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Sciences and Technology Faculty
Department of Common Core Sciences and Technology

Course handout

Maths 2 (Analysis & Algebra 2)

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This course is intended for first year LMD sciences and technology students.

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Preface

This Analysis and Algebra course is intended primarily for first-year LMD Science and Technology students, as well as first-year LMD Material Sciences and Mathematics and Computer Science students.

It covers the official Analysis and Algebra program, namely:

- ▷ Matrices and determinants.
- ▷ Systems of linear equations.
- ▷ The integrals.
- ▷ Differential equations.
- ▷ Functions of several variables.

Each chapter re-establishes the essential bases for approaching scientific studies, and introduces some new concepts, most of which will be covered during this year.

This course is covered in detail with many examples. Most of the theorems and propositions are demonstrated.

At the end of each chapter we provide a list of exercises with their solutions.

As well as courses that I taught from 2017 to 2025 for first-year LMD Science and Technology students within the Science and Technology Common Core Department of the Faculty of Science and Technology.

Finally, errors may be found, please report them to the author.

The author

CHAPTER

1

MATRICES AND DETERMINANTS

In this chapter, $(\mathbb{K}, +, \cdot)$ denotes a commutative field, in practice $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

1.1 Generalities on Matrices

Definition 1.1. *An $m \times n$ matrix is an array of numbers with m rows and n columns. The numbers that make up the matrix are called the elements of the matrix (or also the coefficients). A matrix with m rows and n columns is called a matrix of order (m, n) or of dimension $m \times n$ and can be written as*

$$A = (a_{ij})_{\substack{1\leq i\leq m \\ 1\leq j\leq n}} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{pmatrix}.$$

Notation 1.

(∗) *The set of matrices of m rows and n columns with coefficients in $\mathbb{K}$ is denoted by $M_{m,n}(\mathbb{K})$.*

(∗) *If $m = n$, the matrix A is called a square matrix of order m and the set of square matrices of order m is denoted by $M_m(\mathbb{K})$.*

(∗) *A square matrix whose elements outside the diagonal are all zero (some elements of the diagonal may also be zero) is called a diagonal matrix.*

(∗) *For all* $(m,n) \in (\mathbb{N}^*)^2, 1 \leq i \leq m, 1 \leq j \leq n$, *the matrix of* $M_{m,n}(\mathbb{K})$ *whose* $(ij)^{th}$ *term is* 1 *and all the others are zero is called an elementary matrix, we denote it by* E_{ij}.

(∗) *We define the trace of the matrix* A, *denoted* $tr(A)$ *by:* $tr(A) = \sum_{i=1}^{n} a_{ii}$.

Example 1.2.

1 *The following matrix*

$$A = \begin{pmatrix} 4 & 5 & -1 & 0 \\ -1 & 0 & 2 & 0 \\ \sqrt{2} & 0 & 5 & -3 \end{pmatrix}$$

is a matrix of 3 *rows and* 4 *columns,* $A \in M_{3,4}(\mathbb{R})$, *and we have:* $a_{13} = -1$ *and* $a_{31} = \sqrt{2}$.

2 *The following matrix*

$$B = \begin{pmatrix} \mathbf{4} & -1 & 0 \\ 2 & -\mathbf{7} & 3 \\ 0 & \sqrt{5} & -\mathbf{6} \end{pmatrix}$$

is a square matrix of order 3, *the diagonal of the matrix* B *is the sequence of elements in Bold.*

3 *The matrix* C *below is a diagonal matrix*

$$C = \begin{pmatrix} 4 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & \sqrt{3} \end{pmatrix}.$$

4 *For the matrix* D *given by*

$$D = \begin{pmatrix} -4 & 2 & 1 \\ 0 & 5 & \sqrt{7} \\ 1 & 4 & 3 \end{pmatrix},$$

we provide its trace by

$$tr(D) = \sum_{i=1}^{3} a_{ii} = a_{11} + a_{22} + a_{33} = -4 + 5 + 3 = 4.$$

1.1.1 Matrices Algebra

▷ **Equality of two matrices:** Let $A = (a_{ij})$ and $B = (b_{ij}) \in M_{m,n}(\mathbb{K})$, we say that $A = B$ if all elements of A are equal to the corresponding elements of B:

$$A = B \Longleftrightarrow a_{ij} = b_{ij}, \quad \forall 1 \leq i \leq m, \forall 1 \leq j \leq n.$$

Example 1.3. *We give*

$$E = \begin{pmatrix} 2x+3 & 5 \\ 3 & 2y-3 \end{pmatrix} \text{ and } F = \begin{pmatrix} -1 & 5 \\ 3 & 5 \end{pmatrix}.$$

Let us determine x and y so that the two matrices E and F are equal.

$$E = F \Longleftrightarrow \begin{cases} 2x+3 = -1 \\ 2y-3 = 5 \end{cases} \Longleftrightarrow \begin{cases} x = -2 \\ y = 4 \end{cases}$$

▷ **Sum of two matrices:** Let $A = (a_{ij})$ and $B = (b_{ij}) \in M_{m,n}(\mathbb{K})$. We define the sum of A and B and we denote $A + B$ the matrix:

$$A + B = (c_{ij}) \in M_{m,n}(\mathbb{K}) \text{ such that } c_{ij} = a_{ij} + b_{ij}, \quad \forall 1 \leq i \leq m, \forall 1 \leq j \leq n.$$

Example 1.4.

$$\begin{pmatrix} 4 & -1 & 0 \\ 2 & -3 & 7 \end{pmatrix} + \begin{pmatrix} -3 & -1 & 4 \\ 0 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 4 \\ 2 & -1 & 6 \end{pmatrix}.$$

▷ **Multiplication of a matrix by a scalar:** Let $A = (a_{ij}) \in M_{m,n}(\mathbb{K})$ and $\lambda \in \mathbb{K}$. The product of A with λ is a matrix of the same dimension as A whose each element is the product of λ with the corresponding element of A:

$$\lambda A = \lambda(a_{ij}) = (\lambda a_{ij}), 1 \leq i \leq m, 1 \leq j \leq n.$$

Example 1.5. *Let*

$$A = \begin{pmatrix} 4 & a \\ b & -1 \end{pmatrix} \text{ and } \lambda \in \mathbb{R}, \text{ then, } \lambda A = \begin{pmatrix} 4\lambda & a\lambda \\ b\lambda & -\lambda \end{pmatrix}.$$

Remark 1.6. *By taking $\lambda = -1$, we can define the opposite matrix of a matrix A. It is the matrix $(-1) \times A$ which we also denote by $-A$. Similarly, we define the difference of two matrices A and B : $A - B = A + (-1) \times B$.*

Example 1.7. *Let A and B be the matrices defined by:*

$$A = \begin{pmatrix} 2 & -1 \\ 0 & -4 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 1 \\ -5 & -3 \end{pmatrix}.$$

The opposite of B : $-B = \begin{pmatrix} 0 & -1 \\ 5 & 3 \end{pmatrix}$ and the difference between A and B is

$$A - B = \begin{pmatrix} 2 & -2 \\ 5 & -1 \end{pmatrix}.$$

Example 1.8. *We are given:* $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & -3 \\ 1 & 5 \end{pmatrix}$.
Let $X \in M_2(\mathbb{R})$ *such that* $2X + 3A = B$. *Determine the matrix* X.

$$\begin{aligned} 2X + 3A = B &\iff 2X = B - 3A \\ &\iff X = \frac{1}{2}(B - 3A) \\ &\iff X = \frac{1}{2}\begin{pmatrix} -2 & 0 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}. \end{aligned}$$

▷ **Product of two matrices:** Let $A = (a_{ij}) \in M_{m,n}(\mathbb{K})$ and $B = (b_{jk}) \in M_{n,p}(\mathbb{K})$. (That is, the number of columns of A is equal to the number of rows of B). Then, we define the product $A \times B$ as the matrix of dimension $m \times p$ obtained by multiplying each row of A with each column of B. More precisely, the coefficient of the i^{th} row and the j^{th} column of $A \times B$ is obtained by multiplying the i^{th} row of A with the j^{th} column of B.

$$C = (c_{ik}) \in M_{m,n}(\mathbb{K}) = A \times B \in M_{m,p}(\mathbb{K}) \text{ such that } C = (c_{ik}) = \sum_{j=1}^{p} a_{ij} b_{jk}.$$

Example 1.9. *Let*

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 2 & 3 \\ 4 & 1 \end{pmatrix} \text{ and } C = \begin{pmatrix} 0 & 1 & 2 \\ 5 & 4 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$

So,

$$\begin{aligned} A \times B &= \begin{pmatrix} 1 \times 1 + 2 \times 2 + 3 \times 4 & 1 \times 0 + 2 \times 3 + 3 \times 1 \\ 4 \times 1 + 5 \times 2 + 6 \times 4 & 4 \times 0 + 5 \times 3 + 6 \times 1 \end{pmatrix} \\ &= \begin{pmatrix} 17 & 9 \\ 38 & 21 \end{pmatrix}. \end{aligned}$$

$$\begin{aligned} B \times A &= \begin{pmatrix} 1 \times 1 + 0 \times 4 & 1 \times 2 + 0 \times 5 & 1 \times 3 + 0 \times 6 \\ 2 \times 1 + 3 \times 4 & 2 \times 2 + 3 \times 5 & 2 \times 3 + 3 \times 6 \\ 4 \times 1 + 1 \times 4 & 4 \times 2 + 1 \times 5 & 4 \times 3 + 1 \times 6 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 \\ 14 & 19 & 24 \\ 8 & 13 & 18 \end{pmatrix}. \end{aligned}$$

$$\begin{aligned} A \times C &= \begin{pmatrix} 1 \times 0 + 2 \times 5 + 3 \times 2 & 1 \times 1 + 2 \times 4 + 3 \times 1 & 1 \times 2 + 2 \times 3 + 3 \times 3 \\ 4 \times 0 + 5 \times 5 + 6 \times 2 & 4 \times 1 + 5 \times 4 + 6 \times 1 & 4 \times 2 + 5 \times 3 + 6 \times 3 \end{pmatrix} \\ &= \begin{pmatrix} 16 & 12 & 17 \\ 37 & 30 & 41 \end{pmatrix}. \end{aligned}$$

$C \times A$ is undefined because the number of columns of C is different from the number of rows of A.

Remark 1.10.

* *If the number of columns of A is different from the number of rows of B, then the product $A \times B$ is not defined.*
* *In general, when the product is well defined, we have: $A \times B \neq B \times A$.*
* *The product of square matrices of order m is always defined.*

Definition 1.11. *Let A be a square matrix of order m and let p be a non-zero natural integer. We denote by A^p the matrix defined by:*

$$A^p = \underbrace{A \times A \times \cdots \times A}_{p \text{ times the matrix } A}.$$

Warning !!! The calculation of A^2, for example, does not consist of squaring the elements of A !

Example 1.12. *Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, we have:*

$$\begin{aligned} A^2 &= A \times A \\ &= \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \times \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix} \neq \begin{pmatrix} 1^2 & 2^2 \\ 3^2 & 4^2 \end{pmatrix}. \end{aligned}$$

Properties 1.13. *Let A, B and C be three matrices. When the product is well defined, we have:*

(i) *$A + B = B + A$, the commutativity of matrix addition.*

(ii) *$(A + B) + C = A + (B + C)$, the associativity of matrix addition.*

(iii) *$\lambda(A + B) = \lambda A + \lambda B, \quad (\lambda + \lambda')A = \lambda A + \lambda' A, \quad \lambda(\lambda' A) = (\lambda\lambda')A$.*

(iv) *$A \times (B + C) = A \times B + A \times C$, the left distributivity of matrix multiplication of matrices on addition.*

(v) *$(A + B) \times C = A \times C + B \times C$, the right distributivity of matrix multiplication of matrices on addition.*

(vi) *$(A \times B) \times C = A \times (B \times C)$, associativity of matrix multiplication.*

(vii) *$tr(A \times B) = tr(B \times A)$.*

1.1.2 Special matrices

Definition 1.14. *The matrix whose elements are all zero is called a* **zero** *matrix:*

$$A = \begin{pmatrix} 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 \end{pmatrix}.$$

Definition 1.15. *We call the* **identity** *matrix of order m, the square matrix whose diagonal terms a_{ii} are all equal to 1 and $a_{ij} = 0$ if $i \neq j$. We denote it by I_m and we write:*

$$I_m = \begin{pmatrix} 1 & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 1 \end{pmatrix}.$$

We have for all $A \in M_m(\mathbb{R})$, $AI_m = I_mA = A$.

Example 1.16.

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Definition 1.17. *We call a* **lower triangular** *matrix of order m, the square matrix whose coefficients $a_{ij} = 0$ for $i < j$.*

Example 1.18.

$$A = \begin{pmatrix} 6 & 0 & 0 & 0 \\ -2 & 5 & 0 & 0 \\ \sqrt{3} & 5 & 1 & 0 \\ 7 & 11 & 0 & \frac{1}{6} \end{pmatrix}.$$

Definition 1.19. *We call an* **upper triangular** *matrix of order m, the square matrix whose coefficients $a_{ij} = 0$ for $i > j$.*

Example 1.20.

$$A = \begin{pmatrix} -3 & 2 & 1 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Definition 1.21. *Let $A = (a_{ij}) \in M_{m,n}(\mathbb{K})$. We call* **the transpose** *of A the matrix denoted ${}^tA$ in $M_{n,m}(\mathbb{K})$ defined by: ${}^tA = (a_{ji})$.*

Example 1.22. *Let*

$$A = \begin{pmatrix} 4 & 6 & -1 \\ 2 & 3 & 5 \end{pmatrix}, \ then, \ {}^tA = \begin{pmatrix} 4 & 2 \\ 6 & 3 \\ -1 & 5 \end{pmatrix}.$$

Definition 1.23. *A square matrix A of order m is called* **symmetric** *if and only if ${}^tA = A$.*

Example 1.24. *Let*

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 5 & 6 \\ 3 & 5 & 4 & -1 \\ 4 & 6 & -1 & 3 \end{pmatrix}.$$

Definition 1.25. *A square matrix A of order m is called* **anti-symmetric** *if and only if ${}^tA = -A$.*

Example 1.26. *Let*

$$A = \begin{pmatrix} 0 & 4 & 2 \\ -4 & 0 & -5 \\ -2 & 5 & 0 \end{pmatrix}.$$

Definition 1.27. *A square matrix A of order m is called* **invertible** *if and only if there exists a square matrix A' of order m such that $AA' = A'A = I_m$.*

If A is invertible, then, A' is unique and it is called the inverse of A, denoted A^{-1}.

Example 1.28. *Let*

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

We have $A \in M_3(\mathbb{R})$, therefore $A^{-1} \in M_3(\mathbb{R})$ i.e. it is of the following form:

$$A^{-1} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

We have:

$$AA^{-1}=I_3 \iff \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\iff \begin{pmatrix} a_{11}+2a_{21} & a_{12}+2a_{22} & a_{13}+2a_{23} \\ a_{21}+a_{31} & a_{22}+a_{32} & a_{23}+a_{33} \\ a_{11}+a_{31} & a_{12}+a_{32} & a_1+a_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\iff (i)\begin{cases} a_{11}+2a_{21}=1 \\ a_{21}+a_{31}=0 \\ a_{11}+a_{31}=0 \end{cases} \quad (ii)\begin{cases} a_{12}+2a_{22}=0 \\ a_{22}+a_{32}=1 \\ a_{12}+a_{32}=0 \end{cases} \quad (iii)\begin{cases} a_{13}+2a_{23}=0 \\ a_{23}+a_{33}=0 \\ a_1+a_{33}=1 \end{cases}$$

$$\iff (i)\begin{cases} a_{11}=\frac{1}{3} \\ a_{21}=\frac{1}{3} \\ a_{31}=-\frac{1}{3} \end{cases} \quad (ii)\begin{cases} a_{12}=-\frac{2}{3} \\ a_{22}=\frac{1}{3} \\ a_{32}=\frac{2}{3} \end{cases} \quad (iii)\begin{cases} a_{13}=\frac{2}{3} \\ a_{23}=-\frac{1}{3} \\ a_{33}=\frac{1}{3} \end{cases}$$

Finally,

$$A^{-1}=\frac{1}{3}\begin{pmatrix} 1 & -2 & -2 \\ 1 & 1 & -1 \\ -1 & 2 & 1 \end{pmatrix}.$$

Remark 1.29.

(i) *A non-invertible square matrix is called a* **singular** *matrix.*

(ii) *Let A be an invertible matrix. Then, A^{-1} is also invertible and we have:*

$$(A^{-1})^{-1}=A.$$

Proposition 1.30. *Let A and B be two invertible matrices of the same order (size). Then, AB is invertible and we have: $(AB)^{-1}=B^{-1}A^{-1}$.*

Proof: It is sufficient to show that $(B^{-1}A^{-1})(AB)=I$ and that $(AB)(B^{-1}A^{-1})=I$:

$$(B^{-1}A^{-1})(AB)=B^{-1}(AA^{-1})B=B^{-1}IB^{-1}B^{-1}B=I,$$

and

$$(AB)(B^{-1}A^{-1})=A(BB^{-1})A^{-1}=AIA^{-1}=AA^{-1}=I.$$

Proposition 1.31.

- $\triangleright$ $\forall A\in M_{m,n}(\mathbb{K}),\ {}^t({}^tA)=A.$
- $\triangleright$ $\forall \lambda\in\mathbb{K},\ \forall A\in M_{m,n}(\mathbb{K}),\ {}^t(\lambda A)=\lambda({}^tA).$
- $\triangleright$ $\forall A,B\in M_{m,n}(\mathbb{K}),\ {}^t(A+B)={}^tA+{}^tB.$
- $\triangleright$ $\forall A\in M_{m,n}(\mathbb{K}),\ \forall B\in M_{n,p}(\mathbb{K}),\ {}^t(AB)={}^tB\,{}^tA.$
- $\triangleright$ $\forall A\in M_m(\mathbb{K})$ *invertible*, ${}^t(A^{-1})=({}^tA)^{-1}.$

1.2 Determinants

Let $A = (a_{ij}) \in M_m(\mathbb{K})$, the determinant of A is an element of $\mathbb{K}$, that caracterises some properties of A. it is denoted $det(A)$ or $|A|$.

How to calculate $|A|$?

- Determinant of order 2: Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_2(\mathbb{K})$, then,

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

Example 1.32.

$$A = \begin{pmatrix} 1 & -2 \\ 3 & 5 \end{pmatrix} \Longrightarrow |A| = (1 \times 5) - (-2 \times 3) = 11.$$

- Determinant of order 3: Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in M_3(\mathbb{K})$, then,

▷ **First method:** Development according the first row:

$$|A| = \begin{vmatrix} \overset{+}{a_{11}} & \overset{-}{a_{12}} & \overset{+}{a_{13}} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

▷ **Second method:** Development according to the first column:

$$|A| = \begin{vmatrix} \overset{+}{a_{11}} & a_{12} & a_{13} \\ \overset{-}{a_{21}} & a_{22} & a_{23} \\ \overset{+}{a_{31}} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}.$$

Example 1.33. *Let* $A = \begin{pmatrix} 2 & 1 & 4 \\ 3 & 0 & 5 \\ 1 & 7 & 6 \end{pmatrix} \in M_3(\mathbb{R})$, *then,*

$$\begin{aligned} det(A) &= \begin{vmatrix} 2^+ & 1^- & 4^+ \\ 3 & 0 & 5 \\ 1 & 7 & 6 \end{vmatrix} = 2 \begin{vmatrix} 0 & 5 \\ 7 & 6 \end{vmatrix} - 1 \begin{vmatrix} 3 & 5 \\ 1 & 6 \end{vmatrix} + 4 \begin{vmatrix} 3 & 0 \\ 1 & 7 \end{vmatrix} \\ &= 2[(0)(6) - (5)(7)] - [(3)(6) - (5)(1)] + 4[(3)(7) - (0)(1)] = 1. \end{aligned}$$

Or

$$\begin{aligned} det(A) &= \begin{vmatrix} 2^+ & 1 & 4 \\ 3^- & 0 & 5 \\ 1^+ & 7 & 6 \end{vmatrix} = 2 \begin{vmatrix} 0 & 5 \\ 7 & 6 \end{vmatrix} - 3 \begin{vmatrix} 1 & 4 \\ 7 & 6 \end{vmatrix} + 1 \begin{vmatrix} 1 & 4 \\ 0 & 5 \end{vmatrix} \\ &= 2[(0)(6) - (5)(7)] - 3[(1)(6) - (4)(7)] + [(1)(5) - (4)(0)] = 1. \end{aligned}$$

Remark 1.34. *It is preferable to calculate the determinant following the row or column that contains many zeros.*

▷ **Sarrus method:** here is the formula for the determinant:

$$det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}.$$

There is an easy way to remember this formula: copy the first two columns to the right of the matrix (gray columns), then add the products of three terms by grouping them according to the direction of the descending diagonal, and then subtract the products of three terms grouped according to the direction of the ascending diagonal.

Example 1.35.

$$A = \begin{pmatrix} 2 & 1 & 4 \\ 3 & 0 & 5 \\ 1 & 7 & 6 \end{pmatrix} \in M_3(\mathbb{R}), \quad then, \quad \begin{pmatrix} 2 & 1 & 4 & \mathbf{2} & \mathbf{1} \\ 3 & 0 & 5 & \mathbf{3} & \mathbf{0} \\ 1 & 7 & 6 & \mathbf{1} & \mathbf{7} \end{pmatrix}.$$

$$det(A) = (2)(0)(6)+(1)(5)(1)+(4)(3)(7)-(1)(0)(4)-(7)(5)(2)-(6)(3)(1) = 1.$$

Warning !!! This method does not apply to matrices larger than 3. We will see other methods that apply to square matrices of all sizes and therefore also to 3×3 matrices.

- General case: we denote by A_{ij} the matrix of order $(n-1)$ deduced from A by removing the i^{th} row and the j^{th} column. We call the determinant of A developed according to the i^{th} row the scalar:

$$\begin{aligned} det(A) &= (-1)^{i+1}a_{i1}det(A_{i1}) + (-1)^{i+2}a_{i2}det(A_{i2}) + \cdots + (-1)^{i+n}a_{in}det(A_{in}) \\ &= \sum_{k=1}^{n}(-1)^{i+k}a_{ik}det(A_{ik}). \end{aligned}$$

We designate by the determinant of A developed according to the j^{th} column the scalar:

$$\begin{aligned} det(A) &= (-1)^{1+j}a_{1j}det(A_{1j}) + (-1)^{2+j}a_{2j}det(A_{2j}) + \cdots + (-1)^{n+j}a_{in}det(A_{nj}) \\ &= \sum_{k=1}^{n}(-1)^{k+j}a_{kj}det(A_{kj}). \end{aligned}$$

Properties 1.36.

1/ $det(I_m) = 1$.

2/ $\forall \alpha \in \mathbb{K}, \forall A \in M_m(\mathbb{K}) : det(\alpha A) = \alpha^m det(A)$.

3/ $\forall A, B \in M_m(\mathbb{K}) : det(AB) = det(A)det(B)$.

4/ *A is invertible* $\Longleftrightarrow det(A) \neq 0$ *and* $det(A^{-1}) = \dfrac{1}{det(A)}$.

5/ $\forall A \in M_m(\mathbb{K}) : det({}^tA) = det(A)$.

