu^2=1} \sup_{\theta\in\mathbb{R}} w\big(\mu e^{i\theta}B + \nu e^{-i\theta}C^*\big) \\ &\geq \sup_{\mu^2+\nu^2=1} \sup_{\theta\in\mathbb{R}} \frac{1}{2}\|e^{i\theta}\mu B + e^{-i\theta}\nu C^*\| \text{ (by the first inequality in (2.1))} \\ &= \sup_{\mu^2+\nu^2=1} w\left(\begin{bmatrix} 0 & \mu B \\ \nu C & 0 \end{bmatrix}\right) \text{ (by Lemma 3.3.1)}. \end{aligned}$$

By a similar argument we prove the second inequality. ∎

It is known that (see e.g., [1]) if B and C are positive operators, then $w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) = \dfrac{\|B+C\|}{2}$.

Theorem 3.3.2. *Let $B, C \in \mathcal{B}(\mathcal{H})$ be positive operators. Then*

$$\frac{1}{2}w(B+iC) \leq w_e(B,C) \leq w(B+iC).$$

Proof . *Using the identity $\sup_{\mu^2+\nu^2=1}\|\mu X + \nu Y\| = w(X+iY)$, where X and Y are positive operators, we have*

$$\begin{aligned} w_e(B,C) &\geq \sup_{\mu^2+\nu^2=1} w\left(\begin{bmatrix} 0 & \mu B \\ \nu C & 0 \end{bmatrix}\right) \\ &= \sup_{\mu^2+\nu^2=1} \frac{1}{2}\|\mu B + \nu C\| \\ &= \frac{1}{2}w(B+iC). \end{aligned}$$

By a similar argument we prove the second inequality.

∎

Theorem 3.3.3. *Let $B, C \in \mathcal{B}(\mathcal{H})$. Then*

$$w_e^2(B,C) \leq \frac{1}{2}\Big(\|B\|^2 + \|C\|^2 + \sqrt{(\|B\|^2 - \|C\|^2)^2 + 4w^2(C^*B)}\Big). \tag{3.45}$$

Proof . *We have*

$$\begin{aligned}
w_e^2(B,C) &= \sup_{\theta\in\mathbb{R}}\sup_{\mu^2+\nu^2=1} w^2(e^{i\theta}\mu B+e^{-i\theta}\nu C)\\
&\le \sup_{\theta\in\mathbb{R}}\sup_{\mu^2+\nu^2=1} \|e^{i\theta}\mu B+e^{-i\theta}\nu C\|^2\\
&= \sup_{\theta\in\mathbb{R}}\sup_{\mu^2+\nu^2=1} \left\|(e^{-i\theta}\mu B^*+e^{i\theta}\nu C^*)(e^{i\theta}\mu B+e^{-i\theta}\nu C)\right\|\\
&= \sup_{\mu^2+\nu^2=1}\sup_{\theta\in\mathbb{R}} \left\|2\mu\nu\Re(e^{2i\theta}C^*B)+\mu^2B^*B+\nu^2C^*C\right\|\\
&\le \sup_{\mu^2+\nu^2=1}\left(\sup_{\theta\in\mathbb{R}}\left\|2\mu\nu\Re(e^{2i\theta}C^*B)\right\|+\left\|\mu^2B^*B+\nu^2C^*C\right\|\right)\\
&= \sup_{\mu^2+\nu^2=1}\left(2\mu\nu w(C^*B)+\|\mu^2B^*B+\nu^2C^*C\|\right)\\
&\le \sup_{\mu^2+\nu^2=1}\left(\mu^2\|B\|^2+\nu^2\|C\|^2+2\mu\nu w(C^*B)\right)\\
&\le \left\|\begin{bmatrix}\|B\|^2 & w(C^*B)\\ w(C^*B) & \|C\|^2\end{bmatrix}\right\|\\
&= \frac{1}{2}\left(\|B\|^2+\|C\|^2+\sqrt{(\|B\|^2-\|C\|^2)^2+4w^2(C^*B)}\right),
\end{aligned}$$

as required. ■

Remark 3.3.1. *The inequality (3.45) is stronger than the inequality (3.17). Indeed, setting* $\Theta=\frac{1}{2}\Big(\|B\|^2+\|C\|^2+\sqrt{(\|B\|^2-\|C\|^2)^2+4w^2(C^*B)}\Big)$, *yields*

$$\begin{aligned}
\Theta &\le \frac{1}{2}\Big(\|B\|^2+\|C\|^2+\big|\|B\|^2-\|C\|^2\big|+2w(C^*B)\Big)\\
&= \max\left\{\|B\|^2,\|C\|^2\right\}+w(C^*B).
\end{aligned}$$

If we take $B=\Re(A)$ *and* $C=\Im(A)$ *in Theorem (3.3.3), then we obtain*

$$w^2(A)\le\frac{1}{2}\Big(\|\Re(A)\|^2+\|\Im(A)\|^2+\sqrt{(\|\Re(A)\|^2-\|\Im(A)\|^2)^2+4w^2(\Re(A)\Im(A))}\Big). \tag{3.46}$$

It is clear that the inequality (3.46) is a refinement of the inequality (2.3).

If we choose $B=A$ *and* $C=A^*$ *in Theorem (3.3.3), then we get*

$$w^2(A)\le\frac{1}{2}\left(\|A\|^2+w(A^2)\right). \tag{3.47}$$

It should be mentioned here that the inequality (3.47) was given in [18]. Now, if we apply Theorem 3.3.3 for $B-C$ *and* $B+C$ *instead of* B *and* C*, respectively, and we use the identity*

(3.14), then we obtain the following corollary.

Corrollary 3.3.2. *Let $B, C \in \mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned} w_e^2(B,C) \;&\leq\; \frac{1}{4}\Big(\|B-C\|^2+\|B+C\|^2 \\ &\quad + \sqrt{\left(\|B-C\|^2-\|B+C\|^2\right)^2+4w^2\big((B^*-C^*)(B+C)\big)}\Big). \end{aligned} \tag{3.48}$$

Using an argument similar to that used for Remark 3.3.1, we can easily prove that the inequality (3.48) is better than the inequality (3.18). Also, if we take $B = A$ and $C = A^*$ in the inequality (3.48), then we obtain the following corollary.

