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جامعة مصطفى اسطمبولي معسكر

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*Laboratory of Differential Geometry and it’s Applications*

# DOCTORATE THESIS IN MATHEMATICS

**Speciality : Differential Geometry**

**Entitled**

# Solitons de Ricci et quelques déformations
# Ricci Solitons and Certain Deformations

*Presented by:* **Adel DELLOUM**.

**The 20th November, 2024.**

**In the presence of th honorable jury :**

| | | | |
|---|---|---|---|
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# إهداء

إلى أمي وأبي، لما لهما من الفضل ما يبلغ عنان السماء، حفظهما الله وأدامهما نورًا لدربي. بفضل دعمهما وتشجيعهما المستمرين، تمكنت من تجاوز التحديات وتحقيق هذا الإنجاز. أود أن أعبّر عن شكري العميق لهما على تضحياتهما غير المحدودة وحبهما الذي لا يتزعزع.
في الختام، أعرب عن تقديري العميق لجميع الأفراد، سواء مباشرة أو بشكل غير مباشر، الذين ساهموا في إنجاز هذه الأطروحة. كانت إرشاداتهم الجماعية وخبرتهم ودعمهم وروح الأخوة لا تقدر بثمن. كان لي الشرف أن أتعاون مع أساتذة وزملاء وأصدقاء رائعين، الذين لم يبخلوا بوقتهم ومعرفتهم لتوجيه ودعمي خلال هذه الرحلة الأكاديمية.
أود أن أشكر عائلتي و أصدقائي على دعمهم المعنوي والتشجيع المستمر، الذي كان له دور كبير في إكمال هذا المشروع بنجاح. تقديري العميق لكل واحد منهم، فبدون دعمهم ومساهماتهم لما تمكنت من الوصول إلى هذه المرحلة.

# *Dedication*

To my mother and father, whose kindness knows no bounds, may God preserve them and keep them a guiding light on my path. It is through their unwavering support and encouragement that I have overcome challenges and achieved this milestone. I express my deepest gratitude to them for their limitless sacrifices and unshakeable love.
In conclusion, I extend my heartfelt appreciation to all individuals, whether directly or indirectly, who contributed to the completion of this thesis. Their collective guidance, expertise, support, and spirit of camaraderie have been invaluable. It has been an honor to collaborate with exceptional professors, colleagues, and friends who generously shared their time and knowledge to guide and support me throughout this academic journey.
I would like to thank my family and friends for their moral support and continuous encouragement, which played a significant role in successfully completing this project. I deeply appreciate each and every one of them, as without their support and contributions, I would not have reached this stage.

# الشكر والتقدير

أنعم **الله** تعالى على الإنسان بنعم كثيرة لا تعد و لا تحصى من بينها العلم، وكان على الإنسان يشكر ربه على نعمه كي تدوم ويبارك له فيها ويحفظها من الزوال، قال عَزَّ وَجَلَّ في كتابه:

"**وَإِذْ تَأَذَّنَ رَبُّكُمْ لَئِن شَكَرْتُمْ لَأَزِيدَنَّكُمْ ۖ وَلَئِن كَفَرْتُمْ إِنَّ عَذَابِي لَشَدِيدٌ** (7)" سورة إبراهيم-آية 7

فمن داوم على شكر **الله** تعالى على نعمه الذي أنعمها عليه، زاده **الله** عليها وبارك له فيها، فالحمد و الشكر **لله.**
و يمتد الشكر إلى السيد **غريسي بلجيلالي**، الاستاذ الموقر في جامعة مصطفي اسطمبولي بمعسكر، الذي كان تفانيه الثابتة و ثقته الراسخة و توجيهاته البصيرة لا غنى عنها في تحقيق هذا العمل. تحت ارشاده، كانت هذه الرحلة العلمية غنية و ملهمة، مميزة بالاستفسار العلمي الصارم و الكرم الأكاديمي، له جزيل الشكر على الثروة العلمية التي شاركني إياها خلال هذه الفترة.
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كما توجه الأفكار المماثلة للسيد **بلعربي لكحل**، الأستاذ الموقر في جامعة مستغانم، الذي زاد نقده البناء وخبرته إلى حد كبير من جودة البحث.
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# *Acknowledgments*

Praise be to **Allah**, the Most Gracious, the Most Merciful, who has bestowed upon humanity the profound gift of science, illuminating our path towards understanding the intricate workings of His creation. In the Quran, **Allah** declares:

”**Remember that He promised, “If you are thankful, I will give you more, but if you are thankless, My punishment is terrible indeed**” *Surah Ibrahim Ayat 7 (14:7)*

Through the pursuit of knowledge, we uncover the signs of **Allah**'s wisdom and majesty, from the smallest atom to the vastness of the cosmos, all the praise and thanks **to Allah**.

