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Mustapha Stambouli University of Mascara
Sciences and Technology Faculty
Department of Common Core Sciences and Technology

Course handout

Maths 1 (Analysis & Algebra 1)

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This course is intended for first year LMD sciences and technology students.

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Preface

This course in Analysis and Algebra is intended above all for students in the first year LMD Sciences and Technology, as well as for students in the first year LMD Material sciences and mathematics and computer science.

It covers the official syllabus of Analysis and Algebra, namely:

- ▷ Logic and Reasoning.
- ▷ Sets, relations and maps.
- ▷ Real functions to a real variable.
- ▷ Applications to elementary functions.
- ▷ Limited development.
- ▷ Linear algebra.

Each chapter puts in place the essential bases for approaching scientific studies, and introduces a few new notions, which will for the most part be dealt with during this year.

This course is treated in detail with numerous examples. Most theorems and propositions are proved.

At the end of each chapter we propose corrected exercises.

Finally, errors may be found, please point them out to the author.

The author

CHAPTER

1

LOGIC AND REASONING

The purpose of this chapter is to specify certain rules of logic on which we will support to justify the reasonings used in our demonstrations.

1.1 Logical elements

In this section, we will present the elementary notions of classical logic:

Definition 1.1. *A logical proposition (statement) is any relation that is either true or false.*

- *When the proposition is true (T), it is assigned the value* 1.
- *When the proposition is false* (F)*, it is assigned the value* 0.

These values are called **the truth values of the proposition**.

It is customary for our a proposal using a capital letter P, Q, R, $\cdots$
Thus, to define a logical proposition, it suffices to give its truth values. In general, we put these values in a table which we will call **Table of truths**.

Remark 1.2. *The fact that a proposition can only take the values* 0 *or* 1 *comes from a fundamental principle of classical logic which is: the principle of excluded middle, namely that a logical proposition cannot be true and false. at a time.*

Example 1.3.

a) *«Paris is the capital of France» is a true proposition.*

b) *«Three is an even number» is a false proposition.*

c) *«The number x is odd» is not a proposition since it is impossible to decide whether it is true or false until we know x.*

1.1.1 Logical connectors

a) **The Negation $\neg$:** denoted not P ($\neg P$) or $\overline{P}$, we call negation of a proposition P a proposition that is true when P est false and false when P is true. It can be represented as follows:

P	0	1
$\neg P$	1	0

Example 1.4.

- ▷ *The negation of the propostion* $2 \geq 0$ *is the proposition* $2 < 0$.
- ▷ $\overline{2+4=6}$ *is* $2+4 \neq 6$.

b) **The conjunction $\wedge$:** denoted P and Q ($P \wedge Q$), we call conjunction of propositions P and Q a proposition which is true when P and Q are both true. Its truth table is given by:

$P \setminus Q$	0	1
0	0	0
1	0	1

or

P	0	0	1	1
Q	0	1	0	1
$P \wedge Q$	0	0	0	1

Example 1.5.

- ▷ $2^3 = 8 \wedge 4+4 > 11$ *is a false proposition.*
- ▷ $2+2 = 4 \wedge 2 \times 3 = 6$ *is a true proposition.*

c) **The disjunction $\vee$:** denoted P or Q ($P \vee Q$), we call disjunction of the propositions P or Q a proposition which is true if one of the propositions P or Q is true. Its truth table is given by:

$P \setminus Q$	0	1
0	0	1
1	1	1

or

P	0	0	1	1
Q	0	1	0	1
$P \vee Q$	0	1	1	1

Example 1.6.

- ▷ $\sqrt{2+2} = 4 \vee 45 \div 3 = 15$ *is a true proposition..*
- ▷ $4^3 = 8 \vee 5 + 1 = 7$ *is a false proposition.*

d) **The implication ⇒:** denoted $P \Rightarrow Q$ and reads P implies Q. The proposition $P \Rightarrow Q$ is false when P is true and Q is false and true in all other cases. Its truth table is given by:

$P \setminus Q$	0	1
0	1	1
1	0	1

or

P	0	0	1	1
Q	0	1	0	1
$P \Rightarrow Q$	1	1	0	1

e) **The Equivalence ⇔:** two propositions P ,Q are equivalent if $P \Rightarrow Q$ and $Q \Rightarrow P$ and we write $P \Leftrightarrow Q$, reads P is equivalent to Q. The proposition $P \Leftrightarrow Q$ is a true one when P and Q are simultaneously true or false and false in all other cases. Its truth table is given by:

$P \setminus Q$	0	1
0	1	0
1	0	1

or

P	0	0	1	1
Q	0	1	0	1
$P \Leftrightarrow Q$	1	0	0	1

Example 1.7. *Consider the propositions P: "I have my driver's license" and Q: "I am 18 years old".*
We have: $P \Longrightarrow Q$ is true, on the other hand $Q \Longrightarrow P$ is false. So, we conclude that these two propositions are not equivalent.

f) **The converse:** given P and Q two logical propositions, we call the converse of the implication $P \Rightarrow Q$ the proposition $Q \Rightarrow P$.

All the previous definitions can be summarized in the table below called table of truths:

P	Q	$\neg P$	$\neg Q$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$Q \Rightarrow P$	$P \Leftrightarrow Q$
1	1	0	0	1	1	1	1	1
1	0	0	1	0	1	0	1	0
0	1	1	0	0	1	1	0	0
0	0	1	1	0	0	1	1	1

Properties 1.8. *Let P, Q, R be three propositions. we have the following (true) equivalences:*

1. $\neg(\neg P) \Longleftrightarrow P$.

2. *$(P \wedge Q) \Longleftrightarrow (Q \wedge P)$. (commutative laws)*
3. *$(P \vee Q) \Longleftrightarrow (Q \vee P)$. (commutative laws)*
4. *$\neg(P \wedge Q) \Longleftrightarrow (\neg P) \vee (\neg Q)$. (De Morgan's laws)*[1]
5. *$\neg(P \vee Q) \Longleftrightarrow (\neg P) \wedge (\neg Q)$. (De Morgan's laws)*[2]
6. *$P \wedge (Q \vee R) \Longleftrightarrow (P \wedge Q) \vee (P \wedge R)$. (distributive laws).*
7. *$P \vee (Q \wedge R) \Longleftrightarrow (P \vee Q) \wedge (P \vee R)$. (distributive laws).*
8. *$(P \Rightarrow Q) \Longleftrightarrow (\neg Q \Rightarrow \neg P)$.*

Proof:

1. The truth table of $\neg(\neg P)$ is the following:

P	0	1
$\neg P$	1	0
$\neg(\neg P)$	0	1

we see that it is identical to that of P.

2. In the following table,

P	Q	$P \wedge Q$	$Q \wedge P$	$P \vee Q$	$Q \vee P$
1	1	1	1	1	1
1	0	0	0	1	1
0	1	0	0	1	1
0	0	0	0	0	0

we notice that the propositions $(P \wedge Q)$ and $(Q \wedge P)$ have the same truth values, therefore they are equivalent. Likewise for the propositions $(P \vee Q)$ and $(Q \vee P)$.

3. We establish the proof of De Morgan's rules by giving the truth values of the corresponding logical propositions.

P	Q	$P \wedge Q$	$P \vee Q$	$\neg P$	$\neg Q$	$\neg(P \wedge Q)$	$\neg P \vee \neg Q$	$\neg(P \vee Q)$	$\neg P \wedge \neg Q$
1	1	1	1	0	0	0	0	0	0
1	0	0	1	0	1	1	1	0	0
0	1	0	1	1	0	1	1	0	0
0	0	0	0	1	1	1	1	1	1

[1] Also known as: **duality laws**.

[2] **De Morgan Auguste:** British mathematician (Madurai Tamil Nadu (India) 1806-London 1871). He is the founder with Boole of modern logic.

We see that the logical propositions $\neg(P \wedge Q)$ and $\neg P \vee \neg Q$ have the same truth values, so they are equivalent. Similarly for $\neg(P \vee Q)$ and $\neg P \wedge \neg Q$.

4. In the following table, we notice that the propositions $P\wedge(Q\vee R)$ and $(P\wedge Q)\vee(P\wedge R)$ have the same truth values.

P	Q	R	$Q \vee R$	$P \wedge (Q \vee R)$	$P \wedge Q$	$P \wedge R$	$(P \wedge Q) \vee (P \wedge R)$
1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1
1	0	1	1	1	0	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

therefore they are equivalent. Similarly for $P \vee (Q \wedge R)$ and $(P \vee Q) \wedge (P \vee R)$, we have:

P	Q	R	$Q \wedge R$	$P \vee (Q \wedge R)$	$P \vee Q$	$P \vee R$	$(P \vee Q) \wedge (P \vee R)$
1	1	1	1	1	1	1	1
1	1	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	0	0	0	1	1	1	1
0	1	1	1	1	1	1	1
0	1	0	0	0	1	0	0
0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	0

We deduce that $P \vee (Q \wedge R) \iff (P \vee Q) \wedge (P \vee R)$.

5. To show that the last proposition is true, it suffices to show that the propositions $P \Rightarrow Q$ and $\neg Q \Rightarrow \neg P$ have the same truth values. We have:

P	Q	$\neg P$	$\neg Q$	$P \Rightarrow Q$	$\neg Q \Rightarrow \neg P$
1	1	0	0	1	1
1	0	0	1	0	0
0	1	1	0	1	1
0	0	1	1	1	1

This shows the veracity of our proposition.

1.1.2 Quantifiers

Let $P(x)$ be a proposition defined on a set E:

(i) **Universal quantifier** denoted $\forall$, reads "for all" or "whatever":

$$\forall x \in E, \quad P(x)$$

which is true when all the elements x of E verify $P(x)$.

(ii) **Existential quantifier** denoted $\exists$, reads "it exists at least":

$$\exists x \in E, \quad P(x)$$

which is true when we can find at least one element x in E such that $P(x)$ is true.

Example 1.9.

- $\forall x \in [1, +\infty[; x^2 \geq 1$ *is true.*
- $\exists x \in \mathbb{R}; x^2 = -1$ *is false (no squared real will result in a negative number).*
- $\forall x \in \mathbb{R}_+, \forall n \in \mathbb{N} : (1+x)^n \geq 1$ *is true.*

Properties 1.10.

1. $\overline{\forall x, P(x)} \Longleftrightarrow \exists x, \overline{P(x)}$.
2. $\overline{\exists x, P(x)} \Longleftrightarrow \forall x, \overline{P(x)}$.
3. $\forall x, [P(x) \wedge Q(x)] \Rightarrow [\forall x, P(x)] \wedge [\forall x, Q(x)]$.
4. $\exists x, [P(x) \wedge Q(x)] \Rightarrow [\exists x, P(x)] \wedge [\forall x, Q(x)]$.
5. $\forall x, [P(x) \vee Q(x)] \Rightarrow [\forall x, P(x)] \vee [\forall x, Q(x)]$.
6. $\exists x, [P(x) \vee Q(x)] \Rightarrow [\exists x, P(x)] \vee [\forall x, Q(x)]$.

Corrected exercise:

1. Write the following sentence using quantifiers:

   (i) For any real number, its square is nonnegative: $\forall x \in \mathbb{R}, x^2 \geq 0$.

   (ii) Foe any integer n, there exists a unique real such that $\exp(x)$ equals n: $\forall n \in \mathbb{N}, \exists x \in \mathbb{R} / \exp(x) = n$.

2. Write the negation of the following propositions:

   (i) $\forall x \in \mathbb{R}, \exists y > 0 : x + y > 10 \rightarrow \exists x \in \mathbb{R}, \forall y > 0 : x + y \leq 10$,

   (ii) $\exists x \in \mathbb{R}, \forall y < 0 : x + y \leq 8 \rightarrow \forall x \in \mathbb{R}, \exists y < 0 : x + y > 8$,

   (iii) $P \wedge (\overline{P} \vee Q) \rightarrow \overline{P} \vee (P \wedge \overline{Q})$.

1.2 Method of mathematical reasoning

There are several types of mathematical reasoning, we will deal with the most used in this part:

1.2.1 Direct reasoning

We want to show that the proposition $P \Rightarrow Q$ is true. We assume that P is true and we show that Q is true. This is the method you are most used to.

Example 1.11. *Show that if $a, b \in \mathbb{Q}$ then $a + b \in \mathbb{Q}$.*

Proof: Take $a \in \mathbb{Q}, b \in \mathbb{Q}$. Recall that the rational numbers Q are the set of real numbers written $\dfrac{p}{q}$ with $p \in \mathbb{Z}$ and $q \in \mathbb{N}^*$.

Then $a = \dfrac{p}{q}$ for some $p \in \mathbb{Z}$ and some $q \in \mathbb{N}^*$. Similarly $b = \dfrac{p'}{q'}$ with $p' \in \mathbb{Z}$ and $q' \in \mathbb{N}^*$.
Now,

$$a + b = \frac{p}{q} + \frac{p'}{q'} = \frac{pq' + qp'}{qq'}.$$

The numerator $pq' + qp'$ is indeed an element of $\mathbb{Z}$, the denominator qq' is an element of $\mathbb{N}^*$. So $a + b$ can be written as $a + b = \dfrac{p''}{q''}$ with $p'' \in \mathbb{Z}, q'' \in \mathbb{N}^*$. Thus $a + b \in \mathbb{Q}$.

1.2.2 Reasoning by contraposition

Reasoning by contrapositive is based on the following equivalence (see the Properties 1.8):

$$P \Rightarrow Q \Longleftrightarrow \overline{Q} \Rightarrow \overline{P}.$$

Instead of showing that the implication $P \Rightarrow Q$ is true, the reasoning by contrapositive consists to showing that the $\overline{Q} \Rightarrow \overline{P}$ is true.

Example 1.12. *Let $n \in \mathbb{N}$. Show that if n^2 is even then n is even.*

Proof: We assume that n is not even. Then we want to show that n^2 is not even. Since n is not even, it is odd and therefore there exists $k \in \mathbb{N}$ such that $n = 2k + 1$. Then $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2\ell + 1$ with $\ell = 2k^2 + 2k \in \mathbb{N}$. Thus n^2 is odd.

Conclusion: we have shown that if n is odd then n^2 is odd. By contrapositive this is equivalent to if n^2 is even then n is even.

1.2.3 Contradiction reasoning

To prove that a proposition P is true, we assume that it is false and we leads to a contradiction.

Example 1.13. *Let $a, b \geq 0$. Show that if $\dfrac{a}{1+b} = \dfrac{b}{1+a}$ then $a = b$.*

Proof: By using the proof by contradiction, we prove that $\dfrac{a}{1+b} = \dfrac{b}{1+a}$ and $a \neq b$. As $\dfrac{a}{1+b} = \dfrac{b}{1+a}$ then $a(1+a) = b(1+b)$ so $a + a^2 = b + b^2$ thus $a^2 - b^2 = b - a$. It leads to $(a-b)(a+b) = -(a-b)$. As $a \neq b$ then $a - b \neq 0$ and therefore dividing by $a - b$ we obtain $a + b = -1$. The sum of two positive numbers cannot be negative. We get a contradiction.

Conclusion: if $\dfrac{a}{1+b} = \dfrac{b}{1+a}$ then $a = b$.

1.2.4 Reasoning by counterexample

To prove that the proposition $\forall x \in E, P(x)$ is false, it suffices to find an x_0 of E such that $\overline{P(x_0)}$ is true.

Example 1.14. *Show that the following proposition is false "Every positive integer is sum of three squares". (The squares are $0^2, 1^2, 2^2, 3^2, \cdots$ For example $6 = 1^2 + 1^2 + 2^2$).*

Proof: A counterexample is 7: squares less than 7 are $0, 1, 4$ but with three of these numbers we cannot get 7.

1.2.5 Inductive reasoning

Many results are expressed in the form $\forall n \in \mathbb{N}, P(n)$. A proof by induction allows us to show that such a proposition is true.

The methodology consists of:

(i) Checking that the property $P(0)$ is true.

(ii) Proof that if the property $P(n)$ is true then $P(n+1)$ is true.

The property $P(n)$ assumed to be true is called the induction hypothesis **induction hypothesis**.

Example 1.15. *Show that for all $n \in \mathbb{N}^*$, $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.*

Proof: For $n \geq 1$, Let $P(n)$ denote the following proposition: $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

We will prove by induction that $P(n)$ is true for all $n \geq 1$.

- For $n = 1$ we have $1 = \dfrac{1(1+1)}{2}$. So $P(1)$ is true.
- Fix $n \geq 1$. Suppose $P(n)$ is true. We will show that $P(n+1)$ is true i.e.,

$$\sum_{k=1}^{n+1} k = \frac{(n+1)(n+2)}{2}.$$

We have:

$$\begin{aligned} \sum_{k=1}^{n+1} k &= \sum_{k=1}^{n} k + (n+1) \\ &= \frac{n(n+1)}{2} + (n+1) \\ &= \frac{n(n+1) + 2(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2}. \end{aligned}$$

Thus $P(n+1)$ is true.

Conclusion: by the principle of induction $P(n)$ is true i.e.,

$$\forall n \geq 1, \qquad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

CHAPTER

2

SETS, RELATIONS AND MAPS

2.1 Set theory

Definition 2.1. *We call set E any collection of objects satisfying the same property, each object is an element of the set E.*

Remark 2.2.

- ✓ *To define a set:*
  - (i) *Either we know the list of all its elements, we then say that the set is given by "Extension".*
  - (ii) *Either we only know the relations which link the elements and which allow us to find them all, we then say that the set is given by "Compréhension".*
- ✓ *We put the objects that make up the set between two braces.*
- ✓ *If the number of these objects is finite, we call it cardinal of E and we note it $card(E)$, if E has an infinity of elements, we say that it has infinite cardinality and we denote $Card(E) = \infty$.*
- ✓ *There exists a set, called the empty set and denoted $\emptyset$, which contains any element, so $Card(\emptyset) = 0$.*
- ✓ *A set containing a single element is called "Singleton", so with a cardinality equal to 1. We write $\exists\ !\ x$ and for read it "There is a unique x".*

Example 2.3.

(*i*) *Let* $A = \{1, 3, a, y, \triangle, 2\}$. *A* A *is defined by extension because we know all its elements. The cardinality of* A *is equal to 6* $(Card(A) = 6)$.

(*ii*) *Let* B *be the set of first-year students in the Science and Technology common core. We do not know all these students but we can find them, so* B *is a set given by comprehension.*

(*iii*) *Let* E *be the set of integers that divide* $20, E = \{1, 2, 4, 5, 10, 20\}$.

(*iv*) *Some important sets are the following:*

- ▷ $\mathbb{N} = \{0, 1, 2, 3, \cdots\}$ *the set of natural numbers.*
- ▷ $\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$ *the set of integers.*
- ▷ $\mathbb{Q}$ *the set of rational numbers.*
- ▷ $\mathbb{R}$ *the set of real numbers.*
- ▷ $\mathbb{C}$ *the set of complex numbers.*

2.1.1 Operations on sets

Let E and F be two sets. We notice:

- **Membership:** $x \in E$, means that the element x belongs to E. If x is not an element of E, we say that x does not belong to E and we write $x \notin E$.

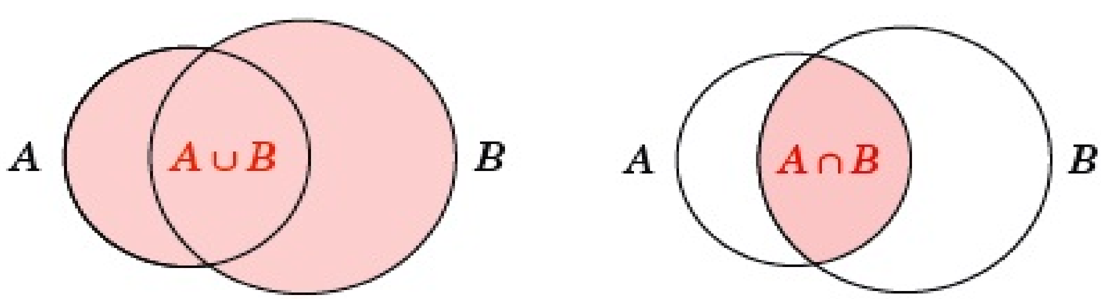
- **Intersection:** the intersection of two sets E and F is the set of their common elements and we write:

$$E \cap F = \{x/x \in E \textit{ and } x \in F\}.$$

  If $E \cap F = \emptyset$, we say that E and F are disjoint.

- **Union:** the union of two ests E and F is the set of their elements counted only once and we write:

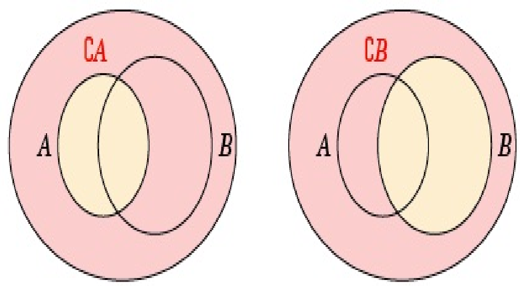
$$E \cup F = \{x/x \in E \textit{ or } x \in F\}.$$



- **Inclusion:** E is included in F if every element of E is an element of F and we have:
$$E \subset F \Longleftrightarrow \forall x, x \in E \Rightarrow x \in F.$$
We also say that E is a part of F or that E is a subset of F.

- **Equlity:** E and F are equal if E is included in F and F is included in E and we writing:
$$\begin{aligned} E = F &\Longleftrightarrow (E \subset F) \wedge (F \subset E) \\ &\Longleftrightarrow \forall x, (x \in E \Longleftrightarrow x \in F). \end{aligned}$$

- **Complement:** Let E be a set and A a subset of $E, (A \subset E)$. We call complement of A in E the set C_E^A of elements in E which are not in A and we write:
$$C_E^A = E \backslash A = E - A = \{x / x \in E \ et \ x \notin A\}.$$
The set $E - A$ is called **difference** of two sets.



- **Symmetric difference:** We call symmetric difference of two sets E and F and we denote by $E \Delta F$ the set defined by:
$$E \Delta F = (E - F) \cup (F - E).$$

Properties 2.4. *Let E, F and G three sets, then the following relations are true:*

∗ $(E \cap F) \subset E \wedge (E \cap F) \subset F)$ *and* $E \subset (E \cup F) \wedge F \subset (E \cup F)$.

∗ $E \cap F = F \cap E$ *and* $E \cup F = F \cup E$. *(Commutativity)*

∗ $(E \cap F) \cap G = E \cap (F \cap G)$ *and* $(E \cup F) \cup G = E \cup (F \cup G)$. *(Associativity)*

∗ $(E \cap F) \cup G = (E \cup G) \cap (F \cup G)$ *et* $(E \cup F) \cap G = (E \cap G) \cup (F \cap G)$. *(Distributivity)*

∗ $E - (F \cap G) = (E - F) \cup (E - G)$ *and* $E - (F \cup G) = (E - F) \cap (E - G)$.

∗ *Si* $F \subset E$ *and* $G \subset E$, *then* $C_E^{F \cap G} = C_E^F \cup C_E^G$ *and* $C_E^{F \cup G} = C_E^F \cap C_E^G$.

∗ $E \cap \emptyset = \emptyset$ *and* $E \cup \emptyset = E$.

∗ $E \cap (F \Delta G) = (E \cap F) \Delta (E \cap G)$.

∗ $E \Delta \emptyset = E$ *and* $E \Delta E = \emptyset$.

2.1.2 Parts of a set

Definition 2.5. *We say that a set E is included in a set F, or that E is a part of the set F, or that E is a subset of F if every element of E is an element of F. We denote $E \subset F$ and we have:*

$$E \subset F \Longleftrightarrow \forall x / x \in E \Rightarrow x \in F).$$

The set of all parts of a set E is denoted by $\mathcal{P}(E)$.

Example 2.6. *Let $E = \{a, b, c\}$, so $\mathcal{P}(E) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.*

Remark 2.7. *The empty set and E are elements of $\mathcal{P}(E)$.*

2.1.3 Partition of a set

Let E be a set and A a family of subsets of E. We say that A is a partition of E if:

(i) Every element of A is not empty.

(ii) The elements of A are pairwise disjoint.

(iii) The union of the elements of A is equal to E.

Example 2.8. *Let $E = \{1, a, \ell, 3, b, c, d, \alpha, \beta, \gamma\}$, then $\mathcal{F} = \{\{a, \gamma\}, \{d, \alpha, \beta\}, \{c, 1\}, \{3, \ell\}, \{b\}\}$ is a partition of the set E.*

2.1.4 Product set (Cartesian product)

Definition 2.9. *Let E and F be two non-empty sets, we denote by $E \times F$ the set of pairs (x, y) such that $x \in E andt$ $y \in F$ is called Cartesian product of[1] of E and F defined by*

$$E \times F = \{(x, y) / x \in E \text{ and } y \in F\}.$$

By definition, we have:

$$\forall (x, y), (x', y') \in E \times F, \quad (x, y) = (x', y') \Longleftrightarrow (x = x') \wedge (y = y').$$

Example 2.10. *Let $E = \{1, 5, \square\}$ and $F = \{a, \alpha, \ell, \triangle, \spadesuit\}$, then*

$$\begin{aligned} E \times F \;=\; & \{(1, a), (5, a), (\square, a), (1, \alpha), (5, \alpha), (\square, \alpha), (1, \ell), (5, \ell), \\ & (\square, \ell), (1, \triangle), (5, \triangle), (\square, \triangle), (1, \spadesuit), (5, \spadesuit), (\square, \spadesuit)\}. \end{aligned}$$

[1] **DESCARTES René:** French philosopher, physicist and mathematician (The Hague 1596-Stockholm 1650). He created the algebra of polynomials, with Fermat he founded analytic geometry. Stated the fundamental properties of algebraic equations and simplified algebraic notation by adopting the first letters of the alphabet to denote constants and the last letters to denote the variables. Published "The Discourse on Method", which is a reference for logical reasoning. Also discovered the principles (rules) of geometric optics.

and

$$F \times E = \{(a,1),(\alpha,1),(\ell,1),(\triangle,1),(\spadesuit,1),(a,5),(\alpha,5),(\ell,5), (\triangle,5),(\spadesuit,5),(a,\square),(\alpha,\square),(\ell,\square),(\triangle,\square),(\spadesuit,\square)\}.$$

Remark 2.11. *$E \times F = F \times E$ if and only if $E = F$.*

Properties 2.12. *Let E, F, G and H be four sets, then the following relations are true:*

1. $E \times F = \emptyset \Rightarrow E = \emptyset$ *or* $F = \emptyset$.
2. $E \times F = F \times E \Longleftrightarrow E = \emptyset$ *or* $F = \emptyset$ *or* $E = F$.
3. $E \times (F \cup G) = (E \times F) \cup (E \times G)$.
4. $(E \cup G) \times F = (E \times F) \cup (G \times F)$.
5. $(E \times F) \cap (G \times H) = (E \cap G) \times (F \cap H)$.
6. $(E \times F) \cup (G \times H) \neq (E \cup G) \times (F \cup H)$.

2.2 Binary relations in a set

Definition 2.13. *A binary relation is a set of ordered pairs. A binary relation on a set E is a set of ordered pairs of elements of E.*

Definition 2.14. *Let E be a set, x and y two elements of E. If there is a link that links x and y we say that they are linked by a relation $\mathcal{R}$, we write $x\mathcal{R}y$ or $\mathcal{R}(x,y)$ and we read "x is related with y".*

Example 2.15. $E = \mathbb{R}, \forall x, y \in E, x\mathcal{R}y \Longleftrightarrow |x| - |y| = x - y$.

Definition 2.16. *Given a binary relation $\mathcal{R}$ between the elements of a non-empty set E, we say that:*

1. $\mathcal{R}$ *is* **Reflexive** $\Longleftrightarrow \forall x \in E; (x\mathcal{R}x)$.
2. $\mathcal{R}$ *is* **Transitive** $\Longleftrightarrow \forall x, y, z \in E; (x\mathcal{R}y) \wedge (y\mathcal{R}z) \Rightarrow x\mathcal{R}z$.
3. $\mathcal{R}$ *is* **Symmetric** $\Longleftrightarrow \forall x, y \in E; x\mathcal{R}y \Rightarrow y\mathcal{R}x$.
4. $\mathcal{R}$ *is* **Antisymmetric** $\Longleftrightarrow \forall x, y \in E; (x\mathcal{R}y) \wedge (y\mathcal{R}x) \Rightarrow x = y$.

2.2.1 Equivalence relations

Definition 2.17. *We say that a binary relation $\mathcal{R}$ in a set E is an equivalence relation if it is reflexive, symmetric et transitive.*

Let $\mathcal{R}$ be an equivalence relation in a set E.