Corrollary 3.3.3. *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned} w^2(A) \;&\leq\; \frac{1}{8}\Big(\|A-A^*\|^2+\|A+A^*\|^2 \\ &\quad + \sqrt{\left(\|A-A^*\|^2-\|A-A^*\|^2\right)^2+4w^2(A^*-A)(A+A^*)}\Big). \end{aligned} \tag{3.49}$$

Using an argument similar to that used for Remark 3.3.1, we can prove that the inequality (3.49) is sharper than the inequality (3.19).

3.4 Inequalities for Euclidean radius of the sums and the products of two operators

In this section, we give some bounds for the Euclidean operator radii of sums and products of two operators.

Theorem 3.4.1. *[20] Let $A, B, C, D \in \mathcal{B}(\mathcal{H})$ and $p, q \geq 1$ with $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$w_e^2(B^*A, D^*C) \leq \|(A^*A)^p+(C^*C)^p\|^{\frac{1}{p}}.\|(B^*B)^q+(D^*D)^q\|^{\frac{1}{q}}. \tag{3.50}$$

Proof . *Let $x \in \mathcal{H}$ be any unit vector. Then*

$$\begin{aligned} |\langle B^*Ax,x\rangle|^2+|\langle D^*Cx,x\rangle|^2 &\leq \langle A^*Ax,x\rangle.\langle B^*Bx,x\rangle+\langle C^*Cx,x\rangle.\langle D^*Dx,x\rangle \\ &\leq (\langle A^*Ax,x\rangle^p+\langle C^*Cx,x\rangle^p)^{\frac{1}{p}}.(\langle B^*Bx,x\rangle^q+\langle D^*Dx,x\rangle^q)^{\frac{1}{q}} \\ &\quad (\textit{by the Young's inequality}) \\ &\leq (\langle (A^*A)^px,x\rangle+\langle (C^*C)^px,x\rangle)^{\frac{1}{p}}(\langle (B^*B)^qx,x\rangle+\langle (D^*D)^qx,x\rangle)^{\frac{1}{q}} \\ &\leq \langle [(A^*A)^p+(C^*C)^p]x,x\rangle^{\frac{1}{p}}\langle [(B^*B)^q+(D^*D)^q]x,x\rangle^{\frac{1}{q}}. \end{aligned}$$

Taking the supremum of both sides in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$ and observing that the operators
$(A^*A)^p + (C^*C)^P$ *and* $(B^*B)^q + (D^*D)^q$ *are self-adjoint, we deduce the desired inequality.* ■
Next, we obtain the following particular case.

Corollary 3.4.1. *Let $A, C \in \mathcal{B}(\mathcal{H})$ and $p, q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$w_e^2(A, C) \leq 2^{\frac{1}{q}} \|(A^*A)^p + (C^*C)^p\|^{\frac{1}{p}}.$$

Proof . *The result follows from (3.50) by taking $B = D = I$.* ■

Corollary 3.4.2. *Let $A, D \in \mathcal{B}(\mathcal{H})$ and $p, q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$w_e^2(A, D) \leq \|(A^*A)^p + I\|^{\frac{1}{p}} . \|(D^*D)^q + I\|^{\frac{1}{q}}.$$

Theorem 3.4.2. *Let $A_1, A_2, B_1, B_2 \in \mathcal{B}(\mathcal{H})$ and $r \geq \frac{1}{2}$. Then*

$$w_e^{2r}(A_1^*A_2, B_1^*B_2) \leq 2^{r-1} \left\| |A_1|^{4r} + |B_2|^{4r} \right\|^{\frac{1}{2}} \left\| |A_2|^{4r} + |B_1|^{4r} \right\|^{\frac{1}{2}}.$$

Proof . *Let $x \in \mathcal{H}$ be any unit vector and let $r \geq \frac{1}{2}$. Setting*

$$e_4 = (|\langle A_1^*A_2x, x\rangle|^2 + |\langle B_1^*B_2x, x\rangle|^2)^r,$$

it follows that

$$\begin{aligned}
e_4 &\leq 2^{r-1} \left(|\langle A_1^*A_2x, x\rangle|^{2r} + |\langle B_1^*B_2x, x\rangle|^{2r} \right) \\
&= 2^{r-1} \left(|\langle A_2x, A_1x\rangle|^{2r} + |\langle B_2x, B_1x\rangle|^{2r} \right) \\
&\leq 2^{r-1} \left(\|A_2x\|^{2r} \|A_1x\|^{2r} + \|B_2x\|^{2r} \|B_1x\|^{2r} \right) \\
&\quad \textit{(by the Cauchy-Schwarz inequality)} \\
&\leq 2^{r-1} \Big(\left(\|A_2x\|^{4r} + \|B_1x\|^{4r} \right) \left(\|A_1x\|^{4r} + \|B_2x\|^{4r} \right) \Big)^{\frac{1}{2}} \\
&\quad \textit{(by the Cauchy-Schwarz inequality)} \\
&\leq 2^{r-1} \Big(\left(\langle |A_2|^2x, x\rangle^{2r} + \langle |B_1|^2x, x\rangle^{2r} \right) \left(\langle |A_1|^2x, x\rangle^{2r} + \langle |B_2|^2x, x\rangle^{2r} \right) \Big)^{\frac{1}{2}} \\
&\leq 2^{r-1} \Big(\left\langle \left(|A_2|^{4r} + |B_1|^{4r} \right)x, x \right\rangle \Big)^{\frac{1}{2}} \Big(\left\langle \left(|A_1|^{4r} + |B_2|^{4r} \right)x, x \right\rangle \Big)^{\frac{1}{2}} \\
&\quad \textit{(by Lemma 1.4.3 (a))} \\
&\leq 2^{r-1} \left\| |A_2|^{4r} + |B_1|^{4r} \right\|^{\frac{1}{2}} \left\| |A_1|^{4r} + |B_2|^{4r} \right\|^{\frac{1}{2}}.
\end{aligned}$$

Taking the supremum in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$, we obtain the desired inequality. ■

Remark 3.4.1. *If we take $A_1 = B_1$ and $A_2 = B_2$ in Theorem 3.4.2 and we use the fact that $w_e^{2r}(A_1^*A_2, A_1^*A_2) = 2^r w^{2r}(A_1^*A_2)$, then we obtain the inequality*

$$w^{2r}(A_1^*A_2) \leq \frac{1}{2}\||A_1|^{4r} + |A_2|^{4r}\| \quad \text{for } r \geq \frac{1}{2},$$

which was already given in [20].

