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Course manual

Analysis 1: Course and corrected exercises

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Dedication

I dedicate this work to:
My dear parents,
My family,
My friends.

Preface

This course is intended primarily for students in higher education who may need mathematical analysis, such as first-year students of the natural sciences (biology, agronomy) and first-year mathematics and informatics (MI) and matter sciences (SM) students. They will find most of the tools and notions of calculus in analysis that they need here. The course is described in detail, including theorems and propositions. All exercises, complete with corrections, will help students to consolidate their learning. This document contains four chapters in an easy-to-read style, covering the following topics:

1) The first chapter discusses sequences of real numbers and their properties.

2) The second chapter is devoted to the properties of series with positive terms, in which we introduce the various convergence criteria.

3) The third chapter looks at real functions with one real variable, focusing on the notion of limits, continuity at a point and differentiability.

4) The last chapter covers the calculation of integrals and primitives, deepening the definitions and methods used in high school, and introducing new tools: integration by parts, change of variable and the notion of indefinite integral for the calculation of primitives.

A. Benkhaled

1

Sequences

1.1 Definitions

Definition 1.1.1. *A sequence is simply an ordered list of numbers. For example, here is a sequence: 0, 1, 2, 3, 4, 5, ... This is different from the set* $\mathbb{N}$ *because, while the sequence is a complete list of every element in the set of natural numbers, in the sequence we very much care what order the numbers come in. For this reason, when we use variables to represent terms in a sequence they will look like this:*

$$a_0, a_1, a_2, a_3, \ldots.$$

To refer to the entire sequence at once, we will write $(a_n)_{n\in\mathbb{N}}$ *or* $(a_n)_{n\geq 0}$*, or sometimes if we are being sloppy, just* (a_n) *(in which case we assume we start the sequence with* a_0*)(The term* a_n *is said to be the general term). We might replace the a with another letter, and* *sometimes we omit a_0, starting with a_1, in which case we would use $(a_n)_{n\geq 1}$ to refer to the sequence as a whole. The numbers in the subscripts are called indices (the plural of index).*

Remark 1.1.1. *While we often just think of a sequence as an ordered list of numbers, it is really a type of function. Specifically, the sequence $(a_n)_{n\geq 0}$ is a function with domain $\mathbb{N}$ where (a_n) is the image of the natural number n. Later we will manipulate sequences in much the same way you have manipulated functions in algebra or calculus. We can shift a sequence up or down, add two sequences, or ask for the rate of change of a sequence. These are done exactly as you would for functions.*

That said, while keeping the rigorous mathematical definition in mind is helpful, we often describe sequences by writing out the first few terms

Example 1.1.1. *Can you find the next term in the following sequences* ?

1- 3, 2, 1, 0, -1, . . .

2- 1, 2, 4, 8, 16, 32, . . .

3- 1, 3, 6, 10, 15, 21, . . .

No you cannot. You might guess that the next terms are:

1- -2

2- 64

3- 34

In fact, those are the next terms of the sequences I had in mind when I made up the example, but there is no way to be sure they are correct.

Given that no number of initial terms in a sequence is enough to say for certain which sequence we are dealing with, we need to find another way to specify a sequence. We consider two ways to do this:

Definition 1.1.2. *A* ***closed formula*** *for a sequence $(a_n)_{n\in\mathbb{N}}$ is a formula for (a_n) using a fixed finite number of operations on n. This is what you normally think of as a formula in n, just as if you were defining a function in terms of n.*

Example 1.1.2. *Here are a few closed formulas for sequences:*

- $a_n = n^2$
- $a_n = \dfrac{n(n+1)}{2}$.

Note in each formula, if you are given n, you can calculate (a_n) directly: just plug in n. For example, to find a_3 in the first sequence, just compute $a_3 = 3^2 = 9$.

Definition 1.1.3. *A **recursive definition** (sometimes called an inductive definition) for a sequence $(a_n)_{n\in\mathbb{N}}$ consists of a recurrence relation : an equation relating a term of the sequence to previous terms (terms with smaller index) and an initial condition: a list of a few terms of the sequence (one less than the number of terms in the recurrence relation).*

Example 1.1.3. *Here are a few recursive definitions for sequences:*

▶ $a_n = 2a_{n-1}$ *with* $a_0 = 1$

▶ $a_n = a_{n-1} + a_{n-2}$ *with* $a_0 = 0$ *and* $a_1 = 1$.

In these formulas, if you are given n, you cannot calculate (a_n) directly, you first need to find (a_{n-1}) (or (a_{n-1}) and (a_{n-2})). In the second sequence, to find a_3 you would find $a_1 = 54$, $a_2 = 108$ and finally $a_3 = 216$.

Definition 1.1.4. *Let $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ be two sequences and $\lambda \in \mathbb{R}$.*

- *The sum of $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ is the sequence of general term $a_n + b_n$.*
- *The product of $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ is the sequence of general term $a_n b_n$.*
- *If for all $n \in \mathbb{N}$: $b_n \neq 0$, the quotient of $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ is that of the sequence of general term: $\dfrac{a_n}{b_n}$.*
- *$\lambda(a_n)_{n\in\mathbb{N}}$ is the sequence of general term λa_n.*

1.2 Arithmetic and geometric sequences

Definition 1.2.1. *If the terms of a sequence differ by a constant, we say the sequence is **arithmetic**. If the initial term (a_0) of the sequence is a and the common difference is d, then we have,*

- Recursive definition: $a_n = a_{n-1} + d$ with $a_0 = a$.

- Closed formula: $a_n = a + dn$.

Example 1.2.1. *Find recursive definitions and closed formulas for the arithmetic sequences below. Assume the first term listed is a_0.*

1. 2, 5, 8, 11, 14, ...

2. 50, 43, 36, 29, ...

First we should check that these sequences really are arithmetic by taking differences of successive terms. Doing so will reveal the common difference d.

1. $5-2=3$, $8-5=3$, etc. To get from each term to the next, we add three, so $d=3$. The recursive definition is therefore $a_n = a_{n-1}+3$ with $a_0 = 2$. The closed formula is $a_n = 2+3n$.

2. Here the common difference is -7, since we add -7 to 50 to get 43, and so on. Thus we have a recursive definition of $a_n = a_{n-1}-7$ with $a_0 = 50$. The closed formula is $a_n = 50-7n$.

Definition 1.2.2. *A sequence is called **geometric** if the ratio between successive terms is constant. Suppose the initial term a_0 is a and the common ratio is r. Then we have,*

- Recursive definition: $a_n = ra_{n-1}$ with $a_0 = a$.

- Closed formula: $a_n = ar^n$.

Example 1.2.2. *Find the recursive and closed formula for the geometric sequences below. Again, the first term listed is a_0.*

1. 3, 6, 12, 24, 48, . . .

2. 27, 9, 3, 1, 1/3, . . .

Start by checking that these sequences really are geometric by dividing each term by its previous term. If this ratio really is constant, we will have found r.

1. $6/3 = 2$, $12/6 = 2$, $24/12 = 2$, etc. Yes, to get from any term to the next, we multiply by $r = 2$. So the recursive definition is $a_n = 2a_{n-1}$ with $a_0 = 3$. The closed formula is $a_n = 3.2n$.

2. The common ratio is $r = 1/3$. So the sequence has recursive definition $a_n = \frac{1}{3}a_{n-1}$ with $a_0 = 27$ and closed formula $a_n = 27.\left(\frac{1}{3}\right)^n$.

Proposition 1.2.1. *1- Let $(a_n)_{n\in\mathbb{N}}$ an arithmetic sequence of ratio r and first term a_0, then*

$$\sum_{k=0}^{n} a_n = a_0 + a_1 + ... + a_n = \frac{(n+1)(a_0+a_n)}{2}.$$

2- Let $(a_n)_{n\in\mathbb{N}}$ an geometric sequence of ratio r and first term a_0, then

$$\sum_{k=0}^{n} a_n = a_0 + a_1 + ... + a_n = a_0\frac{1-r^{n+1}}{1-r}.$$

1.3 Monotone sequences

In this section we consider a particular class of sequences which often occur in applications.

Definition 1.3.1. *We say that the sequence* $(a_n)_{n\in\mathbb{N}}$ *is*

(i) ***increasing*** *if*

$$\forall n \in \mathbb{N} \quad a_n \leq a_{n+1}$$

(ii) ***decreasing*** *if*

$$\forall n \in \mathbb{N} \quad a_n \geq a_{n+1}$$

(iii) ***monotone*** *if it is either increasing or decreasing.*

Example 1.3.1. *(i)* $a_n = n^2$ *is increasing.*
(ii) $a_n = \dfrac{1}{n}$ *is decreasing.*
(iii) $a_n = (-1)^n \dfrac{1}{n}$ *is not monotone.*

1.4 Bounded sequences

Definition 1.4.1. *We say that a sequence* $(a_n)_{n\in\mathbb{N}}$ *is*

1. bounded above if there is a constant $M \in \mathbb{R}$ *such that* $\forall n \in \mathbb{N}\ a_n \leq M$

2. bounded below if there is a constant $m \in \mathbb{R}$ *such that* $\forall n \in \mathbb{N}\ a_n \geq m$

3. bounded if there is a constant $C > 0$ *such that* $|a_n| < C$ *holds for all* n*. If a sequence is not bounded it is said to be unbounded.*

Example 1.4.1. *The sequence* $a_n = (-1)^n + (-1)^n \dfrac{1}{n+1}$ *is bounded by* $C = 2$*.*
On the other hand, the sequence $a_n = n$ *is unbounded.*

1.5 Limit of a sequence

Definition 1.5.1. *A sequence* $(a_n)_{n\in\mathbb{N}}$ *is said to have limit* $L \in \mathbb{R}$ *if for any neighborhood* U *of* L *the sequence lies in this neighborhood eventually.*
We denote this symbolically as

$$a_n \longrightarrow L \quad as \quad n \longrightarrow \infty$$

Definition 1.5.2. *A sequence $(a_n)_{n\in\mathbb{N}}$ lies in a set S eventually if there is an $n_0 \in \mathbb{N}$ such that $a_n \in S$ for all $n \in \mathbb{N}$ with $n \geq n_0$.*

Example 1.5.1. *Let us look at some examples of sequences and try to see what their limits are.*

1- The sequence

$$-1, 4, 5, 7, 8, 8, 8, 8, 8, 8, ...$$

which eventually stabilizes at the constant value 8 has limit 8. For, again, given any neighborhood of 8 the sequence falls inside this neighborhood eventually and stays there.

2- In contrast, the sequence

$$1, 3, 4, 1, 3, 4, 1, 3, 4, 1, 3, 4, ...$$

does not have a limit. For example, the point 3 cannot be the limit of the sequence because, for instance,

$$(2.5, 3.5)$$

is a neighborhood of 3, but the sequence keeps falling outside this neighborhood (when it hits 1 or 4).

3- The sequence

$$1, 3, 5, 7, ...$$

has limit ∞. If you take any neighborhood of ∞, an interval of the form

$$(t, \infty]$$

then eventually the sequence falls inside the neighborhood and stays in there.

1.6 Convergent sequences

Definition 1.6.1. *A sequence $(a_n)_{n\in\mathbb{N}}$ is said to converge to a number L, in symbols*

$$\lim_{n \longrightarrow +\infty} a_n = L$$

if for every $\varepsilon > 0$ there is natural number N such that, for all natural numbers n, if

$$n > N, \text{ then } |a_n - L| < \varepsilon$$

A sequence that converges is said to be convergent. Otherwise, we say the sequence diverges or that it is divergent. That's to say

$$\exists \varepsilon > 0, \forall N \in \mathbb{N}, \exists n \geq N : |a_n - L| \geq \varepsilon$$

Example 1.6.1. *The sequence* $a_n = \dfrac{1}{n+1}$ *is convergent and the limit is* 0.
Let $\varepsilon > 0$*, we have*

$$|a_n - 0| = \left|\frac{1}{n+1}\right| = \frac{1}{n+1} < \varepsilon$$

it means that $n + 1 > \dfrac{1}{\varepsilon}$*, we can say that*

$$\forall n \in \mathbb{N},\ n > \frac{1}{\varepsilon} - 1.$$

So we can choose for example $N = E\left(\dfrac{1}{\varepsilon}\right) - 1$ *where* E *denotes the function of the integer part. Then we have*

$$\forall n \in \mathbb{N},\ n > N \longrightarrow |a_n| < \varepsilon$$

Proposition 1.6.1. *1- A convergent sequence has a unique limit.*
2- A convergent sequence is bounded.

Definition 1.6.2. *1- The sequence* $(a_n)_{n\in\mathbb{N}}$ *diverges to* ∞ *if*

$$(\forall M \in \mathbb{R})(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})[(n > N) \Longrightarrow (a_n > M)].$$

2- The sequence $(a_n)_{n\in\mathbb{N}}$ *diverges to* $-\infty$ *if*

$$(\forall M \in \mathbb{R})(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})[(n > N) \Longrightarrow (a_n < M)].$$

Example 1.6.2. *1- The sequence* $a_n = n^2$ *diverges to* ∞.
2- The sequence $a_n = -n^2$ *diverges to* $-\infty$.
3- The sequence $a_n = (-1)^n$ *diverges.*

Proposition 1.6.2. *1- Every convergent sequence is bounded.*
2- Any real sequence that tends to $+\infty$ *is lower bounded.*
3- Any real sequence that tends to $-\infty$ *is upper bounded.*

Theorem 1.6.1. (**Monotone Convergence Theorem**) *1- If a sequence of real numbers is bounded above and increasing then it is convergent to* $L = sup\{a_n : n \in \mathbb{N}\}$.
2- If a sequence of real numbers is bounded below and decreasing then it is convergent to $L = inf\{a_n : n \in \mathbb{N}\}$.

Example 1.6.3. *Let* $(a_n)_{n\in\mathbb{N}}$ *be a sequence defined by* $a_n = \sqrt{a_{n-1}+2},\ a_1 = \sqrt{2}$.
1- $(a_n)_{n\in\mathbb{N}}$ *is bounded. Indeed, we prove that* $(\forall n \in \mathbb{N})(0 < a_n \leq 2)$.
Positivity is obvious. For the upper bound we use induction. For $n = 1$, $a_n = \sqrt{2} \leq 2$, *and the statement is true. Suppose now that it is true for* $n = k (k \geq 1)$; *that is* $a_k \leq 2$. *For* $n = k+1$ *we have*

$$a_{k+1} = \sqrt{a_k + 2} \leq \sqrt{2+2} = 2,$$

and by the principle of induction the statement is proved.
2- $(a_n)_{n\in\mathbb{N}}$ *is increasing. We have to prove that*

$$(\forall n \in \mathbb{N})(a_{n+1} \geq a_n).$$

This is equivalent to proving that

$$[(\forall n \in \mathbb{N})(\sqrt{a_{n-1}+2} \geq a_n)] \Longleftrightarrow [(\forall n \in \mathbb{N})(a_{n-1}+2 \geq a_n^2)].$$

But,

$$a_n + 2 - a_n^2 = (2 - a_n)(a_n + 1) \geq 0.$$

We conclude that a_n *is convergent.*
Let $\lim\limits_n a_n = a$; *then,* $\lim\limits_n a_{n-1} = a$ *also. Write*

$$a_n^2 = a_{n-1} + 2$$

and take limits in both sides to obtain

$$a^2 = a + 2 \text{ or } (2-a)(a+1) = 0.$$

As $a > 0$, *we conclude that* $a = 2$.

1.7 Limits and inequalities

A basic lemma about limits and inequalities is the so-called squeeze lemma. It allows us to show convergence of sequences in difficult cases if we find two other simpler convergent sequences that "squeeze" the original sequence.

Lemma 1.7.1. *(Sandwich rule).*
Let $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, *and* $(c_n)_{n\in\mathbb{N}}$ *be sequences such that*

$$a_n \leq c_n \leq b_n \ \forall n \in \mathbb{N}.$$

Suppose $(a_n)_{n\in\mathbb{N}}$ *and* $(b_n)_{n\in\mathbb{N}}$ *converge and*

$$\lim_{n\longrightarrow\infty} a_n = \lim_{n\longrightarrow\infty} b_n = \alpha.$$

Then $(c_n)_{n\in\mathbb{N}}$ *converges and*

$$\lim_{n\longrightarrow\infty} c_n = \alpha.$$

Example 1.7.1. *Consider the sequence* $\left(\frac{1}{n\sqrt{n}}\right)_{n\in\mathbb{N}}$. *Since* $\sqrt{n} \geq 1$ *for all* $n \in \mathbb{N}$, *we have*

$$0 \leq \frac{1}{n\sqrt{n}} \leq \frac{1}{n}$$

We already know $\lim \frac{1}{n} = 0$. *Hence,* $\lim_{n\longrightarrow\infty} \frac{1}{n\sqrt{n}} = 0$.

Example 1.7.2. *For any* $a > 1$ *and* $n > a$, *we have* $1 < \sqrt[n]{a} < \sqrt[n]{n}$. *Then by the limit* $\lim_{n\longrightarrow\infty} \sqrt[n]{n} = 1$ *and the sandwich rule, we have* $\lim_{n\longrightarrow\infty} \sqrt[n]{a} = 1$. *On the other hand, for* $0 < a < 1$, *we have* $b = \frac{1}{a} > 1$ *and*

$$\lim_{n\longrightarrow\infty} \sqrt[n]{a} = \lim_{n\longrightarrow\infty} \frac{1}{\sqrt[n]{b}} = \frac{1}{\lim_{n\longrightarrow\infty} \sqrt[n]{b}} = 1$$

Combining all the cases, we get $\lim_{n\longrightarrow\infty} \sqrt[n]{a} = 1$ *for any* $a > 0$.

Limits, when they exist, preserve non-strict inequalities.

Lemma 1.7.2. *Let* $(a_n)_{n\in\mathbb{N}}$ *and* $(b_n)_{n\in\mathbb{N}}$ *be convergent sequences and*

$$a_n \leq b_n$$

for all $n \in \mathbb{N}$. *Then*

$$\lim_{n\longrightarrow\infty} a_n \leq \lim_{n\longrightarrow\infty} b_n$$

1.8 Continuity of algebraic operations

Limits interact nicely with algebraic operations.

Proposition 1.8.1. *Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be sequences. If*

$$\lim_{n\longrightarrow\infty} a_n \quad and \quad \lim_{n\longrightarrow\infty} b_n$$

both exist, then

1-

$$\lim_{n\longrightarrow\infty}(a_n+b_n)=\lim_{n\longrightarrow\infty} a_n+\lim_{n\longrightarrow\infty} b_n$$

2-

$$\lim_{n\longrightarrow\infty}(a_n.b_n)=\lim_{n\longrightarrow\infty} a_n.\lim_{n\longrightarrow\infty} b_n$$

3- Moreover, if $\lim_{n\longrightarrow\infty} b_n\neq 0$, then $b_n\neq 0$ for all $n>N$ for some N, and

$$\lim_{n\longrightarrow\infty}\frac{a_n}{b_n}=\frac{\lim_{n\longrightarrow\infty} a_n}{\lim_{n\longrightarrow\infty} b_n}$$

Example 1.8.1. *Let*

$$a_n=\frac{3n^3+2n^2+13n}{2n^3+16n^2+5}$$

Then

$$a_n=\frac{3+\frac{2}{n}+\frac{13}{n^2}}{2+\frac{16}{n}+\frac{5}{n^3}}.$$

Thus we get

$$\lim_{n\longrightarrow\infty} a_n=\frac{\lim_{n\longrightarrow\infty}(3+\frac{2}{n}+\frac{13}{n^2})}{\lim_{n\longrightarrow\infty}(2+\frac{16}{n}+\frac{5}{n^3})}=\frac{2}{3}$$

Remark 1.8.1. *By plugging in constant sequences. If $k\in\mathbb{R}$ and $(a_n)_{n\in\mathbb{N}}$ is a convergent sequence, then*

$$\lim_{n\longrightarrow\infty} ka_n=k(\lim_{n\longrightarrow\infty} a_n) \quad and \quad \lim_{n\longrightarrow\infty}(k+a_n)=k+\lim_{n\longrightarrow\infty} a_n.$$

Similarly, we find such equalities for constant subtraction and division.

Proposition 1.8.2. *Let $(a_n)_{n\in\mathbb{N}}$ be a convergent sequence such that $a_n\geq 0$. Then*

$$\lim_{n\longrightarrow\infty}\sqrt{a_n}=\sqrt{\lim_{n\longrightarrow\infty} a_n}$$

Proposition 1.8.3. *If $(a_n)_{n\in\mathbb{N}}$ is a convergent sequence, then $(|a_n|)_{n\in\mathbb{N}}$ is convergent and*

$$\lim_{n\longrightarrow\infty}|a_n|=|\lim_{n\longrightarrow\infty} a_n|$$

1.9 Some convergence tests

Proposition 1.9.1. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence. Suppose there is an $a \in \mathbb{R}$ and a convergent sequence $(b_n)_{n\in\mathbb{N}}$ such that*

$$\lim_{n\longrightarrow\infty} b_n = 0$$

and

$$|a_n - a| \leq b_n \quad \forall n \in \mathbb{N}.$$

Then $(a_n)_{n\in\mathbb{N}}$ converges and $\lim\limits_{n\longrightarrow\infty} a_n = a$.

Proposition 1.9.2. *Let $c > 0$*

1- If $c < 1$, then $\lim\limits_{n\longrightarrow\infty} c^n = 0$

2- If $c > 1$, then $(c^n)_{n\in\mathbb{N}}$ is unbounded.

Lemma 1.9.1. *(Ratio test for sequences)*

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence such that $a_n \neq 0$ for all n and such that the limit

$$L := \lim_{n\longrightarrow\infty} \frac{|a_{n+1}|}{|a_n|} \quad \textit{exists.}$$

1- If $L < 1$, then $(a_n)_{n\in\mathbb{N}}$ converges and $\lim\limits_{n\longrightarrow\infty} a_n = 0$

2- If $L > 1$, then $(a_n)_{n\in\mathbb{N}}$ is unbounded (hence diverges).

Example 1.9.1. *Prove that $\lim\limits_{n\longrightarrow\infty} \dfrac{2^n}{n!} = 0$*

Compute $\lim\limits_{n\longrightarrow\infty} \dfrac{2^{n+1}/(n+1)!}{2^n/n!} = \lim\limits_{n\longrightarrow\infty} \dfrac{2^{n+1}}{2^n}.\dfrac{n!}{(n+1)!} = \lim\limits_{n\longrightarrow\infty} \dfrac{2}{n+1} = 0 < 1.$

Then $\lim\limits_{n\longrightarrow\infty} a_n = 0$

Example 1.9.2. *Prove that $\lim\limits_{n\longrightarrow\infty} n^{\frac{1}{n}} = 1$*

Let $\varepsilon > 0$ be given. Consider the sequence $\left(\dfrac{n}{(1+\varepsilon)^n}\right)_{n\in\mathbb{N}}$. Compute

$$\frac{(n+1)/(1+\varepsilon)^{n+1}}{n/(1+\varepsilon)^n} = \frac{n+1}{n}.\frac{1}{1+\varepsilon}.$$

So

$$\lim_{n\longrightarrow\infty} \frac{(n+1)/(1+\varepsilon)^{n+1}}{n/(1+\varepsilon)^n} = \frac{1}{1+\varepsilon} < 1.$$

Therefore, $\left(\dfrac{n}{(1+\varepsilon)^n}\right)_{n\in\mathbb{N}}$ *converges to* 0. *In particular, there exists an* M *such that for* $n \geq M$, *we have*

$$\frac{n}{(1+\varepsilon)^n} < 1, \quad or \quad n^{\frac{1}{n}} < 1+\varepsilon.$$

As $n \geq 1$, *then* $n^{\frac{1}{n}} \geq 1$, *and so* $0 \leq n^{\frac{1}{n}} - 1 < \varepsilon$. *Consequently* $\lim\limits_{n\longrightarrow\infty} n^{\frac{1}{n}} = 1$.

1.10 Adjacent sequences

Definition 1.10.1. *Two sequences* $(a_n)_{n\in\mathbb{N}}$ *and* $(b_n)_{n\in\mathbb{N}}$ *are adjacent if one is increasing and the other is decreasing and if the sequence* $(a_n - b_n)_{n\in\mathbb{N}}$ *converges to* 0.

Theorem 1.10.1. *If* $(a_n)_{n\in\mathbb{N}}$ *and* $(b_n)_{n\in\mathbb{N}}$ *are two adjacent sequences, then they converge to the same limit* l . *Moreover, in the case where* $(a_n)_{n\in\mathbb{N}}$ *is the increasing sequence and* $(b_n)_{n\in\mathbb{N}}$ *is the decreasing sequence, we have*

$$\forall n \in \mathbb{N},\ a_n \leq l \leq b_n.$$

Example 1.10.1. *Let us show that the two general term sequences below are adjacent*

$$a_n = \sum_{k=1}^{n} \frac{1}{k^2} \text{ and } b_n = a_n + \frac{1}{n}$$

We have

$$a_{n+1} - a_n = \sum_{k=1}^{n+1} \frac{1}{k^2} - \sum_{k=1}^{n} \frac{1}{k^2} = \frac{1}{(n+1)^2} > 0$$

hence $(a_n)_{n\in\mathbb{N}^*}$ *is a increasing sequence.*
Likewise, We have

$$b_{n+1} - b_n = \sum_{k=1}^{n+1} \frac{1}{k^2} + \frac{1}{n+1} - \left(\sum_{k=1}^{n} \frac{1}{k^2} + \frac{1}{n}\right) = \frac{-1}{n(n+1)^2} < 0$$

hence $(b_n)_{n\in\mathbb{N}^*}$ *is a decreasing sequence.*
Moreover $\lim\limits_{n\longrightarrow\infty}(b_n - a_n) = \lim\limits_{n\longrightarrow\infty} \dfrac{1}{n} = 0$. *Therefore ;* $(a_n)_{n\in\mathbb{N}^*}$ *and* $(b_n)_{n\in\mathbb{N}^*}$ *are adjacent sequences.*

1.11 Subsequences

It is useful to sometimes consider only some terms of a sequence. A subsequence of $(a_n)_{n\in\mathbb{N}}$ is a sequence that contains only some of the numbers from $(a_n)_{n\in\mathbb{N}}$ in the same order.

Definition 1.11.1. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence. Let $(n_i)_{i\in\mathbb{N}}$ be a strictly increasing sequence of natural numbers, that is, $n_i < n_{i+1}$ for all i. The sequence*

$$(a_{n_i})_{i\in\mathbb{N}}$$

is called a subsequence of $(a_n)_{n\in\mathbb{N}}$.

Example 1.11.1. *Consider the sequence $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$. The sequence $\left(\frac{1}{3n}\right)_{n\in\mathbb{N}}$ is a subsequence. To see how these two sequences fit in the definition, take $n_i := 3i$. The numbers in the subsequence must come from the original sequence.*

Proposition 1.11.1. *If $(a_n)_{n\in\mathbb{N}}$ is a convergent sequence, then every subsequence $(a_{n_i})_{i\in\mathbb{N}}$ is also convergent, and*

$$\lim_{n\longrightarrow\infty} a_n = \lim_{i\longrightarrow\infty} a_{n_i}$$

Remark 1.11.1. *Existence of a convergent subsequence does not imply convergence of the sequence itself.*

Example 1.11.2. *Take the sequence 0, 1, 0, 1, 0, 1, ... That is, $a_n = 0$ if n is odd, and $a_n = 1$ if n is even. The sequence $(a_n)_{n\in\mathbb{N}}$ is divergent; however, the subsequence $(a_{2n})_{n\in\mathbb{N}}$ converges to 1 and the subsequence $(a_{2n+1})_{n\in\mathbb{N}}$ converges to 0.*

Proposition 1.11.2. *A bounded sequence $(a_n)_{n\in\mathbb{N}}$ is convergent and converges to a if and only if every convergent subsequence $(a_{n_i})_{i\in\mathbb{N}}$ converges to a.*

Theorem 1.11.1. *(Bolzano-Weierstrass).*
Suppose a sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers is bounded. Then there exists a convergent subsequence $(a_{n_i})_{i\in\mathbb{N}}$.