- We call **equivalence class** of an element $x \in E$ denoted $\bar{x}, \dot{x}$ or C_x, the set of elements y of E which are in relation $\mathcal{R}$ with x. We write:

$$\dot{x} = \{y \in E, y\mathcal{R}\dot{x}\}.$$

- We define **the quotient set** of E by the relation $\mathcal{R}$ the set of equivalence class of all the elements of E, denoted $E_{/\mathcal{R}}$ and we have:

$$E_{/\mathcal{R}} = \dot{x}, x \in E.$$

Example 2.18. *In $\mathbb{R}$, we define the binary relation $\mathcal{R}$ by*

$$x, y \in \mathbb{R}, x\mathcal{R}y \Longleftrightarrow x^2 - y^2 = x - y.$$

- *Let us show that $\mathcal{R}$ is an equivalence relation:*

  (a) $\forall x \in \mathbb{R}, x^2 - x^2 = x - x = 0 \Rightarrow x\mathcal{R}x \Rightarrow \mathcal{R}$ *is reflexive.*

  (b) $\forall x, y \in \mathbb{R}$,

$$\begin{aligned} x\mathcal{R}y &\Longleftrightarrow x^2 - y^2 = x - y \\ &\Longleftrightarrow -(y^2 - x^2) = -(y - x) \\ &\Longleftrightarrow y^2 - x^2 = y - x \\ &\Longleftrightarrow y\mathcal{R}x, \end{aligned}$$

  donc $\mathcal{R}$ so symmetric.

  (c) $\forall x, y, z \in \mathbb{R}$

$$x\mathcal{R}y \Longleftrightarrow x^2 - y^2 = x - y \tag{2.1}$$

  and

$$y\mathcal{R}z \Longleftrightarrow y^2 - z^2 = y - z \tag{2.2}$$

$$\begin{aligned} (2.1) + (2.2) &\Longleftrightarrow x^2 - y^2 + y^2 - z^2 = x - y + y - z \\ &\Longleftrightarrow x^2 - z^2 = x - z \\ &\Longleftrightarrow x\mathcal{R}z, \end{aligned}$$

  so $\mathcal{R}$ is transitive.

From $(a), (b)$ and (c), we have $\mathcal{R}$ an equivalence relation.

- *Let us specify the class of a for all a for each a in* $\mathbb{R}$*:*

$$\begin{aligned}\dot{a} &= \{x \in \mathbb{R}, x\mathcal{R}a\} \\ &= \{x \in \mathbb{R}, x^2 - a^2 = x - a\} \\ &= \{x \in \mathbb{R}, (x-a)(x+a) = x - a\} \\ &= \{x \in \mathbb{R}, (x-a)(x+a-1) = 0\} \\ &= \{x \in \mathbb{R}, x = a \text{ ou } x = 1-a\} \\ &= \{a, 1-a\}.\end{aligned}$$

2.2.2 Ordering relation

Definition 2.19. *We say that a binary relation* $\mathcal{R}$ *in a set* E *is an ordering relation if it is reflexive, transitive and antisymmetric.*

Let $\mathcal{R}$ be an ordering relation on a set E.

- We say that $\mathcal{R}$ is **total ordering** or **linear ordering** if:

$$\forall x, y \in E, x\mathcal{R}y \ \lor \ y\mathcal{R}x.$$

- We say that $\mathcal{R}$ is **partial ordering** if it is not total ordering, that is:

$$\exists x, y \in E, \text{ neither } \ x\mathcal{R}y \ \land \text{ neither } \ y\mathcal{R}x.$$

Example 2.20. *Let* $E = \{a, b, c\}$*, we denote by* $\mathcal{P}(E)$ *the set of subsets of* E*. In* $\mathcal{P}(E)$*, we define the binary relation* $\mathcal{R}$ *by:*

$$\forall A, B \in \mathcal{P}(E), A\mathcal{R}B \iff A \subset B.$$

- *Let us show that* $\mathcal{R}$ *is an ordering relation:*

  (a) *Let* $A \in \mathcal{P}(E)$*, then it is clear that* $A \subset A$ *hence* $A\mathcal{R}A$ *i.e.,* $\mathcal{R}$ *is reflexive.*

  (b) *Let* $A, B \in \mathcal{P}(E)$*,*

$$\begin{aligned}A\mathcal{R}B \ \land \ B\mathcal{R}A &\iff A \subset B \ \land \ B \subset A \\ &\iff A = B,\end{aligned}$$

  so $\mathcal{R}$ *is antisymmetric.*

  (c) *Let* $A, B, C \in \mathcal{P}(E)$

$$\begin{aligned}A\mathcal{R}B \ \land \ B\mathcal{R}C &\iff A \subset B \ \land \ B \subset C \\ &\iff A \subset C,\end{aligned}$$

  so $\mathcal{R}$ *is transitive.*

  From $(a), (b)$ *and* (c)*, we have* $\mathcal{R}$ *an ordering relation.*

- *Is this total ordering?*
  We have $E = \{a, b, c\}$*, so* $\mathcal{P}(E) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$*. The order of this relation is partial ordering because* $\exists A = \{a\} \in \mathcal{P}(E), \exists B = \{b\} \in \mathcal{P}(E)$ *:* A *is not included in* B *and* B *is also not included in* A*.*

2.3 Maps and Functions

Definition 2.21. *Let E and F two sets.*

- ✓ *We call* **function** *from set E to set F a relation from E to F of which to any element x of E we correspond to at most one element y of F.*
- ✓ *We call* **map** *from E to F a relation from E to F whose to every element x of E we correspond to one and only one element y of F.*

x is said antecedent, E the starting set or antecedants set, y is said the image, F the ending set or images set.

Example 2.22.

$$\begin{array}{rcl} f:\mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & f(x)=\frac{x}{x-1} \end{array} \quad \textit{and} \quad \begin{array}{rcl} g:\mathbb{R}-\{1\} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & g(x)=\frac{x}{x-1} \end{array}$$

In this example g is a map but f is a function and is not a map because the element 1 does not have an image in $\mathbb{R}$.

2.3.1 Characteristic of a map

- ▷ In general, we schematize a function or a map f by:

$$\begin{array}{rcl} f:E & \longrightarrow & F \\ x & \longmapsto & y=f(x). \end{array}$$

$$\Gamma=\{(x,f(x)),x\in E\}=\{(x,y)\in E\times F: y=f(x)\}.$$

is called the graph of f.

- ▷ Two maps are equal if their starting sets are equal, their ending sets are equal and their values are equal.

2.3.2 Composition of maps

Definition 2.23. *Let E, F and G be three sets and $f: E\longrightarrow F, g: F\longrightarrow G$ two maps. We denote by $g\circ f$ the map from E to G defined by:*

$$\forall x\in E, (g\circ f)(x)=g(f(x)).$$

This map[2] *is called* **composed** *of maps f and g.*

[2] $g\circ f$ is a map because for $x,x'\in E$ if $x=x'$, then $f(x)=f(x')$ because f is a map and since g is a map, then $g(f(x))=g(f(x'))$, so $(g\circ f)(x)=(g\circ f)(x')$.

Example 2.24.

$$f : \mathbb{R} \longrightarrow \mathbb{R}_+ \quad x \longmapsto f(x) = x^2 \qquad \text{and} \qquad g : \mathbb{R}_+ \longrightarrow [-1;1] \quad x \longmapsto g(x) = \sin x,$$

then,

$$g \circ f : \mathbb{R} \longrightarrow [-1;1] \quad x \longmapsto (g \circ f)(x) = g[f(x)] = g(x^2) = \sin x^2.$$

Remark 2.25.

(*i*) *In general,* $g \circ f \neq f \circ g$*: noncommutative.*

(*ii*) $f \circ g \circ h = (f \circ g) \circ h = f \circ (g \circ h)$*: the operation "composition of maps" is associative.*

2.3.3 Restricting and extending a map

Let E_1 be a subset of E and $f : E \longrightarrow F$ a map. The map $g : E_1 \longrightarrow F$ such that $\forall x \in E_1, g(x) = f(x)$ is called **the restriction** of f in E_1 and we write $g = f/E_1$ and we also say that f is **the extention** of g in E.

Example 2.26.

$$f : \mathbb{R} \longrightarrow \mathbb{R} \quad x \longmapsto f(x) = \sin x \qquad \text{and} \qquad g : [-\Pi/2;\Pi/2] \longrightarrow \mathbb{R} \quad x \longmapsto g(x) = \sin x$$

In this example, we have: g *is the restriction of* f *in the part* $[-\Pi/2;\Pi/2]$ *or* f *is the extention of* g *in* $\mathbb{R}$*. We write:* $g = f/[-\Pi/2;\Pi/2]$.

2.3.4 Injectives, surjectives, bijectives maps

Definition 2.27. *Let* $f : E \longrightarrow F$ *a map. We say that:*

(*a*) f *is* **injective** *any element of* F *has at most one element of* E*,*

$$\forall x_1, x_2 \in E, f(x_1) = f(x_2) \Rightarrow x_1 = x_2,$$

or in an equivalent way (the logical negation):

$$\forall x_1, x_2 \in E, x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2).$$

(*b*) f *is* **surjective** *every element of* F *has at least one element of* E*,*

$$\forall y \in F, \exists x \in E : y = f(x).$$

(*c*) f *is* **bijective** *if it is injective and surjective, i.e., if every element of* F *has a unique element in* E *by* f*,*

$$\forall y \in F; \exists\,!\; x \in E : y = f(x).$$

Example 2.28. *Let*

$$\begin{array}{rcl} f:\mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & f(x)=x^2+1 \end{array} \quad \textit{and} \quad \begin{array}{rcl} g:\mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & g(x)=2x+1 \end{array}$$

Study the injectivity, the surjectivity and the bijectivity of f and g:

- ▷ *f is not injective because $f(-1)=f(1)=2$ does not imply $-1=1$.*
- ▷ *f is not surjective because $x^2+1=-3$ does not admit a solution.*
- ▷ *f is not bijective because f non injective.*
- ▷ *g is injective because*

$$\begin{array}{rcl} g(x)=g(y) & \Longrightarrow & 2x+1=2y+1 \\ & \Longrightarrow & 2x=2y \\ & \Longrightarrow & x=y. \end{array}$$

- ▷ *g is surjective because $y=2x+1 \Longrightarrow x=\dfrac{y-1}{2}$ i.e., $\forall y\in\mathbb{R}; \exists x\in\mathbb{R}$ such that $x=\dfrac{y-1}{2}$*
- ▷ *g is bijective because g is injective and surjective.*

Properties 2.29. *Let $f:E\longrightarrow F$ and $g:F\longrightarrow G$ be two maps, then we have:*

1. *$(f$ surjective$)\wedge(g$ surjective$)\Rightarrow g\circ f$ surjective.*
2. *$(f$ injective$)\wedge(g$ injective$)\Rightarrow g\circ f$ injective.*
3. *$(f$ bijective$)\wedge(g$ bijective$)\Rightarrow g\circ f$ bijective.*

Proof: We have: $g\circ f:E\longrightarrow G$.

1. Assume that f and g are surjectives and show that $g\circ f$ is surjective. Let $z\in G, g$ being surjective, there exists $y\in F$ such that $z=g(y)$, since $y\in F$ and f is surjective, then there exists $x\in E$ such that $y=f(x)$, therefore $z=g[f(x)]$ and we deduce that:
$$\forall z\in G, \exists x\in E: z=(g\circ f)(x),$$
which shows that $g\circ f$ is surjective.

2. Assume that f and g are injectives and show that $g\circ f$ is injective. Let $x_1,x_2\in E$, then:
$$\begin{array}{rcl} x_1\neq x_2 & \Rightarrow & f(x_1)\neq f(x_2) \quad \textit{because } f \textit{ injective} \\ & \Rightarrow & g[f(x_1)]\neq g[f(x_2)] \quad \textit{because } g \textit{ injective} \\ & \Rightarrow & (g\circ f)(x_1)\neq (g\circ f)(x_2), \end{array}$$
which shows that $g\circ f$ is injective.

3. From 1. and 2., we deduce that if f and g are bijectives, then $g\circ f$ is bijective.

2.4 Direct image and reciprocal image

Definition 2.30. *Let $f : E \longrightarrow F$ a map, $A \subset E$ and $B \subset F$.*

- *We define* **the direct image** *of A by the map f the subset of F denoted by $f(A)$:*

$$f(A) = \{y \in F, \forall x \in A, y = f(x)\} = \{f(x), x \in A\} \subset F.$$

Example 2.31. *Let*

$$\begin{array}{rcl} f : \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & f(x) = x^2 \end{array}$$

and $A = [-2, 1]$. We have:

$$\begin{array}{rcl} f(A) & = & \{f(x), x \in A\} \\ & = & \{x^2, x \in [-2, 1]\} \\ & = & [0, 4]. \end{array}$$

- *We define* **the reciprocal image** *of B by the map f the subset of E denoted by $f^{-1}(B)$:*

$$f^{-1}(B) = \{x \in E, f(x) \in B\} \subset E.$$

Example 2.32. *Let*

$$\begin{array}{rcl} f : \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & f(x) = x^2 \end{array}$$

and $B = [0, 4]$. We have:

$$\begin{array}{rcl} f^{-1}(B) & = & \{x \in \mathbb{R}, f(x) \in [0, 4]\} \\ & = & \{x \in \mathbb{R}, x^2 \in [0, 4]\} \\ & = & \{x \in \mathbb{R}, 0 \leq x^2 \leq 4\} \\ & = & \{x \in \mathbb{R}, x^2 - 4 \leq 0\} \\ & = & \{x \in \mathbb{R}, (x-2)(x+2) \leq 0\} \\ & = & [-2, 2]. \end{array}$$

- *Let f be a map bijective, then there exists a map denoted by f^{-1} which defined by $f^{-1} : F \to E$,*

$$y = f(x) \Longleftrightarrow x = f^{-1}(y),$$

called **reciprocal map** *of f.*

Proposition 2.33. *Let $f : E \longrightarrow F, A, B \subset E$ and $M, N \subset F$, then*

1. $f(A \cup B) = f(A) \cup f(B)$.
2. $f(A \cap B) \subset f(A) \cap f(B)$.
3. $f^{-1}(M \cup N) = f^{-1}(M) \cup f^{-1}(N)$.
4. $f^{-1}(M \cap N) = f^{-1}(M) \cap f^{-1}(N)$.
5. $f^{-1}(C_F M) = C_E f^{-1}(M)$.

Proof:

1. Let $y \in F$, then

$$\begin{aligned} y \in f(A \cup B) &\iff \exists x \in (A \cup B) : y = f(x) \\ &\iff (\exists x \in A \vee \exists x \in B) : y = f(x) \\ &\iff (\exists x \in A : y = f(x)) \vee (\exists x \in B : y = f(x)) \\ &\iff y \in f(A) \vee y \in f(B) \\ &\iff y \in (f(A) \cup f(B)), \end{aligned}$$

which shows that $f(A \cup B) = f(A) \cup f(B)$.

2. Let $y \in F$, then

$$\begin{aligned} y \in f(A \cap B) &\iff \exists x \in A \cap B : y = f(x) \\ &\iff (\exists x \in A \wedge \exists x \in B) : y = f(x) \\ &\iff (\exists x \in A : y = f(x)) \wedge (\exists x \in B : y = f(x) \\ &\Longrightarrow y \in (f(A) \wedge y \in f(B)) \\ &\Longrightarrow y \in (f(A) \cap f(B)), \end{aligned}$$

which shows that $f(A \cap B) \subset f(A) \cap f(B)$.

3. Let $x \in E$, then

$$\begin{aligned} x \in f^{-1}(M \cup N) &\iff f(x) \in (M \cup N) \\ &\iff (f(x) \in M) \vee (f(x) \in N) \\ &\iff (x \in f^{-1}(M)) \vee (x \in f^{-1}(N)) \\ &\iff x \in (f^{-1}(M) \cup f^{-1}(N)), \end{aligned}$$

which shows that $f^{-1}(M \cup N) = f^{-1}(M) \cup f^{-1}(N)$.

4. Let $x \in E$, then

$$\begin{aligned} x \in f^{-1}(M \cap N) &\iff f(x) \in (M \cap N) \\ &\iff (f(x) \in M) \wedge (f(x) \in N) \\ &\iff (x \in f^{-1}(M)) \wedge (x \in f^{-1}(N)) \\ &\iff x \in (f^{-1}(M) \cap f^{-1}(N)), \end{aligned}$$

which shows that $f^{-1}(M \cap N) = f^{-1}(M) \cap f^{-1}(N)$.

5. Let $x \in E$, then

$$\begin{aligned} x \in f^{-1}(C_F M) &\iff f(x) \in C_F M \\ &\iff (f(x) \in F) \wedge (f(x) \notin M) \\ &\iff (x \in E) \wedge (x \notin f^{-1}(M)) \\ &\iff x \in C_E f^{-1}(M), \end{aligned}$$

which shows that $f^{-1}(C_F M) = C_E f^{-1}(M)$.

Exercise 1. *Is the following relation reflexive? Symmetric? Antisymmetric? Transitive? in* $\mathbb{R}$.

$$x\mathcal{R}y \iff (\cos x)^2 + (\sin y)^2 = 1.$$

Solution:

▷ $\mathcal{R}$ is a reflexive relation because:

$$(\cos x)^2 + (\sin x)^2 = 1 \Longrightarrow x\mathcal{R}x.$$

▷ $\mathcal{R}$ is a symmetric relation because:

$$\begin{aligned} x\mathcal{R}y &\iff (\cos x)^2 + (\sin y)^2 = 1 \\ &\iff 1 - (\sin x)^2 + 1 - (\cos y)^2 = 1 \\ &\iff -(\cos y)^2 - (\sin x)^2 = -1 \\ &\iff (\cos y)^2 + (\sin x)^2 = 1 \\ &\iff y\mathcal{R}x. \end{aligned}$$

▷ $\mathcal{R}$ is not an antisymmetric relation because:

$$\begin{cases} x\mathcal{R}y \\ \wedge \\ y\mathcal{R}x \end{cases} \Rightarrow \begin{cases} (\cos x)^2 + (\sin y)^2 = 1 \\ \wedge \\ (\cos y)^2 + (\sin x)^2 = 1 \end{cases}$$

which does not imply that $x = y$.

On the other hand, for example: if $x = 0$ and $y = 2\Pi$:

$$\begin{cases} (\cos 0)^2 + (\sin 2\Pi)^2 = 1 \\ \wedge \\ (\cos 2\Pi)^2 + (\sin 0)^2 = 1 \end{cases}$$

which implies that $0 \neq 2\Pi$.

▷ $\mathcal{R}$ is a transitive relation because: $\forall x, y, z \in \mathbb{R}$,

$$\begin{aligned} \begin{cases} x\mathcal{R}y \\ \wedge \\ y\mathcal{R}z \end{cases} &\Rightarrow \begin{cases} (\cos x)^2 + (\sin y)^2 = 1 \\ \wedge \\ (\cos y)^2 + (\sin z)^2 = 1 \end{cases} \\ &\Rightarrow (\cos x)^2 + (\sin y)^2 + (\cos y)^2 + (\sin z)^2 = 2 \\ &\Rightarrow (\cos x)^2 + (\sin z)^2 = 1 \\ &\Rightarrow x\mathcal{R}z. \end{aligned}$$

Exercise 2. *We consider the map:*

$$\begin{aligned} f : \mathbb{R} - \{2\} &\longrightarrow F \\ x &\longmapsto f(x) = \frac{x+5}{x-2}, \end{aligned}$$

with F a subset of $\mathbb{R}$.

Determine F so that the map f be bijective and give the inverse map of f.

Solution: showing that f is bijective amounts to examining the existence of a solution to the equation $y = f(x)$, for all $y \in F$.

Let $y \in F$, then,

$$\begin{aligned} y = f(x) &\iff y = \frac{x+5}{x-2} \\ &\iff y(x-2) = x+5 \\ &\iff yx - x = 2y + 5 \\ &\iff x(y-1) = 2y+5 \\ &\iff x = \frac{2y+5}{y-1} \quad if\ y \neq 1, \end{aligned}$$

which shows that:

$$\forall y \in \mathbb{R} - \{1\}, \exists!\ x = \frac{2y+5}{y-1}; \quad y = f(x).$$

To show that f is bijective, it remains to see if $x = \dfrac{2y+5}{y-1} \in \mathbb{R} - \{2\}$?

We have:

$$\frac{2y+5}{y-1} = 2 \iff 2y+5 = 2y-2$$
$$\iff 5 = -2 \quad \textit{which is impossible},$$

which shows that $\dfrac{2y+5}{y-1} \in \mathbb{R} - \{2\}$, hence:

$$\forall y \in \mathbb{R} - \{1\}, \exists !\ x = \frac{2y+5}{y-1} \in \mathbb{R} - \{2\}; \quad y = f(x).$$

So f is bijective if $F = \mathbb{R} - \{1\}$ and the inverse of f is:

$$\begin{aligned} f^{-1} : \mathbb{R} - \{1\} &\longrightarrow \mathbb{R} - \{2\} \\ y &\longmapsto f^{-1}(y) = \frac{2y+5}{y-1}. \end{aligned}$$

Exercise 3. *Let the map $f : \mathbb{R} \longrightarrow [5, +\infty[$ defined by:*

$$f(x) = (x^2 - 8)^2 + 5, \qquad \forall x \in \mathbb{R}.$$

We define in $\mathbb{R}$ the relation $\mathcal{R}$ by:

$$\forall x, y \in \mathbb{R}, \quad x\mathcal{R}y \iff f(x) = f(y).$$

1. *Verify que $\mathcal{R}$ is an equivalence relation.*
2. *Calculate $\dot{0}$ and $\dot{2}$.*

Solution: let the map $f : \mathbb{R} \longrightarrow [5, +\infty[$ defined by:

$$f(x) = (x^2 - 8)^2 + 5, \qquad \forall x \in \mathbb{R}.$$

We define in $\mathbb{R}$ the relation $\mathcal{R}$ by:

$$\forall x, y \in \mathbb{R}, \quad x\mathcal{R}y \iff f(x) = f(y).$$

1. Verify que $\mathcal{R}$ is an equivalence relation:

   (a) $\forall x \in \mathbb{R}, f(x) = f(x) \Rightarrow x\mathcal{R}x \Rightarrow \mathcal{R}$ is reflexive.

   (b) $\forall x, y \in \mathbb{R}$,

   $$\begin{aligned} x\mathcal{R}y &\iff f(x) = f(y) \\ &\iff f(y) = f(x) \\ &\iff y\mathcal{R}x, \end{aligned}$$

   so $\mathcal{R}$ is symmetric.

(c) $\forall x, y, z \in \mathbb{R}$,

$$x\mathcal{R}y \Longleftrightarrow f(x) = f(y)$$

and

$$y\mathcal{R}z \Longleftrightarrow f(y) = f(z)$$

$$\begin{aligned} &\Longleftrightarrow \quad f(x) = f(z) \\ &\Longleftrightarrow \quad x\mathcal{R}z, \end{aligned}$$

so $\mathcal{R}$ is transitive.

From $(a), (b)$ and (c), we have $\mathcal{R}$ an equivalence relation.

2. Calculate $\dot{0}$ and $\dot{2}$:

$$\begin{aligned} \dot{0} &= \{x \in \mathbb{R}, x\mathcal{R}0\} \\ &= \{x \in \mathbb{R}, f(x) = f(0)\} \\ &= \{x \in \mathbb{R}, (x^2 - 8)^2 + 5 = 8^2 + 5\} \\ &= \{x \in \mathbb{R}, (x^2 - 8) = \pm 8\} \\ &= \{x \in \mathbb{R}, x^2 = 0 \vee x^2 = 16\} \\ &= \{-4, 0, 4\}. \end{aligned}$$

$$\begin{aligned} \dot{2} &= \{x \in \mathbb{R}, x\mathcal{R}2\} \\ &= \{x \in \mathbb{R}, f(x) = f(2)\} \\ &= \{x \in \mathbb{R}, (x^2 - 8)^2 + 5 = (-4)^2 + 5\} \\ &= \{x \in \mathbb{R}, (x^2 - 8) = \pm 4\} \\ &= \{x \in \mathbb{R}, x^2 = 4 \vee x^2 = 12\} \\ &= \{\pm 2, \pm 2\sqrt{3}\}. \end{aligned}$$

CHAPTER

3

REAL FUNCTIONS WITH ONE REAL VARIABLE

3.1 Definitions and Properties

Definition 3.1. *a* **function** *of a real-valued real variable is a map $f : E \to \mathbb{R}$, where E is a subset of $\mathbb{R}$. In general, E is an interval or a union of intervals. We call E the* **domain of definition** *of the function f. We denote the set of these functions by: $F(E; \mathbb{R})$.*
We call **graph** *of a function f the geometric locus of the points $M(x, y)$ where $x \in E$ and $y = f(x)$ and we write:*

$$G_f = \{(x, y) : x \in E, y = f(x)\}.$$

3.1.1 Arithmetic operations on functions

Let $f : E \to \mathbb{R}$ and $g : E \to \mathbb{R}$ be two functions defined on the same subset E of $\mathbb{R}$. We can then define the following functions:

(i) the **sum** of f and g is a function $f + g : E \to \mathbb{R}$ defined by

$$(f + g)(x) = f(x) + g(x) \quad for\ all \quad x \in E.$$

(ii) the **product** of f and g is a function $f \times g : E \to \mathbb{R}$ defined by

$$(f \times g)(x) = f(x) \times g(x) \quad and \quad (\alpha f)(x) = \alpha f(x) \quad for\ all \quad x \in E,\ \alpha \in \mathbb{R}.$$

(iii) the **rational** of f and g is a function $\dfrac{f}{g}$ defined by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \quad for\ all \quad x \in E,\ g(x) \neq 0.$$

3.1.2 Upper, lower and bounded functions

Let $f : E \to \mathbb{R}$, we say that:

- f is **bounded above (upper)** in E if there exists a constant $M \in \mathbb{R}$ which satisfies:
$$\forall x \in E; f(x) \leq M.$$
- f is **bounded below (lower)** in E if there exists a constant $m \in \mathbb{R}$ which satisfies:
$$\forall x \in E; f(x) \geq m.$$
- f is **bounded** in E if it is bounded above and below at the same time.

3.1.3 Increasing, decreasing functions

Let $f : E \to \mathbb{R}$, we say that:

1. f is **increasing** in E if and only if:
$$\forall x_1; x_2 \in E; x_1 \leq x_2 \Longrightarrow f(x_1) \leq f(x_2)$$
and it is **non-decreasing** if instead of $\leq$ we have $<$.
2. f is **decreasing** in E if and only if:
$$\forall x_1; x_2 \in E; x_1 \leq x_2 \Longrightarrow f(x_1) \geq f(x_2)$$
and it is **non-increasing** if instead of $\geq$ we have $>$.
3. f is **constant** in E if and only if:
$$\forall x_1; x_2 \in E; x_1 \neq x_2 \Longrightarrow f(x_1) = f(x_2).$$
4. a **monotone** function (resp. **strictly monotone**) is a function which is either increasing or decreasing (resp. strictly increasing or strictly decreasing).

Proposition 3.2. *A sum of two increasing (decreasing) functions is an increasing (decreasing) function.*

Example 3.3.

✓ *The square root function* $\sqrt{x} : [0, +\infty[\longrightarrow \mathbb{R}$ *is increasing.*

✓ *The exponential function* $\exp : \mathbb{R} \longrightarrow \mathbb{R}$ *and logarithm* $\ln :]0, +\infty[\longrightarrow \mathbb{R}$ *are non-decreasing.*

✓ *The absolute value function* $|x| : \mathbb{R} \longrightarrow \mathbb{R}$ *is neither increasing nor decreasing. On the other, the function* $|x| : [0, +\infty[\longrightarrow \mathbb{R}$ *is increasing.*

3.1.4 Parity and periodicity

A set $E \in \mathbb{R}$ is said to be symmetric with respect to the origin if: $x \in E \Rightarrow -x \in E$. Let $f : E \to \mathbb{R}$ be a function defined on this interval. We say that:

(a) f is **even** if $\forall x \in E, f(-x) = f(x)$.

(b) f is **odd** if $\forall x \in E, f(-x) = -f(x)$.

(c) f is **periodic** in E of period T if and only if:

$$\exists T > 0; \forall x \in E; f(x+T) = f(x).$$

Graphic interpretation:

(i) f is even if and only if its graph is symmetric with respect to the y axis.

(ii) f is odd if and only if its graph is symmetric with respect to the origin.

(iii) f is periodic of period T if and only if its graph is invariant under the translation of vector $T\vec{i}$, where $\vec{i}$ is the first coordinate vector.

Example 3.4.