Theorem 3.4.3. *Let $B_1, B_2, C_1, C_2 \in \mathcal{B}(\mathcal{H})$ and let $0 < \alpha \leq 1$. Then*

$$\begin{aligned} w_e^2(B_1 + B_2, C_1 + C_2) \quad &\leq \quad \frac{1}{2} w_e^2\Big(|B_1 + B_2|^{2\alpha} + i|C_1 + C_2|^{2\alpha}, \\ &\qquad |(B_1 + B_2)^*|^{2(1-\alpha)} + i|(C_1 + C_2)^*|^{2(1-\alpha)}\Big). \end{aligned}$$

Proof . *Let $x \in \mathcal{H}$ be any unit vector. Setting*

$$e_5 = |\langle (B_1 + B_2)x, x\rangle|^2 + |\langle (C_1 + C_2)x, x\rangle|^2,$$

it follows that

$$\begin{aligned} e_5 \quad &\leq \quad \langle |B_1 + B_2|^{2\alpha}x, x\rangle \langle |(B_1 + B_2)^*|^{2(1-\alpha)}x, x\rangle \\ &\quad + \langle |C_1 + C_2|^{2\alpha}x, x\rangle \langle |(C_1 + C_2)^*|^{2(1-\alpha)}x, x\rangle \quad (\text{by Lemma 1.4.2}) \\ &\leq \quad \{(\langle |B_1 + B_2|^{2\alpha}x, x\rangle^2 + \langle |C_1 + C_2|^{2\alpha}x, x\rangle^2) \\ &\quad \times (\langle |(B_1 + B_2)^*|^{2(1-\alpha)}x, x\rangle^2 + \langle |(C_1 + C_2)^*|^{2(1-\alpha)}x, x\rangle^2)\}^{\frac{1}{2}} \\ &\quad (\text{by the Cauchy-Schwarz inequality}) \\ &\leq \quad |\langle |B_1 + B_2|^{2\alpha}x, x\rangle + i\langle |C_1 + C_2|^{2\alpha}x, x\rangle| \\ &\quad \times |\langle |(B_1 + B_2)^*|^{2(1-\alpha)}x, x\rangle + i\langle |(C_1 + C_2)^*|^{2(1-\alpha)}x, x\rangle| \\ &\leq \quad \frac{1}{2}[|\langle (|B_1 + B_2|^{2\alpha} + i|C_1 + C_2|^{2\alpha})x, x\rangle|^2 \\ &\quad + |\langle (|(B_1 + B_2)^*|^{2(1-\alpha)} + i|(C_1 + C_2)^*|^{2(1-\alpha)})x, x\rangle|^2] \\ &\quad (\text{by the arithmetic-geometric mean inequality}). \end{aligned}$$

Taking the supremum of both sides in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$, we obtain the desired inequality. ■

If we take $B_2 = C_2 = 0$ *and* $\alpha = \frac{1}{2}$ *in Theorem 3.4.3, then we obtain the following result.*

$$w_e^2(B, C) \leq \frac{1}{2} w_e^2(|B| + i|C|, |B^*| + i|C^*|). \tag{3.51}$$

Now, if we choose B, C *to be normal in the inequality (3.51), then we obtain the following corollary.*

Corrollary 3.4.1. *Let* $B, C \in \mathcal{B}(\mathcal{H})$ *be normal. Then*

$$w_e(B, C) \leq w(|B| + i|C|). \tag{3.52}$$

Note that a closely related result to the inequality (3.52) has recently appeared in [27].

Remark 3.4.2. *Using Lemma 3.1.3, we can prove that the inequality (3.52) is a refinement of the second inequality in (3.3).*

If we take $B = \Re(A)$ *and* $C = \Im(A)$ *in the inequality (3.52), then we deduce that*

$$w(A) \leq w\big(|\Re(A)| + i|\Im(A)|\big). \tag{3.53}$$

Clearly the inequality (3.53) is sharper than the inequality (2.3). It should be mentioned here that the inequality (3.53) has been given in [27].

Chapter 4

Weighted numerical radius inequalities for operator and 2×2 operator matrices

The main aim of this chapter is to present the notion of weighted numerical radius. Correspondingly, we give some bounds for the weighted numerical radius of one operator as well as for 2×2 operator matrices. For the particular cases, we reobtain some well known inequalities for the classical numerical radius.

4.1 Weighted numerical radius inequalities for operator

Definition 4.1.1. *[16] Let $A \in \mathcal{B}(\mathcal{H})$ and let $0 \leq t \leq 1$. The weighted real and imaginary parts of A are defined by*

$$\Re_t(A) = tA + (1-t)A^* \quad \text{and} \quad \Im_t(A) = \frac{(1-t)A - tA^*}{i},$$

respectively.

Definition 4.1.2. *[16] Let $A \in \mathcal{B}(\mathcal{H})$ and let $0 \leq t \leq 1$. The weighted numerical radius of A is defined by*

$$w_t(A) = \sup_{\substack{x \in \mathcal{H} \\ \|x\|=1}} |\langle (\Re_t(A) + i\Im_t(A))x, x\rangle| = w\big((1-2t)A^* + A\big).$$

Similarly, the weighted operator norm of A is defined by

$$\|A\|_t = \sup_{\substack{x,y \in \mathcal{H} \\ \|x\|=\|y\|=1}} |\langle (\Re_t(A) + i\Im_t(A))x, y\rangle| = \|(1-2t)A^* + A\|.$$

Proposition 4.1.1. *[16] Let $A \in \mathcal{B}(\mathcal{H})$ and let $0 \leq t \leq 1$. Then*

$$\frac{1}{4t\,(1-t)}\left((\Re_t(A))^2 - \Re_t(A^2)\right) = (\Im(A))^2\,. \tag{4.1}$$

In particular,

$$(\Re(A))^2 - \Re(A^2) = (\Im(A))^2.$$

Proprieties 4.1.1. *[16] Let $A \in \mathcal{B}(\mathcal{H})$ and let $0 \leq t \leq 1$. Then*

1. $w_{\frac{1}{2}}(A) = w(A)$ *and* $\|A\|_{\frac{1}{2}} = \|A\|$.

2. $w_0(A) = 2\|\Re(A)\|$ *and* $w_1(A) = 2\|\Im(A)\|$.

3. $\frac{\|A\|_t}{2} \leq w_t(A) \leq \|A\|_t$.

4. $w_t(A) \leq 2w(A)$.

Another definition of $w_t(\cdot)$ was introduced in [38].