6/ $det(A) = 0$ *if A has two equal columns (rows).*

7/ $det(A) = 0$ *if one of the columns (rows) of A is a linear combination of several other columns (rows).*

8/ *The value of the determinant of A does not change if we add to a column (row) a linear combination of other columns (rows).*

9/ *A determinant is zero if one of its columns (rows) is zero.*

Example 1.37.

$$(i)\ \begin{vmatrix} 2 & 4 & 2 \\ 3 & 4 & 6 \\ 2 & 4 & 2 \end{vmatrix} = 0 \quad \textit{because} \quad L_1 = L_3.$$

$$(ii)\ \begin{vmatrix} 2 & 4 & 2 \\ -3 & 13 & -3 \\ 5 & -24 & 5 \end{vmatrix} = 0 \quad \textit{because} \quad C_1 = C_3.$$

$$(iii)\ \begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 2 & 4 & 6 \end{vmatrix} = 0 \quad \textit{because} \quad L_3 = 2L_1 \ \textit{(Linear combination)}.$$

Definition 1.38. *Let* $A = (a_{ij}) \in M_m(\mathbb{K})$. *We call* **the cofactor** *of the place* (i, j) *in A and we denote by* c_{ij} *the number*

$$c_{ij} = (-1)^{1+j} det(A_{ij}),$$

where A_{ij} *is the matrix of order* $(m-1)$ *deduced from A by removing the* i^{th} *row and the* j^{th} *column.*

Definition 1.39. *Let* $A = (a_{ij}) \in M_m(\mathbb{K})$. *We call* **comatrix** *of A the square matrix of order m defined by:*

$$com(A) = \begin{pmatrix} c_{11} & \dots & c_{m1} \\ \vdots & \dots & \vdots \\ c_{m1} & \dots & c_{mm} \end{pmatrix}.$$

Where c_{ij} *is the cofactor of the place* (i, j) *in A.*

Theorem 1.40. *Let $A = (a_{ij}) \in M_m(\mathbb{K})$ be invertible, then,*

$$A^{-1} = \frac{1}{det(A)}\ {}^t[com(A)].$$

Example 1.41. *Let the matrix A of $M_3(\mathbb{R})$ defined by:*

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 1 & 2 \\ 0 & -1 & 0 \end{pmatrix}.$$

Calculate A^{-1} if it exists.

$$|A| = \begin{vmatrix} 1 & 2 & 0 \\ 3 & 1 & 2 \\ 0 & -1 & 0 \end{vmatrix} = 2 \neq 0, \quad then, \quad A^{-1} \quad exists.$$

$$com(A) = \begin{pmatrix} +\begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} & -\begin{vmatrix} 3 & 2 \\ 0 & 0 \end{vmatrix} & +\begin{vmatrix} 3 & 1 \\ 0 & -1 \end{vmatrix} \\ -\begin{vmatrix} 2 & 0 \\ -1 & 0 \end{vmatrix} & +\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \\ +\begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 2 & 0 & -3 \\ 0 & 0 & 1 \\ 4 & -2 & -5 \end{pmatrix}.$$

Finally,

$$A^{-1} = \frac{1}{det(A)}\ {}^t[com(A)] = \frac{1}{2}\begin{pmatrix} 2 & 0 & 4 \\ 0 & 0 & -2 \\ -3 & 1 & -5 \end{pmatrix}.$$

1.3 Matrices associated with a linear map

1.3.1 Linear maps

Definition 1.42. *Let E and F be two vector spaces in $\mathbb{K}$ and $f : E \longrightarrow F$ is a map. We say that f is a linear map if and only if*

- ▷ $\forall (x, y) \in E^2, f(x + y) = f(x) + f(y)$,
- ▷ $\forall \alpha \in \mathbb{K}, \forall x \in E, f(\alpha x) = \alpha f(x)$.

This is equivalent to saying:

$$\forall \alpha, \beta \in \mathbb{K}, \forall (x, y) \in E^2, \quad f(\alpha x + \beta y) = \alpha f(x) + \beta f(y).$$

We denote by $\mathcal{L}(E, F)$ the set of linear maps from E to F.

Example 1.43.

- *The map* $f:\mathbb{R}\longrightarrow\mathbb{R}$ *defined by:* $f(x)=x+4$ *is not linear because*

$$f(x+y)=x+y+4\neq f(x)+f(y)=x+xy+8.$$

- *The map from* $g:\mathbb{R}^2\longrightarrow\mathbb{R}$ *defined by:*

$$g(x,y)=|x-y|$$

is not linear because $g(X+Y)\neq g(X)+g(Y)$, *where* $X=(x,y)$ *and* $Y=(x',y')$ *in* $\mathbb{R}^2$.

- *The map* $h:\mathbb{R}^3\longrightarrow\mathbb{R}^2$ *defined by:*

$$h(x,y,z)=(x-y,y+2z)$$

is a linear map because:

▷ $\forall(x,y,z),(x',y',z')\in\mathbb{R}^3$, *we have:*

$$\begin{aligned}h[(x,y,z)+(x',y',z')] &= h(x+x',y+y',z+z')\\ &= (x+x'-y-y',y+y'+2z+2z')\\ &= (x-y,y+2z)+(x'-y',y'+2z')\\ &= h(x,y,z)+h(x',y',z').\end{aligned}$$

▷ $\forall\alpha\in\mathbb{R}$, *we have:*

$$\begin{aligned}h[\alpha(x,y,z)] &= h(\alpha x,\alpha y,\alpha z)\\ &= (\alpha x-\alpha y,\alpha y+2\alpha z)\\ &= \alpha(x-y,y+2z)\\ &= \alpha h(x,y,z).\end{aligned}$$

- *The map* $k:\mathbb{R}^2\longrightarrow\mathbb{R}^4$ *defined by:*

$$k(x,y)=(x-2y,2x,y,y-x)$$

is a linear map because:

▷ $\forall(x,y),(x',y')\in\mathbb{R}^2$, *we have:*

$$\begin{aligned}k[(x,y)+(x',y')] &= k(x+x',y+y')\\ &= (x+x'-2y-2y',2x+2x',y+y',y+y'-x-x')\\ &= (x-2y,2x,y,y-x)+(x'-2y',2x',y',y'-x')\\ &= k(x,y)+k(x',y').\end{aligned}$$

▷ $\forall \alpha \in \mathbb{R}$, *we have:*

$$\begin{aligned} k[\alpha(x,y)] &= k(\alpha x, \alpha y) \\ &= (\alpha x - 2\alpha y, 2\alpha x, \alpha y, \alpha y - \alpha x) \\ &= \alpha(x - 2y, 2x, y, y - x) \\ &= \alpha k(x,y). \end{aligned}$$

- *The map from* $\mathbb{R}^3$ *to* $\mathbb{R}^2$ *defined by:* $\varphi(x,y,z) = (x+2y-4z, 2x+3y+z)$ *is a linear map because:*

  ▷ $\varphi((x,y,z)+(x',y',z')) = \varphi(x+x', y+y', z+z')$

$$\begin{aligned} &= ((x+x')+2(y+y')-4(z+z'), 2(x+x')+3(y+y')+(z+z')) \\ &= ((x+2y-4z)+(x'+2y'-4z'), (2x+3y+z)+(2x'+3y'+z')) \\ &= ((x+2y-4z)+(x'+2y'-4z'), (2x+3y+z)+(2x'+3y'+z')) \\ &= ((x+2y-4z), (2x+3y+z)) + (((x'+2y'-4z'), (2x'+3y'+z')) \\ &= \varphi(x,y,z) + \varphi(x',y',z'). \end{aligned}$$

  ▷ $\varphi(\lambda(x,y,z)) = \varphi(\lambda x, \lambda y, \lambda z)$

$$\begin{aligned} &= (\lambda x + 2\lambda y - 4\lambda z, 2\lambda x + 3\lambda y + \lambda z) \\ &= (\lambda(x+2y-4z), \lambda(2x+3y+z)) \\ &= \lambda((x+2y-4z), (2x+3y+z)) \\ &= \lambda\varphi(x,y,z). \end{aligned}$$

Remark 1.44.

∗ *If* f *is a linear map, then,* $f(0_E) = 0_F$.

∗ *If* $E = F$*, the linear map* $f : E \longrightarrow F$ *is called* **endomorphism**.

∗ *If* f *is bijective and linear from* E *to* F*, it is called* **isomorphism**.

∗ *If* f *is a bijective endomorphism then it is an* **automorphism**.

We can write any linear map $f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ as follows:

$$f\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots \quad + \quad \vdots \quad + \quad \vdots \quad + \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{pmatrix}.$$

1.3.2 The associated matrix

The matrix associated with the map f is:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}.$$

We remark that:

$$f\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Example 1.45.

$$\begin{aligned} f : \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ x &\longmapsto f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x + 3y + 2z \\ x - y + 4z \\ 7x - 3z \end{pmatrix}. \end{aligned}$$

The matrix associated with the map f is:

$$A = \begin{pmatrix} 2 & 3 & 2 \\ 1 & -1 & 4 \\ 7 & 0 & -3 \end{pmatrix}.$$

1.4 Linear maps associated with a matrix

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \quad \text{a given matrix.}$$

The linear map associated with the matrix A defined by:

$$f\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Example 1.46.

$$A = \begin{pmatrix} 2 & 3 & 2 \\ 1 & -1 & 4 \\ 7 & 0 & -3 \end{pmatrix}.$$

The linear map associated with the matrix A is:

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & 3 & 2 \\ 1 & -1 & 4 \\ 7 & 0 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x+3y+2z \\ x-y+4z \\ 7x-3z \end{pmatrix}.$$

1.5 Change of bases, Passage matrix

Definition 1.47. *A family of vectors $\{x_i\}_{1\leq i\leq n}$ of a $\mathbb{K}$-vector space $(E,+,\cdot)$ is* **free** *or* **linearly independent** *if for all $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{K}$,*

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0_E \Longrightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0.$$

Example 1.48. *In $\mathbb{R}^3$ the vectors $x_1 = (0,1,3), x_2 = (2,0,-1)$ and $x_3 = (2,0,1)$ are free because:*

$$\begin{aligned} \forall \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}, \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0 \ &\Longrightarrow \begin{cases} 2\alpha_2 + \alpha_3 = 0 \\ \alpha_1 = 0 \\ 3\alpha_1 - \alpha_2 + \alpha_3 = 0 \end{cases} \\ &\Longrightarrow \ \alpha_1 = \alpha_2 = \alpha_3 = 0. \end{aligned}$$

Definition 1.49. *A family of vectors $\{x_i\}_{1\leq i\leq n}$ of a $\mathbb{K}$-vector space $(E,+,\cdot)$ is* **linked** *or* **linearly dependent** *if it exists $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{K}$ not all zero such that*

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0_E.$$

Example 1.50. *In $E = \mathbb{R}_2[x]$ (the vector space of polynomial functions of degree less than or equal to 2 and with real coefficients), the functions f_1, f_2, f_3 defined for all $x \in \mathbb{R}$ by:*

$$f_1(x) = x^2 + 1, \quad f_2(x) = x^2 - 1, \quad f_3(x) = x^2$$

are linked. Indeed, let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that

$$\alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 = 0 \Longrightarrow \begin{cases} \alpha_1 + \alpha_2 + \alpha_3 = 0, \\ \alpha_1 - \alpha_2 = 0 \end{cases}$$

hence $\alpha_1 = \alpha_2 = -\dfrac{\alpha_3}{2}$, therefore there are an infinity of solutions $\left(-\dfrac{\alpha_3}{2}, -\dfrac{\alpha_3}{2}, \alpha_3\right)$ with α_3 arbitrary real for example, $(1,1,-2)$.

Definition 1.51. *A family of vectors $\{x_1, x_2, \cdots, x_n\}$ of $\mathbb{K}$-vector space $(E,+,\cdot)$ is called a* **generating set** *of E or* **generates** *E if every element x of E is a linear combination of $(x_1, x_2, \cdots, x_n)$ i.e.,*

$$\forall\ x \in E, \exists\ \alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{K} \text{ such that } x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n.$$

Example 1.52. *In* $\mathbb{R}^2$ *the two vectors* $x_1 = (2,3)$ *and* $x_2 = (-1,5)$ *is a generating family because:* $\forall\ (x,y) \in \mathbb{R}^2, \exists\ \alpha_1, \alpha_2 \in \mathbb{R}$ *such that:*

$$\begin{aligned} (x,y) &= \alpha_1 x_1 + \alpha_2 x_2 \\ &= \alpha_1(2,3) + \alpha_2(-1,5) \\ &= (2\alpha_1 - \alpha_2, 3\alpha_1 + 5\alpha_2) \end{aligned}$$

$$\Longrightarrow \begin{cases} x = 2\alpha_1 - \alpha_2 \\ y = 3\alpha_1 + 5\alpha_2 \end{cases}$$

$$\Longrightarrow \begin{cases} \alpha_1 = \dfrac{5x+y}{13} \\ \\ \alpha_2 = \dfrac{-3x+2y}{13} \end{cases}$$

therefore (α_1, α_2) *exists for all* $(x,y) \in \mathbb{R}^2$.

Definition 1.53. *A family of vectors* $\{x_1, x_2, \cdots, x_n\}$ *of* $\mathbb{K}$*-vector space* $(E, +, \cdot)$ *is a* **basis** *of* E *if it is both free and generating set.*

Example 1.54. $B_0 = \{(1,0,0), (0,1,0), (0,0,1)\}$ *is a basis of* $\mathbb{R}^3$. *Indeed,*

(*i*) B_0 *is free because:*

$$\begin{aligned} \alpha(1,0,0) + \beta(0,1,0) + \gamma(0,0,1) = (0,0,0) &\Rightarrow (\alpha, \beta, \gamma) = (0,0,0) \\ &\Rightarrow \alpha = \beta = \gamma = 0. \end{aligned}$$

(*ii*) B_0 *is generating set of* $\mathbb{R}^3$ *because:*

$$\forall (x,y,z) \in \mathbb{R}^3, (x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1).$$

Definition 1.55. *Let* B_1, B_2 *be two bases of* E. *We call the* **passage** *matrix from* B_1 *to* B_2 *the matrix of* $M_n(\mathbb{K})$ *whose columns are formed from the components of the vectors of* B_2 *expressed in the base* B_1, *we denote it* $Pass(B_1, B_2)$, *i.e., if* $B_1 = \{e_1, e_2, \cdots, e_n\}$ *and* $B_2 = \{e'_1, e'_2, \cdots, e'_n\}$, *then,*

$$\begin{aligned} e'_1 &= a_{11}e_1 + a_{21}e_2 + \cdots + a_{n1}e_n, \\ e'_2 &= a_{12}e_1 + a_{22}e_2 + \cdots + a_{n2}e_n, \\ &\vdots \\ e'_n &= a_{1n}e_1 + a_{2n}e_2 + \cdots + a_{nn}e_n, \end{aligned}$$

and

$$Pass(B_1, B_2) = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{n1} \\ a_{21} & a_{22} & \dots & a_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}.$$

Example 1.56. *Let* $E = \mathbb{R}^3$,

$$\begin{aligned} B_1 &= \{e_1 = (1,1,0), e_2 = (0,1,1), e_3 = (1,1,1)\}, \\ B_2 &= \{e'_1 = (2,1,0), e'_2 = (0,2,1), e'_3 = (1,1,1)\} \end{aligned}$$

be two bases of $\mathbb{R}^3$ *(to be checked).*

Determine the passage matrix from B_1 *to* B_2*, we have:*

$$\begin{cases} e'_1 = a_{11}e_1 + a_{21}e_2 + a_{31}e_3 \\ e'_2 = a_{12}e_1 + a_{22}e_2 + a_{32}e_3 \\ e'_3 = a_{13}e_1 + a_{23}e_2 + a_{33}e_3 \end{cases} \Longrightarrow \begin{cases} e'_1 = e_1 - e_2 + e_3 \\ e'_2 = e_1 + 2e_2 - e_3 \\ e'_3 = e_3 \end{cases}$$

Then,

$$Pass(B_1, B_2) = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 2 & 0 \\ 1 & -1 & 1 \end{pmatrix}.$$

Properties 1.57. *Let* B_1, B_2, B_3 *be three bases of* E*, then,*

1 $Pass(B_1, B_3) = Pass(B_1, B_2) \times Pass(B_2, B_3)$.

2 $Pass(B_1, B_1) = I_n$ *(identity matrix).*

3 $P = Pass(B_1, B_2)$ *is invertible and* $P^{-1} = Pass(B_2, B_1)$.

Exercise 1. *Let*

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \end{pmatrix}.$$

Calculate $A^t, A^tA, AA^t, tr(A^tA)$ *and* $tr(AA^t)$.

Solution:

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \end{pmatrix}, \text{ then, } {}^tA = \begin{pmatrix} 1 & 3 \\ 2 & -1 \\ 0 & 4 \end{pmatrix}.$$

$$A^tA = \begin{pmatrix} 10 & -1 & 12 \\ -1 & 5 & -4 \\ 12 & -4 & 16 \end{pmatrix}, \qquad AA^t = \begin{pmatrix} 5 & 1 \\ 1 & 26 \end{pmatrix}.$$

$$tr(A^tA) = \sum_{i=1}^{3} a_{ii} = 10 + 5 + 16 = 31$$

and

$$tr(AA^t) = \sum_{i=1}^{2} a_{ii} = 5 + 26 = 31.$$

Exercise 2. *Let A and B be two matrices such that*

$$A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & -3 \\ 3 & 0 & -3 \\ -3 & -3 & 6 \end{pmatrix}.$$

We set $M = 3A + B$.

1) *Calculate $M^2, |M^2|, tr(M^2), f(M)$ such that $f(x) = x^2 + 2x - 15$.*

2) *Deduce that M is invertible and calculate its inverse M^{-1}.*

Solution:

$$\begin{aligned} M = 3A + 2B &= 3\begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -3 \\ 3 & 0 & -3 \\ -3 & -3 & 6 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} = 3I_3. \end{aligned}$$

1) $M^2 = 9I_3, |M^2| = 729, tr(M^2) = 27, f(M) = M^2 + 2M - 15I_3 = 9I_3 + 6I_3 - 15I_3 = 0.$

2) We have:

$$\begin{aligned} M^2 + 2M - 15I_3 = 0 &\implies M(M + 2I_3) = 15I_3 \\ &\implies M\left(\frac{M + 2I_3}{15}\right) = I_3, \end{aligned}$$

therefore M is invertible and its inverse

$$M^{-1} = \frac{1}{15}(M + 2I_3) = \frac{1}{3}I_3.$$

Exercise 3. *Let A and B be two matrices such that:*

$$A = \begin{pmatrix} \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{2}{3} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{1}{2} \end{pmatrix}, \qquad B = \begin{pmatrix} 5 & 0 & -1 \\ -1 & \frac{16}{3} & -1 \\ 0 & \frac{-4}{3} & 6 \end{pmatrix}.$$

We set $C = A + \frac{1}{4}B$.

1) *Calculate $C^2, C^3, C^n, S = C + C^2 + C^3 + \cdots + C^n, |C^2|$ and $f(C)$ such that $f(x) = x^3 + x^2 - 12$.*

2) *Deduce that M is invertible and calculate its inverse is C^{-1}.*

Solution:

$$C = A + \frac{1}{4}B = \begin{pmatrix} \frac{3}{4} & 1 & \frac{1}{4} \\ \frac{1}{4} & \frac{2}{3} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{1}{2} \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 5 & 0 & -1 \\ -1 & \frac{16}{3} & -1 \\ 0 & \frac{-4}{3} & 6 \end{pmatrix}$$
$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 2I_3.$$

1) $C^2 = 4I_3, C^3 = 8I_3, C^n = 2^n I_3$,
S is a sum of the terms of a geometric sequence with general term $U_n = 2^n$, the first term is $U_1 = 1$ and the reason is $q = 2$, then,

$$\begin{aligned} S &= C + C^2 + C^3 + \cdots + C^n \\ &= 2I_3 + 4I_3 + 8I_3 + \cdots + 2^n I_3 \\ &= (2 + 4 + 8 + \cdots + 2^n)I_3 \\ &= 2(2^n - 1)I_3, \end{aligned}$$

$|C^2| = 64$ and $f(C) = C^3 + C^2 - 12I_3 = 8I_3 + 4I_3 - 12I_3 = 0.$

2) We have:

$$\begin{aligned} C^3 + C^2 - 12I_3 = 0 &\implies C(C^2 + C) = 12I_3 \\ &\implies C\left(\frac{C^2 + C}{12}\right) = I_3. \end{aligned}$$

Therfore C is invertible and its inverse is $C^{-1} = \frac{1}{12}(C^2 + C) = \frac{1}{2}I_3$

Exercise 4. *Let A, B and C be three matrices such that:*

$$A = \begin{pmatrix} -1 & 1 & 1 & -2 \\ 1 & -1 & -2 & 1 \\ 1 & -2 & -1 & 1 \\ -2 & 1 & 1 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

1) *Calculate C^2 and deduce C^{-1}.*

2) *Show that $AC = CA$ (The commutativity of matrix product).*

3) *Calculate B^2 and check that $B = A + 2I_4 + 3C$.*

4) *Determine α and β such that $A^2 + \alpha AC + \beta I_4 = 0$. ($I_4$ is an identity matrix of order 4).*

5) *Deduce that A is invertible and calculate its inverse.*

Solution:

1) Calculate C^2 and deduce C^{-1}:

$$\begin{aligned} C^2 = C \times C &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I_4. \end{aligned}$$

we have: $C \times C^{-1} = I_4$, then, by identification, we obtain, $C^{-1} = C$.

2) Show that $AC = CA$: We have:

$$\begin{aligned} AC &= \begin{pmatrix} -1 & 1 & 1 & -2 \\ 1 & -1 & -2 & 1 \\ 1 & -2 & -1 & 1 \\ -2 & 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -2 & 1 & 1 & -1 \\ 1 & -2 & -1 & 1 \\ 1 & -1 & -2 & 1 \\ -1 & 1 & 1 & -2 \end{pmatrix} \end{aligned}$$

et

$$\begin{aligned} CA &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 & -2 \\ 1 & -1 & -2 & 1 \\ 1 & -2 & -1 & 1 \\ -2 & 1 & 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -2 & 1 & 1 & -1 \\ 1 & -2 & -1 & 1 \\ 1 & -1 & -2 & 1 \\ -1 & 1 & 1 & -2 \end{pmatrix}. \end{aligned}$$

Then, the product is commutative.

3) Calculate B^2 and check that $B = A + 2I_4 + 3C$:

$$B^2 = B \times B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \end{pmatrix}.$$

We have:

$$A + 2I_4 + 3C = \begin{pmatrix} -1 & 1 & 1 & -2 \\ 1 & -1 & -2 & 1 \\ 1 & -2 & -1 & 1 \\ -2 & 1 & 1 & -1 \end{pmatrix} + 2\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + 3\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = B.$$

4) Determine α and β such that $A^2 + \alpha AC + \beta I_4 = 0$:

$$A^2 + \alpha AC + \beta I_4 = 0$$

$$\iff \begin{pmatrix} 7 & -6 & -6 & 6 \\ -6 & 7 & 6 & -6 \\ -6 & 6 & 7 & -6 \\ 6 & -5 & -6 & 7 \end{pmatrix} + \begin{pmatrix} -2\alpha & \alpha & \alpha & -\alpha \\ \alpha & -2\alpha & -\alpha & \alpha \\ \alpha & -\alpha & -2\alpha & \alpha \\ -\alpha & \alpha & \alpha & -2\alpha \end{pmatrix} + \begin{pmatrix} \beta & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\iff \begin{pmatrix} 7-2\alpha+\beta & -6+\alpha & -6+\alpha & 6-\alpha \\ -6+\alpha & 7-2\alpha+\beta & 6-\alpha & -6+\alpha \\ -6+\alpha & 6-\alpha & 7-2\alpha+\beta & -6+\alpha \\ 6-\alpha & -6+\alpha & -6+\alpha & 7-2\alpha+\beta \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

by identification, we obtain,

$$\begin{cases} 7 - 2\alpha + \beta = 0 \\ 6 - \alpha = 0 \end{cases} \iff \begin{cases} \alpha = 6 \\ \beta = 5. \end{cases}$$

5) Deduce that A is invertible and calculate its inverse: We have:

$$A^2 + 6AC + 5I_4 = 0 \Longleftrightarrow A(A + 6C) = -5I_4 \Longleftrightarrow A\left(-\frac{A + 6C}{5}\right) = I_4,$$

therefore A is invertible and its inverse is

$$A^{-1} = \frac{-1}{5}(A + 6C) = \frac{-1}{5}\begin{pmatrix} -1 & 1 & 1 & 4 \\ 1 & -1 & 4 & 1 \\ 1 & 4 & -1 & 1 \\ 4 & 1 & 1 & -1 \end{pmatrix}.$$

Exercise 5. *For all number real θ, we set:*

$$A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

1. *Check that A_θ is invertible and calculate its inverse.*
2. *Determine the values of θ so that:*

$$\frac{1}{2}tr(A_\theta) = \det(A_\theta).$$

Solution:

$$A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

1. Check that A_θ is invertible and calculate its inverse:

   We have: $\det(A_\theta) = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1.$

   Since $\det(A_\theta) = 1 \neq 0$, then, A_θ is invertible. Its inverse is given by:

$$A_\theta^{-1} = \frac{1}{\det(A)}[com(A)]^t$$

   with

$$com(A_\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \Rightarrow [com(A_\theta)]^t = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

   Finally,

$$A_\theta^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

2. Determine the values of θ:

$$\begin{aligned} \frac{1}{2}tr(A_\theta) = \det(A_\theta) &\Longleftrightarrow \frac{1}{2}2\cos\theta = 1 \\ &\Longleftrightarrow \cos\theta = 1 \\ &\Longleftrightarrow \theta = 2k\pi, \qquad k \in \mathbb{Z}. \end{aligned}$$

CHAPTER

2

SYSTEMS OF LINEAR EQUATIONS

2.1 General

Definition 2.1. *A system of linear equations with m equations and n unknowns is a system of the form:*

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{cases} \tag{2.1}$$

where the coefficients a_{ij} and b_i are given and the x_i are unknowns in $\mathbb{R}$ or $\mathbb{C}$.