1. *The function defined on* $\mathbb{R}$ *by* $x \longmapsto x^{2n} (n \in \mathbb{N})$ *is even.*

2. *The function defined on* $\mathbb{R}$ *by* $x \longmapsto x^{2n+1} (n \in \mathbb{N})$ *is odd.*

3. *The function* $\cos : \mathbb{R} \longrightarrow \mathbb{R}$ *is even and* $\sin : \mathbb{R} \longrightarrow \mathbb{R}$ *is odd.*

4. *The sine and cosine functions are* 2π*-periodic and the tangent function is* π*-periodic.*

3.2 Limit of a function

Let $a \in \mathbb{R}$. Throughout this chapter, we will say that a function f with domain of definition D_f is defined in the neighborhood of a if there exists a real $h > 0$ such that we are in one of the following three cases:

- ▷ $D_f \cap [a-h; a] \setminus \{a\} = [a-h; a[$ i.e., f is defined in a neighborhood to the left of a and possibly undefined at a.

- ▷ $D_f \cap [a; a+h] \setminus \{a\} =]a; a+h]$ i.e., f is defined in a right neighborhood of a and possibly undefined at a.

- ▷ $D_f \cap [a-h; a+h] \setminus \{a\} = [a-h; a+h] \setminus \{a\}$ i.e., f is defined in a neighborhood of a and possibly undefined at a.

3.2.1 Limit of a function at a point

Definition 3.5. *Let $a, \ell \in \mathbb{R}$ and let f be a function defined in the neighborhood of a. The number ℓ is said to be the limit of f when x tends to a and we write $\lim_{x \to a} f(x) = \ell$ if*

$$\forall \varepsilon > 0, \exists \alpha > 0, \forall x \in D_f, |x - a| < \alpha \Longrightarrow |f(x) - \ell| < \varepsilon.$$

We also say that $f(x)$ tends to ℓ when x tends to a.

Example 3.6. $\lim_{x \to 0}(3x + 1) = 1$.

$$\forall \varepsilon > 0, \exists \alpha = \frac{\varepsilon}{3} > 0, \forall x \in D_f, |x - 0| < \frac{\varepsilon}{3} \Longrightarrow |(3x + 1) - 1| < \varepsilon.$$

3.2.2 One-sided limits

Definition 3.7. *Let $a, \ell \in \mathbb{R}$ and let f be a function defined in the neighborhood of a.*

1. *We say that f has ℓ a left-hand limit at a if the restriction of f in $D_f \cap]-\infty; a[$ has ℓ a limit at a. In this case, this limit is unique and we denote it $\lim_{x \to a^-} f(x) = \ell$,*

$$\forall \varepsilon > 0, \exists \alpha > 0, \forall x \in D_f, a - \alpha < x < a \Longrightarrow |f(x) - \ell| < \varepsilon.$$

2. *We say that f has ℓ a right-hand limit at a if the restriction of f à $D_f \cap]a; +\infty[$ admet ℓ a limit at a. In this case, this limit is unique and we denote it $\lim_{x \to a^+} f(x) = \ell$,*

$$\forall \varepsilon > 0, \exists \alpha > 0, \forall x \in D_f, a < x < a + \alpha \Longrightarrow |f(x) - \ell| < \varepsilon.$$

Example 3.8.

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto f(x) = \begin{cases} 4x + 5 & si\ x < 0, \\ 2x + 1 & si\ x \geq 0. \end{cases} \end{aligned}$$

We have: $\lim_{x \to 0^-} f(x) = 5$ and $\lim_{x \to 0^+} f(x) = 1$.
In this case we say that f hasn't a limit at 0.

Proposition 3.9.

$$\lim_{x \to a} f(x) = \ell \Longleftrightarrow \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = \ell.$$

3.2.3 Uniqueness of the limit

Theorem 3.10. *Let f be a function defined in the neighborhood of $a \in \mathbb{R}$.*

(i) If f has a limit ℓ at a, it is unique, we then denote $\lim_{x \to a} f(x) = \ell$.

(ii) If f is defined at a and has a limit at a, then $\lim_{x \to a} f(x) = f(a)$.

3.2.4 Sequential characterization of the limit

Theorem 3.11. *Let f be a function defined in the neighborhood of $a \in \mathbb{R}$ and let $\ell \in \mathbb{R}$. The following propositions are equivalent:*

(*i*) $\lim\limits_{x\to a} f(x) = \ell$.

(*ii*) *For every sequence (u_n) in D_f has a limit a for all n, then, $f(u_n)$ has a limit ℓ.*

Method: to show that a function f hasn't a limit at a, it suffices to find two sequences (u_n) and (v_n) have a same limit at a such that $f(u_n)$ and $f(v_n)$ have different limits.

Example 3.12. *The function $x \longmapsto \sin\dfrac{1}{x}$ hasn't a limit at 0: let*

$$u_n = \frac{1}{\frac{\Pi}{2} + 2\Pi n} \qquad \text{and} \qquad v_n = \frac{1}{\Pi + 2\Pi n}.$$

Then,

$$f(u_n) \longrightarrow 1, \qquad f(v_n) \longrightarrow 0$$

have different limits, hence, $\lim\limits_{x\to 0} \sin\dfrac{1}{x}$ does not exist.

Theorem 3.13. *Let f and g be two functions defined in the neighborhood of $a \in \mathbb{R}$ and let $\ell, \ell' \in \mathbb{R}$. If $\lim\limits_{x\to a} f(x) = \ell$ and $\lim\limits_{x\to a} g(x) = \ell'$. Then,*

1. $\lim\limits_{x\to a}[f(x) + g(x)] = \ell + \ell'$.

2. $\lim\limits_{x\to a}[f(x)g(x)] = \ell\ell'$.

3. $\lim\limits_{x\to a}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\ell}{\ell'}, \qquad \ell' \neq 0.$

Proposition 3.14. *(Composition of limits).*
Let f be a function defined in the neighborhood of $a \in \mathbb{R}$ and g be a function defined in the neighborhood of $b \in \mathbb{R}$ and let $\ell \in \mathbb{R}$.
If $\lim\limits_{x\to a} f(x) = b$ and $\lim\limits_{x\to b} g(x) = \ell$, then, $\lim\limits_{x\to a}(gof)(x) = \ell$.

3.2.5 Passing to the limit

Let f and g be two functions defined in the neighborhood of $a \in \mathbb{R}$ and let $\ell, \ell', m, M \in \mathbb{R}$.

(*i*) If $\lim\limits_{x\to a} f(x) = \ell$ and $\lim\limits_{x\to a} g(x) = \ell'$ and if $f \leq g$ in the neighborhood of a, then, $\ell \leq \ell'$.

(*ii*) If $\lim\limits_{x\to a} f(x) = \ell$ and $f \leq M$ in the neighborhood of a, then, $\ell \leq M$.

(*iii*) If $\lim\limits_{x\to a} f(x) = \ell$ and $f \geq m$ in the neighborhood of a, then, $\ell \geq m$.

3.2.6 Pinching Theorem, Decrease Theorem and Increase Theorem

Let $a, \ell \in \mathbb{R}$ and let f, g and h be three functions defined in the neighborhood of a.

Theorem 3.15. *(Pinching Theorem)*
If $\lim_{x\to a} h(x) = \ell, \lim_{x\to a} g(x) = \ell$ and $h \le f \le g$ in the neighborhood of a, then, f has a limit at a and this one is worth ℓ.

Example 3.16. *Study the limit of $f(x) = x\sin\frac{1}{x}$ at 0. We have: $\forall x \in \mathbb{R}^*$,*

$$-1 \le \sin\frac{1}{x} \le 1 \Longleftrightarrow \begin{cases} x \le \sin\frac{1}{x} \le -x, & if\ x < 0, \\ -x \le \sin\frac{1}{x} \le x, & if\ x > 0. \end{cases}$$

Using the previous theorem, we get $\lim_{x\to 0} x\sin\frac{1}{x} = 0$.

Theorem 3.17. *(Decrease Theorem)*
If $\lim_{x\to a} h(x) = +\infty$ and $h \le f$ in the neighborhood of a, then, f has a limit at a and this one is worth $+\infty$.

Theorem 3.18. *(Increase Theorem)*
If $\lim_{x\to a} g(x) = -\infty$ and $f \le g$ in the neighborhood of a, then, f has a limit at a and this one is worth $-\infty$.

Example 3.19. *Study the limit of $f(x) = \frac{x}{2+\sin x}$ at $\pm\infty$. We have: $\forall x \in \mathbb{R}$,*

$$-1 \le \sin x \le 1 \Rightarrow 1 \le 2 + \sin x \le 3 \Rightarrow \frac{1}{3} \le \frac{1}{2+\sin x} \le 1.$$

- ▷ *If $x \in]-\infty, 0[\Longrightarrow \frac{x}{3} \ge \frac{x}{2+\sin x}$ and using the previous theorem increase, we get:*

$$\lim_{x\to-\infty} f(x) = -\infty \quad because \quad \lim_{x\to-\infty} \frac{x}{3} = -\infty.$$

- ▷ *If $x \in]0, +\infty[\Longrightarrow \frac{x}{3} \le \frac{x}{2+\sin x}$ and using the previous theorem decrease, we get:*

$$\lim_{x\to+\infty} f(x) = +\infty \quad because \quad \lim_{x\to+\infty} \frac{x}{3} = +\infty.$$

3.3 Continuous functions

Definition 3.20. *Let a be a real number and f a function defined in the neighborhood of a. We say that f is continuous at a if f is defined at a and $\lim_{x\to a} f(x) = f(a)$, i.e.,*

$$\forall \varepsilon > 0, \exists \alpha > 0, \forall x \in D_f, |x - a| < \alpha \Rightarrow |f(x) - f(a)| < \varepsilon.$$

Example 3.21. *$f(x) = 3x + 1$ on $\mathbb{R}$, f is continuous at $x_0 = 2$:*

$$\forall \varepsilon > 0, \exists \alpha = \frac{\varepsilon}{3} > 0, \forall x \in D_f, |x - 2| < \frac{\varepsilon}{3} \Rightarrow |(3x + 1) - 7| < \varepsilon.$$

Definition 3.22. *Let a be a real number and f a function defined in the neighborhood of a. We say that:*

(i) f is left continuous at a if $\lim_{x \to a^-} f(x) = f(a)$, i.e.,

$$\forall \varepsilon > 0, \exists \alpha > 0, \forall x \in D_f, 0 < a - x < \alpha \Rightarrow |f(x) - f(a)| < \varepsilon.$$

Example 3.23. *Let $f(x) = \sqrt{3 - x}$; $D_f =]-\infty, 3]$.*
f is left continuous at $a = 3$ because $\lim_{x \to 3^-} f(x) = 0 = f(3)$.

(ii) f is right continuous at a if $\lim_{x \to a^+} f(x) = f(a)$. C'est à dire:

$$\forall \varepsilon > 0, \exists \alpha > 0, \forall x \in D_f, 0 < x - a < \alpha \Rightarrow |f(x) - f(a)| < \varepsilon.$$

Example 3.24. *Let $f(x) = \sqrt{1 - x^2}$; $D_f = [-1, 1]$.*
f is right continuous at $a = -1$ because $\lim_{x \to -1^+} f(x) = 0 = f(-1)$.

Example 3.25. *The function:*

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto f(x) = \begin{cases} 3x & if\ x < 1, \\ 2 & si\ x = 1, \\ 3x - 1 & if\ x > 1. \end{cases} \end{aligned}$$

f is not left continuous at 1 but it is right continuous at 1 because:

$$\lim_{x \to 1^-} f(x) = 3 \neq f(1) \quad et \quad \lim_{x \to 1^+} f(x) = 2 = f(1).$$

In this case f is not continuous at 1.

Proposition 3.26. *f is continuous at a if and only if f is left continuous and right continuous at a.*

3.3.1 Continuous of composed functions

Let $f : I \longrightarrow I'$ and $g : I' \longrightarrow \mathbb{R}$ be two functions continuous at a and $f(a)$ respectively. Then, $gof : I \longrightarrow \mathbb{R}$ is continuous at a.

3.3.2 Continuous over an interval

Let $f : I \longrightarrow \mathbb{R}$. We say that f is continuous on I (I denotes an interval) if f is continuous at any point of I.

We denote $C(I;\mathbb{R})$ or $C^0(I;\mathbb{R})$ the set of continuous functions on I with values in $\mathbb{R}$.

Example 3.27. *Let $\alpha \in \mathbb{R}$, we define the function:*

$$\begin{array}{rcl} f:\mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & f(x) = \begin{cases} x^2 + (\alpha^2-1)^2 & si\ x<0, \\ x^3 & si\ x \geq 0. \end{cases} \end{array}$$

f is continuous on $\mathbb{R}$ if:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0).$$

We have: $\lim_{x\to 0^-} f(x) = (\alpha^2-1)^2$ and $\lim_{x\to 0^+} f(x) = 0$, then, by identification, we get: $(\alpha^2-1)^2 = 0$, which gives $\alpha = \pm 1$.

3.3.3 Operations on continuous functions

Let f and g be two functions continuous at a. Then,

- $\forall \alpha, \beta \in \mathbb{R} : \alpha f + \beta g$ is continuous at a.
- $f \cdot g$ is continuous at a.
- $\frac{f}{g}$ is continuous at a if $g(a) \neq 0$.
- $|f|$ is continuous at a.

Remark 3.28. *f is called discontinuous at a if:*

(a) *f is not defined at a.*

(b) *the limit exists but different from $f(a)$.*

(c) *the limit does not exist.*

3.3.4 The Intermediate-Value Theorem

Theorem 3.29. *Let f be a continuous function on interval $[a;b]$. For any real k between $f(a)$ and $f(b)$, there exists $x_0 \in [a,b]$ such that $f(x_0) = k$, and if moreover f is strictly monotone, then, the x_0 is unique.*

Example 3.30. *Show that $\ln x - \frac{1}{x} = 0$ has a unique solution on $]1,2[$.*

Let $f(x) = \ln x - \frac{1}{x}$. We have:

(i) *f is continuous on* $[1,2]$.

(ii) $f(1)=-1$ *and* $f(2)\simeq 0,19$.

the IVT, implies $\exists x_0 \in]1,2[: f(x_0)=0$.

Uniqueness: $f'(x)=\dfrac{1}{x}+\dfrac{1}{x^2}$, *hence f is non-decreasing. Then, the solution x_0 is unique.*

3.3.5 Extension by continuous

Let f be a function in the neighborhood of a but not defined at a. We say that f is extendable by continuous at a if f has a finite limit ℓ at a. The extension $\tilde{f}$ of f defined by:

$$\tilde{f}(x)=\begin{cases} f(x) & if \quad x\neq a,\\ \ell & if \quad x=a \end{cases}$$

then, is continuous at a.

Example 3.31. *Find an extension by continuous to $\mathbb{R}$ of the following function:*

$$f(x)=\frac{x^3+5x+6}{x^3+1}, \qquad D_f=\mathbb{R}-\{-1\}.$$

We have: $\lim\limits_{x\to -1} f(x)=\lim\limits_{x\to -1}\dfrac{x^3+5x+6}{x^3+1}=\dfrac{8}{3}$. *The extension $\tilde{f}$ of f defined by:*

$$\tilde{f}(x)=\begin{cases} \dfrac{x^3+5x+6}{x^3+1} & if \quad x\neq -1,\\ \dfrac{8}{3} & if \quad x=-1. \end{cases}$$

3.4 Derivative and differentiability of a function

Definition 3.32. *Let f be a function and $a\in D_f$. We say f is differentiable at a if:*

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a} \quad exists\ and\ finite.$$

This limit is called the derivative of f at a and denoted $f'(a)$, or $\frac{df}{dx}(a)$. Another writing of the derivative at a:

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=f'(a).$$

Example 3.33. *Let f be a function defined by:*

$$\begin{array}{rcl} f:\mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & f(x)=x^3. \end{array}$$

Find the derivative of f at a of $\mathbb{R}$. We have:

$$\begin{array}{rcl} f'(a) & = & \lim\limits_{x\to a} \dfrac{f(x)-f(a)}{x-a} \\ & = & \lim\limits_{x\to a} \dfrac{x^3-a^3}{x-a} \\ & = & \lim\limits_{x\to a} \dfrac{(x-a)(x^2+ax+a^2)}{x-a} \\ & = & \lim\limits_{x\to a}(x^2+ax+a^2)=3a^2. \end{array}$$

3.4.1 Left-hand Derivative, Right-hand Derivative

We define the left-hand derivative of f at a by

$$f'_g(a)=\lim_{x\to a^-}\frac{f(x)-f(a)}{x-a}.$$

Similarly, we define right-hand derivative of f at a by:

$$f'_d(a)=\lim_{x\to a^+}\frac{f(x)-f(a)}{x-a},$$

and

$$f \text{ is differentiable at } a \Longleftrightarrow f'_g(a)=f'_d(a)=f'(a).$$

Example 3.34. *Let f be the function defined by:*

$$\begin{array}{rcl} f:\mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & f(x)=\begin{cases} 1-2x & if\ x<0, \\ x+1 & if\ x\geq 0. \end{cases} \end{array}$$

is f differentiable at 0?

We have:

$$\begin{array}{rcl} f'_g(0) & = & \lim\limits_{x\to 0^-} \dfrac{f(x)-f(0)}{x} \\ & = & \lim\limits_{x\to 0^-} \dfrac{(1-2x)-1}{x} \\ & = & \lim\limits_{x\to 0^-} \dfrac{-2x}{x}=-2, \end{array}$$

et

$$\begin{aligned} f'_d(0) &= \lim_{x \to 0^+} \frac{f(x) - f(0)}{x} \\ &= \lim_{x \to 0^+} \frac{(x+1) - 1}{x} \\ &= \lim_{x \to 0^+} \frac{x}{x} = 1. \end{aligned}$$

Finally, f is not differentiable at 0 because $f'_g(0) \neq f'_d(0)$.

Definition 3.35. *f is differentiable on I if it is differentiable at every point of I and the map:*

$$\begin{aligned} f' : I &\rightarrow \mathbb{R} \\ x &\mapsto f'(x) \end{aligned}$$

is called the derivative function of f.

3.4.2 Geometric interpretation of the derivative

Let f be a differentiable function at a and (G) the graph representing of f. The equation of the tangent (T) of the graph (G) at $M(a, f(a))$ is

$$(T) : y = f'(a)(x - a) + f(a),$$

where $f'(a)$ represents the slope of the line tangent to the graph(G).

3.4.3 Operations on derivatives

Let f and g be two differentiable functions at a. Then $\alpha f, \alpha \in \mathbb{R}, f + g, f \times g$ are differentiable at a, as well as $\frac{f}{g}$ if $g(a) \neq 0$. Moreover, we have the formulas:

- $(\alpha f)'(x) = \alpha f'(x)$.
- $(f + g)'(x) = f'(x) + g'(x)$.
- $(f \times g)'(x) = f'(x)g(x) + f(x)g'(x)$.
- $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$.

Theorem 3.36. Derivative of a composed function.

Let $f : I \longrightarrow I'$ and $g : I' \longrightarrow \mathbb{R}$ be two differentiable functions at a and $f(a)$ respectively. Then, $gof : I \longrightarrow \mathbb{R}$ is differentiable at a and we have:

$$(gof)'(a) = f'(a) \cdot g'[f(a)].$$

Proof: We have:

$$\begin{aligned}(gof)'(a) &= \lim_{x\to a}\frac{(gof)(x)-(gof)(a)}{x-a}\\ &= \lim_{x\to a}\frac{g[f(x)]-g[f(a)]}{x-a}\\ &= \lim_{x\to a}\frac{g[f(x)]-g[f(a)]}{f(x)-f(a)}\cdot\frac{f(x)-f(a)}{x-a}\\ &= \lim_{x\to a}\frac{g[f(x)]-g[f(a)]}{f(x)-f(a)}\cdot\lim_{x\to a}\frac{f(x)-f(a)}{x-a}.\\ &= f'(a)\cdot g'[f(a)].\end{aligned}$$

Example 3.37. *Consider the functions f and g defined by:*

$$\begin{array}{rcl} f:\mathbb{R} & \longrightarrow & \mathbb{R}_+ \\ x & \longmapsto & f(x)=x^2 \end{array} \quad \textit{and} \quad \begin{array}{rcl} g:\mathbb{R}_+ & \longrightarrow & [-1;1] \\ x & \longmapsto & g(x)=\sin x, \end{array}$$

then,

$$\begin{array}{rcl} g\circ f:\mathbb{R} & \longrightarrow & [-1;1] \\ x & \longmapsto & (g\circ f)(x)=\sin x^2 \end{array}$$

and

$$\begin{aligned}(g\circ f)'(x) &= f'(x)\cdot g'[f(x)]\\ &= 2x\cos x^2.\end{aligned}$$

Theorem 3.38. Derivative of a reciprocal function.

If f is differentiable at x_0, then, f^{-1} is differentiable at $f(x_0)$ and we have:

$$(f^{-1})'(y)=\frac{1}{f'(x)}.$$

Proof: We have:

$$\begin{aligned}(f^{-1})'(y_0) &= \lim_{y\to y_0}\frac{(f^{-1})(y)-(f^{-1})(y_0)}{y-y_0}\\ &= \lim_{y\to y_0}\frac{1}{\dfrac{y-y_0}{(f^{-1})(y)-(f^{-1})(y_0)}}\\ &= \lim_{x\to x_0}\frac{1}{\dfrac{f(x)-f(x_0)}{x-x_0}}\\ &= \frac{1}{f'(x_0)}.\end{aligned}$$

Example 3.39. *Consider the function f defined by:*

$$\begin{array}{rcl} f:\mathbb{R} & \longrightarrow &]0,+\infty[\\ x & \longmapsto & f(x)=e^x \end{array}$$

is bijective so has a reciprocal map:

$$\begin{array}{rcl} f^{-1}:]0,+\infty[& \longrightarrow & \mathbb{R} \\ y & \longmapsto & f^{-1}(y)=\ln y, \end{array}$$

where

$$y=f(x)\Longleftrightarrow y=e^x\Longleftrightarrow x=\ln y.$$

$$\begin{aligned} (f^{-1})'(y) &= \frac{1}{f'(x)} \\ &= \frac{1}{e^x} \\ &= \frac{1}{y}. \end{aligned}$$

3.4.4 Higher order derivative

Let $f:I\to\mathbb{R}$ be a differentiable function on I, then, f' is called the derivative of order 1 of f. If f' is differentiable on I then, its derivative is called the derivative of order 2 of f. We note it f'' or f^2:

$$f^2=f''=(f')'.$$

In general, we define the derivative of order n of f by:

$$f^n=(f^{n-1})'; \qquad \forall n\geq 1, f^0=f.$$

We say that f is of class C^1 on I if f is differentiable on I and f' is continuous on I.

We say that f is of class C^n on I and we write $f\in C^n(I)$ if f is n^{th} differentiable on I and f^n is continuous on I.

f is called to be of class C^∞ on I if it is class $C^n, \forall n\in\mathbb{N}$.

3.4.5 n^{th} Derivative of a product (Leibniz formula)

Theorem 3.40. *Let $f,g:[a,b]\to\mathbb{R}$ n^{th} differentiable, then $f\cdot g$ is n^{th} differentiable and we have:*

$$\forall x\in[a,b];\quad (f\cdot g)^n(x)=\sum_{k=0}^{n}C_k^n f^{n-k}(x)g^k(x)\quad \textit{where } C_k^n=\frac{n!}{k!(n-k)!}.$$

Proof: if $n = 1$, we have:

$$\begin{aligned}(f \cdot g)^1(x) = (f \cdot g)'(x) &= \sum_{k=0}^{1} C_k^1 f^{1-k}(x) g^k(x) \\ &= C_0^1 f^1(x) g^0(x) + C_1^1 f^0(x) g^1(x) \\ &= f'(x)g(x) + f(x)g'(x).\end{aligned}$$

Example 3.41. *Calculate* $(x^3 \sin 4x)^3$. *We have:*

$$\begin{aligned}(x^3 \sin 4x)^3 &= \sum_{k=0}^{3} C_k^3 (x^3)^{3-k} (\sin 4x)^k \\ &= C_0^3 (x^3)^3 (\sin 4x)^0 + C_1^3 (x^3)^2 (\sin 4x)^1 \\ &+ C_2^3 (x^3)^1 (\sin 4x)^2 + C_3^3 (x^3)^0 (\sin 4x)^3,\end{aligned}$$

continue $\cdots$

3.4.6 Derivability and continuous

If f is differentiable at x_0 then, f is continuous at x_0. The converse is false in general.

Example 3.42. $f(x) = |x|, \quad x \in \mathbb{R}$. *$f$ is continuous at $x_0 = 0$ but it is not differentiable at $x_0 = 0$ because* $f'_g(0) = -1 \neq f'_d(0) = 1$.

3.4.7 Fundamental theorems on differentiable functions

Theorem 3.43. *(Rolle's Theorem).*

Let $f : [a, b] \to \mathbb{R}$ be a function verifying:

1. *f is continuous on $[a, b]$,*
2. *f is differentiable on $]a, b[$,*
3. *$f(a) = f(b)$.*

Then, $\exists c \in]a, b[$: $f'(c) = 0$.

Rolle's theorem tells us that there is a point c at which the tangent is parallel to the x axis.

Example 3.44. *To show that the equation* $\sin x + \cos x = 0$ *has at least one solution in the interval* $]0, \Pi[$*, we use the function* $f(x) = e^x \sin x - 1$ *which is continuous on* $[0; \Pi]$*, differentiable on* $]0, \Pi[$ *and* $f(0) = f(\Pi) = -1$.

So by **the Rolle's theorem** $\exists c \in]0; \Pi[$ *such that*

$$f'(c) = 0 \Rightarrow e^c \sin c + e^c \cos c = 0 \Rightarrow \sin c + \cos c = 0.$$

Theorem 3.45. *(Lagrange's Theorem or finite increments).*

Let $f : [a, b] \to \mathbb{R}$ be a function satisfying:

1. *f is continuous on $[a, b]$,*
2. *f is differentiable on $]a, b[$.*

Then, $\exists c \in]a, b[: f(b) - f(a) = (b - a)f'(c)$.

Proof: Consider the function $g : [a, b] \to \mathbb{R}$ defined by:

$$\forall x \in [a, b], g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

The function g is:

- Continuous on $[a, b]$ and differentiable on $]a, b[$ because it is the product and the sum of the functions continuous on $[a, b]$ and differentiable on $]a, b[$.
- $g(a) = g(b) = 0$.

So by the Rolle's theorem, $\exists c \in]a, b[: g'(c) = 0$. We have:

$$\begin{aligned} g'(c) = 0 &\iff f'(c) - \frac{f(b) - f(a)}{b - a} = 0 \\ &\iff f'(c) = \frac{f(b) - f(a)}{b - a} \\ &\iff \exists c \in]a, b[: f(b) - f(a) = (b - a)f'(c). \end{aligned}$$

Lagrange's theorem tells us that there is a point $c \in]a, b[$ in which the tangent to the graph is parallel to the line joining the two points $(a, f(a)), (b, f(b))$.

3.5 Hospital Rules

(i) Let $f, g : I \longrightarrow \mathbb{R}$ be two continuous functions on I, differentiable on $I - \{x_0\}$ and verifying the following conditions:

▷ $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0$,

▷ $g'(x) \neq 0, \forall x \in I - \{x_0\}$,

then,

$$\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = \ell \Longrightarrow \lim_{x \to x_0} \frac{f(x)}{g(x)} = \ell.$$

Example 3.46.

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1.$$

The converse is generally false.

Remark 3.47. *The Hospital Rule is true when $x \to +\infty$. Let's $x = \dfrac{1}{t}$,*

$$\begin{aligned}\lim_{x\to+\infty}\frac{f(x)}{g(x)} &= \lim_{x\to t}\frac{f\left(\frac{1}{t}\right)}{g\left(\frac{1}{t}\right)} = \lim_{x\to t}\frac{\frac{-1}{t^2}f'\left(\frac{1}{t}\right)}{\frac{-1}{t^2}g'\left(\frac{1}{t}\right)}\\ &= \lim_{x\to t}\frac{f'\left(\frac{1}{t}\right)}{g'\left(\frac{1}{t}\right)} = \lim_{x\to t} f'(x)g'(x).\end{aligned}$$

$\lim\limits_{x\to t} f'(x)g'(x) = \dfrac{0}{0}$ *and f', g' verify the conditions of the theorem, then we can apply once again the Hospital's rule.*

(*ii*) Let $f, g : I \longrightarrow \mathbb{R}$ be two continuous functions on I, differentiable on $I - \{x_0\}$ and verifying the following conditions:

- ▷ $\lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} g(x) = +\infty$,
- ▷ $g'(x) \neq 0, \forall x \in I - \{x_0\}$,

then,

$$\lim_{x\to x_0}\frac{f'(x)}{g'(x)} = \ell \Longrightarrow \lim_{x\to x_0}\frac{f(x)}{g(x)} = \ell.$$

The previous remark is true in this case.