Thanks also extend to Mr. **Gherici Beldjilali**, the esteemed professor at the University of Mustapha Stambouli in Mascara, whose unwavering dedication, steadfast confidence, and insightful guidance were indispensable in achieving this work. Under his mentorship, this academic journey has been enriching and inspiring, marked by rigorous scientific inquiry and academic generosity. My deepest gratitude goes to him for the wealth of knowledge he shared with me during this period. Furthermore, my heartfelt appreciation extends Mr. **BELKHELFA Mohammed**, distinguished Professor and scholar at the University of Mustapha Stamboli of Mascara, whose gracious acceptance of chairing the jury amidst his myriad professional commitments reflects his exemplary character. To him, I offer my utmost respect and sincere gratitude. Similar sentiments are directed to Mr. **BELARBI Lakhal**, esteemed Professor at Mostaganem's university, whose insightful critique and expertise have significantly enhanced the quality of my research. Gratitude also encompasses Mr. Dr. **BOUZIR Habib** at the University of Mustapha Stambouli in Mascara, whose thoughtful examination and valuable insights have enriched this thesis immeasurably. Likewise, appreciation is extended to Mr. **AKIF AKYOL Mehmet**, distinguished Professor at the Bingol University, Turkey, whose diligent evaluation has further elevated the scholarly merit of this work. Finally, I would like to express my gratitude to Mr. Dr. **Si Nasser Moulay** at the University of Mustapha Stambouli in Mascara, for his presence. I thank **Allah** for meeting him, as his kindness has been a source of inspiration and motivation to continue on the path.

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In conclusion, I extend profound gratitude to all individuals, directly or indirectly, who have contributed to the fruition of this thesis. Their collective guidance, expertise, support, and camaraderie have been invaluable assets. It has been an honor to collaborate with such remarkable mentors, colleagues, and friends. My deepest appreciation to all who have played a part in this intellectual odyssey.

# ملخص


هذا العمل تم إنجازه للحصول على درجة الدكتوراه في الرياضيات، في مجال الهندسة التفاضلية في جامعة معسكر.
عنوان الرسالة هو **ريتشي-سوليتون و تحولاته** ويتكون من كلمتين مفتاحيتين مهمتين. الأولى، هي ريتشي-سوليتون، وهذه هي الهياكل الهندسية التي تشكل حلاً ذاتي التشابه لتدفق ريتشي، وقد كانت مجالًا للبحث للعديد من الرياضيين منذ استخدام غريغوري بيرلمان تدفق ريتشي لحل فرضية بوانكاريه. بالتحولات نعني تغييرات المترية، وهذه الإجراءات هي أداة رئيسية لإنشاء أمثلة لهياكل نظرية جديدة في الهندسة، وكذلك دراسة صلابتها.
غرض هذه الرسالة هو التحقيق في تحول ريتشي-سوليتون وصلابته من خلال تحولات مترية مختلفة ضمن إطار الهندسة الريمانية.
نتائجنا المنشورة الأولى تغوص في التحقيق حول التحولات $\mathcal{D}$-تحاكي على فئة من المنوعات الريمانية وكيف يؤثر ذلك على هيكل السوليتون عليها.
النتيجة الثانية تحقق بتفصيل كبير في المنوعات المترية التلامسية تقريبا من الصنف $C_{12}$ و التي سمّاها الاستاذ غ. **بالجيلالي** في أبحاثه "**المنوّعات ذات الرُكن**"، حيث ندرس خصائصها الهندسية وسوليتون عليها بالإضافة إلى تشوهات متعددة مثل التشوهات $\eta$-تحاكي و $\omega$-تحاكي. كما نقدم نتائج حول هيكلها المرتبط بلورينتز $C_{12}$.