Definition 4.1.3. *Let $A \in \mathcal{B}(\mathcal{H})$ and let $0 \leq t \leq 1$. The weighted numerical radius of A is defined by*

$$w_t(A) = \sup_{\theta \in \mathbb{R}} \|\Re_t(e^{i\theta}A)\|$$

.

For $t = \dfrac{1}{2}$, we get $w_{\frac{1}{2}}(A) = w(A)$.

Theorem 4.1.1. *[38] The function $w_t(\cdot) : \mathcal{B}(\mathcal{H}) \to [0, \infty)$ is a norm on $\mathcal{B}(\mathcal{H})$.*

Proposition 4.1.2. *[16] Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$w_t(A) = w_t(A^*). \tag{4.2}$$

Proof . *For $0 \leq t \leq 1$,*

$$\begin{aligned} w_t(A^*) &= w\big((1-2t)A + A^*\big) \\ &= w\big((1-2t)(\Re(A) + i\Im(A)) + \Re(A) - i\Im(A)\big) \\ &= 2w\big((1-t)\Re(A) - ti\Im(A)\big) \\ &= 2w\big((1-t)\Re(A) + ti\Im(A)\big), \end{aligned} \tag{4.3}$$

where the last identity follows because $w(A) = w(A^*)$. Moreover,

$$\begin{aligned} w_t(A) &= w((1-2t)A^* + A) \\ &= w((1-2t)(\Re(A) - i\Im(A)) + \Re(A) - i\Im(A)) \\ &= 2w((1-t)\Re(A) + ti\Im(A)). \end{aligned} \tag{4.4}$$

From (4.3) and (4.4), it follows that $w_t(A) = w_t(A^*)$.

■

Theorem 4.1.2. *[38] For every $A \in \mathcal{B}(\mathcal{H})$, the function $f(t) = w_t(A)$ is a convex function on $[0,1]$.*

Proof .*Let $A \in \mathcal{B}(\mathcal{H})$, and $0 \le t_1, t_2, \lambda \le 1$. Then we have*

$$\begin{aligned} f(t_1\lambda + (1-\lambda)t_2) &= w_{t_1\lambda+(1-\lambda)t_2}(A) \\ &= \sup_{\theta\in\mathbb{R}} \|(t_1\lambda + (1-\lambda)t_2)e^{i\theta}A + (1-\lambda t_1 - t_2 + \lambda t_2)e^{-i\theta}A^*\| \\ &= \sup_{\theta\in\mathbb{R}} \|(t_1\lambda)e^{i\theta}A + \lambda(1-t_1)e^{-i\theta}A^* + (1-\lambda)t_2e^{i\theta} + (1-\lambda)(1-t_2)e^{-i\theta}A^*\| \\ &\le \sup_{\theta\in\mathbb{R}} \lambda\|t_1e^{i\theta}A + (1-t_1)e^{-i\theta}A^*\| + (1-\lambda)\sup_{\theta\in\mathbb{R}} \|t_2e^{i\theta} + (1-t_2)e^{-i\theta}A^*\| \\ &= \lambda w_{t_1}(A) + \lambda w_{t_2}(A) \\ &= \lambda f(t_1) + (1-\lambda)f(t_2) \end{aligned}$$

■

In the sequel, we set $S = ((1-2t)A^* + A$, $r = \min\{(1-t), t\}$ and $R = \max\{(1-t), t\}$, where $t \in [0,1]$. In the following theorem, we give a characterization of $w_t(\cdot)$.

Theorem 4.1.3. *Let $A \in \mathcal{B}(\mathcal{H})$. Then for any $0 \le t \le 1$,*

$$\frac{w_r(A)}{2R} \le w(A) \le \frac{w_R(A)}{2r}.$$

Proof . *For $0 \le t \le \frac{1}{2}$, we have*

$$\begin{aligned} w_t(A) &= w((1-2t)A^* + A) \\ &\le (1-2t)w(A^* + w(A) \\ &= 2(1-t)w(A). \end{aligned}$$

For $\frac{1}{2} \leq t \leq 1$, we have

$$\begin{aligned} w((1-2t)A^* + A) &= w(A - (2t-1)A^*) \\ &\geq w(A) - (2t-1)w(A^*) \\ &= w(A) - (2t-1)w(A^*) \\ &= 2(1-t)w(A). \end{aligned}$$

By combining the above inequalities, we obtain the desired inequality.

■

Theorem 4.1.4. *Let $A \in \mathcal{B}(\mathcal{H})$ and let $A = \Re_t(A) + i\Im_t(A)$ be the generalization Cartesian decomposition of A. Then for $\alpha, \beta \in \mathbb{R}$*

$$w_t(A) = \sup_{\alpha^2+\beta^2=1} \|\alpha\Re_t(A) + \beta\Im_t(A^*)\|.$$

In particular,

$$\|\Re_t(A)\| \leq w_t(A), \ \|\Im_t(A^*)\| \leq w_t(A) \text{ and } \frac{1}{\sqrt{2}}\|\Re_t((1+i)A)\| \leq w_t(A).$$

Proof . *We have*

$$\begin{aligned} \Re_t(e^{i\theta}A) &= (1-t)e^{-i\theta}A^* + te^{i\theta}A \\ &= (1-t)(\cos\theta - i\sin\theta)A^* + t(\cos\theta + i\sin\theta)A \\ &= \cos\theta((1-t)A^* + tA) + \frac{(1-t)(\sin\theta)A^* - t(\sin\theta)A}{i} \\ &= \cos\theta((1-t)A^* + tA) - \sin\theta\Big(\frac{(1-t)A^* - tA}{-i}\Big) \\ &= \cos\theta\Re_t(A) - \sin\theta\Im_t(A^*). \end{aligned}$$

Therefore, by putting $\alpha = \cos\theta$ and $\beta = -\sin\theta$ in the above inequality and using the definition of the weighted numerical radius, we obtain the desired equality.
Especially, by letting $(\alpha,\beta) = (1,0)$, $(\alpha,\beta) = (0,1)$ and $(\alpha,\beta) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, we get the required inequalities. ■