Example 3.48.

$$\begin{aligned}\lim_{x\to+\infty}\frac{x^n}{e^x} &= \lim_{x\to+\infty}\frac{nx^{n-1}}{e^x} = \lim_{x\to+\infty}\frac{n(n-1)x^{n-2}}{e^x}\\ &= \cdots = \lim_{x\to+\infty}\frac{n(n-1)(n-2)\cdots x^0}{e^x} = 0.\end{aligned}$$

Exercise 1. *Let α be a non-negative real, consider the function f defined by:*

$$f(x) = \begin{cases} x^2 + x + \dfrac{1}{\alpha} & if\ x \leq 0,\\ \dfrac{\sin(\alpha x)}{x} - \sqrt{x} & si\ 0 < x \leq \alpha.\end{cases}$$

1. *Determine the domain of definition of f.*
2. *For which value of α the function f is-it continuous at $x_0 = 0$.*
3. *For the value of α found in question 2., show that there exists at least one real c in the interval $]0, \alpha[$ solution of the equation $f(x) = 0$.*

Solution: Let α be a non-negative real, consider the function f defined by:

$$f(x) = \begin{cases} x^2 + x + \dfrac{1}{\alpha} & if\ x \leq 0, \\ \dfrac{\sin(\alpha x)}{x} - \sqrt{x} & if\ 0 < x \leq \alpha. \end{cases}$$

1. Determine the domain of definition of f:

$$D_f =]-\infty; 0] \cup]0, \alpha] =]-\infty, \alpha].$$

2. For which value of α the function f is-it continuous at $x_0 = 0$:

$$f\ is\ continuous\ at\ x_0 \Longleftrightarrow \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0).$$

We have:

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x^2 + x + \frac{1}{\alpha} = \frac{1}{\alpha}$$

and

$$\begin{aligned} \lim_{x \to 0^+} f(x) &= \lim_{x \to 0^+} \frac{\sin(\alpha x)}{x} - \sqrt{x} \\ &= \lim_{x \to 0^+} \alpha \cdot \frac{\sin(\alpha x)}{\alpha x} - \sqrt{x} \\ &= \alpha. \end{aligned}$$

By identification, we get: $\dfrac{1}{\alpha} = \alpha$, which gives $\alpha = \pm 1$, since α be a non-negative real, then, $\alpha = 1$.

3. Show that $\exists c \in]0, \alpha[$ solution of the equation $f(x) = 0$:

$$\alpha = 1, \qquad f(x) = \frac{\sin(\alpha x)}{x} - \sqrt{x}.$$

Let us apply the Intermediate-Value Theorem to f on $]0, 1[$:

- f is continuous on $[0, 1]$.
- $f(0) = 1 > 0$ and $f(1) = \sin 1 - 1 < 0$.

The IVT, implies $\exists c \in]0, 1[$: $f(c) = 0$.

Exercise 2. *Consider the map:*

$$\begin{aligned} f : \mathbb{R}^* &\longrightarrow \mathbb{R} \\ x &\longmapsto f(x) = 1 - \frac{\sin(2x)}{x}. \end{aligned}$$

1. *Calculate $f(-\Pi)$ et $f(\Pi)$. The function f is it injective? justify.*

2. *Calculate* $\lim_{x\to 0} f(x)$.
3. *Deduce that the function* f *is extendable by continuous at* $x_0 = 0$. *Let* g *be the extension by continuous of* f, *write the expression for* $g(x)$.
4. *Show that the equation* $g(x) = 0$ *has at least one real solution in the interval* $[0, \Pi]$.
5. *Study the differentiability* g *at* $x_0 = 0$.

Solution:

$$\begin{array}{rcl} f : \mathbb{R}^* & \longrightarrow & \mathbb{R} \\ x & \longmapsto & f(x) = 1 - \dfrac{\sin(2x)}{x}. \end{array}$$

1. We have: $f(-\Pi) = 1 - \dfrac{\sin(-2\Pi)}{\Pi} = 1$ and $f(\Pi) = 1 - \dfrac{\sin(2\Pi)}{\Pi} = 1$. (Remark that the map f is even).
   The map f is not injective because $-\Pi \neq \Pi$ and $f(-\Pi) = f(\Pi)$.
2. $\lim_{x\to 0} f(x) = \lim_{x\to 0}\left[1 - \dfrac{\sin(2x)}{x}\right] = \lim_{x\to 0}\left[1 - 2\dfrac{\sin(2x)}{2x}\right] = -1$, because $\lim_{t\to 0}\dfrac{\sin t}{t} = 1$.
3. The limit of $f(x)$ at $x_0 = 0$ exists and is finite so the function f is extendable by continuous at this point. Its extension g is defined by:

$$g(x) = \begin{cases} f(x) & if\ x \neq 0 \\ -1 & if\ x = 0. \end{cases}$$

4. Let us show that the equation $g(x) = 0$ has at least one real solution in the interval $[0, \Pi]$:
   - (i) The function g is continuous on $[0, \Pi]$.
   - (ii) $g(0)g(\Pi) = g(0)f(\Pi) = -1 < 0$.

   By the Intermediate-Value Theorem: $\exists c \in]0, \Pi[; g(c) = 0$.
5. Differentiability of g at $x_0 = 0$:

$$\lim_{x\to 0}\frac{g(x) - g(0)}{x - 0} = \lim_{x\to 0}\frac{2 - \dfrac{\sin(2x)}{2x}}{x} = \lim_{x\to 0}\frac{2x - \sin(2x)}{x^2},$$

   the direct calculation leads to the indeterminate form $\frac{0}{0}$, we can apply the Hospital's rule because the functions $x \longmapsto 2x - \sin(2x)$ et $x \longmapsto x^2$ are differentiable at 0. We have:

$$\lim_{x\to 0}\frac{[2x - \sin(2x)]'}{[x^2]'} = \lim_{x\to 0}\frac{2 - 2\cos(2x)}{2x}$$

also gives the indeterminate form $\frac{0}{0}$. Let us apply once again the Hospital's rule:

$$\lim_{x\to 0}\frac{[2-2\cos(2x)]'}{[2x]'}=\lim_{x\to 0}\frac{4\sin(2x)}{2}=0.$$

So,

$$\lim_{x\to 0}\frac{g(x)-g(0)}{x-0}=0.$$

Therefore the function g is differentiable at 0 and $g'(0)=0$.

CHAPTER

4

APPLICATIONS TO ELEMENTARY FUNCTIONS

4.1 Power function

Definition 4.1. *The power function of exponent* $a, (a \in \mathbb{R}^*)$*:*

$$x \longmapsto x^a = \underbrace{x \times x \times \cdots \times x}_{a \ times}$$

is a continuous map on $]0, +\infty[$ *and strictly monotone (increasing if* $a > 0$ *and decreasing if* $a < 0$*).*

It is differentiable on $]0, +\infty[$ *of derivative::* $x \longmapsto ax^{a-1}$.

We have:

$$\lim_{x \to 0} x^a = \begin{cases} +\infty & a < 0, \\ 1 & a = 0, \\ 0 & a > 0. \end{cases}$$

and

$$\lim_{x \to +\infty} x^a = \begin{cases} 0 & a < 0, \\ 1 & a = 0, \\ +\infty & a > 0. \end{cases}$$

Let's look at the function $x \longmapsto x^{\frac{1}{a}}$ *on* $\mathbb{R}^+$ *is the reciprocal function of* $f : x \longmapsto x^a$ *on* $\mathbb{R}^+$.

4.2 Logarithmic function

Definition 4.2. *There exists a unique function, denoted* $\ln :]0,+\infty[\longrightarrow \mathbb{R}$ *such that:*

$$\ln'(x) = \frac{1}{x}, \qquad for\ all\ x > 0\ and\ \ln 1 = 0.$$

Proposition 4.3.

(*i*) $\ln$ *is a continuous, non-decreasing and defines a bijection of* $]0,+\infty[$ *on* $\mathbb{R}$.

(*ii*) *The function* $\ln$ *is concave and* $\ln x \leq x - 1$ *for all* $x > 0$.

(*iii*) *Moreover this function* $\ln$ *satisfies for all* $a, b > 0$:

- ▷ $\ln(a \times b) = \ln a + \ln b$,
- ▷ $\ln \frac{1}{a} = -\ln a$,
- ▷ $\ln a^n = n \ln a, \forall n \in \mathbb{N}$.

(*iv*) *Three limits must be known:*

$$\lim_{x \to 0^+} \ln x = -\infty, \quad \lim_{x \to +\infty} \ln x = +\infty, \quad \lim_{x \to 0} \frac{\ln(1+x)}{x} = 1.$$

Remark 4.4. $\ln x$ *is called the natural logarithm or also natural logarithm. It is characterized by* $\ln e = 1$. *We define the logarithm to base a by:*

$$\log_a x = \frac{\ln x}{\ln a}, \qquad so\ that\ \log_a a = 1.$$

For $a = 10$ *we obtain the decimal logarithm* $\log_{10}$ *which verifies* $\log_{10} 10 = 1$, *hence* $\log_{10}(10)^n = n$. *Therefore, we have the equivalence:*

$$x = 10^y \Longleftrightarrow y = \log_{10} x.$$

4.3 Exponential function

Definition 4.5. *The reciprocal bijection of* $\ln :]0,+\infty[\longrightarrow \mathbb{R}$ *is called the exponential function, denoted* $\exp : \mathbb{R} \longrightarrow]0,+\infty[$.

For $x \in \mathbb{R}$, *we also denote* e^x *for* $\exp x$.

Proposition 4.6.

(*i*) $\exp : \mathbb{R} \longrightarrow]0,+\infty[$ *is a continuous, non-decreasing.*

(*ii*) *The exponential function is differentiable and* $\exp' x = \exp x$, *for all* $x \in \mathbb{R}$. *It is convex et* $\exp x \geq 1 + x$.

(*iii*) *Furthermore this function* ln *satisfies for all* $a, b > 0$*:*

- $\exp(\ln x) = x$ *for all* $x > 0$ *et* $\ln(\exp x) = x$ *for all* $x \in \mathbb{R}$.
- $\exp(a+b) = \exp a) \times \exp(b)$, $\qquad \exp(a-b) = \dfrac{\exp(a)}{\exp(b)}$.
- $(\exp x)^n = \exp(nx)$.

(*iv*) *Three limits must be know:*

$$\lim_{x\to-\infty} \exp x = 0, \quad \lim_{x\to+\infty} \exp x = +\infty, \quad \lim_{x\to 0} \frac{\exp x - 1}{x} = 1.$$

Remark 4.7. *The exponential function is the unique function which satisfies:*

$$\exp'(x) = \exp(x) \quad \textit{for all } x \in \mathbb{R} \quad \textit{and} \quad \exp(1) = e,$$

where $e = 2,718\cdots$ *is the number which satisfies* $\ln e = 1$.

We call the exponential function to $a (a > 0)$*, the function denoted* $\exp_a$ *defined on* $\mathbb{R}$ *by:*

$$\forall x \in \mathbb{R} : \exp_a(x) = e^{x\ln(a)} = a^x.$$

Power and comparison: by definition, for $x > 0$ and $\alpha \in \mathbb{R} : x^\alpha = \exp(\alpha \ln x)$, let's compare the functions $\ln x, \exp x$ with x:

$$\lim_{x\to 0^+} x\ln x = 0, \quad \lim_{x\to-\infty} x\exp x = 0, \quad \lim_{x\to+\infty} \frac{\ln x}{x} = 0, \quad \lim_{x\to+\infty} \frac{\exp x}{x} = +\infty.$$

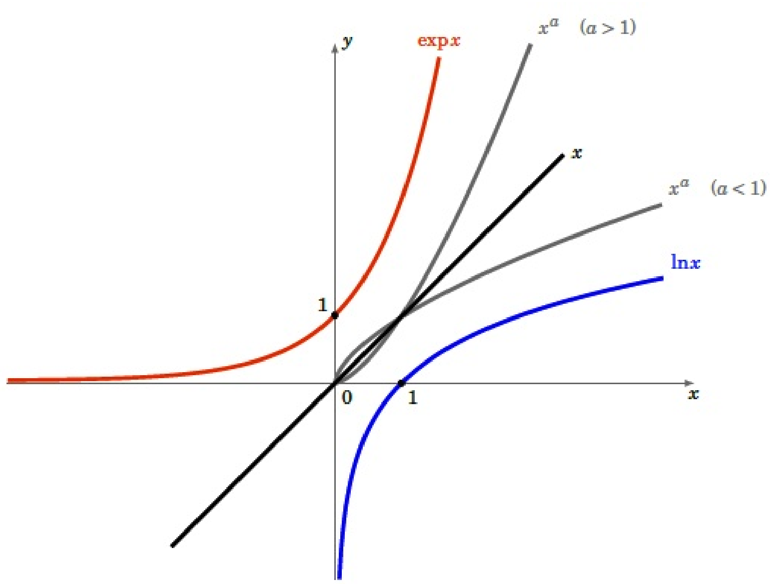


Figure 4.1: Compare $x^a, \ln x$ and $\exp x$ functions.

4.4 Hyperbolic functions and their inverses

The four hyperbolic functions are defined as follows:

▷ **hyperbolic sine:**

$$\begin{aligned} sh : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto shx = \frac{e^x - e^{-x}}{2}. \end{aligned}$$

▷ **hyperbolic cosine:**

$$\begin{aligned} ch : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto chx = \frac{e^x + e^{-x}}{2}. \end{aligned}$$

▷ **hyperbolic tangent:**

$$\begin{aligned} th : \mathbb{R} &\longrightarrow]-1, +1[\\ x &\longmapsto thx = \frac{shx}{chx} = \frac{e^x - e^{-x}}{e^x + e^{-x}}. \end{aligned}$$

▷ **hyperbolic cotangent:**

$$\begin{aligned} coth : \mathbb{R}^* &\longrightarrow]-\infty, -1[\cup]1, +\infty[\\ x &\longmapsto cothx = \frac{chx}{shx} = \frac{e^x + e^{-x}}{e^x - e^{-x}}. \end{aligned}$$

The functions are defined on $\mathbb{R}$ except $cothx$ which defines on $\mathbb{R} - \{0\}$, continuous, differentiable and satisfy the following properties:

1. The function $x \longmapsto shx$ is odd and non-decreasing on $\mathbb{R}$, $\lim\limits_{x \to \mp\infty} shx = \mp\infty$, its derivative is $(shx)' = chx$.

   Its reciprocal function, denoted $x \longmapsto \arg shx$ is also continuous non-decreasing from $\mathbb{R}$ to $\mathbb{R}$ with $\lim\limits_{x \to \mp\infty} \arg shx = \mp\infty$:

   $$y = \arg shx \Longleftrightarrow x = shy.$$

   Another expression of the function $\arg sh$, we have:

   $$ch^2y - sh^2y = 1 \Longrightarrow chy = \sqrt{1 + sh^2y} = \sqrt{1 + x^2}.$$

   On the other hand, if $x = shy$, then, $\dfrac{dx}{dy} = chy$ and

   $$(\arg shx)' = \frac{1}{chy} = \frac{1}{\sqrt{1 + sh^2y}} = \frac{1}{\sqrt{1 + x^2}}, \quad \forall x \in \mathbb{R}.$$

   So

   $$chy + shy = e^y = x + \sqrt{1 + x^2},$$

   which implies:

   $$\arg shx = y = \ln e^y = \ln(x + \sqrt{1 + x^2}).$$

2. The function $x \longmapsto chx$ is even and non-increasing on $\mathbb{R}^-$ and and non-decreasing on $\mathbb{R}^+$, $\lim\limits_{x\to\mp\infty} chx = \mp\infty$, its derivative is $(chx)' = shx$.

   Its reciprocal function, denoted $x \longmapsto \arg chx$ is also continuous non-decreasing from $]1, +\infty[$ to $\mathbb{R}^*$ with $\lim\limits_{x\to+\infty} \arg chx = +\infty$:

$$y = \arg chx \Longleftrightarrow x = chy.$$

   On the other hand, if $x = chy$, then, $\dfrac{dx}{dy} = shy$ and

$$(\arg chx)' = \frac{1}{shy} = \frac{1}{\sqrt{ch^2y - 1}} = \frac{1}{\sqrt{x^2-1}}, \quad \forall x \geq 1.$$

   Another expression of the function $\arg ch$, we have:

$$ch^2y - sh^2y = 1 \Longrightarrow shy = \sqrt{ch^2y - 1} = \sqrt{x^2-1}.$$

   So

$$chy + shy = e^y = x + \sqrt{x^2-1},$$

   which implies:

$$\arg chx = y = \ln e^y = \ln(x + \sqrt{x^2-1}).$$

   Here are the graphs of these functions:

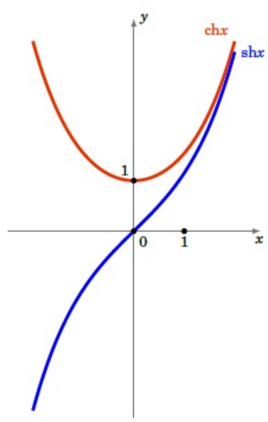


Figure 4.2: shx and chx functions.

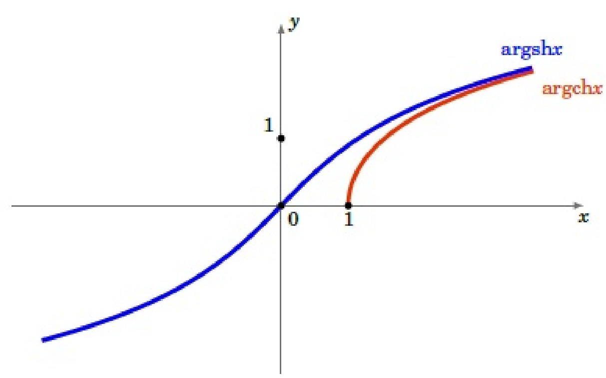


Figure 4.3: $\arg shx$ and $\arg chx$ inverse functions.

3. The function $x \longmapsto thx$ is odd and non-decreasing on $\mathbb{R}$, $\lim\limits_{x\to\mp\infty} thx = \mp 1$, its derivative is

$$(th)'(x) = 1 - th^2x = \frac{1}{ch^2x}.$$

Its reciprocal function, denoted $x \longmapsto \arg thx$ is also continuous non-decreasing from $]-1,+1[$ to $\mathbb{R}$ with $\lim_{x\to -1^+} \arg thx = -\infty$ and $\lim_{x\to +1^-} \arg thx = +\infty$:

$$y = \arg thx \Longleftrightarrow x = thy.$$

On the other hand, if $x = thy$, then, $\dfrac{dx}{dy} = 1 - th^2 y$ and

$$(\arg thx)' = \frac{1}{1 - th^2 y} = \frac{1}{1 - x^2}.$$

Another expression of the function $\arg th$, we have:

$$x = \frac{e^y - e^{-y}}{e^y + e^{-y}} \Longleftrightarrow e^{2y} = \frac{1+x}{1-x},$$

which implies:

$$\arg thx = y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right).$$

4. The function $x \longmapsto cothx$ is odd and non-increasing and bijective from $\mathbb{R}^*$ to $]-\infty,-1[\cup]1,+\infty[$, $\lim_{x\to\mp\infty} cothx = \mp 1$ and $\lim_{x\to 1^\mp} cothx = \mp\infty$, its derivative is

$$(coth)'(x) = 1 - coth^2 x = \frac{-1}{sh^2 x}.$$

Its reciprocal function, denoted $x \longmapsto \arg cothx$ is also strictly continuous increasing from $]-1,+1[$ to $\mathbb{R}$,

$$y = \arg cothx \Longleftrightarrow x = cothy.$$

We can show that:

$$\arg cothx = y = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|.$$

Here are the graphs of these functions:

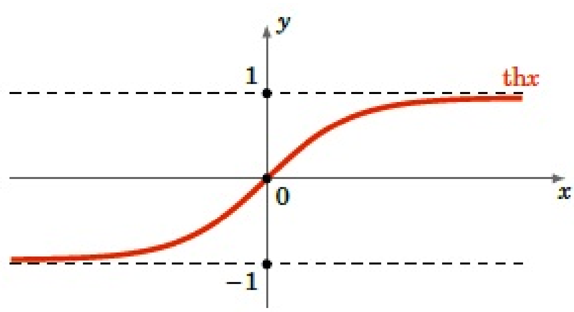


Figure 4.4: thx function.

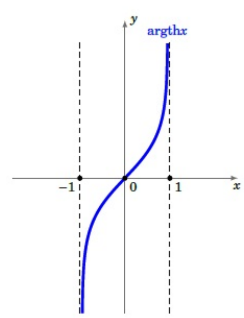


Figure 4.5: $\arg thx$ inverse function.

4.5 Trigonometric functions and their inverses

Definition 4.8. *The function* $\mathbf{y} = \sin \mathbf{x}$ *is an odd function of period* 2Π. *This function is continuous and defined on* $\mathbb{R}$ *and verifies the following relations:*

- ▷ $-1 \leq \sin x \leq 1$,
- ▷ $\sin(-x) = -\sin(x)$,
- ▷ $\sin(\Pi \pm x) = \mp \sin(x)$,
- ▷ $\sin\left(\frac{\Pi}{2} \pm x\right) = \cos(x)$.

Its derivative which is deduced by a rotation in the opposite trigonometric direction on the circle (clockwise direction) is equal to:

$$(\sin(x))' = \cos(x).$$

Definition 4.9. *The function* $\mathbf{y} = \cos \mathbf{x}$*is an even function of period* 2Π. *This function is continuous and defined on* $\mathbb{R}$ *and verifies the following relations:*

- ▷ $-1 \leq \cos x \leq 1$,
- ▷ $\cos(-x) = \cos(x)$,
- ▷ $\cos(\Pi \pm x) = -\cos(x)$,
- ▷ $\cos\left(\frac{\Pi}{2} \pm x\right) = \mp \sin(x)$.

Its derivative which is deduced by a rotation in the opposite trigonometric direction on the circle (clockwise direction) is equal to:

$$(\cos(x))' = -\sin(x).$$

Remark 4.10. *The Sinus and Cosine functions represent the coordinates of the point* M *belonging to the trigonometric circle. These functions verify the following relations:*

$$\cos^2(x) + \sin^2(x) = 1.$$

Definition 4.11. *The function* $\mathbf{y} = \tan \mathbf{x} = \frac{\sin \mathbf{x}}{\cos \mathbf{x}}$ *is an odd function of period* Π. *This function is continuous and defined on* $\mathbb{R} - \left\{(2k+1)\frac{\Pi}{2}, k \in \mathbb{Z}\right\}$ *and verifies the following relations:*

- ▷ $\tan(-x) = -\tan(x)$,

▷ $\tan(\Pi \pm x) = \pm \tan(x)$,

▷ $\tan\left(\frac{\Pi}{2} \pm x\right) = \mp \frac{1}{\tan}$.

Its derivative is equal to:

$$(\tan(x))' = \frac{1}{\cos^2(x)} = 1 + \tan^2(x).$$

Definition 4.12. *The function* $\mathbf{y} = \mathbf{co}\tan\mathbf{x} = \frac{\mathbf{1}}{\tan}$ *is an odd function of period* Π. *This function is continuous and defined on* $\mathbb{R} - \{k\Pi, k \in \mathbb{Z}\}$ *and verifies the following relations:*

▷ $co\tan(-x) = -co\tan(x)$,

▷ $co\tan(\Pi \pm x) = \pm co\tan(x)$,

▷ $co\tan\left(\frac{\Pi}{2} \pm x\right) = \mp co\tan(x)$.

Its derivative is equal to:

$$(co\tan(x))' = \frac{-1}{\sin^2(x)} = -(1 + co\tan^2(x)).$$

We call **reciprocal circular functions** the following four functions:

1. **Arcsine** function is a reciprocal function of the sine function on the interval $\left[\frac{-\Pi}{2}, \frac{\Pi}{2}\right]$:

$$\begin{aligned} \arcsin : [-1,1] &\longrightarrow \left[\frac{-\Pi}{2}, \frac{\Pi}{2}\right] \\ x &\longmapsto \arcsin x, \end{aligned}$$

and verifies the following relations:

▷ if $x \in \left[\frac{-\Pi}{2}, \frac{\Pi}{2}\right]$: $\sin x = y \Leftrightarrow x = \arcsin y$,

▷ $\sin(\arcsin x) = x, \quad \forall x \in [-1,1]$,

▷ $\arcsin(\sin x) = x, \quad \forall x \in \left[\frac{-\Pi}{2}, \frac{\Pi}{2}\right]$,

▷ $\cos(\arcsin x) = \sqrt{1 - x^2}, \quad \forall x \in [-1,1]$.

It is odd, continuous, differentiable on $]-1,1[$ and non-decreasing, its derivative is :

$$(\arcsin(x))' = \frac{1}{\sqrt{1 - x^2}}.$$

2. **Arccosine** function is a reciprocal function of the cosine function on $[0, \Pi]$:

$$\begin{aligned} \arccos : [-1, 1] &\longrightarrow [0, \Pi] \\ x &\longmapsto \arccos x, \end{aligned}$$

   and verifies the following relations:

   - ▷ if $x \in [0, \Pi] : \cos x = y \Leftrightarrow x = \arccos y$,
   - ▷ $\cos(\arccos x) = x, \quad \forall x \in [-1, 1]$,
   - ▷ $\arccos(\cos x) = x, \quad \forall x \in [0, \Pi]$,
   - ▷ $\sin(\arccos x) = \sqrt{1 - x^2}, \quad \forall x \in [-1, 1]$.

   It is continuous, differentiable on $]-1, 1[$ and non-increasing, its derivative is:

$$(\arccos(x))' = \frac{-1}{\sqrt{1 - x^2}}.$$

3. **Arctangent** function is a reciprocal function of the tangent function on $\left]\frac{-\Pi}{2}, \frac{\Pi}{2}\right[$:

$$\begin{aligned} \arctan :]-\infty, +\infty[&\longrightarrow \left]\frac{-\Pi}{2}, \frac{\Pi}{2}\right[\\ x &\longmapsto \arctan x, \end{aligned}$$

   and verifies the following relations:

   - ▷ if $x \in \left]\frac{-\Pi}{2}, \frac{\Pi}{2}\right[: \tan x = y \Leftrightarrow x = \arctan y$,
   - ▷ $\tan(\arctan x) = x, \quad \forall x \in \mathbb{R}$,
   - ▷ $\arctan(\tan x) = x, \quad \forall x \in \left]\frac{-\Pi}{2}, \frac{\Pi}{2}\right[$.

   It is odd, continuous, differentiable and non-decreasing, its derivative is:

$$(\arctan(x))' = \frac{1}{1 + x^2}.$$

4. **Arccotangent** function is a reciprocal function of the cotangent function on $]0, \Pi[$:

$$\begin{aligned} arccotan :]-\infty, +\infty[&\longrightarrow]0, \Pi[\\ x &\longmapsto arccotan x. \end{aligned}$$

   It is continuous, differentiable and non-increasing, its derivative is:

$$(arccotan(x))' = -\frac{1}{1 + x^2}.$$

Here are the graphs of these functions:

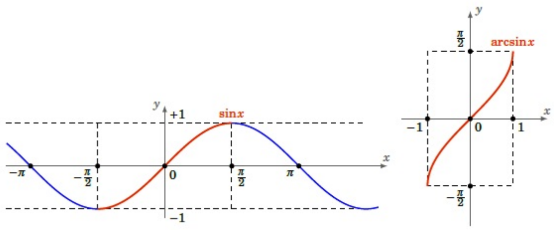


Figure 4.6: $\sin x$ and her inverse $\arcsin x$.

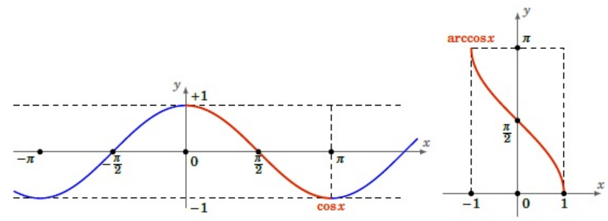


Figure 4.7: $\cos x$ and her inverse $\arccos x$.

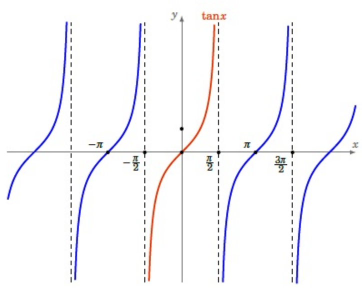


Figure 4.8: $\tan x$ function.