# *Abstract*


This work was completed to obtain a Doctorate in Mathematics, specializing in Differential Geometry at the University of Mascara.
The title of the dissertation is **Ricci-Soliton and its Transformations** and it comprises two significant keywords. The first, Ricci-Soliton, refers to the geometric structures that form a self-similar solution to the Ricci flow, a topic that has been extensively researched by mathematicians since **Grigori Perelman** used the Ricci flow to solve the Poincaré conjecture. By transformations, we mean metric changes, which are crucial tools for constructing examples of new theoretical structures in geometry and studying their rigidity.
The purpose of this dissertation is to investigate the transformation and rigidity of Ricci-Solitons through various metric transformations within the framework of Riemannian geometry.
Our first published results delve into the investigation of $\mathcal{D}$-isometric transformations on a class of Riemannian manifolds and how these transformations affect the soliton structure on them.
The second result provides a detailed examination of almost contact metric manifolds of class $C_{12}$, termed "**corner manifolds**" by Professor **Gherici Beldjilali** in his research. We study their geometric properties and solitons, as well as various deformations such as $\eta$-homothetic and $\omega$-homothetic deformations. Additionally, we present results on their structure related to Lorentzian $C_{12}$.

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# Introduction

PART

I

Section 1

## Prelude



**Table 1**. Contents for Part I

One of the earliest contributions to topology was made by **Leonhard Euler** in the $18^{th}$ century. Euler's investigation into the **Königsberg** bridge problem [35] and his development of the Euler characteristic laid the foundational concepts for the field.

In addition, **Bernhard Riemann**'s Figure 6 mid-$19^{\text{th}}$ century research on Riemann surfaces [36], inspired by his mentor **Johann Carl Friedrich Gauss**, greatly advanced the understanding of geometric topology.
Numerous prominent mathematicians have contributed to the study of Riemann surfaces, with a particular focus on the geometric classification of manifolds, which are regarded as natural generalizations of surfaces in higher dimensions. There are two common methods of classification: explicit enumeration and implicit classification using invariants. Manifolds possess a rich set of invariants, including:

| **Point Set Topology** | **Algebraic Topology** | **Geometric Topology** |
|---|---|---|
| Compactness | Homotopy Groups | Fundamental Group |
| Connectedness | Homology | Orientability |
| | Cohomology | Surgery Theory |

**Table 2**. Some Invariants on Manifolds.

**Figure 1**. **Leonhard Euler 1707-1783.**

**Figure 2**. **Johann Carl Friedrich Gauss 1777-1855.**

Low-dimensional manifolds are classified based on their geometric structure, while high-dimensional manifolds are classified algebraically using surgery theory. **Low dimensions** refer to dimensions up to 4. "**High dimensions**" encompass dimensions 5 and higher. The case of dimension 4 is particularly intriguing, as it exhibits low-dimensional behavior smoothly (but not topologically). To date, we recognize a unique connected 0-dimensional manifold, which is the point.
Disconnected 0-dimensional manifolds are merely discrete sets, classified by their cardinality, devoid of any geometric properties, and their study falls under combinatorics.
A connected, compact 1-dimensional manifold without boundary is homeomorphic (or diffeomorphic, if smooth) to the circle. A second countable, non-compact 1-dimensional manifold is homeomorphic or diffeomorphic to the real line.
According to the **uniformization theorem**, every connected, closed 2-dimensional manifold (surface) admits a constant curvature metric. There are three possible curvatures: **positive**, **zero**, and **negative**. Consequently, every surface, or 2-dimensional manifold, can be categorized as either:

$$\begin{cases} \text{A sphere with curvature } 1, \\ \text{A Euclidean plane with curvature } 0, \\ \text{A hyperbolic plane with curvature } -1. \end{cases}$$

Section 2

## 3-Dimensional Case and Poincaré Conjecture

**Figure 3. Henri Poincaré 1854-1912.**

Continuing the classification up to 3-dimensional manifolds, **Henri Poincaré** conjectured in 1904 that spaces which locally resemble ordinary 3-dimensional space but are finite in extent have an intriguing property. Poincaré hypothesized that if such a space is simply connected—meaning each loop in the space can be continuously contracted to a point—then it must be a 3-dimensional sphere. Efforts to resolve this conjecture spurred significant progress in geometric topology throughout the $20^{\text{th}}$ century.