The system (2.1) can be written in matrix form:

$$AX = B$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

Example 2.2.

$$\begin{cases} 2x + 3y + 2z = 10 \\ x - y + 4z = -5 \\ 7x - 3z = 11 \end{cases} \iff \begin{pmatrix} 2 & 3 & 2 \\ 1 & -1 & 4 \\ 7 & 0 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 \\ -5 \\ 11 \end{pmatrix}.$$

Where

$$A = \begin{pmatrix} 2 & 3 & 2 \\ 1 & -1 & 4 \\ 7 & 0 & -3 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} 10 \\ -5 \\ 11 \end{pmatrix}.$$

2.1.1 Rank of a system of linear equations

Definition 2.3. *Let $A \in M_{m,n}(\mathbb{K})$. We call the rank of A the maximum number of column vectors of A that are linearly independent and we have: $rg(A) \leq \min(m, n)$.*

Definition 2.4. *The rank of a matrix A is the order of the highest non-zero determinant extracted from A.*

Definition 2.5. *The rank of the linear system (S) is the rank of its associated matrix A, we have:*

$$rg(S) = rg(A) = rg(f).$$

Example 2.6.

$$A = \begin{pmatrix} 3 & 1 & 2 & 3 \\ 5 & -1 & -3 & 4 \\ 2 & 4 & 9 & 3 \\ 1 & -2 & -1 & 4 \end{pmatrix},$$

we find: $|A| = 0$, therefore A is of rank $r < 4$. Among the matrices of order 3 extracted from A, we find:

$$A_1 = \begin{pmatrix} 3 & 1 & 2 \\ 5 & -1 & -3 \\ 2 & 4 & 9 \end{pmatrix},$$

with determinant equal 2. It follows that $rg(A) = 3$.

Example 2.7.

$$B = \begin{pmatrix} 1 & 1 & 3 & 5 \\ 1 & 2 & 5 & 9 \\ 2 & 3 & 8 & 14 \end{pmatrix},$$

the four determinants of order 3 are zero, on the other hand the determinant of order 2 extracted from B, we find:

$$B_1 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},$$

is not zero, it follows that $rg(B) = 2$.

2.2 Study of the set of solutions

Let the system (2.1) that we assume to be of rank r be written in such a way that the determinant Δ of the coefficients of the first r unknowns and first r equations is non-zero.

Characteristic determinant: We call the characteristic determinant of (2.1) the determinant of the form:

$$D_k = \begin{vmatrix} a_{11} & \cdots & a_{1r} & b_1 \\ \vdots & \cdots & \vdots & \vdots \\ a_{r1} & \cdots & a_{rr} & b_r \\ a_{k1} & \cdots & a_{kr} & b_k \end{vmatrix}, \qquad k = r+1, r+2, \cdots, m.$$

- ▷ If $r = m = n$, the system (2.1) admits only one solution.
- ▷ If $r < m < n$, the system (2.1) is indeterminate with $(n-r)$ parameters.
- ▷ If $r < m$ and if at least one of the characteristic determinants of (2.1) is non-zero, (2.1) has no solution.
- ▷ If $r < m$ and if the characteristic determinants of (2.1) are zero, (2.1) reduces to the r equations and is solved as in the second case.

Example 2.8.

$$\begin{cases} x + y + 2z = -2 \\ x + 2y + 3z = a \\ 3x + 5y + 8z = 2 \\ 5x + 9y + 14z = b \end{cases} \iff \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 5 & 8 \\ 3 & 9 & 14 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ a \\ 2 \\ b \end{pmatrix}$$
$$\iff AX = B.$$

The four determinants of order 3 *are zero, but the determinant of order* 2 *extracted* $\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}$ *is not zero,* A *is of rank* 2.

The characteristic determinants:

$$D_1 = \begin{vmatrix} 1 & 1 & -2 \\ 1 & 2 & a \\ 3 & 5 & 2 \end{vmatrix} = -2a + 4 \qquad \text{and} \qquad D_2 = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & a \\ 5 & 9 & b \end{vmatrix} = -4a + b + 2.$$

- *If* $D_1 \neq 0$ *or* $D_2 \neq 0$*, then, this system has no solution.*
- *If* $D_1 = D_2 = 0$*, then, this indeterminate system has a parameter* z*:*

$$\begin{cases} x = -3z - 6 \\ y = z + 4 \end{cases} \qquad z \in \mathbb{R}.$$

2.3 Methods for solving a linear system

2.3.1 Solving by Cramer's method

Let (S) be a square linear system, i.e., its matrix A is square, with the matrix interpretation $AX = B$. If the matrix A is invertible, we can solve this system using Cramer's method.

We will denote A_i the matrix A of coefficients in which we have replaced the i^{th} column by the matrix B.

Solving the system using Cramer's method gives

$$x_i = \frac{det(A_i)}{det(A)}, \qquad i = 1, \cdots, n.$$

Example 2.9.

$$\begin{cases} 3x - y = 4 \\ -5x + 2y = -2 \end{cases} \iff \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}.$$

This will give us

$$A_1 = \begin{pmatrix} 4 & -1 \\ -2 & 2 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 3 & 4 \\ -5 & -2 \end{pmatrix}$$

and

$$x = \frac{\begin{vmatrix} 4 & -1 \\ -2 & 2 \end{vmatrix}}{\begin{vmatrix} 3 & -1 \\ -5 & 2 \end{vmatrix}} = 6, \qquad y = \frac{\begin{vmatrix} 3 & 4 \\ -5 & -2 \end{vmatrix}}{\begin{vmatrix} 3 & -1 \\ -5 & 2 \end{vmatrix}} = 14.$$

Then, $X = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ 14 \end{pmatrix}$ *is a unique solution of this system.*

Example 2.10.

$$\begin{cases} 2x + 3y - z = 5 \\ 4x + 4y - 3z = 3 \\ -2x + 3y - z = 1 \end{cases} \iff \begin{pmatrix} 2 & 3 & -1 \\ 4 & 4 & -3 \\ -2 & 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix}.$$

This will give us

$$A_1 = \begin{pmatrix} 5 & 3 & -1 \\ 3 & 4 & -3 \\ 1 & 3 & -1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 2 & 5 & -1 \\ 4 & 3 & -3 \\ -2 & 1 & -1 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 2 & 3 & 5 \\ 4 & 4 & 3 \\ -2 & 3 & 1 \end{pmatrix}$$

and

$$x = \frac{\begin{vmatrix} 5 & 3 & -1 \\ 3 & 4 & -3 \\ 1 & 3 & -1 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & -1 \\ 4 & 4 & -3 \\ -2 & 3 & -1 \end{vmatrix}} = 1, \qquad y = \frac{\begin{vmatrix} 2 & 5 & -1 \\ 4 & 3 & -3 \\ -2 & 1 & -1 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & -1 \\ 4 & 4 & -3 \\ -2 & 3 & -1 \end{vmatrix}} = 2, \qquad z = \frac{\begin{vmatrix} 2 & 3 & 5 \\ 4 & 4 & 3 \\ -2 & 3 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & -1 \\ 4 & 4 & -3 \\ -2 & 3 & -1 \end{vmatrix}} = 3.$$

Then, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ *is a unique solution of this system.*

2.3.2 Solving by the inverse matrix method

Let (S) be a square linear system, with the matrix interpretation: $AX = B$. If the matrix A is invertible, we can solve this system by the inverse matrix method as follows:

$$AX = B \Longleftrightarrow X = A^{-1}B.$$

Example 2.11.

$$\begin{cases} 3x - y = 4 \\ -5x + 2y = -2 \end{cases} \Longleftrightarrow \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}.$$

We have:

$$|A| = 1 \neq 0 \Longleftrightarrow A \textit{ is invertible}.$$

We can calculate A^{-1} *as follows:*

$$com(A) = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} \Longleftrightarrow [com(A)]^t = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}.$$

This will give us

$$A^{-1} = \frac{1}{|A|}[com(A)]^t = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}.$$

Then,

$$X = A^{-1}B \Longleftrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 6 \\ 14 \end{pmatrix}.$$

Example 2.12.

$$\begin{cases} 2x + 3y - z = 5 \\ 4x + 4y - 3z = 3 \\ -2x + 3y - z = 1 \end{cases} \Longleftrightarrow \begin{pmatrix} 2 & 3 & -1 \\ 4 & 4 & -3 \\ -2 & 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix}.$$

$$|A| = 20 \neq 0, \quad \textit{then}, \quad A^{-1} \quad \textit{exists}.$$

We can calculate A^{-1} as follows:

$$com(A) = \begin{pmatrix} +\begin{vmatrix} 4 & -3 \\ 3 & -1 \end{vmatrix} & -\begin{vmatrix} 4 & -3 \\ -2 & -1 \end{vmatrix} & +\begin{vmatrix} 4 & 4 \\ -2 & 3 \end{vmatrix} \\ \\ -\begin{vmatrix} 3 & -1 \\ 3 & -1 \end{vmatrix} & +\begin{vmatrix} 2 & -1 \\ -2 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ -2 & 3 \end{vmatrix} \\ \\ +\begin{vmatrix} 3 & -1 \\ 4 & -3 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 4 & -3 \end{vmatrix} & +\begin{vmatrix} 2 & 3 \\ 4 & 4 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 5 & 10 & 20 \\ 0 & -4 & -12 \\ -5 & 2 & -4 \end{pmatrix}.$$

Finally,

$$A^{-1} = \frac{1}{det(A)}\ {}^{t}[com(A)] = \frac{1}{20}\begin{pmatrix} 5 & 0 & -5 \\ 10 & -4 & 2 \\ 20 & -12 & -4 \end{pmatrix}.$$

Then,

$$X = A^{-1}B \Longleftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{20}\begin{pmatrix} 5 & 0 & -5 \\ 10 & -4 & 2 \\ 20 & -12 & -4 \end{pmatrix}\begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

2.3.3 Solving by the Gauss elimination method

The principle of this method is to transform the system:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n \end{cases} \tag{2.2}$$

to a system:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ \qquad\quad a'_{22}x_2 + \ldots + a'_{2n}x_n = b'_2 \\ \vdots \\ \qquad\qquad\qquad\qquad a'_{nn}x_n = b'_n \end{cases}$$

Application of the Gauss method: To simplify the calculation, we set $n = 3$, then, the system (4.11) becomes:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 & \longrightarrow\ E_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 & \longrightarrow\ E_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 & \longrightarrow\ E_3 \end{cases}$$

Step 1: Eliminate x_1 in E_2 and E_3

$$\begin{cases} E_1^1 = E_1 \\ E_2^1 = E_2 - \dfrac{a_{21}}{a_{11}}E_1 \\ E_3^1 = E_3 - \dfrac{a_{31}}{a_{11}}E_1 \end{cases} \Longrightarrow \begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 & \longrightarrow\ E_1^1 \\ \qquad\quad a'_{22}x_2 + a'_{23}x_3 = b'_2 & \longrightarrow\ E_2^1 \\ \qquad\quad a'_{32}x_2 + a'_{33}x_3 = b'_3 & \longrightarrow\ E_3^1 \end{cases}$$

Step 2: Eliminate x_2 in E_3^1

$$\begin{cases} E_1^2 = E_1^1 \\ E_2^2 = E_2^1 \\ E_3^2 = E_3^1 - \dfrac{a'_{32}}{a'_{22}} E_2^1 \end{cases} \Longrightarrow \begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 & \longrightarrow E_1^2 \\ a'_{22}x_2 + a'_{23}x_3 = b'_2 & \longrightarrow E_2^2 \\ a''_{33}x_3 = b''_3 & \longrightarrow E_3^2 \end{cases}$$

Then, we obtain the solution, starting by $x_3 = \dfrac{b''_3}{a''_{33}}$, x_2 and x_1.

Remark 2.13. *The Gauss method is applicable in the case where* $a_{kk}^k \neq 0, \ \forall k \leq n$.

Example 2.14.

$$\begin{cases} 2x + 3y - z = 5 & \longrightarrow E_1 \\ 4x + 4y - 3z = 3 & \longrightarrow E_2 \\ -2x + 3y - z = 1 & \longrightarrow E_3 \end{cases}$$

Step 1: *Eliminate x in E_2 and E_3*

$$\begin{cases} E_1^1 = E_1 \\ E_2^1 = E_2 - \dfrac{a_{21}}{a_{11}} E_1 = E_2 - 2E_1 \\ E_3^1 = E_3 - \dfrac{a_{31}}{a_{11}} E_1 = E_3 + E_1 \end{cases} \Longrightarrow \begin{cases} 2x + 3y - z = 5 & \longrightarrow E_1^1 \\ -2y - z = -7 & \longrightarrow E_2^1 \\ 6y - 2z = 6 & \longrightarrow E_3^1 \end{cases}$$

Step 2: *Eliminate y in E_3^1*

$$\begin{cases} E_1^2 = E_1^1 \\ E_2^2 = E_2^1 \\ E_3^2 = E_3^1 - \dfrac{a'_{32}}{a'_{22}} E_1 = E_3^1 + 3E_2^1 \end{cases} \Longrightarrow \begin{cases} 2x + 3y - z = 5 & \longrightarrow E_1^2 \\ -2y - z = -7 & \longrightarrow E_2^2 \\ -5z = -15 & \longrightarrow E_3^2 \end{cases}$$

Then, E_3^2 *gives* $z = \dfrac{15}{5} = 3$, E_2^2 *gives* $y = 2$ *and* E_1^2 *gives* $x = 1$.

Finally, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ *is a unique solution of this system.*

2.4 Discussion of a system of linear equations

In some cases, the coefficients of the unknowns of the system depend on parameters. The solutions of the system will therefore obviously also depend on these laters. For certain particular values of the parameters, the system may even be impossible or indeterminate (The system will has fewer equations than unknowns).

To discuss a parametric system is to seek the set of solutions based on the values of the parameters.

Example 2.15. *Solve and discuss the system of unknowns x_1, x_2 and x_3 as a function of parameter m.*

$$\begin{cases} x_1 + mx_2 + x_3 = 2m \\ mx_1 + x_2 + x_3 = 0 \\ x_1 + mx_2 + (m+1)x_3 = m \end{cases} \tag{2.3}$$

The matrix interpretation of this system is given by:

$$\begin{cases} x_1 + mx_2 + x_3 = 2m \\ mx_1 + x_2 + x_3 = 0 \\ x_1 + mx_2 + (m+1)x_3 = m \end{cases} \iff \begin{pmatrix} 1 & m & 1 \\ m & 1 & 1 \\ 1 & m & m+1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2m \\ 0 \\ m \end{pmatrix}$$
$$\iff AX = B.$$

Let us first calculate the determinant of the matrix A of the coefficients of the unknowns (check).

$$|A| = \begin{vmatrix} 1 & m & 1 \\ m & 1 & 1 \\ 1 & m & m+1 \end{vmatrix} = m(1-m^2).$$

This determinant has three roots: $m = -1, m = 0$ and $m = 1$.

We know that det(A) is the denominator in Cramer's formulas. Therefore, it will be necessary to distinguish the case where $|A| \neq 0$ and the case where $|A| = 0$.

- **First case:** $|A| \neq 0$ *i.e., $m \neq -1, m \neq 0$ and $m \neq 1$.*

  In this case, the system admits a unique solution given by the Cramer's formulas.

$$x_1 = \frac{\begin{vmatrix} 2m & m & 1 \\ 0 & 1 & 1 \\ m & m & m+1 \end{vmatrix}}{|A|} = \frac{m(m+1)}{m(1-m^2)} = \frac{m+1}{(1-m)(1+m)} = \frac{1}{1-m}.$$

$$x_2 = \frac{\begin{vmatrix} 1 & 2m & 1 \\ m & 0 & 1 \\ 1 & m & m+1 \end{vmatrix}}{|A|} = \frac{m(-2m^2-m+1)}{m(1-m^2)} = \frac{m(-2)(m+1)\left(m-\frac{1}{2}\right)}{m(1-m)(1+m)}$$
$$= \frac{m(-2m+1)}{1-m}.$$

$$x_3 = \frac{\begin{vmatrix} 1 & m & 2m \\ m & 1 & 0 \\ 1 & m & m \end{vmatrix}}{|A|} = \frac{m(m^2-1)}{m(1-m^2)} = \frac{(m-1)(m+1)}{(1-m)(1+m)} = -1.$$

This system admits the set of solutions:

$$S = \left\{ \left(\frac{1}{1-m}, \frac{m(1-2m)}{1-m}, -1 \right) \right\}.$$

- **Second case:** $|A| = 0$ *i.e.,* $m = -1, m = 0$ *and* $m = 1$.

  ✓ *If* $m = -1$. *The system* (2.3) *is written in the form:*

$$\begin{cases} x_1 - x_2 + x_3 = -2 \\ -x_1 + x_2 + x_3 = 0 \\ x_1 - x_2 = -1. \end{cases}$$

  ▷ *If we subtract the second equation from the first, we obtain an equation equivalent to the third.*

  ▷ *If we add the first two equations member by member, we find* $x_3 = -1$.

  Therefore, the system (2.3) *reduces to:*

$$\begin{cases} x_1 - x_2 = -1 \\ x_3 = -1. \end{cases}$$

  It is simply indeterminate and admits as a set of solutions:

$$S = \{(\lambda, \lambda + 1, -1), \quad \lambda \in \mathbb{R}\}.$$

  ✓ *If* $m = 0$. *The system* (2.3) *is written in the form:*

$$\begin{cases} x_1 + x_3 = 0 \\ x_2 + x_3 = 0 \\ x_1 + x_3 = 0. \end{cases}$$

  Therefore, it comes down to:

$$\begin{cases} x_1 + x_3 = 0 \\ x_2 + x_3 = 0. \end{cases}$$

  It is simply indeterminate and admits as a set of solutions:

$$S = \{(\lambda, \lambda, -\lambda), \quad \lambda \in \mathbb{R}\}.$$

  ✓ *If* $m = 1$. *The system* (2.3) *is written in the form:*

$$\begin{cases} x_1 + x_2 + x_3 = 2 \\ x_1 + x_2 + x_3 = 0 \\ x_1 + x_2 + 2x_3 = 1. \end{cases}$$

  The first two equations being incompatible, the system is impossible, $S = \emptyset$.

Exercise 1. *For all* $\alpha \in \mathbb{R}$, *we consider the following system of linear equations:*

$$\begin{cases} \alpha x + z = 2 \\ y + z = 3 \\ y + 2z = 5 \end{cases}$$

1. Write this system in matrix form.

2. Show that this system has a unique solution.

3. Let $\alpha = 1$. Solve this system by three methods (Cramer, Gauss and the inverse matrix).

Solution:

1. Write the system in matrix form:

$$\begin{cases} \alpha x + z = 2 \\ y + z = 3 \\ y + 2z = 5 \end{cases} \iff \begin{pmatrix} \alpha & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$$

$$\iff A \cdot X = B.$$

2. Show that this system has a unique solution:

$$\det(A) = \begin{vmatrix} \alpha & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix} = \alpha(2-1) + (0) = \alpha.$$

This system admits a unique solution if and only if $\det(A) \neq 0$, i.e.,

$$\det(A) \neq 0 \Rightarrow \alpha \neq 0 \Rightarrow \alpha \in \mathbb{R}^*.$$

3. For $\alpha = 1$, solve this system:

$$\begin{cases} x + z = 2 \\ y + z = 3 \\ y + 2z = 5 \end{cases} \iff \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$$

$$\iff A \cdot X = B.$$

▷ **Cramer's method:** Let

$$A_1 = \begin{pmatrix} 2 & 0 & 1 \\ 3 & 1 & 1 \\ 5 & 1 & 2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 5 & 2 \end{pmatrix} \quad \textit{and} \quad A_3 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 1 & 5 \end{pmatrix}.$$

Then,

$$x = \frac{|A_1|}{|A|} = 0, \quad y = \frac{|A_2|}{|A|} = 1 \quad \textit{and} \quad z = \frac{|A_3|}{|A|} = 2.$$

Finally, the solution is given by:

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

▷ **Gauss method:**

$$\begin{cases} x+z=2 \longrightarrow E_1 \\ y+z=3 \longrightarrow E_2 \\ y+2z=5 \longrightarrow E_3 \end{cases}$$

Eliminate y in E_3:

$$\begin{cases} E_1'=E_1 \\ E_2'=E_2 \\ E_3'=E_3-E_2 \end{cases} \Longrightarrow \begin{cases} x+z=2 \longrightarrow E_1' \\ y+z=3 \longrightarrow E_2' \\ z=2 \longrightarrow E_3' \end{cases}$$

Then, E_3' gives $z=2, E_2'$ gives $y=1$ and E_1' gives $x=0$.

Finally, $X=\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ is a unique solution of this system.

▷ **Inverse matrix method:**

$$\begin{cases} \alpha x+z=2 \\ y+z=3 \\ y+2z=5 \end{cases} \Longleftrightarrow \begin{pmatrix} \alpha & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$$

$$\Longleftrightarrow A\cdot X=B$$

$$\Longleftrightarrow X=A^{-1}\cdot B.$$

We have:

$$|A|=1\neq 0, \quad then, \quad A^{-1} \quad exists.$$

We can calculate A^{-1} as follows:

$$com(A)=\begin{pmatrix} +\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix} & +\begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} \\ -\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \\ +\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & -1 \\ -1 & -1 & 1 \end{pmatrix}.$$

Finally,

$$A^{-1}=\frac{1}{det(A)}[com(A)]^t=\begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

Then,

$$X=A^{-1}B \Longleftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}=\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

Exercise 2. *We consider the matrix C defined by:*

$$C = \begin{pmatrix} \alpha+2 & 1 & -1 \\ 12 & 2\alpha+2 & 2 \\ \alpha & 2 & \alpha \end{pmatrix}$$

1. *Calculate the determinant of C.*
2. *Let $P(\alpha) = 2\alpha^3 + 8\alpha^2 - 8\alpha - 32$. Calculate $P(2)$.*
3. *For which values of α, C is invertible.*
4. *We consider the following system of linear equations:*

$$\begin{pmatrix} \alpha+2 & 1 & -1 \\ 12 & 2\alpha+2 & 2 \\ \alpha & 2 & \alpha \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}. \tag{2.4}$$

   Solve the system (4.11) *for $\alpha \in \mathbb{R} - \{-4, -2, 2\}$ by Cramer's method.*

Solution: We consider the matrix C defined by:

$$C = \begin{pmatrix} \alpha+2 & 1 & -1 \\ 12 & 2\alpha+2 & 2 \\ \alpha & 2 & \alpha \end{pmatrix}$$

1. Calculate the determinant of C:

$$\begin{aligned} |C| &= (\alpha+2)\begin{vmatrix} 2\alpha+2 & 2 \\ 2 & \alpha \end{vmatrix} - \begin{vmatrix} 12 & 2 \\ \alpha & \alpha \end{vmatrix} - \begin{vmatrix} 12 & 2\alpha+2 \\ \alpha & 2 \end{vmatrix} \\ &= 2\alpha^3 + 8\alpha^2 - 8\alpha - 32. \end{aligned}$$

2. Let $P(\alpha) = 2\alpha^3 + 8\alpha^2 - 8\alpha - 32$. Calculate $P(2)$: $P(2) = 0$.
3. The values of α for C to be invertible:

$$C \;\; is \;\; invertible \iff |C| \neq 0.$$

$$P(\alpha) = 0 \iff (\alpha - 2)(a\alpha^2 + b\alpha + c) = 0.$$

   By identification or Euclidean division, we find:

$$\begin{aligned} P(\alpha) = 0 &\iff (\alpha-2)(2\alpha^2 + 12\alpha + 16) = 0 \\ &\iff (\alpha-2)(\alpha+2)(\alpha+4) = 0 \\ &\iff \alpha = 2 \vee \alpha = -2 \vee \alpha = -4. \end{aligned}$$

   Consequently, C is invertible if $\alpha \in \mathbb{R} - \{-4, -2, 2\}$.