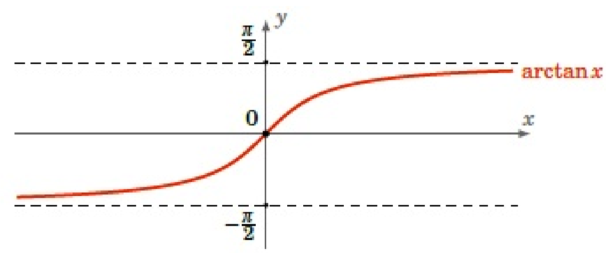


Figure 4.9: $\arctan x$ function inverse.

Exercise 1. *By using the definition of the derivative of a function, calculate the following limits*

$$\lim_{x\longrightarrow 0}\frac{1-\cos\sqrt{x}}{x},\qquad \lim_{x\longrightarrow 1}\frac{\arctan x-\dfrac{\Pi}{4}}{x-1}.$$

Solution:

- Let $f(x)=\cos\sqrt{x}\Rightarrow f(0)=1$, then,

$$\lim_{x\longrightarrow 0}\frac{1-\cos\sqrt{x}}{x}=-\lim_{x\longrightarrow 0}\frac{f(x)-f(0)}{x-0}=-f'(0).$$

  Its derivative is:

$$f'(x)=-\frac{1}{2\sqrt{x}}\sin\sqrt{x}\Longrightarrow f'(0)=-\frac{1}{2}\lim_{x\longrightarrow 0}\frac{\sin\sqrt{x}}{\sqrt{x}}=-\frac{1}{2}.$$

  Then,

$$\lim_{x\longrightarrow 0}\frac{1-\cos\sqrt{x}}{x}=\frac{1}{2}.$$

- Let $g(x)=\arctan x\Rightarrow g(1)=\dfrac{\Pi}{4}$, which give:

$$\lim_{x\longrightarrow 1}\frac{\arctan x-\dfrac{\Pi}{4}}{x-1}=\lim_{x\longrightarrow 1}\frac{g(x)-g(1)}{x-1}=-g'(1).$$

Its derivative is:

$$g'(x) = \frac{1}{1+x^2} \Longrightarrow g'(1) = \lim_{x \longrightarrow 1} g'(x) = \frac{1}{2}.$$

Then,

$$\lim_{x \longrightarrow 1} \frac{\arctan x - \dfrac{\Pi}{4}}{x-1} = \frac{1}{2}.$$

Exercise 2. *Calculate the derivatives of the following functions:*

$$f(x) = (\sin x + \ln(4+x^2))^{\frac{3}{7}}, \qquad g(x) = (ch)^{\cos^2 x}, x \in \mathbb{R}.$$

Solution: We use the usual derivation rules in each case:

(i) Let $u(x) = \sin x + \ln(4+x^2)$, so $f(x) = (u(x))^{\frac{3}{7}}$, then,

$$f'(x) = \frac{3}{7}u'(x)(u(x))^{\frac{3}{7}-1} = \frac{3}{7}u'(x)(u(x))^{-\frac{4}{7}},$$

which implies,

$$f'(x) = \frac{3}{7}\left(\cos x + \frac{2x}{4+x^2}\right)\left(\sin x + \ln(4+x^2)\right)^{-\frac{4}{7}}.$$

(ii) The function g can be written $g(x) = (chx)^{\cos^2 x} = e^{v(x)}$ with $v(x) = \cos^2 x \ln chx$. Then, $g'(x) = v'(x)e^{v(x)}$. We have:

$$v'(x) = \left(-2\cos x \sin x \ln chx + \cos^2 x \frac{shx}{chx}\right).$$

So

$$g'(x) = \left(-2\sin 2x \ln chx + \cos^2 x thx\right)(chx)^{\cos^2 x}.$$

Exercise 3. *Let f be the function defined on $\mathbb{R}$ by:*

$$f(x) = \begin{cases} \dfrac{x}{\Pi}\arctan\left(\dfrac{1}{x}\right) & if\ x \neq 0, \\ 0 & if\ x = 0. \end{cases}$$

1. *Show that f is continuous at $x = 0$.*

2. *Determine the right derivative $f'_d(0)$ and the left derivative $f'_g(0)$ of f at $x = 0$. Is the function f differentiable at $x = 0$?*

Solution:

1. For all $x \in \mathbb{R}$, we have:

$$\left|\arctan\left(\frac{1}{x}\right)\right| \leq \frac{\Pi}{2} \Longrightarrow |f(x)| = \frac{|x|}{\Pi}\left|\arctan\left(\frac{1}{x}\right)\right| \leq \frac{|x|}{2},$$

then, $\lim_{x \longrightarrow 0} f(x) = 0 = f(0)$. Consequently the function f is continuous at $x = 0$.

2. When $x \longrightarrow 0^+, \arctan\left(\frac{1}{x}\right) \longrightarrow \frac{\Pi}{2}$, then,

$$f_d'(0) = \lim_{x \longrightarrow 0^+} \frac{f(x) - f(0)}{x - 0} = \frac{1}{\Pi} \lim_{x \longrightarrow 0^+} \arctan\left(\frac{1}{x}\right).$$

The right derivative of the function f is $f_d'(0) = \frac{1}{2}$.

Likewise, when $x \longrightarrow 0^-, \arctan\left(\frac{1}{x}\right) \longrightarrow -\frac{\Pi}{2}$, then,

$$f_g'(0) = \lim_{x \longrightarrow 0^-} \frac{f(x) - f(0)}{x - 0} = \frac{1}{\Pi} \lim_{x \longrightarrow 0^-} \arctan\left(\frac{1}{x}\right).$$

The left derivative of the function f is $f_g'(0) = -\frac{1}{2}$. As $f_d'(0) \neq f_g'(0)$, the function f is not differentiable at $x = 0$.

Exercise 4.

1. *Write a function of x the two functions:* $\cos(\arcsin x)$ *and* $\sin(\arccos x)$.
2. *Solve the equation on $\mathbb{R}$:*

$$\arcsin x = \arcsin\frac{4}{5} + \arcsin\frac{3}{5}.$$

3. *Show that:*

$$\arccos x + \arcsin x = \frac{\Pi}{2}.$$

Solution:

1. We have:

$$\cos^2 y + \sin^2 y = 1 \Longrightarrow \cos y = \pm\sqrt{1 - \sin^2 y},$$

since

$$-\frac{\Pi}{2} \leq \arcsin x \leq \frac{\Pi}{2} \Longrightarrow \cos(\arcsin x) \geq 0,$$

$$\Longrightarrow \cos(\arcsin x) = \sqrt{1 - \sin^2(\arcsin x)} = \sqrt{1 - x^2}.$$

Likewise,

$$\sin y = \pm\sqrt{1 - \cos^2 y} \quad \Longrightarrow \quad \sin(\arccos x) = \sqrt{1 - \cos^2(\arccos x)}$$

$$\Longrightarrow \quad \sin(\arccos x) = \sqrt{1 - x^2} \quad \textit{because}: \ 0 \leq \arccos x \leq \Pi.$$

2. Solve the equation on $\mathbb{R}$:

$$\arcsin x = \arcsin \frac{4}{5} + \arcsin \frac{3}{5}.$$

We have:

$$\sin(a+b) = \sin a \cos b + \cos a \sin b.$$

$$\begin{aligned}
\arcsin x &= \arcsin \frac{4}{5} + \arcsin \frac{3}{5} \Rightarrow \sin(\arcsin x) = \sin\left(\arcsin \frac{4}{5} + \arcsin \frac{3}{5}\right) \\
&\Rightarrow x = \sin\left(\arcsin \frac{4}{5}\right) \cos\left(\arcsin \frac{3}{5}\right) + \cos\left(\arcsin \frac{4}{5}\right) \sin\left(\arcsin \frac{3}{5}\right) \\
&\Rightarrow x = \frac{4}{5}\sqrt{1 - \frac{9}{25}} + \sqrt{1 - \frac{16}{25}}\frac{3}{5} \\
&\Rightarrow x = \frac{16}{25} + \frac{9}{25} = 1.
\end{aligned}$$

3. Show that: $\arccos x + \arcsin x = \frac{\Pi}{2}$. We pose:

$$\begin{aligned}
f(x) = \arccos x + \arcsin x &\Longrightarrow f'(x) = \frac{-1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}} = 0 \\
&\Longrightarrow f(x) = C, \quad C \textit{ is a constant}, \quad \textit{but } f(0) = \frac{\Pi}{2} \\
&\Longrightarrow f(x) = \arccos x + \arcsin x = \frac{\Pi}{2}.
\end{aligned}$$

Exercise 5. *Let's show the following inequality:*

$$\forall x \in]0;1[, \qquad \arcsin x < \frac{x}{\sqrt{1-x^2}}.$$

Solution: *We apply* **the finite increment theorem** *to the function* $\arcsin x$ *on* $[0;x] \subset [0;1]$. *Then, there exist a constant* $c \in]0;x[$ *such that:*

$$f(x) - f(0) = (x-0)f'(c) \Longrightarrow f'(c) = \frac{1}{\sqrt{1-c^2}},$$

but,

$$\frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-x^2}}, \qquad \textit{because } c < x;$$

which implies:

$$\arcsin x < \frac{x}{\sqrt{1-x^2}}.$$

CHAPTER

5

LIMITED DEVELOPMENT

5.1 Taylor formulas

A function f is continuous on $[a,b] \subset \mathbb{R}$ and differentiable at $x_0 \in]a,b[$ can be written in the neighborhood of x_0:

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + (x - x_0)\varepsilon(x),$$

with $\lim_{x \to x_0} \varepsilon(x) = 0$. This amounts to saying that f is approximated by a polynomial of degree 1:

$$x \longmapsto P(x) = f(x_0) + (x - x_0)f'(x_0).$$

The error committed $R(x) = (x - x_0)\varepsilon(x)$ tends to 0 when x tends to x_0. Taylor's formula generalizes this result to functions n differentiable which can be approximated (in the neighborhood of x_0) by polynomials of degree n. More exactly,

$$f(x) = \underbrace{f(x_0) + \frac{(x - x_0)}{1!}f'(x_0) + \frac{(x - x_0)^2}{2!}f^{(2)}(x_0) + \cdots + \frac{(x - x_0)^n}{n!}f^{(n)}(x_0)}_{P_n(x)} + R_n(x_0, x).$$

$P_n(x)$ is a polynomial of degree n at $(x - x_0)$ which approximates f with a precision R_n and $R_n(x_0, x)$ is called remainder of order n. Various forms of $R_n(x_0, x)$ exist, the most classic form is the following:

5.1.1 Taylor formula with integral remainder

Theorem 5.1. *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a function of class C^{n+1} and let $x, x_0 \in [a, b]$. Then,*

$$\begin{aligned} f(x) &= f(x_0) + \frac{(x - x_0)}{1!} f'(x_0) + \frac{(x - x_0)^2}{2!} f^{(2)}(x_0) \\ &+ \cdots + \frac{(x - x_0)^n}{n!} f^{(n)}(x_0) + \int_{x_0}^{x} \frac{f^{(n+1)}(t)}{n!} (x - t)^n dt. \end{aligned}$$

Example 5.2. *The function $f(x) = e^x$ is of class C^{n+1} on $\mathbb{R}$ for all n. Let us fix $x_0 \in \mathbb{R}$. As $f'(x) = e^x, f''(x) = e^x, \cdots$ then, for all $x \in \mathbb{R}$:*

$$e^x = e^{x_0} + (x - x_0)e^{x_0} + \frac{(x - x_0)^2}{2!} e^{x_0} + \cdots + \frac{(x - x_0)^n}{n!} e^{x_0} + \int_{x_0}^{x} \frac{e^t}{n!} (x - t)^n dt.$$

5.1.2 Taylor formula with Lagrange remainder

Theorem 5.3. *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a function of class C^{n+1} and let $x, x_0 \in [a, b]$. There exists a real c between x_0 and x such that:*

$$\begin{aligned} f(x) &= f(x_0) + \frac{(x - x_0)}{1!} f'(x_0) + \frac{(x - x_0)^2}{2!} f^{(2)}(x_0) \\ &+ \cdots + \frac{(x - x_0)^n}{n!} f^{(n)}(x_0) + \frac{(x - x_0)^{n+1}}{(n+1)!} f^{(n+1)}(c). \end{aligned}$$

Example 5.4. *Let $x_0, x \in \mathbb{R}$. For all integer $n \geq 0$, there exist c between x_0 and x such that:*

$$e^x = e^{x_0} + (x - x_0)e^{x_0} + \frac{(x - x_0)^2}{2!} e^{x_0} + \cdots + \frac{(x - x_0)^n}{n!} e^{x_0} + \frac{(x - x_0)^n}{n!} e^{x_0} + \frac{(x - x_0)^{n+1}}{(n+1)!} e^c.$$

5.1.3 Taylor Mac-Laurin formula

When $x_0 = 0$ in the Taylor-Lagrange formula, we pose $c = \theta x, 0 < \theta < 1, c \in]0, x[$ and we obtain

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f^{(2)}(0) + \cdots + \frac{x^n}{n!} f^{(n)}(0) + \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(\theta x).$$

Remark 5.5. *The Taylor Mac-Laurin formula is often used in the calculation of approximate values.*

Example 5.6. *Using the Mac-Laurin formula of order 2 to the function $x \longmapsto e^x$, show that we have $\frac{8}{3} < e < 3$. We have:*

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} e^{\theta x},$$

and for $x = 1$,

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{6}e^{\theta}, \quad i.e., \quad e = \frac{5}{2} + \frac{1}{6}e^{\theta}.$$

Using the fact that $0 < \theta < 1$, *we obtain:*

$$\frac{8}{3} < \frac{5}{2} + \frac{1}{6}e^{\theta} < \frac{5}{2} + \frac{1}{6}e.$$

Hence

$$\begin{aligned} e < \frac{5}{2} + \frac{1}{6}e \quad &\Rightarrow \quad \frac{5}{6}e < \frac{5}{2} \\ &\Rightarrow \quad e < 3. \end{aligned}$$

Finally, $\frac{8}{3} < e < 3$.

5.1.4 Taylor Young formula

We will restrict the hypotheses by assuming only that $f^{(n)}(x_0)$ exists.

Theorem 5.7. *Let* $f : [a, b] \longrightarrow \mathbb{R}$ *and* $x_0 \in [a, b]$. *Suppose* $f^{(n)}(x_0)$ *exists (finite), then,* $\forall x \in \bigvee(x_0)$,

$$f(x) = f(x_0) + \frac{(x - x_0)}{1!}f'(x_0) + \frac{(x - x_0)^2}{2!}f^{(2)}(x_0) + \cdots + \frac{(x - x_0)^n}{n!}f^{(n)}(x_0) + o(x - x_0)^n,$$

where $o(x - x_0)^n = (x - x_0)^n \varepsilon(x)$ *with* $\lim_{x \to x_0} \varepsilon(x) = 0$.

Remark 5.8. *The Taylor Lagrange formula gives a global study of the function on the interval, while the Taylor Young formula gives a local study of the function in the neighborhood of* x_0.

The Taylor Young formula is useful for calculating limits.

Example 5.9. *Let* $f :]-1, +\infty[\longrightarrow \mathbb{R}, x \longmapsto \ln(1 + x), f$ *is infinitely differentiable. We have:* $f(0) = 0$ *and*

$$\begin{array}{lll} f'(x) = \dfrac{1}{1+x}, & f''(x) = \dfrac{-1}{(1+x)^2}, & f'''(x) = \dfrac{2}{(1+x)^3}, \\ f'(0) = 1, & f''(0) = -1, & f'''(0) = 2. \end{array}$$

Then,

$$\ln(1 + x) = x - \frac{x^2}{2!} + \frac{x^3}{3} + x^3\varepsilon(x), \ avec \ \lim_{x \to 0} \varepsilon(x) = 0$$

is a limited development of order 3 *of* $\ln(1 + x)$ *in the neighborhood of* x_0.

5.2 Limited development

5.2.1 Limited development in the neighborhood of zero

We have seen that in a neighborhood of x_0 we can approximate $f(x)$ by a polynomial P_n of order n so that $f(x) - P_n(x) = o(x - x_0)^n$. This when $f^{(n)}(x)$ exists.

Now, we will see that such a polynomial can exist even if $f^{(n)}$ does not exist and even if f is not continuous at x_0.

Definition 5.10. *Let f be a function defined in the neighborhood of zero. We call that f admits a limited development of order n in the neighborhood of 0 if there exists an open I of center 0 and the constants $a_0, a_1, \cdots, a_n$ such that $\forall x \in I, x \neq 0$*

$$\begin{aligned} f(x) &= a_0 + a_1 x + \cdots + a_n x^n + x^n \varepsilon(x) \text{ with } \lim_{x \to 0} \varepsilon(x) = 0 \\ &= \underbrace{a_0 + a_1 x + \cdots + a_n x^n}_{P_n(x)} + o(x^n). \end{aligned}$$

- *$P_n(x) = a_0 + a_1 x + \cdots + a_n x^n$ is the regular part of limited development.*
- *$o(x^n) = x^n \varepsilon(x)$ with $\lim_{x \to 0} \varepsilon(x) = 0$ is the remainder.*

Example 5.11.

1. *$f(x) = 1 + \frac{5}{2}x + 3x^2 + x^3 \sin \frac{1}{x}, x \in \mathbb{R}^*$ is a limited development of order 2 in the neighborhood of 0.*

   If f admits a limited development of order n in the neighborhood of 0, then, $\lim_{x \to 0} f(x)$ exists. Indeed, $f(x) = a_0 + a_1 x + \cdots + a_n x^n + x^n \varepsilon(x)$, hence $\lim_{x \to 0} f(x) = a_0$. This does not mean that f is continuous at 0 because $f(0)$ may not exist.

2. *$f(x) = \frac{1}{x}, x \neq 0$ does not admit a limited development in the neighborhood of 0 because $\lim_{x \to 0} f(x) = \infty$.*

   If f admits a limited development of order n in the neighborhood of 0 and $a_0 = f(0)$, then, f is differentiable at 0. Indeed, $\forall x \neq 0, f(x) = f(0) + a_1 x + \cdots + a_n x^n + x^n \varepsilon(x)$ with $\lim_{x \to 0} \varepsilon(x) = 0$.

   $$\frac{f(x) - f(0)}{x - 0} = a_1 + a_2 x + \cdots + a_{n-1} x^{n-1} + x^{n-1} \varepsilon(x) \text{ with } \lim_{x \to 0} \varepsilon(x) = 0.$$

   So $\lim_{x \to 0} \dfrac{f(x) - f(0)}{x - 0} = a_1 = f'(0)$.

Proposition 5.12. *(Uniqueness). If f admits a limited development of order n in the neighborhood of 0, then, this limited development is unique.*

Proof: Suppose that f admits two limited development of order n in the neighborhood of 0, i.e.,

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n + x^n \varepsilon_1(x) \ \textit{with} \ \lim_{x\to 0} \varepsilon_1(x) = 0$$

and

$$f(x) = b_0 + b_1 x + \cdots + b_n x^n + x^n \varepsilon_2(x) \ \textit{with} \ \lim_{x\to 0} \varepsilon_2(x) = 0.$$

Which give

$$(a_0 - b_0) + (a_1 - b_1)x + \cdots + (a_n - b_n)x^n = x^n[\varepsilon_1(x) - \varepsilon_2(x)].$$

Passing to the limit when $x \to 0$, we have: $a_0 = b_0$, hence

$$(a_1 - b_1)x + \cdots + (a_n - b_n)x^n = x^n[\varepsilon_1(x) - \varepsilon_2(x)].$$

If $x \neq 0$, we obtain

$$(a_1 - b_1) + (a_2 - b_2)x \cdots + (a_n - b_n)x^{n-1} = x^{n-1}[\varepsilon_1(x) - \varepsilon_2(x)].$$

Passing to the limit when $x \to 0$, we have: $a_1 = b_1$. In this way, we have $a_n = b_n, \forall n$. Hence the uniqueness of the limited development.

Theorem 5.13. *If $f^{(n)}(0)$ exists, then the limited development of f is*

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f^{(2)}(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) + x^n\varepsilon(x) \ \textit{with} \ \lim_{x\to 0}\varepsilon(x) = 0.$$

Corollary 5.14. *If $f^{(n)}(0)$ exists and f admits a limited development of order n in the neighborhood of 0, then,*

$$a_0 = f(0), \ a_1 = \frac{f'(0)}{1!}, \ a_2 = \frac{f^{(2)}(0)}{2!}, \cdots, \ \frac{f^{(n)}(0)}{n!}.$$

Proof: It is thanks to the uniqueness of the limited development.

Example 5.15. *(Limited development obtained by division according to increasing power).*

$$\begin{aligned} f(x) &= \frac{1}{1-x} \\ &= 1 + x + x^2 + \cdots + x^n\varepsilon(x) \end{aligned}$$

such that $\varepsilon(x) = \dfrac{x}{1-x} \to 0$ *when* $x \to 0$.

We can deduce $f^{(n)}(0)$, indeed,

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f^{(2)}(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) + x^n\varepsilon(x) \ \textit{with} \ \lim_{x\to 0}\varepsilon(x) = 0,$$

and by identification, we have: $\forall k : 0 \leq k \leq n, \dfrac{f^{(k)}(0)}{k!} = 1.$

Remark 5.16. *The existence of a limited development does not imply the existence of derivatives. Indeed, let*

$$g(x) = 1 + x + x^2 + \cdots + x^n + x^{n+1}\sin\frac{1}{x},$$

it is clear that g admits a limited development of order n in the neighborhood of 0 but it is not differentiable at 0 since it is not defined at 0.

5.2.2 Limited development of usual functions at the origin

Throughout this subsection, ε denotes a zero limit function at 0.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + x^n\varepsilon(x).$$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + (-1)^n x^n + x^n\varepsilon(x).$$

$$\frac{1}{1+x} = 1 - x + x^2 + \cdots + (-1)^n x^n + x^n\varepsilon(x).$$

$$chx = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2n}}{(2n)!} + x^{2n+1}\varepsilon(x).$$

$$shx = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} + x^{2n+2}\varepsilon(x).$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + x^{2n+1}\varepsilon(x).$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + x^{2n+2}\varepsilon(x).$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n+1}\frac{x^n}{n} + x^n\varepsilon(x).$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots + \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}x^n + x^n\varepsilon(x).$$

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + \cdots + (-1)^n\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}x^n + x^n\varepsilon(x).$$

$$\frac{1}{\sqrt{1-x}} = 1 + \frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + \cdots + \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}x^n + x^n\varepsilon(x).$$

5.2.3 Limited development in the neighborhood of x_0 and of infinity

Definition 5.17. *We call that f defined in the neighborhood of x_0 admits a limited development of order n in $\bigvee(x_0)$ if the function*

$$F : x \longmapsto F(x) = f(x_0 + x)$$

admits a limited development of order n in $\bigvee(x_0)$. We have:

$$F(x) = a_0 + a_1x + \cdots + a_nx^n + x^n\varepsilon(x),$$

so

$$f(x_0 + x) = a_0 + a_1x + \cdots + a_nx^n + x^n\varepsilon(x),$$

i.e.,

$$f(y) = a_0 + a_1(y - x_0) + \cdots + a_n(y - x_0)^n + (y - x_0)^n\varepsilon((y - x_0)),$$

or in an equivalent way:

$$f(x) = a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)^n + (x - x_0)^n\varepsilon((x - x_0)).$$

Finally, we return from the neighborhood of x_0 to the neighborhood of 0 by the change of variable $z = x - x_0$.

Likewise the limited development in the neighborhood of the inifnity is done by the change of variable $y = \dfrac{1}{x}$.

Definition 5.18. *We call that a numerical function admits a limited development of order n in $\bigvee(+\infty)$ if there exists a polynomial P of degree less than or equal to n such that we have in $\bigvee(+\infty)$,*

$$f(x) = P\left(\frac{1}{x}\right) + o\left(\frac{1}{x^n}\right).$$

Example 5.19.

(*i*) *The limited development of $x \longmapsto e^x$ in $\bigvee(1)$. We put $u = x - 1$, so in $\bigvee(0)$ we have:*

$$e^u = 1 + u + \frac{u^2}{2} + \frac{u^3}{3!} + \cdots + \frac{u^n}{n!} + o(u^n).$$

By consequently,

$$e^{x-1} = 1 + (x - 1) + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3!} + \cdots + \frac{(x-1)^n}{n!} + o(u(x-1)^n).$$

Finally,

$$e^x = e\left[1 + (x - 1) + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3!} + \cdots + \frac{(x-1)^n}{n!} + o(u(x-1)^n)\right].$$

(*ii*) *The limited development of* $x \longmapsto e^x$ *in* $\bigvee(+\infty)$*: the limited development of* $x \longmapsto e^x$ *in the neighborhood of the inifnity, it suffices to put* $y = \frac{1}{x}$*. We have:*

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + o(x^n),$$

so

$$e^{\frac{1}{y}} = 1 + \frac{1}{y} + \frac{1}{2y^2} + \frac{1}{3!y^3} + \cdots + \frac{1}{n!y^n} + o\left(\frac{1}{y^n}\right).$$

5.2.4 Operations on limited development

1. **Limited development obtained by restriction:** If f admits a limited development of order n in the neighborhood of 0, then, $\forall k \leq n$, f a limited development of order k. Indeed, we have:

$$\begin{aligned} f(x) &= a_0 + a_1 x + \cdots + a_n x^n + x^n \varepsilon(x) \textit{ with } \lim_{x\to 0} \varepsilon(x) = 0 \\ &= a_0 + a_1 x + \cdots + a_k x^k + a_{k+1} x^{k+1} + a_{k+2} x^{k+2} + \cdots \\ & \quad + a_n x^n + x^n \varepsilon(x) \textit{ with } \lim_{x\to 0} \varepsilon(x) = 0 \\ &= a_0 + a_1 x + \cdots + a_k x^k \\ &= + x^k [a_{k+1} x + a_{k+2} x^2 + \cdots + a_n x^{n-k} + x^{n-k} \varepsilon(x)]. \end{aligned}$$

such that $\varepsilon_2(x) = [a_{k+1}x + a_{k+2}x^2 + \cdots + a_n x^{n-k} + x^{n-k}\varepsilon(x)] \to 0$ when $x \to 0$.

2. **Algebraic operations on the limited development:**

Theorem 5.20. *If* f *and* g *admit limited development of order* n *in the neighborhood of* 0*, then,* $f + g$*,* $f \times g$ *admit limited development of order* n *in the neighborhood of* 0 *and* $\frac{f}{g}$ *admits a limited development of order* n *if* $\lim_{x\to 0} g(x) \neq 0$*.*

▷ **Sum and product:** We suppose that f and g are two functions which admit limited development of order n in the neighborhood of 0:

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n + x^n"\varepsilon_1(x), g(x) = b_0 + b_1 x + \cdots + b_n x^n + x^n"\varepsilon_2(x).$$

Properties 5.21.

- $f + g$ *admits a limited development of order* n *which is:*

$$(f+g)(x) = f(x) + g(x) = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n + x^n \varepsilon(x).$$

Example 5.22. *Let* $h(x) = \ln(1 + x) + \cos x$*. We have:*

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + x^4\varepsilon_1(x), \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + x^4\varepsilon_2(x).$$

Then,

$$\begin{aligned} h(x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + x^4\varepsilon_1(x) + 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + x^4\varepsilon_2(x) \\ &= 1 + x - x^2 + \frac{x^3}{3} - \frac{5}{4}x^4 + x^4\varepsilon(x), \end{aligned}$$

is a limited development of order 4 *of* h *in the neighborhood of* 0.

- $f \times g$ *admits a limited development in the neighborhood of* 0 *of order* n *which is:*

$$(f \times g)(x) = f(x) \times g(x) = T_n(x) + x^n\varepsilon(x),$$

where $T_n(x)$ *is the polynomial* $(a_0 + a_1x + \cdots + a_nx^n) \times (b_0 + b_1x + \cdots + b_nx^n)$ *truncated to order* n. *(truncated a polynomial to order* n *means that we only keep monomes of degree* $\leq n$.*)*

Example 5.23. *Let* $f(x) = \sin x . \cos x$. *We have:*

$$\sin x = x - \frac{x^3}{3!} + x^3\varepsilon_1(x),$$

$$\cos x = 1 - \frac{x^2}{2!} + x^3\varepsilon_2(x).$$

Then,

$$f(x) = x - \frac{2}{3}x^3 + x^3\varepsilon(x).$$

▷ **Division:** Here is how to calculate the limited development of a quotient f/g. Let

$$f(x) = a_0 + a_1x + \cdots + a_nx^n + x^n\text{''}\varepsilon_1(x), g(x) = b_0 + b_1x + \cdots + b_nx^n + x^n\text{''}\varepsilon_2(x).$$

We will use the limited development of $\frac{1}{1+u} = 1 - u + u^2 - u^3 + \cdots$.