Corollary 4.1.1. *Let $A \in \mathcal{B}(\mathcal{H})$ and let $A = \Re_t(A) + i\Im_t(A)$ be the generalization Cartesian decomposition of A. Then*

$$w_t(A) \leq \sqrt{\|(\Re_t(A)\|^2 + \|\Im_t(A^*)\|^2}.$$

Proof . *From Theorem 4.1.4, we have*

$$\begin{aligned}
w_t(A) &= \sup_{\alpha^2+\beta^2=1} \|\alpha\Re_t(A) + \beta\Im_t(A^*)\| \\
&\leq \sup_{\alpha^2+\beta^2=1} (\alpha\|\Re_t(A)\| + \beta\|\Im_t(A^*)\|) \\
&\leq (\alpha^2+\beta^2)^{\frac{1}{2}}(\|\Re_t(A)\|^2 + \|\Im_t(A^*)\|^2)^{\frac{1}{2}} \\
&\quad (\textit{by Cauchy-Schwarz inequality}).
\end{aligned}$$

Hence, we get the desired inequality. ■

Theorem 4.1.5. *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$2r\|\Re(A)\| \leq w(\Re_t(A)) \leq 2R\|\Re(A)\|. \tag{4.5}$$

Proof . *For $t = 0$ or $t = 1$, the inequalities (4.5) are satisfied. Let $t \in]0, 1[$, then*

$$\begin{aligned}
\|\Re(A)\| &= \frac{1}{2}\sup_{\|x\|=1} |\langle (A^* + A)x, x\rangle| \\
&= \frac{1}{2}\sup_{\|x\|=1} \left|\frac{1-t}{1-t}\langle A^*x, x\rangle + \frac{t}{t}\langle Ax, x\rangle\right| \\
&\leq \frac{1}{2}\sup_{\|x\|=1} \left|\frac{1-t}{r}\langle A^*x, x\rangle + \frac{t}{r}\langle Ax, x\rangle\right| \\
&= \frac{1}{2r}\sup_{\|x\|=1} |\langle ((1-t)A^* + tA)x, x\rangle| \\
&= \frac{1}{2r} w(\Re_t(A)).
\end{aligned}$$

Therefore, we obtain the first inequality.

To prove the second inequality, we have

$$\begin{aligned}
\|\Re(A)\| &= \frac{1}{2}\sup_{\|x\|=1}|\langle (A^*+A)x,x\rangle| \\
&= \frac{1}{2}\sup_{\|x\|=1}\left|\frac{1-t}{1-t}\langle A^*x,x\rangle+\frac{t}{t}\langle Ax,x\rangle\right| \\
&\geq \frac{1}{2}\sup_{\|x\|=1}\left|\frac{1-t}{R}\langle A^*x,x\rangle+\frac{t}{R}\langle Ax,x\rangle\right| \\
&= \frac{1}{2R}\sup_{\|x\|=1}|\langle ((1-t)A^*+tA)x,x\rangle| \\
&= \frac{1}{2R}w(\Re_t(A)).
\end{aligned}$$

Therefore, we get the second inequality. ■

Theorem 4.1.6. *Let $A\in\mathcal{B}(\mathcal{H})$. Then*

$$\frac{1}{2}r^2\|(A^*A+AA^*\|\leq w_t^2(A)\leq 2R^2\|A^*A+AA^*\|.$$

Proof *.Let $x\in\mathcal{H}$ be any unit vector. Setting $S=(1-2t)A^*+A$. Then $\Re(S)=2(1-t)\Re(A)$ and $\Im(S)=2t\Im(A)$. Thus,*

$$\begin{aligned}
|\langle Sx,x\rangle|^2 &= |\langle 2(1-t)\Re(A)x,x\rangle|^2+|\langle 2t\Im(A)x,x\rangle|^2 \\
&\leq 4(1-t)^2|\langle (\Re(A))^2x,x\rangle|+4t^2|\langle (\Im(A))^2x,x\rangle| \text{ (by Lemma \textbf{??}(a))} \\
&\leq 4R^2|\langle \big((\Re(A))^2+(\Im(A))^2\big)x,x\rangle| \\
&= 2R^2|\langle (AA^*+A^*A)x,x\rangle|
\end{aligned}$$

By taking the supremum on both sides in the above inequality over $x\in\mathcal{H}$ with $\|x\|=1$, we obtain the second inequality.

By the convexity of the function $f(t)=t^2$ on $[0,\infty)$, we have

$$\begin{aligned}
|\langle Sx,x\rangle|^2 &= |\langle 2(1-t)\Re(A)x,x\rangle|^2+|\langle 2t\Im(A)x,x\rangle|^2 \\
&\geq \frac{1}{2}\big(2(1-t)|\langle \Re(A)x,x\rangle|+2t|\langle \Im(A)x,x\rangle|\big)^2 \\
&\geq 2r^2|\langle \big(\Re(A)\pm\Im(A)\big)x,x\rangle|^2.
\end{aligned}$$

By taking the supremum on both sides in the above inequality over $x\in\mathcal{H}$ with $\|x\|=1$, we obtain

$$w_t^2(A)\geq 2r^2\|\Re(A)\pm\Im(A)\|^2=2r^2\big\|\big(Re(A)\pm\Im(A)\big)^2\big\|.$$

Hence,

$$\begin{aligned}
2w_t^2(A) &\geq 2r^2\left(\left\|\big(\Re(A)+\Im(A)\big)^2\right\|+\left\|\big(\Re(A)-\Im(A)\big)^2\right\|\right) \\
&\geq 2r^2\left\|\big(\Re(A)+\Im(A)\big)^2+\big(\Re(A)-\Im(A)\big)^2\right\| \\
&= 2r^2\left\|\big(\Re(A)\big)^2+\big(\Im(A)\big)^2\right\| \\
&= r^2\|A^*A+AA^*\|.
\end{aligned}$$

Therefore, we get the first inequality. ■

Theorem 4.1.7. *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$\|(1-t)^2A^*A+t^2AA^*\|+2t(1-t)c(A^2) \leq w_t^2(A) \leq \|(1-t)^2A^*A+t^2AA^*\|+2t(1-t)w(A^2).$$