4. We consider the following system of linear equations:

$$\begin{pmatrix} \alpha+2 & 1 & -1 \\ 12 & 2\alpha+2 & 2 \\ \alpha & 2 & \alpha \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}. \tag{2.5}$$

Solve the system (4.11) for $\alpha \in \mathbb{R} - \{-4, -2, 2\}$ by Cramer's method:

$$C_1 = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 2\alpha+2 & 2 \\ 0 & 2 & \alpha \end{pmatrix} \Longrightarrow |C_1| = 2\alpha^2 + 3\alpha - 2,$$

$$C_2 = \begin{pmatrix} \alpha+2 & 1 & -1 \\ 12 & -1 & 2 \\ \alpha & 0 & \alpha \end{pmatrix} \Longrightarrow |C_2| = -\alpha^2 - 13\alpha,$$

$$C_3 = \begin{pmatrix} \alpha+2 & 1 & 1 \\ 12 & 2\alpha+2 & -1 \\ \alpha & 2 & 0 \end{pmatrix} \Longrightarrow |C_3| = -2\alpha^2 - \alpha + 28.$$

$$\begin{aligned} x &= \frac{|C_1|}{|C|} = \frac{2\alpha^2 + 3\alpha - 2}{(\alpha-2)(\alpha+2)(\alpha+4)}, \\ y &= \frac{|C_2|}{|C|} = \frac{-\alpha^2 - 13\alpha}{(\alpha-2)(\alpha+2)(\alpha+4)}, \\ z &= \frac{|C_3|}{|C|} = \frac{-2\alpha^2 - \alpha + 28}{(\alpha-2)(\alpha+2)(\alpha+4)}. \end{aligned}$$

Exercise 3. *Consider the following matrix:*

$$M = \begin{pmatrix} 2 & 1 & 0 \\ -3 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

1. *Calculate $A = I_3 - M$ and $B = I_3 + M + M^2$. Recall that $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.*

2. *Calculate $A \times B$ and $B \times A$.*

3. *Deduce that B is invertible and give its inverse.*

4. *Solve the following system of linear equations:*

$$\begin{cases} 4x + 2y + z = 1 \\ -5x - 2y - z = 2 \\ 2x + y + z = -1 \end{cases}$$

(a) *using question* 3.

(b) *using Cramer's method.*

Solution:

$$M = \begin{pmatrix} 2 & 1 & 0 \\ -3 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

1. Let us calculate $A = I_3 - M$ and $B = I_3 + M + M^2$:

$$\begin{aligned} A = I_3 - M &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 1 & 0 \\ -3 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & -1 & 0 \\ 3 & 2 & -1 \\ -1 & 0 & 2 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} B = I_3 + M + M^2 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 1 & 0 \\ -3 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 1 \\ -2 & -2 & -2 \\ 1 & 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 2 & 1 \\ -5 & -2 & -1 \\ 2 & 1 & 1 \end{pmatrix}. \end{aligned}$$

2. Let us calculate $A \times B$ and $B \times A$:

$$\begin{aligned} A \times B &= \begin{pmatrix} -1 & -1 & 0 \\ 3 & 2 & -1 \\ -1 & 0 & 2 \end{pmatrix} \times \begin{pmatrix} 4 & 2 & 1 \\ -5 & -2 & -1 \\ 2 & 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3 \end{aligned}$$

and

$$\begin{aligned} B \times A &= \begin{pmatrix} 4 & 2 & 1 \\ -5 & -2 & -1 \\ 2 & 1 & 1 \end{pmatrix} \times \begin{pmatrix} -1 & -1 & 0 \\ 3 & 2 & -1 \\ -1 & 0 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3. \end{aligned}$$

3. Deduce that B is invertible and give its inverse B^{-1}: We have:$A \times B = B \times A = I_3$, therefore, B is invertible ($|B| \neq 0$) and its inverse is given by:

$$B^{-1} = A = \begin{pmatrix} -1 & -1 & 0 \\ 3 & 2 & -1 \\ -1 & 0 & 2 \end{pmatrix}.$$

4. We will solve the following system of linear equations:

$$\begin{cases} 4x + 2y + z = 1 \\ -5x - 2y - z = 2 \\ 2x + y + z = -1 \end{cases}$$

(a) Using question 3:

$$\begin{aligned}
\begin{cases} 4x + 2y + z = 1 \\ -5x - 2y - z = 2 \\ 2x + y + z = -1 \end{cases} &\iff \begin{pmatrix} 4 & 2 & 1 \\ -5 & -2 & -1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \\
&\iff B \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \\
&\iff \begin{pmatrix} x \\ y \\ z \end{pmatrix} = B^{-1} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \\
&\iff \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \\
&\iff \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & -1 & 0 \\ 3 & 2 & -1 \\ -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \\
&\iff \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 8 \\ -3 \end{pmatrix}
\end{aligned}$$

(b) Using Cramer's method:

$$\begin{aligned}
\begin{cases} 4x + 2y + z = 1 \\ -5x - 2y - z = 2 \\ 2x + y + z = -1 \end{cases} &\iff \begin{pmatrix} 4 & 2 & 1 \\ -5 & -2 & -1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \\
&\iff B \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.
\end{aligned}$$

We have: $|B| \neq 0$, therefore, this system admits only one solution.

This will give us

$$B_1 = \begin{pmatrix} 1 & 2 & 1 \\ 2 & -2 & -1 \\ -1 & 1 & 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 4 & 1 & 1 \\ -5 & 2 & -1 \\ 2 & -1 & 1 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 4 & 2 & 1 \\ -5 & -2 & 2 \\ 2 & 1 & -1 \end{pmatrix}$$

and

$$\begin{aligned} x &= \frac{\begin{vmatrix} 1 & 2 & 1 \\ 2 & -2 & -1 \\ -1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 4 & 2 & 1 \\ -5 & -2 & -1 \\ 2 & 1 & 1 \end{vmatrix}} = -3, \\ y &= \frac{\begin{vmatrix} 4 & 1 & 1 \\ -5 & 2 & -1 \\ 2 & -1 & 1 \end{vmatrix}}{\begin{vmatrix} 4 & 2 & 1 \\ -5 & -2 & -1 \\ 2 & 1 & 1 \end{vmatrix}} = 8, \\ z &= \frac{\begin{vmatrix} 4 & 2 & 1 \\ -5 & -2 & 2 \\ 2 & 1 & -1 \end{vmatrix}}{\begin{vmatrix} 4 & 2 & 1 \\ -5 & -2 & -1 \\ 2 & 1 & 1 \end{vmatrix}} = -3. \end{aligned}$$

Then, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 8 \\ -3 \end{pmatrix}$ is a unique solution of this system.

CHAPTER

3

THE INTEGRALS

The origin of the current theory of integration dates back to antiquity with the concern for measuring areas, lengths and volumes, "geometer" means in ancient Greek "surveyor" that is to say "one who measures the earth".

Three ideas found the concept of measurement, the first is the definition of measurement for simple sets: $b-a$ for the length of the segment $[a,b], L_1L_2$ for the area of a rectangle whose sides measure L_1, L_2 and $L_1L_2L_3$ for the volume of a rectangular parallelepiped whose sides have lengths L_1, L_2 and L_3. The second idea is that of addition: the measure of the union of two disjoint sets is equal to the sum of the measures of these sets. The third idea is that of continuity of measurement: thus the length of the circle is the limit of the lengths of the regular polygons inscribed in this circle.

This is the path we will follow to construct the integral of a continuous function: we begin by defining the integral of a constant function, then that of a step function and finally that of a limit of a sequence of step functions.

The modern form of this construction is due to Georg Friedrich Bernhard Riemann, 1826-1866 and Jean Gaston Darboux, 1842-1917. The theory of measure will find its completed form thanks to the work of Henri Léon Lebesgue, 1875-1941.

In this chapter, we seek to construct the reciprocal operator of the derivation operator.

3.1 Indefinite integral, property

Definition 3.1. *Let f be a continuous function on I. We call that $F : I \to \mathbb{R}$ is a primitive of f on I if and only if the derivative of F gives f $(F' = f)$. Then, we get into the habit of writing any primitive of f in the form:*

$$F(x) = \int f(x)dx$$

and is also called the indefinite integral of f.

Example 3.2. *Let $f : \mathbb{R} - \{-\frac{3}{2}\} \to \mathbb{R} - \{-\frac{3}{2}\}$ be defined by:*

$$f(x) = x^3 + 1 - \frac{2}{(2x+3)^2}.$$

Then, $F : \mathbb{R} - \{-\frac{3}{2}\} \to \mathbb{R} - \{-\frac{3}{2}\}$ defined by:

$$F(x) = \frac{1}{4}x^4 + x + \frac{1}{2x+3}$$

is a primitive of f.

The function defined by:

$$F_1(x) = \frac{1}{4}x^4 + x - \sqrt{3} + \frac{1}{2x+3}$$

and the function defined by:

$$F_2(x) = \frac{1}{4}x^4 + x + \frac{1}{2} + \frac{1}{2x+3}$$

are also primitives of f.

Remark 3.3. *The primitive of a function if it exists is not unique.*

Proposition 3.4. *Let f and g be two continuous functions and $\lambda \in \mathbb{R}$, we have:*

- ▷ $\displaystyle\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx.$
- ▷ $\displaystyle\int \lambda f(x)dx = \lambda \int f(x)dx.$

3.1.1 Primitives of usual functions

- If f and g are two continuous functions on I and if F and G are primitives on I of f and g respectively, $F + G$ is a primitive of $f + g$ on I.
- If f is a continuous function on I, if F is a primitive of f on I and if λ is a real number, λF is a primitive of λf on I.

- Otherwise, we have the following table in which f systematically designates a derivable function on an interval I whose derivative f' is continuous on I:

$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$
$x^n,\ n \neq -1$	$\frac{1}{n+1}x^{n+1} + c$	$e^{nx},\ n \in \mathbb{R}^*$	$\frac{1}{n}e^{nx} + c$
$\frac{1}{x^2}$	$-\frac{1}{x} + c$	$\frac{1}{x}$	$\ln\|x\| + c$
$\frac{1}{\sqrt{x}}$	$2\sqrt{x} + c$	$\frac{f'(x)}{\sqrt{f(x)}}$	$2\sqrt{f(x)} + c$
$\sin x$	$-\cos x + c$	$\cos x$	$\sin x + c$
$\sin(ax+b)$	$-\frac{1}{a}\cos(ax+b) + c$	$\cos(ax+b)$	$\frac{1}{a}\sin(ax+b) + c$
$\tan x$	$-\ln\|\cos x\| + c$	$\cot x$	$\ln\|\sin x\| + c$
$\tan(ax+b)$	$-\frac{1}{a}\ln\|\cos(ax+b)\| + c$	$\cot(ax+b)$	$\frac{1}{a}\ln\|\sin(ax+b)\| + c$
$\sinh x$	$\cosh x + c$	$\cosh x$	$\sinh x + c$
$\tanh x$	$\ln(\cosh x) + c$	$\coth x$	$\ln(\sinh x) + c$
$\frac{1}{\sin x}$	$\ln\left\|\tan\frac{x}{2}\right\| + c$	$\frac{1}{\cos x}$	$\ln\left\|\tan\left(\frac{x}{2} + \frac{\Pi}{4}\right)\right\| + c$
$\frac{1}{\sin^2 x}$	$-\cot x + c$	$\frac{1}{\cos^2 x}$	$\tan x + c$

$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$
$\dfrac{1}{\sinh x}$	$\ln\left\lvert\tanh\dfrac{x}{2}\right\rvert + c$	$\dfrac{1}{\cosh x}$	$2\arctan e^x + c$
$\dfrac{1}{\sinh^2 x}$	$-\cot x + c$	$\dfrac{1}{\cosh^2 x}$	$\tanh x + c$
$\dfrac{1}{1+x^2}$	$\arctan x + c$	$\dfrac{1}{1-x^2}$	$\dfrac{1}{2}\ln\left\lvert\dfrac{1+x}{1-x}\right\rvert + c$
$\dfrac{1}{\sqrt{1-x^2}}$	$\arcsin x + c$	$\dfrac{1}{\sqrt{x^2+\alpha}}$	$\ln\lvert x+\sqrt{x^2+\alpha}\rvert + c$
$f'(x)f^n(x)$	$\dfrac{1}{n+1}f^{n+1}(x) + c$	$\dfrac{f'(x)}{1+f^2(x)}$	$\arcsin f(x) + c$
$\dfrac{f'(x)}{f(x)}$	$\ln\lvert f(x)\rvert + c$	$\dfrac{f'(x)}{f^n(x)},\ n\geq 2$	$\dfrac{-1}{n-1}f^{n-1} + c,\ c\in\mathbb{R}$

Example 3.5. *c is a real constant.*

1.

$$\begin{aligned} I_1 &= \int\left(x^4-\sqrt{x^3}+x+\sqrt{2}\right)dx \\ &= \int\left(x^4-x^{\frac{3}{2}}+x+\sqrt{2}\right)dx \\ &= \frac{1}{5}x^5-\frac{2}{5}x^{\frac{5}{2}}+\sqrt{2}x+c. \end{aligned}$$

2.

$$\begin{aligned} I_2 &= \int \frac{x^4 - 3x^2 + 4}{x^2} dx \\ &= \int \left(x^2 - 3 + \frac{4}{x^2} \right) dx \\ &= \frac{1}{3}x^3 - 3x - \frac{4}{x} + c. \end{aligned}$$

3.

$$\begin{aligned} I_3 &= \int \left(\sqrt{x} - \sqrt[3]{x^2} + \frac{1}{\sqrt[4]{x^3}} \right) dx \\ &= \int \left(x^{\frac{1}{2}} - x^{\frac{2}{3}} + x^{-\frac{3}{4}} \right) dx \\ &= \frac{2}{3}x^{\frac{3}{2}} - \frac{3}{5}x^{\frac{5}{3}} + 4x^{\frac{1}{4}} + c \\ &= \frac{2}{3}\sqrt{x^3} - \frac{3}{5}\sqrt[3]{x^5} + 4\sqrt[4]{x} + c. \end{aligned}$$

4.

$$\begin{aligned} I_4 &= \int (2x+5)^7 \, dx \\ &= \frac{1}{2} \int 2\,(2x+5)^7 \, dx \\ &= \frac{1}{2} \left[\frac{1}{8}(2x+5)^8 \right] + c \\ &= \frac{1}{16}(2x+5)^8 + c. \end{aligned}$$

5.

$$\begin{aligned} I_5 &= \int \frac{x}{(3x^2 - 5)^3} dx \\ &= \frac{1}{6} \int \frac{6x}{(3x^2 - 5)^3} dx \\ &= \frac{1}{6} \left[\frac{-1}{2(3x^2 - 5)^2} \right] + c \\ &= \frac{-1}{12(3x^2 - 5)^2} + c. \end{aligned}$$

6.

$$\begin{aligned} I_6 &= \int \frac{dx}{(3x-4)^5} \\ &= \frac{1}{3}\int \frac{3}{(3x-4)^5}dx \\ &= \frac{1}{3}\left[\frac{-1}{4(3x-4)^4}\right] + c \\ &= \frac{-1}{12(3x-4)^4} + c. \end{aligned}$$

7.

$$\begin{aligned} I_7 &= \int \frac{-2x-1}{x^2+x-2}dx \\ &= -\int \frac{2x+1}{x^2+x-2}dx \\ &= \ln(x^2+x-2) + c. \end{aligned}$$

8.

$$\begin{aligned} I_8 &= \int \frac{2x+1}{\sqrt{x^2+x+1}}dx \\ &= 2\int \frac{2x+1}{2\sqrt{x^2+x+1}}dx \\ &= 2\sqrt{x^2+x+1} + c. \end{aligned}$$

3.2 Defined integration

Definition 3.6. *Let f be a continuous function on $I = [a, b]$. The definite integral of f between a and b is the real number defined by:*

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a),$$

where F is a primitive of f on I.

Properties 3.7. *Let f and g be positive step functions on $[a, b]$, we have:*

- $\int_a^b f(x)dx \geq 0.$

- $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|\, dx.$

- *If k is a positive real verifying $|f(x)| \leq k$ on $[a,b]$, then,* $\left|\int_a^b f(x)dx\right| \leq k(b-a).$

- *If $f \geq g$, then,* $\int_a^b f(x)dx \geq \int_a^b g(x)dx.$

- *If $c \in [a,b]$, then,* $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$

Example 3.8.

1. *Let us calculate* $\int_0^{\frac{\Pi}{4}} \cos(2x)dx$*:*

$$\int_0^{\frac{\Pi}{4}} \cos(2x)dx = \left[\frac{1}{2}\sin(2x)\right]_0^{\frac{\Pi}{4}} = \frac{1}{2}\sin\frac{\Pi}{2} - \frac{1}{2}\sin 0 = \frac{1}{2}.$$

2. *Let us calculate* $\int_0^{\frac{\Pi}{3}} \frac{\sin x}{\cos^2 x}dx$*:*

$$\int_0^{\frac{\Pi}{3}} \frac{\sin x}{\cos^2 x}dx = -\left[\frac{-1}{\cos x}\right]_0^{\frac{\Pi}{3}} = -\left[\frac{-1}{\cos\frac{\Pi}{3}} + \frac{1}{\cos 0}\right] = 1.$$

3.2.1 Methods of integrations

1. **Integration by parts:**

Theorem 3.9. *Let u and v be two differentiable functions on an interval I. We have:*

$$\int u'(x)v(x)dx = u(x)v(x) - \int u(x)v'(x)dx.$$

Proof: We have: $[u(x)v(x)]' = u'(x)v(x) + u(x)v'(x)$ and we integrate, then,

$$\begin{aligned}
&\int [u(x)v(x)]'dx = \int u'(x)v(x)dx + \int u(x)v'(x)dx \\
\iff\ & u(x)v(x) = \int u'(x)v(x)dx + \int u(x)v'(x))dx \\
\iff\ & \int u'(x)v(x)dx = u(x)v(x) - \int u(x)v'(x))dx.
\end{aligned}$$

Remark 3.10. *Integration by parts is used in the case where the function to be integrated is an elementary function or product of the following elementary functions: trigonometric functions, polynomials, inverse functions of trigonometric functions,* $\ln[f(x)], e^{f(x)}, \cdots$

The method of integration by parts is frequently used in the calculation of integrals of the form: $\int x^k \sin x dx$, $\int x^k \cos x dx$, $\int x^k \ln x dx$, $\int x^k e^{ax} dx$.

Or in the reduction formulas which introduce for each integral a new integral, of the same form as the initial integral, but having a reduced or increased exponent.

Example 3.11.

(a) Let us calculate $\int x^n \ln x dx$. *We set:*

$$u(x) = \ln x \Longleftrightarrow u'(x) = \frac{1}{x}, \qquad v'(x) = x^n \Longleftrightarrow v(x) = \frac{1}{n+1}x^{n+1}.$$

$$\begin{aligned}
\int x^n \ln x dx &= \frac{1}{n+1}x^{n+1}\ln x - \int \frac{1}{n+1}x^{n+1}\frac{1}{x}dx \\
&= \frac{1}{n+1}x^{n+1}\ln x - \frac{1}{n+1}\int x^n dx \\
&= \frac{x^{n+1}}{n+1}\left(\ln x - \frac{1}{n+1}\right) + c, \qquad c \in \mathbb{R}.
\end{aligned}$$

(b) Let us calculate $\int x \sin x dx$. *We set:*

$$u(x) = x \Longleftrightarrow u'(x) = 1, \qquad v'(x) = \sin x \Longleftrightarrow v(x) = -\cos x.$$

$$\begin{aligned}
\int x \sin x dx &= -x\cos x + \int \cos x dx \\
&= -x\cos x + \sin x + c \\
&= x(\sin x - \cos x) + c, \qquad c \in \mathbb{R}.
\end{aligned}$$

(c) $\int \ln(ax+b)dx$. *We set:*

$$u'(x) = 1 \Longleftrightarrow u(x) = x, \qquad v(x) = \ln(ax+b) \Longleftrightarrow v'(x) = \frac{a}{ax+b}.$$

$$\begin{aligned}\int \ln(ax+b)dx &= x\ln(ax+b) - \int \frac{ax}{ax+b}dx \\ &= x\ln(ax+b) - \int \left[\frac{ax+b}{ax+b} - \frac{b}{ax+b}\right]dx \\ &= x\ln(ax+b) - x + \frac{b}{a}\int \frac{a}{ax+b}dx \\ &= x\ln(ax+b) - x + \frac{b}{a}\ln(ax+b) + c \\ &= \left(x+\frac{b}{a}\right)\ln(ax+b) - x + c, \qquad c\in\mathbb{R}.\end{aligned}$$

(d) $\int \arctan x$. *We set:*

$$u'(x) = 1 \Longleftrightarrow u(x) = x, \qquad v(x) = \arctan x \Longleftrightarrow v'(x) = \frac{1}{x^2+1}.$$

$$\begin{aligned}\int \arctan x dx &= x\arctan x - \int \frac{x}{x^2+1}dx \\ &= x\arctan x - \frac{1}{2}\int \frac{2x}{x^2+1}dx \\ &= x\arctan x - \frac{1}{2}\ln|x^2+1| + c, \qquad c\in\mathbb{R}.\end{aligned}$$

2. **Change of variable:**

If the calculation of $\int f(x)dx$ proves difficult, we replace x by $\varphi(t)$ derivable and therefore $dx = \varphi'(t)dt$ and we will have:

$$\int f(x)dx = \int f[\varphi(t)]\varphi'(t)dt.$$

Example 3.12.

(a) To calculate the integral $\int (x+3)^4dx$, *replace* $x+3$ *by* t. *Which gives* $x = t-3 \Rightarrow dx = dt$. *Then,*

$$\int (x+3)^4dx = \int t^4dt = \frac{1}{5}t^5 + c = \frac{1}{5}(x+3)^5 + c, \qquad c\in\mathbb{R}.$$

(b) Let us calculate $\int \sqrt{\sin x}\cos x dx$.

We set:

$$t = \sin x \Longleftrightarrow dt = \cos x dx \Longleftrightarrow dx = \frac{dt}{\cos x}.$$

$$\int \sqrt{\sin x}\cos x dx = \int t^{\frac{1}{2}}\cos x\frac{dt}{\cos x} = \int t^{\frac{1}{2}}dt = \frac{2}{3}t^{\frac{3}{2}} + c = \frac{2}{3}\sqrt{\sin^3 x} + c, \quad c\in\mathbb{R}.$$

(c) *Let us calculate* $\int \tan x dx = \int \frac{\sin x}{\cos x} dx$.

We set:

$$t = \cos x \Longleftrightarrow dt = -\sin x dx \Longleftrightarrow dx = -\frac{dt}{\sin x}.$$

$$\int \tan x dx = \int \frac{\sin x}{t} \frac{dt}{-\sin x} = -\int \frac{dt}{t} = -\ln|t| + c = -\ln|\cos x| + c, \quad c \in \mathbb{R}.$$

(d) *Let us calculate* $\int \tan^3 x dx = \int \frac{\sin^3 x}{\cos^3 x} dx = \int \frac{(1-\cos^2 x)\sin x}{\cos^3 x} dx$.

We set:

$$t = \cos x \Longleftrightarrow dt = -\sin x dx \Longleftrightarrow dx = -\frac{dt}{\sin x}.$$

$$\begin{aligned} \int \tan^3 x dx &= \int \frac{(1-t^2)\sin x}{t^3} \frac{dt}{-\sin x} = -\int \frac{dt}{t^3} + \int \frac{dt}{t} \\ &= -\frac{1}{2t^2} + \ln|t| + c = -\frac{1}{2\cos^2 x} + \ln|\cos x| + c, \quad c \in \mathbb{R}. \end{aligned}$$

(e) *Let us calculate* $\int \frac{\cos^3 x}{\sin^4 x} dx = \int \frac{(1-\sin^2 x)\cos x}{\sin^4 x} dx$.