1.12 Cauchy sequences

Definition 1.12.1. *A sequence $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence if for every $\varepsilon > 0$ there exists an $M \in \mathbb{N}$ such that for all $n \geq M$ and all $k \geq M$, we have*

$$|a_n - a_k| < \varepsilon$$

Example 1.12.1. *The sequence $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ is a Cauchy sequence.*
Given $\varepsilon > 0$, find M such that $M > \frac{2}{\varepsilon}$. Then for $n,\ k \geq M$, we have $\frac{1}{n} < \frac{\varepsilon}{2}$ and $\frac{1}{k} < \frac{\varepsilon}{2}$. Therefore, for $n,\ k \geq M$, we have

$$\left|\frac{1}{n} - \frac{1}{k}\right| \leq \left|\frac{1}{n}\right| + \left|\frac{1}{k}\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Example 1.12.2. *The sequence $\left(\frac{n+1}{n}\right)_{n\in\mathbb{N}}$ is a Cauchy sequence.*
Given $\varepsilon > 0$, find M such that $M > \frac{2}{\varepsilon}$. Then for $n,\ k \geq M$, we have $\frac{1}{n} < \frac{\varepsilon}{2}$ and $\frac{1}{k} < \frac{\varepsilon}{2}$. Therefore, for $n,\ k \geq M$, we have

$$\begin{aligned} \left|\frac{n+1}{n} - \frac{k+1}{k}\right| &= \left|\frac{k(n+1) - n(k+1)}{kn}\right| \\ &= \left|\frac{k-n}{kn}\right| \\ &\leq \left|\frac{k}{kn}\right| + \left|\frac{n}{kn}\right| \\ &= \left|\frac{1}{n}\right| + \left|\frac{1}{k}\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Proposition 1.12.1. *Every Cauchy sequence is bounded.*

Lemma 1.12.1. *A Cauchy sequence that has a convergent subsequence is convergent.*

Theorem 1.12.1. *A convergent sequence is a Cauchy sequence.*

Theorem 1.12.2. *A sequence of real numbers is Cauchy if and only if it converges.*

Definition 1.12.2. *A sequence $(a_n)_{n\in\mathbb{N}}$ is called contractive if there exists $k \in [0, 1)$ such that*

$$|a_{n+2} - a_{n+1}| \leq k|a_{n+1} - a_n| \ for\ all\ n \in \mathbb{N}.$$

Remark 1.12.1. *The condition $k < 1$ in definition is crucial. Consider the following example. Let $a_n = \ln n$ for all $n \in \mathbb{N}$. Since $1 < \frac{n+2}{n+1} < \frac{n+1}{n}$ for all $n \in \mathbb{N}$ and the natural logarithm is an increasing function, we have*

$$|a_{n+2} - a_{n+1}| = \ln\left(\frac{n+2}{n+1}\right) < \ln\left(\frac{n+1}{n}\right) = |a_{n+1} - a_n|$$

Therefore, the inequality in Definition is satisfied with $k = 1$, yet the sequence $(\ln n)_{n\in\mathbb{N}}$ does not converge.

Theorem 1.12.3. *Every contractive sequence is convergent.*

1.13 Exercises

Exercice 1.13.1. *Study the convergence of the following sequence, for all* $n \in \mathbb{N}$, $a_n = n + (-1)^n n + \frac{1}{n}$.

Proof 1.13.1. *If* n *is even* $\Longrightarrow a_n = 2n + \frac{1}{n} \longrightarrow +\infty$
If n *is odd* $\Longrightarrow a_n = \frac{1}{n} \longrightarrow 0$.
Then the sequence $(a_n)_{n\in\mathbb{N}}$ *is divergent since it is not bounded.*

Exercice 1.13.2. *Prove that the sequence of the general term, for all* $n \in \mathbb{N}$, $a_n = \int_1^n \frac{\cos t}{t^2} dt$. *is a Cauchy sequence.*

Proof 1.13.2. *Given* n, k *if* $n = k$ *then* $a_n = a_k$, *so* $|a_n - a_k| = 0 < \varepsilon$.
If $n \neq k$, *assume* $k < n$.

$$\begin{aligned} |a_n - a_k| &= \left| \int_1^n \frac{\cos t}{t^2} dt - \int_1^k \frac{\cos t}{t^2} dt \right| \\ &= \left| \int_k^n \frac{\cos t}{t^2} dt \right| \\ &\leq \int_k^n \left| \frac{\cos t}{t^2} \right| dt \\ &\leq \int_k^n \frac{1}{t^2} dt \\ &= -\frac{1}{n} + \frac{1}{k} \leq \frac{1}{k} \leq \frac{1}{M} < \varepsilon. \end{aligned}$$

Given $\varepsilon > 0$, *choose* $M \in \mathbb{N}$ *such that* $\frac{1}{\varepsilon} < M$. *Then for all* $n,k \in \mathbb{N}$ *with* $M \leq k \leq n$ *we have* $|a_n - a_k| < \varepsilon$.

Exercice 1.13.3. *Let* $A > 0$ *be fixed. Start with any* $a_n > 0$ *and define*

$$a_n = \frac{1}{2}\left(a_{n-1} + \frac{A}{a_{n-1}}\right) \quad n = 2,\ 3,\ 4, ...$$

Calculate $\lim_{n\longrightarrow\infty} a_n$.

Proof 1.13.3. *We will show* $a_2 \geq a_3 \geq a_4 \geq ...$ *For* $n \geq 2$

$$\begin{aligned} a_n^2 - A &= \frac{1}{4}\left(a_{n-1}^2 + \frac{A^2}{a_{n-1}^2} + 2A\right) - A \\ &= \frac{1}{4}\left(a_{n-1}^2 + \frac{A^2}{a_{n-1}^2} - 2A\right) \\ &\leq \frac{1}{4}\left(a_{n-1} - \frac{A}{a_{n-1}}\right)^2 \geq 0. \end{aligned}$$

So $a_n^2 \geq A$ *for all* $n \geq 2$. *Since all* $a_n > 0$, $a_n \geq \sqrt{A}$ *for all* $n \geq 2$. *For* $n \geq 2$

$$\begin{aligned} a_n - a_{n+1} &= a_n - \frac{1}{2}\left(a_n + \frac{A}{a_n}\right) \\ &= \frac{1}{2}\left(a_n - \frac{A}{a_n}\right) \\ &= \frac{1}{2}\frac{a_n^2 - A}{a_n} \geq 0. \end{aligned}$$

So $a_n \geq a_{n+1}$ *for all* $n \geq 2$. *So* $(a_n)_{n\geq 2}$ *is decreasing and bounded. So* $\lim\limits_{n\longrightarrow\infty} a_n = a$ *exists. Then we solve for* a. *We have that*

$$\begin{aligned} a_n &= \frac{1}{2}\left(a_{n-1} + \frac{A}{a_{n-1}}\right) \\ \downarrow & \qquad\quad \downarrow \\ a &= \frac{1}{2}\left(a + \frac{A}{a}\right). \end{aligned}$$

Then $2a = a + \dfrac{A}{a} \Longleftrightarrow a = \pm\sqrt{A}$.
Since all $a_n > 0$, *limit* a *cannot be negative. So* $a = \sqrt{A}$.

Exercice 1.13.4. *Prove that*
1- $\lim\limits_{n\longrightarrow\infty} \dfrac{1}{n^k} = 0$ *for* $k > 0$
2- $\lim\limits_{n\longrightarrow\infty} \sqrt[n]{n} = 1$.
3- If $p > 0$ *and* $\alpha \in \mathbb{R}$ *are constants then* $\lim\limits_{n\longrightarrow\infty} \dfrac{n^\alpha}{(1+p)^n} = 0$.

Proof 1.13.4. *1- Another way of expressing the same limit is*

$$\lim_{n\longrightarrow\infty} n^k = 0 \ for \ k < 0$$

To establish the limit, we note that the inequality $\left|\dfrac{1}{n^k} - 0\right| = \left|\dfrac{1}{n^k}\right| < \varepsilon$ *is the same as* $n > \varepsilon^{-\frac{1}{p}}$. *Therefore choosing* $N = \varepsilon^{-\frac{1}{p}}$.

2- Let $a_n = \sqrt[n]{n} = 1$ *Then* $a_n > 0$ *and*

$$n = (1 + a_n)^n = 1 + na_n + \frac{n(n-1)}{2}a_n^2 + ... > \frac{n(n-1)}{2}a_n^2.$$

This implies $a_n^2 < \dfrac{2}{n-1}$. *In order to get* $|\sqrt[n]{n} - 1| = a_n < \varepsilon$, *it is sufficient to have* $\dfrac{2}{n-1} < \varepsilon^2$, *which is the same as* $n > \dfrac{2}{\varepsilon^2} + 1$. *Therefore we may choose* $N = \dfrac{2}{\varepsilon^2} + 1$.

3- If $\alpha \leq 0$ *we have* $\lim\limits_{n\longrightarrow\infty} \dfrac{n^\alpha}{(1+p)^n} = 0$. *So assume* $\alpha > 0$. *Fix a natural number* k *such that* $\alpha < k$. *Then for* $n \geq 2k$

$$\underbrace{n(n-1)...(n-k+1)}_{k\ terms} > \underbrace{\frac{n}{2}.\frac{n}{2}...\frac{n}{2}}_{k\ terms} = \left(\frac{n}{2}\right)^k$$

and

$$\begin{aligned}(1+p)^n &= \sum_{l=0}^{n}\binom{n}{l}p^l \\ &> \binom{n}{k}p^k = \frac{n(n-1)...(n-k+1)}{k!}p^k \\ &> \frac{\left(\frac{n}{2}\right)^k}{k!}p^k = \frac{n^k}{2^k k!}p^k = \frac{n^{k-\alpha}n^\alpha}{2^k k!}p^k.\end{aligned}$$

Then

$$0 < \frac{n^\alpha}{(1+p)^n} < \frac{2^k k!}{p^k}\frac{1}{n^{k-\alpha}}.$$

By sandwich property, we have $\lim\limits_{n\longrightarrow\infty} \dfrac{n^\alpha}{(1+p)^n} = 0$.

Exercice 1.13.5. *Prove that*

1- $\lim\limits_{n\longrightarrow\infty} a^n = 0\ for\ |a| < 1$.

2- $\lim\limits_{n\longrightarrow\infty} n^p a^n = 0\ for\ |a| < 1$ *and any* p.

3- $\lim\limits_{n\longrightarrow\infty} \dfrac{a^n}{n!} = 0\ for\ any\ a$.

Proof 1.13.5. *1- Another way of expressing the same limit is* $\lim\limits_{n\longrightarrow\infty} \dfrac{1}{a^n} = 0\ for\ |a| > 1$. *Let* $\dfrac{1}{|a|} = 1 + b$. *Then* $b > 0$ *and* $\dfrac{1}{|a^n|} = (1+b)^n = 1 + nb + \dfrac{n(n-1)}{2}b^2 + ... > nb$. *This implies* $|a^n| < \dfrac{1}{nb}$. *In order to get* $|a^n| < \varepsilon$, *it is sufficient to have* $\dfrac{1}{nb} < \varepsilon$. *This suggests us to choose* $N = \dfrac{1}{b\varepsilon}$.

2- Fix a natural number $P > p + 1$. *For* $n > 2P$, *we have*

$$\begin{aligned}\frac{1}{|a^n|} &= 1 + nb + \frac{n(n-1)}{2}b^2 + ... + \frac{n(n-1)...(n-P+1)}{P!}b^p + ... \\ &> \frac{n(n-1)...(n-P+1)}{P!}b^p \\ &> \frac{\left(\frac{n}{2}\right)^p}{P!}b^p.\end{aligned}$$

This implies

$$|n^p a^n| < \frac{n^p|a^n|}{n} < \frac{2^P P!}{b^2}\frac{1}{n}$$

and suggests us to choose $N = \max\left\{2P, \dfrac{2^P P!}{2b^2\varepsilon}\right\}$.

3- Fix a natural number $P > |a|$. *For* $n > P$, *we have*

$$\left|\frac{a^n}{n!}\right| = \frac{|a|^P}{P!}\frac{|a|}{P+1}\frac{|a|}{P+2}\cdots\frac{|a|}{n-1}\frac{|a|}{n} \leq \frac{|a|^P}{P!}\frac{|a|}{n}.$$

In order to get $\left|\dfrac{a^n}{n!}\right| < \varepsilon$, *we only need to make sure* $\dfrac{|a|^P}{P!}\dfrac{|a|}{n} < \varepsilon$. *This leads to the choice* $N = \max\left\{P, \dfrac{|a|^{P+1}}{P!\varepsilon}\right\}$.

Exercice 1.13.6. *Prove that the sequence* $(a_n)_{n\in\mathbb{N}}$, *where* $a_n = \left(1+\dfrac{1}{n}\right)^n$ *converges.*

Proof 1.13.6. *The binomial expansion tells us*

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &= 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3 \\ &+ \ldots + \frac{n(n-1)\ldots 1}{n!}\left(\frac{1}{n}\right)^n \\ &= 1 + \frac{1}{1!} + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) \\ &+ \ldots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{n-1}{n}\right).\end{aligned}$$

By comparing the similar formula for a_{n+1}, *we find the sequence is strictly increasing. The formula also tells us*

$$\begin{aligned}a_n &< 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} \\ &< 1 + \frac{1}{1!} + \frac{1}{1.2} + \frac{1}{2.3} + \ldots + \frac{1}{(n-1)n} \\ &= 2 + \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots + \left(\frac{1}{n-1} - \frac{1}{n}\right) \\ &= 2 + \frac{1}{1} - \frac{1}{n} < 3.\end{aligned}$$

Therefore the sequence converges.

Exercice 1.13.7. *1- Show, by induction on* n, *that* $a_n = 2n-1$ *solves the recurrence* $a_1 = 1$, $a_{n+1} = 2a_n + 1$.

2- Prove, by induction on n, *that* $\dfrac{1}{1.4} + \dfrac{1}{4.7} + \ldots + \dfrac{1}{(3n-2)(3n+1)} = \dfrac{n}{(3n+1)}$ *for* $n \geq 1$.

Proof 1.13.7. *1- The base case is $n = 1$: $2^1 - 1 = 1 = a_1$. Suppose the result for $n = k$. Then the right hand side of the recurrence $2a_k + 1$, substituting the inductive hypothesis, equals $2(2^k - 1) + 1$, which equals $2^{k+1} - 1$ as required. That completes the induction.*
2- The base case is $n = 1$: the left hand side is $\frac{1}{4}$ and the right hand side is $\frac{1}{3.1+1}$. For the inductive step, suppose that

$$\frac{1}{1.4} + \frac{1}{4.7} + ... + \frac{1}{(3k-2)(3k+1)} = \frac{k}{(3k+1)}.$$

Now compute

$$\frac{1}{1.4} + \frac{1}{4.7} + ... + \frac{1}{(3k-2)(3k+1)} + \frac{1}{(3k+1)(3k+4)} = \frac{k}{(3k+1)} + \frac{1}{(3k+1)(3k+4)}.$$

Simplifying, this equals

$$\frac{k(3k+4)+1}{(3k+1)(3k+4)} = \frac{k+1}{3(k+1)+1}$$

This is the required result for $k + 1$. That completes the proof by induction.

Exercice 1.13.8. *Find the following limits: $\lim_{n \longrightarrow \infty} a_n := \sqrt[n]{b}$, where $b > 0$.*

Proof 1.13.8. *Consider the case where $b > 1$. In this case, $a_n > 1$ for every n. By the binomial theorem,*

$$b = a_n^n = (a_n - 1 + 1)^n \geq 1 + n(a_n - 1).$$

This implies

$$0 < a_n - 1 \leq \frac{b-1}{n}.$$

For each $\varepsilon > 0$, choose $N > \frac{b-1}{\varepsilon}$. It follows that for $n \geq N$,

$$|a_n - 1| = a_n - 1 < \frac{b-1}{n} \leq \frac{b-1}{N} < \varepsilon.$$

Thus, $\lim_{n \longrightarrow \infty} a_n = 1$.
In the case where $b = 1$, it is obvious that $a_n = 1$ for all n and, hence, $\lim_{n \longrightarrow \infty} a_n = 1$.
If $0 < b < 1$, let $k = \frac{1}{b}$ and define $s_n = \sqrt[n]{k} = \frac{1}{a_n}$.
Since $k > 1$, it has been shown that $\lim_{n \longrightarrow \infty} s_n = 1$. This implies

$$\lim_{n \longrightarrow \infty} a_n = \lim_{n \longrightarrow \infty} \frac{1}{s_n} = 1$$

Exercice 1.13.9. *Find the following limits if they exist:*

$$(a)\ \lim_{n\longrightarrow\infty}\sqrt{n+1}-\sqrt{n} \qquad (b)\ \lim_{n\longrightarrow\infty}\frac{\sqrt{n+1}-\sqrt{n}}{n} \qquad (c)\ \lim_{n\longrightarrow\infty}\sqrt{n^2+n}-n$$

$$(d)\ \lim_{n\longrightarrow\infty}\sqrt[3]{n^3+3n^2}-\sqrt{n^2+n} \quad (e)\ \lim_{n\longrightarrow\infty}\sqrt[3]{n^3+3n^2}-n \qquad (f)\ \lim_{n\longrightarrow\infty}\frac{\sqrt{3n}+1}{\sqrt{n}+\sqrt{3}}$$

Proof 1.13.9. (a)

$$\begin{aligned}\lim_{n\longrightarrow\infty}(\sqrt{n+1}-\sqrt{n}) &= \lim_{n\longrightarrow\infty}\frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}}\\ &= \lim_{n\longrightarrow\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}}\\ &= 0.\end{aligned}$$

(b) *in the same way as* (a). $\lim_{n\longrightarrow\infty}\frac{\sqrt{n+1}-\sqrt{n}}{n}=0.$

(c)

$$\begin{aligned}\lim_{n\longrightarrow\infty}(\sqrt{n^2+n}-n) &= \lim_{n\longrightarrow\infty}\frac{(\sqrt{n^2+n}-n)(\sqrt{n^2+n}+n)}{\sqrt{n^2+n}+n}\\ &= \lim_{n\longrightarrow\infty}\frac{n}{\sqrt{n^2+n}+n}\\ &= \lim_{n\longrightarrow\infty}\frac{1}{\sqrt{1+\frac{1}{n}}+1}=\frac{1}{2}.\end{aligned}$$

(e)

$$\begin{aligned}\lim_{n\longrightarrow\infty}(\sqrt[3]{n^3+3n^2}-n) &= \lim_{n\longrightarrow\infty}\frac{(\sqrt[3]{n^3+3n^2}-n)(\sqrt[3]{(n^3+3n^2)^2}+n\sqrt[3]{n^3+3n^2}+n^2)}{\sqrt[3]{(n^3+3n^2)^2}+n\sqrt[3]{n^3+3n^2}+n^2}\\ &= \lim_{n\longrightarrow\infty}\frac{3n^2}{\sqrt[3]{(n^3+3n^2)^2}+n\sqrt[3]{n^3+3n^2}+n^2}\\ &= \lim_{n\longrightarrow\infty}\frac{3n^2}{\sqrt[3]{n^6(1+3/n)^2}+n\sqrt[3]{n^3(1+3/n)}+n^2}\\ &= \lim_{n\longrightarrow\infty}\frac{3n^2}{n^2(\sqrt[3]{(1+3/n)^2}+\sqrt[3]{(1+3/n)}+1)}\\ &= \lim_{n\longrightarrow\infty}\frac{3}{(\sqrt[3]{(1+3/n)^2}+\sqrt[3]{(1+3/n)}+1)}=1\end{aligned}$$

(d)

$$\begin{aligned}\lim_{n\longrightarrow\infty}(\sqrt[3]{n^3+3n^2}-\sqrt{n^2+n}) &= \lim_{n\longrightarrow\infty}(\sqrt[3]{n^3+3n^2}-n+n-\sqrt{n^2+n})\\ &= \lim_{n\longrightarrow\infty}(\sqrt[3]{n^3+3n^2}-n)+\lim_{n\longrightarrow\infty}(n-\sqrt{n^2+n})\\ &= 1-\frac{1}{2}=\frac{1}{2}.\end{aligned}$$

(f) $\lim\limits_{n\longrightarrow\infty} \dfrac{\sqrt{3n}+1}{\sqrt{n}+\sqrt{3}} = \lim\limits_{n\longrightarrow\infty} \dfrac{\sqrt{3n}}{\sqrt{n}} = \sqrt{3}.$

Exercice 1.13.10. *Let $a_1 = 2$. Define $a_{n+1} = \dfrac{a_n + 5}{3}$ for $n \geq 1$.*

(a) Prove that a_n is an increasing sequence.

(b) Prove that $a_n \leq 3$ for all $n \in \mathbb{N}$.

(c) Find the limit of a_n.

Proof 1.13.10. *(a) We prove by induction that for all $n \in \mathbb{N}$, $a_n < a_{n+1}$. Since $a_2 = \dfrac{a_1+5}{3} = \dfrac{7}{3} > 2 = a_1$, the statement is true for $n = 1$. Next, suppose $a_k < a_{k+1}$ for some $k \in \mathbb{N}$. Then $a_k + 5 < a_{k+1} + 5$ and $\dfrac{a_k+5}{3} < \dfrac{a_{k+1}+5}{3}$. Therefore,*

$$a_{k+1} = \frac{a_k+5}{3} < \frac{a_{k+1}+5}{3} = a_{k+2}.$$

It follows by induction that the sequence is increasing.

(b) Again, we proceed by induction. The statement is clearly true for $n = 1$. Suppose that $a_k \leq 3$ for some $k \in \mathbb{N}$. Then

$$a_{k+1} = \frac{a_k+5}{3} \leq \frac{3+5}{3} = \frac{8}{3} \leq 3.$$

It follows that $a_n \leq 3$ for all $n \in \mathbb{N}$.

(c) From the Monotone Convergence Theorem, we deduce that there is $l \in \mathbb{R}$ such that $\lim\limits_{n\longrightarrow\infty} a_n = l$. Since the subsequence $(a_{k+1})_{k\geq 1}$ also converges to l, taking limits on both sides of the equation $a_{n+1} = \dfrac{a_n+5}{3}$, we obtain $l = \dfrac{l+5}{3}$. Therefore, $3l = l + 5$ and, hence, $l = \dfrac{5}{2}$.

Exercice 1.13.11. *Prove that each of the following sequences is convergent and find its limit.*

(a) $a_{n+1} = \dfrac{a_n+3}{2}$ *and* $a_1 = 1$ *for* $n \geq 1$ $\qquad$ (b) $a_{n+1} = \sqrt{a_n+6}$ *and* $a_1 = \sqrt{6}$ *for* $n \geq 1$

(c) $a_{n+1} = \dfrac{1}{3}\left(2a_n + \dfrac{1}{a_n^2}\right)$ *and* $a_1 > 0$ *for* $n \geq 1$ $\qquad$ (d) $a_{n+1} = \dfrac{1}{2}\left(a_n + \dfrac{b}{a_n}\right)$, $b > 0$.

Proof 1.13.11. *The limit of (a) and (b) is 3.*

(c) We use the well known inequality

$$\frac{a+b+c}{3} \geq \sqrt[3]{abc} \text{ for } a, b, c \geq 0.$$

By induction, we see that $a_n > 0$ *for all* $n \in \mathbb{N}$. *Moreover,*

$$a_{n+1} = \frac{1}{3}\left(2a_n + \frac{1}{a_n^2}\right) = \frac{1}{3}\left(a_n + a_n + \frac{1}{a_n^2}\right) \geq \sqrt[3]{a_n.a_n.\frac{1}{a_n^2}} = 1.$$

We also have, for $n \geq 2$,

$$a_{n+1} - a_n = \frac{1}{3}\left(2a_n + \frac{1}{a_n^2}\right) - a_n = \frac{-a_n^3 + 1}{3a_n^2} = \frac{-(a_n - 1)(a_n^2 + a_n + 1)}{3a_n^2} < 0.$$

Thus, $(a_n)_{n\in\mathbb{N}}$ *is monotone deceasing (for* $n \geq 2$ *) and bounded below. We can show that* $\lim\limits_{n\longrightarrow\infty} a_n = 1$.

(d) Use the inequality $\dfrac{a+b}{2} \geq \sqrt{ab}$ *for* $a, b \geq 0$ *to show that* $a_{n+1} \geq \sqrt{b}$ *for all* $n \in \mathbb{N}$. *And using again induction to show that* $(a_n)_{n\in\mathbb{N}}$ *is monotone decreasing. Thus* $\lim\limits_{n\longrightarrow\infty} a_n = \sqrt{b}$.

Exercice 1.13.12. *Let* a *and* b *be two positive real numbers with* $a < b$. *Define* $a_1 = a$, $b_1 = b$, *and*

$$a_{n+1} = \sqrt{a_n b_n} \text{ and } b_{n+1} = \frac{a_n + b_n}{2} \text{ for } n \geq 1$$

Show that $(a_n)_{n\in\mathbb{N}}$ *and* $(b_n)_{n\in\mathbb{N}}$ *are convergent to the same limit.*

Proof 1.13.12. *Observe that*

$$b_{n+1} = \frac{a_n + b_n}{2} \geq \sqrt{a_n.b_n} = a_{n+1} \text{ for all } n \in \mathbb{N}.$$

Thus,

$$a_{n+1} = \sqrt{a_n.b_n} \geq \sqrt{a_n.a_n} = a_n \text{ for all } n \in \mathbb{N},$$
$$b_{n+1} = \frac{a_n + b_n}{2} \leq \frac{b_n + b_n}{2} = b_n \text{ for all } n \in \mathbb{N}.$$

It follows that $(a_n)_{n\in\mathbb{N}}$ *is monotone increasing and bounded above by* b_1, *and* $(b_n)_{n\in\mathbb{N}}$ *is decreasing and bounded below by* a_1. *Let* $l_1 = \lim\limits_{n\longrightarrow\infty} a_n$ *and* $l_2 = \lim\limits_{n\longrightarrow\infty} b_n$. *Then*

$$\left(l_1 = \sqrt{l_1 l_2} \text{ and } l_2 = \frac{l_1 + l_2}{2}\right) \Longrightarrow l_1 = l_2.$$

Exercice 1.13.13. *Let* $(a_n)_{n\in\mathbb{N}}$ *be defined by* $a_1 = 2$ *and*

$$a_{n+1} = a_n - \frac{a_n^2 - 2}{2a_n}.$$

- Prove that the sequences $(a_n)_{n\in\mathbb{N}}$ *is convergent and find its limit.*

Proof 1.13.13. *We must first find out if this sequence is well defined. So let us prove* a_n *exists and* $a_n > 0$ *for all* n *(so the sequence is well defined and bounded below). Let us show this by induction. We know that* $a_1 = 2 > 0$*. For the induction step, suppose* $a_n > 0$*. Then*

$$a_{n+1} = a_n - \frac{a_n^2 - 2}{2a_n} = \frac{2a_n^2 - a_n^2 + 2}{2a_n} = \frac{a_n^2 + 2}{2a_n}$$

It is always true that $a_n^2 + 2 > 0$*, and as* $a_n > 0$*, then* $\frac{a_n^2+2}{2a_n} > 0$ *and hence* $a_{n+1} > 0$*. Next let us show that the sequence is monotone decreasing. If we show that* $a_n^2 - 2 \geq 0$ *for all* n*, then* $a_{n+1} \leq a_n$ *for all* n*. Obviously* $a_1^2 - 2 = 4 - 2 = 2 > 0$*. For an arbitrary* n*, we have*

$$a_{n+1}^2 - 2 = \left(\frac{a_n^2 + 2}{2a_n}\right)^2 - 2 = \frac{a_n^4 + 4a_n^2 + 4 - 8a_n^2}{4a_n^2} = \frac{a_n^4 - 4a_n^2 + 4}{4a_n^2} = \frac{(a_n^2 - 2)^2}{4a_n^2}$$

Since squares are nonnegative, $a_{n+1}^2 - 2 \geq 0$ *for all* n*. Therefore,* $(a_n)_{n\in\mathbb{N}}$ *is monotone decreasing and bounded* ($a_n > 0$ *for all* n*), and so the limit exists.*
Let us define $l = \lim\limits_{n\longrightarrow\infty} a_n$*. Take the limit of both sides in equation* $a_{n+1} = a_n - \frac{a_n^2 - 2}{2a_n}$ *we obtain*

$$l^2 - 2 = 0 \Longleftrightarrow l = \pm\sqrt{2}.$$

As $a_n > 0$ *for all* n *we get* $l \geq 0$*, and therefore* $l = \sqrt{2}$*.*

Exercice 1.13.14. *Using the Cauchy criterion show that the sequence* $(a_n)_{n\in\mathbb{N}^*}$ *is convergent and the sequence* $(b_n)_{n\geq 2}$ *is divergent*

$$1)\ a_n = \sum_{k=1}^{n} \frac{\sin k}{2^k} \quad \text{and} \quad 2)\ b_n = \sum_{k=2}^{n} \frac{1}{\ln k}$$

Proof 1.13.14. *1) Given* p*,* q*, assume* $q < p$*.*

$$\begin{aligned} |a_p - a_q| &= \left|\sum_{k=1}^{p} \frac{\sin k}{2^k} - \sum_{k=1}^{q} \frac{\sin k}{2^k}\right| \\ &= \left|\sum_{k=q+1}^{p} \frac{\sin k}{2^k}\right| \\ &\leq \sum_{k=q+1}^{p} \left|\frac{\sin k}{2^k}\right| \\ &\leq \sum_{k=q+1}^{p} \frac{1}{2^k}. \end{aligned}$$

Observe that $\sum_{k=q+1}^{p} \frac{1}{2^k}$ *is the sum of the* $p-q$ *terms of a geometric sequence of ratio* $\frac{1}{2}$*, thus*

$$\sum_{k=q+1}^{p} \frac{1}{2^k} = \frac{1}{2^{q+1}} \left(\frac{1-\frac{1}{2^{p-q}}}{1-\frac{1}{2}} \right) = \frac{1}{2^q} \left(1 - \frac{1}{2^{p-q}} \right) \leq \frac{1}{2^q}.$$

Then

$$|a_p - a_q| \leq \frac{1}{2^q} \leq \frac{1}{M} < \varepsilon.$$

Given $\varepsilon > 0$*, choose* $M \in \mathbb{N}$ *such that* $\frac{1}{\varepsilon} < M$*. Then for all* $p,q \in \mathbb{N}$ *with* $M \leq 2^q$ *and* $q < p$ *we have* $|a_p - a_q| < \varepsilon$.