In the 1930s, **J. H. C. Whitehead** [78, 79] claimed to have proved the conjecture but later retracted his claim. During this process, he discovered examples of simply-connected non-compact 3-manifolds not homeomorphic to $\mathbb{R}^3$, the prototype of which is now known as the **Whitehead manifold**. In the 1950s and 1960s, numerous mathematicians attempted proofs of the conjecture, only to uncover flaws in their arguments. Notable figures such as **Georges de Rham, R. H. Bing, Wolfgang Haken, Edwin E. Moise,** and **Christos Papakyriakopoulos** made significant efforts to prove the conjecture. In 1958, **R. H. Bing** established a weakened version of the Poincaré conjecture: if every simple closed curve in a compact 3-manifold lies within a 3-ball, then the manifold is homeomorphic to the 3-sphere [12]. Bing also elucidated some of the challenges in proving the Poincaré conjecture [13].

**Wlodzimierz Jakobsche** demonstrated in 1978 that if the Bing–Borsuk conjecture holds in dimension 3, then the Poincaré conjecture must also hold true [45].

**Figure 4. William Thurston 1946-2012.**

Over time, the conjecture gained renown for its difficulty. **John Milnor** [52] noted that errors in incorrect proofs could be "rather subtle and difficult to detect." Efforts to prove the conjecture advanced the understanding of 3-manifolds. Experts often hesitated to announce proofs and viewed such claims with skepticism. The 1980s and 1990s witnessed several high-profile but ultimately flawed proofs, which were not published in peer-reviewed journals [73].

A comprehensive examination of attempts to prove this conjecture can be found in the accessible book **Poincaré's Prize** by **George Szpiro** [72].

The eventual proof relied on **Richard S. Hamilton**'s approach using the Ricci flow [43]. By developing numerous novel techniques and results in Ricci flow theory, **Grigori Perelman** adapted and completed Hamilton's program. In papers posted to the arXiv repository in 2002 and 2003 [63–65], Perelman presented his work proving the Poincaré conjecture and the more general geometrization conjecture of **William Thurston**. Over subsequent years, numerous mathematicians scrutinized his papers and produced detailed expositions of his work.

Section 3

## A Door Wide Open

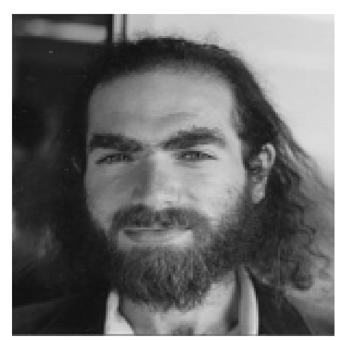

**Figure 5. Grigori Perelmann 1966-.**

After the widespread success of **Grigori Perelman**, many mathematicians turned their attention to investigating the Ricci flow. For example, detailed proofs of the conjecture can be found in [26]. Another important concept to study is the Ricci soliton, which represents a self-similar solution to the Ricci flow. Over the past two decades, numerous mathematicians have investigated Ricci solitons on manifolds.

**Sharma** [69] initiated the investigation of Ricci solitons in contact Riemannian geometry. **Ghosh and Sharma** [41, 61] derived results by considering K-contact, Kenmotsu, Sasakian, and $(\kappa, \mu)$-contact metrics as Ricci solitons. **Bejan and Crasmare-**