(a) If $b_0 = 1$, we put $u = b_1x + b_nx^n + x^n\varepsilon_2(x)$ and the quotient is written:

$$\frac{f}{g} = f \cdot \frac{1}{g} = f \cdot \frac{1}{1+u}.$$

Example 5.24. *Limited development of* $\tan x$ *in the neighborhood of* 0 *of order* 5. *We have:*

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + x^5\varepsilon_1(x), \quad \cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + x^5\varepsilon_2(x).$$

By putting $u = -\frac{x^2}{2} + \frac{x^4}{24} + x^5\varepsilon_2(x)$. *We will need to*

$$u^2 = \left(-\frac{x^2}{2} + \frac{x^4}{24} + x^5\varepsilon_2(x)\right)^2 = \frac{x^4}{4} + x^5\varepsilon_2(x), \quad u^3 = x^5\varepsilon(x).$$

So,

$$\begin{aligned}\frac{1}{\cos x} &= \frac{1}{1+u} = 1 - u + u^2 - u^3 + u^3\varepsilon_2(x)\\ &= 1 + \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^4}{4} + x^5\varepsilon_2(x) = 1 + \frac{x^2}{2} - \frac{5}{24}x^4 + x^5\varepsilon_2(x).\end{aligned}$$

Finally,

$$\begin{aligned}\tan x &= \sin x \times \frac{1}{\cos x} = \left(x - \frac{x^3}{6} + \frac{x^5}{120} + x^5\varepsilon_1(x)\right)\\ &\times \left(1 + \frac{x^2}{2} - \frac{5}{24}x^4 + x^5\varepsilon_2(x)\right) = x + \frac{x^3}{3} + \frac{2}{15}x^5 + x^5\varepsilon_1(x).\end{aligned}$$

(b) If b_0 is arbitrary with $b_0 \neq 1$, then, we return to the previous case in writing:

$$\frac{1}{g(x)} = \frac{1}{b_0}\frac{1}{1 + \frac{b_1}{b_0}x + \cdots + \frac{b_n}{b_0}x^n + \frac{x^n\varepsilon(x)}{b_0}}.$$

Example 5.25. *Limited development of* $\frac{1+x}{2+x}$ *in the neighborhood of* 0 *of order* 4*:*

$$\begin{aligned}\frac{1+x}{2+x} &= (1+x)\frac{1}{2}\frac{1}{1+\frac{x}{2}}\\ &= \frac{1}{2}(1+x)\left(1 - \frac{x}{2} + (\frac{x}{2})^2 - \left(\frac{x}{2}\right)^3 + \left(\frac{x}{2}\right)^4 + o(x^4)\right)\\ &= \frac{1}{2} + \frac{x}{4} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{x^4}{32} + o(x^4).\end{aligned}$$

(c) If $b_0 = 0$ then, we factor by x^k (for a certain k) in order to reduce to.

Example 5.26. *If we wish to calculate the limited development of* $\frac{\sin x}{shx}$ *in the neighborhood of* 0 *of order* 4*, then, we write:*

$$\begin{aligned}\frac{\sin x}{shx} &= \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^5)}{x + \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^5)} = \frac{x\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} + o(x^4)\right)}{x\left(1 + \frac{x^2}{3!} + \frac{x^4}{5!} + o(x^4)\right)}\\ &= \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} + o(x^4)\right) \times \frac{1}{1 + \frac{x^2}{3!} + \frac{x^4}{5!} + o(x^4)}\\ &= \cdots = 1 - \frac{x^2}{2} + \frac{x^4}{18} + o(x^4).\end{aligned}$$

▷ **Limited development of a composite function:**

Theorem 5.27. *If f and g admit limited development of order n in the neighborhood of 0 and if $g(0) = 0$, then, $f \circ g$ admits a limited development of order n in the neighborhood of 0.*

Remark 5.28. *The regular part of $f \circ g$ is obtained by replacing in the regular part of f, the regular part of g keeping that the powers are less than or equal to n.*

Example 5.29. *Let $h(x) = e^{\sin x}$. We put $f(u) = e^u$ and $g(x) = \sin x$. We have $g(0) = \sin 0 = 0$,*

$$e^u = 1 + u + \frac{u^2}{2} + \frac{u^3}{3!} + o(u^3),$$

$$\sin x = x - \frac{x^3}{6} + o(x^3),$$

so

$$\begin{aligned}(f \circ g)(x) &= e^{\sin x} \\ &= 1 + \left(x - \frac{x^3}{6} + o(x^3)\right) + \frac{1}{2}\left(x - \frac{x^3}{6} + o(x^3)\right)^2 \\ &\quad \frac{1}{6}\left(x - \frac{x^3}{6}3 + o(x^3)\right)^3 \\ &= 1 + x + \frac{1}{2}x^2 + o(x^3).\end{aligned}$$

5.2.5 Derivation of limited development

We have seen that the existence of limited development does not require the existence of the derivative. So we cannot say anything regarding the limited development of the derivative.

Theorem 5.30. *Let f be a differentiable function in the neighborhood of 0 and admits a limited development of order n in the neighborhood of 0:*

$$f(x) = P(x) + x^n \epsilon(x) \text{ with } \lim_{x \to 0} \epsilon(x) = 0.$$

If the derivative f' admits a limited development of order $(n-1)$ in the neighborhood of 0, then,

$$f'(x) = Q(x) + x^{n-1}\eta(x) \text{ with } \lim_{x \to 0} \eta(x) = 0.$$

such that

$$Q(x) = P'(x).$$

Example 5.31. *Let* $f(x) = \dfrac{1}{1-x}$*. We know that:*

$$f(x) = 1 + x + x^2 + \cdots + x^n + x^n\epsilon(x) \text{ with } \lim_{x\to 0}\epsilon(x) = 0.$$

Therefore

$$\begin{aligned} \frac{1}{(1-x)^2} &= \left(\frac{1}{1-x}\right)' = f'(x) \\ &= 1 + 2x + \cdots + nx^{n-1} + x^{n-1}(x)\eta(x) \text{ with } \lim_{x\to 0}\eta(x) = 0. \end{aligned}$$

5.2.6 Integration of limited development

Theorem 5.32. *Let* f *be a numerical differentiable function in the interval* $I =]-\alpha, \alpha[$, $\alpha > 0$, *its derivative is* f'*. If* f' *admits a limited development of order* n *in the neighborhood of* 0,

$$f'(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + x^n\epsilon(x) \text{ with } \lim_{x\to 0}\epsilon(x) = 0,$$

then, f *admits a limited development of order* $(n+1)$ *in the neighborhood of* 0,

$$f(x) = f(0) + a_0x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \cdots + \frac{a_n}{n+1}x^{n+1} + x^{n+1}\eta(x) \text{ with } \lim_{x\to 0}\eta(x) = 0,$$

here $x^{n+1}\eta(x) = \int_0^x t^n\epsilon(t)dt$.

Example 5.33. *We have:*

$$\frac{1}{1+x} = 1 - x + x^2 + \cdots + (-1)^nx^n + x^n\epsilon(x) \text{ with } \lim_{x\to 0}\epsilon(x) = 0,$$

by consequently,

$$\ln(1+x) = \ln 1 + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^n}{n+1}x^{n+1} + x^{n+1}\eta(x) \text{ with } \lim_{x\to 0}\eta(x) = 0.$$

5.2.7 Generalized limited development

If f defined in $\bigvee(0)$ does not admit a limited development in $\bigvee(0)$ but $x^\alpha f(x), \alpha > 0$ admits a limited development, then, we can write in $\bigvee(0)$ for $x \neq 0$,

$$x^\alpha f(x) = a_0 + a_1x + \cdots + a_nx^n + x^n\epsilon(x) \text{ with } \lim_{x\to 0}\epsilon(x) = 0,$$

hence

$$f(x) = \frac{1}{x^\alpha}\left[a_0 + a_1x + \cdots + a_nx^n + x^n\epsilon(x)\right]$$

it is a generalized limited development of f in $\bigvee(0)$.

Example 5.34. *Let f defined by*

$$f(x) = \frac{1}{x - x^2}.$$

f does not admit a limited development in the neighborhood of 0 because $\lim_{x \to 0} f(x) = +\infty$, but we have:

$$\begin{aligned} xf(x) &= \frac{1}{1-x} \\ &= 1 + x + x^2 + \cdots + x^n \epsilon(x). \end{aligned}$$

By consequently,

$$f(x) = \frac{1}{x}\left[1 + x + x^2 + \cdots + x^n \epsilon(x)\right]$$

is the generalized limited development of f.

5.3 Applications

The limited development are very useful in finding the limits of functions and the study of indeterminate forms.

Example 5.35. *Let*

$$f(x) = \frac{\ln(1+x) - x + \dfrac{x^2}{2} - \dfrac{x^3}{3}}{(\sin x)^4}.$$

By using the Young's formula to order 4 to the function $x \longmapsto g(x) = \ln(1+x)$, calculate $\lim_{x \to 0} f(x)$. We have:

$$g(x) = g(0) + \frac{x}{1!}g'(0) + \frac{x^2}{2!}g^{(2)}(0) + \frac{x^3}{3!}g^{(3)}(0) + \frac{x^4}{4!}g^{(4)}(0) + o(x)^4,$$

i.e.,

$$\ln(1+x) = 0 + x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + x^4\epsilon(x), \textit{ with } \lim_{x \to 0} \epsilon(x) = 0.$$

By consequently,

$$f(x) = \frac{-\dfrac{x^4}{4} + x^4\epsilon(x)}{(\sin x)^4},$$

which admits as a limit $\dfrac{-1}{4}$ when x tends to 0.

Example 5.36.

▷ *Find the limit when x tends to 0 of the following function:*

$$f(x) = \frac{\sin x - x\cos x}{x(1-\cos x)}.$$

We have:

$$\sin x = x - \frac{x^3}{3!} + o(x^3)$$

and

$$\cos x = 1 - \frac{x^2}{2!} + o(x^3).$$

By consequently,

$$f(x) = \frac{\frac{x^3}{3} + o(x^3)}{\frac{x^2}{3} + \circ(x^3)}.$$

Finally,

$$\lim_{x\to 0} f(x) = \frac{2}{3}.$$

▷ *Determine the limit, when x tends to $+1$ of the following function:*

$$f(x) = \frac{x^2}{x-1} - \sqrt[3]{x^3+x^2+1}.$$

We have an indeterminate form $+\infty - \infty$. We saw that by putting $y = \frac{1}{x}$, we come back in the neighborhood of 0, we find:

$$\begin{aligned} g(y) &= f\left(\frac{1}{y}\right) \\ &= \frac{1}{y-y^2} - \frac{1}{y}(1+y+y^3)^{\frac{1}{3}}. \end{aligned}$$

After calculations we find:

$$g(y) = \frac{2}{3} + \frac{10}{9} + o(y).$$

Finally,

$$\lim_{x\to+\infty} f(x) = \lim_{y\to 0} g(y) = \frac{2}{3}.$$

Exercise 1. *Calculate the following limits:*

(a)

$$\lim_{x\to 0} \frac{e^{x^2} - \cos x}{x^2},$$

(b)

$$\lim_{x\to 0} \frac{\ln(1+x)-\sin x}{x},$$

(c)

$$\lim_{x\to 0} \frac{\cos x-\sqrt{1-x^2}}{x^4}.$$

Solution:

(a) We have:

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + o(x^4)$$

and

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + o(x^4).$$

Hence

$$e^{x^2} - \cos x = \frac{3}{2}x^2 + o(x^2).$$

By consequently,

$$\lim_{x\to 0} \frac{e^{x^2}-\cos x}{x^2} = \frac{3}{2}.$$

(b) We have:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)$$

and

$$\sin x = x - \frac{x^3}{3!} + o(x^3).$$

Hence

$$\ln(1+x) - \sin x = -\frac{x^2}{2} + o(x^2).$$

By consequently,

$$\lim_{x\to 0} \frac{\ln(1+x)-\sin x}{x} = 0.$$

(c) We have:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + o(x^4)$$

and

$$\sqrt{1-x^2} = 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 + o(x^4).$$

Hence

$$\cos x - \sqrt{1-x^2} = \frac{x^4}{6}o(x^4).$$

By consequently,

$$\lim_{x\to 0} \frac{\cos x-\sqrt{1-x^2}}{x^4} = \frac{1}{6}.$$

Exercise 2. *Calculate the following limit:*

$$\lim_{x\to-\infty} \sqrt{x^2+3x+2}+x.$$

Solution: We have:

$$\begin{aligned}
\sqrt{x^2+3x+2}+x &= |x|\left(\sqrt{1+\frac{3}{x}+\frac{2}{x^2}}-1\right)\\
&= |x|\left(1+\frac{1}{2}\left(\frac{3}{x}+\frac{2}{x^2}\right)+o\left(\frac{1}{x}\right)-1\right)\\
&= |x|\left(1+\frac{1}{2}\frac{3}{x}+o\left(\frac{1}{x}\right)\right).
\end{aligned}$$

By consequently,

$$\lim_{x\to-\infty} \sqrt{x^2+3x+2}+x = \frac{-3}{2}.$$

Exercise 3.

1. *Let g be the function defined by*

$$g(x) = \sqrt{1+x}.$$

   Calculate the limited development of the function g of order 2 in the neighborhood of zero.

2. *Let f be the function defined by*

$$f(x) = x\sqrt{\frac{x+3}{x}}.$$

   Show that in the neighborhood of infinity, we have:

$$f(x) = \alpha x+\beta+\frac{\gamma}{x}+\frac{1}{x}\varepsilon\left(\frac{1}{x}\right).$$

3. *Interpret this result geometrically.*

Solution:

$$g(x) = \sqrt{1+x}, \quad f(x) = x\sqrt{\frac{x+3}{x}}.$$

1. Calculate the limited development of the function g of order 2 in the neighborhood of zero:

$$\begin{aligned}
g(x) &= g(0)+g'(0)x+\frac{g''(0)}{2}x^2+x^2\varepsilon(x)\\
&= 1+\frac{1}{2}x-\frac{1}{8}x^2+x^2\varepsilon(x).
\end{aligned}$$

2. Calculate the limited development of the function f in the neighborhood of infinity:

$$\begin{aligned} f(x) &= x\sqrt{\frac{x+3}{x}} \\ &= x\sqrt{x+\frac{3}{x}} \\ &= x\left[1+\frac{1}{2}\left(\frac{3}{x}\right)-\frac{1}{8}\left(\frac{3}{x}\right)^2+\frac{1}{x^2}\varepsilon\left(\frac{1}{x}\right)\right] \\ &= x+\frac{3}{2}-\frac{9}{8x}+\frac{1}{x}\varepsilon\left(\frac{1}{x}\right). \end{aligned}$$

Then,

$$\alpha = 1, \qquad \beta = \frac{3}{2}, \qquad \gamma = -\frac{9}{8}.$$

3. Interpretation geometrically:

$$y = x + \frac{3}{2}$$

is a graph asymptote equation (C_f).

CHAPTER

6

LINEAR ALGEBRA

6.1 Algebraic structures

The notions which follow are of interest on a terminological and structural level before tackling the study of vector spaces.

6.1.1 Laws and internal composition

Definition 6.1. *We call internal composition law (i.c.l) in a set E, any map:*

$$\begin{array}{rcl} \star : E \times E & \longrightarrow & E \\ (a,b) & \longmapsto & a \star b. \end{array}$$

A subset F of E is said to be stable with respect to the law $\star$ if:

$$\forall a, b \in F, a \star b \in F.$$

We note it: $a \star b,\ a \triangle b$ or $a \perp b,\ \cdots$

Example 6.2. *Addition and multiplication are internal composition laws in $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ but subtraction is not internal in $\mathbb{N}$.*

Definition 6.3. *Let E and Ω be sets. We call external composition law (e.c.l) in E any map:*

$$\begin{array}{rcl} \perp : \Omega \times E & \longrightarrow & E \\ \alpha \cdot x & \longmapsto & \alpha \perp x. \end{array}$$

In this general framework, the elements of Ω are called operators and we say that E is provided with a composition law external to operators in Ω.

Remark 6.4. *The law can be written multiplicatively using a point.*

Definition 6.5. *Let $\star$ and $\bullet$ be two internal composition laws in E, we say that:*

1. **Commutativity:** *$\star$ is commutative if: $\forall a, b \in E, a \star b = b \star a$.*

   Example 6.6. *Intersection and union are commutative internal composition laws in the set of parts of a set.*

2. **Associativity:** *$\star$ is associative if: $\forall a, b, c \in E, (a \star b) \star c = a \star (b \star c)$.*

   Example 6.7. *The composition of applications is an associative internal composition law.*

   On the other hand, the law of composition $\star$ defined in $\mathbb{Q}$ by: $x \star y = \dfrac{x+y}{2}$ is not associative.

3. **Distributivity:** *$\star$ is distributive with respect to $\bullet$ if:*

   $$\forall a, b, c \in E, a \star (b \bullet c) = (a \star b) \bullet (a \star c) \quad \textit{and} \quad (b \bullet c) \star a = (b \star a) \bullet (c \star a).$$

   If, moreover, the law $\star$ is commutative, it suffices to show one of the two equality.

   Example 6.8. *Multiplication is distributive with respect to addition in $\mathbb{C}$.*

4. **Neutral element:** *$e \in E$ is a neutral element of the loi $\star$ if:*

   $$\forall a \in E, a \star e = e \star a = a.$$

   If, furthermore, the law $\star$ is commutative, it suffices to show that: $\forall a \in E; a \star e = a$ or $e \star a = a$.

   Example 6.9. *1 is a neutral element of multiplication in $\mathbb{R}$.*

   Proposition 6.10. *If the neutral element exists then it is unique.*

5. **Symmetric element (inverse):** *Let $\star$ admits a neutral element e. Two elements a and a' are symmetric for the law $\star$ if: $a \star a' = a' \star a = e$.*

   Example 6.11. *The symmetric of a in $\mathbb{Z}$ provided with the addition is: $-a$.*

   Proposition 6.12. *If the law $\star$ is associative, then, if the symmetric element exists, it is unique.*

6. **Regular element:** *We say that α is a regular element for the law $\star$ if it satisfies:*

$$\forall a; b \in E; (a \star \alpha = b \star \alpha) \Longrightarrow a = b \quad \text{and} \quad \forall a; b \in E; (\alpha \star a = \alpha \star b) \Longrightarrow a = b.$$

If, moreover, the law $\star$ is commutative, it suffices to verify one of the two implications.

Example 6.13. *In $\mathbb{C}$ provided with the addition, every element is regular.*

7. **Stable part:** *A parte A is said to be stable of E for the law $\star$, if*

$$\forall a; b \in A; a \star b \in A.$$

Example 6.14. *The set of even natural integers is stable for addition, on the other hand the set of odd integers is not stable for addition because: $3 + 5 = 8$ which is even.*

Example 6.15. *Let F be a set and $E = \mathcal{P}(F)$. We consider in E the internal composition laws $\cap$ and $\cup$, then, it is very easy to show that:*

- ▷ $\cap$ *and* $\cup$ *are associative.*
- ▷ $\cap$ *and* $\cup$ *are commutative.*
- ▷ $\emptyset$ *is the neutral element of* $\cup$.
- ▷ F *is the neutral element of* $\cap$.

Example 6.16. *Let $E = \{a, b, \gamma\}$, we define a i.c.l in E by:*

$\star$	a	b	γ
a	a	b	γ
b	b	γ	a
γ	γ	a	a

i.e.,:

$$\begin{cases} a \star a = a, & a \star b = b, & a \star \gamma = \gamma, \\ b \star a = b, & b \star b = \gamma, & b \star \gamma = a, \\ \gamma \star a = \gamma, & \gamma \star b = a, & \gamma \star \gamma = a. \end{cases}$$

We remark that:

(i) *a is the neutral element of $\star$.*

(ii) *All elements of E are invertible such that:*

- ▷ *a is the inverse of a.*
- ▷ *γ is the inverse of b.*

▷ *b and γ are reciprocals of γ.*

Conventions: given a law of internal associative composition in a set E:

- If the law is denoted $+$, its neutral element is denoted 0_E or 0, and we denote it $a' = -a$.
- If the law is denoted multiplicatively, its neutral element is denoted 1_E or 1, and we denote it $a' = a^{-1}$.

6.1.2 Group structure

Definition 6.17. *We call a group any non-empty set G provided with an internal composition law $\star$ such that:*

(i) *$\star$ is associative.*

(ii) *$\star$ has a neutral element e.*

(iii) *Every element of G is symmetrizable.*

If moreover $\star$ is commutative, we say that $(G, \star)$ is a commutative group, or Abelien[1] group.

Example 6.18. $(\mathbb{Z}, +)$ *is a commutative group.*

Example 6.19. *We define the operation $\star$ by:*

$$\forall x, y \in]-1, 1[, \quad x \star y = \frac{x+y}{1+xy}.$$

Show that $(]-1, 1[, \star)$ is an Abelien group.

1. *$\star$ is an internal composition law in $]-1, 1[$:*

$$\begin{aligned} \forall x, y \in]-1, 1[: |x| < 1 \wedge |y| < 1 &\iff |x||y| = |xy| < 1 \\ &\iff 0 < 1 + xy < 2. \end{aligned}$$

[1]**ABEL Niels Henrik:** Norwegian mathematician (Île de Finnoy 1802-Arendal 1829). Algebraist, he created the theory of elliptic functions. He died of tuberculosis

So,

$$\forall x,y\in]-1,1[:\left|\frac{x+y}{1+xy}\right|<1 \iff \frac{|x+y|}{|1+xy|}<1$$
$$\iff |x+y|<|1+xy|$$
$$\iff |x+y|<1+xy \quad \textit{because} \quad 1+xy>0$$
$$\iff -(1+xy)<x+y<1+xy$$
$$\iff \begin{cases} x+y-1-xy<0 \\ x+y+1+xy>0 \end{cases}$$
$$\iff \begin{cases} x(1-y)+y-1<0 \\ x(1+y)+y+1>0 \end{cases}$$
$$\iff (*)\begin{cases} (x-1)(1-y)<0 \\ (x+1)(1+y)>0. \end{cases}$$

Since $-1<x,y<1$, *then,*

$$(1-y>0)\wedge(x-1<0) \ \textit{et} \ (1+y>0)\wedge(x+1>0),$$

so

$$[(1-y)(x-1)<0] \ \textit{and} \ [(1+y)(x+1)>0],$$

hence, we deduce that $(*)$ *is true for all* $x,y\in]-1,1[$, *and therefore,*

$$\forall x,y\in]-1,1[: |x\star y|=\left|\frac{x+y}{1+xy}\right|<1,$$

which shows that $\star$ *is an internal composition law in* $]-1,1[$.

2. $\star$ *is commutative: according to the commutativity of addition and multiplication in* $\mathbb{R}$ *we have:*

$$\forall x,y\in]-1,1[: x\star y=\frac{x+y}{1+xy}=\frac{y+x}{1+yx}=y\star x,$$

which shows that $\star$ *is commutative.*

3. $\star$ *is associative: let* $x,y,z\in]-1,1[$, *then,*

$$\begin{aligned}(x\star y)\star z &= \frac{(x\star y)+z}{1+(x\star y)z}=\frac{\frac{x+y}{1+xy}+z}{1+\frac{x+y}{1+xy}z} \\ &= \frac{(x+y)+z(1+xy)}{(1+xy)+(x+y)z}=\frac{x+y+z+xyz}{1+xy+xz+yz},\end{aligned}$$

et on a:

$$\begin{aligned} x\star(y\star z) &= \frac{x+(y\star z)}{1+x(y\star z)}=\frac{x+\dfrac{y+z}{1+yz}}{1+x\dfrac{y+z}{1+yz}} \\ &= \frac{x(1+yz)+(y+z)}{(1+yz)+x(y+z)}=\frac{x+y+z+xyz}{1+xy+xz+yz}, \end{aligned}$$

by comparing the two expressions we obtain:

$$\forall x,y,z\in]-1,1[,\quad (x\star y)\star z=x\star(y\star z)$$

hence, we deduce that $\star$ is associative.

4. $\star$ *admits a neutral element: let $e\in\mathbb{R}$, then,*

$$e \text{ is a neutral element of } \star \iff \forall x\in]-1,1[,\quad e\star x=x\star e=x,$$

as $\star$ is commutative and

$$\begin{aligned} x\star e=x &\iff \frac{x+e}{1+xe}=x \\ &\iff x+e=x+x^2e \\ &\iff e(1-x^2)=0 \\ &\iff e=0\vee x=\pm1. \end{aligned}$$

We deduce that $e=0\in]-1,1[$ is the neutral element of $\star$.

5. *Every element of $]-1,1[$ is symmetrizable: let $x\in]-1,1[$ and $x\in\mathbb{R}$, then,*

$$\begin{aligned} x\star x'=e &\iff \frac{x+x'}{1+xx'}=0 \\ &\iff x+x'=0 \\ &\iff x'=-x, \end{aligned}$$

as $\star$ is commutative, we deduce that every element $x\in]-1,1[$ is symmetrizable and its symmetric is $x'=-x\in]-1,1[$.

From 1., 2., 3., 4. and 5. we deduce that $(]-1,1[,\star)$ is an Abelien group.

Exercise 1. *On $\mathbb{R}-\{1\}$ we define the law $\star$ as follows: $x\star y=x+y-xy$.*

(*i*) *Verify that $\star$ is an internal composition law.*

(*ii*) *Show that $(\mathbb{R}-\{1\},\star)$ is a commutative group.*

(iii) *Solve the equation:* $2 \star 3 \star x \star 5 = 5 \star 3$.

Solution:

(i) Verify that $\star$ is an internal composition law: let us show that: $\forall x, y \in \mathbb{R} - \{1\}$, then, $x \star y \in \mathbb{R} - \{1\} \Longleftrightarrow x + y - xy \neq 1$.

If

$$\begin{aligned} x + y - xy = 1 &\Longrightarrow x + y - xy - 1 = 0 \\ &\Longrightarrow (1 - x)(y - 1) = 0 \\ &\Longrightarrow x = 1 \vee y = 1, \quad \textit{contradiction}. \end{aligned}$$

Then, $\star$ is an internal composition law.

(ii) Show that $(\mathbb{R} - \{1\}, \star)$ is a commutative group:

(a) Let us show that $\star$ is commutative, $\forall x, y \in \mathbb{R} - \{1\} : x \star y = y \star x$.

Let $x, y \in \mathbb{R} - \{1\}$,

$$x \star y = x + y - xy = y + x - yx = y \star x.$$

Then, $\star$ is a commutative law.

(b) Let us show that $\star$ is associative, $\forall x, y, z \in \mathbb{R} - \{1\} : (x \star y) \star z = x \star (y \star z)$.

Let $x, y, z \in \mathbb{R} - \{1\}$:

$$\begin{aligned} (x \star y) \star z = (x + y - xy) \star z &= (x + y - xy) + z - (x + y - xy)z \\ &= x + y + z - xy - xz - yz + xyz, \end{aligned}$$

and

$$\begin{aligned} x \star (y \star z) = x \star (y + z - yz) &= x + (y + z - yz) - x(y + z - yz) \\ &= x + y + z - xy - xz - yz + xyz. \end{aligned}$$

By identifying the two results, we obtain, $(x \star y) \star z = x \star (y \star z)$.

Then $\star$ is an associative law.

(c) Let us show that $\star$ admits a neutral element, $\exists e \in \mathbb{R} - \{1\}, \forall x \in \mathbb{R} - \{1\}, x \star e = e \star x = x$. Since $\star$ is commutative then it suffices to show that: $x \star e = x$.

We have:

$$x \star e = x \Leftrightarrow x + e - xe = x \Leftrightarrow e(1 - x) = 0 \Leftrightarrow e = 0 \textit{ because } x \in \mathbb{R} - \{1\}.$$

(*d*) Let us show that each element $x \in \mathbb{R}-\{1\}$ admits a symmetric element denoted x^{-1} such that: $x \star x^{-1} = x^{-1} \star x = e = 0$. Since $\star$ is commutative then, il suffices to solve the equation:

$$x \star x^{-1} = 0 \Leftrightarrow x + x^{-1} - xx^{-1} = 0 \Leftrightarrow x + x^{-1}(1-x) = 0 \Leftrightarrow x^{-1} = \frac{x}{x-1},$$

which is well defined because $x \in \mathbb{R} - \{1\}$.
Conclusion: $(\mathbb{R} - \{1\}, \star)$ is a commutative group.