Proof . Let $x \in \mathcal{H}$ be any unit vector and let θ be a real number such that

$$e^{2i\theta}\langle A^2x,x\rangle = |\langle A^2x,x\rangle|.$$

We have

$$\begin{aligned}
w_t^2(A) &\geq \|\Re_t(e^{i\theta}A)\|^2 \\
&= \|(1-t)e^{-i\theta}A^*+te^{i\theta}A\|^2 \\
&= \|((1-t)e^{-i\theta}A^*+te^{i\theta}A)((1-t)e^{i\theta}A+te^{-i\theta}A^*)\| \\
&= \|(1-t)^2A^*A+t^2AA^*+2t(1-t)\Re(e^{i2\theta}A^2)\| \\
&\geq \left|\left\langle\big((1-t)^2A^*A+t^2AA^*+2t(1-t)\Re(e^{i2\theta}A^2)\big)x,x\right\rangle\right| \\
&= \left|\left\langle\big((1-t)^2A^*A+t^2AA^*\big)x,x\right\rangle+2t(1-t)\langle\Re(e^{i2\theta}A^2)x,x\rangle\right| \\
&= \left|\left\langle\big((1-t)^2A^*A+t^2AA^*\big)x,x\right\rangle+2t(1-t)\Re\big(e^{i2\theta}\langle A^2x,x\rangle\big)\right| \\
&= \left|\left\langle\big((1-t)^2A^*A+t^2AA^*\big)x,x\right\rangle+2t(1-t)|\langle A^2x,x\rangle|\right| \\
&\geq \left|\left\langle\big((1-t)^2A^*A+t^2AA^*\big)x,x\right\rangle+2t(1-t)c(A^2)\right|.
\end{aligned}$$

Thus,

$$\begin{aligned}
w_t^2(A) &\geq \sup_{\|x\|=1}\left|\langle((1-t)^2A^*A+t^2AA^*)x,x\rangle+2t(1-t)c(A^2)\right| \\
&= \|(1-t)^2A^*A+t^2AA^*\|+2t(1-t)c(A^2),
\end{aligned}$$

as required.

To prove the second inequality, we have

$$\begin{aligned}
w_t^2(A) &= \sup_{\theta\in\mathbb{R}} \|\Re_t(e^{i\theta}A)\|^2 \\
&= \sup_{\theta\in\mathbb{R}} \|(1-t)e^{-i\theta}A^* + te^{i\theta}A\|^2 \\
&= \sup_{\theta\in\mathbb{R}} \|((1-t)e^{-i\theta}A^* + te^{i\theta}A)((1-t)e^{i\theta}A + te^{-i\theta}A^*)\| \\
&= \sup_{\theta\in\mathbb{R}} \|(1-t)^2A^*A + t^2AA^* + 2t(1-t)\Re(e^{i2\theta}A^2)\| \\
&\le \|(1-t)^2A^*A + t^2AA^*\| + 2t(1-t)\sup_{\theta\in\mathbb{R}} \|\Re(e^{i2\theta}A^2)\| \\
&= \|(1-t)^2A^*A + t^2AA^*\| + 2t(1-t)w(A^2).
\end{aligned}$$

Hence, we get the second inequality. ■

Remark 4.1.1. *If we take* $t = \frac{1}{2}$ *in Theorem 4.1.7, then we obtain*

$$\frac{1}{4}\|A^*A + AA^*\| + \frac{1}{2}c(A^2) \le w^2(A) \le \frac{1}{4}\|A^*A + AA^*\| + \frac{1}{2}w(A^2).$$

which was already given in [2].

Theorem 4.1.8. *Let* $A \in \mathcal{B}(\mathcal{H})$*. Then*

$$w_t^2(A) \le \frac{1}{2}\big(w_t^2(\Re(A)) + w_t^2(\Im(A))\big).$$

Proof . Let $x \in \mathcal{H}$ be any unit vector. Then

$$\begin{aligned}
|\langle Sx, x\rangle|^2 &= \frac{1}{2}\big(|\langle((1-2t)A^* + A)x, x\rangle|^2 + |\langle((1-2t)A + A^*)x, x\rangle|^2\big) \\
&= \frac{1}{4}\Big(\Big|\Big\langle\big(((1-2t)A^* + A) + ((1-2t)A + A^*)\big)x, x\Big\rangle\Big|^2 \\
&\quad + \Big|\Big\langle\big(((1-2t)A^* + A) - ((1-2t)A + A^*)\big)x, x\Big\rangle\Big|^2\Big) \\
&= \frac{1}{4}\Big(\Big|\Big\langle\big((1-2t)(A^* + A) + (A + A^*)\big)x, x\Big\rangle\Big|^2 \\
&\quad + \Big|\Big\langle\big((1-2t)(A^* - A) + (A - A^*)\big)x, x\Big\rangle\Big|^2\Big).
\end{aligned}$$

By taking the supremum on both sides in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$, we obtain the desired inequality. ■

4.2 Weighted numerical radius inequalities for 2×2 operator matrices

We start this section by the following lemmas for weighted numerical radius inequalities of 2×2 operator matrices.

Lemme 4.2.1. *[35] Let $A, D \in \mathcal{B}(\mathcal{H})$. Then*

$$w_t\left(\begin{bmatrix} A & D \\ D & A \end{bmatrix}\right) = \max\left\{w_t(A+D), w_t(A-D)\right\}.$$

Lemme 4.2.2. *[3] Let $x, y \in \mathcal{H}$. Then for any $\alpha \geq 0$*

$$|\langle x, y\rangle|^2 \leq \frac{1}{\alpha+1}\|x\|\|y\||\langle x, y\rangle| + \frac{\alpha}{1+\alpha}\|x\|^2\|y\|^2.$$

Corollary 4.2.1. *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$w_t(A) \leq w_t\left(\begin{bmatrix} A^* & A \\ A & A^* \end{bmatrix}\right).$$

Proof . The result follows by using Theorem 4.1.8 and Lemma 4.2.1. ■

In the sequel, we set $S = (1-2t)T^* + T$.

Theorem 4.2.1. *Let $A, D, B, C \in \mathcal{B}(\mathcal{H})$. Then*

$$w_t^2\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leq 8R^2 \max\left\{\left\||A|^2+|C|^2\right\|, \left\||D|^2+|B|^2\right\|\right\}.$$

Proof . Let $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, $T_1 = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$ and $T_2 = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$. Let $x \in \mathcal{H}\oplus\mathcal{H}$ with

$\|x\| = 1$. Then

$$\begin{aligned}
|\langle Sx, x\rangle|^2 &\leq 4R^2|\langle Tx\, x\rangle|)^2 \\
&\leq 4R^2\big(|\langle T_1x\, x\rangle| + |\langle T_2x\, x\rangle|\big)^2 \\
&\leq 8R^2(|\langle T_1x\, x\rangle|^2 + |\langle T_2x\, x\rangle|^2) \\
&\quad \text{(by the convexity of the function } f(t) = t^2 \text{ on } [0, +\infty)) \\
&\leq 8R^2\big(\|T_1x\|^2 + \|T_2x\|^2\big)\text{(by the Cauchy-Schwarz inequality)} \\
&\leq 8R^2\big(\langle(|T_1|^2 + |T_1|^2)x, x\rangle\big) \\
&\leq 8R^2\big(\||T_1|^2 + |T_2|^2\|\big) \\
&\leq 8R^2 \max\{\||A|^2 + |C|^2\|, \||D|^2 + |B|^2\|\}.
\end{aligned}$$

By taking the supremum on both sides in the above inequality over $x \in \mathcal{H} \oplus \mathcal{H}$ with $\|x\| = 1$, we obtain the desired inequality. ■

Corollary 4.2.2. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned}
\omega_t^2\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right) &= \max\big\{w_t^2(A+B), w_t^2(A-B)\big\} \\
&\leq 8R^2\big\||A|^2 + |B|^2\big\|.
\end{aligned}$$

Remark 4.2.1. *If we take $t = \frac{1}{2}$ and $A = B$ in Corollary 4.2.2, then we reobtain the second inequality in* (2.1).