We set:

$$t = \sin x \Longleftrightarrow dt = \cos x dx \Longleftrightarrow dx = \frac{dt}{\cos x}.$$

$$\begin{aligned} \int \frac{(1-\sin^2 x)\cos x}{\sin^4 x} dx &= \int \frac{(1-t^2)\cos x}{t^4} \frac{dt}{\cos x} \\ &= \int \frac{dt}{t^4} - \int \frac{dt}{t^2} = -\frac{1}{3t^3} + \frac{1}{t} + c \\ &= -\frac{1}{3\sin^3 x} + \frac{1}{\sin x} + c, \quad c \in \mathbb{R}. \end{aligned}$$

3.3 The integral of polynomials

Definition 3.13. *A polynomial with coefficients in* $\mathbb{K}(\mathbb{K} = \mathbb{R}$ *or* $\mathbb{K} = \mathbb{C})$ *is an expression of the forme:*

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} \cdots + a_2 x^2 + a_1 x + a_0,$$

where $n \in \mathbb{N}$ *and* $a_0, a_1, \cdots, a_n \in \mathbb{K}$.

- ▷ *The set of polynomials is denoted* $\mathbb{K}[X]$.
- ▷ *The* a_i *are called the coefficients of the polynomial.*
- ▷ *We call the degree of* P *the largest integer* i *such that* $a_i \neq 0$, *we denote it* $degP$.

We have:

$$\begin{aligned}\int P(x)dx &= \int [a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} \cdots + a_2 x^2 + a_1 x + a_0]dx \\ &= \frac{a_n}{n+1}x^{n+1} + \frac{a_{n-1}}{n}x^n + \frac{a_{n-2}}{n-1}x^{n-1} \cdots + \frac{a_2}{3}x^3 + \frac{a_1}{2}x^2 + a_0 x + c, \qquad c \in \mathbb{R}.\end{aligned}$$

Example 3.14. $\int (x^3 - x^2 + 2x + 3)dx = \frac{1}{4}x^4 - \frac{1}{3}x^3 + x^2 + 3x + c, \quad c \in \mathbb{R}.$

3.4 Integration of exponential and trigonometric functions

3.4.1 Integration of exponential functions

▷ To calculate the primitives of the form $\int f(e^x)dx$, we can choose the change of variable $e^x = t, dx = \frac{dt}{e^x}$, therefore

$$\int f(e^x)dx = \int \frac{1}{t}f(t)dt.$$

Example 3.15.

1. $\int \frac{e^x}{e^x+2}dx = \int \frac{1}{t}\frac{t}{t+2}dt = \ln|t+2| + c = \ln|e^x+2| + c, \quad c \in \mathbb{R}.$
2.

$$\begin{aligned}\int \frac{dx}{5\cosh x + 3\sinh x + 4} &= \int \frac{dx}{5\left(\frac{e^x+e^{-x}}{2}\right) + 3\left(\frac{e^x-e^{-x}}{2}\right) + 4} \\ &= \int \frac{e^x}{4e^{2x}+4e^x+1}dx = \int \frac{1}{t}\frac{t}{4t^2+4t+1}dt \\ &= \int \frac{dt}{(2t+1)^2} = \frac{1}{2}\int \frac{2dt}{(2t+1)^2} \\ &= \frac{1}{2}\left(\frac{-1}{2t+1}\right) + c = \frac{-1}{4e^x+2} + c, \qquad c \in \mathbb{R}.\end{aligned}$$

▷ Calculate the primitives of the form $I_n = \int x^n e^x dx$, we use the integral by part as follows:

$$u(x) = x^n \Longleftrightarrow u'(x) = nx^{n-1}, \qquad v'(x) = e^x \Longleftrightarrow v(x) = e^x.$$

Then,

$$I_n = x^n e^x - nI_{n-1}.$$

Example 3.16. *Let us calculate* $\int x^2 e^x dx$. *(Integration by part).*

We set:

$$u(x) = x^2 \Longleftrightarrow u'(x) = 2x, \qquad v'(x) = e^x \Longleftrightarrow v(x) = e^x.$$

Then,

$$\int x^2 e^x dx = x^2 e^x dx - 2\int xe^x dx.$$

Now let us calculate $\int xe^x dx$.

We set:

$$u(x) = x \Longleftrightarrow u'(x) = 1, \qquad v'(x) = e^x \Longleftrightarrow v(x) = e^x.$$

Then,

$$\begin{aligned}\int xe^x dx = xe^x - \int e^x dx \quad &\Longleftrightarrow \quad \int xe^x dx = xe^x - e^x + c\\ &\Longleftrightarrow \quad \int xe^x dx = (x-1)e^x + c, \qquad c \in \mathbb{R}.\end{aligned}$$

Finally,

$$\int x^2 e^x dx = x^2 e^x dx - 2[(x-1)e^x] + c = (x^2 - 2x + 2)e^x + c, \quad c \in \mathbb{R}.$$

3.4.2 Integration of trigonometric functions

Calculating the primitives of trigonometric functions requires a good knowledge of ance of trigonometry formulas.

Example 3.17. *Let us calculate* $I = \int \frac{\sin(2x)}{\sin^2 x - 5\sin x + 6} dx$.

We have:

$$I = \int \frac{\sin(2x)}{\sin^2 x - 5\sin x + 6} dx = \int \frac{2\sin x \cos x}{\sin^2 x - 5\sin x + 6} dx$$

Let us set:

$$t = \sin x \Longrightarrow dt = \cos x dx \Longrightarrow dx = \frac{dt}{\cos x}.$$

Then,

$$\begin{aligned}
I &= \int \frac{2t}{t^2-5t+6}dt \\
&= \int \frac{2t-5}{t^2-5t+6}dt + 5\int \frac{1}{t^2-5t+6}dt \\
&= \ln|t^2-5t+6| + 5\int \left[\frac{-1}{t-2} + \frac{1}{t-3}\right] dt \\
&= \ln|t^2-5t+6| - 5\ln|t-2| + 5\ln|t-3| + c \\
&= \ln|\sin^2 x - 5\sin x + 6| + 5\ln\left|\frac{\sin x - 3}{\sin x - 2}\right| + c, \qquad c \in \mathbb{R}.
\end{aligned}$$

▷ Calculate the primitives of the form $\int f(\sin x, \cos x, \tan x)dx$.

We can choose the change of variable $t = \tan\frac{x}{2}$, therefore we get:

$$\begin{cases} dx = \dfrac{2}{1+t^2}dt \\ \sin x = \dfrac{2t}{1+t^2} \\ \cos x = \dfrac{1-t^2}{1+t^2}. \end{cases}$$

Then,

$$\int f(\sin x, \cos x, \tan x)dx = \int f\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}, \frac{2t}{1-t^2}\right)\frac{2}{1+t^2}dt.$$

Proof:

- We have:

$$\begin{aligned}
t = \tan\frac{x}{2} &\iff \frac{x}{2} = \arctan t \\
&\iff x = 2\arctan t \\
&\iff dx = \frac{2}{t^2+1}dt.
\end{aligned}$$

- We have:

$$\begin{aligned} x = 2\arctan t &\iff \sin x = \sin(2\arctan t)\\ &\iff \sin x = 2\sin(\arctan t)\cos(\arctan t)\\ &\iff \sin x = 2\tan(\arctan t)\cos^2(\arctan t)\\ &\iff \sin x = 2t\cos^2(\arctan t)\\ &\iff \sin x = 2t\left(\frac{1}{1+t^2}\right) = \frac{2t}{1+t^2}, \end{aligned}$$

because

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha,$$

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} \Rightarrow \sin\alpha = \tan\alpha\cos\alpha,$$

and

$$\begin{aligned} 1+\tan^2\alpha &= 1+\frac{\sin^2\alpha}{\cos^2\alpha}\\ &= \frac{1}{\cos^2\alpha}\\ &\Rightarrow \cos^2\alpha = \frac{1}{1+\tan^2}. \end{aligned}$$

- We have:

$$\cos x = \sqrt{1-\sin^2 x} = \sqrt{1-\left(\frac{2t}{1+t^2}\right)^2} = \frac{1-t^2}{1+t^2}.$$

Example 3.18.

1. *Let us Calculate* $\int \frac{dx}{\sin^3 x}$*:*

$$\begin{aligned} \int\frac{dx}{\sin^3 x} &= 2\int\frac{\frac{dt}{1+t^2}}{\left(\frac{2t}{1+t^2}\right)^3}\\ &= \frac{1}{4}\int\frac{(1+t^2)^2}{t^3}dt\\ &= \frac{1}{4}\int\left(\frac{1}{t^3}+\frac{2}{t}+t\right)dt\\ &= \frac{1}{4}\left(-\frac{1}{2t^2}+2\ln|t|+\frac{1}{2}t^2\right)+c, \quad c\in\mathbb{R}. \end{aligned}$$

2. *Let us Calculate* $\int_{-\frac{\Pi}{2}}^{0} \frac{dx}{1-\sin x}$*:*

we have: $t = \tan\frac{x}{2}$*, therefore if* $x = \frac{-\Pi}{2}$*, then* $t = -1$ *and if* $x = 0$*, then* $t = 0$.

$$\begin{aligned}\int_{-\frac{\Pi}{2}}^{0} \frac{dx}{1-\sin x} &= 2\int_{-1}^{0} \frac{\frac{dt}{1+t^2}}{1-\frac{2t}{1+t^2}} \\ &= 2\int_{-1}^{0} \frac{1}{(1-t)^2}dt \\ &= -2\left[\frac{-1}{1-t}\right]_{-1}^{0} = 1.\end{aligned}$$

▷ Calculate the primitives of the form $\int \sin^n x \cos^m x dx$:

we can choose the change of variable $\sin x = t, \cos x = \sqrt{1-t^2}, dx = \frac{dt}{\cos x}$, therefore we get:

$$\int \sin^n x \cos^m x dx = \int t^n (1-t^2)^{\frac{m-1}{2}} dt.$$

Example 3.19. $\int \sin^2 x \cos^3 x dx = \int t^2(1-t^2)dt = \frac{1}{3}t^3 - \frac{1}{5}t^5 + c, \quad c \in \mathbb{R}.$

3.5 Integration of rational functions

A fraction function where rational function is a function $f(x)$ quotient of two polynomial functions $f(x) = \frac{P(x)}{Q(x)}$.

Example 3.20.

$$f_1(x) = \frac{1}{x^3+1}, \qquad f_2(x) = \frac{x+2}{x^3+3}, \qquad f_3(x) = \frac{x+3}{(x-1)^4(x^2-2x+3)^3}.$$

3.5.1 Decomposition into simple elements of a fraction $f(x) = \frac{P(x)}{Q(x)}$ on $\mathbb{R}$

▷ 1^{st} **case:** If *deg* $(P) <$ *deg* (Q).

Let $Q(x) = (x-a)^k(x-b)^m \cdots (x-c)^n(x^2+px+q)^t \cdots (x^2+rx+s)^l$ where $a, b, \cdots, c$ are real roots and the polynomial of second degree not admit solutions

($\Delta_1 = p^2 - 4q < 0$ et $\Delta_2 = r^2 - 4s < 0$). Then,

$$\begin{aligned}
\frac{P(x)}{Q(x)} &= \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \frac{A_3}{(x-a)^3} + \cdots + \frac{A_k}{(x-a)^k} \\
&\quad + \frac{B_1}{x-b} + \frac{B_2}{(x-b)^2} + \frac{B_3}{(x-b)^3} + \cdots + \frac{B_m}{(x-b)^m} \\
&\quad + \frac{C_1}{x-c} + \frac{C_2}{(x-c)^2} + \frac{C_3}{(x-c)^3} + \cdots + \frac{C_n}{(x-c)^n} \\
&\quad + \frac{M_1x+N_1}{x^2+px+q} + \frac{M_2x+N_2}{(x^2+px+q)^2} + \frac{M_3x+N_3}{(x^2+px+q)^3} + \cdots + \frac{M_\ell x+N_\ell}{(x^2+px+q)^\ell} \\
&\quad + \frac{U_1x+V_1}{x^2+rx+s} + \frac{U_2x+V_2}{(x^2+rx+s)^2} + \frac{U_3x+V_3}{(x^2+rx+s)^3} + \cdots + \frac{U_tx+V_t}{(x^2+rx+s)^t}.
\end{aligned}$$

Example 3.21.

1.

$$\begin{aligned}
f(x) &= \frac{1}{x^5-x^2} = \frac{1}{x^2(x^3-1)} = \frac{1}{x^2(x-1)(x^2+x+1)} \\
&= \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x-1} + \frac{dx+e}{x^2+x+1} \\
&= \frac{(a+c+d)x^4 + (b+c-d+e)x^3 + (-a+c-e)x^2 - b}{x^5-x^2}.
\end{aligned}$$

By identification, we obtain:

$$\begin{cases} a+c+d=0, \\ b+c-d+e=0, \\ -a+c-e=0, \\ -b=1, \end{cases} \iff \begin{cases} a=0, \\ b=-1, \\ c=e=\frac{1}{3} \\ d=\frac{-1}{3}. \end{cases}$$

Then,

$$f(x) = -\frac{1}{x^2} + \frac{1}{3(x-1)} - \frac{x-1}{3(x^2+x+1)}.$$

2.

$$\begin{aligned}
g(x) &= \frac{x}{(x^2-1)(x-2)} = \frac{x}{(x-1)(x+1)(x-2)} \\
&= \frac{a}{x-1} + \frac{b}{x+1} + \frac{c}{x-2} \\
&= \frac{(a+b+c)x^2 + (-a-3b)x - 2a + 2b - c}{(x^2-1)(x-2)}.
\end{aligned}$$

By identification, we obtain:

$$\begin{cases} a+b+c=0, \\ -a-3b=1, \\ -2a+2b-c=0, \end{cases} \iff \begin{cases} a=-\frac{1}{2}, \\ b=-\frac{1}{6}, \\ c=\frac{2}{3}. \end{cases}$$

Then,

$$g(x) = -\frac{1}{2(x-1)} - \frac{1}{6(x+1)} + \frac{2}{3(x+2)}.$$

3.

$$\begin{aligned} h(x) &= \frac{2x^2-3x+3}{x^3-2x^2+x} = \frac{2x^2-3x+3}{x(x-1)^2} \\ &= \frac{a}{x} + \frac{b}{x-1} + \frac{c}{(x-1)^2} \\ &= \frac{(a+b)x^2+(-2a-b+c)x+a}{x^3-2x^2+x}. \end{aligned}$$

By identification, we obtain:

$$\begin{cases} a+b=2, \\ -2a-b+c=-3, \\ a=3, \end{cases} \iff \begin{cases} a=3, \\ b=-1, \\ c=2. \end{cases}$$

Then,

$$h(x) = \frac{3}{x} - \frac{1}{x-1} + \frac{2}{(x-1)^2}.$$

So it is enough to know the values of the integrals of these types to deduce those of the integrals of the rationals.

▷ 2^{nd} **case:** If $deg\ (P) \geq deg\ (Q)$.

We devise $P(x)$ by $Q(x)$ such that the remainder $R(x)$ has $deg\ (R) < deg\ (Q)$. If $S(x)$ is the solution to the devising, then, $P(x) = Q(x) \cdot S(x) + R(x)$. Finally,

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)},$$

where $\dfrac{R(x)}{Q(x)}$ treated as 1^{st} case.

Example 3.22.

1.

$$f(x) = \frac{x^3+3x^2+5x+7}{x^2+2} = x+3+\frac{3x+1}{x^2+2}.$$

2.

$$g(x) = \frac{x^6+2x^4+2x^3-1}{x(x^2+1)^2} = x + \frac{2x^3-x^2-1}{x(x^2+1)^2}$$

$$\begin{aligned}\frac{2x^3-x^2-1}{x(x^2+1)^2} &= \frac{a}{x} + \frac{bx+c}{x^2+1} + \frac{dx+e}{(x^2+1)^2} \\ &= \frac{(a+b)x^4+cx^3+(2a+b+d)x^2+(c+e)x+a}{x(x^2+1)^2}.\end{aligned}$$

By identification, we obtain:

$$\begin{cases} a+b=0, \\ c=2, \\ 2a+b+d=-1, \\ c+e=0, \\ a=-1, \end{cases} \iff \begin{cases} a=-1, \\ b=1, \\ c=2, \\ d=0, \\ e=-2. \end{cases}$$

Then,

$$g(x) = x - \frac{1}{x} + \frac{x+2}{x^2+1} - \frac{2}{(x^2+1)^2}.$$

3.

$$h(x) = \frac{x^3+1}{x^3-5x^2+6x} = 1 + \frac{5x^2-6x+1}{x^3-5x^2+6x}$$

$$\begin{aligned}\frac{5x^2-6x+1}{x^3-5x^2+6x} &= \frac{5x^2-6x+1}{x(x-2)(x-3)} = \frac{a}{x} + \frac{b}{x-2} + \frac{c}{x-3} \\ &= \frac{(a+b+c)x^2+(-5a-3b-2c)x+6a}{x^3-5x^2+6x}.\end{aligned}$$

By identification, we obtain:

$$\begin{cases} a+b+c=5, \\ -5a-3b-2c=-6, \\ 6a=1, \end{cases} \iff \begin{cases} a=\frac{1}{6}, \\ b=\frac{-9}{2}, \\ c=\frac{28}{3}. \end{cases}$$

Then,

$$h(x) = 1 + \frac{1}{6x} - \frac{9}{2(x-2)} + \frac{28}{3(x-3)}.$$

4.

$$k(x) = \frac{x^4}{x^4+3x^2+2} = 1 + \frac{-3x^2-2}{x^4+3x^2+2}$$

$$\begin{aligned}\frac{-3x^2-2}{x^4+3x^2+2} &= \frac{-3x^2-2}{(x^2+1)(x^2+2)} = \frac{ax+b}{x^2+1}+\frac{cx+d}{x^2+2}\\ &= \frac{(a+c)x^3+(b+d)x^2+(2a+c)x+2b+d}{x^4+3x^2+2}.\end{aligned}$$

By identification, we obtain:

$$\begin{cases} a+c=0,\\ b+d=-3,\\ 2a+c=0,\\ 2b+d=-2 \end{cases} \iff \begin{cases} a=0,\\ b=1,\\ c=0,\\ d=-4. \end{cases}$$

Then,

$$k(x) = 1+\frac{1}{x^2+1}-\frac{4}{x^2+2}.$$

3.5.2 Integration of type $\int \frac{dx}{x-a}, \int \frac{dx}{(x-a)^k}$

$$\int \frac{dx}{(x-a)^k} = \begin{cases} \ln|x-a|+c, & \text{if } k=1\\ \dfrac{-1}{(k-1)(x-a)^{k-1}}+c, & \text{if } k\geq 2. \end{cases} \qquad c\in\mathbb{R}.$$

Example 3.23.

1. *Let us calculate* $\int \frac{x}{(x^2-1)(x-2)}dx$

$$\begin{aligned}\int \frac{x}{(x^2-1)(x-2)}dx &= -\int \frac{dx}{2(x-1)}-\int\frac{dx}{6(x+1)}+\int\frac{2dx}{3(x+2)}\\ &= -\frac{1}{2}\int\frac{dx}{x-1}-\frac{1}{6}\int\frac{dx}{x+1}+\frac{2}{3}\int\frac{dx}{x+2}\\ &= -\frac{1}{2}\ln|x-1|-\frac{1}{6}\ln|x+1|+\frac{2}{3}\ln|x+2|+c, \quad c\in\mathbb{R}.\end{aligned}$$

2. *Let us calculate* $\int \frac{2x^2-3x+3}{x^3-2x^2+x}dx$.

$$\begin{aligned}\int \frac{2x^2-3x+3}{x^3-2x^2+x}dx &= \int\frac{3dx}{x}-\int\frac{dx}{x-1}-\int\frac{2dx}{(x-1)^2}\\ &= 3\int\frac{dx}{x}-\int\frac{dx}{x-1}-2\int\frac{dx}{(x-1)^2}\\ &= 3\ln|x|-\ln|x-1|+\frac{2}{x-1}+c, \qquad c\in\mathbb{R}.\end{aligned}$$

3. *Let us calculate* $\int \frac{x^3+1}{x^3-5x^2+6x}dx$.

$$\begin{aligned}\int \frac{x^3+1}{x^3-5x^2+6x}dx &= \int dx + \int \frac{5x^2-6x+1}{x^3-5x^2+6x}dx \\ &= \int dx + \int \frac{dx}{6x} - \int \frac{9dx}{2(x-2)} + \int \frac{28dx}{3(x-3)} \\ &= \int dx + \frac{1}{6}\int \frac{dx}{x} - \frac{9}{2}\int \frac{dx}{x-2} + \frac{28}{3}\int \frac{dx}{x-3} \\ &= x + \frac{1}{6}\ln|x| - \frac{9}{2}\ln|x-2| + \frac{28}{3}\ln|x-3| + c, \qquad c \in \mathbb{R}.\end{aligned}$$

4. *Let us calculate* $\int \frac{\tan x}{1+\cos x}dx = \int \frac{\sin x}{\cos x(1+\cos x)}dx$. *(By change of variable). We set:*

$$t = \cos x \Longleftrightarrow dt = -\sin x dx \Longleftrightarrow dx = -\frac{dt}{\cos x}.$$

$$\begin{aligned}\int \frac{\sin x}{\cos x(1+\cos x)}dx &= -\int \frac{dt}{t(t+1)} \\ &= -\int \frac{dt}{t} + \int \frac{dt}{t+1} = -\ln|t| + \ln|t+1| + c \\ &= \ln\left|\frac{t+1}{t}\right| + c = \ln\left|1 + \frac{1}{\cos}\right| + c, \quad c \in \mathbb{R}.\end{aligned}$$

3.5.3 Integration of type $\int \frac{Mx+N}{x^2+px+q}dx$

We remark that $x^2+px+q = \left(x+\frac{p}{2}\right)^2 + \left(q - \frac{p^2}{4}\right)$, if we set $x + \frac{p}{2} = t$ and $q - \frac{p^2}{4} = \alpha^2 > 0$,

$$\begin{aligned}\int \frac{Mx+N}{x^2+px+q}dx &= \int \frac{M\left(t-\frac{p}{2}\right)+N}{t^2+\alpha^2}dt \\ &= M\int \frac{tdt}{t^2+\alpha^2} + \left(N - \frac{Mp}{2}\right)\int \frac{dt}{t^2+\alpha^2} \\ &= \frac{M}{2}\ln|t^2+\alpha^2| + \frac{2N-Mp}{2\alpha}\arctan\frac{t}{\alpha} + c \\ &= \frac{M}{2}\ln|x^2+px+q| + \frac{2N-Mp}{2\alpha}\arctan\frac{2x+p}{2\alpha} + c, \qquad c \in \mathbb{R}.\end{aligned}$$

Example 3.24.