(*iii*) Solve the equation: $2 \star 3 \star x \star 5 = 5 \star 3$, (by using the notion of the symmetry element):

$$\begin{aligned} 2 \star 3 \star x \star 5 = 5 \star 3 \iff & x = \frac{3}{2} \star 2 \star 5 \star 3 \star \frac{5}{4} \\ \iff & x = 2. \end{aligned}$$

6.1.3 Subgroups

Definition 6.20. *Let $(G, \star)$ be a group, we call subgroup of $(G, \star)$ any non-empty subset G' of G such that the restriction of $\star$ to G' makes it a group.*

As $\star$ is associative in G then, its restriction to G' is also associative, consequently $G' \neq \emptyset$ is a subgroup of $(G, \star)$ if it is stable with respect to $\star$ and to the inversion operation, i.e.,

- ▷ $G' \neq \emptyset$.
- ▷ $\forall a, b \in G', a \star b \in G'$.
- ▷ $\forall a \in G', a^{-1} \in G'$.

It is clear that if $(G, \star)$ is a group, then, $(G', \star)$ is a subgroup of G.

Properties 6.21. *Let $(G, \star)$ be a group and $G' \subset G$, then,*

$$G' \textit{ is a subgroup of } G \iff \begin{cases} G' \neq \emptyset, \\ \forall a, b \in G', \quad a \star b^{-1} \in G'. \end{cases}$$

Proof:

1. Let $(G', \star)$ be a subgroup of $(G, \star)$, then,

   (*i*) $\star$ has a neutral element in G', therefore $G' \neq \emptyset$.

   (*ii*) Let $a, b \in G'$, as G' provided with the restriction of $\star$ is a group, then b^{-1} exists in G' and as G' is stable with respect to $\star$ we deduce that $a \star b^{-1} \in G'$.

2. Conversely, let G' be a subset of G such that: $\begin{cases} G' \neq \emptyset, \\ \forall a, b \in G', \quad a \star b^{-1} \in G'. \end{cases}$ Let us show that G' with the restriction of $\star$ is a group.

(i) As $G' \neq \emptyset$ then, it exists $a \in G'$ and according to the second hypothesis we will have:

$$e = a \star a^{-1} \in G',$$

which shows that the restriction of $\star$ admits a neutral element e in G'.

(ii) Let $a \in G'$, as $e \in G'$, then, according to the second hypothesis we will have:

$$a^{-1} = e \star a^{-1} \in G',$$

which shows that every element a of G' is invertible in G' with respect to the restriction of $\star$ to G'.

(iii) The restriction of $\star$ to G' is an internal composition law, because for all a and b in G', from (ii) we have: $b^{-1} \in G'$ and by using the second hypothesis, we deduce that:

$$a \star b = a \star (b^{-1})^{-1} \in G'.$$

(iv) The restriction of $\star$ to G' is associative, because $\star$ is associative in G.

Example 6.22. *Let $(G, \star)$ be a group and $G' = \{x \in G; \forall y \in G, x \star y = y \star x\}$, then G' is a subgroup of G.*

Indeed,

(i) *If e is the neutral element of $\star$, then, $e \in G'$ because:*

$$\forall x \in G, \quad e \star x = x \star e = x.$$

(ii) *let $x, y \in G'$, then,*

$$\begin{aligned} \forall z \in G, \qquad (x \star y^{-1}) \star z &= (x \star y^{-1}) \star (z^{-1})^{-1} \\ &= x \star (y^{-1} \star (z^{-1})^{-1}) \quad \text{because } \star \text{ is associative} \\ &= x \star (z^{-1} \star y)^{-1} \\ &= x \star (y \star z^{-1})^{-1} \quad \text{because } y \in G' \\ &= x \star ((z^{-1})^{-1} \star y^{-1}) \\ &= x \star (z \star y^{-1}) \\ &= (x \star z) \star y^{-1} \quad \text{because } \star \text{ is associative} \\ &= (z \star x) \star y^{-1} \quad \text{because } x \in G' \\ &= z \star (x \star y^{-1}) \quad \text{because } \star \text{ is associative,} \end{aligned}$$

which shows that $x \star y^{-1} \in G'$.

From (i) and (ii), we deduce that G' is a subgroup of G.

6.1.4 Quotient groups

Let $(G, \star)$ be a group and G' a subgroup of G. We define a binary relation $\mathcal{R}$ on G by:

$$\forall a, b \in G, a\mathcal{R}b \Longleftrightarrow a \star b^{-1} \in G'$$

Properties 6.23. *$\mathcal{R}$ is an equivalence relation on G.*

Proof:

(i) $\mathcal{R}$ is reflexive, because: $\forall x \in G$, as G' is a subgroup of G, then, $x \star x^{-1} = e \in G'$, so

$$\forall x \in G, x\mathcal{R}x.$$

(ii) $\mathcal{R}$ is symmetric, because: $\forall x, y \in G$,

$$\begin{aligned} x\mathcal{R}y &\Longleftrightarrow x \star y^{-1} \in G' \\ &\Longrightarrow (x \star y^{-1})^{-1} \in G' \\ &\Longrightarrow y \star x^{-1} \in G' \\ &\Longrightarrow y\mathcal{R}x. \end{aligned}$$

(iii) $\mathcal{R}$ is transitive, because: $\forall x, yz \in G$,

$$\begin{aligned} x\mathcal{R}y \wedge y\mathcal{R}z &\Longleftrightarrow x \star y^{-1} \in G' \wedge y \star z^{-1} \in G' \\ &\Longrightarrow (x \star y^{-1}) \star (y \star z^{-1}) \in G' \quad \textit{because } G' \textit{ is a subgroup} \\ &\Longrightarrow x \star (y^{-1} \star y) \star z^{-1} \in G' \quad \textit{because } \star \textit{ is associative} \\ &\Longrightarrow x \star z^{-1} \in G' \\ &\Longrightarrow x\mathcal{R}z. \end{aligned}$$

From (i), (ii) and (iii), we deduce that $\mathcal{R}$ is an equivalence relation.

We denote $G/_{G'}$ the quotient set $G/_{\mathcal{R}}$. We define on $G/_{G'} \times G/_{G'}$ the operation $\oplus$ by:

$$\forall (\dot{a}, \dot{b}) \in G/_{G'} \times G/_{G'}, \dot{a} \oplus \dot{b} = \dot{\overline{a \star b}}.$$

6.1.5 Group Homomorphisms

We consider $(G, \bullet)$ and $(H, \star)$ be two groups, with e and e' their respective neutral elements

Definition 6.24. *A map $f : G \longrightarrow H$ is called homomorphism of groups from G to H if:*

$$\forall a, b \in G, \quad f(a \bullet b) = f(a) \star f(b).$$

▷ *If f is bijective, we say that f is an isomorphism (of group) of G on H. Then, we say that G is isomorphic to H, or that G and H are isomorphic.*

▷ *If $G = H$, we say that f is an endomorphism of G, and if moreover f is bijective, we say that f is an automorphism (of group) of G.*

Example 6.25. *Given the groups $(\mathbb{R}, +)$ and $(\mathbb{R}^*, \cdot)$, then, the maps:*

$$\begin{array}{rcl} f : (\mathbb{R}, +) & \longrightarrow & (\mathbb{R}^*, \cdot) \\ x & \longmapsto & f(x) = \exp x \end{array} \qquad et \qquad \begin{array}{rcl} g : (\mathbb{R}^*, \cdot) & \longrightarrow & (\mathbb{R}, +) \\ x & \longmapsto & g(x) = \ln |x| \end{array}$$

are homomorphisms.

Definition 6.26. *Let $f : G \longrightarrow H$ be a homomorphism of groups. We call the kernel of f the set:*

$$\ker f = f^{-1}(e') = \{a \in G; f(a) = e'\},$$

and the image of f the set:

$$\mathfrak{Im} f = f(G) = \{f(a), a \in G\}.$$

Properties 6.27. *Let $f : G \longrightarrow H$ be a homomorphism of groups, then,*

(i) $f(e) = e'$,

(ii) $\forall a \in G, f^{-1}(a) = f(a^{-1})$.

6.1.6 Ring structure

Definition 6.28. *We call a ring any set A provided with two laws of composition $\star$ and $\triangle$ such that:*

1. *$(A, \star)$ is an Abelian group (we denote 0 or 0_A neutral element of $\star$),*
2. *$\triangle$ is associative and distributive with respect to $\star$.*

If moreover $\triangle$ is commutative, we say that $(A, \star, \triangle)$ is a commutative ring.

Definition 6.29. *We call subring of $(A, \star, \triangle)$, any subset A' of A such that provided with the restrictions of the laws $\star$ and $\triangle$ is ring.*

If A is a unit ring and $1_A \in A'$, we say that A' is a unit subring.

Exercise 2. *Let $(\mathbb{R}, +, \cdot)$ be the ring of real numbers. We define two new laws $\star$ and $\bullet$ on $\mathbb{R}$ as the following:*

$$\forall (x, y) \in \mathbb{R}^2; \quad x \star y = x + y - 2, \qquad x \bullet y = xy - 2x - 2y + 6.$$

1. *Show that $(\mathbb{R}, \star)$ is an Abelian group.*
2. *Show that $(\mathbb{R}, \star, \bullet)$ is a commutative ring.*

Solution:

1. Let us show that $(\mathbb{R}, \star)$ is an Abelian group:

   (a) Let us show that $\star$ is commutative, $\forall x, y \in \mathbb{R} : x \star y = y \star x$.

   Let $x, y \in \mathbb{R}$,
$$x \star y = x + y - 2 = y + x - 2 = y \star x.$$
   Then, $\star$ is commutative law.

   (b) Let us show that $\star$ is associative, $\forall x, y, z \in \mathbb{R} : (x \star y) \star z = x \star (y \star z)$.

   Let $x, y, z \in \mathbb{R}$:
$$(x \star y) \star z = (x + y - 2) \star z = (x + y - 2) + z - 2 = x + y + z - 4,$$
   and
$$x \star (y \star z) = x \star (y + z - 2) = x + (y + z - 2) - 2 = x + y + z - 4.$$
   By identifying the two results, we obtain, $(x \star y) \star z = x \star (y \star z)$.

   Then, $\star$ is an associative law.

   (c) Let us show that $\star$ admits a neutral element, $\exists e \in \mathbb{R}, \forall x \in \mathbb{R}, x \star e = e \star x = x$. Since $\star$ is commutative, then, it suffices to show that: $x \star e = x$, indeed,
$$x \star e = x \Leftrightarrow x + e - 2 = x \Leftrightarrow e - 2 = 0 \Leftrightarrow e = 2.$$

   (d) Let us show that each element $x \in \mathbb{R}$ admits a symmetric element denoted x^{-1} such that: $x \star x^{-1} = x^{-1} \star x = e = 2$. Since $\star$ is commutative then, it suffices to solve the equation:
$$x \star x^{-1} = 2 \Leftrightarrow x + x^{-1} - 2 = 2 \Leftrightarrow x + x^{-1} = 4 \Leftrightarrow x^{-1} = 4 - x,$$
   which is well defined because $x \in \mathbb{R}$.

   Conclusion: $(\mathbb{R}, \star)$ is a commutative group.

2. Let us show that $(\mathbb{R}, \star, \bullet)$ is a commutative ring:

   (a) Let us show that $\bullet$ is commutative, $\forall x, y \in \mathbb{R} : x \bullet y = y \bullet x$.

   Let $x, y \in \mathbb{R}$,
$$x \bullet y = xy - 2x - 2y + 6 = yx - 2y - 2x + 6 = y \bullet x.$$
   Then, $\bullet$ is a commutative law.

(b) Let us show that $\bullet$ is associative, $\forall x, y, z \in \mathbb{R} : (x \bullet y) \bullet z = x \bullet (y \bullet z)$.

Let $x, y, z \in \mathbb{R}$:

$$\begin{aligned}(x \bullet y) \bullet z &= (xy - 2x - 2y + 6) \bullet z \\ &= (xy - 2x - 2y + 6)z - 2(xy - 2x - 2y + 6) - 2z + 6 \\ &= xyz - 2xz - 2xy - 2yz + 4x + 4y + 4z - 6,\end{aligned}$$

and

$$\begin{aligned}x \bullet (y \bullet z) &= x \bullet (yz - 2y - 2z + 6) \\ &= x(yz - 2y - 2z + 6) - 2x - 2(yz - 2y - 2z + 6) + 6 \\ &= xyz - 2xz - 2xy - 2yz + 4x + 4y + 4z - 6.\end{aligned}$$

By identifying the two results, we obtain, $(x \bullet y) \bullet z = x \bullet (y \bullet z)$.

Then $\bullet$ is a associative law.

(c) Let us show that $\bullet$ is distributive with respect to $\star$,

$$\forall x, y, z \in \mathbb{R}, x \bullet (y \star z) = (x \bullet y) \star (x \bullet z) \ et \ (y \star z) \bullet x = (y \bullet x) \star (z \bullet x).$$

Let $x, y, z \in \mathbb{R}$:

$$x \bullet (y \star z) = x \bullet (y + z - 2) = xy + xz - 4x - 2y - 2z + 10,$$

and

$$\begin{aligned}(x \bullet y) \star (x \bullet z) &= (xy - 2x - 2y + 6) \star (xz - 2x - 2z + 6) \\ &= xy + xz - 4x - 2y - 2z + 10.\end{aligned}$$

By identifying the two results, we obtain, $x \bullet (y \star z) = (x \bullet y) \star (x \bullet z)$. Let $x, y, z \in \mathbb{R}$:

$$\begin{aligned}(y \star z) \bullet x &= (y + z - 2) \bullet x \\ &= (y + z - 2)x - 2(y + z - 2) - 2x + 6 \\ &= xy + xz - 4x - 2y - 2z + 10,\end{aligned}$$

and

$$\begin{aligned}(y \bullet x) \star (z \bullet x) &= (yx - 2y - 2x + 6) \star (zx - 2z - 2x + 6) \\ &= (yx - 2y - 2x + 6) + (zx - 2z - 2x + 6) - 2 \\ &= yx + zx - 4x - 2y - 2z + 10 \\ &= xy + xz - 4x - 2y - 2z + 10.\end{aligned}$$

By identifying the two results, we obtain, $(y \star z) \bullet x = (y \bullet x) \star (z \bullet x)$. Then, $\bullet$ is distributive with respect to $\star$.

Conclusion: $(\mathbb{R}, \star, \bullet)$ is a commutative ring.

6.1.7 Field

Definition 6.30. *An element $x \in \mathbb{K}$ is invertible with respect to the law $\triangle$ if there is an element $y \in \mathbb{K}$ such that:*

$$x \triangle y = y \triangle x = e', \quad (e' \text{ is the neutral element with respect to } \triangle).$$

Definition 6.31. *We say that $(\mathbb{K}, *, \triangle)$ is a field if:*

1. *$(\mathbb{K}, *, \triangle)$ is a ring.*
2. *Any element distinct to e (operation $*$) is invertible for the law $\triangle$.*

If moreover $\triangle$ is commutative, we speak of a commutative field.

Example 6.32. *$(\mathbb{R}, +, \times)$ is a commutative field, but $(\mathbb{Z}, +, \times)$ is not a field.*

6.2 Vector space

The vector space structure is involved in a large part of mathematics: it provides a fundamental link between algebra and geometry. It will be used in linear algebra, analysis and geometry.

In this section, $\mathbb{K}$ will designate a commutative field, in practice $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Definition 6.33. *We call $\mathbb{K}$-vector space any set E provided with an internal law denoted $+$ and an external law defined by:*

$$\begin{array}{rcl} \mathbb{K} \times E & \longrightarrow & E \\ \alpha, x & \longmapsto & \alpha \times x. \end{array}$$

such that:

(i) *$(E, +)$ is an Abelian group,*

(ii) $\forall (\alpha, \beta) \in \mathbb{K}^2, \forall x \in E, (\alpha + \beta)x = \alpha x + \beta x,$

(iii) $\forall \alpha \in \mathbb{K}, \forall (x, y) \in E^2, \alpha(x + y) = \alpha x + \alpha y,$

(iv) $\forall (\alpha, \beta) \in \mathbb{K}^2, \forall x \in E, \alpha(\beta x) = (\alpha\beta)x,$

(v) $\forall x \in E, 1x = x.$

Remark 6.34.

- *We will abbreviate vector space to v. s and $\mathbb{K}$-vector space to $\mathbb{K}$-v.s.*
- *The elements of a $\mathbb{K}$-v.s will be called vectors, the elements of $\mathbb{K}$ will be called scalars.*

Example 6.35. *Let* $\mathbb{K} = \mathbb{R}, E = \mathbb{C}$*, then,* $(\mathbb{C}, +, \times)$ *is a* $\mathbb{R}$*-v.s because:*

- ▷ $(\mathbb{C}, +)$ *is an Abelian group.*
- ▷ *As* $(\mathbb{C}, +, \times)$ *is a commutative field, then* $\times$ *is distributive with respect to addition, we deduce that conditions (ii) and (iii) are true for* $\alpha, \beta \in \mathbb{R}$.
- ▷ *Likewise, as* $(\mathbb{C}, +, \times)$ *is a field, then,* $\times$ *is associative. So,*

$$\forall \alpha, \beta \in \mathbb{C}, \forall x \in \mathbb{C}, (\alpha \times \beta) \times x = \alpha(\beta \times x),$$

hence,

$$\forall \alpha, \beta \in \mathbb{R}, \forall x \in \mathbb{C}, (\alpha \times \beta) \times x = \alpha(\beta \times x), \quad \text{because} \ \ \mathbb{R} \subset \mathbb{C}.$$

Finally, we have: $\forall x \in \mathbb{C}, 1 \times x = x$.

Proposition 6.36. *(Calculation rules)*

Let $(E, +, \cdot)$ *be a* $\mathbb{K}$*-vector space. Then,*

- $\forall \vec{x} \in E, 0_{\mathbb{K}} \cdot \vec{x} = \vec{0}$, *($\vec{0}$ is the neutral element of + in $(E, +)$).*
- $\forall \alpha \in \mathbb{K}, \alpha \cdot \vec{0} = \vec{0}$.
- $\forall \alpha \in \mathbb{K}, \forall \vec{x} \in E, \alpha(-\vec{x}) = (-\alpha) \cdot \vec{x} = -(\alpha \cdot \vec{x})$.
- $\alpha x = 0_{\mathbb{K}} \Leftrightarrow \alpha = 0$ *or* $\vec{x} = \vec{0}$.

6.2.1 Vector subspace

Definition 6.37. *Let E be a $\mathbb{K}$-vector space and F a subset of E. We say that F is a vector subspace of E if and only if:*

(i) $F \neq \emptyset, \quad (\vec{0} \in F)$,

(ii) $\forall x, y \in F, x + y \in F$,

(iii) $\forall \alpha \in \mathbb{K}, \forall x \in F, \alpha \cdot x \in F$.

Example 6.38.

1. *Let* $E = \mathbb{R}^3$, $F = \{\vec{x} = (x, y, z) \in \mathbb{R}^3 : x + y = 1\}$ *is not vector subspace of E because* $\vec{0} = (0, 0, 0) = (x, y, z) \in \mathbb{R}^3$*, so,*

$$\vec{0} \in F \Leftrightarrow x + y = 1 \Leftrightarrow 0 + 0 = 1, \quad \text{impossible}$$

hence, we deduce that $\vec{0} \notin F$.

2. *Let* $E = \mathbb{R}^3$, $F = \{\vec{x} = (x, y, z) \in \mathbb{R}^3 : x + 2y - z = 0\}$ *is a vector subspace of* E. *Indeed,*

   - ▷ *let* $\vec{0} = (0, 0, 0)$ *and* $(x, y, z) \in \mathbb{R}^3$, *then,* $x + 2y - z = 0 + 2 \cdot 0 - 0 = 0$, *so,* $\vec{0} \in F$.
   - ▷ *let* $\vec{x} = (x, y, z)$ *and* $\vec{y} = (x', y', z') \in F$, *then,* $\vec{x} + \vec{y} = (x + x', y + y', z + z') = (u, v, w)$ *and*

$$\left.\begin{array}{l} \vec{x} \in F \Rightarrow x + 2y - z = 0 \\ \vec{y} \in F \Rightarrow x' + 2y' - z' = 0 \end{array}\right\} \begin{array}{l} \Longrightarrow (x + x') + 2(y + y') - (z + z') = 0 \\ \Longrightarrow u + 2v - w = 0 \\ \Longrightarrow \vec{x} + \vec{y} \in F. \end{array}$$

   - ▷ *let* $\alpha \in \mathbb{R}$, *then,*

$$\vec{x} \in F \Rightarrow \alpha(x + 2y - z) = 0 \Rightarrow \alpha\vec{x} \in F.$$

   Hence, we deduce that F *is a vector subspace of* $\mathbb{R}^3$.

Exercise 3. *Which of these sets are vector subspaces of* $E = \mathbb{R}^3$

1. $F_1 = \{(x, y, z) \in \mathbb{R}^3 : x + 2y = 1\}$;
2. $F_2 = \{(x, y, z) \in \mathbb{R}^3 : x + y^2 = 0\}$;
3. $F_3 = \{(x, y, z) \in \mathbb{R}^3 : x + y = -2z\}$;
4. $F_4 = \{= (x, y, z) \in \mathbb{R}^3 : x + 2z = 0\}$.

6.2.2 Linear combinations

Definition 6.39. *Let* E *be a* $\mathbb{K}$*-vector space and* $B = \{\vec{x_1}, \vec{x_2}, \cdots, \vec{x_n}\} \subset E$. *We call linear combinations of* $\vec{x_1}, \vec{x_2}, \cdots, \vec{x_n}$, *any vector* $\vec{x} = \alpha_1\vec{x_1} + \alpha_2\vec{x_2} + \cdots + \alpha_n\vec{x_n}$ *with* $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{K}$.

Example 6.40. *In* $\mathbb{R}^3$, *we have:* $(1, 2, 3) = (1, 0, 0) + 2(0, 1, 0) + 3(0, 0, 1) = \vec{e_1} + 2\vec{e_2} + 3\vec{e_3}$, *then,* $\vec{x} = (1, 2, 3)$ *is a linear combinations of* $\vec{e_1}, \vec{e_2}, \vec{e_3}$.

$\vec{y} = (5, 2, -4) = 5\vec{e_1} + 2\vec{e_2} - 4\vec{e_3}$ *is a linear combinations of* $\vec{e_1}, \vec{e_2}, \vec{e_3}$.

In general, $\forall \vec{x} = (x, y, z) \in \mathbb{R}^3 : \vec{x} = x\vec{e_1} + y\vec{e_2} + z\vec{e_3}$, *therefore any element of* $\mathbb{R}^3$ *is a linear combinations of the vectors* $\vec{e_1} = (1, 0, 0), \vec{e_2} = (0, 1, 0), \vec{e_3} = (0, 0, 1)$.

6.2.3 Intersection and the union of two subspaces

Proposition 6.41. *Let E be a $\mathbb{K}$-vector space and F, F' two vector subspaces of E, then, $F \cap F'$ is a vector subspaces of E, indeed,*

(*i*) *let $\vec{0} \in E$, as F and F' are two vector subspaces of E, then, $\vec{0} \in F$ et $\vec{0} \in F'$, hence, we deduce that $\vec{0} = F \cap F'$.*

(*ii*) *let $\vec{x}, \vec{y} \in F \cap F'$, then, $(\vec{x}, \vec{y} \in F)$ and $(\vec{x}, \vec{y} \in F')$. As F and F' are two vector subspaces of E, then, $[(\vec{x} + \vec{y}) \in F]$ and $[(\vec{x} + \vec{y}) \in F']$, hence, we deduce that $(\vec{x} + \vec{y}) \in F \cap F'$. two vector subspaces of E, then, $\alpha\vec{x} \in F$ and $\alpha\vec{x} \in F']$, hence, we deduce that $\alpha\vec{x} \in F \cap F'$.*

From (i), (ii) and (iii), $F \cap F'$ is a vector subspaces of E.

Remark 6.42. *The union of two vector subspaces is not always a vector subspace.*

For example, let $E = \mathbb{R}^2, F = \{(x, y), x = 0\}, F' = \{(x, y), y = 0\}$. we have: F and F' are two vector subspaces of $\mathbb{R}^2$ but $F \cup F'$ is not a vector subspace of $\mathbb{R}^2$ because

$$F \cup F' = \{(x, y) \in \mathbb{R}^2, x = 0 \vee y = 0\},$$

$\vec{x} = (1.0), \vec{y} = (0, 1) \in F \cup F'$ and $\vec{x} + \vec{y} = (1, 1) \notin F \cup F'$. Hence, we deduce that $F \cup F'$ is not a vector subspace of $\mathbb{R}^2$.

6.2.4 Sum of subspaces. Direct sum

- ▷ **Sum of subspaces:** if F and G are two vector subspaces of E then, the sum of F and G is defined by:

$$F + G = \{x \in E \textit{ such that } \vec{x} = \vec{x_1} + \vec{x_2} \textit{ with } \vec{x_1} \in F \textit{ and } \vec{x_2} \in G\}.$$

- ▷ **Direct sum:** we say that the sum $F + G$ is direct, or that F and G are supplementary with respect to E, if the decomposition $\vec{x} = \vec{x_1} + \vec{x_2}$ of any one element of E in sum of two elements of F and G is unique. We note $E = F \oplus G$, otherwise we have:

$$E = F \oplus G \Longleftrightarrow \begin{cases} E = F + G \\ \wedge \\ F \cap G = \{0_E\} \end{cases}$$

Example 6.43. *In $\mathbb{R}^3$, the following two vector subspaces:*

$$F = \{(x, y, z); x = y = z\} \quad \textit{and} \quad G = \{(x, y, 0), x, y \in \mathbb{R}\}$$

sont supplementary. Indeed,

1. *We have: $\mathbb{R}^3 = F + G$ because:*

(*i*) $F \subset \mathbb{R}^3$ *and* $G \subset \mathbb{R}^3 \Rightarrow F + G \subset \mathbb{R}^3$.

(*ii*) $\forall \vec{x} \in \mathbb{R}^3, \vec{x} = (x, y, z) = (z, z, z) + (x - z, y - z, 0) \in F + G \Rightarrow \mathbb{R}^3 \subset F + G$.

2. (*i*) *We have:* $0_E \in F$ *and* $0_E \in G$ *because* F *and* G *are two vector subspaces of* $E \Rightarrow 0_E \in F \cap G \Rightarrow \{0_E\} \subset F \cap G$.

   (*ii*) *If* $\vec{x} \in F \cap G \Rightarrow \vec{x} \in F$, $\vec{x} \in G \Rightarrow \vec{x} = (x, x, x)$, $\vec{x} = (x, y, 0) \Rightarrow x = y$ *and* $x = 0 \Rightarrow \vec{x} = (0, 0, 0) \Rightarrow F \cap G \subset \{0_E\}$.

6.2.5 Family of vectors of a vector space

1. **Linked families:** a family $\{x_i\}_{1 \le i \le n}$ of vectors of a $\mathbb{K}$-vector space $(E, +, \cdot)$ is linked or linearly dependent if it exists $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{K}$ not all zero such that, $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0_E$.

   Example 6.44. *In* $E = \mathbb{R}_2[x]$ *(the vector space of polynomial functions of degree less than or equal to 2 and with real coefficients), the functions* f_1, f_2, f_3 *defined for all* $x \in \mathbb{R}$ *by:*
   $$f_1(x) = x^2 + 1, \quad f_2(x) = x^2 - 1, \quad f_3(x) = x^2$$
   are linked. Indeed, let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ *such that:*
   $$\alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 = 0 \Longrightarrow \begin{cases} \alpha_1 + \alpha_2 + \alpha_3 = 0, \\ \alpha_1 - \alpha_2 = 0 \end{cases}$$
   hence $\alpha_1 = \alpha_2 = -\frac{\alpha_3}{2}$, *therefore there are an infinity of solutions* $(-\frac{\alpha_3}{2}, -\frac{\alpha_3}{2}, \alpha_3)$ *with* α_3 *arbitrary real for example,* $(1, 1, -2)$.