**anu** also contributed significant results in this field. In [5], they extended the study of

Ricci solitons to paracontact manifolds. **De** and collaborators [30] explored Ricci solitons in $f$-Kenmotsu manifolds, analyzing the behavior of generalized Sasakian space form and generalized $(\kappa, \mu)$ space form under generalized $\mathcal{D}$-conformal deformation. Several researchers, including **Nagaraja** and **Premalatha** [55], **De** and **Ghosh** [28], and **Shaikh et al.** [68], examined the behavior of normal almost contact metric, $(\kappa, \mu)$ contact metric, and trans-Sasakian manifolds under $\mathcal{D}$-homothetic deformations. We utilize the constancy of specific contact structures under generalized $\mathcal{D}$-conformal and $\mathcal{D}$-homothetic deformations to investigate Ricci solitons.
Recently, $C_{12}$-manifolds have emerged as a prominent and extensively studied topic in differential geometry. Recent works of **Beldjilali**, **Bouzir** and **Bayour** [6, 8, 16, 27] provide a comprehensive overview of the results obtained in this context.
What sets $C_{12}$-manifolds apart from other almost contact metric structures is their unique nature; they are neither contact nor normal. They exhibit significant characteristics to well-known manifolds such as Sasaki, Kenmotsu, and cosymplectic manifolds. Thus, it is imperative to explore various concepts studied on these well-known manifolds on $C_{12}$-manifolds and compare the resulting insights.
The study conducted by the authors in [6] delves into Ricci solitons and generalized Ricci solitons on 3-dimensional $C_{12}$-manifolds. Notably, they establish that any 3-dimensional $C_{12}$-manifold meeting specific conditions conforms to the generalized Ricci soliton equation.

Section 4

# Our Part

This research endeavor aims to achieve a Ph.D. degree in the field of mathematics, specifically focusing on differential geometry, at the University of Mascara.
The thesis, titled **Ricci-Soliton and Deformations**, encompasses two pivotal themes. The term *Ricci-soliton* denotes a geometric structure that constitutes a self-similar solution to the Ricci flow, a subject that has garnered significant attention among mathematicians, particularly since Grigory Perelman employed Ricci flow to resolve the Poincaré conjecture. On the other hand, *deformations* refer to alterations in metrics, serving as a primary tool for generating instances of novel theoretical constructs in geometry and examining their rigidity.
The primary objective of this thesis is to scrutinize the transformation of Ricci solitons and their rigidity through diverse metric deformations within the framework of Riemannian geometry.
Our inaugural publication, as referenced in [31], delves into the investigation of $\mathcal{D}$-isometric deformations on compact gradient Riemannian manifolds and their implications on the soliton structure therein.
A subsequent study extensively explores a class of almost contact metric manifolds denoted as $C_{12}$, analyzing its geometric properties, solitons, and various deformations such as $\eta$-conformal and $\omega$-conformal deformations. Additionally, findings are presented on its associated Lorentz-$C_{12}$ structure.
**In summary, the following inquiries have been addressed:**

1. What are the necessary conditions for preserving a $C_{12}$ structure under generalized $\mathcal{D}$-conformal deformation, contingent upon deformation functions?
2. Is it plausible for a $C_{12}$ structure to exhibit a trivial Ricci soliton structure?
3. Under what circumstances does a compact gradient Riemannian manifold, admitting a Ricci soliton, maintain this attribute under $\mathcal{D}$-isometric deformation?

We have also combined the concepts of generalized Ricci-soliton and Ricci-Yamabe soliton and classified this new structure on 3-dimensional Lie groups.

# Riemannian Geometry

PART

II

In this part, we explore the domain of Riemannian geometry, revisiting fundamental concepts essential for our subsequent discussions. We review indispensable constructs such as the metric tensor, affine connections, derivatives, and curvatures, laying the groundwork for the topics ahead.



**Table 3**. Contents for Part II

**Figure 6**. **Georg Friedrich Bernhard Riemann 1826-1866.**

Section 5

## Metric Tensor

The metric tensor is a cornerstone of Riemannian geometry, defining the notion of distance, angles, and curvature on a manifold. It provides a way to measure lengths and angles between tangent vectors at each point of the manifold, thus endowing the space with a notion of geometry.

**Definition 1** A metric $g$ on a smooth manifold $M$ is a tensor of type $(0,2)$ that assigns a smooth function to pairs of tangent vectors at each point on the manifold

$$g : \Gamma(TM) \times \Gamma(TM) \longrightarrow C^\infty(M),$$

where $\Gamma(TM) \times \Gamma(TM)$ is the Cartesian product of the space of smooth sections of $TM$ with itself, and $C^\infty(M)$ is the space of smooth real-valued functions on $M$.
The metric tensor $g$ exhibits the ensuing properties, for any vector fields $X, Y, Z$ and a function $f$:

1. $C^\infty(M)$**-Bilinearity**:

$$\begin{cases} g(X+Y,Z) = g(X,Z) + g(Y,Z), \\ g(X,Y+Z) = g(X,Y) + g(X,Z), \\ g(fX,Y) = g(X,fY) = fg(X,Y). \end{cases}$$

2. **Symmetry**: $g(X,Y) = g(Y,X)$.
3. **Non-degenerancy**: if $g(X,Y) = 0$, for all vector fields $X$, then $Y = 0$.
4. **Definite Positive**: for any non-zero vector field $X$, $g(X,X) > 0$.