Theorem 4.2.2. *Let $A, D, B, C \in \mathcal{B}(\mathcal{H})$. Then*

$$w_t^2\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leq 8R^2 \max\big\{w(|A|+i|B^*|), w(|D|+i|C^*|)\big\} \max\big\{w(|A^*|+i|C|), w(|D^*|+i|B|)\big\}.$$

Proof . Let T, T_1, T_2 be as described in the proof of Theorem 4.2.1 and let $x \in \mathcal{H} \oplus \mathcal{H}$

with $\|x\| = 1$. Then

$$\begin{aligned}
|\langle Sx, x\rangle|^2 &\le 4R^2(|\langle Tx\, x\rangle|)^2 \\
&\le 4R^2\big(|\langle T_1 x\, x\rangle| + |\langle T_2 x\, x\rangle|\big)^2 \\
&\le 8R^2(|\langle T_1 x\, x\rangle|^2 + |\langle T_2 x\, x\rangle|^2) \\
&\text{(by the convexity of the function } f(t) = t^2 \text{ on } [0, +\infty)\text{)} \\
&\le 8R^2[\langle |T_1|x, x\rangle\langle |T_1^*|x, x\rangle + \langle |T_2|x, x\rangle\langle |T_2^*|x, x\rangle]\text{(by Lemma1.4.2)} \\
&\le 8R^2\{(\langle |T_1|x, x\rangle^2 + \langle |T_2^*|x, x\rangle^2)(\langle |T_1^*|x, x\rangle^2 + \langle |T_2|x, x\rangle^2)\}^{\frac{1}{2}} \\
&\text{(by the inequality } (ab + cd)^2 \le (a^2 + c^2)(b^2 + d^2) \text{ for } a, b, c, d \in \mathbb{R}\text{)} \\
&= 8R^2\big(|\langle |T_1|x, x\rangle + i\langle |T_2^*|x, x\rangle|^2 |\langle |T_1^*|x, x\rangle + i\langle |T_2|x, x\rangle|^2\big)^{\frac{1}{2}} \\
&\le 8R^2 w(|T_1| + i|T_2^*|) w(|T_1^*| + i|T_2|) \\
&= 8R^2 w\left(\begin{bmatrix} |A| + i|B^*| & 0 \\ 0 & |D| + i|C^*| \end{bmatrix}\right) w\left(\begin{bmatrix} |A^*| + i|C| & 0 \\ 0 & |D^*| + i|B| \end{bmatrix}\right).
\end{aligned}$$

By taking the supremum in the above inequality over $x \in \mathcal{H} \oplus \mathcal{H}$ with $\|x\| = 1$, we obtain the desired inequality. ■

If we put $A = D$ and $B = C$ in Theorem 4.2.2, then we obtain the following Corollary.

Corollary 4.2.3. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned}
\omega_t^2\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right) &= \max\left\{w_t^2(A + B), w_t^2(A - B)\right\} \\
&\le 8R^2 w(|A| + i|B^*|) w(|A^*| + i|B|).
\end{aligned}$$

Remark 4.2.2. *If we take $t = \dfrac{1}{2}$ and $A = B$ in Corollary 4.2.3, then we reobtain the inequality(2.10).*

Next, we present another upper bound for $w_t(\cdot)$.

Theorem 4.2.3. *Let $A, D, B, C \in \mathcal{B}(\mathcal{H})$, $t \in [0, 1]$ and $\beta \ge 0$. Then*

$$w_t^2\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \le 8R^2\left(\frac{1}{\beta + 1}\sqrt{\max\{\||A|^2 + |C|^2\|, \||D|^2 + |B|^2\|\}} \times \sqrt{w^2\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right) + w^2\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)} + \frac{\beta}{1 + \beta}\max\{\||A|^2 + |C|^2\|, \||D|^2 + |B|^2\|\}\right).$$

Proof . Let T, T_1, T_2 be as described in the proof of Theorem 4.2.1 and let $x \in \mathcal{H} \oplus \mathcal{H}$ with $\|x\| = 1$. Then

$$\begin{aligned}
|\langle Sx, x\rangle|^2 &\leq 4R^2(|\langle Tx\, x\rangle|)^2 \\
&\leq 4R^2(|\langle T_1x\, x\rangle| + |\langle T_2x\, x\rangle|)^2 \\
&\leq 8R^2(|\langle T_1x\, x\rangle|^2 + |\langle T_2x\, x\rangle|^2) \\
&\leq 8R^2\left(\frac{1}{1+\beta}\|T_1x\||\langle T_1x, x\rangle| + \frac{\beta}{1+\beta}\|T_1x\|^2\right. \\
&+ \left.\frac{1}{\beta+1}\|T_2x\|\langle T_2x, x\rangle| + \frac{\beta}{1+\beta}\|T_2x\|^2\right) \text{ (by Lemma 3.1.2)} \\
&\leq 8R^2\left(\frac{1}{\alpha+1}(\|T_1x\|^2 + \|T_2x\|^2)^{\frac{1}{2}}(|\langle T_1x, x\rangle|^2\right. \\
&+ \left.|\langle T_2x, x\rangle|^2)^{\frac{1}{2}} + \frac{\beta}{1+\beta}(\|T_1x\|^2 + \|T_2x\|^2)\right) \\
&\leq 8R^2\left(\frac{1}{\beta+1}\sqrt{\||T_1|^2 + |T_2|^2\|}\sqrt{w(T_1)^2 + w(T_2)^2}\right. \\
&+ \left.\frac{\beta}{1+\beta}\||T_1|^2 + |T_2|^2\|\right) \\
&\leq 8R^2\left(\frac{1}{\beta+1}\sqrt{\max\{\||A|^2 + |C|^2\|, \||D|^2 + |B|^2\|\}}\right. \\
&\times \sqrt{w^2\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right) + w^2\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)} \\
&+ \left.\frac{\beta}{1+\beta}\max\{\||A|^2 + |C|^2\|, \||D|^2 + |B|^2\|\}\right).
\end{aligned}$$

By taking the supremum in the above inequality over $x \in \mathcal{H} \oplus \mathcal{H}$ with $\|x\| = 1$, we obtain the required inequality. ■

If we take $A = D$ and $B = C$ in Theorem 4.2.3, then we obtain the following Corollary.