2. **Free families:** a family $\{x_i\}_{1 \le i \le n}$ of vectors of a $\mathbb{K}$-vector space $(E, +, \cdot)$ is free or linearly independent if for all $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{K}$,
   $$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0_E \Longrightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0.$$

   Example 6.45. *In* $\mathbb{R}^3$ *the vectors* $x_1 = (0, 1, 3), x_2 = (2, 0, -1)$ *and* $x_3 = (2, 0, 1)$ *are free because:*
   $$\begin{aligned} \forall \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}, \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0 \quad &\Longrightarrow \quad \begin{cases} 2\alpha_2 + \alpha_3 = 0 \\ \alpha_1 = 0 \\ 3\alpha_1 - \alpha_2 + \alpha_3 = 0 \end{cases} \\ &\Longrightarrow \quad \alpha_1 + \alpha_2 + \alpha_3 = 0. \end{aligned}$$

3. **Generating family:** a family of vectors $\{x_1, x_2, \cdots, x_n\}$ of $\mathbb{K}$-vector space $(E, +, \cdot)$ is said to be a generating of E or generates E if every element x of E is a linear combination of $(x_1, x_2, \cdots, x_n)$ i.e.,
   $$\forall x \in E, \exists \alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{K} \textit{ such that } x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n.$$

Example 6.46. *In* $\mathbb{R}^2$ *the two vectors* $x_1 = (2,3)$ *and* $x_2 = (-1,5)$ *is a generating family because:* $\forall (x,y) \in \mathbb{R}^2, \exists \alpha_1, \alpha_2 \in \mathbb{R}$ *tel que:*

$$\begin{aligned}
(x,y) &= \alpha_1 x_1 + \alpha_2 x_2 = \alpha_1(2,3) + \alpha_2(-1,5) = (2\alpha_1 - \alpha_2, 3\alpha_1 + 5\alpha_2 \\
&\Longrightarrow \begin{cases} x = 2\alpha_1 - \alpha_2 \\ y = 3\alpha_1 + 5\alpha_2 \end{cases} \Longrightarrow \begin{cases} \alpha_1 = \dfrac{5x+y}{13} \\ \\ \alpha_2 = \dfrac{-3x+2y}{13} \end{cases}
\end{aligned}$$

therefore (α_1, α_2) *exists for all* $(x,y) \in \mathbb{R}^2$.

4. **Basis:** a family of vectors $\{x_1, x_2, \cdots, x_n\}$ of $\mathbb{K}$-vector space $(E, +, \cdot)$ is a basis of E if it is both free and generating.

   Example 6.47. $B_0 = \{(1,0,0), (0,1,0), (0,0,1)\}$ *is a basis of* $\mathbb{R}^3$. *Indeed,*

   (*i*) B_0 *is free because:*

   $$\begin{aligned}
   \alpha(1,0,0) + \beta(0,1,0) + \gamma(0,0,1) = (0,0,0) &\Rightarrow (\alpha, \beta, \gamma) = (0,0,0) \\
   &\Rightarrow \alpha = \beta = \gamma = 0.
   \end{aligned}$$

   (*ii*) B_0 *is generating of* $\mathbb{R}^3$ *because:*

   $$\forall (x,y,z) \in \mathbb{R}^3, (x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1).$$

5. **Dimension of a vector space:** the finite dimension n of vector space E is the maximum number of vectors that a free system extracted from E can contain and we note $\dim E = n$, by convention we set: $\dim(\{0_E\}) = 0$.

   In other words the dimension of a vector space E is the number of vectors which form the basis of E. If the number of elements of a system free of E is not increased, we say that E is of infinite dimension.

   Remark 6.48. *If* F *is a vector subspace of a vector space* E *of dimension* n *then,*

   $$F \subset E \Longrightarrow \dim F \leq \dim E.$$

   Example 6.49. $\dim(\mathbb{R}^2) = 2$.

6. **Rank of a system of vectors:** we call rank of a system of p vectors $(x_1, x_2, \cdots, x_p)$ of E with $\dim E = n$, the dimension r of the vector subspace $(x_1, x_2, \cdots, x_p)$. In other words, r is the maximum number of vectors that a free system extracted from the given system can contain.

6.2.6 Subspace generated by a set

Definition 6.50. *Let A be a part of a vector space E. We define the vector subspace generated by a set A, the smallest vector subspace containing the set A. We note it $S(A)$.*

Example 6.51. *If A is a vector subspace of E then, $S(A) = A, S(\emptyset) = \{0_E\}$.*

Exercise 4. *In $\mathbb{R}^3$, we consider the following subsets:*

$$E_1 = \{(a+b, b-3a, a) \in \mathbb{R}^3; a, b \in \mathbb{R}\} \;\; \textit{and} \;\; E_2 = \{(c, -2c, c) \in \mathbb{R}^3; c \in \mathbb{R}\}$$

with E_2 is a vector subspace of $\mathbb{R}^3$.

1. *Show that E_1 is a vector subspace of $\mathbb{R}^3$.*
2. *Determine a basis B_1 of E_1 and a basis B_2 of E_2.*
3. *Deduce $\dim E_1$ and $\dim E_2$.*
4. *Show that $\mathbb{R}^3 = E_1 + E_2$.*
5. *Deduce whether the sum is direct or not.*

Solution:

$$E_1 = \{(a+b, b-3a, a) \in \mathbb{R}^3; a, b \in \mathbb{R}\} \;\; \textit{and} \;\; E_2 = \{(c, -2c, c) \in \mathbb{R}^3; c \in \mathbb{R}\}.$$

1. Let us show that E_1 is a vector subspace of $\mathbb{R}^3$:

   (i) $(0,0,0) \in E_1 \Longrightarrow E_1 \neq \emptyset$.

   (ii) $\forall x_1, x_2 \in E_1 \Longrightarrow x_1 + x_2 \in E_1$?
   Let $x_1, x_2 \in E_1 \Longrightarrow x_1 = (a_1 + b_1, b_1 - 3a_1, a_1)$ and $x_2 = (a_2 + b_2, b_2 - 3a_2, a_2)$
   $\Longrightarrow x_1 + x_2 = [(a_1 + a_2) + (b_1 + b_2), (b_1 + b_2) - 3(a_1 + a_2), (a_1 + a_2)] \in E_1$.

   (iii) $\forall x \in E_1, \forall \alpha \in \mathbb{R} \Longrightarrow \alpha x \in E_1$?
   Let $x \in E_1$ and $\alpha \in \mathbb{R} \Longrightarrow \alpha x = (a+b, b-3a, a) \Longrightarrow \alpha x = (\alpha a + \alpha b, \alpha b - 3\alpha a, \alpha a) \in E_1$.

   Conclusion: E_1 is a vector subspace of $\mathbb{R}^3$.

2. Let us determine a basis B_1 of E_1 and a basis B_2 of E_2:

   ▷ $x \in E_1 \Rightarrow x = (a+b, b-3a, a) = a(1,-3,1) + b(1,1,0) \Rightarrow B_1 = \{(1,-3,1); (1,1,0)\}$ generates E_1, but

   $$\alpha(1,-3,1) + \beta(1,1,0) = (0,0,0) \Longrightarrow \alpha = \beta = 0,$$

   then, the two vectors of B_1 are linearly independent. Which implies that $B_1 = \{(1,-3,1); (1,1,0)\}$ is a basis of E_1.

▷ $x \in E_2 \Longrightarrow x = (c, -2c, c) = c(1, -2, 1)$ then $B_2 = \{(1, -2, 1)\}$ generates E_2, but

$$\alpha(1, -2, 1) = (0, 0, 0) \Longrightarrow \alpha = 0.$$

Which implies that $B_2 = \{(1, -2, 1)\}$ is a basis of E_2.

3. Deduce $\dim E_1$ and $\dim E_2$:

$$\dim E_1 = 2 \quad \textit{and} \quad \dim E_2 = 1.$$

4. Let us show that: $\mathbb{R}^3 = E_1 + E_2$:

   * $E_1 \subset \mathbb{R}^3$ and $E_2 \subset \mathbb{R}^3 \Longrightarrow E_1 + E_2 \subset \mathbb{R}^3$.
   * Let $u \in \mathbb{R}^3 \Longrightarrow u = (x, y, z) = (a + b, b - 3a, a) + (c, -2c, c)$

$$\Longrightarrow \begin{cases} x = a + b + c \\ y = b - 3a - 2c \\ z = a + c \end{cases} \Longrightarrow \begin{cases} a = x - y - 3z \\ b = x - z \\ c = -x + y + 4z \end{cases}$$

$$\begin{aligned} \Longrightarrow u &= (x, y, z) = (2x - y - 4z, -2x + 3y + 8z, x - y - 3z) \\ &+ (-x + y + 4z, 2x - 2y - 8z, -x + y + 4z) \\ &\in E_1 + E_2. \end{aligned}$$

   Hence, $\mathbb{R}^3 = E_1 + E_2$.

5. Deduce that $\mathbb{R}^3 = E_1 \oplus E_2$:

   - $\dim E_1 = 2$ and $\dim E_2 = 1$, therefore $\dim E_1 + \dim E_2 = \dim \mathbb{R}^3 = 3$ or we have: $\mathbb{R}^3 = E_1 + E_2$.
   - $E_1 \cap E_2 = \{(0, 0, 0)\}$ because E_1 and E_2 are two vector subspaces. Additionally, if $u \in E_1 \cap E_2 \Longrightarrow u = (a + b, b - 3a, a)$ and $u = (c, -2c, c)$

$$\Longrightarrow \begin{cases} a + b = c \\ b - 3a = -2c \\ a = c \end{cases} \Longrightarrow \begin{cases} a = 0 \\ b = 0 \\ c = 0 \end{cases}$$

$$\Longrightarrow \begin{cases} \mathbb{R}^3 = E_1 + E_2 \\ E_1 \cap E_2 = \{(0, 0, 0)\} \end{cases} \quad Or \quad \begin{cases} \dim E_1 + \dim E_2 = \dim \mathbb{R}^3 \\ E_1 \cap E_2 = \{(0, 0, 0)\} \end{cases}$$

   The sum is direct $\mathbb{R}^3 = E_1 \oplus E_2$.

Exercise 5. *Let*

$$E_1 = \{(a, b, c) \in \mathbb{R}^3; a = c\}, \ \ E_2 = \{(a, b, c) \in \mathbb{R}^3; a + b + c = 0\} \ \textit{and} \ \ E_3 = \{(0, 0, c); c \in \mathbb{R}\}.$$

1. *Show that $E_i; i = 1, 2, 3$ are vector subspaces of $\mathbb{R}^3$.*

2. *Show that* $\mathbb{R}^3 = E_1 + E_2, \mathbb{R}^3 = E_2 + E_3$ *and* $\mathbb{R}^3 = E_1 + E_3$.

3. *In which case the sum is direct?*

Solution:

$E_1 = \{(a, b, c) \in \mathbb{R}^3; a = c\}$, $E_2 = \{(a, b, c) \in \mathbb{R}^3; a+b+c = 0\}$ *and* $E_3 = \{(0, 0, c); c \in \mathbb{R}\}$.

1. Let us show that $E_i; i = 1, 2, 3$ are vector subspaces of $\mathbb{R}^3$: for E_1 by example,

   (i) $(0, 0, 0) \in E_1 \Longrightarrow E_1 \neq \emptyset$.

   (ii) $\forall u_1, u_2 \in E_1 \Longrightarrow u_1 + u_2 \in E_1$?
   Let $u_1, u_2 \in E_1 \Longrightarrow u_1 = (x_1, y_1, x_1)$ and $u_2 = (x_2, y_2, x_2)$
   $\Longrightarrow u_1 + u_2 = (x_1 + x_2, y_1 + y_2, x_1 + x_2 \in E_1$.

   (iii) $\forall u \in E_1, \forall \alpha \in \mathbb{R} \Longrightarrow \alpha u \in E_1$?
   Let $u \in E_1$ and $\alpha \in \mathbb{R} \Longrightarrow \alpha u = (a, b, a) \Longrightarrow \alpha u = (\alpha a, \alpha b, \alpha a) \in E_1$.

   Then, E_1 is a vector subspace of $\mathbb{R}^3$.

   The same applies to the last two cases.

2. Let us show that $\mathbb{R}^3 = E_1 + E_2, \mathbb{R}^3 = E_2 + E_3$ and $\mathbb{R}^3 = E_1 + E_3$:

   ▷ Let us show that $\mathbb{R}^3 = E_1 + E_2$?

   We have: $E_1 \subset \mathbb{R}^3$ and $E_2 \subset \mathbb{R}^3 \Longrightarrow E_1 + E_2 \subset \mathbb{R}^3$.
   If $u \in \mathbb{R}^3 \Longrightarrow u = (x, y, z) = (\alpha, \beta, \alpha) + (\tau, \eta, -\tau - \eta)$

   $$= (\alpha + \tau, \beta + \eta, \alpha - \tau - \eta) \Longrightarrow \begin{cases} x = \alpha + \tau \\ y = \beta + \eta \\ z = \alpha - \tau - \eta \end{cases}$$

   $$\text{Just take for example, } \beta = 0 \Longrightarrow \begin{cases} \alpha = \dfrac{x + y + z}{2} \\ \tau = \dfrac{x - y - z}{2} \\ \eta = y \end{cases}.$$

   Hence,

   $$\begin{aligned} u = (x, y, z) &= \left(\frac{x + y + z}{2}, 0, \frac{x + y + z}{2}\right) \\ &+ \left(\frac{x - y - z}{2}, y, -\frac{x - y - z}{2} - y\right) \in E_1 + E_2. \end{aligned}$$

   ▷ Let us show that $\mathbb{R}^3 = E_2 + E_3$?

   We have: $E_1 \subset \mathbb{R}^3$ and $E_3 \subset \mathbb{R}^3 \Longrightarrow E_1 + E_3 \subset \mathbb{R}^3$.
   If $u \in \mathbb{R}^3 \Longrightarrow u = (x, y, z) = (x, y, x) + (0, 0, z - x) \in E_1 + E_3$.

▷ Let us show that $\mathbb{R}^3 = E_1 + E_3$?

We have: $E_3 \subset \mathbb{R}^3$ and $E_2 \subset \mathbb{R}^3 \Longrightarrow E_2 + E_3 \subset \mathbb{R}^3$.
If $u \in \mathbb{R}^3 \Longrightarrow u = (x, y, z) = (x, y, -x-y) + (0, 0, x+y+z) \in E_2 + E_3$.

Hence, $\mathbb{R}^3 = E_1 + E_2, \mathbb{R}^3 = E_2 + E_3$ and $\mathbb{R}^3 = E_1 + E_3$.

3. In which case is the sum direct?

(i) It is enough to check if we have $E_1 \cap E_2 = \{(0,0,0)\}$?

- $(0,0,0) \in E_1 \cap E_2$.
- If $u \in E_1 \cap E_2 \Rightarrow \begin{cases} u \in E_1 \\ and \\ u \in E_2 \end{cases} \Rightarrow u = \begin{cases} (a,b,a) \\ (a,b,-a-b) \end{cases} \Rightarrow \begin{cases} a = -a-b \\ b = -2a. \end{cases}$

So, for example, $(1,-2,1) \in E_1 \cap E_2 \Rightarrow E_1 \cap E_2 \neq \{(0,0,0)\}$, which implies that the sum is not direct.

(ii) It is enough to check if we have $E_2 \cap E_3 = \{(0,0,0)\}$?

- $(0,0,0) \in E_2 \cap E_3$.
- If $u \in E_2 \cap E_3 \Rightarrow \begin{cases} u \in E_2 \\ and \\ u \in E_3 \end{cases} \Rightarrow u = \begin{cases} (a,b,-a-b) \\ (0,0,c) \end{cases} \Rightarrow a = b = c = 0$

$$\Rightarrow u = (0,0,0) \Rightarrow E_2 \cap E_3 = \{(0,0,0)\},$$

which implies that the sum is not direct.

(iii) It is enough to check if we have $E_1 \cap E_3 = \{(0,0,0)\}$?

- $(0,0,0) \in E_1 \cap E_3$
- If $u \in E_1 \cap E_3 \Rightarrow \begin{cases} u \in E_1 \\ and \\ u \in E_3 \end{cases} \Rightarrow u = \begin{cases} (a,b,a) \\ (0,0,c) \end{cases} \Rightarrow a = b = c = 0$

$$\Rightarrow u = (0,0,0) \Rightarrow E_1 \cap E_3 = \{(0,0,0)\},$$

which implies that the sum is not direct.

6.3 Linear map

Definition 6.52. *Let E and F be two vector spaces in $\mathbb{K}$ and $f : E \longrightarrow F$ is a map. We say that f is a linear map if and only if*

▷ $\forall (x,y) \in E^2, f(x+y) = f(x) + f(y),$

▷ $\forall \alpha \in \mathbb{K}, \forall x \in E, f(\alpha x) = \alpha f(x).$

This is equivalent to saying:

$$\forall \alpha, \beta \in \mathbb{K}, \forall (x, y) \in E^2, \quad f(\alpha x + \beta y) = \alpha f(x) + \beta f(y).$$

We denote by $\mathcal{L}(E, F)$ *the set of linear maps from* E *to* F*.*

Example 6.53.

- *The map* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *defined by:* $f(x) = x + 4$ *is not linear because* $f(x+y) = x + y + 4 \neq f(x) + f(y)$*.*
- *The map from* $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *defined by:*

$$g(x, y) = |x - y|$$

  is not linear because $g(X + Y) \neq g(X) + g(Y)$*.*
- *The map* $h : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ *defined by:*

$$h(x, y, z) = (x - y, y + 2z)$$

  is a linear map because:

  ▷ $\forall (x, y, z), (x', y', z') \in \mathbb{R}^3$*, we have:*

$$\begin{aligned} h[(x, y, z) + (x', y', z')] &= h(x + x', y + y', z + z') \\ &= (x + x' - y - y', y + y' + 2z + 2z') \\ &= (x - y, y + 2z) + (x' - y', y' + 2z') \\ &= h(x, y, z) + h(x', y', z'). \end{aligned}$$

  ▷ $\forall \alpha \in \mathbb{R}$*, we have:*

$$\begin{aligned} h[\alpha(x, y, z)] &= h(\alpha x, \alpha y, \alpha z) = (\alpha x - \alpha y, \alpha y + 2\alpha z) \\ &= \alpha(x - y, y + 2z) = \alpha h(x, y, z). \end{aligned}$$

- *The map* $k : \mathbb{R}^2 \longrightarrow \mathbb{R}^4$ *defined by:*

$$k(x, y) = (x - 2y, 2x, y, y - x)$$

  is a linear map because:

  ▷ $\forall (x, y), (x', y') \in \mathbb{R}^2$*, we have:*

$$\begin{aligned} k[(x, y) + (x', y')] &= k(x + x', y + y') \\ &= (x + x' - 2y - 2y', 2x + 2x', y + y', y + y' - x - x') \\ &= (x - 2y, 2x, y, y - x) + (x' - 2y', 2x', y', y' - x') \\ &= k(x, y) + k(x', y'). \end{aligned}$$

▷ $\forall \alpha \in \mathbb{R}$, *we have:*

$$\begin{aligned} k[\alpha(x,y)] &= k(\alpha x, \alpha y) = (\alpha x - 2\alpha y, 2\alpha x, \alpha y, \alpha y - \alpha x) \\ &= \alpha(x-2y, 2x, y, y-x) = \alpha k(x,y). \end{aligned}$$

Remark 6.54.

* *If f is a linear map, then,* $f(0_E) = 0_F$.
* *If $E = F$, the linear map $f : E \longrightarrow F$ is called* **endomorphism**.
* *If f is bijective and linear from E to F, it is called* **isomorphism**.
* *If f is a bijective endomorphism then it is an* **automorphism**.

6.3.1 Kernel of a linear map

Definition 6.55. *Let E, F be two $\mathbb{K}$-vector spaces and f be a linear map from E to F. Then, to find the kernel of f, we solve the equation $f(x) = 0_F$. So,*

$$\ker f = \{x \in E, f(x) = 0_F\},$$

which is a vector subspace of E.

Example 6.56. *The map f from $\mathbb{R}^3$ to $\mathbb{R}^2$ defined by:*

$$f(x,y,z) = (x-y, y+2z)$$

is a linear application. Then the kernel of f is

$$\ker f = \{u \in \mathbb{R}^3, f(u) = 0_{\mathbb{R}^2}\}.$$

Let $u = (x,y,z) \in \mathbb{R}^3$. We have:

$$u \in \ker f \iff f(u) = (0,0) \iff (x-y, y+2z) = (0,0)$$

$$\iff \begin{cases} x - y = 0 \\ y + 2z = 0 \end{cases} \iff \begin{cases} x = y \\ z = -\dfrac{y}{2} \end{cases} \iff u = y\left(1, 1, -\frac{1}{2}\right),$$

and therefore $\ker f$ *is the vector subspace generated by the vector* $\left(1, 1, -\frac{1}{2}\right)$ *noted,*

$$\ker f = Vect\left\{\left(1, 1, -\frac{1}{2}\right)\right\}.$$

6.3.2 Image of a linear map

Definition 6.57. *Let E, F be two $\mathbb{K}$-vector spaces and f be a linear map from E to F. The image of f is the set of all images of the elements of E by f. So,*

$$\mathfrak{Im} f = \{f(u), u \in E\}.$$

Moreover if $\{e_1, e_2, \cdots, e_n\}$ is a basis of E, then, $\mathfrak{Im} f = Vect\{f(e_1), f(e_2), \cdots, f(e_n)\}$ i.e., the subspace generated by the vectors $f(e_1), f(e_2), \cdots, f(e_n)$.

Example 6.58. *Let E be a $\mathbb{R}$-vector space of dimension 3, $B = \{\vec{i}, \vec{j}, \vec{k}\}$ a basis of E and f the endomorphism of E defined by:*

$$f(\vec{i}) = -\vec{i} + \vec{k}, \quad f(\vec{j}) = \vec{j} + \vec{k}, \quad f(\vec{k}) = \vec{i} + \vec{j}.$$

Then, the image of f is defined as follows:

$$\begin{aligned} \mathfrak{Im} f &= Vect\{f(\vec{i}), f(\vec{j}), f(\vec{k})\} \qquad \textit{but } f(\vec{k}) = f(\vec{j}) - f(\vec{i}) \\ &= Vect\{f(\vec{i}), f(\vec{j})\} \\ &= \{x(-\vec{i} + \vec{k}) + y(\vec{j} + \vec{k}); x, y \in \mathbb{R}\}. \end{aligned}$$

6.3.3 Rank of a linear map

Definition 6.59. *Let E, F be two $\mathbb{K}$-vector spaces and f be a linear map from E to F. The rank of a linear map f is the dimension of the image of this map. We have:*

$$rg(f) = \dim(\mathfrak{Im} f),$$

moreover if E has finite dimension, we have the rank theorem:

$$\dim E = rg(f) + \dim(\ker f).$$

Example 6.60. *Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a linear map defined by: $f(x, y) = (4x - 2y, 6x - 3y)$. Then, we have:*

$$\begin{aligned} \mathfrak{Im} f &= \{f(x, y); x, y \in \mathbb{R}\} = \{(4x - 2y, 6x - 3y); x, y \in \mathbb{R}\} \\ &= \{(2x - y)(2, 3); x, y \in \mathbb{R}\} = \alpha(2, 3); x, y \in \mathbb{R}\} \\ &= Vect\{(2, 3)\}, \end{aligned}$$

the vector $(2, 3)$ is a basis of $\mathfrak{Im} f$ and therefore $rg(f) = 1$.

6.3.4 Injectivity of a linear map

Let E, F be two $\mathbb{K}$-vector spaces and f be a linear map from E to F. Note that f is injective is injective if and only if:

$$\forall x_1, x_2 \in E; x_1 \neq x_2 \Longrightarrow f(x_1) \neq f(x_2), or \quad f(x_1) = f(x_2) \Longrightarrow x_1 = x_2.$$

But for linear maps, it is enough to show that $\ker f = \{0_E\}$. In fact we have:

$$f \text{ is injective} \Longleftrightarrow \ker f = \{0_E\}.$$

Example 6.61. *Let f be a linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$ defined by:*

$$f(x, y) = (x - y, x + y).$$

Then, f is injective because:

$$\begin{aligned} u = (x, y) \in \ker f &\Longleftrightarrow f(u) = 0_{\mathbb{R}^2} \\ &\Longleftrightarrow (x - y, x + y) = (0, 0) \\ &\Longleftrightarrow \begin{cases} x - y = 0 \\ x + y = 0 \end{cases} \Longleftrightarrow x = y = 0. \end{aligned}$$

therefore $\ker f = \{(0, 0)\}$ *and consequently f is injective.*

6.3.5 Projector

Let f be an endomorphism of a $\mathbb{K}$-vector space. We will say that f is a projector, if we have: $f \circ f = f$ or $\mathfrak{Im} f$ and $\ker f$ are supplementary and that $\forall x \in \mathfrak{Im} f, f(x) = x$, we will say that f is the projection on $\mathfrak{Im} f$ parallel to $\ker f$.

Example 6.62. *Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear map defined by:*

$$f(x, y) = (4x - 2y, 6x - 3y).$$

Then, we have:

$$\begin{aligned} (f \circ f)(u) &= f(f(u)) = f(x, y) = f(4x - 2y, 6x - 3y) \\ &= (4x - 2y, 6x - 3y) = f(u), \end{aligned}$$

and therefore f is a projector.

6.3.6 Symmetry

Let f be an endomorphism of a $\mathbb{K}$-vector space. We will say that f is a symmetry, if we have: $f \circ f = Id_E$ or $\ker(f - Id_E)$ and $\ker(f + Id_E)$ are supplementary. We will say that f is the symmetry of E with respect to $\ker(f - Id_E)$ and parallel to $\ker(f + Id_E)$.

Example 6.63. *Let* $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *be the linear map defined by:* $f(x, y) = (y, x)$*. Then, we have:*

$$(f \circ f)(u) = f(f(u)) = f(y, x) = (x, y),$$

and therefore f *is a symmetry.*

Exercise 6. *Let the following map:*

$$\begin{array}{rcl} f : \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ (x, y, z) & \longmapsto & f(x, y, z) = (3x, x + 2y - 2z, 3x - 3y + 3z) \end{array}.$$

1. *Show that* f *is a linear map.*
2. *Determine a basis of* $\ker f$*.*
3. *Calculate* $\dim(\ker f)$ *and* $\dim(\mathfrak{Im} f)$*.*
4. *The map* f *is it injective ?*

Solution: Let the following map:

$$\begin{array}{rcl} f : \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ (x, y, z) & \longmapsto & f(x, y, z) = (3x, x + 2y - 2z, 3x - 3y + 3z) \end{array}.$$

1. Let us show that f is a linear map:

   ▷ $\forall (x, y, z), (x', y', z') \in \mathbb{R}^3$, we have:

$$\begin{array}{rcl} f[(x, y, z) + (x', y', z')] & = & f(x + x', y + y', z + z') \\ & = & (x + x' - y - y', y + y' + z + z') \\ & = & (x - y, y + z) + (x' - y', y' + z') \\ & = & f(x, y, z) + f(x', y', z'). \end{array}$$

   ▷ $\forall \alpha \in \mathbb{R}$, we have:

$$\begin{array}{rcl} f[\alpha(x, y, z)] & = & f(\alpha x, \alpha y, \alpha z) = (\alpha x - \alpha y, \alpha y + \alpha z) \\ & = & \alpha(x - y, y + z) = \alpha f(x, y, z). \end{array}$$

   Then, f is a linear map.

2. Determine a basis of $\ker f$:

$$\ker f = \{(x, y, z) \in \mathbb{R}^3, f(x, y, z) = 0_{\mathbb{R}^3}\}.$$

$$\begin{array}{rcl} f(x, y, z) = (0, 0, 0) & \Longleftrightarrow & \left\{ \begin{array}{l} x - y = 0 \\ y + z = 0 \end{array} \right. \\ & \Longleftrightarrow & \left\{ \begin{array}{l} x = y \\ z = -y, \end{array} \right. \end{array}$$

therefore $(x, y, z) = (y, y, -y) = y(1, 1, -1)$ with $y \in \mathbb{R}$.

Since $(1, 1, -1) \in \ker f$, then, $\{(1, 1, -1)\}$ is a generating part of $\ker f$ and us $(1, 1, -1) \neq 0_{\mathbb{R}^3}$, then, $\{(1, 1, -1)\}$ is free and therefore is a basis of $\ker f$.

3. Calculate $\dim(\ker f)$ and $\dim(\mathfrak{Im} f)$:

   - ▷ $\dim(\ker f) = 1$.
   - ▷ We know that $\dim(\mathbb{R}^3) = \dim(\ker f) + \dim(\mathfrak{Im} f)$, therefore, $\dim(\mathfrak{Im} f) = \dim(\mathbb{R}^3) - \dim(\ker f) = 3 - 1 = 2$.

4. The map f is it injective ?

   f is not injective because $(\ker f) \neq 0_{\mathbb{R}^3}$.

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