The existence of the metric tensor on smooth manifolds is guaranteed by the **Unit Partition Theorem**, see [75]. This theorem ensures that any smooth manifold can be covered by a set of coordinate charts such that the transition maps between overlapping charts are smooth.
Consequently, the metric tensor can be defined locally on each chart and smoothly patched together across the manifold. This essential property underpins the foundational framework of Riemannian geometry, allowing for the rigorous study of distances, angles, and curvature on curved spaces. For more on this, see [34, 49].
In local system of coordinates, we can write

$$g = g_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j, \quad i,j \in \{1,2,...,n\},$$

where $n = \dim(M)$ and $g_{ij}$ represent differentiable functions, referred to as the components of the metric tensor relative to a local chart.
Likewise, if $X = X^m \partial_m$ and $Y = Y^l \partial_l$, then

$$\begin{aligned} g(X,Y) &= g_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j \left(X^m \partial_m, Y^l \partial_l\right) = g_{ij} X^m Y^l \mathrm{d}x^i(\partial_m) \mathrm{d}x^j(\partial_l) = g_{ij} X^m Y^l \delta_{im} \delta_{jl} \\ &= g_{ij} X^i Y^j. \end{aligned}$$

Hence, the metric tensor $g$ provides a means to compute the length of a vector field $X$ defined on $M$ via the subsequent expression

$$|X|^2 = g(X,X). \tag{5.1}$$

Thus, it is possible to express any vector field $X \in \Gamma(TM)$ utilizing the metric $g$ through a linear combination of $X$'s projection onto an orthonormal frame $\{e_i\}_{1 \le i \le n}$ as follows

$$X = \sum_{i=1}^{i=n} g(X, e_i) e_i. \tag{5.2}$$

**Definition 2** A Riemannian manifold is a pair $(M, g)$, where $M$ is a differentiable manifold, and $g$ is a Riemannian metric on the tangent bundle $TM$.

According to Definition 2, every differentiabl manifold M possesses at least one Riemannian metric $g$ . Nevertheless, this metric is not unique.

Below are illustrations of several Riemannian manifolds accompanied by their respective metric tensors:

**Example 5.1.** The parametrization of a sphere of radius $R$ Figure 7 is given by

$$\begin{cases} x(\theta,\phi) = R\sin\theta\cos\phi, \\ y(\theta,\phi) = R\sin\theta\sin\phi, \\ z(\theta,\phi) = R\cos\theta, \end{cases}$$

where $0 \le \theta \le \pi$ is the polar angle and $0 \le \phi < 2\pi$ is the azimuthal angle.
Calculate the differentials of the parametrization

$$\begin{cases} \mathrm{d}x = R\cos\theta\cos\phi\,\mathrm{d}\theta - R\sin\theta\sin\phi\,\mathrm{d}\phi, \\ \mathrm{d}y = R\cos\theta\sin\phi\,\mathrm{d}\theta + R\sin\theta\cos\phi\,\mathrm{d}\phi, \\ \mathrm{d}z = -R\sin\theta\,\mathrm{d}\theta. \end{cases}$$

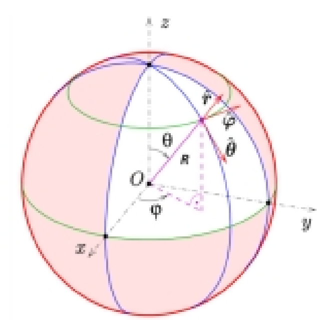


**Figure 7. Unit Vectors in Spherical Coordinates.**

Next, we need find the line element $ds^2$ on the sphere

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \\ &= R^2\left(\cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta\right)\mathrm{d}\theta^2 + R^2\left(\sin^2\theta\sin^2\phi + \sin^2\theta\cos^2\phi)\right)\mathrm{d}\phi^2. \end{aligned}$$

And so the metric of a 2-sphere is

$$\mathrm{d}s^2 = R^2\mathrm{d}\theta^2 + R^2\sin^2\theta\mathrm{d}\phi^2. \tag{5.3}$$