1. *Let us calculate* $\int \frac{x^3+3x^2+5x+7}{x^2+2}dx$.

$$\begin{aligned}\int \frac{x^3+3x^2+5x+7}{x^2+2}dx &= \int (x+3)dx + \int \frac{3x+1}{x^2+2}dx \\ &= \frac{1}{2}x^2 + \frac{3}{2}\ln|x^2+2| + \frac{\sqrt{2}}{2}\arctan\frac{\sqrt{2}x}{2} + c, \qquad c \in \mathbb{R}.\end{aligned}$$

2. *Let us calculate* $\int \frac{dx}{x^5-x^2}$.

$$\begin{aligned}\int \frac{dx}{x^5-x^2} &= -\int \frac{dx}{x^2} + \frac{1}{3}\int \frac{dx}{x-1} - \frac{1}{3}\int \frac{x-1}{x^2+x+1}dx \\ &= \frac{1}{x} + \frac{1}{3}\ln|x-1| + \frac{1}{6}\ln|x^2+x+1| \\ &\quad + \frac{\sqrt{3}}{3}\arctan\frac{\sqrt{3}(2x+1)}{3} + c, \qquad c \in \mathbb{R}.\end{aligned}$$

3. *Let us calculate* $\int \frac{x^4}{x^4+3x^2+2}dx$.

$$\begin{aligned}\int \frac{x^4}{x^4+3x^2+2}dx &= \int dx + \int \frac{-3x^2-2}{x^4+3x^2+2}dx \\ &= \int dx + \int \frac{dx}{x^2+1} - 4\frac{dx}{x^2+2} \\ &= x + \arctan x - 2\sqrt{2}\arctan\frac{\sqrt{2}x}{2} + c, \qquad c \in \mathbb{R}.\end{aligned}$$

3.5.4 Integration of type $\int \frac{Mx+N}{(x^2+px+q)^\ell}dx$

If we set as above, $x + \frac{p}{2} = t$ and $q - \frac{p^2}{4} = \alpha^2 > 0$, we obtain:

$$\int \frac{Mx+N}{(x^2+px+q)^\ell}dx = M\int \frac{tdt}{(t^2+\alpha^2)^\ell} + \frac{2N-Mp}{2}\int \frac{dt}{(t^2+\alpha^2)^\ell}.$$

Let us study separately the two integrals obtained in the second member.

The first is calculated very simply:

$$\int \frac{tdt}{(t^2+\alpha^2)^\ell} = \frac{1}{2}\int \frac{2tdt}{(t^2+\alpha^2)^\ell} = \frac{-1}{2(\ell-1)(t^2+\alpha^2)^{\ell-1}}.$$

The calculation of the second integral is a little more complicated. Let

$$I_\ell = \int \frac{dt}{(t^2+\alpha^2)^\ell}, \qquad \ell = 1,2,3,\cdots.$$

If we integrate by parts, by seting,

$$u(t) = \frac{1}{(t^2+\alpha^2)^\ell} \Longleftrightarrow u'(t) = \frac{2\ell t}{(t^2+\alpha^2)^{\ell+1}}, \qquad v'(t) = 1 \Longleftrightarrow v(t) = t.$$

We obtain:

$$\begin{aligned} I_\ell &= \frac{t}{(t^2+\alpha^2)^\ell} + 2\ell \int \frac{t^2}{(t^2+\alpha^2)^{\ell+1}} dt \\ &= \frac{t}{(t^2+\alpha^2)^\ell} + 2\ell \int \frac{(t^2+\alpha^2)-\alpha^2}{(t^2+\alpha^2)^{\ell+1}} dt \\ &= \frac{t}{(t^2+\alpha^2)^\ell} + 2\ell \left[\int \frac{dt}{(t^2+\alpha^2)^\ell} - \alpha^2 \int \frac{dt}{(t^2+\alpha^2)^{\ell+1}} \right] + c, \qquad c \in \mathbb{R}. \end{aligned}$$

i.e.,

$$I_\ell = \frac{t}{(t^2+\alpha^2)^\ell} + 2\ell I_\ell - 2\ell\alpha^2 I_{\ell+1}.$$

Thus, we obtain the recurrence relation:

$$I_{\ell+1} = \frac{1}{2\ell\alpha^2}\frac{t}{(t^2+\alpha^2)^\ell} + \frac{2\ell-1}{2\ell\alpha^2} I_\ell \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (*)$$

which allows us to calculate I_ℓ for $\ell > 1$, knowing that I_1 is easy to calculate $\left(I_1 = \arctan \dfrac{t}{\alpha}\right)$.

Example 3.25.

1) *Let us calculate* $\int \frac{x^6+2x^4+2x^3-1}{x(x^2+1)^2} dx$*:*

$$\int \frac{x^6+2x^4+2x^3-1}{x(x^2+1)^2} = \int x dx - \int \frac{dx}{x} + \int \frac{x+2}{x^2+1} dx - 2\int \frac{dx}{(x^2+1)^2}.$$

We have:

$$\begin{aligned} \int \frac{x+2}{x^2+1} dx &= \int \frac{x}{x^2+1} dx + \int \frac{2}{x^2+1} dx \\ &= \frac{1}{2} \ln|x^2+1| + 2\arctan x + c, \qquad c \in \mathbb{R}. \end{aligned}$$

We can calculate

$$I_2 = \int \frac{1}{(x^2+1)^2} dx$$

by using the recurrence relation $(*)$ *where*

$$I_1 = \int \frac{1}{x^2+1}dx = \arctan x + c, \quad c \in \mathbb{R},$$

we obtain:

$$\begin{aligned} I_2 &= \frac{1}{2}\frac{x}{x^2+1} + \frac{1}{2}I_1 \\ &= \frac{1}{2}\frac{x}{x^2+1} + \frac{1}{2}\arctan x + c, \qquad c \in \mathbb{R}. \end{aligned}$$

Finally,

$$\int \frac{x^6+2x^4+2x^3-1}{x(x^2+1)^2} =$$

$$\begin{aligned} &= \int xdx - \int \frac{dx}{x} + \int \frac{x+2}{x^2+1}dx - 2\int \frac{dx}{(x^2+1)^2} \\ &= \frac{x^2}{2} - \ln|x| + \frac{\ln|x^2+1|}{2} + 2\arctan x - \frac{x}{x^2+1} - \arctan x + c \\ &= \frac{x^2}{2} - \ln|x| + \frac{\ln|x^2+1|}{2} + \arctan x + \frac{x}{x^2+1} + c, \qquad c \in \mathbb{R}. \end{aligned}$$

2) *Let us calculate* $\int \frac{3x+6}{(x^2+x+1)^2}dx$*:*

$$\begin{aligned} \int \frac{3x+6}{(x^2+x+1)^2}dx &= 3\int \frac{x+2}{(x^2+x+1)^2}dx \\ &= \frac{3}{2}\int \frac{2x+4}{(x^2+x+1)^2}dx \\ &= \frac{3}{2}\int \frac{2x+1}{(x^2+x+1)^2}dx + \frac{9}{2}\int \frac{1}{(x^2+x+1)^2}dx. \end{aligned}$$

▷ *Let us calculate* $\int \frac{2x+1}{(x^2+x+1)^2}dx$*:*

$$\begin{aligned} \int \frac{2x+1}{(x^2+x+1)^2}dx &= \int \frac{u'(x)}{[u(x)]^n}dx \\ &= \frac{-1}{x^2+x+1} + c, \qquad c \in \mathbb{R}. \end{aligned}$$

▷ *Let us calculate* $\int \frac{1}{(x^2+x+1)^2}dx$*:*

$$\begin{aligned}\int \frac{1}{(x^2+x+1)^2}dx &= \int \frac{1}{\left[\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right]^2}dx\\ &= \int \frac{1}{\left(t^2+\frac{3}{4}\right)^2}dt\\ &= \int \frac{1}{\left[\frac{3}{4}\left(\frac{4t^2}{3}+1\right)\right]^2}dt\\ &= \frac{16}{9}\int \frac{1}{\left[\left(\frac{2t}{\sqrt{3}}\right)^2+1\right]^2}\end{aligned}$$

and by change of variable $s=\frac{2t}{\sqrt{3}}$, *we obtain:*

$$\int \frac{1}{(x^2+x+1)^2}dx = \frac{16}{9}\frac{2}{\sqrt{3}}\int \frac{ds}{(s^2+1)^2}.$$

We can calculate

$$I_2 = \int \frac{1}{(s^2+1)^2}ds$$

by using the recurrence relation $(*)$ *with*

$$I_1 = \int \frac{1}{s^2+1}ds = \arctan s + c, \quad c\in\mathbb{R},$$

we obtain:

$$\begin{aligned}I_2 &= \frac{1}{2}\frac{s}{s^2+1}+\frac{1}{2}I_1\\ &= \frac{1}{2}\frac{s}{s^2+1}+\frac{1}{2}\arctan s + c\\ &= \frac{1}{2}\frac{2t}{\sqrt{3}\left(\frac{4t^2}{3}+1\right)}+\frac{1}{2}\arctan\frac{2t}{\sqrt{3}}+c\\ &= \frac{2x+1}{2\sqrt{3}\left(\frac{4\left(x+\frac{1}{2}\right)^2}{3}+1\right)}+\frac{1}{2}\arctan\frac{2x+1}{\sqrt{3}}+c, \qquad c\in\mathbb{R}.\end{aligned}$$

Finally,

$$\int \frac{3x+6}{(x^2+x+1)^2}dx = \frac{-1}{x^2+x+1} + \frac{2x+1}{2\sqrt{3}\left(\frac{4\left(x+\frac{1}{2}\right)^2}{3}+1\right)} + \frac{1}{2}\arctan\frac{2x+1}{\sqrt{3}} + c, \qquad c \in \mathbb{R}.$$

Geometric interpretation:

▷ Area of a domain between two curves: Let f and g be two continuous functions such that $f(x) \leq g(x)$ on the interval $[a,b]$. The algebraic area of the domain delimited by the graph C_f and C_g and the straight lines with equations $x = a$ and $x = b$ is

$$A = \int_a^b [g(x) - f(x)]dx.$$

▷ Calculation of volumes, in the space provided with the orthogonal plane $(O, \vec{i}, \vec{j}, \vec{k})$, the unit of volume is the volume of the right block constructed from the points O, I, J and K where $\overrightarrow{OI} = \vec{i}, \overrightarrow{OJ} = \vec{j}$ and $\overrightarrow{OK} = \vec{k}$. Let Σ be a solid limited by the planes of equations $z = a$ and $z = b$ where $a < b$.

If the intersection of Σ awith a plane of coast z is a surface whose area is given by $S(z)$, then, the volume of Σ is

$$V = \int_a^b S(z)dz.$$

Exercise 1.

1. *Determine the real constants a and b such that:*

$$\frac{1}{x^2-5x+6} = \frac{a}{x-2} + \frac{b}{x-3}.$$

2. *Calculate the following indefinite integral:*

$$I = \int \frac{1}{x^2-5x+6}dx$$

and deduce the value of the definite integral

$$I = \int_0^1 \frac{1}{x^2-5x+6}dx.$$

3. *EUsing a suitable change of variable, calculate*

$$J=\int_0^{\frac{\pi}{2}}\frac{2\cos t}{\sin^2 t-5\sin t+6}dt.$$

Solution:

1. Determine the real constants a and b:

$$\begin{aligned}\frac{1}{x^2-5x+6} &= \frac{a}{x-2}+\frac{b}{x-3}\\ &= \frac{a(x-3)+b(x-2)}{(x-2)(x-3)}\\ &= \frac{(a+b)x-3a-2b}{x^2-5x+6}.\end{aligned}$$

By identification, we obtain,

$$\begin{cases} a+b=0\\ -3a-2b=1\end{cases} \Longrightarrow \begin{cases} a=-b\\ b=1\end{cases} \Longrightarrow \begin{cases} a=-1\\ b=1.\end{cases}$$

Then,

$$\frac{1}{x^2-5x+6}=\frac{-1}{x-2}+\frac{1}{x-3}.$$

2. Calculate the following indefinite integral:

$$\begin{aligned}I=\int\frac{1}{x^2-5x+6}dx &= \int\frac{-1}{x-2}dx+\int\frac{1}{x-3}dx\\ &= -\ln|x-2|+\ln|x-3|+c, \qquad c\in\mathbb{R}.\end{aligned}$$

and

$$\begin{aligned}I=\int_0^1\frac{1}{x^2-5x+6}dx &= -\left[\ln|x-2|\right]_0^1+\left[\ln|x-3|\right]_0^1\\ &= 2\ln 2-\ln 3.\end{aligned}$$

3. Using a suitable change of variable, let us calculate:

$$J=\int_0^{\frac{\pi}{2}}\frac{2\cos t}{\sin^2 t-5\sin t+6}dt.$$

We set

$$x=\sin t\Longrightarrow dx=\cos t dt.$$

If

$$t = 0 \Longrightarrow x = \sin 0 = 0 \quad and \quad t = \frac{\Pi}{2} \Longrightarrow x = \sin\frac{\Pi}{2} = 1.$$

Then,

$$J = \int_0^1 \frac{1}{x^2 - 5x + 6} dx = 2I = 2\ln 2 - 2\ln 3.$$

Exercise 2. *Let*

$$I_1 = \int_0^{\Pi} (x\cos x)^2 dx, \quad I_2 = \int_0^{\Pi} (x\sin x)^2 dx.$$

1. *Calculate $I_1 + I_2$ and $I_1 - I_2$.*
2. *Deduce I_1 and I_2.*

Solution:

1. Let us calculate $I_1 + I_2$ and $I_1 - I_2$:

   ▷ Let us calculate $I_1 + I_2$:

$$\begin{aligned} I_1 + I_2 &= \int_0^{\Pi} (x\cos x)^2 dx + \int_0^{\Pi} (x\sin x)^2 dx \\ &= \int_0^{\Pi} x^2 dx = \left[\frac{1}{3}x^3\right]_0^{\Pi} = \frac{\Pi^3}{3}. \end{aligned}$$

   ▷ Let us calculate $I_1 - I_2$:

$$I_1 - I_2 = \int_0^{\Pi} (x\cos x)^2 dx - \int_0^{\Pi} (x\sin x)^2 dx = \int_0^{\Pi} x^2 \cos 2x dx.$$

   By integration by parts twice, let us set

$$u = x^2 \Rightarrow du = 2x, \quad dv = \cos(2x) \Rightarrow v = \frac{1}{2}\sin(2x).$$

$$I_1 - I_2 = \left[\frac{1}{2}x^2\sin(2x)\right]_0^{\Pi} - \int_0^{\Pi} x\sin(2x)dx,$$

   let us set

$$u = x \Rightarrow du = 1, \quad dv = \sin(2x) \Rightarrow v = -\frac{1}{2}\cos(2x).$$

$$\begin{aligned} I_1 - I_2 &= \left[\frac{1}{2}x^2\sin(2x)\right]_0^{\Pi} + \left[\frac{1}{2}x\cos(2x)\right]_0^{\Pi} - \frac{1}{2}\int_0^{\Pi}\cos(2x)dx \\ &= \left[\frac{1}{2}x^2\sin(2x) + \frac{1}{2}x\cos(2x) - \frac{1}{4}\sin(2x)\right]_0^{\Pi} = \frac{\Pi}{2}. \end{aligned}$$

2. Deduce I_1 and I_2:

$$\begin{cases} I_1 + I_2 = \dfrac{\Pi^3}{3} \\ \\ I_1 - I_2 = \dfrac{\Pi}{2} \end{cases} \Longrightarrow \begin{cases} I_1 = \dfrac{\Pi^3}{6} + \dfrac{\Pi}{4} \\ \\ I_2 = \dfrac{\Pi^3}{6} - \dfrac{\Pi}{4} \end{cases}$$

Exercise 3. *We set:*

$$H = \int_0^{\frac{\Pi}{8}} e^{-2x}\cos(2x)dx, \quad I = \int_0^{\frac{\Pi}{8}} e^{-2x}\cos^2(x)dx \quad et \quad J = \int_0^{\frac{\Pi}{8}} e^{-2x}\sin^2(x)dx.$$

1. Calculate H, $I+J$ and $I-J$.

2. Deduce I and J.

Solution: We set:

$$H = \int_0^{\frac{\Pi}{8}} e^{-2x}\cos(2x)dx, \quad I = \int_0^{\frac{\Pi}{8}} e^{-2x}\cos^2(x)dx \quad and \quad J = \int_0^{\frac{\Pi}{8}} e^{-2x}\sin^2(x)dx.$$

1. Let us calculate H, $I+J$ and $I-J$:

   - Let us calculate H by integration by part, we set

   $$U = e^{-2x} \Rightarrow dU = -2e^{-2x}, \quad dV = \cos(2x) \Rightarrow V = \frac{1}{2}\sin(2x).$$

   $$\begin{aligned} H &= \left[\frac{1}{2}e^{-2x}\sin(2x)\right]_0^{\frac{\Pi}{8}} + \int_0^{\frac{\Pi}{8}} e^{-2x}\sin(2x)dx \\ &= \frac{\sqrt{2}}{4}e^{-\frac{\Pi}{4}} + \int_0^{\frac{\Pi}{8}} e^{-2x}\sin(2x)dx. \end{aligned}$$

   By integration by part, we set

   $$U = e^{-2x} \Rightarrow dU = -2e^{-2x}, \quad dV = \sin(2x) \Rightarrow V = -\frac{1}{2}\cos(2x).$$

   $$\begin{aligned} H &= \frac{\sqrt{2}}{4}e^{-\frac{\Pi}{4}} + \left[-\frac{1}{2}e^{-2x}\cos(2x)\right]_0^{\frac{\Pi}{8}} - \int_0^{\frac{\Pi}{8}} e^{-2x}\cos(2x)dx \\ &= \frac{\sqrt{2}}{4}e^{-\frac{\Pi}{4}} - \frac{\sqrt{2}}{4}e^{-\frac{\Pi}{4}} + \frac{1}{2} - H. \end{aligned}$$

   $$\Longrightarrow 2H = \frac{1}{2} \Longrightarrow H = \frac{1}{4}$$

- Calculate $I + J$:

$$\begin{aligned} I + J &= \int_0^{\frac{\Pi}{8}} e^{-2x}[\cos^2(x) + \sin^2(x)]dx \\ &= \int_0^{\frac{\Pi}{8}} e^{-2x}dx = \left[-\frac{1}{2}e^{-2x}\right]_0^{\frac{\Pi}{8}} = \frac{1}{2}(1 - e^{-\frac{\Pi}{4}}) \end{aligned}$$

- Calculate $I - J$:

$$\begin{aligned} I - J &= \int_0^{\frac{\Pi}{8}} e^{-2x}[\cos^2(x) - \sin^2(x)]dx \\ &= \int_0^{\frac{\Pi}{8}} e^{-2x}\cos(2x)dx = H = \frac{1}{4} \end{aligned}$$

2. Deduce I and J:

$$\begin{cases} I + J = \dfrac{1}{2}(1 - e^{-\frac{\Pi}{4}}) \\ \\ I - J = \dfrac{1}{4} \end{cases} \Longrightarrow \begin{cases} I = \dfrac{3}{8} - \dfrac{1}{4}e^{-\frac{\Pi}{4}} \\ \\ J = \dfrac{1}{8} - \dfrac{1}{4}e^{-\frac{\Pi}{4}} \end{cases}$$

CHAPTER

4

DIFFERENTIAL EQUATIONS

In this chapter we will learn how to solve the most basic cases of first-order and second-order differential equations with constant coefficients.

4.1 Ordinary differential equations

4.1.1 Linear differential equations

Definition 4.1. *Let I be an open interval of $\mathbb{R}, A_0, \cdots, A_n$ and f be continuous functions in I. An equation of the form:*

$$A_0(x)y(x) + A_1(x)y'(x) + \cdots + A_n(x)y^{(n)}(x) = f(x), \qquad x \in I, \tag{4.1}$$

is called a linear differential equation.

The associated homogeneous equation is:

$$A_0(x)y(x) + A_1(x)y'(x) + \cdots + A_n(x)y^n(x) = 0, \qquad x \in I, \tag{4.2}$$

The function y is the unknown in these equations.

We call a solution of the equation (4.1) *any function y differentiable in I which verifies the equality $A_0(x)y(x) + A_1(x)y'(x) + \cdots + A_n(x)y^n(x) = f(x)$ for all $x \in I$.*

Example 4.2.

- $y' - y^2 - x = 0$ *is a* **homogeneous** *equation of the first order.*

- $y'' + xy + y^3 = x^2$ *is a* **non-homogeneous** *equation of the second order.*
- $y_1(x) = x$ *and* $y_2(x) = 5$ *are two solutions of the equation* $y''' + y'' = 0$, *then,* $y_3(x) = x + 5, y_4(x) = 10x$ *are also solutions.*
- $y = \cos x$ *is a solution defined in* $\mathbb{R}$ *of the differential equation* $y'' + y = 0$.

Proposition 4.3. *If* S_0 *is the set of solutions of* (4.2) *and* y_p *is a* **particular** *(obvious) solution of* (4.1), *then, the set of solutions of* (4.1) *is given by:*

$$S = \{y_p + y_h \quad \text{such that} \quad y_h \in S_0\}.$$

Proof: We have: y_p is a particular solution of (4.1) and y_h is a solution of (4.2), then,

$$A_0(x)y_p(x) + A_1(x)y_p'(x) + \cdots + A_n(x)y_p^n(x) = f(x),$$

and

$$A_0(x)y_h(x) + A_1(x)y_h'(x) + \cdots + A_n(x)y_h^n(x) = 0,$$

therefore

$$A_0(x)[y_p(x) + y_h(x)] + A_1(x)[y_p(x) + y_h(x)]' + \cdots + A_n(x)[y_p(x) + y_h(x)]^n = f(x).$$

Which implies that $y_p + y_h$ is a solution of (4.1).

Remark 4.4. *To solve or integrate a differential equation is to find all of its solutions when they exist.*

4.1.2 Differential equations with separate variables

Let I and J be two intervals of $\mathbb{R}$, $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ two continuous functions.

A differential equation is called to have separate or separable variables if it is of the form:

$$g(y)y' = f(x).$$

Solving: We have:

$$\begin{aligned} g(y)y' = f(x) \quad &\Longleftrightarrow \quad g(y)\frac{dy}{dx} = f(x) \\ &\Longleftrightarrow \quad g(y)dy = f(x)dx \\ &\Longleftrightarrow \quad \int g(y)dy = \int f(x)dx \\ &\Longleftrightarrow \quad G(y) = F(x) + c, \end{aligned}$$

where c is a constant.

Finally, we obtain,

$$y = G^{-1}[F(x) + c]$$

Where G is a primitive of g in J and F is a primitive of f in I.

Example 4.5.

1) *Consider the following differential equation:* $y' = xy$.

We remark that $y = 0$ *is a trivial (obvious) solution. Assume that* $y \neq 0$, *therefore* $\frac{y'}{y} = x$ *which is a separate variable with* $f(x) = x$ *and* $g(y) = \frac{1}{y}$. *We have:*

$$\begin{aligned}
y' = xy &\iff \frac{dy}{dx} = xy \\
&\iff \frac{dy}{y} = xdx \\
&\iff \ln|y| = \frac{1}{2}x^2 + c \\
&\iff |y| = e^{\frac{1}{2}x^2+c} \\
&\iff y = \pm e^{\frac{1}{2}x^2+c} \\
&\iff y = Ke^{\frac{1}{2}x^2}, \qquad K = \pm e^c.
\end{aligned}$$

Finally, the solutions are $y(x) = Ke^{\frac{1}{2}x^2}, \qquad K \in \mathbb{R}$. *Which are defined in* $\mathbb{R}$.

2) *Solve* $y' = y^2$. *We have:*

$$\begin{aligned}
y' = y^2 &\iff \frac{dy}{dx} = y^2 \\
&\iff \frac{dy}{y^2} = dx \\
&\iff -\frac{1}{y} = x + c \\
&\iff y = -\frac{1}{x+c}, \qquad c \in \mathbb{R}.
\end{aligned}$$

3) *Solve* $y' = x^2 + 1$. *We have:*

$$\begin{aligned}
y' = x^2 + 1 &\iff \frac{dy}{dx} = x^2 + 1 \\
&\iff dy = (x^2 + 1)dx \\
&\iff y = \frac{1}{3}x^3 + x + c, \qquad c \in \mathbb{R}.
\end{aligned}$$

4) *Solve in* $I =]1, \infty[$ *the following differential equation:*

$$xy' \ln x = (3\ln x + 1)y.$$

We can separate the variables x *and* y *by dividing by* $yx\ln x$*,*

$$\begin{aligned} xy' \ln x = (3\ln x + 1)y &\iff \frac{y'}{y} = \frac{3\ln x + 1}{x \ln x} \\ &\iff \frac{dy}{y} = \frac{3\ln x + 1}{x\ln x} dx \\ &\iff \int \frac{1}{y} dy = \int \left(\frac{3}{x} + \frac{1}{x\ln x}\right) dx \\ &\iff \ln|y| = 3\ln|x| + \ln|\ln x| + c \\ &\iff y = \pm e^{3\ln|x| + \ln|\ln x| + c} = K e^{3\ln|x| + \ln|\ln x|}, \qquad K = \pm e^c. \end{aligned}$$

4.1.3 Homogeneous differential equations in x and y

It is an equation of the form:

$$y' = f\left(\frac{y}{x}\right), \qquad x \neq 0, \tag{4.3}$$

where $f : I \longrightarrow \mathbb{R}$ is a continuous function.