Corollary 4.2.4. *Let $A, B \in \mathcal{B}(\mathcal{H})$ and $\beta \geq 0$. Then*

$$\begin{aligned}
\omega_t^2\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right) &= \max\left\{w_t^2(A+B), w_t^2(A-B)\right\} \\
&\leq 8R^2\left(\frac{1}{\beta+1}\sqrt{\||A|^2 + |B|^2\|} \times \sqrt{w^2(A) + w^2(B)} + \frac{\beta}{1+\beta}\||A|^2 + |B|^2\|\right).
\end{aligned}$$

Remark 4.2.3. *If we take $t = \dfrac{1}{2}$ and $A = B$ in Corollary 4.2.4, then we get the following* *inequality*

$$w^2(A) \leq \frac{1}{1+\beta}\|A\|w(A) + \frac{\beta}{1+\beta}\|A\|^2.$$

It clear that this inequality refines the second inequality in (2.1).

Theorem 4.2.4. *Let $A, D, B, C \in \mathcal{B}(\mathcal{H})$ and $0 \leq \alpha \leq 1$. Then*

$$\begin{aligned} w_t^4\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) &\leq 128R^4\Big(\max\{\omega^4(A), \omega^4(D)\} \\ &+ \frac{1}{8}(3+\alpha)\max\{\||C|^4 + |B^*|^4\|, \||B|^4 + |C^*|^4\|\} \\ &+ \frac{1}{4}(1-\alpha)\max\{w^2(BC), w^2(CB)\}\Big). \end{aligned}$$

Proof . Let T, T_1, T_2 be as described in the proof of Theorem 4.2.1 and let $x \in \mathcal{H} \oplus \mathcal{H}$ with $\|x\| = 1$. Then

$$\begin{aligned} |\langle Sx, x\rangle|^4 &\leq 16R^4\,|\langle Tx, x\rangle|^4 \\ &\leq 16R^4\big(|\langle T_1x\,x\rangle| + |\langle T_2x\,x\rangle|\big)^4 \\ &\leq 128R^4\left(|\langle T_1x\,x\rangle|^4 + |\langle T_2x, x\rangle\langle x, T_2^*x\rangle|^2\right) \\ &\leq 128R^4\Big(|\langle T_1x\,x\rangle|^4 + \frac{1}{4}(3+\alpha)\|T_2x\|^2\|T_2^*x\|^2 + \frac{1}{4}(1-\alpha)|\langle T_2x, T_2^*x\rangle|^2\Big) \\ &\text{(by Lemma 2.1.6)} \\ &\leq 128R^4\Big(|\langle T_1x\,x\rangle|^4 + \frac{1}{8}(3+\alpha)\big(\|T_2x\|^4 + \|T_2^*x\|^4\big) + \frac{1}{4}(1-\alpha)|\langle T_2^2x, x\rangle|^2\Big) \\ &\text{(by the arithmetic-geometric mean inequality).} \end{aligned}$$

Thus,

$$
\begin{aligned}
|\langle Sx,x\rangle|^4 &\leq 128R^4\Big(|\langle T_1x\,x\rangle|^4+\frac{1}{8}(3+\alpha)\langle(|T_2|^4+|T_2^*|^4)x,x\rangle+\frac{1}{4}(1-\alpha)\big\langle T_2^2x,x\big\rangle|^2\Big)\\
&= 128R^4\Big(|\langle T_1x\,x\rangle|^4+\frac{1}{8}(3+\alpha)\left\langle\begin{bmatrix}|C|^4+|B^*|^4 & 0\\ 0 & |B|^4+|C^*|^4\end{bmatrix}x,x\right\rangle\\
&+\frac{1}{4}(1-\alpha)\left|\left\langle\begin{bmatrix}BC & 0\\ 0 & CB\end{bmatrix}x,x\right\rangle\right|^2\Big)\\
&\leq 128R^4\Big(\max\{\omega^4(A),\omega^4(D)\}+\frac{1}{8}(3+\alpha)w\left(\begin{bmatrix}|C|^4+|B^*|^4 & 0\\ 0 & |B|^4+|C^*|^4\end{bmatrix}\right)\\
&+\frac{1}{4}(1-\alpha)w^2\left(\begin{bmatrix}BC & 0\\ 0 & CB\end{bmatrix}\right)\Big)\\
&= 128R^4\Big(\max\{\omega^4(A),\omega^4(D)\}+\frac{1}{8}(3+\alpha)\max\{\||C|^4+|B^*|^4\|,\||B|^4+|C^*|^4\|\}\\
&+\frac{1}{4}(1-\alpha)\max\{w^2(BC),w^2(CB)\}\Big).
\end{aligned}
$$

By taking the supremum in the above inequality over $x\in\mathcal{H}\oplus\mathcal{H}$ with $\|x\|=1$, we obtain the desired inequality . ■

If we take $A=D$ and $B=C$ in Theorem 4.2.4, then we have

Corollary 4.2.5. *Let $A,\ B\in\mathcal{B}(\mathcal{H})$ and $0\leq\alpha\leq 1$. Then*

$$
w_t^4\left(\begin{bmatrix}A & B\\ B & A\end{bmatrix}\right)\leq 128R^4\Big(\omega^4(A)+\frac{1}{8}(3+\alpha)\||B|^4+|B^*|^4\|+\frac{1}{4}(1-\alpha)w^2(B^2)\Big).
$$

Remark 4.2.4. *If we take $t=\frac{1}{2}$ and $A=B$ in Corollary 4.2.5 and using Lemma 4.2.1, then we obtain*

$$
w^4(A)\leq\frac{3+\alpha}{8}\||A|^4+|A^*|^4\|+\frac{1-\alpha}{4}w^2(A^2),
$$

which has been already given in [4].

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