2) b_n *is not a Cauchy sequence if and only if*

$$\exists \varepsilon > 0, \forall n \in \mathbb{N}; \exists p, q \in \mathbb{N} : n \leq p, n \leq q \text{ and } |b_p - b_q| \geq \varepsilon.$$

Let $p = 2n$ *and* $q = n$ *then*

$$|b_p - b_q| = \sum_{k=n+1}^{2n} \frac{1}{\ln k} >^{\ln k < k} \sum_{k=n+1}^{2n} \frac{1}{k} \geq^{k \leq 2n} \sum_{k=n+1}^{2n} \frac{1}{2n} = \frac{1}{2}.$$

Therefore, we take $\varepsilon = \frac{1}{2}$.

Exercice 1.13.15. *Using the bounding principle of a sequence, show that the sequence* $(a_n)_{n \in \mathbb{N}^*}$ *converges to a limit* l *to be determined in each case:*

1) $a_n = \sum_{k=1}^{n} \frac{n}{n^3+k}$ 2) $a_n = \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}}$

3) $a_n = \sum_{k=1}^{n} \frac{1}{3+|\sin k|\sqrt{k}}$ 4) $a_n = \frac{\lfloor \sqrt{n} \rfloor}{n}$*, where* $\lfloor . \rfloor$ *denotes the whole part.*

Proof 1.13.15. *1) We have for any* $k = 1 : n$

$$n^3 + 1 \leq n^3 + k \leq n^3 + n \Longleftrightarrow \frac{n}{n^3+n} \leq \frac{n}{n^3+k} \leq \frac{n}{n^3+1}.$$

Thus

$$\sum_{k=1}^{n} \frac{n}{n^3+n} \leq \sum_{k=1}^{n} \frac{n}{n^3+k} \leq \sum_{k=1}^{n} \frac{n}{n^3+1} \Longleftrightarrow \frac{n^2}{n^3+n} \leq a_n \leq \frac{n^2}{n^3+1}.$$

AS

$$\lim_{n \longrightarrow \infty} \frac{n^2}{n^3+n} = \lim_{n \longrightarrow \infty} \frac{n^2}{n^3+1} = 0.$$

Then $\lim_{n\longrightarrow\infty} a_n = 0$.

2) We have for any $k = 1 : n$

$$\begin{aligned} n^2+1 \le n^2+k \le n^2+n &\iff \sqrt{n^2+1} \le \sqrt{n^2+k} \le \sqrt{n^2+n}. \\ &\iff \frac{1}{\sqrt{n^2+n}} \le \frac{1}{\sqrt{n^2+k}} \le \frac{1}{\sqrt{n^2+1}} \end{aligned}$$

Thus

$$\sum_{k=1}^{n} \frac{1}{\sqrt{n^2+n}} \le \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}} \le \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+1}} \iff \frac{n}{\sqrt{n^2+n}} \le a_n \le \frac{n}{\sqrt{n^2+1}}.$$

AS

$$\lim_{n\longrightarrow\infty} \frac{n}{\sqrt{n^2+n}} = \lim_{n\longrightarrow\infty} \frac{n}{\sqrt{n^2+1}} = 1.$$

Then $\lim_{n\longrightarrow\infty} a_n = 1$.

3) we say that for any $k = 1 : n$, $|\sin k| \le 1$, *thus*

$$\begin{aligned} 3+|\sin k|\sqrt{k} \le 3+\sqrt{k} \le 3+\sqrt{n} &\iff \frac{1}{3+\sqrt{n}} \le \frac{1}{3+|\sin k|\sqrt{k}} \\ &\iff \sum_{k=1}^{n} \frac{1}{3+\sqrt{n}} \le \sum_{k=1}^{n} \frac{1}{3+|\sin k|\sqrt{k}} \\ &\iff \frac{n}{3+\sqrt{n}} \le a_n. \end{aligned}$$

As $\lim_{n\longrightarrow\infty} \frac{n}{3+\sqrt{n}} = +\infty$. *Then* $\lim_{n\longrightarrow\infty} a_n = +\infty$.

4) we say that for any $x \in \mathbb{R}$

$$\lfloor x \rfloor \le x < \lfloor x \rfloor + 1.$$

Thus for $x = \sqrt{n}$ *we have*

$$\lfloor \sqrt{n} \rfloor \le \sqrt{n} < \lfloor \sqrt{n} \rfloor + 1.$$

Let $m = \lfloor \sqrt{n} \rfloor$, *so*

$$\begin{aligned} m \le \sqrt{n} < m+1 &\iff m^2 \le n < (m+1)^2 \\ &\iff \frac{1}{(m+1)^2} < \frac{1}{n} \le \frac{1}{m^2} \\ &\iff \frac{m}{(m+1)^2} < a_n \le \frac{1}{m}. \end{aligned}$$

AS

$$\lim_{n\longrightarrow\infty} \frac{m}{(m+1)^2} = \lim_{n\longrightarrow\infty} \frac{1}{m} = 0.$$

Then $\lim_{n\longrightarrow\infty} a_n = 0$.

Exercice 1.13.16. *Let* $a_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$, *for all* $n \in \mathbb{N}^*$

1) Prove that $\lim_{n\longrightarrow\infty} a_n = +\infty$

2) Prove that $\frac{1}{2\sqrt{n+1}} \leq \sqrt{n+1} - \sqrt{n} \leq \frac{1}{2\sqrt{n}}$

3) Deduce that $2(\sqrt{n+1} - 1) \leq a_n \leq 2\sqrt{n} - 1$

4) Let $b_n = \frac{a_n}{\sqrt{n}}$*; show that* $(b_n)_{n\in\mathbb{N}^*}$ *is convergent towards a limit to be specified.*

Proof 1.13.16. *1) We have for any* $k = 1 : n$

$$\begin{aligned} k \leq n &\iff \sqrt{k} \leq \sqrt{n} \\ &\iff \frac{1}{\sqrt{n}} \leq \frac{1}{k} \\ &\iff \sum_{k=1}^{n} \frac{1}{\sqrt{n}} \leq \sum_{k=1}^{n} \frac{1}{k} \\ &\iff \sqrt{n} \leq a_n. \end{aligned}$$

Then $\lim_{n\longrightarrow\infty} a_n = +\infty$.

2) We have for any $n \in \mathbb{N}^*$

$$\begin{aligned} \sqrt{n} \leq \sqrt{n+1} &\iff 2\sqrt{n} \leq \sqrt{n} + \sqrt{n+1} \leq 2\sqrt{n+1} \\ &\iff \frac{1}{2\sqrt{n+1}} \leq \sqrt{n+1} - \sqrt{n} \leq \frac{1}{2\sqrt{n}}. \end{aligned}$$

3) We have from 2), for any $k = 1 : n$

$$\frac{1}{2\sqrt{k+1}} \leq \sqrt{k+1} - \sqrt{k} \leq \frac{1}{2\sqrt{k}}.$$

So

$$\begin{array}{ccccc} \sum_{k=1}^{n} \frac{1}{2\sqrt{k+1}} & \leq & \sum_{k=1}^{n} (\sqrt{k+1} - \sqrt{k}) & \leq & \sum_{k=1}^{n} \frac{1}{2\sqrt{k}} \\ \shortparallel & & \shortparallel & & \shortparallel \\ \frac{1}{2}(a_{n+1} - 1) & \leq & \sqrt{n+1} - 1 & \leq & \frac{1}{2} a_n. \end{array}$$

Thus

$$a_n \geq 2(\sqrt{n+1} - 1).$$

Likewise for $k = 1 : n-1$, we have

$$\begin{array}{ccccc} \displaystyle\sum_{k=1}^{n-1} \frac{1}{2\sqrt{k+1}} & \leq & \displaystyle\sum_{k=1}^{n-1} (\sqrt{k+1} - \sqrt{k}) & \leq & \displaystyle\sum_{k=1}^{n-1} \frac{1}{2\sqrt{k}} \\ \shortparallel & & \shortparallel & & \shortparallel \\ \frac{1}{2}(a_n - 1) & \leq & \sqrt{n} - 1 & \leq & \frac{1}{2} a_{n-1}. \end{array}$$

So

$$a_n \leq 2\sqrt{n} - 1.$$

4) We have for any $n \in \mathbb{N}^*$

$$2(\sqrt{n+1} - 1) \leq a_n \leq 2\sqrt{n} - 1 \Longleftrightarrow \frac{2(\sqrt{n+1} - 1)}{\sqrt{n}} \leq b_n \leq \frac{2\sqrt{n} - 1}{\sqrt{n}}.$$

As

$$\lim_{n \longrightarrow \infty} \frac{2(\sqrt{n+1} - 1)}{\sqrt{n}} = \lim_{n \longrightarrow \infty} \frac{2\sqrt{n} - 1}{\sqrt{n}} = 2.$$

Then $\lim_{n \longrightarrow \infty} b_n = 2$.

Exercice 1.13.17. *Consider the sequence $(a_n)_{n \in \mathbb{N}}$ defined by:* $\begin{cases} a_0 = 0 \\ a_{n+1} = \sqrt{a_n + 2} \end{cases}$

1) Prove that $0 \leq a_n < 2$, for any $n \in \mathbb{N}$.

2) Deduce the monotony of $(a_n)_{n \in \mathbb{N}}$.

3) Consider the sequence $(b_n)_{n \in \mathbb{N}}$ defined by: $b_n = 2 - a_n$.

(a) What is the sign of $(b_n)_{n \in \mathbb{N}}$?

(b) Prove that for any $n \in \mathbb{N}$, $\dfrac{b_{n+1}}{b_n} \leq \dfrac{1}{2}$.

(c) Prove that for any $n \in \mathbb{N}^*$, $b_n \leq \left(\dfrac{1}{2}\right)^{n-1}$.

(d) Deduce the limit of the sequence $(b_n)_{n \in \mathbb{N}}$, then that of $(a_n)_{n \in \mathbb{N}}$.

Proof 1.13.17. *1) We prove by induction that for all $n \in \mathbb{N}$, $0 \leq a_n < 2$. Since $0 \leq a_0 < 2$, the statement is true for $n = 0$. Next, suppose $0 \leq a_n < 2$. So*

$$2 \leq a_n + 2 < 4 \Longleftrightarrow \sqrt{2} \leq a_{n+1} < 2 \Longrightarrow 0 \leq a_{n+1} < 2$$

Then $0 \leq a_n < 2$ for all $n \in \mathbb{N}$.

2) We have

$$a_{n+1} - a_n = \sqrt{a_n + 2} - a_n = \frac{a_n + 2 - a_n^2}{\sqrt{a_n + 2} + a_n} = \frac{(a_n + 1)(2 - a_n)}{\sqrt{a_n + 2} + a_n}$$

and $0 \leq a_n < 2$, *then* $(a_n+1)(2-a_n) > 0$. *Therefore* $(a_n)_{n\in\mathbb{N}}$ *is increasing.*

3)

(a) We remarque from question 1) that $a_n < 2$, *then* $b_n > 0$ *for all* $n \in \mathbb{N}$.

(b) We have

$$\frac{b_{n+1}}{b_n} = \frac{2-a_{n+1}}{2-a_n} = \frac{2-\sqrt{a_n+2}}{2-a_n} = \frac{(2-a_n)}{(2-a_n)(2+\sqrt{a_n+2})} = \frac{1}{2+\sqrt{a_n+2}}.$$

And as $0 \leq a_n$, *we can show that*

$$\frac{1}{2+\sqrt{a_n+2}} \leq \frac{1}{2}.$$

Then $\dfrac{b_{n+1}}{b_n} \leq \dfrac{1}{2}$.

(c) Again, we proceed by induction. The statement is clearly true for $n = 1$. *Suppose that* $b_n \leq \left(\frac{1}{2}\right)^{n-1}$ *for some* $n \in \mathbb{N}$. *So from question(b) we have*

$$b_{n+1} \leq \frac{1}{2} b_n \leq \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} = \left(\frac{1}{2}\right)^{n}.$$

It follows that $b_n \leq \left(\frac{1}{2}\right)^{n-1}$ *for any* $n \in \mathbb{N}^*$.

(d) As $0 < b_n \leq \left(\frac{1}{2}\right)^{n-1}$ *for any* $n \in \mathbb{N}^*$ *and* $\lim\limits_{n\longrightarrow\infty} \left(\frac{1}{2}\right)^{n-1} = 0$, *as a result* $\lim\limits_{n\longrightarrow\infty} b_n = 0$ *and* $\lim\limits_{n\longrightarrow\infty} a_n = 2$.

Exercice 1.13.18. *Let consider the sequence* $(a_n)_{n\in\mathbb{N}}$ *defined by:* $\begin{cases} a_0 = 1 \\ a_{n+1} = a_n e^{-a_n} \end{cases}$

1) Prove that $a_n > 0$, *for any* $n \in \mathbb{N}$.

2) Deduce the monotony of $(a_n)_{n\in\mathbb{N}}$.

3) Deduce that $(a_n)_{n\in\mathbb{N}}$ *is convergent then calculate its limit.*

4) Let $b_n = \sum\limits_{k=1}^{n} a_k$, *prove that* $a_{n+1} = e^{-b_n}$, *for any* $n \in \mathbb{N}$.

5) Deduce that $\lim\limits_{n\longrightarrow\infty} b_n = +\infty$.

Proof 1.13.18. *1) By induction, the statement is clearly true for* $n = 0$. *Suppose that* $a_n > 0$. *So* $a_{n+1} = a_n e^{-a_n} > 0$. *Then* $a_n > 0$, *for any* $n \in \mathbb{N}$.

2) We have

$$a_{n+1} - a_n = a_n(e^{-a_n} - 1).$$

As

$$a_n > 0 \Longleftrightarrow -a_n < 0 \Longleftrightarrow e^{-a_n} - 1 < 0$$

and consequently

$$a_{n+1} - a_n < 0.$$

Then $(a_n)_{n\in\mathbb{N}}$ *is decreasing.*

3) $(a_n)_{n\in\mathbb{N}}$ *is a lower bound and decreasing sequence then it is convergent. Let us define* $l = \lim\limits_{n\longrightarrow\infty} a_n$*, take the limit of both sides in equation* $a_{n+1} = a_n e^{-a_n}$ *we obtain*

$$l = le^{-l} \Longleftrightarrow l(e^{-l} - 1) = 0 \Longleftrightarrow l = 0$$

Then $\lim\limits_{n\longrightarrow\infty} a_n = 0$.

4) Again by induction, for $n = 0$ *we have* $a_1 = e^{-b_0}$*. Suppose that* $a_{n+1} = e^{-b_n}$*. So*

$$a_{n+2} = a_{n+1}e^{-a_{n+1}} = e^{-b_n}e^{-a_{n+1}} = e^{-b_{n+1}}.$$

Then $a_{n+1} = e^{-b_n}$*, for any* $n \in \mathbb{N}$.

5) We have $b_n = -\ln a_{n+1}$ *and* $\lim\limits_{n\longrightarrow\infty} a_{n+1} = 0$*. Then* $\lim\limits_{n\longrightarrow\infty} b_n = +\infty$.

Exercice 1.13.19. *We define the two real sequences* $(a_n)_{n\in\mathbb{N}^*}$ *and* $(b_n)_{n\in\mathbb{N}^*}$ *by:*

$$\begin{cases} a_1 = 1 \\ a_{n+1} = \dfrac{a_n + 2b_n}{3} \end{cases} \quad \text{and} \quad \begin{cases} b_1 = 12 \\ b_{n+1} = \dfrac{a_n + 3b_n}{4} \end{cases}$$

1) We pose $\forall n \in \mathbb{N}^*$, $c_n = b_n - a_n$*. Express the sequence* $(c_n)_{n\in\mathbb{N}^*}$ *as a function of* n *then calculate its limit.*

2) Show that the sequences $(a_n)_{n\in\mathbb{N}^*}$ *and* $(b_n)_{n\in\mathbb{N}^*}$ *are adjacent.*

Proof 1.13.19. *1) We have*

$$c_{n+1} = b_{n+1} - a_{n+1} = \frac{a_n + 3b_n}{4} - \frac{a_n + 2b_n}{3} = \frac{1}{12}c_n.$$

We deduce that the sequence $(c_n)_{n\in\mathbb{N}^*}$ *is a geometric sequence with ratio* $q = \dfrac{1}{12}$*, hence*

$$c_n = c_1\left(\frac{1}{12}\right)^{n-1} = \frac{11}{(12)^{n-1}}$$

and consequently $\lim_{n \longrightarrow \infty} c_n = 0$.

2) We have

$$a_{n+1} - a_n = \frac{a_n + 2b_n}{3} - a_n = \frac{2}{3}(b_n - a_n) = \frac{2}{3}c_n > 0$$

hence $(a_n)_{n\in\mathbb{N}^*}$ *is a increasing sequence.*

Likewise, We have

$$b_{n+1} - b_n = \frac{a_n + 3b_n}{4} - b_n = \frac{-1}{4}(b_n - a_n) = \frac{-1}{4}c_n < 0$$

hence $(b_n)_{n\in\mathbb{N}^*}$ *is a decreasing sequence.*

Moreover $\lim_{n \longrightarrow \infty}(b_n - a_n) = \lim_{n \longrightarrow \infty} c_n = 0$. *Therefore ;* $(a_n)_{n\in\mathbb{N}^*}$ *and* $(b_n)_{n\in\mathbb{N}^*}$ *are adjacent sequences.*

2

Numerical series with positive terms

2.1 Basic information on numerical series

Definition 2.1.1. *Given a numerical sequence* $(U_n)_n$*, we define the sequence* $(S_n)_n$ *of partial sums by*

$$S_n = \sum_{k=0}^{k=n} U_k.$$

The series of general term U_n *and of partial sum* S_n *denoted by the symbol*

$$\sum_{n=0}^{n=\infty} U_n, \ \sum_{n\geq O} U_n \ orjust \ \sum U_n$$

is by definition the value (if this one exists) of the limit $s = \lim_{n\to\infty} S_n$.

If s *exists we say the series is convergent. We write* $\sum_{n=0}^{n=\infty} U_n = s$.

If $s = +\infty$ *or* s *does not exist we say that the series is divergent.*

Example 2.1.1. *The series*

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)^2 - 1}$$

converges and the limit is $\frac{3}{4}$*. in fact,*

$$\frac{1}{(n+1)^2 - 1} = \frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}.$$

Then $A = \frac{1}{2}$ *and* $B = -\frac{1}{2}$*. We get* $U_n = \frac{1}{2}\left(\frac{1}{n} + \frac{1}{n+2}\right)$*. So*

$$\begin{aligned} S_n &= \sum_{k=1}^{k=n} U_k \\ &= \frac{1}{2}\left(\frac{1}{1} + \frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{2} + \frac{1}{4}\right) + \cdots + \frac{1}{2}\left(\frac{1}{n} + \frac{1}{n+2}\right) \\ &= \frac{1}{2}\left(\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right). \end{aligned}$$

Then $\lim_{n\to\infty} S_n = \frac{3}{4}$

Example 2.1.2. *The series of the general term* $U_n = \frac{1}{2^n}$ *converges because its partial sum is written as follows*

$$\begin{aligned} S_n &= \sum_{k=0}^{k=n} U_k \\ &= 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} \\ &= \frac{1 - \frac{1}{2^{n+1}}}{1 - \frac{1}{2}} \\ &= 2 - \frac{1}{2^n} \end{aligned}$$

It is clear that $\lim_{n\to\infty} S_n = 2$

Example 2.1.3. *The series of the general term* $U_n = n$ *diverges because its partial sum is written as follows*

$$\begin{aligned} S_n &= \sum_{k=1}^{k=n} U_k \\ &= 1 + 2 + 3 + \cdots + n \\ &= \frac{n(n+1)}{2} \end{aligned}$$

It is clear that $\lim_{n\to\infty} S_n = \infty$

2.2 Operations on series

Properties 2.2.1. *If* $\sum a_n$ *and* $\sum b_n$ *are both convergent series then,*

1. $\sum ca_n$ *where* c *is any number, is also convergent and*

$$\sum ca_n = c\sum a_n.$$

2. $\sum a_n \pm \sum b_n$ *is also convergent and*

$$\sum a_n \pm \sum b_n = \sum (a_n \pm b_n).$$

Corollary 2.2.1. *let* $\sum a_n$ *and* $\sum b_n$ *be two series.*

1. *If* $\sum a_n$ *converges and* $\sum b_n$ *diverges, then* $\sum (a_n + b_n)$ *diverges.*
2. *If* $\sum a_n$ *and* $\sum b_n$ *diverge, we can say nothing about* $\sum (a_n + b_n)$*.*

Theorem 2.2.1. *If* $\sum U_n$ *converges then* $\lim_{n\to\infty} U_n = 0$*.*

Example 2.2.1. *1) The series of the general term* $U_n = \dfrac{1}{n^2-1}$ *converges then* $\lim_{n\to\infty} \dfrac{1}{n^2-1} = 0$*.*
2) The series of the general term $U_n = \dfrac{1}{3^{n-1}}$ *converges then* $\lim_{n\to\infty} \dfrac{1}{3^{n-1}} = 0$*.*

Corollary 2.2.2. *If* $\lim_{n\to\infty} U_n \neq 0$ *then* $\sum U_n$ *will diverge.*

Example 2.2.2. *1) The following series is divergent.*

$$\sum_{n=0}^{\infty} \frac{4n^2 - n^3}{11 + 2n^3}$$

because $\lim_{n\to\infty} \dfrac{4n^2 - n^3}{11 + 2n^3} = -\dfrac{1}{2} \neq 0$*.*
2) The series of the general term $U_n = (-1)^n$ *is divergent because* $\lim_{n\to\infty} (-1)^n$ *doesn't exist.*
3) The series of the general term $U_n = n^{\frac{1}{n}}$ *is divergent because*

$$\lim_{n\to\infty} n^{\frac{1}{n}} = \lim_{n\to\infty} e^{\ln(n^{\frac{1}{n}})} = \lim_{n\to\infty} e^{\frac{\ln n}{n}} = e^0 = 1 \neq 0.$$

Remark 2.2.1. *In the theorem 2.2.1, the fact that the sequence $(U_n)_n$ tends towards 0 is a necessary condition for the series to converge but it is not sufficient.*

Example 2.2.3. *We consider the series of general term $U_n = \ln\left(1+\frac{1}{n}\right)$.*
We observe that $\lim_{n\to\infty} U_n = 0$. On the other hand, we can easily verify that the sequence of partial sums $(S_n)_n$ diverges.
Indeed,

$$\begin{aligned} S_n &= \sum_{k=1}^{k=n} \ln\left(1+\frac{1}{k}\right) \\ &= \sum_{k=1}^{k=n} \ln\left(\frac{k+1}{k}\right) \\ &= \ln\left(\prod_{k=1}^{k=n}\left(\frac{k+1}{k}\right)\right) \\ &= \ln\frac{2\times 3\times\cdots\times(n+1)}{1\times 2\times\cdots\times n} \\ &= \ln(n+1) \longrightarrow \infty. \end{aligned}$$

Therefore, the series $\sum \ln\left(1+\frac{1}{n}\right)$ diverges.

Theorem 2.2.2. *(Cauchy series). The series $\sum_{n=0}^{n=\infty} U_n$ converges if and only if*

$$\forall\epsilon>0, \exists N\in\mathbb{N}, \forall p\geq q\geq N, \left|\sum_{k=q}^{k=p} U_k\right| \leq \epsilon. \tag{2.2.1}$$

Theorem 2.2.3. *(The boundedness test). Let $\sum_{n=0}^{n=\infty} U_n$ be a series with nonnegative terms. Then the series is convergent if and only if the partial sums sequence $(S_n)_n$ is bounded.*

2.3 Convergence of positive-term series

2.3.1 Geometric series

Theorem 2.3.1. *(Geometric series). A Geometric series is any series that can be written in the form,*

$$\sum_{n=1}^{n=\infty} ar^{n-1}.$$

We can show that the partial sums of these series are,

$$S_n = \frac{a}{1-r} - \frac{ar^n}{1-r}.$$

So let's take the limit of the partial sums.

$$\lim_{n\to\infty} S_n = \frac{a}{1-r} - \frac{a}{1-r}\lim_{n\to\infty} r^n.$$

Therefore, a Geometric series will converge if $|r| < 1$ *and we have*

$$\sum_{n=1}^{n=\infty} ar^{n-1} = \frac{a}{1-r}$$

Example 2.3.1. *The series* $\sum_{n=1}^{n=\infty} 9^{-n+2}4^{n+1}$ *converges.*
Let's first start by rewriting the series

$$\begin{aligned}\sum_{n=1}^{n=\infty} 9^{-n+2}4^{n+1} &= \sum_{n=1}^{n=\infty} 9^{-(n-2)}4^{n+1}\\ &= \sum_{n=1}^{n=\infty} \frac{4^{n+1}}{9^{n-2}}\\ &= \sum_{n=1}^{n=\infty} \frac{4^{n-1}4^2}{9^{n-1}9^{-1}}\\ &= \sum_{n=1}^{n=\infty} 144\left(\frac{4}{9}\right)^{n-1}\end{aligned}$$

So, this is a Geometric series with $a = 144$ *and* $r = \frac{4}{9} < 1$. *Therefore, it will converges and its value will be,*

$$\sum_{n=1}^{n=\infty} 9^{-n+2}4^{n+1} = \frac{1296}{5}.$$

2.3.2 Comparison test

Theorem 2.3.2. *(Comparison test). Let* $\sum_{n=0}^{n=\infty} U_n$ *and* $\sum_{n=0}^{n=\infty} V_n$ *be series such that* $0 \le U_n \le V_n$ *for all* $n \in \mathbb{N}$.

1. *If* $\sum_{n=0}^{n=\infty} V_n$ *converges, then so does* $\sum_{n=0}^{n=\infty} U_n$.
2. *If* $\sum_{n=0}^{n=\infty} U_n$ *diverges, then so does* $\sum_{n=0}^{n=\infty} V_n$.

Properties 2.3.1. *(p-series or the p-test). For $p \in \mathbb{R}$, the series*

$$\sum_{n=1}^{n=\infty} \frac{1}{n^p}$$

converges if and only if $p > 1$.

Example 2.3.2. *The series* $\sum_{n=0}^{n=\infty} \frac{1}{n^2+1}$ *converges. First we have,*

$$\frac{1}{n^2+1} < \frac{1}{n^2}, \quad \forall n \in \mathbb{N}.$$

The series $\sum_{n=1}^{n=\infty} \frac{1}{n^2}$ *converges by the p-series test. Therefore, by the Comparison test,* $\sum_{n=0}^{n=\infty} \frac{1}{n^2+1}$ *converges.*

Example 2.3.3. *The series* $\sum_{n=1}^{n=\infty} \left(\sqrt{1+n^4} - \sqrt{n^4-1}\right)$ *converges. First we have,*

$$\sqrt{1+n^4} - \sqrt{n^4-1} = \frac{1}{\sqrt{1+n^4} + \sqrt{n^4-1}}$$

and

$$\frac{1}{\sqrt{1+n^4} + \sqrt{n^4-1}} < \frac{1}{n^2}, \quad \forall n \in \mathbb{N}.$$

The series $\sum_{n=1}^{n=\infty} \frac{1}{n^2}$ *converges by the p-series test. Therefore, by the Comparison test* $\sum_{n=1}^{n=\infty} \left(\sqrt{1+n^4} - \sqrt{n^4-1}\right)$ *converges.*

2.3.3 Cauchy condensation test

Theorem 2.3.3. *(Cauchy condensation test). Suppose $U_1 \geq U_2 \geq \cdots \geq 0$. Then the series* $\sum_{n=1}^{n=\infty} U_n$ *is convergent if and only if* $\sum_{k=0}^{n=\infty} 2^k U_{2^k} = U_1 + 2U_2 + 4U_4 + 8U_8 + \cdots$ *is convergent.*

Example 2.3.4. *Let $p > 1$ be constant and consider the p-series* $\sum_{n=1}^{n=\infty} \frac{1}{n^p}$ *. We have $U_n = \frac{1}{n^p}$ and $U_1 \geq U_2 \geq \cdots \geq 0$. Then*

$$\begin{aligned} \sum_{k=0}^{k=\infty} 2^k U_{2^k} &= \sum_{k=0}^{n=\infty} 2^k \frac{1}{(2^k)^p} \\ &= \sum_{k=0}^{n=\infty} 2^{k-kp} \\ &= \sum_{k=0}^{n=\infty} (2^{1-p})^k \end{aligned}$$

is Geometric series. It is convergent because $p > 1$*, i.e.* $2^{1-p} < 1$.
Therefore $\sum_{n=1}^{n=\infty} \frac{1}{n^p}$ *is convergent.*

2.3.4 Equivalence test

Theorem 2.3.4. *(Equivalence test). Suppose that we have two series* $\sum_{n=0}^{n=\infty} U_n$ *and* $\sum_{n=0}^{n=\infty} V_n$ *with* $U_n \geq 0,\ V_n > 0$ *for all* n*. Define,*

$$c = \lim_{n\to\infty} \frac{U_n}{V_n}$$

If c is positive and is finite then either both series converge or both series diverge.