Solving: We use this change of unknown $z = \dfrac{y}{x}$ which gives $y' = z + xz'$. Consequently,

$$\begin{aligned} (4.3) &\iff z + xz' = f(z) \\ &\iff z + x\frac{dz}{dx} = f(z) \\ &\iff zdx + xdz = f(z)dx \\ &\iff xdz = [f(z) - z]dx \\ &\iff \frac{x}{dx} = \frac{f(z) - z}{dz} \\ &\iff \int \frac{dx}{x} = \int \frac{dz}{f(z) - z} \\ &\iff \ln|x| = \phi(z) + c, \qquad \phi \textit{ is a primitive of } f(z) - z \\ &\iff x = \pm e^{\phi(z) + c} = K e^{\phi(z)}, \qquad K = \pm e^c, \end{aligned}$$

then, we determine y the solution of (4.3) using the relation $y = xz$.

Example 4.6. *Solve $xyy' = y^2 - x^2$. It is of the form $y' = f\left(\frac{y}{x}\right)$ where*

$$f\left(\frac{y}{x}\right) = \frac{y}{x} - \frac{x}{y}.$$

The change of unknown $z = \frac{y}{x}$ implies to the equation $z + xz' = z - \frac{1}{z}$

$$\begin{aligned} z + xz' = z - \frac{1}{z} &\iff z + x\frac{dz}{dx} = z - \frac{1}{z} \\ &\iff x\frac{dz}{dx} = -\frac{1}{z} \\ &\iff \frac{x}{dx} = -\frac{1}{zdz} \\ &\iff \frac{dx}{x} = -zdz \\ &\iff \ln|x| = -\frac{1}{2}z^2 + c \\ &\iff x = \pm e^{-\frac{1}{2}z^2 + c} \\ &\iff x = Ke^{-\frac{1}{2}z^2}, \qquad K = \pm e^c. \end{aligned}$$

Hence

$$y(x) = \pm\sqrt{x(K - x)}.$$

4.2 First-order differential equations

A first order linear differential equation is a differential equation of the form:

$$a(x)y' + b(x)y = f(x), \qquad or \qquad y' + b(x)y = f(x), \tag{4.4}$$

where a, b and f are real functions defined on an interval I. (4.4) is called a non-homogeneous differential equation (or with second member).

The following differential equation:

$$a(x)y' + b(x)y = 0, \qquad or \qquad y' + b(x)y = 0, \tag{4.5}$$

is called a homogeneous differential equation (or without a second member). (4.5) is called the homogeneous equation associated with equation (4.4).

4.2.1 Solving the homogeneous equation (4.5)

Let $y' + b(x)y = 0$. If $y \neq 0$, we have:

$$\begin{aligned} y' + b(x)y = 0 &\iff \frac{dy}{dx} = -b(x)y \\ &\iff \frac{dy}{y} = -b(x)dx \\ &\iff \ln|y| = -B(x) + c, \qquad B \textit{ is a primitive of } b(x) \\ &\iff y = \pm e^{-B(x)+c} = Ke^{-B(x)}, \qquad K = \pm e^c. \end{aligned}$$

Remark 4.7. $y = 0$ *is a trivial (obvious) solution of* (4.5). *Finally,*

$$y(x) = Ke^{-B(x)}, \qquad K \in \mathbb{R}$$

is the homogeneous solution of (4.5).

4.2.2 Solving the non-homogeneous equation (4.4)

If y_h is the homogeneous solution of (4.5) and y_p is a particular solution of (4.4).

Method of variation of the constant:

Let $y_h(x) = Ke^{-B(x)}$ be the homogeneous solution of (4.5). We vary the constant K, and the particular solution of (4.4) will be $y_p(x) = K(x)e^{-B(x)}$. We have:

$$y_p'(x) = K'(x)e^{-B(x)} - K(x)B'(x)e^{-B(x)}.$$

By replacing $y_p(x)$ and $y_p'(x)$ in (4.4), we obtain:

$$(4.4) \iff K'(x)e^{-B(x)} - K(x)B'(x)e^{-B(x)} + b(x)K(x)e^{-B(x)} = f(x),$$

we obtain $K(x)$ and finally, the general solution of (4.4) is given by:

$$y_g(x) = y_h(x) + y_p(x).$$

Example 4.8.

1) *Solve the differential equation:*

$$y' - y = e^x \cdots\cdots\cdots\cdot (E).$$

The associated homogeneous equation is

$$y' - y = 0 \cdots\cdots\cdots\cdot (EH).$$

For $y \neq 0$,

$$\begin{aligned} y' - y = 0 &\iff \frac{dy}{y} = dx \\ &\iff \ln|y| = x + c \\ &\iff y = Ke^x, \qquad K = \pm e^c. \end{aligned}$$

$y = 0$ is an obvious solution of (EH). Finally, the general solution of (EH) is $y(x) = Ke^x$, $K \in \mathbb{R}$. We vary the constant K and the general solution of (E) will be $y(x) = K(x)e^x$.

We have: $y'(x) = K'(x)e^x + K(x)e^x$. By replacing $y(x)$ and $y'(x)$ in (E), we obtain:

$$\begin{aligned} (E) &\iff K'(x)e^x + K(x)e^x - K(x)e^x = e^x \\ &\iff K'(x) = 1 \\ &\iff K(x) = x + d, \qquad d \in \mathbb{R}. \end{aligned}$$

Therefore the general solution of (E) is $y(x) = (x + d)e^x$.

2) *Solve the differential equation:*

$$y' + 2xy = 2xe^{-x^2} \cdots\cdots\cdots\cdot (E).$$

The associated homogeneous equation is

$$y' + 2xy = 0 \cdots\cdots\cdots\cdot (EH).$$

For $y \neq 0$,

$$\begin{aligned} y' + 2xy = 0 &\iff \frac{dy}{y} = -2xdx \\ &\iff \ln|y| = -x^2 + c \\ &\iff y = Ke^{-x^2}, \qquad K = \pm e^c. \end{aligned}$$

$y = 0$ is an obvious solution of (EH). Finally, the general solution of (EH) is $y(x) = Ke^{-x^2}$, $K \in \mathbb{R}$. We vary the constant K and the general solution of (E) will be $y(x) = K(x)e^{-x^2}$.

We have: $y'(x) = K'(x)e^{-x^2} - 2xK(x)e^{-x^2}$. By replacing $y(x)$ and $y'(x)$ in (E), we obtain:

$$\begin{aligned} (E) &\iff K'(x)e^{-x^2} - 2xK(x)e^{-x^2} + 2xK(x)e^{-x^2} = 2xe^{-x^2} \\ &\iff K'(x) = 2x \\ &\iff K(x) = x^2 + d, \qquad d \in \mathbb{R}. \end{aligned}$$

Therefore the general solution of (E) is $y(x) = (x^2 + d)e^{-x^2}$.

3) *Solve the differential equation:*

$$y' - \frac{1}{x}y = x \cdots\cdots\cdots\cdot (E).$$

The associated homogeneous equation is

$$y' - \frac{1}{x}y = 0 \cdots\cdots\cdots\cdot (EH).$$

For $y \neq 0$,

$$\begin{aligned} y' - \frac{1}{x}y = 0 &\iff \frac{dy}{y} = \frac{1}{x}dx \\ &\iff \ln|y| = \ln|x| + c \\ &\iff y = Kx, \qquad K = \pm e^c. \end{aligned}$$

$y = 0$ *is an obvious solution of* (EH)*. Finally, the general solution of* (EH) *is* $y(x) = Kx,\ K \in \mathbb{R}$*. We vary the constant* K *and the general solution of* (E) *will be* $y(x) = K(x)x$*.*

We have: $y'(x) = K'(x)x + K(x)$*. By replacing* $y(x)$ *and* $y'(x)$ *in* (E)*, we obtain:*

$$\begin{aligned} (E) &\iff K'(x)x + K(x) - \frac{1}{x}K(x)x = x \\ &\iff K'(x) = 1 \\ &\iff K(x) = x + d, \qquad d \in \mathbb{R}. \end{aligned}$$

Therefore the general solution of (E) *is* $y(x) = (x + d)x$*.*

4) *Solve in* $\mathbb{R}^*$ *the differential equation:*

$$xy' + 2y = \frac{x^2}{x^2+1} \cdots\cdots\cdots\cdot (E).$$

The associated homogeneous equation is

$$y' + \frac{2}{x}y = 0 \cdots\cdots\cdots\cdot (EH).$$

For $y \neq 0$,

$$\begin{aligned} y' + \frac{2}{x}y = 0 &\iff \frac{dy}{y} = -\frac{2}{x}dx \\ &\iff \ln|y| = -2\ln|x| + c \\ &\iff y = \frac{K}{x^2}, \qquad K = \pm e^c. \end{aligned}$$

$y = 0$ is an obvious solution of (EH). Finally, the general solution of (EH) is $y(x) = \dfrac{K}{x^2}$, $K \in \mathbb{R}$. We vary the constant K and the general solution of (E) will be $y(x) = \dfrac{K(x)}{x^2}$.

We have: $y'(x) = \dfrac{K'(x)x^2 - 2xK(x)}{x^4}$. By replacing $y(x)$ and $y'(x)$ in (E), we obtain:

$$\begin{aligned}
(E) \iff & \frac{K'(x)x^2 - 2xK(x)}{x^4} + \frac{2}{x}\frac{K(x)}{x^2} = \frac{x}{x^2+1} \\
\iff & \frac{K'(x)}{x^2} - \frac{-2K(x)}{x^3} + \frac{2K(x)}{x^3} = \frac{x}{x^2+1} \\
\iff & K'(x) = \frac{x^3}{x^2+1} = x - \frac{x}{x^2+1} \\
\iff & K(x) = \frac{1}{2}x^2 - \frac{1}{2}\ln|x^2+1| + d, \qquad d \in \mathbb{R}.
\end{aligned}$$

Therefore the general solution of (E) is

$$y(x) = \frac{\frac{1}{2}x^2 - \frac{1}{2}\ln|x^2+1| + d}{x^2} = \frac{1}{2} - \frac{\ln|x^2+1| + d}{2x^2}.$$

4.2.3 Bernoulli's differential equation

Let f and g be two continuous functions on an interval I. An equation of the form:

$$y' + f(x)y + g(x)y^\alpha = 0, \qquad \alpha \in \mathbb{R},\ \alpha \neq 0, 1 \tag{4.6}$$

is called Bernoulli's differential equation.

We discard the cases $\alpha = 0$ and $\alpha = 1$ for which the equation is linear. The function y will be assumed to be positive if α is non-integer and furthermore non-zero if α is negative.

To find the solutions of the Bernoulli's differential equation, we divide by y^α, then, we change the unknown function $z = y^{1-\alpha}$. We will have:

$$\frac{y'}{y^\alpha} + \frac{f(x)}{y^{\alpha-1}} + g(x) = 0$$

and therefore,

$$y'y^{-\alpha} + f(x)y^{1-\alpha} + g(x) = 0.$$

This last equation becomes of z (where $z' = (1-\alpha)y^{-\alpha}y'$)

$$\frac{z'}{1-\alpha} + f(x)z + g(x) = 0,$$

which is a first order linear differential equation

Example 4.9.

1) *Let Bernoulli's differential equation:* $y' + xy + xy^4 = 0$. *It is of the form* (4.6) *where* $\alpha = 4, f(x) = g(x) = x$. *By setting* $z = y^{-3}$ *for* $y \neq 0$, *we will have:*

$$z' - 3xz = 3x \cdots\cdots\cdots\cdots (E).$$

The homogeneous equation associated of (E) *is*

$$z' - 3xz = 0 \cdots\cdots\cdots\cdots (EH).$$

$$\begin{aligned} z' - 3xz = 0 &\iff \frac{dz}{z} = 3xdx \\ &\iff \ln|z| = \frac{3}{2}x^2 + C \\ &\iff z = Ke^{\frac{3}{2}x^2}, \qquad K = \pm e^C. \end{aligned}$$

$z = 0$ *is an obvious solution of* (EH). *Finally, the general solution of* (EH) *is* $z(x) = Ke^{\frac{3}{2}x^2}$, $K \in \mathbb{R}$. *We vary the constant* K *and the general solution of* (E) *will be* $z(x) = K(x)e^{\frac{3}{2}x^2}$.

We have: $z'(x) = K'(x)e^{\frac{3}{2}x^2} + 3K(x)e^{\frac{3}{2}x^2}$. *By replacing* $z(x)$ *and* $z'(x)$ *in* (E), *we obtain:*

$$\begin{aligned} (E) &\iff K'(x)e^{\frac{3}{2}x^2} + 3xK(x)e^{\frac{3}{2}x^2} - 3xK(x)e^{\frac{3}{2}x^2} = 3x \\ &\iff K'(x) = 3xe^{-\frac{3}{2}x^2} \\ &\iff K(x) = -e^{-\frac{3}{2}x^2} + d, \qquad d \in \mathbb{R}. \end{aligned}$$

Therefore the general solution of (E) *is* $z(x) = \left(-e^{-\frac{3}{2}x^2} + d\right)e^{\frac{3}{2}x^2} = -1 + de^{\frac{3}{2}x^2}$. *Hence*

$$\frac{1}{y^3(x)} = -1 + d\ e^{\frac{3}{2}x^2} \iff y(x) = \sqrt[3]{\frac{1}{-1 + d\ e^{\frac{3}{2}x^2}}}.$$

2) *Let the Bernoulli's differential equation:* $xy' + y = y^2 \ln x$. *It is of the form* (4.6) *where* $\alpha = 2, f(x) = \frac{1}{x}$ *and* $g(x) = -\frac{\ln x}{x}$. *By setting* $z = y^{-1}$ *pour* $y \neq 0$, *we will have:*

$$z' - \frac{1}{x}z = -\frac{\ln x}{x} \cdots\cdots\cdots\cdots (E).$$

The homogeneous equation associated of (E) *is*

$$z' - \frac{1}{x}z = 0 \cdots\cdots\cdots\cdots (EH).$$

$$\begin{aligned} z' - \frac{1}{x}z = 0 &\iff \frac{dz}{z} = \frac{1}{x}dx \\ &\iff \ln|z| = \ln x + C \\ &\iff z = Kx, \qquad K = \pm e^C. \end{aligned}$$

$z=0$ is an obvious solution of (EH). Finally, the general solution of (EH) is $z(x)=Kx,\ K\in\mathbb{R}$. We vary the constant K and the general solution of (E) will be $z(x)=K(x)x$.

We have: $z'(x)=K'(x)x+K(x)$. By replacing $z(x)$ and $z'(x)$ in (E), we obtain:

$$\begin{aligned}(E) &\iff K'(x)x+K(x)-\frac{1}{x}K(x)x=-\frac{\ln x}{x}\\ &\iff K'(x)=-\frac{\ln x}{x^2}\\ &\iff K(x)=\int -\frac{\ln x}{x^2}dx.\end{aligned}$$

Let us calculate $\int -\frac{\ln x}{x^2}dx$ (Integration by parts). We set:

$$u(x)=\ln x \iff u'(x)=\frac{1}{x}, \qquad v'(x)=-\frac{1}{x^2}\iff v(x)=\frac{1}{x}.$$

$$\begin{aligned}\int -\frac{\ln x}{x^2}dx &= \frac{\ln x}{x}-\int \frac{1}{x}\frac{1}{x}dx\\ &= \frac{\ln x}{x}+\frac{1}{x}+d, \qquad d\in\mathbb{R}.\end{aligned}$$

Therefore the general solution of (E) is $z(x)=\left(\frac{\ln x}{x}+\frac{1}{x}+d\right)x=\ln x+1+d$. Hence,

$$\frac{1}{y(x)}=\ln x+d\ x+1 \iff y(x)=\frac{1}{\ln x+d\ x+1}.$$

4.2.4 Solution verifying an initial condition

The data of an initial condition for equation (4.4) in the open interval I is the data of a point x_0 of I and a real number y_0. A solution satisfying this initial condition is a solution y such that $y(x_0)=y_0$.

The initial condition allows us to determine the exact constant of the general solution y_g of equation (4.4), which shows the existence and uniqueness of the solution verifying the initial condition.

Remark 4.10. *There exists only one solution of equation (4.4) in I satisfying the initial condition $y(x_0)=y_0$.*

Example 4.11. *Solve* $\begin{cases} y'-\dfrac{1}{x}y=x\\ y(1)=0.\end{cases}$

The general solution given by:

$$y(x)=(x+d)x, \qquad d\in\mathbb{R}.$$

We have:

$$y(1) = 0 \Longleftrightarrow 1 + d = 0 \Rightarrow d = -1.$$

Finally, the solution y which is verified the initial condition is given by:

$$y(x) = (x - 1)x.$$

4.3 Second-order differential equations

A second order differential equation is of the form:

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x), \tag{4.7}$$

where a, b and c are given functions, called coefficients of the differential equation and f is called the second member of the differential equation. A solution of (4.7) is a function y of class C^2 on an interval I verifying (4.7) for all $x \in I$.

The general solution of equation (4.7) is written:

$$y_g(x) = y_h(x) + y_p(x),$$

where y_h is the solution of the homogeneous equation associated of the equation (4.7) as follows:

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = 0,$$

and y_p is a particular solution of (4.7).

4.4 The second-order ordinary differential equations with constant coefficient

Let $a, b, c \in \mathbb{R}$ and $f : I \longrightarrow \mathbb{R}$ be a continuous function. The differential equation

$$ay'' + by' + cy = f(x), \tag{4.8}$$

is called a second-order differential equation with constant coefficients with a second member. We associate it with the equation without a second member

$$ay'' + by' + cy = 0. \tag{4.9}$$

Proposition 4.12. *If y_h is a general solution of the homogeneous equation* (4.9) *and y_p is a particular solution of the nonhomogeneous equation* (4.8)*, then, $y_p + y_h$ is a general solution of* (4.8)*.*

4.4.1 Solving the homogeneous equation (4.9)

We seek the solution in the form $y = e^{rx}, r \in \mathbb{R}$.

Hence we have: $y' = re^{rx} = ry$ and $y'' = r^2 e^{rx} = r^2 y$, so the non-homogeneous equation (4.8) becomes

$$y(ar^2 + br + c) = 0.$$

The equation

$$ar^2 + br + c = 0, \tag{4.10}$$

is called the characteristic equation of the differential equation (4.9).

Depending on the sign of $\Delta = b^2 - 4ac$, we have the following results:

- ▷ If $\Delta < 0$: the characteristic equation (4.10) admits two complex conjugate roots $r_1 = \alpha - i\beta$ et $r_2 = \alpha + i\beta, \alpha, \beta \in \mathbb{R}$, then, the general solution of (4.9) is of the form::

$$y(x) = (C_1 \cos \beta x + C_2 \sin \beta x)e^{\alpha x}.$$

- ▷ If $\Delta = 0$: the characteristic equation (4.10) admits a double root r then, the general solution of (4.9) is of the form:

$$y(x) = (C_1 x + C_2)e^{rx}.$$

- ▷ If $\Delta > 0$: the characteristic equation (4.10) admits two distinct real roots $r_1 \neq r_2$, then, the general solution of (4.9) is of the form:

$$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}.$$

Where C_1, C_2 are two real constants.

Example 4.13. *Solve the following differential equations:*

1) $y'' + 2y' + y = 0$, *the characteristic equation is* $r^2 + 2r + 1 = 0 \iff (r+1)^2 = 0$. *We have:* $\Delta = 0$, *therefore this equation admits a double root* $r = -1$, *the general solution is of the form:*

$$y(x) = (C_1 x + C_2)e^{-x}, \qquad C_1, C_2 \in \mathbb{R}.$$

2) $y'' - 4y' + 3y = 0$, *the characteristic equation is* $r^2 - 4r + 3 = 0$. *We have:* $\Delta = 4 > 0$, *therefore this equation admits two distinct roots* $r_1 = 1$ *and* $r_2 = 3$, *the general solution is of the form:*

$$y(x) = C_1 e^{x} + C_2 e^{3x}, \qquad C_1, C_2 \in \mathbb{R}.$$

3) $y'' + 2y' + 4y = 0$, *the characteristic equation is* $r^2 + 2r + 4 = 0$. *We have:* $\Delta = -12 < 0$, *therefore this equation admits two complex conjugate roots* $r_1 = -1 - i\sqrt{3}$ *and* $r_2 = -1 + i\sqrt{3}$, *the general solution is of the form:*

$$y(x) = (C_1 \cos \sqrt{3}x + C_2 \sin \sqrt{3}x)e^{-x}, \qquad C_1, C_2 \in \mathbb{R}.$$

4.4.2 Solving the non-homogeneous equation (4.8)

the general solution of (4.8) is written in the form $y = y_h + y_p$ where y_h is the general solution of the homogeneous equation (4.9) and y_p s a particular solution of the equation with right-hand side.

- **The second member is the sum of two terms:** A particular solution of

$$ay'' + by' + cy = f_1(x) + f_2(x)$$

is the sum of a particular solution of the equation

$$ay'' + by' + cy = f_1(x)$$

and a particular solution of the equation

$$ay'' + by' + cy = f_2(x).$$

- **The second member is a polynomial of degree n:** Let to be solved

$$ay'' + by' + cy = P_n(x)$$

where P_n is a polynomial of degree n. We are seeking for a particular polynomial solution. We distinguish two cases:

▷ If $c \neq 0$, we seek y_p in the form of a polynomial of degree n.

▷ If $c = 0$ and $b \neq 0$, $y_p = xQ_n(x)$ where Q_n is a polynomial of degree n.

Example 4.14. *Let us solve the following equation:*

$$y'' + 2y' - 3y = x^3 + 2x + 1 \cdots\cdots\cdots\cdot (E).$$

We have: $y_h = C_1 e^x + C_2 e^{-3x}$, where C_1, C_2 are two real constants.

Since $c = -3 \neq 0$, we seek a particular solution in the form:

$$y_p = a_0 x^3 + a_1 x^2 + a_2 x + a_3.$$

By replacing $y_p(x), y_p'(x)$ and $y_p''(x)$ in (E) and identifying, we obtain:

$$a_0 = -\frac{1}{3}, \qquad a_1 = -\frac{2}{3}, \qquad a_2 = -\frac{20}{9}, \qquad a_3 = -\frac{61}{27}.$$

The general solution is

$$y(x) = C_1 e^x + C_2 e^{-3x} - \frac{1}{3}x^3 - \frac{2}{3}x^2 - \frac{20}{9}x + -\frac{61}{27}.$$

- **The second member is of the form e^{mx} (m is a constant):** In the search for a particular solution y_p, it is necessary to distinguish three cases according to the values of m.
  - ▷ m is not a root of the characteristic equation. We then seek a particular solution in the form:
  $$y_p = Ke^{mx}.$$
  - ▷ m is a simple root of the characteristic equation. We then seek a particular solution in the form:
  $$y_p = Kxe^{mx}.$$
  - ▷ m is a double root of the characteristic equation. We then seek a particular solution in the form:
  $$y_p = Kx^2e^{mx}.$$

Example 4.15.

1) *Let us solve the following equation:*
$$y'' - 4y' + 4y = e^{2x} \cdots\cdots\cdots\cdot (E).$$
We have: $y_h = (C_1x + C_2)e^{2x}$, *where* C_1, C_2 *are two real constants. We are seeking for a particular solution in the form:*
$$y_p = Kx^2e^{2x},$$
because 2 *is a double root of the characteristic equation. By replacing* $y_p(x), y'_p(x)$ *and* $y''_p(x)$ *in* (E) *and identifying, we find* $K = \frac{1}{2}$, *therefore the general solution*
$$y(x) = (C_1x + C_2)e^{2x} + \frac{1}{2}x^2e^{2x} = (\frac{1}{2}x^2 + C_1x + C_2)e^{2x}, \qquad C_1, C_2 \in \mathbb{R}.$$

2) *Let us solve the following equation:*
$$y'' - 5y' + 6y = 2e^{3x} + e^{4x} \cdots\cdots\cdots\cdot (E).$$
We have: $y_h = C_1e^{2x} + C_1e^{3x}$, *where* C_1, C_2 *are two real constants. We are seeking for a particular solution* $y_p = y_{p_1} + y_{p_2}$ *where* y_{p_1} *is a particular solution of*
$$y'' - 5y' + 6y = 2e^{3x} \cdots\cdots\cdots\cdot (E_1)$$
and y_{p_2} *is a particular solution of*
$$y'' - 5y' + 6y = e^{4x} \cdots\cdots\cdots\cdot (E_2)$$
in the form: $y_{p_1} = K_1xe^{3x}$ *and* $y_{p_2} = K_2e^{4x}$. *By replacing* $y_{p_1}(x), y'_{p_1}(x)$ *and* $y''_{p_1}(x)$ *in* (E_1) *and identifying, we find* $K_1 = \frac{1}{2}$, *then, replacing* $y_{p_2}(x), y'_{p_2}(x)$ *and* $y''_{p_2}(x)$ *in* $(E_)$ *and identifying, we find* $K_2 = 2$. *Finally, the general solution is*
$$y(x) = C_1e^{2x} + C_1e^{3x} + \frac{1}{2}xe^{3x}2)e^{4x}, \qquad C_1, C_2 \in \mathbb{R}.$$

- **The second member is of the form** $\sin mx$ **(or** $\cos mx, m$ **is a constant):** In this situation, it is necessary to distinguish two cases in the search for a solution particular.

  ▷ im is not a root of the characteristic equation. We then seek a solution particular in the form:
  $$y_p = K_1 \cos mx + K_2 \sin mx$$
  and we determine the constants K_1 and K_2 by identification.

  ▷ im is a root of the characteristic equation. We then seek a solution particular in the form:
  $$y_p = x(K_1 \cos mx + K_2 \sin mx)$$
  and as in the previous case, we determine the constants K_1 and K_2 by identification.

Example 4.16.

1) *Let us solve the following equation:*

$$y'' + 4y' = \cos x \cdots\cdots\cdots\cdots (E).$$

We have: $y_h = C_1 \cos 2x + C_2 \sin 2x$, *where* C_1, C_2 *are two real constants and we have:* i *is not a root of the characteristic equation, therefore*

$$y_p = K_1 \cos x + K_2 \sin x.$$

By replacing $y_p(x), y_p'(x)$ *and* $y_p''(x)$ *in* (E) *and identifying, we find* $K_1 = \frac{1}{3}$ *and* $K_2 = 0$. *Finally, the general solution is*

$$y(x) = C_1 \cos 2x + C_2 \sin 2x + \frac{1}{3} \cos x, \qquad C_1, C_2 \in \mathbb{R}.$$

2) *Let us solve the following equation:*

$$y'' + 9y' = \sin 3x \cdots\cdots\cdots\cdots (E).$$

We have: $y_h = C_1 \cos 3x + C_2 \sin 3x$, *where* C_1, C_2 *are two real constants and we have:* $3i$ *is a root of the characteristic equation, therefore*

$$y_p = x(K_1 \cos x + K_2 \sin x).$$

By replacing $y_p(x), y_p'(x)$ *and* $y_p''(x)$ *in* (E) *and identifying, we find* $K_1 = -\frac{1}{6}$ *and* $K_2 = 0$. *Finally, the general solution is*

$$y(x) = C_1 \cos 3x + C_2 \sin 3x - \frac{x}{6} \cos x, \qquad C_1, C_2 \in \mathbb{R}.$$

- **The second member is of the form $P(x)e^{mx}$ (where P is a polynomial and m is a constant):** We seek the particular solution in the form:

$$y_p = Q(x)e^{mx}.$$

where Q is a polynomial whose degree can be specified:

▷ If m is not a root of the characteristic equation, then, $deg\ (Q) = deg\ (P)$.

▷ If m is one of the two roots of the characteristic equation, then, $deg\ (Q) = deg\ (P) + 1$.

▷ If m is a double root of the characteristic equation, then, $deg\ (Q) = deg\ (P) + 2$.

Example 4.17.

1) *Let us solve the following equation:*

$$y'' - 2y' + y = (x + 2)e^x \cdots\cdots\cdots\cdots (E).$$

We have: $y_h = (C_1x + C_2)e^x$, where C_1, C_2 are two real constants and we have: $m = 1$ is a double root of the characteristic equation, therefore $y_p = Q(x)e^x$ where $Q(x) = ax^3 + bx^2 + cx + d$. By replacing $y_p(x), y_p'(x)$ and $y_p''(x)$ in (E) and identifying, we find $a = \frac{1}{6}, b = 1$ and $c = d = 0$. Finally, the general solution is

$$y(x) = \left(\frac{1}{6}x^3 + x^2 + C_1x + C_2\right)e^x, \qquad C_1, C_2 \in \mathbb{R}.$$

2) *Let us solve the following equation:*

$$y'' - 4y' + 4y = (x^2 + 1)e^x \cdots\cdots\cdots\cdots (E).$$

We have: $y_h = (C_1x + C_2)e^{2x}$, where C_1, C_2 are two real constants and we have: $m = 1$ is not a root of the characteristic equation, therefore $y_p = Q(x)e^x$ where $Q(x) = ax^2 + bx + c$. By replacing $y_p(x), y_p'(x)$ and $y_p''(x)$ in (E) and identifying, we find $a = 1, b = 4$ and $c = 7$. Finally, the general solution is

$$y(x) = (x^2 + 4x + 7 + C_1x + C_2)e^x, \qquad C_1, C_2 \in \mathbb{R}.$$

4.4.3 Solution verifying initial conditions

Given two real numbers y_0 and y_1. There exists one and only one solution y of the differential equation such that $y(x_0) = y_0$ and $y'(x_1) = y_1'$.

The initial condition allows us to determine the exact constant of the general solution y_g of the equation (4.4), which shows the existence and uniqueness of the solution verifying the initial condition.

Remark 4.18. *There exists only one solution of equation* (4.4) *on* I *satisfying the initial condition* $y(x_0) = y_0$.

Example 4.19. *Solve* $\begin{cases} y'' - 4y' + 4y = (x^2+1)e^x \\ y(0) = 1, y'(0) = 0. \end{cases}$
The general solution given by:

$$y(x) = (x^2 + 4x + 7 + C_1 x + C_2)e^x, \qquad C_1, C_2 \in \mathbb{R}.$$

$$\begin{cases} y(0) = 1 \\ y'(0) = 0. \end{cases} \Longleftrightarrow \begin{cases} 7 + C_2 = 1 \\ C_1 + 11 + C_2 = 0. \end{cases} \Longleftrightarrow \begin{cases} C_1 = -5 \\ C_2 = -6. \end{cases}$$

Finally, the solution y *which verifies the initial condition is given by:*

$$y(x) = (x^2 - x + 1)e^x.$$

Exercise 1. *Consider the following second-order differential equation:*

$$y'' - 2y' + y = (6x + 2)e^x. \tag{4.11}$$

1. *Solve the homogeneous differential equation associated of* (4.11).
2. *Determine the constants* α *and* β *for that* $y_p = (\alpha x^3 + \beta x^2)e^x$ *be a solution particular of* (4.11).
3. *Determine the general solution of* (4.11).
4. *Find the solution to equation* (4.11) *verifying* $y(0) = 1$ *and* $y'(0) = 2$.

Solution:

1. Resolution of the homogeneous differential equation associated of (4.11) is $y'' - 2y' + y = 0$.

   The characteristic equation is $r^2 - 2r + 1 = 0 \Longleftrightarrow (r-1)^2 = 0$. We have: $\Delta = 0$, this equation admits a double root $r = 11$, the homogeneous solution is of the form:

   $$y_h(x) = (C_1 x + C_2)e^x, \qquad C_1, C_2 \in \mathbb{R}.$$

2. Determine the constants α and β: We have: $y_p = (\alpha x^3 + \beta x^2)e^x$, then,

   $$y_p' = [\alpha x^3 + (3\alpha + \beta)x^2 + 2\beta x]e^x$$

   and

   $$y_p'' = [\alpha x^3 + (6\alpha + \beta)x^2 + (6\alpha + 4\beta)x + 2\beta]e^x.$$

   By replacing $y_p(x), y_p'(x)$ and $y_p''(x)$ in (4.11) and identifying, we find $\alpha = 1$ and $\beta = 1$. Therefore we will have $y_p(x) = (x^3 + x^2)e^x$.

3. Determine the general solution of (4.11):

$$y_g(x) = y_h(x) + y_p(x) = (x^3 + x^2 + C_1 x + C_2)e^x, \qquad C_1, C_2 \in \mathbb{R}.$$

4. Let us seek for the solution to equation (4.11) verifying $y(0) = 1$ and $y'(0) = 2$: We have:

$$\begin{cases} y(0) = 1 \\ y'(0) = 2 \end{cases} \iff \begin{cases} C_2 = 1 \\ C_1 + C_2 = 2 \end{cases}$$
$$\iff \begin{cases} C_2 = 1 \\ C_1 = 1 \end{cases}$$

Finally,

$$y_g(x) = (x^3 + x^2 + x + 1)e^x.$$

Exercise 2. *Let f be a function defined by:*

$$f(x) = \frac{1}{x(1-x^2)}, \quad x \in \mathbb{R} - \{-1, 0, 1\}.$$

1. *Calculate $\int f(x)dx$.*

2. *Solve the following differential equation:*

$$y' - y = \frac{e^x}{x(1-x^2)}.$$

3. *Solve the following differential equation:*

$$y'' - 2y' - 3y = (8x - 8)e^x.$$

Solution: Let f be a function defined by:

$$f(x) = \frac{1}{x(1-x^2)}, \quad x \in \mathbb{R} - \{-1, 0, 1\}.$$

1. Calculate $\int f(x)dx$: We have:

$$f(x) = \frac{1}{x(1-x^2)} = \frac{1}{x(1-x)(1+x)} = \frac{a}{x} + \frac{b}{(1-x)} + \frac{c}{(1+x)}.$$

By identifying in this equation the coefficients of the same powers in x (after having reduced to the same denominator), we find:

$$a = 1, \quad b = \frac{1}{2}, \quad c = -\frac{1}{2}.$$

$$\begin{aligned}\int f(x)dx &= \int \frac{1}{x} + \frac{1}{2}\int \frac{1}{(1-x)} - \frac{1}{2}\int \frac{1}{(1+x)} \\ &= \ln|x| - \frac{1}{2}\ln|1-x| - \frac{1}{2}\ln|1+x| + c \\ &= \ln\left(\frac{|x|}{\sqrt{|1-x^2|}}\right) + c, \quad c \in \mathbb{R}.\end{aligned}$$

2. Solve the following differential equation:

$$y' - y = \frac{e^x}{x(1-x^2)}. \tag{4.12}$$

- Let us solve the homogeneous differential equation: $y' - y = 0$:

$$\begin{aligned} y' - y = 0 &\implies \frac{dy}{y} = dx \\ &\implies \ln|y| = x + c \\ &\implies y_h = ke^x, \quad k = \pm e^c \in \mathbb{R}.\end{aligned}$$

- Particular solution of $y' - y = \dfrac{e^x}{x(1-x^2)}$: we apply the variation method of the constant, then,

$$y_p = k(x)e^x \implies y_p' = k'(x)e^x + k(x)e^x.$$

By replacing y_p and y_p' in (4.12), we find:

$$\begin{aligned} k'(x) = \frac{1}{x(1-x^2)} \Rightarrow k(x) &= \int \frac{1}{x(1-x^2)} \\ &= \ln\left(\frac{|x|}{\sqrt{|1-x^2|}}\right) + c, \quad c \in \mathbb{R}.\end{aligned}$$

Then, $y_p = \left[\ln\left(\frac{|x|}{\sqrt{|1-x^2|}}\right) + c\right] e^x$.

Finally, the general solution of (4.12) is

$$\begin{aligned} y_g &= y_h + y_p \\ &= ke^x + \left[\ln\left(\frac{|x|}{\sqrt{|1-x^2|}}\right) + c\right] e^x \\ &= \left[\ln\left(\frac{|x|}{\sqrt{|1-x^2|}}\right) + c'\right] e^x, \qquad c' = k + c \in \mathbb{R}.\end{aligned}$$

3. Solve the following differential equation:

$$y'' - 2y' - 3y = (8x - 8)e^x \cdots\cdots\cdots\cdots\cdots\cdots (E)$$

- Let us solve the homogeneous differential equation:

$$y'' - 2y' - 3y = 0 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (EH)$$

  the characteristic equation given by: $r^2 - 2r - 3 = 0$ admits two real roots $r_1 = -1$ and $r_2 = 3$ therefore $y_h = c_1 e^{-x} + c_2 e^{3x}$, where c_1 and c_2 are real constants.

- Particular solution of equation (E):

  We are seeking for a particular solution $y_p = q(x)e^x$. We have: $\alpha = 1$ is not a root of (EH), then, $d\check{r}q = d\check{r}p = 1$.

$$y_p = (ax + b)e^x \Rightarrow y_p' = [ax + (a + b)]e^x \Rightarrow y_p'' = [ax + (2a + b)]e^x.$$

  By replacing y_p, y_p' and y_p'' in (E), we find:

$$[ax + (2a + b)]e^x - 2[ax + (a + b)]e^x - 3(ax + b)e^x = (8x - 8)e^x$$

$$\Rightarrow -4ax - 4b = 8x - 8 \Rightarrow \begin{cases} a = -2 \\ b = 2 \end{cases}$$

  Then, $y_p = (-2x + 2)e^x$.

  Finally, the general solution of (E) is:

$$y_g = y_h + y_p = c_1 e^{-x} + c_2 e^{3x} + (-2x + 2)e^x. \quad c_1, c_2 \in \mathbb{R}.$$

CHAPTER

5

FUNCTIONS OF SEVERAL VARIABLES

5.1 Topology of $\mathbb{R}^n$

5.1.1 Norm in a vector space

We call norm in a $\mathbb{R}$-vector space E any map N of E to $\mathbb{R}$ that satisfies:

- ▹ $\forall x \in E, N(x) \geq 0$ and $N(x) = 0 \Leftrightarrow x = 0$.
- ▹ $\forall x \in E, \forall \lambda \in \mathbb{R}, N(\lambda x) = |\lambda| N(x)$.
- ▹ $\forall x \in E, \forall y \in E, N(x+y) \leq N(x) + N(y)$. (Triangular inequality).

$N(x)$ is often denoted $||x||$, which recalls the analogy with the absolute value in $\mathbb{R}$ or the modulus in $\mathbb{C}$.

Properties 5.1.

$$\forall x \in E, \forall y \in E : |||x|| - ||y||| \leq ||x|| + ||y||.$$

Usual norms in $\mathbb{R}^n$: The three usual norms in $\mathbb{R}^n$ defined for $X = (x_1, x_2, \cdots, x_n)$ are:

(i) $||X||_\infty = \sup\{|x_1|, |x_2|, \cdots, |x_n|\}$.

(ii) $||X||_1 = \sum_{i=1}^{n} |x_i|$.

(*iii*) $||X||_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$.

5.1.2 Remarkable parts of $\mathbb{R}^n$

Balls: In $\mathbb{R}^n$ with a norm, we call:

- Open ball with center $\omega \in \mathbb{R}^n$ and $r > 0$, the set

$$B(\omega, r) = \{X \in \mathbb{R}^n, ||X - \omega|| < r\}.$$

- Closed ball with center $\omega \in \mathbb{R}^n$ and $r > 0$, the set

$$B(\omega, r) = \{X \in \mathbb{R}^n, ||X - \omega|| \leq r\}.$$

Bounded parts: A part D of $\mathbb{R}^n$ is bounded if the set of reals $||X - Y||$, where X and Y are any vectors of D is bounded.

D is bounded if and only if, there exist a ball that contains it.

5.2 Generalities

A function f defined on a subset D of $\mathbb{R}^2$ and with real values, corresponds to any vector $X = (x, y)$ of D a unique real $f(X)$. The set

$$S = \{(x, y, f(x, y)), (x, y) \in D\}$$

is the representative surface of f, it is the analogue of the representative curve of f for a function of one variable.

Example 5.2.

1) *The function* $f(x, y) = x^3 + xy + y^2 + 2$ *is defined in* $\mathbb{R}^2$.

2) *The function* $g(x, y) = \sqrt{1 - (x^2 + y^2)}$ *is defined inside the circle* $x^2 + y^2 = 1$.

Let f be a function defined in $D \subset \mathbb{R}^2$ with values in $\mathbb{R}$ and $A = (a_1, a_2) \in D$. We call partial functions associated to f at point A the functions:

$$x_1 \longrightarrow f(x_1, a_1) \qquad \textit{and} \qquad x_2 \longrightarrow f(x_2, a_2)$$

defined on an open interval containing a_1 and a_2.

5.3 Limits, Continuous function

Let us define the notion of a limit of a function $f(x, y)$ of two variables. Suppose that the function f is defined at any point $M(x, y)$ sufficiently close of $M_0(a, b)$.

Definition 5.3. *We call the number ℓ is the limit of $f(x, y)$ when $M(x, y)$ tends to $M_0(a, b)$ and we write*

$$\lim_{x\to a, y\to b} f(x, y) = \ell, \qquad \text{or} \qquad \lim_{M\to M_0} f(x, y) = \ell.$$

If for any ε given positive, there exists a positive δ such that:

$$||(x, y) - (a, b)|| < \delta \Longrightarrow |f(x, y) - \ell| < \varepsilon.$$

Remark 5.4. *The existence and possible value of the limit are independent of the standard chosen in $\mathbb{R}^2$. We call the norms of $\mathbb{R}^2$ are equivalent. When it exists, the limit is unique.*

Suppose that the function $f(x, y)$ is defined at the point $M_0(a, b)$ and at all points of $M_0(a, b)$.

Definition 5.5. *The function $f(x, y)$ is called to be continuous at the point $M_0(a, b)$ if*

$$\lim_{x\to a, y\to b} f(x, y) = f(a, b), \qquad \text{or} \qquad \lim_{M\to M_0} f(x, y) = f(a, b).$$

- ▷ *If f is continuous at every point D, we call f is continuous in D.*
- ▷ *If f is continuous in D, then the partial functions associated with f at a point are continuous in D.*

Algebraic operations: As for functions of one variable, the sum, the product, the quotient (when the denominator does not cancel) of two continuous functions are continuous.

Example 5.6. *For $(x, y) \in \mathbb{R}^2 - \{(0, 0)\}$, we set $f(x, y) = \dfrac{x^2y^2}{x^2 + y^2}$. The function f is defined in $D = \mathbb{R}^2 - \{(0, 0)\}$ with values in $\mathbb{R}$ and $(0, 0)$ is adherent to D. We have:*

$$\lim_{(x,y)\to(0,0)} x^2y^2 = 0 \qquad \text{and} \qquad \lim_{(x,y)\to(0,0)} (x^2 + y^2) = 0.$$

So there is a problem.

For $(x, y) \in \mathbb{R}^2, (|x| - |y|)^2 \geq 0$ and therefore $|xy| \leq \dfrac{1}{2}(x^2 + y^2)$, (we must memorize the previous inequality which is frequently used in practice). Consequently, for $(x, y) \in D$,

$$|f(x, y)| = |xy|\frac{|xy|}{x^2 + y^2} \leq \frac{1}{2}|xy|.$$

Now, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{1}{2}|xy| = 0$ because the function $g : (x, y) \longrightarrow \dfrac{1}{2}|xy|$ is continuous in $\mathbb{R}^2$ and at $(0, 0)$ and therefore $\displaystyle\lim_{(x,y)\to(0,0)} f(x, y) = 0$. The function f has a limit at $(0, 0)$ and this limit is equal to 0.

5.4 Partial derivatives and differentiability of a function

Let $(x_0, y_0) \in D$. The partial derivatives of f at (x_0, y_0) are:

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} \quad \textit{and} \quad \frac{\partial f}{\partial y}(x_0, y_0) = \lim_{k \to 0} \frac{f(x_0, y_0 + k) - f(x_0, y_0)}{k}.$$

We say that f is of class C^1 on D if $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous on D.

The gradient of f at (x_0, y_0) is the vector whose components are the first partial derivatives.

Definition 5.7. *If the partial derivative functions themselves admit partial derivatives at (x_0, y_0), these derivatives are called second partial derivatives of f at (x_0, y_0). We denote them*

$$\frac{\partial^2 f}{\partial x^2}(x_0, y_0) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(x_0, y_0) \quad \textit{and} \quad \frac{\partial^2 f}{\partial y^2}(x_0, y_0) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)(x_0, y_0).$$

$$\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(x_0, y_0) \quad \textit{and} \quad \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(x_0, y_0).$$

Similarly, we can define partial derivatives of order greater than 2 by recurrence.

We say that f is of class C^k on D if the partial derivatives of order k are continuous on D.

We say that f is of class C^1 on D if partial derivatives of all orders exist and continuous on D.

Theorem 5.8. *If $\frac{\partial^2 f}{\partial x \partial y}$ or $\frac{\partial^2 f}{\partial y \partial x}$ is continuous at (x_0, y_0), then,*

$$\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) = \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0).$$

Definition 5.9. *We say that f is differentiable at (x_0, y_0) if there exist real constants A and B such that*

$$f(x_0 + h, y_0 + k) - f(x_0, y_0) = Ah + Bk + ||(h, k)||\varepsilon(h, k)$$

where $\lim_{(h,k) \to (0,0)} \varepsilon(h, k) = 0$.

In this case, we have $A = \frac{\partial f}{\partial x}(x_0, y_0)$ et $B = \frac{\partial f}{\partial y}(x_0, y_0)$.

Theorem 5.10. *If f is differentiable at (x_0, y_0), then, f has partial derivatives at (x_0, y_0). If f is of class C^1 in the neighborhood of (x_0, y_0), then, f is differentiable at (x_0, y_0).*

Both converses are false.

Example 5.11. *If for all* (x, y) *de* $\mathbb{R}^2, f(x, y) = xe^{x^2+y^2}$*, then, for all* $(x, y) \in \mathbb{R}^2$*, we have:*

$$\frac{\partial f}{\partial x}(x, y) = e^{x^2+y^2} + x2xe^{x^2+y^2} = (2x^2 + 1)e^{x^2+y^2},$$

and

$$\frac{\partial f}{\partial y}(x, y) = 2xye^{x^2+y^2}.$$

5.5 Double and triple integrals

In this section, we will only give some elements relating to the calculations of double and triple integrals. Let us consider the set

$$A = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b, \varphi(x) \leq y \leq \psi(x)\},$$

where φ and ψ are two continuous functions in $[a, b]$. Then,

$$\int\int_A f(x, y)dxdy = \int_a^b \left(\int_{\varphi(x)}^{\psi(x)} f(x, y)dy\right) dx.$$

This is Fubini's theorem, which defines the double integral using two simple integrals.

If $f(x, y) = 1$, the double integral $\int\int_A dxdy$ is the area of A.

Change of variables formula: Let f be a continuous function on a closed and bounded domain D in bijection with a closed and bounded domain Δ by means of the functions of class $C^1, x = \varphi(u, v)$ and $y = \psi(u, v)$, then,

$$\int\int_D f(x, y)dxdy = \int\int_\Delta f(x(u, v), y(u, v)) \left|\frac{D(x, y)}{D(u, v)}\right| dudv.$$

The determinant

$$\left|\frac{D(x, y)}{D(u, v)}\right| = \begin{vmatrix} \frac{\partial \varphi}{\partial u}(u, v) & \frac{\partial \varphi}{\partial v}(u, v) \\ \frac{\partial \psi}{\partial u}(u, v) & \frac{\partial \psi}{\partial v}(u, v) \end{vmatrix}$$

is called Jacobian.

Polar coordinates case:

$$\int\int_D f(x, y)dxdy = \int\int_\Delta f(\rho\cos\theta, \rho\sin\theta)\rho d\rho d\theta.$$

Similarly, we can define the triple integral

$$I = \int\int\int_D f(x, y, z)dxdydz$$

for a continuous function f on a closed and bounded domain D of $\mathbb{R}^3$ by using simple successive integrals.

If $f(x, y) = 1$, the triple integral $\int\int_D dxdydz$ is the volume of D.

We can also define (as for double integrals) the formulas for change of variables.
Cylindrical coordinates case:

$$I = \int\int\int_\Delta f(\rho\cos\theta, \rho\sin\theta, z)\rho d\rho d\theta dz.$$

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