Example 2.3.5. *the following series converges* $\sum_{n=0}^{n=\infty} \frac{1}{3^n - n}$.

Let use $\sum_{n=0}^{n=\infty} \frac{1}{3^n}$ *as the second series. We know that this series converges since it's Geometric series.*
let's calculate the limit

$$\begin{aligned} c &= \lim_{n\to\infty} \frac{U_n}{V_n} \\ &= \lim_{n\to\infty} \frac{1}{3^n} \frac{3^n - n}{1} \\ &= \lim_{n\to\infty} \left(1 - \frac{n}{3^n}\right) \\ &= 1 - \lim_{n\to\infty} \frac{n}{3^n} \\ &= 1 \end{aligned}$$

So, c is positive and finite so by the Equivalence test the series $\sum_{n=0}^{n=\infty} \frac{1}{3^n - n}$ *converges.*

2.3.5 Ratio test or d'Alembert test

Theorem 2.3.5. *(Ratio test or d'Alembert test). Suppose we have the series* $\sum_{n=0}^{n=\infty} U_n$*. Define,*

$$l = \lim_{n\to\infty} \frac{U_{n+1}}{U_n}.$$

Then,

1. *If* $l < 1$ *the series is convergent.*

2. *If $l > 1$ the series is divergent.*

3. *If $l = 1$ the series may be divergent, may be convergent.*

Example 2.3.6. *the following series converges* $\sum_{n=1}^{n=\infty} \frac{10^n}{4^{2n+1}(n+1)}$.

We have

$$U_n = \frac{10^n}{4^{2n+1}(n+1)}$$

and

$$U_{n+1} = \frac{10^{n+1}}{4^{2n+3}(n+2)}.$$

Now, let calculate l,

$$\begin{aligned} l &= \lim_{n\to\infty} \frac{U_{n+1}}{U_n} \\ &= \lim_{n\to\infty} \frac{10^{n+1}}{4^{2n+3}(n+2)} \frac{4^{2n+1}(n+1)}{10^n} \\ &= \lim_{n\to\infty} \frac{10(n+1)}{16(n+2)} \\ &= \frac{10}{16} \lim_{n\to\infty} \frac{n+1}{n+2} \\ &= \frac{10}{16} < 1. \end{aligned}$$

So, $l < 1$ and so by the Ratio test the series converges.

Example 2.3.7. *the following series is divergent* $\sum_{n=0}^{n=\infty} \frac{n!}{5^n}$.

We have

$$U_n = \frac{n!}{5^n}$$

and

$$U_{n+1} = \frac{(n+1)!}{5^{n+1}}.$$

Now, let calculate l,

$$\begin{aligned} l &= \lim_{n\to\infty} \frac{U_{n+1}}{U_n} \\ &= \lim_{n\to\infty} \frac{(n+1)!}{5^{n+1}} \frac{5^n}{n!} \\ &= \lim_{n\to\infty} \frac{n+1}{5} \\ &= \frac{1}{5} \lim_{n\to\infty} (n+1) \\ &= \infty > 1. \end{aligned}$$

So, by the Ratio test this series diverges.

2.3.6 The Raabe-Duhamel test

Theorem 2.3.6. *(The Raabe-Duhamel test). Let* $\sum_{n=0}^{n=\infty} U_n$ *be a series with positive terms. Define,*

$$l = \lim_{n\to\infty} n\left(\frac{U_n}{U_{n+1}} - 1\right) \in \mathbb{R}_+ \cup \{\infty\}.$$

Then,

1. *If* $l > 1$ *the series is convergent.*
2. *If* $l < 1$ *the series is divergent.*
3. *If* $l = 1$ *we cannot decide on the nature of this series.*

Example 2.3.8. *Let us find the nature of the series*

$$\sum_{n=1}^{n=\infty} \frac{1.3.5.\cdots.(2n+1)}{2.4.6.\cdots.2n}\frac{1}{2n+3}.$$

Since

$$\lim_{n\to\infty}\frac{U_{n+1}}{U_n} = \lim_{n\to\infty}\frac{(2n+3)^2}{(2n+2)(2n+5)} = 1,$$

let us apply Raabe-Duhamel test. Since

$$\lim_{n\to\infty} n\left(\frac{U_n}{U_{n+1}} - 1\right) = \lim_{n\to\infty}\frac{2n^2+n}{(2n+3)^2} = \frac{1}{2} < 1,$$

then the series is divergent.

2.3.7 The Cauchy root test

Theorem 2.3.7. *(The Cauchy root test). Let* $\sum_{n=0}^{n=\infty} U_n$ *be a series with positive terms. Define,*

$$l = \lim_{n\to\infty}\sqrt[n]{U_n} \in \mathbb{R}_+ \cup \{\infty\}.$$

Then,

1. *If* $l < 1$ *the series is convergent.*

2. *If $l > 1$ the series is divergent.*

3. *If $l = 1$ No information.*

Example 2.3.9. *Let us find the nature of the series*

$$\sum_{n=1}^{n=\infty} \frac{n^n}{3^{1+2n}}.$$

Now, let calculate l,

$$\begin{aligned} l &= \lim_{n\to\infty} \sqrt[n]{U_n} \\ &= \lim_{n\to\infty} \sqrt[n]{\frac{n^n}{3^{1+2n}}} \\ &= \lim_{n\to\infty} \frac{n}{3^{\frac{1}{n}+2}} \\ &= \frac{\infty}{3^2} \\ &= \infty > 1. \end{aligned}$$

So, by the Root test this series is divergent.

Example 2.3.10. *Let us find the nature of the series*

$$\sum_{n=0}^{n=\infty} \left(\frac{5n+3n^3}{7n^3+2}\right)^n.$$

Now, let calculate l,

$$\begin{aligned} l &= \lim_{n\to\infty} \sqrt[n]{U_n} \\ &= \lim_{n\to\infty} \sqrt[n]{\left(\frac{5n+3n^3}{7n^3+2}\right)^n} \\ &= \lim_{n\to\infty} \frac{5n+3n^3}{7n^3+2} \\ &= \frac{3}{7} < 1. \end{aligned}$$

So, by the Root test this series is convergent.

2.3.8 Integral test

Theorem 2.3.8. *(Integral test). Suppose that $f(x)$ is a continuous, positive and decreasing function on the interval $[k, \infty)$ and that $f(n) = U_n$ then,*

1. *If* $\int_k^{\infty} f(x)dx$ *is convergent so is* $\sum_{n=k}^{n=\infty} U_n$.

2. *If* $\int_k^{\infty} f(x)dx$ *is divergent so is* $\sum_{n=k}^{n=\infty} U_n$.

Example 2.3.11. *Let us find the nature of the series*

$$\sum_{n=2}^{n=\infty} \frac{1}{n \ln n}.$$

In this case the function we'll use is,

$$f(x) = \frac{1}{x \ln x}$$

This function is clearly positive and decreasing. Therefore, all we need to do is determine the convergence of the following integral.

$$\begin{aligned} \int_2^{\infty} \frac{1}{x \ln x} dx &= \lim_{y \to \infty} \int_2^{y} \frac{1}{x \ln x} dx \\ &= \lim_{y \to \infty} \ln(\ln(x)) \Big|_2^y \\ &= \lim_{y \to \infty} (\ln(\ln(y)) - \ln(\ln 2)) \\ &= \infty. \end{aligned}$$

The integral is divergent and so the series is also divergent by the Integral test.

2.4 Exercises

Exercice 2.4.1. *Using Root test to determine if the following series converges or diverges.*

$$1)\ \sum_{n=1}^{n=\infty}\left(\frac{3n+1}{4-2n}\right)^{2n} \qquad 2)\ \sum_{n=0}^{n=\infty}\frac{n^{1-3n}}{4^{2n}} \qquad 3)\ \sum_{n=4}^{n=\infty}\frac{(-5)^{1+2n}}{2^{5n-3}}$$

Proof 2.4.1. 1) *We'll need to compute l.*

$$\begin{aligned} l &= \lim_{n\to\infty}\sqrt[n]{\left(\frac{3n+1}{4-2n}\right)^{2n}} \\ &= \lim_{n\to\infty}\left(\frac{3n+1}{4-2n}\right)^{2} \\ &= \frac{9}{4} > 1, \end{aligned}$$

so by the Root test the series diverges.

2) *Let calculate l*

$$\begin{aligned} l &= \lim_{n\to\infty}\sqrt[n]{\frac{n^{1-3n}}{4^{2n}}} \\ &= \lim_{n\to\infty}\frac{n^{\frac{1}{n}-3}}{4^2} \\ &= \lim_{n\to\infty}\frac{n^{\frac{1}{n}}n^{-3}}{4^2} \\ &= \frac{(1)(0)}{16} = 0 < 1, \end{aligned}$$

so by the Root test the series converges.

3) *We have*

$$\begin{aligned} l &= \lim_{n\to\infty}\sqrt[n]{\frac{(-5)^{1+2n}}{2^{5n-3}}} \\ &= \lim_{n\to\infty}\frac{(-5)^{\frac{1}{n}+2}}{2^{5-\frac{3}{n}}} \\ &= \frac{25}{32} < 1, \end{aligned}$$

so by the Root test the series converges.

Exercice 2.4.2. *Using Ratio test to determine if the following series converges or diverges.*

$$1)\ \sum_{n=1}^{n=\infty}\frac{3^{1-2n}}{n^2+1} \qquad 2)\ \sum_{n=0}^{n=\infty}\frac{(2n)!}{5n+1} \qquad 3)\ \sum_{n=2}^{n=\infty}\frac{2^{1+3n}(n+1)}{n^2 5^{1+n}} \qquad 4)\ \sum_{n=3}^{n=\infty}\frac{e^{4n}}{(n-2)!}$$

Proof 2.4.2. 1) *We'll need to compute l.*

$$\begin{aligned} l &= \lim_{n\to\infty} \frac{U_{n+1}}{U_n} \\ &= \lim_{n\to\infty} \frac{3^{1-2(n+1)}}{(n+1)^2+1} \frac{n^2+1}{3^{1-2n}} \\ &= \lim_{n\to\infty} \frac{3^{-1-2n}}{(n+1)^2+1} \frac{n^2+1}{3^{1-2n}} \\ &= \lim_{n\to\infty} \frac{n^2+1}{9[(n+1)^2+1]} \\ &= \frac{1}{9} < 1, \end{aligned}$$

so by the Ratio test the series converges.

2) *Let calculate l*

$$\begin{aligned} l &= \lim_{n\to\infty} \frac{U_{n+1}}{U_n} \\ &= \lim_{n\to\infty} \frac{(2(n+1))!}{5(n+1)+1} \frac{5n+1}{(2n)!} \\ &= \lim_{n\to\infty} \frac{(2n+2)!}{5n+6} \frac{5n+1}{(2n)!} \\ &= \lim_{n\to\infty} \frac{(2n+2)(2n+1)(5n+1)}{5n+6} \\ &= \infty > 1, \end{aligned}$$

so by the Ratio test the series diverges.

3) *We have*

$$\begin{aligned} l &= \lim_{n\to\infty} \frac{U_{n+1}}{U_n} \\ &= \lim_{n\to\infty} \frac{2^{1+3(n+1)}((n+1)+1)}{(n+1)^2 5^{1+(n+1)}} \frac{n^2 5^{1+n}}{2^{1+3n}(n+1)} \\ &= \lim_{n\to\infty} \frac{2^{4+3n}(n+2)}{(n+1)^2 5^{2+n}} \frac{n^2 5^{1+n}}{2^{1+3n}(n+1)} \\ &= \lim_{n\to\infty} \frac{8}{5} \frac{n+2}{(n+1)^2} \frac{n^2}{n+1} \\ &= \frac{8}{5} > 1, \end{aligned}$$

so by the Ratio test the series diverges.

4) *We'll need to compute l.*

$$\begin{aligned} l &= \lim_{n\to\infty} \frac{U_{n+1}}{U_n} \\ &= \lim_{n\to\infty} \frac{e^{4(n+1)}}{((n+1)-2)!} \frac{(n-2)!}{e^{4n}} \\ &= \lim_{n\to\infty} \frac{e^{4n+4}}{(n-1)!} \frac{(n-2)!}{e^{4n}} \\ &= \lim_{n\to\infty} \frac{e^4}{n-1} \\ &= 0 < 1, \end{aligned}$$

so by the Ratio test the series converges.

Exercice 2.4.3. *Use the Comparison test to determine the nature of the following series.*

$$1)\ \sum_{n=3}^{n=\infty} \frac{e^{-n}}{n^2+2n} \qquad 2)\ \sum_{n=0}^{n=\infty} \frac{2^n \sin^2(5n)}{4^n+\cos^2(n)} \qquad 3)\ \sum_{n=1}^{n=\infty} \frac{2n^3+7}{n^4 \sin^2(n)} \qquad 4)\ \sum_{n=2}^{n=\infty} \frac{n-1}{\sqrt{n^6+1}}$$

Proof 2.4.3. 1) *We have*

$$e^{-n} < e^{-3} < 1.$$

Using this we can get the following relationship,

$$\frac{e^{-n}}{n^2+2n} < \frac{1}{n^2+2n}.$$

On the other hand we have,

$$n^2+2n > n^2.$$

Then we can get the following relationship,

$$\frac{e^{-n}}{n^2+2n} < \frac{1}{n^2+2n} < \frac{1}{n^2}.$$

Now, the series $\displaystyle\sum_{n=3}^{n=\infty} \frac{1}{n^2}$ *converges by the p-series test. Therefore, by the Comparison test,* $\displaystyle\sum_{n=3}^{n=\infty} \frac{e^{-n}}{n^2+2n}$ *converges.*

2) *We have*

$$2^n \sin^2(5n) < 2^n(1) < 2^n$$

and

$$4^n + \cos^2(n) > 4^n + 0 = 4^n$$

Then we get the following relationship,

$$\frac{2^n \sin^2(5n)}{4^n + \cos^2(n)} < \frac{2^n}{4^n} = \left(\frac{1}{2}\right)^n.$$

We also know that the series $\sum_{n=0}^{n=\infty} \left(\frac{1}{2}\right)^n$ *will converge because it is a Geometric series. Therefore, by the Comparison test,* $\sum_{n=0}^{n=\infty} \frac{2^n \sin^2(5n)}{4^n + \cos^2(n)}$ *converges.*

3) *We have*

$$2n^3 < 2n^3 + 7$$

and

$$n^4 \sin^2(n) < n^4$$

Then we get the following relationship,

$$\frac{2n^3 + 7}{n^4 \sin^2(n)} > \frac{2n^3}{n^4} = \frac{2}{n}$$

We also know that the series $\sum_{n=1}^{n=\infty} \frac{2}{n}$ *diverge by the p-series test. Therefore, by the Comparison test,* $\sum_{n=1}^{n=\infty} \frac{2n^3 + 7}{n^4 \sin^2(n)}$ *diverge.*

4) *We have*

$$n > n - 1$$

and

$$n^6 < n^6 + 1$$

Then we get the following relationship,

$$\frac{n-1}{\sqrt{n^6+1}} < \frac{n}{\sqrt{n^6}} = \frac{1}{n^2}$$

We also know that the series $\sum_{n=2}^{n=\infty} \frac{1}{n^2}$ *converges by the p-series test. Therefore, by the Comparison test,* $\sum_{n=2}^{n=\infty} \frac{n-1}{\sqrt{n^6+1}}$ *converges.*

Exercice 2.4.4. *Use the Equivalence test to determine the nature of the following series.*

$$1) \sum_{n=7}^{n=\infty} \frac{4}{n^2 - 2n - 3} \qquad 2) \sum_{n=1}^{n=\infty} \frac{4n^2 - n}{n^3 + 9} \qquad 3) \sum_{n=1}^{n=\infty} \frac{\sqrt{2n^2 + 4n + 1}}{n^3 + 9}$$

Proof 2.4.4. 1) *We have*

$$n^2 > n^2 - 2n - 3.$$

Using this we can get the following relationship,

$$\frac{4}{n^2} < \frac{4}{n^2 - 2n - 3}.$$

Let use $\sum_{n=7}^{n=\infty} \frac{4}{n^2}$ *as the second series. So, let's compute the limit,*

$$\begin{aligned} l &= \lim_{n\to\infty} \frac{U_n}{V_n} \\ &= \lim_{n\to\infty} \frac{4}{n^2 - 2n - 3} \frac{n^2}{4} \\ &= \lim_{n\to\infty} \frac{n^2}{n^2 - 2n - 3} \\ &= 1 > 0, \end{aligned}$$

Now, the series $\sum_{n=7}^{n=\infty} \frac{4}{n^2}$ *converges by the p-series test. Therefore, by the Equivalence test,* $\sum_{n=7}^{n=\infty} \frac{4}{n^2 - 2n - 3}$ *converges.*

2) *We have the following inequalities,*

$$4n^2 - n < 4n^2$$

and

$$n^3 + 9 > n^3$$

Then we get the following relationship,

$$\frac{4n^2 - n}{n^3 + 9} < \frac{4n^2}{n^3} = \frac{4}{n}.$$

Let use $\sum_{n=1}^{n=\infty} \frac{4}{n}$ *as the second series. So, let's compute the limit,*

$$\begin{aligned} l &= \lim_{n\to\infty} \frac{U_n}{V_n} \\ &= \lim_{n\to\infty} \frac{4n^2 - n}{n^3 + 9} \frac{n}{4} \\ &= \lim_{n\to\infty} \frac{4n^3 - n^2}{4n^3 + 36} \\ &= 1 > 0, \end{aligned}$$

Now, the series $\sum_{n=1}^{n=\infty} \frac{4}{n}$ *diverge by the p-series test. Therefore, by the Equivalence test,* $\sum_{n=1}^{n=\infty} \frac{4n^2-n}{n^3+9}$ *diverge.*

3) *We have the following inequalities,*

$$2n^2 < 2n^2 + 4n + 1$$

and

$$n^3 < n^3 + 9$$

Let use $\sum_{n=1}^{n=\infty} \frac{\sqrt{2}}{n^2}$ *as the second series. So, let's compute the limit,*

$$\begin{aligned} l &= \lim_{n\to\infty} \frac{U_n}{V_n} \\ &= \lim_{n\to\infty} \frac{\sqrt{2n^2+4n+1}}{n^3+9} \frac{n^2}{\sqrt{2}} \\ &= \lim_{n\to\infty} \frac{n^2\sqrt{n^2\left(2+\frac{4}{n}+\frac{1}{n^2}\right)}}{\sqrt{2}n^3\left(1+\frac{9}{n^3}\right)} \\ &= \frac{\sqrt{2}}{\sqrt{2}} = 1 > 0, \end{aligned}$$

Now, the series $\sum_{n=1}^{n=\infty} \frac{\sqrt{2}}{n^2}$ *converge by the p-series test. Therefore, by the Equivalence test,* $\sum_{n=1}^{n=\infty} \frac{\sqrt{2n^2+4n+1}}{n^3+9}$ *converge.*

Exercice 2.4.5. *Use the Integral test to determine the nature of the following series.*

$$1)\ \sum_{n=3}^{n=\infty} \frac{3}{n^2-3n+2} \qquad 2)\ \sum_{n=0}^{n=\infty} \frac{n^2}{n^3+1} \qquad 3)\ \sum_{n=2}^{n=\infty} \frac{1}{(2n+7)^3}$$

Proof 2.4.5. 1) *Let*

$$f(x) = \frac{3}{x^2-3x+2}.$$

We have for $x \geq 3$

$$x^2 - 3x + 2 \geq 0 \quad \textit{and} \quad f'(x) = \frac{9-6x}{(x^2-3x+2)^2}.$$

Then $f(x)$ *is a continuous, positive and decreasing function on the interval* $[3, \infty[$.

Now, let's compute the integral

$$\int_3^\infty \frac{3}{x^2 - 3x + 2} dx.$$

It is easy to check that

$$\frac{3}{x^2 - 3x + 2} = \frac{-3}{x-1} + \frac{3}{x-2}.$$

So

$$\begin{aligned}
\int_3^\infty \frac{3}{x^2 - 3x + 2} dx &= \int_3^\infty \left(\frac{-3}{x-1} + \frac{3}{x-2} \right) dx \\
&= \lim_{y \to \infty} \int_3^y \left(\frac{-3}{x-1} + \frac{3}{x-2} \right) dx \\
&= \lim_{y \to \infty} (3 \ln |x-2| - 3 \ln |x-1|) \Big|_3^t \\
&= \lim_{y \to \infty} \left[3 \ln \left| \frac{t-2}{t-1} \right| + 3 \ln 2 \right] \\
&= 3 \ln 2.
\end{aligned}$$

Therefore the integral converge and so by the Integral test the series converge.

2) *Let*

$$f(x) = \frac{x^2}{x^3 + 1}.$$

We have for $x \geq 0$

$$\frac{x^2}{x^3+1} \geq 0 \quad \textit{and} \quad f'(x) = \frac{x(2 - x^3)}{(x^3 + 1)^2}.$$

Then $f(x)$ *is a continuous, positive and decreasing function on the interval* $[\sqrt[3]{2}, \infty[$.

Now, let's compute the integral

$$\int_0^\infty \frac{x^2}{x^3 + 1} dx.$$

So

$$\begin{aligned}
\int_0^\infty \frac{x^2}{x^3+1} dx &= \lim_{y \to \infty} \int_0^y \frac{x^2}{x^3+1} dx \\
&= \lim_{y \to \infty} \left(\frac{1}{3} \ln |x^3 + 1| \right) \Big|_0^t \\
&= \frac{1}{3} \lim_{y \to \infty} \ln |y^3 + 1| \\
&= \infty
\end{aligned}$$

Therefore the integral diverge and so by the Integral test the series diverge.

3) *Let*

$$f(x) = \frac{1}{(2x+7)^3}.$$

We have for $x \geq 2$

$$\frac{1}{(2x+7)^3} > 0 \quad \textit{and} \quad f(x) = \frac{1}{(2x+7)^3} > \frac{1}{(2(x+1)+7)^3} = f(x+1).$$

Then $f(x)$ *is a continuous, positive and decreasing function on the interval* $[2, \infty[$.

Now, let's compute the integral

$$\int_2^{\infty} \frac{1}{(2x+7)^3} dx.$$

So

$$\begin{aligned}
\int_2^{\infty} \frac{1}{(2x+7)^3} dx &= \lim_{y \to \infty} \int_2^{y} \frac{1}{(2x+7)^3} dx \\
&= \lim_{y \to \infty} \left(-\frac{1}{4} \frac{1}{(2x+7)^2} \right) \Bigg|_2^t \\
&= \frac{1}{484}.
\end{aligned}$$

Therefore the integral converge and so by the Integral test the series converge.

3

Numerical functions of a real variable

3.1 Generality on functions

3.1.1 Definitions and terminology

Definition 3.1.1. *(Application). we call an application from a set E to a set F, any correspondence f that associates one and only one element $y \in F$ with any element $x \in E$. We write*

$$\begin{array}{rccc} f: & E & \longrightarrow & F \\ & x & \longmapsto & y = f(x). \end{array}$$

Definition 3.1.2. *(Numerical function). We call a numerical function on a set $E \subset \mathbb{R}$, any relation f , which associates to any $x \in E$, at most one element $y \in F \subset \mathbb{R}$; we write*

$$\begin{array}{rccc} f: & E \subset \mathbb{R} & \longrightarrow & F \subset \mathbb{R} \\ & x & \longmapsto & y = f(x). \end{array}$$

The set E is called the start set and F is called the end set.
$y = f(x)$ is called the image of x by f and x is an antecedent of y.

Example 3.1.1. • *The application of a set $E \subset \mathbb{R}$ to itself, which associates x with each element x, is called an identity application, noted as (id_E).*

$$\begin{array}{rccc} id_E: & E \subset \mathbb{R} & \longrightarrow & E \subset \mathbb{R} \\ & x & \longmapsto & id_E(x) = x. \end{array}$$

• *Let*

$$\begin{array}{rccc} f: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & f(x) = \dfrac{1}{x}. \end{array}$$

f is a function on $\mathbb{R}$ but not an application on $\mathbb{R}$, because 0 has no image by f.

• *Let*

$$\begin{array}{rccc} f: & E & \longrightarrow & F \\ & x & \longmapsto & f(x) = a. \end{array}$$

f is a constant application.

• *Let*

$$\begin{array}{rcccl} |.|: & \mathbb{R} & \longrightarrow & \mathbb{R} & \\ & x & \longmapsto & |x| & = \begin{cases} x & si \quad x \geq 0 \\ -x & si \quad x < 0 \end{cases} \end{array}$$

$|.|$ is the absolute value function.

• *Let*

$$\begin{array}{rccc} [.]: & \mathbb{R} & \longrightarrow & \mathbb{Z} \\ & x & \longmapsto & [x] \end{array}$$

defined by

$$[x] \leq x < [x] + 1$$

$[x]$ or $E(x)$ is the integer function of x.

Definition 3.1.3. *(Domain of definition). The domain of definition of a numerical function*

$$f : E \subset \mathbb{R} \longrightarrow F \subset \mathbb{R},$$

is the set of points $x \in E$ where $f(x)$ is well defined. We denote it D_f, and write

$$D_f = \{x \in \mathbb{R} \ / \ f(x) \ \ exists\}.$$

Example 3.1.2. *Find the domain of definition of the following functions:*

$$f_1(x) = \frac{1}{x^2-2}, \quad f_2(x) = \frac{1}{\cos(x)}, \quad f_3(x) = \sqrt{x^2-5x+6}, \quad f_4 = \ln(4-X^2).$$

- *For the function $f_1(x)$, we have*

$$\begin{aligned} D_{f_1} &= \{x \in \mathbb{R} \ / \ f_1(x) \ \ exists\} \\ &= \{x \in \mathbb{R} \ / \ x^2 - 2 \neq 0\} \\ &= \{x \in \mathbb{R} \ / \ x \neq -2 \ \ or \ \ x \neq 2\} \\ &= \mathbb{R} - \{-2, 2\}. \end{aligned}$$

- *For the function $f_2(x)$, we have*

$$\begin{aligned} D_{f_2} &= \{x \in \mathbb{R} \ / \ f_2(x) \ \ exists\} \\ &= \{x \in \mathbb{R} \ / \ \cos(x) \neq 0\} \\ &= \{x \in \mathbb{R} \ / \ x \neq k\frac{\pi}{2} \ \ where \ \ k \in \mathbb{Z}\} \\ &= \mathbb{R} - \{k\frac{\pi}{2} \ \ where \ \ k \in \mathbb{Z}\}. \end{aligned}$$

- *For the function $f_3(x)$, we have*

$$\begin{aligned} D_{f_3} &= \{x \in \mathbb{R} \ / \ f_3(x) \ \ exists\} \\ &= \{x \in \mathbb{R} \ / \ x^2 - 5x + 6 \geq 0\} \\ &= \{x \in \mathbb{R} \ / \ (x-2)(x-3) \geq 0\} \\ &=]-\infty, 2] \cup [3, +\infty[. \end{aligned}$$

- *For the function $f_4(x)$, we have*

$$\begin{aligned} D_{f_4} &= \{x \in \mathbb{R} \ / \ f_4(x) \ \ exists\} \\ &= \{x \in \mathbb{R} \ / \ 4 - X^2 \geq 0\} \\ &= \{x \in \mathbb{R} \ / \ (2-x)(2+x) \geq 0\} \\ &=]-2, 2[. \end{aligned}$$

Definition 3.1.4. *(Curve of a function). The curve (graph) of a function $f : E \subset \mathbb{R} \longrightarrow F \subset \mathbb{R}$ is the set defined by*

$$C_f = \{(x, f(x)) \ / \ x \in E\}$$

3.1.2 Direct image, Reciprocal image:

Definition 3.1.5. *(Direct image). Let $f : E \longrightarrow F$ be an application and A a subset of E. The direct image of A by the application f is the subset of F denoted $f(A)$:*

$$f(A) = \{y = f(x) \in F \ \ / \ x \in A\} \subset F.$$

Example 3.1.3. *Let*

$$\begin{array}{rccccc} f: & \mathbb{R} & \longrightarrow & \mathbb{R} & & \\ & x & \longmapsto & f(x) & = & x^2 \end{array}$$

- *For $A = [-2, 1]$, we have*

$$\begin{aligned} f(A) &= \{f(x),\ x \in A\} \\ &= \{x^2,\ x \in [-2, 1]\} \\ &= [0, 4]. \end{aligned}$$

- *For $A = [-1, 1]$, we have*

$$\begin{aligned} f(A) &= \{f(x),\ x \in A\} \\ &= \{x^2,\ x \in [-1, 1]\} \\ &= [0, 1]. \end{aligned}$$

Definition 3.1.6. *(Reciprocal image). Let $f : E \longrightarrow F$ be an application and B a subset of E. The reciprocal image of B by the application f is the subset of F denoted $f^{-1}(B)$:*

$$f^{-1}(B) = \{x \in E \ \ / \ f(x) \in B\} \subset E.$$

Example 3.1.4. *Let*

$$\begin{array}{rccccc} f: & \mathbb{R} & \longrightarrow & \mathbb{R} & & \\ & x & \longmapsto & f(x) & = & x^2 \end{array}$$

- *For $B = [0, 4]$, we have*

$$\begin{aligned} f^{-1}(B) &= \{x \in \mathbb{R},\ f(x) \in [0, 4]\} \\ &= \{x \in \mathbb{R},\ x^2 \in [0, 4]\} \\ &= \{x \in \mathbb{R},\ (x - 2)(x + 2) \leq 0\} \\ &= [-2, 2]. \end{aligned}$$

• *For* $A = [0,2]$, *we have*

$$\begin{aligned} f^{-1}(B) &= \{x \in \mathbb{R},\ f(x) \in [0,2]\} \\ &= \{x \in \mathbb{R},\ x^2 \in [0,2]\} \\ &= \{x \in \mathbb{R},\ (x-\sqrt{2})(x+\sqrt{2}) \leq 0\} \\ &= [-\sqrt{2}, \sqrt{2}]. \end{aligned}$$

3.1.3 Composite application:

Definition 3.1.7. *Let* E, F *and* G *be three sets and* $f : E \longrightarrow F$, $g : F \longrightarrow G$ *two applications. We denote the composite application of* f *and* g *denoted* $g \circ f$ *by*

$$\forall x \in E, \quad (g \circ f)(x) = g(f(x)).$$

Example 3.1.5. • *given the applications:*

$$\begin{array}{rccclcrcccl} f: & \mathbb{R} & \longrightarrow & \mathbb{R}_+ & & \qquad & g: & \mathbb{R}_+ & \longrightarrow & \mathbb{R}_+ & \\ & x & \longmapsto & f(x) & = x^2 & & & x & \longmapsto & g(x) & = x^3 \end{array}$$

Then

$$\begin{array}{rcccl} g \circ f: & \mathbb{R} & \longrightarrow & \mathbb{R}_+ & \\ & x & \longmapsto & (g \circ f)(x) & = g(f(x)) = (x^2)^3 = x^6 \end{array}$$

and

$$\begin{array}{rcccl} f \circ g: & \mathbb{R}_+ & \longrightarrow & \mathbb{R}_+ & \\ & x & \longmapsto & (f \circ g)(x) & = f(g(x)) = (x^3)^2 = x^6 \end{array}$$

• *given the applications:*

$$\begin{array}{rccclcrcccl} f_1: &]0,+\infty[& \longrightarrow &]0,+\infty[& & \qquad & g_1: &]0,+\infty[& \longrightarrow & \mathbb{R} & \\ & x & \longmapsto & f_1(x) & = \dfrac{1}{x} & & & x & \longmapsto & g_1(x) & = \dfrac{x-1}{x+1} \end{array}$$

Then

$$\begin{array}{rcccl} g_1 \circ f_1: &]0,+\infty[& \longrightarrow & \mathbb{R} & \\ & x & \longmapsto & (g_1 \circ f_1)(x) & = g_1(f_1(x)) = \dfrac{\frac{1}{x}-1}{\frac{1}{x}+1} = \dfrac{1-x}{1+x} \end{array}$$

and

$$\begin{array}{rcccl} f_1 \circ g_1: &]0,+\infty[& \longrightarrow &]0,+\infty[& \\ & x & \longmapsto & (f_1 \circ g_1)(x) & = f_1(g_1(x)) = \dfrac{1}{\frac{x-1}{x+1}} = \dfrac{x+1}{x-1} \end{array}$$

3.1.4 Injective, surjective and bijective functions:

Definition 3.1.8. *Let* $f : E \subset \mathbb{R} \longrightarrow F \subset \mathbb{R}$ *be a function.*

• *We say that* f *is injective if and only if every element* y *in* F *has at most one antecedent in* E*. In other words,*

$$\forall x_1, x_2 \in E : f(x_1) = f(x_2) \Longrightarrow x_1 = x_2$$

• *We say that* f *is surjective if and only if every element* y *in* F *has at least one antecedent in* E*. In other words,*

$$\forall y \in F,\ \exists x \in E : y = f(x)$$

• *We say that* f *is bijective if and only if every element* y *in* F *has a single antecedent in* E*. In other words,*

$$\forall y \in F,\ \exists! x \in E : y = f(x)$$

Remark 3.1.1. *We say that* f *is bijective if and only if it is both injective and surjective.*

Definition 3.1.9. *(reciprocal function): Let*

$$\begin{array}{rccccl} f: & E \subset \mathbb{R} & \longrightarrow & F \subset \mathbb{R} & & \\ & x & \longmapsto & y & = & f(x) \end{array}$$

a bijective function. The reciprocal (inverse) function is the function denoted by

$$\begin{array}{rccccl} f^{-1}: & F & \longrightarrow & E & & \\ & Y & \longmapsto & X & = & f^{-1}(Y) \end{array}$$

which verifies the following properties:

1. $(\forall x \in E : y = f(x)) \Longleftrightarrow (\forall y \in F : x = f^{-1}(y))$.

2. $f \circ f^{-1} = id_F$ *and* $f^{-1} \circ f = id_E$.

Example 3.1.6. *Let's consider the two functions:*

$$\begin{array}{rccccl} g: & \mathbb{N} & \longrightarrow & \mathbb{Q} & & \\ & x & \longmapsto & g(x) & = & \dfrac{1}{1+x} \end{array}$$

and

$$\begin{array}{rccccc} h: & \mathbb{R} & \longrightarrow & \mathbb{R} & & \\ & x & \longmapsto & h(x) & = & 3x \end{array}$$

1. *Are g, h injective, surjective or bijective?*

2. *If g or h is bijective, give its inverse.*

For 1.: let $x_1, x_2 \in \mathbb{N}$

$$\begin{aligned} g(x_1) = g(x_2) &\implies \frac{1}{1+x_1} = \frac{1}{1+x_2} \\ &\implies 1 + x_1 = 1 + x_2 \\ &\implies x_1 = x_2. \end{aligned}$$

Thus g is injective.

We have $\forall x \in \mathbb{N},\ \ g(x) \leq 1.$ then $y = 2$ has no antecedent. So g is not surjective, then g is not bijective.

For 2.: let $x_1, x_2 \in \mathbb{R}$

$$\begin{aligned} h(x_1) = h(x_2) &\implies 3x_1 = 3x_2 \\ &\implies x_1 = x_2. \end{aligned}$$

Thus h is injective.

Let $y \in \mathbb{R}$,

$$\begin{aligned} y = h(x) = 3x &\implies y = 3x \\ &\implies x = \frac{y}{3}. \end{aligned}$$

So

$$\forall y \in \mathbb{R},\ \exists x = \frac{y}{3} \in \mathbb{R}:\ y = h(x).$$

Thus h is surjective.

Consequently, h is a bijective function, and its inverse function h^{-1}, is defined by

$$\begin{array}{rccccc} h^{-1}: & \mathbb{R} & \longrightarrow & \mathbb{R} & & \\ & y & \longmapsto & x & = & h^{-1}(y) = \dfrac{y}{3} \end{array}$$

3.1.5 Parity and periodicity:

Definition 3.1.10. *Let I be an interval of $\mathbb{R}$ symmetrical about 0. Let $f : I \longrightarrow \mathbb{R}$ be a function defined on this interval. We said that:*

• *f is even if*

$$\forall x \in I,\ f(-x) = f(x)$$

• *f is odd if*

$$\forall x \in I,\ f(-x) = -f(x)$$

Example 3.1.7. • *The function*

$$f(x) = e^{\sqrt{1-x^2}}$$

defined on $[-1,1]$ is even because: $\forall x \in [-1,1]$, we have $-x \in [-1,1]$ and

$$f(-x) = e^{\sqrt{1-(-x)^2}} = f(x).$$

• *The function*

$$h(x) = \ln\left(x + \sqrt{x^2+1}\right)$$

defined on $\mathbb{R}$ is odd because: $\forall x \in \mathbb{R}$, we have

$$\begin{aligned}
h(-x) &= \ln\left(-x + \sqrt{(-x)^2+1}\right)\\
&= \ln\left(\frac{(-x+\sqrt{x^2+1})(-x-\sqrt{x^2+1})}{(-x-\sqrt{x^2+1})}\right)\\
&= \ln\left(\frac{(x^2-(x^2+1)}{-(x+\sqrt{x^2+1})}\right)\\
&= \ln\left(\frac{-1}{-(x+\sqrt{x^2+1})}\right)\\
&= \ln\left(\frac{1}{x+\sqrt{x^2+1}}\right)\\
&= -\ln\left(x+\sqrt{x^2+1}\right)\\
&= -h(x)
\end{aligned}$$

Remark 3.1.2. • *The curve of an even function is symmetrical about the y-axis.*

• *The curve of an odd function is symmetrical about the origin.*

Definition 3.1.11. *(Periodicity) Let f be a function defined on $D_f \subseteq \mathbb{R}$.*

- *We say that f is a periodic function if there is a strictly positive real number T such that:*

$$\forall x \in D_f,\ f(x+T) = f(x).$$

- *We call the period of f the smallest positive number T such that:*

$$f(x+T) = f(x)\ \ \forall x \in D_f.$$

Example 3.1.8. • *The function* $\tan x$ *is periodic with period π, in fact*

$$\begin{aligned}\tan(x+\pi) &= \frac{\sin(x+\pi)}{\cos(x+\pi)} \\ &= \frac{-\sin(x)}{-\cos(x)} \\ &= \tan x\end{aligned}$$

- *The function $f(x) = x - [x]$ is periodic with period* 1*, in fact*

$$\begin{aligned}f(x+1) &= x+1-[x+1] \\ &= x+1-[x]-1 \\ &= f(x)\end{aligned}$$

3.1.6 Majorized, minorized and bounded functions:

Definition 3.1.12. *Let $f : E \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a function, we say that*

- *f is majored over E if there exists $M \in \mathbb{R}$,*

$$\forall x \in \mathbb{E}:\ f(x) \leq M.$$

- *f is minorized over E if there exists $m \in \mathbb{R}$,*

$$\forall x \in \mathbb{E}:\ f(x) \geq m.$$

- *f is bounded on E if it is both bounded above and below, i.e.*

$$\exists m,\ M \in \mathbb{R},\ \forall x \in \mathbb{E}:\ m \leq f(x) \leq M$$

or

$$\exists S > 0,\ \forall x \in \mathbb{E}:\ |f(x)| \leq S.$$

Example 3.1.9. *The function* $f(x) = \dfrac{1}{x}$ *is bounded on* $I = [1, 2]$*, because:* $\forall x \in [1, 2]$

$$\begin{aligned} 1 \leq x \leq 2 &\implies \frac{1}{2} \leq \frac{1}{x} \leq 1 \\ &\implies \frac{1}{2} \leq f(x) \leq 1. \end{aligned}$$

But it is not bounded on $J =]0, 1]$*, because:* $\forall x \in]0, 1]$

$$\begin{aligned} 0 < x \leq 1 &\implies 1 \leq \frac{1}{x} \leq +\infty \\ &\implies 1 \leq f(x) \leq +\infty. \end{aligned}$$

Definition 3.1.13. *Let* $f : E \subset \mathbb{R} \longrightarrow \mathbb{R}$ *be a function*

- *We call the upper bound of* f *on* E *the smallest of the majorants, we denote it* $\sup_{x \in E} f(x)$.
- *We call the lower bound of* f *on* E *the largest of the minorants, we denote it* $\inf_{x \in E} f(x)$.

3.1.7 Monotonic functions:

Definition 3.1.14. *Let* $f : E \subset \mathbb{R} \longrightarrow \mathbb{R}$ *be a function.*

- f *is said to be increasing*

$$\forall x, y \in E : x \leq y \Longrightarrow f(x) \leq f(y)$$

- f *is said to be decreasing*

$$\forall x, y \in E : x \leq y \Longrightarrow f(x) \geq f(y)$$

- f *is said to be monotonic if* f *is increasing or decreasing.*
- f *is said to be strictly increasing*

$$\forall x, y \in E : x < y \Longrightarrow f(x) < f(y)$$

- f *is said to be strictly decreasing*

$$\forall x, y \in E : x < y \Longrightarrow f(x) > f(y)$$

- f *is said to be strictly monotonic if* f *is strictly increasing or strictly decreasing.*

Example 3.1.10. *The function defined by:* $f(x) = |x|$ *is increasing on* R_+*, While it is decreasing on* R_-.

3.2 Limits of function:

Definition 3.2.1. *A part $V \subset \mathbb{R}$ is a neighborhood of $x \in \mathbb{R}$, if it contains an open interval of $\mathbb{R}$ containing x, in other words*

$$\exists > 0\gamma : \ x \in]x - \gamma, x + \gamma[\subset V$$

Example 3.2.1. $[-2, 2]$ *is a neighborhood of all these points except the two points -2 and 2.*

Definition 3.2.2. *Let f be a function defined in a neighborhood I of x_0 except perhaps in x_0. The number l is called the limit of f when x tends to x_0 and we write*

$$l = \lim_{x \to x_0} f(x)$$

if

$$\forall \varepsilon > 0, \ \exists \delta(\varepsilon) > 0, \ \forall x \in I : \ |x - x_0| < \delta(\varepsilon) \Longrightarrow |f(x) - l| < \varepsilon.$$

Example 3.2.2. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function, such that $f(x) = x^2 - 1$. we'll show that: $\lim\limits_{x \to 1} f(x) = 0$.*
Let $\varepsilon > 0$, we have

$$\begin{aligned} |f(x) - l| &= |x^2 - 1| \\ &= |x + 1||x - 1| \\ &< (|x| + |1|)|x - 1| \\ &< 2|x - 1| < \varepsilon. \end{aligned}$$

So

$$|x - 1| < \frac{\varepsilon}{2}.$$

Therefore, the best choice of $\delta(\varepsilon)$ is $\delta(\varepsilon) = \dfrac{\varepsilon}{2}$.

Example 3.2.3. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function, such that $f(x) = 3x + 1$. we'll show that: $\lim\limits_{x \to 0} f(x) = 1$.*
Let $\varepsilon > 0$, we have

$$\begin{aligned} |f(x) - l| &= |3x + 1 - 1| \\ &= |3x| < \varepsilon. \end{aligned}$$

So

$$|x| < \frac{\varepsilon}{3}.$$

Therefore, the best choice of $\delta(\varepsilon)$ *is* $\delta(\varepsilon) = \frac{\varepsilon}{3}$.

Theorem 3.2.1. *If* f *has a limit at the point* x_0*, this limit is unique.*

3.2.1 Right limit, left limit:

Definition 3.2.3. *Let* $f : I \longrightarrow \mathbb{R}$ *be a function,*

- *We say that* f *admits a limit* l *when* x *tends to* x_0 *on the left or by lower values* $(x \rightarrow^{<} x_0)$ *and we note*

$$\lim_{x \rightarrow_{<} x_0} f(x) = l$$

if

$$\forall \varepsilon > 0,\ \exists \delta(\varepsilon) > 0,\ \forall x \in I :\ -\delta(\varepsilon) < x - x_0 < 0 \Longrightarrow |f(x) - l| < \varepsilon.$$

- *We say that* f *admits a limit* l *when* x *tends to* x_0 *on the right or by higher values* $(x \rightarrow^{>} x_0)$ *and we note*

$$\lim_{x \rightarrow_{>} x_0} f(x) = l$$

if

$$\forall \varepsilon > 0,\ \exists \delta(\varepsilon) > 0,\ \forall x \in I :\ 0 < x - x_0 < \delta(\varepsilon) \Longrightarrow |f(x) - l| < \varepsilon.$$

Proposition 3.2.1. *Let* $f : I \longrightarrow \mathbb{R}$ *be a function. Then*

$$\lim_{x \rightarrow x_0} f(x) = l \Longleftrightarrow \lim_{x \rightarrow_{<} x_0} f(x) = \lim_{x \rightarrow_{>} x_0} f(x) = l$$

Example 3.2.4. *Let* $f : \mathbb{R} \longrightarrow \mathbb{R}^*$ *be a function, such that*

$$f(x) = \frac{|x|}{3x} = \begin{cases} \frac{1}{3} & if \quad x > 0 \\ -\frac{1}{3} & if \quad x < 0 \end{cases}$$

We have

$$\lim_{x \rightarrow_{<} 0} f(x) = -\frac{1}{3}$$

and

$$\lim_{x \rightarrow_{>} 0} f(x) = \frac{1}{3}$$

As

$$\lim_{x \to_< 0} f(x) \neq \lim_{x \to_> 0} f(x).$$

So f has no limit at the point $x = 0$.

Theorem 3.2.2. *Let $f : I \longrightarrow \mathbb{R}$ be a function. Then $\lim\limits_{x \to x_0} f(x) = l$ if and only if for any sequence $(x_n)_{n \in \mathbb{N}}$, we have*

$$\lim_{n \to +\infty} x_n = x_0 \Longrightarrow \lim_{n \to +\infty} f(x_n) = l$$

Remark 3.2.1. *According to the previous theorem, if there are two sequences $(x_n)_n$, $(y_n)_n$ converging to x_0 such that $\lim\limits_{n \to +\infty} f(x_n) \neq \lim\limits_{n \to +\infty} f(y_n)$, then the limit of f does not exist at x_0.*

Example 3.2.5. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}^*$ be a function, such that*

$$f(x) = \cos \frac{1}{x}$$

f has no limit at the point $x = 0$. Indeed, we consider the two sequences:

$$\forall n \in \mathbb{N}^* : \; x_n = \frac{1}{n\pi + \frac{\pi}{2}}, \quad y_n = \frac{1}{2n\pi}.$$

It is clear that

$$\lim_{n \to +\infty} x_n = \lim_{n \to +\infty} y_n = 0,$$

$$\lim_{n \to +\infty} f(x_n) = \lim_{n \to +\infty} \cos(n\pi + \frac{\pi}{2}) = 0$$

and

$$\lim_{n \to +\infty} f(y_n) = \lim_{n \to +\infty} \cos(2n\pi) = 1.$$

As

$$\lim_{n \to +\infty} f(x_n) \neq \lim_{n \to +\infty} f(y_n).$$

Consequently, f has no limit at point 0.

3.2.2 Infinite limit:

Definition 3.2.4. *Let $f : I \longrightarrow \mathbb{R}$ be a function,*

- *We say that f has limit $+\infty$ at x_0 and we note*

$$\lim_{x \to x_0} f(x) = +\infty$$

if

$$\forall A > 0,\ \exists \delta > 0,\ \forall \in I : |x - x_0| < \delta \Longrightarrow f(x) > A.$$

- *We say that f has limit $-\infty$ at x_0 and we note*

$$\lim_{x \to x_0} f(x) = -\infty$$

if

$$\forall A > 0,\ \exists \delta > 0,\ \forall \in I : |x - x_0| < \delta \Longrightarrow f(x) < -A.$$

- *We say that f has limit $+\infty$ at $+\infty$ and we note*

$$\lim_{x \to +\infty} f(x) = +\infty$$

if

$$\forall A > 0,\ \exists B > 0,\ \forall x \in I : x > B \Longrightarrow f(x) > A.$$

- *We say that f has limit $-\infty$ at $+\infty$ and we note*

$$\lim_{x \to +\infty} f(x) = -\infty$$

if

$$\forall A > 0,\ \exists B > 0,\ \forall x \in I : x > B \Longrightarrow f(x) < -A.$$

- *We say that f has limit $+\infty$ at $-\infty$ and we note*

$$\lim_{x \to -\infty} f(x) = +\infty$$

if

$$\forall A > 0,\ \exists B > 0,\ \forall x \in I : x < -B \Longrightarrow f(x) > A.$$

- *We say that f has limit $-\infty$ at $-\infty$ and we note*

$$\lim_{x \to -\infty} f(x) = -\infty$$

if

$$\forall A > 0,\ \exists B > 0,\ \forall x \in I : x < -B \Longrightarrow f(x) < -A.$$

Theorem 3.2.3. *Let f, g and h be three functions defined on a neighbourhood I of x_0 such that*

$$\forall x \in I :\ f(x) \leq g(x) \leq h(x).$$

If

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = l,$$

then

$$\lim_{x \to x_0} g(x) = l.$$

Example 3.2.6. *Let $f : \mathbb{R}_+^* \longrightarrow \mathbb{R}$ be a function, such that*

$$f(x) = x \cos \frac{1}{x}$$

We have

$$\forall x \in \mathbb{R}_+^* : \quad -1 \leq \cos \frac{1}{x} \leq 1,$$

then

$$\forall x \in \mathbb{R}_+^* : \quad -x \leq x \cos \frac{1}{x} \leq x.$$

As $\lim_{x \to 0} -x = 0$ and $\lim_{x \to 0} x = 0$. So using the previous theorem, we obtain

$$\lim_{x \to 0} x \cos \frac{1}{x} = 0$$

Corollary 3.2.1. *Let f and h be two functions defined on a neighbourhood I of x_0. If f is bounded on the neighbourhood of x_0 and $\lim_{x \to x_0} h(x) = 0$, then*

$$\lim_{x \to x_0} f(x)h(x) = 0$$

Example 3.2.7. *Let $f : \mathbb{R}_+^* \longrightarrow \mathbb{R}$ be a function, such that*

$$f(x) = x \sin \frac{1}{x}$$

We have

$$\lim_{x \to 0} x \sin \frac{1}{x} = 0$$

because $\sin \frac{1}{x}$ is bounded and $\lim_{x \to 0} x = 0$.

3.2.3 Limit operations

Theorem 3.2.4. *Let f and g be two functions defined on an interval I of $\mathbb{R}$, such that $\lim_{x\to x_0} f(x) = l_1$, $\lim_{x\to x_0} g(x) = l_2$ and $\beta \in \mathbb{R}$, we have*

1. $\lim_{x\to x_0} (f+g)(x) = \lim_{x\to x_0} f(x) + \lim_{x\to x_0} g(x) = l_1 + l_2$
2. $\lim_{x\to x_0} (f.g)(x) = \lim_{x\to x_0} f(x). \lim_{x\to x_0} g(x) = l_1.l_2$
3. $\lim_{x\to x_0} (\beta.f)(x) = \beta \lim_{x\to x_0} f(x) = \beta.l_1$
4. $\lim_{x\to x_0} \left(\frac{f}{g}\right)(x) = \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)} = \frac{l_1}{l_2} \quad if \ l_2 \neq 0$

Example 3.2.8. *Let's calculate the following limits.*

1.

$$\begin{aligned} \lim_{x\to+\infty} (\sqrt{x^2+5x-3} - x) &= \lim_{x\to+\infty} \frac{5x-3}{\sqrt{x^2+5x-3}+x} \\ &= \lim_{x\to+\infty} \frac{x\left(5-\frac{3}{x}\right)}{x\left(\sqrt{1+\frac{5}{x}-\frac{3}{x^2}}+1\right)} \\ &= \frac{5}{2}. \end{aligned}$$

2.

$$\begin{aligned} \lim_{x\to-1}(9x^2+6x-3) &= \lim_{x\to-1} 3.(3x^2+2x-1) \\ &= 3 \lim_{x\to-1}(3x^2+2x-1) \\ &= 3.0 = 0. \end{aligned}$$

3.

$$\begin{aligned} \lim_{x\to1}(x^2-4) &= \lim_{x\to1}(x-2).(x+2) \\ &= \lim_{x\to1}(x-2). \lim_{x\to1}(x+2) \\ &= (-1).(3) = -3. \end{aligned}$$

Indeterminate forms

There are situations where we cannot say anything about the limits. We say that we have an indeterminate form (I.F). Here's a list of indeterminate forms.

$$+\infty - \infty, \quad 0. + \infty, \quad \frac{\infty}{\infty}, \quad \frac{0}{0}, \quad 1^\infty, \quad 0^0, \quad \infty^0$$

Example 3.2.9. *Let's calculate the following limits.*

$$\lim_{x\to 0}\frac{\sin(2x)}{2\sin x} = \frac{0}{0}\ I.F.$$

We have

$$\sin(2x) = 2\sin x\cos x$$

then

$$\begin{aligned}\lim_{x\to 0}\frac{\sin(2x)}{2\sin x} &= \lim_{x\to 0}\frac{2\sin x\cos x}{2\sin x}\\ &= \lim_{x\to 0}\cos x\\ &= 1.\end{aligned}$$

3.3 Continuous functions

3.3.1 Continuous functions at a point

Definition 3.3.1. *Let $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a function. We say that f is continuous at $x_0 \in I$ if*

$$\lim_{x\to x_0} f(x) = f(x_0).$$

This is equivalent to

$$\forall\varepsilon > 0,\ \exists\delta(\varepsilon) > 0,\ \forall x \in I:\ |x - x_0| < \delta(\varepsilon) \Longrightarrow |f(x) - f(x_0)| < \varepsilon.$$

Example 3.3.1. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function, such that*

$$f(x) = \begin{cases} x^3\sin\frac{1}{x} & if \quad x \neq 0\\ 0 & if \quad x = 0\end{cases}$$

f is continuous at $x = 0$. because, we have

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x^3 \underbrace{\sin\frac{1}{x}}_{bounded} = 0 = f(0)$$

Definition 3.3.2. *(Continuity on left and right) Let $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a function. We say that f is right-continuous at $x_0 \in I$, respectively left-continuous at $x_0 \in I$ if*

$$\lim_{x \to_> x_0} f(x) = f(x_0),$$

respectively if

$$\lim_{x \to_< x_0} f(x) = f(x_0).$$

Example 3.3.2. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function, such that*

$$f(x) = \begin{cases} x^3 & if \quad x < 0 \\ 2x+3 & if \quad x \geq 0 \end{cases}$$

f is right-continuous at 0, indeed

$$\begin{aligned} \lim_{x \to_> 0} f(x) &= \lim_{x \to_> 0} (2x+3) \\ &= 3 = f(0). \end{aligned}$$

but it is not left-continuous at 0 because

$$\begin{aligned} \lim_{x \to_< 0} f(x) &= \lim_{x \to_> 0} x^3 \\ &= 0 \neq f(0). \end{aligned}$$

3.3.2 Continuity on an interval:

Definition 3.3.3. *We say that f is continuous on an interval I if it is continuous at any point of I. We denote by $C(I)$ the set of continuous functions on I.*

Example 3.3.3. *Let $f : \mathbb{R} \longrightarrow \mathbb{Z}$ be a function, such that $f(x) = [x]$.*
Note that for any point $m \in \mathbb{Z}$, we have:

$$\lim_{x \to_> m} [x] = m \neq \lim_{x \to_< m} [x] = m - 1,$$

which shows the discontinuity of this function at any point $m \in \mathbb{Z}$. In conclusion, the integer function is continuous on $\mathbb{R} - \mathbb{Z}$.

Proposition 3.3.1. *If f and g are two continuous functions at x_0, then*

- $\forall\ \alpha,\ \beta \in \mathbb{R},\ \alpha f + \beta g$ *is continuous at* x_0.
- $f.g$ *is continuous at* x_0.
- *if* $g(x_0) \neq 0$, *then* $\dfrac{f}{g}$ *is continuous at* x_0.
- $|f|$ *is continuous at* x_0.

3.3.3 Continuity of composite functions:

Properties 3.3.1. *Let $f : I \longrightarrow I^{'}$ and $g : I^{'} \longrightarrow \mathbb{R}$ be two functions continuous at x_0 and $f(x_0)$ respectively. Then $g \circ f : I \longrightarrow \mathbb{R}$ is continuous at x_0.*

3.3.4 Characterizing continuity using numerical sequences:

Proposition 3.3.2. *Let $f : I \longrightarrow \mathbb{R}$, be a function and $x_0 \in I$. The following properties are equivalent,*

- f *is continuous at* x_0.
- *For any sequence $(U_n)_n$ of elements of I that converges to x_0, the sequence $(f(U_n))_n$ converges to $f(x_0)$.*

Example 3.3.4. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function, such that*

$$f(x) = \begin{cases} \cos \dfrac{1}{x} & if \quad x \neq 0 \\ 0 & if \quad x = 0 \end{cases}$$

Let

$$U_n = \frac{1}{2n\pi}$$

and

$$V_n = \frac{1}{2n\pi + \frac{\pi}{2}}$$

we have

$$f(U_n) = 1 \neq f(V_n) = 0$$

Consequently, f is not continuous in 0.

3.3.5 Continuity prolongation:

Proposition 3.3.3. *Let f be a function defined on an interval I, except that it can be at $x_0 \in I$, if f has a finite limit l at x_0. i.e. $\lim_{x \to x_0} f(x) = l$, then the function defined by*

$$\widehat{f}(x) = \begin{cases} f(x) & if \quad x \neq x_0 \\ l & if \quad x = x_0 \end{cases}$$

is called the continuity prolongation of f on I.

Example 3.3.5. *Let $f : \mathbb{R}^* \longrightarrow \mathbb{R}$ be a function, such that*

$$f(x) = x \cos\left(\frac{1}{x}\right)$$

As

$$\forall x \in \mathbb{R}^* : \; 0 \leq |f(x)| \leq |x|,$$

we deduce that

$$\lim_{x \to 0} f(x) = 0$$

Therefore, f is prolongable by continuity on 0 and its prolongation is the function $\widehat{f}$ defined on all $\mathbb{R}$ by

$$\widehat{f}(x) = \begin{cases} x \cos\left(\frac{1}{x}\right) & if \quad x \neq 0 \\ 0 & if \quad x = 0 \end{cases}$$

Theorem 3.3.1. *(Intermediate value theorem) Let $f : [a, b] \longrightarrow \mathbb{R}$ be a function, such that*

1. *f is continuous on $[a, b]$*
2. *f(a).f(b)<0. Then*

$$\exists x_0 \in]a, b[: f(x_0) = 0$$

And if f is strictly monotonic, then x_0 is unique.

Example 3.3.6. *Let $f : [1, 2] \longrightarrow \mathbb{R}$ be a function, such that*

$$f(x) = \ln x - \frac{1}{x}.$$

We have

1. *f is continuous on $[1, 2]$*

2. $f(1).f(2) = (-1).(0.19) < 0$.

Then using intermediate value theorem

$$\exists x_0 \in]1,2[: f(x_0) = 0.$$

Uniqueness :

We have

$$f'(x) = \frac{1}{x} + \frac{1}{x^2} > 0$$

so f is strictly increasing. Therefore the solution is unique.

Example 3.3.7. *Let $f : [-2,1] \longrightarrow \mathbb{R}$ be a function, such that*

$$f(x) = x^3 - 2x + 2.$$

We have

1. *f is continuous on $[-2,1]$*

2. $f(-2).f(1) = (-2).(1) < 0$.

Then using intermediate value theorem

$$\exists x_0 \in]-2,1[: f(x_0) = 0.$$

Example 3.3.8. *Let $f : [0,1] \longrightarrow \mathbb{R}$ be a function, such that*

$$f(x) = [x] - \frac{1}{2}.$$

The intermediate value theorem does not apply to f, because it is not continuous at 1.

3.4 Derivative of a function:

Definition 3.4.1. *Let $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a function.*

1. *We say that f is derivable at x_0 if*

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists and is finite. This limit is called the derivative of f at x_0 and is denoted $f'(x_0)$.

Example 3.4.1. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function, such that*

$$f(x) = x^3$$

We have

$$\begin{aligned} f'(x_0) &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \\ &= \lim_{x \to x_0} \frac{x^3 - x_0^3}{x - x_0} \\ &= \lim_{x \to x_0} \frac{(x - x_0)(x^2 + x_0 x + x_0^2)}{x - x_0} \\ &= \lim_{x \to x_0} (x^2 + x_0 x + x_0^2) \\ &= 3x_0^2 \end{aligned}$$

3.4.1 Right derivative and left derivative:

Definition 3.4.2. *Let $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a function.*
The right derivative of f at x_0 is defined by

$$f'_r(x) = \lim_{x \to_> x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

Similarly, we define the left derivative of f at x_0 as

$$f'_l(x) = \lim_{x \to_< x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

and

$$f \text{ is derivable at } x_0 \Longleftrightarrow f'_r(x) = f'_l(x) = f'(x)$$

Example 3.4.2. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function, such that*

$$f(x) = \begin{cases} x + 1 & if \quad x \geq 0 \\ 1 - 2x & if \quad x < 0 \end{cases}$$

We have

$$\begin{aligned} f'_r(0) &= \lim_{x \to_> 0} \frac{f(x) - f(0)}{x - 0} \\ &= \lim_{x \to_> x_0} \frac{x + 1 - 1}{x} \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} f_l'(0) &= \lim_{x \to_< 0} \frac{f(x) - f(0)}{x - 0} \\ &= \lim_{x \to_> x_0} \frac{1 - 2x - 1}{x} \\ &= -2. \end{aligned}$$

Then f is not derivable at 0 because $f_r'(0) \neq f_l'(0)$.

Example 3.4.3. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}_+^*$ be a function, such that*

$$f(x) = \begin{cases} x & if \quad x \geq 0 \\ -x & if \quad x < 0 \end{cases}$$

We have

$$\begin{aligned} f_r'(0) &= \lim_{x \to_> 0} \frac{f(x) - f(0)}{x - 0} \\ &= \lim_{x \to_> x_0} \frac{x}{x} \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} f_l'(0) &= \lim_{x \to_< 0} \frac{f(x) - f(0)}{x - 0} \\ &= \lim_{x \to_> x_0} \frac{-x}{x} \\ &= -1. \end{aligned}$$

Then f is not derivable at 0 because $f_r'(0) \neq f_l'(0)$.

Definition 3.4.3. *Let $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a function.*
f is derivable on I if it is derivable at any point on I and the application

$$\begin{aligned} f' : \; I &\longrightarrow \mathbb{R} \\ x &\longmapsto f'(x) \end{aligned}$$

is called the derivative function of f.

3.4.2 Operations on derivable functions:

Proposition 3.4.1. *Let f, g be two functions derivable at $x_0 \in \mathbb{R}$, then if $g(x_0) \neq 0$ we have:*

1. $(\alpha f)'(x_0) = \alpha f'(x_0),\ \alpha \in \mathbb{R}$
2. $(f+g)'(x_0) = f'(x_0) + g'(x_0)$
3. $(f.g)'(x_0) = f'(x_0).g(x_0) + f(x_0).g'(x_0)$
4. $\left(\dfrac{f}{g}\right)'(x_0) = \dfrac{f'(x_0).g(x_0) - f(x_0).g'(x_0)}{g^2(x_0)}$

3.4.3 Derivative of a composite function:

Proposition 3.4.2. *Let $f : I \longrightarrow I'$ and $g : I' \longrightarrow \mathbb{R}$ be two functions derivable at x_0 and $f(x_0)$ respectively. Then $g \circ f : I \longrightarrow \mathbb{R}$ is derivable at x_0 and we have*

$$(g \circ f)'(x_0) = f'(x_0)g'(f(x_0)).$$

Example 3.4.4. *Let the functions f and g be defined by*

$$\begin{array}{rccccc} f: & \mathbb{R} & \longrightarrow & \mathbb{R} & & \\ & x & \longmapsto & f(x) & = & x^2 \end{array}$$

and

$$\begin{array}{rccccc} g: & \mathbb{R} & \longrightarrow & \mathbb{R} & & \\ & x & \longmapsto & g(x) & = & \cos x. \end{array}$$

Then

$$\begin{array}{rccccc} g \circ f: & \mathbb{R} & \longrightarrow & \mathbb{R} & & \\ & x & \longmapsto & (g \circ f)(x) & = & \cos x^2. \end{array}$$

and

$$(g \circ f)'(x) = f'(x)g'(f(x)) = -2x \sin x^2$$

3.4.4 Derivative of a reciprocal function:

Proposition 3.4.3. *If f is derivable at x_0, then f^{-1} is derivable at $f(x_0)$ and we have*

$$(f^{-1})'(f(x_0)) = \frac{1}{f'(x_0)}$$

Example 3.4.5. *The function*

$$\begin{array}{rcccl} g: & \mathbb{R} & \longrightarrow &]0,+\infty[& \\ & x & \longmapsto & g(x) & = e^x. \end{array}$$

is bijective and therefore has a reciprocal application

$$\begin{array}{rcccl} g^{-1}: &]0,+\infty[& \longrightarrow & \mathbb{R} & \\ & x & \longmapsto & g^{-1}(x) & = \ln x. \end{array}$$

with

$$y = e^x \Longleftrightarrow \ln y = x,$$

and we have

$$\begin{aligned} (g^{-1})'(y) &= (\ln)'(y) \\ &= \frac{1}{f'(x)} \\ &= \frac{1}{e^x} \\ &= \frac{1}{y}. \end{aligned}$$

3.4.5 Fundamental theorems on derivable functions:

Theorem 3.4.1. *(Rolle's theorem) Let $f : [a, b] \longrightarrow \mathbb{R}$ be a function satisfying*

1. *f is continuous on $[a, b]$*
2. *f is derivable on $]a, b[$*
3. *$f(a) = f(b)$. Then*

$$\exists c \in]a, b[: f'(c) = 0.$$

Theorem 3.4.2. *(Lagrange's theorem or finite increase theorem) Let $f : [a, b] \longrightarrow \mathbb{R}$ be a function satisfying*

1. *f is continuous on $[a, b]$*
2. *f is derivable on $]a, b[$. Then*

$$\exists c \in]a, b[: f(b) - f(a) = (b - a)f'(c).$$

Example 3.4.6. *Let's show that*

$$\forall x > 0, \ \sin x \leq x.$$

Consider the function

$$\begin{array}{rcccl} f: & \mathbb{R} & \longrightarrow & \mathbb{R} & \\ & y & \longmapsto & f(y) & = & y - \sin y. \end{array}$$

The function f is continuous on $[0, x]$ and derivable on $]0, x[, \ \forall x > 0$. Then according to the finite increasing theorem,

$$\underbrace{\exists c \in]0, x[: f(x) - f(0) = (x-0) f'(c)}_{\Updownarrow}.$$

$$\exists c \in]0, x[: x - \sin x = x(1 - \cos c).$$

As $x > 0$ and $\cos c \leq 1$, we obtains $\forall x > 0, \ \sin x \leq x$.

Corollary 3.4.1. *(Inequality of finite increments) Let $f : I \longrightarrow \mathbb{R}$ be a function derivable on I. If there exists a constant M such that for all $x \in I$, $|f'(x)| \leq M$. Then*

$$\forall x, y \in I : \ |f(x) - f(y)| \leq M|x - y|.$$

Example 3.4.7. *Let $f(x) = \sin x$. We have*

$$\forall x \in \mathbb{R} : \ f'(x) = \cos x.$$

It is clear that

$$\forall x \in \mathbb{R} : \ |f'(x)| \leq 1.$$

using the inequality of finite increments we find that

$$\forall x, y \in \mathbb{R} : \ |\sin x - \sin y| \leq |x - y|.$$

In particular for $y = 0$, we obtain

$$|\sin x| \leq |x|.$$

Theorem 3.4.3. *(Cauchy's theorem) Let $f, g : [a, b] \longrightarrow \mathbb{R}$ be two functions satisfying*

1. *f, g are continuous on $[a, b]$*
2. *f, g are continuous on $]a, b[$. Then*

$$\exists c \in]a, b[: \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

3.4.6 Higher-order derivative:

Let $f : I \longrightarrow \mathbb{R}$, be a derivable function. If its derivative $f' : I \longrightarrow \mathbb{R}$ is also derivable. We note $f^{(2)} = f'' = (f')'$, the second derivative of f. More generally, we note

$$f^{(0)} = f,\ f^{(1)} = f',\ f^{(2)} = f'',\ \ldots,\ f^{(n+1)} = (f^{(n)})'$$

the successive derivatives of f.

Definition 3.4.4. *(C^n-class function) Let $I \subset \mathbb{R}$ be an interval. For any integer $n \in \mathbb{N}$, we define the space $C^n(I)$, as the set of functions $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ that are n-times derivable and its n-th derivative $f^{(n)}$ is continuous.*

Example 3.4.8. *Let's calculate the n-th derivative of the function $f(x) = e^{3x}$.*
We have

$$f'(x) = 3e^{3x},\ f''(x) = 9e^{3x},\ f^{(3)}(x) = 27e^{3x}.$$

We can show by recurrence that for all $n \in \mathbb{N}$, we have

$$f^{(n)}(x) = 3^n e^{3x}.$$

3.4.7 Monotony criterion:

Proposition 3.4.4. *Let f be a function from I into $\mathbb{R}$, derivable on I, then*

1. $f' > 0$ *on* $I \Longleftrightarrow f$ *is increasing on* I
2. $f' < 0$ *on* $I \Longleftrightarrow f$ *is decreasing on* I.

Proposition 3.4.5. *(Derivability and continuity) If f is derivable at x_0, then f is continuous at x_0. The reciprocal is generally false.*

Example 3.4.9. *Let $f(x) = |x|,\ x \in \mathbb{R}$. f is continuous at 0 but not derivable at 0 because*

$$f_r'(0) = 1 \neq -1 = f_l'(0)$$

Theorem 3.4.4. *(Hospital rule) Let $f, g : I \longrightarrow \mathbb{R}$ be two continuous functions on I, derivable on $I - \{x_0\}$ and satisfying the following conditions:*

1. $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0.$
2. $g'(x) \neq 0, \ \forall x \in I - \{x_0\}$, *then*

$$\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = l \Longrightarrow \lim_{x \to x_0} \frac{f(x)}{g(x)} = l$$

Example 3.4.10.

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1.$$

Remark 3.4.1. *The reciprocal of the hospital rule is generally false.*

Example 3.4.11. *Let*

$$f(x) = x^2 \cos \frac{1}{x}$$

and

$$g(x) = x.$$

We have

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} x \cos \frac{1}{x} = 0.$$

While

$$\lim_{x \to 0} \frac{f'(x)}{g'(x)} = \lim_{x \to 0} \left(2x \cos \frac{1}{x} + \sin \frac{1}{x} \right) \text{ that doesn't exist.}$$

Remark 3.4.2. *Hospital's rule is true when* $x \longrightarrow +\infty$ *and if* $\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = \frac{0}{0}$ *and* $f', \ g'$ *satisfy the conditions of the theorem, then we can again apply the Hospital rule.*

Theorem 3.4.5. *(Hospital rule bis) Let* $f, g : I \longrightarrow \mathbb{R}$ *be two continuous functions on* I*, derivable on* $I - \{x_0\}$ *and satisfying the following conditions:*

1. $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = +\infty.$
2. $g'(x) \neq 0, \ \forall x \in I - \{x_0\}$, *then*

$$\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = l \Longrightarrow \lim_{x \to x_0} \frac{f(x)}{g(x)} = l$$

Example 3.4.12.

$$\begin{aligned}
\lim_{x\to+\infty}\frac{x^n}{e^x} &= \lim_{x\to+\infty}\frac{nx^{n-1}}{e^x} \\
&= \lim_{x\to+\infty}\frac{n(n-1)x^{n-2}}{e^x} \\
&= \dots \\
&= \lim_{x\to+\infty}\frac{n(n-1)\dots 2\times 1x^0}{e^x} \\
&= 0
\end{aligned}$$

3.5 Exercises

Exercice 3.5.1. *Find the domain of definition of the following functions:*

1) $f_1(x) = \dfrac{x+1}{2-e^{\frac{1}{x}}}$ 2) $f_2(x) = \dfrac{1}{[x]}$ 3) $f_3(x) = \sqrt{x^2-1}\left(e^{\frac{1}{1-x}}\right)$ 4) $f_4(x) = (1+\ln x)^{\frac{1}{x}}$

Proof 3.5.1. 1)

$$\begin{aligned} D_{f_1} &= \{x \in \mathbb{R} \ / \ f_1(x) \ exists\} \\ &= \left\{x \in \mathbb{R} \ / \ 2 - e^{\frac{1}{x}} \neq 0\right\} \\ &= \left\{x \in \mathbb{R} \ / \ 2 \neq e^{\frac{1}{x}}\right\} \\ &= \left\{x \in \mathbb{R} \ / \ x \neq \frac{1}{\ln 2}\right\} \\ &= \mathbb{R} - \left\{\frac{1}{\ln 2}\right\}. \end{aligned}$$

2)

$$\begin{aligned} D_{f_2} &= \{x \in \mathbb{R} \ / \ f_2(x) \ exists\} \\ &= \{x \in \mathbb{R} \ / \ [x] \neq 0\}. \end{aligned}$$

We have

$$\forall x \in [0,1[, \ [x] = 0,$$

then

$$\begin{aligned} D_{f_2} &= \mathbb{R} - \{x \in \mathbb{R} \ / \ x \in [0,1[\} \\ &=]-\infty, 0[\cup[1, +\infty[. \end{aligned}$$

3)

$$\begin{aligned} D_{f_3} &= \{x \in \mathbb{R} \ / \ f_3(x) \ exists\} \\ &= \left\{x \in \mathbb{R} \ / \ x^2 - 1 \geq 0 \ and \ 1 - x \neq 0\right\} \\ &= \{x \in \mathbb{R} \ / \ x \in]-\infty, -1] \cup [1, +\infty[\ and \ x \neq 1\} \\ &=]-\infty, -1]\cup]1, +\infty[. \end{aligned}$$

4)

$$\begin{aligned}
D_{f_4} &= \{x \in \mathbb{R} \ / \ f_4(x) \ exists\} \\
&= \{x \in \mathbb{R} \ / \ x \neq 0, \ x > 0 \ and \ 1 + \ln x > 0\} \\
&= \left\{x \in \mathbb{R} \ / \ x > 0 \ and \ x > e^{-1}\right\} \\
&=]e^{-1}, +\infty[.
\end{aligned}$$

Exercice 3.5.2. *Check the limits of the following functions:*

1) $l_1 = \lim\limits_{x \to 0} \dfrac{x - \sin 2x}{x + \sin 3x}$ 2) $l_2 = \lim\limits_{x \to > a} \dfrac{\sqrt{x} - \sqrt{a} + \sqrt{x - a}}{\sqrt{x^2 - a^2}}$ 3) $l_3 = \lim\limits_{x \to > +\infty} \dfrac{\ln(1 + e^{2x})}{x}$

Proof 3.5.2. 1)

$$\begin{aligned}
l_1 &= \lim_{x \to 0} \frac{x - \sin 2x}{x + \sin 3x} \\
&= \lim_{x \to 0} \frac{2x\left(\frac{1}{2} - \frac{\sin 2x}{2x}\right)}{3x\left(\frac{1}{3} + \frac{\sin 3x}{3x}\right)} \\
&= -\frac{1}{4}.
\end{aligned}$$

2)

$$\begin{aligned}
l_2 &= \lim_{x \to > a} \frac{\sqrt{x} - \sqrt{a} + \sqrt{x - a}}{\sqrt{x^2 - a^2}} \\
&= \lim_{x \to > a} \frac{\frac{\sqrt{x} - \sqrt{a}}{\sqrt{x - a}} + 1}{\sqrt{x + a}} \\
&= \lim_{x \to > a} \frac{\frac{\sqrt{x - a}}{\sqrt{x} + \sqrt{a}} + 1}{\sqrt{x + a}} \\
&= \frac{1}{\sqrt{2a}}.
\end{aligned}$$

3)

$$\begin{aligned}
l_3 &= \lim_{x \to +\infty} \frac{\ln(1 + e^{2x})}{x} \\
&= \lim_{x \to +\infty} \frac{\ln(e^{2x}(e^{-2x} + 1))}{x} \\
&= \lim_{x \to +\infty} \frac{\ln(e^{2x}) + \ln(e^{-2x} + 1)}{x} \\
&= \lim_{x \to +\infty} \frac{2x + \ln(e^{-2x} + 1)}{x} \\
&= 2 + \lim_{x \to +\infty} \frac{\ln(e^{-2x} + 1)}{x} \\
&= 2.
\end{aligned}$$

Exercice 3.5.3. *Using the definition of the limit of a function, show that :*

1) $\lim\limits_{x\to 4}(2x-1)=7$ 2) $\lim\limits_{x\to+\infty}\dfrac{3x-1}{2x+1}=\dfrac{3}{2}$ 3) $\lim\limits_{x\to+\infty}\ln x=+\infty$ 4) $\lim\limits_{x\to_{>}-3}\dfrac{4}{x+3}=+\infty$

Proof 3.5.3. 1) *Using the definition of the limit of a function, we have*

$$\left(\lim_{x\to 4}(2x-1)=7\right)\Longleftrightarrow\left(\forall\varepsilon>0,\ \exists\delta(\varepsilon)>0,\ \forall x\in\mathbb{R}:\ |x-4|<\delta(\varepsilon)\Longrightarrow|2x-8|<\varepsilon\right).$$

As

$$|2x-8|<\varepsilon\Longleftrightarrow 2|x-4|<\varepsilon\Longleftrightarrow|x-4|<\frac{\varepsilon}{2}.$$

Then it is sufficient to take $\delta(\varepsilon)=\dfrac{\varepsilon}{2}$.

2) *Using the definition of the limit of a function, we have*

$$\left(\lim_{x\to+\infty}\frac{3x-1}{2x+1}=\frac{3}{2}\right)\Longleftrightarrow\left(\forall\varepsilon>0,\ \exists\delta(\varepsilon)>0,\ \forall x\in\mathbb{R}:\ x>\delta(\varepsilon)\Longrightarrow\left|\frac{3x-1}{2x+1}-\frac{3}{2}\right|<\varepsilon\right).$$

As

$$\left|\frac{3x-1}{2x+1}-\frac{3}{2}\right|<\varepsilon\Longleftrightarrow\frac{5}{4x+2}<\varepsilon\Longleftrightarrow x>\frac{5-2\varepsilon}{4\varepsilon}.$$

Then it is sufficient to take $\delta(\varepsilon)=\left|\dfrac{5-2\varepsilon}{4\varepsilon}\right|$.

3) *Using the definition of the limit of a function, we have*

$$\left(\lim_{x\to+\infty}\ln x=+\infty\right)\Longleftrightarrow\left(\forall A>0,\ \exists\delta(A)>0,\ \forall x\in\mathbb{R}:\ x>\delta(A)\Longrightarrow\ln x>A\right).$$

As

$$\ln x>A\Longleftrightarrow x>e^{A}.$$

Then it is sufficient to take $\delta(A)=e^{A}$.

4) *Using the definition of the limit of a function, we have*

$$\left(\lim_{x\to_{>}-3}\frac{4}{x+3}=+\infty\right)\Longleftrightarrow\left(\forall A>0,\ \exists\delta(A)>0,\ \forall x\in\mathbb{R}:\ -3<x<-3+\delta(A)\Longrightarrow\frac{4}{x+3}>A\right).$$

As

$$\frac{4}{x+3}>A\Longleftrightarrow x<\frac{4}{A}-3.$$

Then it is sufficient to take $\delta(A)=\dfrac{4}{A}$.

Exercice 3.5.4. *Calculate, if existing, the limits*

$$1)\ \lim_{x\to+\infty}\frac{[\ln\sqrt{x}]}{\sqrt{x}} \qquad 2)\ \lim_{x\to 0}\frac{\ln(1+x)-x}{x^2}$$

Proof 3.5.4. 1) *We have* $\forall x \in \mathbb{R}^+$

$$\left[\ln\sqrt{x}\right] \leq \ln\sqrt{x} < \left[\ln\sqrt{x}\right] + 1.$$

Then

$$\underbrace{\ln\sqrt{x} - 1 < \left[\ln\sqrt{x}\right] \leq \ln\sqrt{x}}_{\Downarrow}$$

$$\frac{\ln\sqrt{x}-1}{\sqrt{x}} < \frac{[\ln\sqrt{x}]}{\sqrt{x}} \leq \frac{\ln\sqrt{x}}{\sqrt{x}}.$$

AS $\lim\limits_{x\to+\infty}\dfrac{\ln\sqrt{x}-1}{\sqrt{x}} = 0$ *and* $\lim\limits_{x\to+\infty}\dfrac{\ln\sqrt{x}}{\sqrt{x}} = 0$, *then*

$$\lim_{x\to+\infty}\frac{[\ln\sqrt{x}]}{\sqrt{x}} = 0.$$

2) *We have*

$$\lim_{x\to 0}\frac{\ln(1+x)-x}{x^2} = \frac{0}{0}\ (I.F)$$

Using Hospital's rule, let

$$f(x) = \ln(1+x) - x \quad and \quad g(x) = x^2.$$

So

$$f'(x) = -\frac{x}{1+x} \quad and \quad g'(x) = 2x.$$

Then

$$\lim_{x\to 0}\frac{\ln(1+x)-x}{x^2} = \lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f'(x)}{g'(x)} = \lim_{x\to 0} -\frac{1}{1+x} = -1$$

Exercice 3.5.5. *Let* $f_n : \mathbb{R} \longrightarrow \mathbb{R}$ *be the application defined, for all* $n \in \mathbb{N}$, *by:*

$$f_n(x) = \ln(1+x^n) + x - 1$$

1) *Show that there exists* $c_n \in [0,1]$ *such that* $f_n(c_n) = 0$.

2) *Show that* f_n *is strictly increasing on* $\mathbb{R}^+$, *deduce that* c_n *is unique.*

Proof 3.5.5. 1) *f_n is a continuous function on $[0,1]$, $f_n(0) = -1 < 0$ and $f_n(1) = \ln 2 > 0$, according to the intermediate value theorem, there exists $c_n \in [0,1]$ such that $f_n(c_n) = 0$.*

2) *calculate the derivative of f_n*

$$f_n^{'}(x) = \frac{nx^{n-1}}{1+x^n} + 1 = \frac{nx^{n-1}+1+x^n}{1+x^n} > 0 \quad \forall x \in [0, +\infty[$$

so f_n is strictly increasing. Therefore the solution is unique.

Exercice 3.5.6. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function, such that*

$$f(x) = \begin{cases} \dfrac{3-x^2}{2} & if \quad x \leq 1 \\ \dfrac{1}{x} & if \quad x > 1 \end{cases}$$

1) *Show that there exists $c \in]0,2[$ such that $f(2) - f(0) = (2-0)f'(c)$.*

2) *Determine the possible values of c.*

Proof 3.5.6. 1) *firstly, we show that f is continuous on $[0,2]$.*

If $x \neq 1$, f is continuous.

If $x = 1$ we have

$$\lim_{x \to <1} f(x) = \lim_{x \to <1} \frac{3-x^2}{2} = 1 = f(1) \quad and \quad \lim_{x \to >1} f(x) = \lim_{x \to >1} \frac{1}{x} = 1 = f(1).$$

This shows that the function is continuous at $x = 1$.

secondly, we show that f is derivable on $]0,2[$. Likewise, if $x \neq 1$, f is derivable.

If $x = 1$ we have

$$\begin{aligned} \lim_{x \to <1} \frac{f(x) - f(1)}{x-1} &= \lim_{x \to <1} \frac{\frac{3-x^2}{2} - 1}{x-1} \\ &= \lim_{x \to <1} \frac{1+x}{-2} = -1 = f_l^{'}. \end{aligned}$$

and

$$\begin{aligned} \lim_{x \to >1} \frac{f(x) - f(1)}{x-1} &= \lim_{x \to >1} \frac{\frac{1}{x} - 1}{x-1} \\ &= \lim_{x \to >1} \frac{1}{-x} = -1 = f_r^{'}. \end{aligned}$$

This shows that the function is derivable at $x = 1$. Then using the finite increase theorem, there exists $c \in]0,2[$ such that $f(2) - f(0) = (2-0)f'(c)$.

2) *We have*

$$f(2) = \frac{1}{2} \quad and \quad f(0) = \frac{3}{2}$$

Consequently

$$f(2) - f(0) = (2-0)f'(c) \Longleftrightarrow f'(c) = -\frac{1}{2}$$

If $0 \leq c \leq 1$, *then*

$$f'(c) = -\frac{1}{2} \Longleftrightarrow -c = -\frac{1}{2} \Longleftrightarrow c = \frac{1}{2} \quad \textit{is a solution.}$$

If $1 < c \leq 2$, *then*

$$f'(c) = -\frac{1}{2} \Longleftrightarrow -\frac{1}{c^2} = -\frac{1}{2} \Longleftrightarrow c^2 = 2 \Longleftrightarrow c = \pm\sqrt{2}.$$

As $-\sqrt{2} < 1$, *so* $-\sqrt{2}$ *is not a solution. Therefore there are two solutions* $c = \frac{1}{2}$ *and* $c = \sqrt{2}$.

Exercice 3.5.7. *Let* $f : [0,1] \longrightarrow \mathbb{R}$ *be a function, such that*

$$f(x) = \begin{cases} \dfrac{1}{1+x} & if \quad 0 \leq x < \dfrac{1}{2} \\ 2x + \alpha x^2 & if \quad \dfrac{1}{2} \leq x \leq 1 \end{cases}$$

1) *Find, if they exist, the* $\alpha \in \mathbb{R}$ *so that* f *is continuous.*

2) *Find, if they exist, the* $\alpha \in \mathbb{R}$ *so that* f *is derivable.*

Proof 3.5.7. 1) *We have*

$$\lim_{x \to < \frac{1}{2}} f(x) = \lim_{x \to < \frac{1}{2}} \frac{1}{1+x} = \frac{2}{3}$$

and

$$\lim_{x \to > \frac{1}{2}} f(x) = \lim_{x \to < \frac{1}{2}} (2x + \alpha x^2) = 1 + \frac{\alpha}{4}.$$

Then

$$f \textit{ is continuous } \Longleftrightarrow 1 + \frac{\alpha}{4} = \frac{2}{3} \Longleftrightarrow \alpha = -\frac{4}{3}.$$

2) *A necessary condition for* f *to be derivable is for* f *to be continuous, so if there is a value of* α *for which* f *is derivable, it can only be* $\alpha = -\frac{4}{3}$. *i.e.*

$$f(x) = \begin{cases} \dfrac{1}{1+x} & if \quad 0 \leq x < \dfrac{1}{2} \\ 2x - \dfrac{4}{3}x^2 & if \quad \dfrac{1}{2} \leq x \leq 1 \end{cases}$$

If $0 \leq x < \frac{1}{2}$ *we have*

$$f'(x) = -\frac{1}{(1+x)^2}$$

and

$$\lim_{x \to < \frac{1}{2}} f'(x) = \lim_{x \to < \frac{1}{2}} -\frac{1}{(1+x)^2} = -\frac{4}{9}.$$

If $\frac{1}{2} \leq x \leq 1$ *we have*

$$f'(x) = 2 - \frac{8}{3}x$$

and

$$\lim_{x \to > \frac{1}{2}} f'(x) = \lim_{x \to < \frac{1}{2}} \left(2 - \frac{8}{3}x\right) = \frac{2}{3}.$$

As

$$\lim_{x \to < \frac{1}{2}} f'(x) \neq \lim_{x \to > \frac{1}{2}} f'(x),$$

then f *is not derivable at* $x = \frac{1}{2}$*. Consequently, there is no* $\alpha \in \mathbb{R}$ *such that the function* f *is derivable.*

Exercice 3.5.8. *Let* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *be a function, such that*

$$f(x) = \begin{cases} \dfrac{\sin(ax)}{x} & if \quad x < 0 \\ 1 & if \quad x = 0 \\ e^{bx} - x & if \quad x > 0 \end{cases}$$

1) *Using Hospital's rule, find the following limit*

$$\lim_{x \to 0} \frac{\cos(x)x - \sin(x)}{x^2}$$

2) *Find* a *so that* f *is continuous on* $\mathbb{R}$.

3) *Find* b *so that* f *is derivable on* $\mathbb{R}$.

Proof 3.5.8. 1) *We have*

$$\lim_{x \to 0} \frac{\cos(x)x - \sin(x)}{x^2} = \frac{0}{0} \ (I.F)$$

Using Hospital's rule, let

$$f(x) = \cos(x)x - \sin(x) \quad and \quad g(x) = x^2.$$

So

$$f'(x) = -\sin(x)x + \cos(x) - \cos(x) = -\sin(x)x \quad and \quad g'(x) = 2x.$$

Then

$$\lim_{x\to 0}\frac{\cos(x)x-\sin(x)}{x^2}=\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{f'(x)}{g'(x)}=\lim_{x\to 0}-\frac{\sin(x)}{2}=0$$

2) *We have*

$$\lim_{x\to_< 0} f(x)=\lim_{x\to_< 0}\frac{\sin(ax)}{x}=a$$

and

$$\lim_{x\to_> 0} f(x)=\lim_{x\to_> 0}(e^{bx}-x)=1.$$

Then

$$f \text{ is continuous } \Longleftrightarrow a=1.$$

3) *If* $x<0$, $f(x)=\dfrac{\sin(x)}{x}$. *So*

$$f'(x)=\frac{\cos(x)x-\sin(x)}{x^2}$$

and

$$\lim_{x\to_< 0} f'(x)=\lim_{x\to_< 0}\frac{\cos(x)x-\sin(x)}{x^2}=0.$$

If $x>0$, $f(x)=e^{bx}-x$. *So*

$$f'(x)=be^{bx}-1$$

and

$$\lim_{x\to_> 0} f'(x)=\lim_{x\to_> 0}(be^{bx}-1)=b-1.$$

Then

$$f \text{ is derivable } \Longleftrightarrow b-1=0 \Longleftrightarrow b=1.$$

Exercice 3.5.9. *In each of the following cases, say whether the application f is prolonged by continuity at a and give the prolongation by continuity where appropriate.*

1) $f:[-3,6[\cup]6,+\infty[\longrightarrow\mathbb{R}$ *defined by:*

$$f(x)=\frac{\sqrt{x+3}-3}{x-6} \quad \text{and} \quad a=6.$$

2) $f:]-\infty,0[\cup]0,+\infty[\longrightarrow\mathbb{R}$ *defined by:*

$$f(x)=\frac{\sqrt{1+x}-\sqrt{1-x}}{x} \quad \text{and} \quad a=0.$$

Proof 3.5.9. 1) *We have*

$$\begin{aligned}\lim_{x\to 6}\frac{\sqrt{x+3}-3}{x-6} &= \lim_{x\to 6}\frac{\sqrt{x+3}-3}{x-6}\frac{\sqrt{x+3}+3}{\sqrt{x+3}+3}\\ &= \lim_{x\to 6}\frac{x-6}{(x-6)(\sqrt{x+3}+3)}\\ &= \lim_{x\to 6}\frac{1}{\sqrt{x+3}+3}=\frac{1}{6}.\end{aligned}$$

So the continuity prolongation is defined by:

$$f(x)=\begin{cases}\dfrac{\sqrt{x+3}-3}{x-6} & if \quad x\neq 6\\ \dfrac{1}{6} & if \quad x=6\end{cases}$$

2) *We have*

$$\begin{aligned}\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x} &= \lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\\ &= \lim_{x\to 0}\frac{2x}{x(\sqrt{1+x}+\sqrt{1-x})}\\ &= \lim_{x\to 0}\frac{2}{\sqrt{1+x}+\sqrt{1-x}}=1.\end{aligned}$$

So the continuity prolongation is defined by:

$$f(x)=\begin{cases}\dfrac{\sqrt{1+x}-\sqrt{1-x}}{x} & if \quad x\neq 0\\ 1 & if \quad x=0\end{cases}$$

Exercice 3.5.10. *Let $f:\mathbb{R}\longrightarrow\mathbb{R}$ be a function, such that*

$$f(x)=\begin{cases}\dfrac{|x-1|}{x+1} & if \quad x>0\\ x^2-a & if \quad x\leq 0\end{cases}$$

1) *Find the value of parameter a needed for the function f to be continuous.*

2) *Study the derivability of the function f.*

Proof 3.5.10. 1) *Note that the function f can be written as*

$$f(x)=\begin{cases}\dfrac{x-1}{x+1} & if \quad x\geq 1\\ \dfrac{1-x}{x+1} & if \quad 0<x\leq 1\\ x^2-a & if \quad x\leq 0\end{cases}$$

let's calculate the limits on the right and left at 0. *We obtain*

$$\lim_{x \to <0} f(x) = \lim_{x \to <0} (x^2 - a) = -a$$

and

$$\lim_{x \to >0} f(x) = \lim_{x \to <0} \frac{1-x}{x+1} = 1.$$

So

$$f \text{ is continuous } \Longleftrightarrow -a = 1 \Longleftrightarrow a = -1.$$

2) *Let's calculate the limit of the rate of variation to the right and to the left at* 0 *and* 1. *We obtain*

$$\lim_{x \to <0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to <0} \frac{x^2 + 1 - 1}{x} = 0$$

and

$$\lim_{x \to >0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to >0} \frac{\frac{1-x}{x+1} - 1}{x} = \lim_{x \to >0} \frac{-2x}{x(x+1)} = -2.$$

Since the limits on the right and left of the rate of variation are different, then f is not derivable at 0.

For point 1, *we have*

$$\lim_{x \to <1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to <1} \frac{\frac{1-x}{x+1}}{x - 1} = -\frac{1}{x+1} = -\frac{1}{2}$$

and

$$\lim_{x \to >1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to >1} \frac{\frac{x-1}{x+1}}{x - 1} = \lim_{x \to >1} \frac{1}{x+1} = \frac{1}{2}.$$

Thus, f is not derivable at 1.

4

Integral calculus

4.1 Primitive concept

Definition 4.1.1. *Let* $f : I \longrightarrow \mathbb{R}$ *be a function defined on an interval* I *of* $\mathbb{R}$*. We call a primitive of* f *on* I *any function* $F : I \longrightarrow \mathbb{R}$ *which is derivable on* I *such that*

$$\forall x \in I \quad F'(x) = f(x).$$

Example 4.1.1. 1) *A primitive on* $]0, +\infty[$ *of the function* $f(x) = \dfrac{1}{x}$ *is the function* F *defined by*

$$F(x) = \ln x.$$

2) *A primitive on* $\mathbb{R}$ *of the function* $f(x) = \cos x$ *is the function* F *defined by*

$$F(x) = \sin x.$$

3) *A primitive on* $\mathbb{R}$ *of the exponential function is the exponential function itself.*

4) *A primitive on* $\mathbb{R}$ *of the function* $f(x) = x^n$ *is the function* F *defined by*

$$F(x) = \frac{x^{n+1}}{n+1}, \quad \forall n \in \mathbb{N}.$$

4.1.1 Existence of primitives

Theorem 4.1.1. *Let* $f : I \longrightarrow \mathbb{R}$ *be a function defined on an interval* I *of* $\mathbb{R}$. *If* f *is continuous on* I, *then* f *admits primitives on* I.

Remark 4.1.1. *In the previous theorem, the condition of continuity is sufficient for a function to admit primitives, while this condition is not necessary. For example, consider the function*

$$F(x) = \begin{cases} x \cos \frac{1}{x} & \text{if} \quad x \neq 0, \\ 0 & \text{if} \quad x = 0 \end{cases}$$

It is clear that F *is a primitive of the function*

$$f(x) = \begin{cases} \cos \frac{1}{x} - \frac{1}{x} \sin \frac{1}{x} & \text{if} \quad x \neq 0, \\ 0 & \text{if} \quad x = 0 \end{cases}$$

on any interval $[a, b]$, *while* f, *is not continuous at the point* $x = 0$.

Proposition 4.1.1. *Let* $f : I \longrightarrow \mathbb{R}$ *be a function defined on an interval* I *of* $\mathbb{R}$, *admitting a primitive* F *on* I. *Then another function* $G : I \longrightarrow \mathbb{R}$ *is a primitive of* f *if and only if there exists a constant* $k \in \mathbb{R}$ *such that*

$$G(x) = F(x) + k, \ \forall x \in I.$$

4.1.2 Classical primitives

The table below summarizes some popular primitives.

Table 4.1: Primitives of usual functions

Function f	Primitive F	Interval I
k	$kx+C$	$\mathbb{R}$
x^α, $\alpha \neq -1$	$\dfrac{x^{\alpha+1}}{\alpha+1}+C$	$]0, +\infty[$
$\dfrac{1}{x}$	$\ln\|x\|+C$	$\mathbb{R}^*$
$u'(x)\cdot u^\alpha(x)$, $\alpha \neq -1$	$\dfrac{u^{\alpha+1}(x)}{\alpha+1}+C$	$u(x)>0$
$\dfrac{u'(x)}{u(x)}$	$\ln\|u(x)\|+C$	$u(x)\neq 0$
e^x	e^x+C	$\mathbb{R}$
$\cos x$	$\sin x+C$	$\mathbb{R}$
$\cos(ax+b)$, $a\neq 0$, $b\in\mathbb{R}$	$\dfrac{1}{a}\sin(ax+b)+C$	$\mathbb{R}$
$\sin x$	$-\cos x+C$	$\mathbb{R}$
$\sin(ax+b)$, $a\neq 0$, $b\in\mathbb{R}$	$-\dfrac{1}{a}\cos(ax+b)+C$	$\mathbb{R}$
$\dfrac{1}{\cos^2 x}$	$\tan x+C$	$x\neq\dfrac{\pi}{2}+k\pi$, $k\in\mathbb{Z}$
$\dfrac{1}{\sqrt{x^2-1}}$	$\ln\left(x+\sqrt{x^2-1}\right)+C$	$]1, +\infty[$
$\dfrac{1}{\sqrt{x^2-k^2}}$, $k>0$	$\ln\left(x+\sqrt{x^2-k^2}\right)+C$	$]k, +\infty[$
$\dfrac{1}{\sqrt{x^2+1}}$	$\ln\left(x+\sqrt{x^2+1}\right)+C$	$\mathbb{R}$
$\dfrac{1}{\sqrt{x^2+k^2}}$, $k>0$	$\ln\left(x+\sqrt{x^2+k^2}\right)+C$	$\mathbb{R}$
$\dfrac{1}{x^2-k^2}$, $k>0$	$\dfrac{1}{2k}\ln\left\|\dfrac{x-k}{x+k}\right\|+C$	$\mathbb{R}-\{-1, 1\}$

4.1.3 Indefinite integral

Definition 4.1.2. *The family of all primitives of the function $f: I\subset\mathbb{R}\longrightarrow\mathbb{R}$, is called the indefinite integral of f, let it be denoted by:*

$$\int f(x)dx$$

Furthermore, if F is any primitive of f , we write

$$\int f(x)dx = F(x) + C$$

where C is an arbitrary constant.

Theorem 4.1.2. *(Properties) Let $f : I \longrightarrow \mathbb{R}$ and $g : I \longrightarrow \mathbb{R}$ be two functions defined on an interval I of $\mathbb{R}$. We assume that f admits on I a primitive F and that g admits on I a primitive G. Then, for all real numbers α and β, the function $\alpha F + \beta G$ is a primitive of $\alpha f + \beta g$ on I and write.*

$$\int (\alpha f(x) + \beta g(x))dx = \alpha \int f(x)dx + \beta \int g(x)dx = \alpha F(x) + \beta G(x) + C$$

Example 4.1.2. 1)

$$\int (7x^2 - e^x)dx = 7\int x^2 dx - \int e^x dx = \frac{7}{3}x^3 - e^x + C.$$

2)

$$\int \sqrt{x}dx = \int x^{\frac{1}{2}}dx = \frac{2}{3}x^{\frac{3}{2}}.$$

3)

$$\int \frac{e^{\frac{1}{x}}}{x^2}dx = -\int e^{\frac{1}{x}}\left(\frac{1}{x}\right)' dx = -e^{\frac{1}{x}} + C.$$

4)

$$\int \cos^2(x)dx = \int \left(\frac{1}{2} + \frac{\cos(2x)}{2}\right) dx = \frac{x}{2} + \frac{\sin(2x)}{4} + C.$$

4.1.4 Definite integral

Definition 4.1.3. *Let a and b be two real numbers such that $a \leq b$. Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function on the segment $[a, b]$ of $\mathbb{R}$. We call the integral of f on $[a, b]$, or the integral from a to b of f, denoted $\int_a^b f(t)dt$, the real number:*

$$\int_a^b f(t)dt = F(b) - F(a),$$

where F is a primitive of f on $[a, b]$.

Proposition 4.1.2. *(Chasles relation) Let a, b, c be three real numbers such that $a \leq c \leq b$. Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function on the segment $[a, b]$ of $\mathbb{R}$. We have:*

$$\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt$$

Proposition 4.1.3. *Let f and g be two continuous functions on a segment $[a, b]$ of $\mathbb{R}$. For all real numbers α and β, we have :*

$$\int_a^b (\alpha f(x) + \beta g(x))dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx$$

Example 4.1.3. *let's calculate the following integrals.*

$$\begin{aligned} I_1 &= \int_1^2 (2 - 4e^{3x})dx \\ &= \int_1^2 2dx - 4\int_1^2 e^{3x}dx \\ &= [2x]_1^2 - \frac{4}{3}\left[e^{3x}\right]_1^2 \\ &= 2 - \frac{4}{3}(e^6 - e^3). \end{aligned}$$

$$\begin{aligned} I_2 &= \int_0^1 \frac{x+1}{x^2+2x+5}dx \\ &= \frac{1}{2}\int_0^1 \frac{(x^2+2x+5)'}{x^2+2x+5}dx \\ &= \frac{1}{2}\left[\ln(x^2+2x+5)\right]_0^1 \\ &= \frac{1}{2}(\ln 8 - \ln 5) = \ln\sqrt{\frac{8}{5}}. \end{aligned}$$

$$\begin{aligned} I_3 &= \int_0^1 (2x+3)\sqrt{x^2+3x+4}dx \\ &= \int_0^1 (x^2+3x+4)^{\frac{1}{2}}(x^2+3x+4)'dx \\ &= \frac{2}{3}\left[(x^2+3x+4)^{\frac{3}{2}}\right]_0^1 \\ &= \frac{2}{3}(8^{\frac{3}{2}} - 4^{\frac{3}{2}}) = \frac{32}{3}\sqrt{2} - \frac{16}{3}. \end{aligned}$$

4.2 Methods of calculating integrals

4.2.1 Integration by parts

Theorem 4.2.1. *Let u and v be two functions of class C^1 on an interval $[a,b]$. Then:*

1) *For a primitive calculation*

$$\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx.$$

2) *For calculating a definite integral*

$$\int_a^b u(x)v'(x)dx = [u(x)v(x)]_a^b - \int_a^b u'(x)v(x)dx.$$

Example 4.2.1. *let's calculate the following integrals.*

1)

$$I_1 = \int_0^{\frac{\pi}{2}} e^{-2x}\cos x dx.$$

We do a first integration by parts with:

$$\begin{cases} u'(x) = \cos x \\ v(x) = e^{-2x} \end{cases} \text{ hence } \begin{cases} u(x) = \sin x \\ v'(x) = -2e^{-2x} \end{cases}$$

It comes

$$I_1 = \left[e^{-2x}\sin x\right]_0^{\frac{\pi}{2}} + 2\int_0^{\frac{\pi}{2}} e^{-2x}\sin x dx = e^{-\pi} + 2J_1, \text{ avec } J_1 = \int_0^{\frac{\pi}{2}} e^{-2x}\sin x dx.$$

We do a second integration by parts with:

$$\begin{cases} w'(x) = \sin x \\ v(x) = e^{-2x} \end{cases} \text{ hence } \begin{cases} w(x) = -\cos x \\ v'(x) = -2e^{-2x} \end{cases}$$

It comes

$$J_1 = \left[-e^{-2x}\cos x\right]_0^{\frac{\pi}{2}} - 2\int_0^{\frac{\pi}{2}} e^{-2x}\cos x dx = 1 - 2I_1.$$

So

$$I_1 = e^{-\pi} + 2J_1 = e^{-\pi} + 2(1 - 2I_1),$$

then

$$I_1 = \frac{1}{5}(e^{-\pi} + 2).$$

2)

$$I_2 = \int x^2 e^x dx.$$

We do a first integration by parts with:

$$\begin{cases} u'(x) = e^x \\ v(x) = x^2 \end{cases} \text{hence} \begin{cases} u(x) = e^x \\ v'(x) = 2x \end{cases}$$

It comes

$$I_2 = x^2 e^x - 2\int xe^x dx = x^2 e^x - 2J_2, \text{ avec } J_2 = \int xe^x dx.$$

We do a second integration by parts with:

$$\begin{cases} w'(x) = e^x \\ v(x) = x \end{cases} \text{hence} \begin{cases} w(x) = e^x \\ v'(x) = 1 \end{cases}$$

It comes

$$J_2 = xe^x - \int e^x dx = (x-1)e^x + c.$$

Then

$$I_2 = x^2 e^x - 2(x-1)e^x + c = (x^2 - 2x + 2)e^x + c.$$

4.2.2 Change of variable

Theorem 4.2.2. *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function on $[a, b]$ and*

$$\phi : [\alpha, \beta] \longrightarrow [a, b]$$

a bijective function of class C^1 on $[\alpha, \beta]$. Then the composite function

$$t \longmapsto f(\phi(t))\phi'(t)$$

is integrable on $[\alpha, \beta]$, moreover

1) *For a primitive calculation*

$$\int f(x)dx = \int f(\phi(t))\phi'(t)dt.$$

2) *For calculating a definite integral*

$$\int_a^b f(x)dx = \int_{\phi^{-1}(a)}^{\phi^{-1}(b)} f(\phi(t))\phi'(t)dt.$$

Example 4.2.2. *let's calculate the following integrals.*

1)

$$I_1 = \int_0^1 \frac{x}{\sqrt{x^2+1}}dx.$$

Let's consider $\phi : [0,1] \longrightarrow [1,2]$ *defined by* $\phi(x) = x^2+1$, *it is of class* C^1 *and verifies* $\phi'(x) = 2x$. *So*

$$I_1 = \int_0^1 \frac{\phi'(x)}{2\sqrt{\phi(x)}}dx = \left[\sqrt{\phi(x)}\right]_0^1 = \sqrt{2}-1.$$

2)

$$I_2 = \int_{-1}^1 \sqrt{1-x^2}dx.$$

Let's consider $\phi : [-\frac{\pi}{2}, \frac{\pi}{2}] \longrightarrow [-1,1]$ *defined by* $\phi(t) = \sin t$, *it is of class* C^1 *and verifies* $\phi'(t) = \cos t$. *So*

$$I_2 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1-\sin^2 t}\cos t dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}(\cos 2t+1)dt = \left[\frac{1}{4}\sin 2t + \frac{1}{2}t\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi}{2}.$$

3)

$$I_3 = \int \frac{e^x}{(e^x+1)^2}dx.$$

Let's consider

$$t = e^x + 1 \quad \Rightarrow \quad dt = e^x dx$$

This gives

$$I_3 = \int \frac{1}{t^2}\,dt = \int t^{-2}\,dt = -\frac{1}{t},$$

Returning to the initial variable, we obtain

$$I_3 = -\frac{1}{e^x+1} + C.$$

4.2.3 Rational fraction integration

▶ **Rational fractions of the form** $\frac{c}{(x-a)^n}$

If the rational function is of the form $f(x) = \frac{c}{(x-a)^n}$ with $a, c \in \mathbb{R}$, $n \in \mathbb{N}^*$. The primitives are of the form:

$$F(x) = \begin{cases} \frac{c}{(1-n)(x-a)^{n-1}} + C & \text{if} \quad n > 1, \\ c\ln|x-a| + C & \text{if} \quad n = 1 \end{cases}$$

on each of the intervals $]-\infty, a[$ and $]a, +\infty[$.

Example 4.2.3. *let's calculate the following integral.*

$$I = \int \frac{2}{(x-2)^2}\,dx,$$

we have $n = 2 > 1$ *and* $c = a = 2$*. Then*

$$I = \frac{2}{-(x-2)} + C = \frac{2}{2-x} + C.$$

▶ **Rational fractions of the form** $\frac{ax+b}{cx+d}$

If the rational function is of the form $f(x) = \frac{ax+b}{cx+d} = \frac{a}{c} + \frac{bc-ad}{c(cx+d)}$ with $a, b, c, d \in \mathbb{R}$, $a \neq 0,\ c \neq 0$. The primitives are of the form:

$$F(x) = \frac{a}{c}x + \frac{bc-ad}{c^2}\ln\left|x + \frac{d}{c}\right| + C,$$

on each of the intervals $]-\infty, -\frac{d}{c}[$ and $]-\frac{d}{c}, +\infty[$.

Example 4.2.4. *let's calculate the following integral.*

$$I = \int \frac{6x}{3x+9}\,dx,$$

we have $a = 6,\ b = 0,\ c = 3$ *and* $d = 9$*. Then*

$$I = \frac{6}{3}x - \frac{54}{9}\ln|x + \frac{9}{3}| + C = 2x - 6\ln|x+3| + C.$$

▶ **Rational fractions of the form** $\frac{1}{ax^2+bx+c}$

If the rational function is of the form $f(x) = \frac{1}{ax^2+bx+c}$ with $a, b, c \in \mathbb{R},\ a \neq 0$. We distinguish three cases:

• If $ax^2 + bx + c$ has two distinct real roots $r_1 < r_2$, then there exist $\lambda, \mu \in \mathbb{R}$ such that:

$$f(x) = \frac{\lambda}{x - r_1} + \frac{\mu}{x - r_2},$$

and therefore

$$F(x) = \lambda \ln|x - r_1| + \mu \ln|x - r_2| + C,$$

on each of the intervals $]-\infty, r_1[$, $]r_1, r_2[$ and $]r_2, +\infty[$.

- If $ax^2 + bx + c$ has a double root $r \in \mathbb{R}$, then

$$f(x) = \frac{1}{a(x-r)^2},$$

and therefore

$$F(x) = \frac{-1}{a(x-r)} + C,$$

on each of the intervals $]-\infty, r[$ and $]r, +\infty[$.

- If $ax^2 + bx + c$ has no real root, there are two real numbers

$$\alpha = \frac{b}{2a}, \text{ and } \beta = \frac{\sqrt{-(b^2-4ac)}}{2a}$$

such that:

$$ax^2 + bx + c = a[(x+\alpha)^2 + \beta^2].$$

We obtain:

$$f(x) = \frac{1}{a\beta^2\left[\left(\frac{x+\alpha}{\beta}\right)^2 + 1\right]},$$

and therefore

$$F(x) = \frac{1}{a\beta}\arctan\left(\frac{x+\alpha}{\beta}\right) + C, \ \forall x \in \mathbb{R}.$$

Example 4.2.5. *let's calculate the following integral.*

$$I = \int \frac{1}{x^2+2x-3}\,dx,$$

Note that the denominator has two distinct roots $x_1 = 1$ and $x_2 = -3$. So

$$I = \int \frac{1}{(x-1)(x+3)}\,dx,$$

Decomposing the fraction $\frac{1}{(x-1)(x+3)}$ into simple elements, we obtain

$$\frac{1}{(x-1)(x+3)} = \frac{1/4}{x-1} - \frac{1/4}{x+3}.$$

As a consequence,

$$\begin{aligned} I &= \int \frac{1/4}{x-1}dx - \int \frac{1/4}{x+3}dx \\ &= \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+3| + C. \end{aligned}$$

▶ Rational fractions of the form $\dfrac{cx+d}{x^2+px+q}$

If the rational function is of the form $\dfrac{cx+d}{x^2+px+q}$ with $c, d, p, q \in \mathbb{R}$ such that $p^2 - 4q < 0$. The idea is to make the derivative of the denominator appear in the numerator by writing:

$$cx + d = \frac{c}{2}(2x+p) + d - \frac{cp}{2},$$

so that

$$f(x) = \frac{c}{2}\frac{2x+p}{x^2+px+q} + \left(d - \frac{cp}{2}\right)\frac{1}{x^2+px+q},$$

and therefore

$$F(x) = \frac{c}{2}\ln(x^2+px+q) + \frac{1}{\beta}\left(d - \frac{cp}{2}\right)\arctan\left(\frac{x+\alpha}{\beta}\right) + C,\ \forall x \in \mathbb{R},$$

with $\alpha = \dfrac{p}{2},\ \beta = \dfrac{\sqrt{-p^2+4q}}{2}$.

Example 4.2.6. *let's calculate the following integral.*

$$I = \int \frac{x-1}{x^2-5x+6}\,dx,$$

Note that the denominator has two distinct roots $x_1 = 2$ and $x_2 = 3$. So

$$I = \int \frac{x-1}{(x-2)(x-3)}\,dx,$$

Decomposing the fraction $\dfrac{x-1}{(x-2)(x-3)}$ into simple elements, we obtain

$$\frac{x-1}{(x-2)(x-3)} = \frac{-1}{x-2} + \frac{2}{x-3}.$$

As a consequence,

$$\begin{aligned} I &= \int \frac{-1}{x-2}dx + \int \frac{2}{x-3}dx \\ &= \ln|2-x| + 2\ln|x-3| + C. \end{aligned}$$

▶ **Rational fractions of the form** $\dfrac{cx+d}{(x^2+qx+p)^n}$

If the rational function is of the form $\dfrac{cx+d}{(x^2+px+q)^n}$ with $c,d,p,q\in\mathbb{R}$, $n\in\mathbb{N}^*$ and $p^2-4q<0$. We start by showing the derivative $2x+p$ of the polynomial x^2+px+q in the numerator. By identification, we have

$$cx+d=\frac{c}{2}(2x+p)+d-\frac{c}{2}p.$$

It comes

$$\frac{cx+d}{(x^2+px+q)^n}=\frac{\frac{c}{2}(2x+p)}{(x^2+px+q)^n}+\frac{d-\frac{c}{2}p}{(x^2+px+q)^n}$$

Therefore,

$$\int\frac{cx+d}{(x^2+px+q)^n}\,dx=\frac{c}{2}\underbrace{\int\frac{2x+p}{(x^2+px+q)^n}\,dx}_{①}+\left(d-\frac{c}{2}p\right)\underbrace{\int\frac{1}{(x^2+px+q)^n}\,dx}_{②}$$

The integral ① is of the form $\displaystyle\int\frac{u'(x)}{u^n(x)}\,dx$, so

$$①=\int\frac{2x+p}{(x^2+px+q)^n}\,dx=\begin{cases}\ln\left(x^2+px+q\right)+C & \text{if} \quad n=1,\\[2ex] \dfrac{-1}{(n-1)\left(x^2+px+q\right)^{n-1}}+C & \text{if} \quad n\neq 1.\end{cases}$$

To calculate the integral ②, first write the trinomial x^2+px+q, in the canonical form, i.e.

$$x^2+px+q=\left(x+\frac{p}{2}\right)^2+q-\frac{p^2}{4}=(x+\alpha)^2+\beta^2,$$

where

$$\alpha=\frac{p}{2},\quad \beta=\sqrt{q-\frac{p^2}{4}}.$$

It is clear that

$$②=\int\frac{1}{(x^2+px+q)^n}\,dx=\int\frac{1}{\left((x+\alpha)^2+\beta^2\right)^n}\,dx$$

Consequently, the integral ② is reduced, after the change of variable $x+\alpha=\beta t$, to integral calculation

$$J_n=\int\frac{1}{(t^2+1)^n}\,dt.$$

The calculation of J_n is performed by integration by parts. Indeed,

$$\begin{cases} u(t) = \dfrac{1}{(t^2+1)^n} \\ \\ v'(t) = 1 \end{cases} \Rightarrow \begin{cases} u'(t) = -2n\dfrac{t}{(t^2+1)^{n+1}} \\ \\ v(t) = t \end{cases}$$

We'll have

$$\begin{aligned} J_n &= \frac{t}{(t^2+1)^n} + 2n\int \frac{t^2}{(t^2+1)^{n+1}}\,dt \\ &= \frac{t}{(t^2+1)^n} + 2n\int \frac{(t^2+1)-1}{(t^2+1)^{n+1}}\,dt \\ &= \frac{t}{(t^2+1)^n} + 2n\left(J_n - J_{n+1}\right) \end{aligned}$$

Consequently, we find the recurrence relation:

$$2nJ_{n+1} = \frac{t}{(t^2+1)^n} + (2n-1)\,J_n.$$

So the whole calculation comes down to

$$J_1 = \int \frac{1}{t^2+1}\,dt = \arctan t + C.$$

Example 4.2.7. *let's calculate the following integral.*

$$I = \int \frac{1}{(x^2+1)^2}\,dx,$$

Note that

$$\frac{1}{(x^2+1)^2} = \frac{x^2+1}{(x^2+1)^2} - \frac{x^2}{(x^2+1)^2}$$

. So

$$I = \int \frac{x^2+1}{(x^2+1)^2}\,dx - \int \frac{x^2}{(x^2+1)^2}\,dx = I_1 - I_2.$$

Where

$$I_1 = \int \frac{x^2+1}{(x^2+1)^2}\,dx = \int \frac{1}{x^2+1}\,dx = \arctan x + C_1$$

and

$$I_2 = \int \frac{x^2}{(x^2+1)^2}\,dx.$$

The calculation of I_2 is performed by integration by parts. Indeed,

$$\begin{cases} u(t) = x \\ v'(t) = \dfrac{x}{(x^2+1)^2} \end{cases} \Rightarrow \begin{cases} u'(t) = 1 \\ v(t) = -\dfrac{1}{2}\dfrac{1}{x^2+1} \end{cases}$$

We'll have

$$I_2 = -\frac{1}{2}\frac{x}{x^2+1} + \int \frac{1}{2}\frac{1}{x^2+1}dx = -\frac{1}{2}\frac{x}{x^2+1} + \frac{1}{2}\arctan x + C_2$$

As a consequence,

$$I = \arctan x + \frac{1}{2}\frac{x}{x^2+1} - \frac{1}{2}\arctan x + C = \frac{1}{2}\frac{x}{x^2+1} + \frac{1}{2}\arctan x + C.$$

▶ **Rational fractions in** $\cos$ **and** $\sin$

Let

$$I = \int R(\cos x, \sin x)\, dx,$$

where R is a rational function in $\cos x$ and $\sin x$.
The integral is reduced, after the change of variable $t = \tan\dfrac{x}{2}$, to the integration of a rational function. Then we find the formula,

$$\cos x = \frac{1-t^2}{1+t^2}, \quad \sin x = \frac{2t}{1+t^2}, \quad dx = \frac{2}{1+t^2}dt.$$

Consequently, the integral I becomes

$$I = 2\int \frac{1}{1+t^2} \cdot R\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right) dt.$$

Example 4.2.8. *let's calculate the following integral.*

$$I = \int \frac{1}{\cos x + 1}\, dx,$$

we pose:

$$t = \tan\frac{x}{2} \quad \Rightarrow \quad \begin{cases} \cos x = \dfrac{1-t^2}{1+t^2} \\ dx = \dfrac{2}{1+t^2}dt \end{cases}$$

This leads to

$$I = \int \frac{1}{\frac{1-t^2}{1+t^2}+1}\frac{2}{1+t^2}dt = \int dt = t + C = \tan\frac{x}{2} + C.$$

▶ **Special cases:**

• **First case**

$$I_1 = \int R(\cos x) \sin x \, dx,$$

where R is a rational function in $\cos x$. We make the change of variable

$$t = \cos x \quad \Rightarrow \quad dt = -sinxdx$$

Consequently, the integral I_1 becomes

$$I_1 = -\int R(t) \, dt.$$

Example 4.2.9. *let's calculate the following integral.*

$$I = \int \frac{-\sin x}{\cos x - 1} \, dx,$$

we pose:

$$t = \cos x \quad \Rightarrow \quad dt = -\sin x dx,$$

This leads to

$$\begin{aligned} I &= \int \frac{-\sin x}{t-1} \frac{dt}{-\sin x} \\ &= \ln|t-1| + C \\ &= \ln|\cos x - 1| + C. \end{aligned}$$

• **Second case**

$$I_2 = \int R(\sin x) \cos x \, dx,$$

where R is a rational function in $\sin x$. We make the change of variable

$$t = \sin x \quad \Rightarrow \quad dt = \cos x dx$$

Consequently, the integral I_2 becomes

$$I_2 = \int R(t) \, dt.$$

Example 4.2.10. *let's calculate the following integral.*

$$I = \int \frac{\cos x}{2\sin x + 3}\,dx,$$

we pose:

$$t = \sin x \quad \Rightarrow \quad dt = \cos x dx,$$

This leads to

$$\begin{aligned} I &= \int \frac{\cos x}{2t+3}\frac{dt}{\cos x} \\ &= \frac{1}{2}\ln|2t+3| + C \\ &= \frac{1}{2}\ln|2\sin x + 3| + C. \end{aligned}$$

▶ **Rational fractions in** e^x

Let

$$I = \int R\left(e^x\right)\,dx,$$

where R is a rational function in e^x.

We make the change of variable

$$t = e^x \quad \Rightarrow \quad x = \ln t, \quad \text{et} \quad dx = \frac{1}{t}dt$$

Consequently, the integral I becomes

$$I = \int \frac{R(t)}{t}\,dt.$$

Example 4.2.11. *let's calculate the following integral.*

$$I = \int \frac{e^{2x}}{2e^x + 4}\,dx,$$

we pose:

$$t = e^x \quad \Rightarrow \quad x = \ln t, \quad et \quad dx = \frac{1}{t}dt,$$

This leads to

$$\begin{aligned}
I &= \int \frac{t^2}{2t+4}\frac{dt}{t} \\
&= \frac{1}{2}\int \frac{2t}{2t+4}dt \\
&= \frac{1}{2}\left[\int \frac{2t+4}{2t+4}dt - \int \frac{4}{2t+4}dt\right] \\
&= \frac{t}{2} - 2\ln|2t+4| + C \\
&= \frac{e^x}{2} - 2\ln|2e^x+4| + C.
\end{aligned}$$

4.3 Exercises

Exercice 4.3.1. *Evaluate each of the following integrals:*

$$I_1 = \int \left(3\sqrt[4]{x^3} + \frac{7}{x^5} + \frac{1}{6\sqrt{x}}\right) dx, \quad I_2 = \int (x + \sqrt[3]{x})(4 - x^2)dx, \quad I_3 = \int \sin 3x dx,$$

$$I_4 = \int_0^1 \frac{x+1}{x^2+2x+5} dx, \quad I_5 = \int \sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) dx, \quad I_6 = \int_{\frac{\pi}{2}}^{\pi} (1+\cos x)dx.$$

Proof 4.3.1. 1)

$$\begin{aligned} I_1 &= \int \left(3\sqrt[4]{x^3} + \frac{7}{x^5} + \frac{1}{6\sqrt{x}}\right) dx \\ &= \int \left(3x^{\frac{3}{4}} + 7x^{-5} + \frac{1}{6}x^{-\frac{1}{2}}\right) dx \\ &= \frac{12}{7}x^{\frac{7}{4}} - \frac{7}{4}x^{-4} + \frac{1}{3}x^{\frac{1}{2}} + C. \end{aligned}$$

2)

$$\begin{aligned} I_2 &= \int (x + \sqrt[3]{x})(4 - x^2)dx \\ &= \int (4x - x^3 + 4x^{\frac{1}{3}} - x^{\frac{7}{3}})dx \\ &= 2x^2 - \frac{1}{4}x^4 + 3x^{\frac{4}{3}} - \frac{3}{10}x^{\frac{10}{3}} + C. \end{aligned}$$

3)

$$\begin{aligned} I_3 &= \int \sin 3x dx \\ &= -\frac{1}{3}\cos 3x + C. \end{aligned}$$

4)

$$\begin{aligned} I_4 &= \int_0^1 \frac{x+2}{x^2+4x+3} dx \\ &= \frac{1}{2}\int_0^1 \frac{(x^2+4x+3)'}{x^2+4x+3} dx \\ &= \frac{1}{2}\left[\ln(x^2+4x+3)\right]_0^1 = \ln\frac{8}{3}. \end{aligned}$$

5)

$$I_5 = \int \sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) dx$$

Recall the following double angle formula.

$$\sin 2x = 2\sin x\cos x.$$

A small rewrite of this formula gives,

$$\sin x\cos x = \frac{1}{2}\sin 2x.$$

If we now replace all the x's with $\frac{x}{2}$ *we get,*

$$\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) = \frac{1}{2}\sin x.$$

Then the integral becomes

$$I_5 = \int \frac{1}{2}\sin x dx = -\frac{1}{2}\cos x + C$$

6)

$$\begin{aligned} I_6 &= \int_{\frac{\pi}{2}}^{\pi}(1+\cos x)dx \\ &= [x+\sin x]_{\frac{\pi}{2}}^{\pi} \\ &= \frac{\pi}{2} - 1. \end{aligned}$$

Exercice 4.3.2. *Using integration by parts, evaluate each of the following integrals:*

$$I_1 = \int (3x+5)\cos\left(\frac{x}{4}\right)dx, \quad I_2 = \int x^2\sin(10x)dx, \quad I_3 = \int x^5\sqrt{x^3+1}dx,$$

$$I_4 = \int_1^e x^n \ln x dx \ (n\in\mathbb{N}), \quad I_5 = \int_0^1 (x^2+x)e^{2x}dx, \quad I_6 = \int_2^3 \frac{x}{\sqrt{x-1}}dx.$$

Proof 4.3.2. 1)

$$I_1 = \int (3x+5)\cos\left(\frac{x}{4}\right)dx.$$

Let's use the following choices:

$$\begin{cases} u(x) = 3x+5 \\ v'(x) = \cos\left(\frac{x}{4}\right) \end{cases} \Rightarrow \begin{cases} u'(x) = 3 \\ v(x) = 4\sin\left(\frac{x}{4}\right) \end{cases}$$

The integral is then,

$$\begin{aligned} I_1 &= 4\,(3x+5)\sin\left(\frac{x}{4}\right) - 12\int \sin\left(\frac{x}{4}\right)dx \\ &= 4\,(3x+5)\sin\left(\frac{x}{4}\right) + 48\cos\left(\frac{x}{4}\right) + C. \end{aligned}$$

2)

$$I_2 = \int x^2 \sin(10x)dx.$$

Let's use the following choices:

$$\begin{cases} u(x) = x^2 \\ v'(x) = \sin(10x) \end{cases} \Rightarrow \begin{cases} u'(x) = 2x \\ v(x) = -\dfrac{1}{10}\cos(10x) \end{cases}$$

The integral is then,

$$\int x^2 \sin(10x)dx = -\frac{x^2}{10}\cos(10x) + \frac{1}{5}\int x\cos(10x)dx.$$

The new integral will also require integration by parts. For this second integral we will use the following choices:

$$\begin{cases} u(x) = x \\ v'(x) = \cos(10x) \end{cases} \Rightarrow \begin{cases} u'(x) = 1 \\ v(x) = \dfrac{1}{10}\sin(10x) \end{cases}$$

So, the integral becomes,

$$\begin{aligned} I_2 &= -\frac{x^2}{10}\cos(10x) + \frac{1}{5}\left(\frac{x}{10}\sin(10x) - \frac{1}{10}\int \sin(10x)dx\right) \\ &= -\frac{x^2}{10}\cos(10x) + \frac{1}{5}\left(\frac{x}{10}\sin(10x) - \frac{1}{100}\cos(10x)dx\right) + C \\ &= -\frac{x^2}{10}\cos(10x) + \frac{x}{50}\sin(10x) - \frac{1}{500}\cos(10x)dx + C. \end{aligned}$$

3)

$$I_3 = \int x^5\sqrt{x^3+1}dx.$$

Let's use the following choices:

$$\begin{cases} u(x) = x^3 \\ v'(x) = x^2\sqrt{x^3+1} \end{cases} \Rightarrow \begin{cases} u'(x) = 3x \\ v(x) = \dfrac{2}{9}(x^3+1)^{\frac{3}{2}} \end{cases}$$

The integral is then,

$$\begin{aligned} I_3 &= \frac{2}{9}x^3(x^3+1)^{\frac{3}{2}} - \frac{2}{3}\int x^2(x^3+1)^{\frac{3}{2}}dx \\ &= \frac{2}{9}x^3(x^3+1)^{\frac{3}{2}} - \frac{4}{45}(x^3+1)^{\frac{5}{2}} + C. \end{aligned}$$

4)

$$I_4 = \int_1^e x^n \ln x dx.$$

Let's use the following choices:

$$\begin{cases} u(x) = \ln x \\ v'(x) = x^n \end{cases} \Rightarrow \begin{cases} u'(x) = \dfrac{1}{x} \\ v(x) = \dfrac{x^{n+1}}{n+1} \end{cases}$$

The integral is then,

$$\begin{aligned} I_4 &= \left[\frac{x^{n+1}}{n+1}\ln x\right]_1^e - \int_1^e \frac{x^{n+1}}{n+1}\frac{1}{x}dx \\ &= \frac{e^{n+1}}{n+1} - \left[\frac{x^{n+1}}{(n+1)^2}\right]_1^e \\ &= \frac{e^{n+1}}{n+1} - \frac{e^{n+1}-1}{(n+1)^2} = \frac{ne^{n+1}+1}{(n+1)^2}. \end{aligned}$$

5)

$$I_5 = \int_0^1 (x^2+x)e^{2x}dx$$

Let

$$J_n = \int_0^1 x^n e^{2x}dx.$$

Let's use integration by parts to calculate J_n. So our choices for u and v' would be the following.

$$\begin{cases} u(x) = x^n \\ v'(x) = e^{2x} \end{cases} \Rightarrow \begin{cases} u'(x) = nx^{n-1} \\ v(x) = \dfrac{e^{2x}}{2} \end{cases}$$

The integral is then,

$$\begin{aligned} J_n &= \left[x^n\frac{e^{2x}}{2}\right]_0^1 - \int_0^1 nx^{n-1}\frac{e^{2x}}{2}dx \\ &= \frac{e^2}{2} - \frac{n}{2}J_{n-1} \end{aligned}$$

First, we calculate J_0, then using the recurrence relation. So

$$J_0 = \int_0^1 e^{2x} dx = \frac{e^2 - 1}{2}$$

$$J_1 = \frac{e^2}{2} - \frac{J_0}{2} = \frac{e^2}{4} + \frac{1}{4}$$

$$J_2 = \frac{e^2}{2} - \frac{2J_1}{2} = \frac{e^2}{4} - \frac{1}{4}$$

Finally we note that

$$I_5 = J_2 + J_1 = \frac{e^2}{2}.$$

6)

$$I_6 = \int_2^3 \frac{x}{\sqrt{x-1}} dx.$$

Let's use the following choices:

$$\begin{cases} u(x) = x \\ v'(x) = (x-1)^{-\frac{1}{2}} \end{cases} \Rightarrow \begin{cases} u'(x) = 1 \\ v(x) = 2(x-1)^{\frac{1}{2}} \end{cases}$$

The integral is then,

$$\begin{aligned} I_6 &= \left[2x(x-1)^{\frac{1}{2}}\right]_2^3 - 2\int_2^3 (x-1)^{\frac{1}{2}} dx \\ &= \left[2x(x-1)^{\frac{1}{2}}\right]_2^3 - \frac{4}{3}\left[(x-1)^{\frac{3}{2}}\right]_2^3 \\ &= \frac{10}{3}\sqrt{2} - \frac{8}{3}. \end{aligned}$$

Exercice 4.3.3. *Using the change of variable, calculate the following integrals:*

$$I_1 = \int_1^{e^2} \frac{\ln x}{x + x\ln^2 x} dx, \quad I_2 = \int_0^1 \frac{e^{2x}}{e^x + 1} dx, \quad I_3 = \int_1^e \frac{1}{x\sqrt{\ln x + 1}} dx,$$

$$I_4 = \int_0^{\frac{1}{2}} \frac{x}{(1-x^2)^{\frac{3}{2}}} dx, \quad I_5 = \int_1^2 \frac{1}{x(x^3+1)} dx, \quad I_6 = \int \sin(1-x)(2-\cos(1-x))^4 dx.$$

Proof 4.3.3. 1)

$$I_1 = \int_1^{e^2} \frac{\ln x}{x + x\ln^2 x} dx.$$

Let's use the following substitution:

$$t = \ln x \quad \Rightarrow \quad dt = \frac{dx}{x} \quad \Rightarrow \quad dx = e^t dt,$$

then

$$\begin{cases} x = e^2 \\ x = 1 \end{cases} \quad \Rightarrow \quad \begin{cases} t = 2 \\ t = 0 \end{cases}$$

The integral is then,

$$\begin{aligned} I_1 &= \int_0^2 \frac{t}{e^t + t^2 e^t} e^t dt \\ &= \int_0^2 \frac{t}{1+t^2} dt \\ &= \frac{1}{2}\left[\ln(1+t^2)\right]_0^2 = \ln\sqrt{5} \end{aligned}$$

2)

$$I_2 = \int_0^1 \frac{e^{2x}}{e^x + 1} dx.$$

Let's use the following substitution:

$$t = e^x \quad \Rightarrow \quad dt = e^x dx = t dx \quad \Rightarrow \quad dx = \frac{dt}{t},$$

then

$$\begin{cases} x = 1 \\ x = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} t = e \\ t = 1 \end{cases}$$

The integral is then,

$$\begin{aligned} I_2 &= \int_1^e \frac{t^2}{t+1} \frac{dt}{t} \\ &= \int_1^e \frac{t}{t+1} dt \\ &= \int_1^e \frac{t+1-1}{t+1} dt \\ &= \int_1^e \left(1 - \frac{1}{t+1}\right) dt \\ &= [t - \ln(t+1)]_1^e = e - \ln(e+1) - 1 + \ln 2. \end{aligned}$$

3)

$$I_3 = \int_1^e \frac{1}{x\sqrt{\ln x + 1}} dx.$$

Let's use the following substitution:

$$t = \sqrt{\ln x + 1} \quad \Rightarrow \quad dt = \frac{1}{2}\frac{1}{t}\frac{dx}{x} \quad \Rightarrow \quad dx = 2xtdt,$$

then

$$\begin{cases} x = e \\ x = 1 \end{cases} \quad \Rightarrow \quad \begin{cases} t = \sqrt{2} \\ t = 1 \end{cases}$$

The integral is then,

$$\begin{aligned} I_3 &= \int_1^{\sqrt{2}} \frac{1}{xt} 2xtdt \\ &= 2\int_1^{\sqrt{2}} dt = [2t]_1^{\sqrt{2}} = 2(\sqrt{2} - 1). \end{aligned}$$

4)

$$I_4 = \int_0^{\frac{1}{2}} \frac{x}{(1 - x^2)^{\frac{3}{2}}} dx.$$

Let's use the following substitution:

$$t = 1 - x^2 \quad \Rightarrow \quad dt = -2xdx \quad \Rightarrow \quad dx = -\frac{1}{2x}dt,$$

then

$$\begin{cases} x = \frac{1}{2} \\ x = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} t = \frac{3}{4} \\ t = 1 \end{cases}$$

The integral is then,

$$\begin{aligned} I_4 &= \int_1^{\frac{3}{4}} \frac{x}{t^{\frac{3}{2}}} \frac{-1}{2x} dt \\ &= \frac{-1}{2}\int_1^{\frac{3}{4}} \frac{1}{t^{\frac{3}{2}}} dt = \frac{-1}{2}\left[-2t^{-\frac{1}{2}}\right]_1^{\frac{3}{4}} = \frac{2}{\sqrt{3}} - 1. \end{aligned}$$

5)

$$I_5 = \int_1^2 \frac{1}{x(x^3 + 1)} dx.$$

Let's use the following substitution:

$$t = x^3 + 1 \quad \Rightarrow \quad x^3 = t - 1 \quad \Rightarrow \quad dt = 3x^2 dx \quad \Rightarrow \quad dx = \frac{1}{3x^2} dt,$$

then

$$\begin{cases} x = 2 \\ x = 1 \end{cases} \Rightarrow \begin{cases} t = 9 \\ t = 2 \end{cases}$$

The integral is then,

$$\begin{aligned} I_5 &= \int_2^9 \frac{1}{xt}\frac{1}{3x^2} dt \\ &= \frac{1}{3}\int_2^9 \frac{1}{tx^3} dt \\ &= \frac{1}{3}\int_2^9 \frac{1}{t(t-1)} dt. \end{aligned}$$

Since

$$\frac{1}{t(t-1)} = \frac{1}{t-1} - \frac{1}{t},$$

then

$$\begin{aligned} I_5 &= \frac{1}{3}\left(\int_2^9 \frac{1}{t-1} dt - \int_2^9 \frac{1}{t} dt\right) \\ &= \frac{1}{3}\left([\ln|t-1|]_2^9 - [\ln|t|]_2^9\right) \\ &= \frac{1}{3}\left(4\ln 2 - 2\ln 3\right). \end{aligned}$$

6)

$$I_6 = \int \sin(1-x)(2-\cos(1-x))^4 dx.$$

Let's use the following substitution:

$$t = 2 - \cos(1-x) \quad \Rightarrow \quad dt = -\sin(1-x)dx \quad \Rightarrow \quad dx = -\frac{dt}{\sin(1-x)},$$

The integral is then,

$$\begin{aligned} I_6 &= \int \sin(1-x) t^4 \frac{-dt}{\sin(1-x)} \\ &= -\int t^4 dt = -\frac{1}{5}t^5 + C = -\frac{1}{5}(2-\cos(1-x))^5 + C. \end{aligned}$$

Exercice 4.3.4. *Evaluate each of the following integrals:*

$$I_1 = \int \frac{x^2}{(x-2)(x-3)}dx, \quad I_2 = \int \frac{2x-1}{x(x-1)^2}dx, \quad I_3 = \int \frac{x^7+1}{x^2-1}dx,$$

$$I_4 = \int \frac{5x^2-2x+3}{(x^2+1)(x-1)}dx, \quad I_5 = \int \frac{2x+1}{(x-1)(x-2)^2}dx, \quad I_6 = \int \frac{x^2+1}{x(x-1)(x^2-2x+4)}dx.$$

Proof 4.3.4. 1)

$$I_1 = \int \frac{x^2}{(x-2)(x-3)}dx.$$

We can see that

$$\frac{x^2}{(x-2)(x-3)} = \frac{x^2-5x+6+5x-6}{x^2-5x+6} = 1 + \frac{5x-6}{x^2-5x+6}$$

Using the simple element decomposition for $\frac{5x-6}{x^2-5x+6}$*, we find*

$$\frac{5x-6}{x^2-5x+6} = \frac{5x-6}{(x-2)(x-3)} = \frac{a}{x-2} + \frac{b}{x-3}$$

where $a = -4$ *and* $b = 9$*. The integral is then,*

$$\begin{aligned} I_1 &= \int \left(1 - \frac{4}{x-2} + \frac{9}{x-3}\right) dx \\ &= x - 4\ln|x-2| + 9\ln|x-3| + C. \end{aligned}$$

2)

$$I_2 = \int \frac{2x-1}{x(x-1)^2}dx.$$

Using the simple element decomposition for $\frac{2x-1}{x(x-1)^2}$*, we find*

$$\frac{2x-1}{x(x-1)^2} = \frac{a}{x} + \frac{b}{x-1} + \frac{c}{(x-1)^2}$$

where $a = -1$, $b = 1$ *and* $c = 1$*. The integral is then,*

$$\begin{aligned} I_2 &= \int \left(\frac{-1}{x} + \frac{1}{x-1} + \frac{1}{(x-1)^2}\right) dx \\ &= -\ln|x| + \ln|x-1| - \frac{1}{x-1} + C. \end{aligned}$$

3)

$$I_3 = \int \frac{x^7+1}{x^2-1}dx.$$

Using the Euclidean division of x^7+1 by x^2-1, we obtain

$$x^7+1 = (x^2-1)Q + R,$$

where $Q = x^5 + x^3 + x$ and $R = x+1$. The integral is then,

$$\begin{aligned} I_3 &= \int \left(x^5 + x^3 + x + \frac{1}{x-1}\right) dx \\ &= \frac{x^6}{6} + \frac{x^4}{4} + \frac{x^2}{2} + \ln|x-1| + C. \end{aligned}$$

4)

$$I_4 = \int \frac{5x^2-2x+3}{(x^2+1)(x-1)}dx.$$

Using the simple element decomposition for $\frac{5x^2-2x+3}{(x^2+1)(x-1)}$, we find

$$\frac{5x^2-2x+3}{(x^2+1)(x-1)} = \frac{a}{x-1} + \frac{bx+c}{x^2+1}$$

where $a = 3$, $b = 2$ and $c = 0$. The integral is then,

$$\begin{aligned} I_4 &= \int \left(\frac{3}{x-1} + \frac{2x}{x^2+1}\right) dx \\ &= 3\ln|x-1| + \ln|x^2+1| + C. \end{aligned}$$

5)

$$I_5 = \int \frac{2x+1}{(x-1)(x-2)^2}dx.$$

Using the simple element decomposition for $\frac{2x+1}{(x-1)(x-2)^2}$, we find

$$\frac{2x+1}{(x-1)(x-2)^2} = \frac{a}{x-1} + \frac{b}{x-2} + \frac{c}{(x-2)^2}$$

where $a = 3$, $b = -3$ and $c = 5$. The integral is then,

$$\begin{aligned} I_5 &= \int \left(\frac{3}{x-1} - \frac{3}{x-2} + \frac{5}{(x-2)^2}\right) dx \\ &= 3\ln|x-1| - 3\ln|x-2| - \frac{5}{x-2} + C. \end{aligned}$$

6)

$$I_6 = \int \frac{x^2+1}{x(x-1)(x^2-2x+4)}dx.$$

Using the simple element decomposition for $\dfrac{x^2+1}{x(x-1)(x^2-2x+4)}$*, we find*

$$\frac{x^2+1}{x(x-1)(x^2-2x+4)} = \frac{a}{x} + \frac{b}{x-1} + \frac{cx+d}{x^2-2x+4}$$

where $a = -\frac{1}{4}$, $b = \frac{2}{3}$, $c = -\frac{5}{12}$ *and* $d = \frac{7}{6}$*. The integral is then,*

$$\begin{aligned} I_6 &= \int \left(\frac{-\frac{1}{4}}{x} + \frac{\frac{2}{3}}{x-1} + \frac{-\frac{5}{12}x + \frac{7}{6}}{x^2-2x+4} \right) dx \\ &= -\frac{1}{4}\ln|x| + \frac{2}{3}\ln|x-1| - \frac{5}{12}\int \frac{x - \frac{14}{5}}{x^2-2x+4}dx. \end{aligned}$$

Now let's calculate the integral $\int \frac{x-\frac{14}{5}}{x^2-2x+4}dx$*. We can show that*

$$\begin{aligned} \frac{x-\frac{14}{5}}{x^2-2x+4} &= \frac{1}{2}\frac{2x-\frac{28}{5}}{x^2-2x+4} = \frac{1}{2}\frac{2x-2+2-\frac{28}{5}}{x^2-2x+4} \\ &= \frac{1}{2}\left(\frac{2x-2}{x^2-2x+4} + \frac{2-\frac{28}{5}}{x^2-2x+4} \right) \\ &= \frac{1}{2}\frac{2x-2}{x^2-2x+4} - \frac{9}{5}\frac{1}{x^2-2x+4}. \end{aligned}$$

Then

$$\int \frac{x-\frac{14}{5}}{x^2-2x+4}dx = \frac{1}{2}\ln|x^2-2x+4| - \frac{9}{5}\int \frac{1}{x^2-2x+4}dx$$

We still have to calculate the integral $\int \frac{1}{x^2-2x+4}dx$*. Let's use the following substitution:*

$$t = \frac{x-1}{\sqrt{3}} \quad \Rightarrow \quad dt = \frac{dx}{\sqrt{3}} \quad \Rightarrow \quad dx = \sqrt{3}dt,$$

so

$$\int \frac{1}{x^2-2x+4}dx = \frac{2}{\sqrt{12}}\int \frac{1}{t^2+1}dt = \frac{1}{\sqrt{3}}\arctan t + C = \frac{1}{\sqrt{3}}\arctan\frac{x-1}{\sqrt{3}} + C.$$

Finally

$$I_6 = -\frac{1}{4}\ln|x| + \frac{2}{3}\ln|x-1| - \frac{5}{24}\ln|x^2-2x+4| + \frac{\sqrt{3}}{4}\arctan\frac{x-1}{\sqrt{3}} + C.$$

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