Write down the components of the metric tensor $g_{ij}$ using the line element $ds^2$ via the formula

$$\mathrm{d}s^2 = g_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j.$$

To obtain

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{pmatrix}.$$

**Example 5.2.** The parametric equations for a torus with major radius $R$ and minor radius $r$ are

$$\begin{cases} x(\theta,\phi) = (R + r\cos\phi)\cos\theta, \\ y(\theta,\phi) = (R + r\cos\phi)\sin\theta, \\ z(\theta,\phi) = r\sin\phi. \end{cases}$$

The differentials of the parameters are

$$\begin{cases} \mathrm{d}x = -(R + r\cos\phi)\sin\theta\mathrm{d}\theta - r\sin\phi\cos\theta\mathrm{d}\phi, \\ \mathrm{d}y = (R + r\cos\phi)\cos\theta\mathrm{d}\theta - r\sin\phi\sin\theta\mathrm{d}\phi, \\ \mathrm{d}z = r\cos\phi\mathrm{d}\phi. \end{cases}$$

The metric tensor is thus given by the formula

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \\ &= \Big((R + r\cos\phi)^2\sin^2\theta + (R + r\cos\phi)^2\cos^2\theta\Big)\mathrm{d}\theta^2 + \Big(r^2\sin^2\phi\cos^2\theta + r^2\sin^2\phi\sin^2\theta + r^2\cos^2\phi\Big)\mathrm{d}\phi^2. \end{aligned}$$

This yields, subsequent to simplifications

$$\mathrm{d}s^2 = (R + r\cos\phi)^2\mathrm{d}\theta^2 + r^2\mathrm{d}\phi^2. \tag{5.4}$$

The metric tensor in matrix form is

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} (R + r\cos\phi)^2 & 0 \\ 0 & r^2 \end{pmatrix}.$$

Subsection 5.3

## Musical Isomorphism ♯ and ♭

The metric tensor $g$ allows us to establish a bijective correspondence between 1-forms and vector fields on the Riemannian manifold $M$. The map sharp, denoted as $\sharp$

$$\begin{aligned} \sharp : \Gamma(TM^*) &\longrightarrow \Gamma(TM) \\ \omega &\to \sharp\omega \end{aligned}$$

defined by

$$\boxed{g(\sharp\omega, X) = \omega(X),} \tag{5.5}$$

where $X$ is a vector field on $M$ and $\Gamma(TM^*)$ denotes the dual bundle of the tangent bundle $\Gamma(TM)$ of the manifold $M$.
Locally, if $\omega = \omega_i \mathrm{d}x^i$ and $X = X^j \partial_j$, then with the help of (5.2), we obtain

$$\sharp\omega = \omega_i g^{ij} \partial_j,$$

in this context, $g^{ij}$ signifies the constituent elements of the inverse metric tensor of $g$.
Conversely, the flat operation, symbolized by $\flat$, constitutes a linear mapping that converts vectors into covectors (one-forms), represented by

$$\begin{gathered} \flat : \Gamma(TM) \longrightarrow \Gamma(TM^*) \\ X \to X^\flat \end{gathered}$$

this operation is characterized by the equation

$$\boxed{X^\flat(Y) = g(X, Y),} \tag{5.6}$$

where $X$ and $Y$ denote vector fields defined on $M$.
The maps $\sharp$ and $\flat$ play significant roles in differential geometry, often termed as the **musical isomorphism** or **canonical isomorphism**. These mappings, commonly known for their application in Ricci calculus as **raising and lowering indices**, facilitate various computations and transformations.
For more comprehensive insights into these concepts, interested readers may refer to authoritative texts such as [49, 76], which delve deeper into the theoretical underpinnings and practical applications of these isomorphisms.

Section 6

# Affine Connection

**Figure 8. Professor Élie Joseph Cartan 1869-1951.**

Affine connections are fundamental mathematical structures in differential geometry that generalize the notion of partial differentiation of vector fields on smooth manifolds. They provide a systematic way to differentiate vector fields along curves and are essential tools in studying curvature and geometric properties of manifolds. Upon considering a manifold, it becomes evident that various concepts can be discussed upon its definition. These include the establishment of functions, differentiation, examination of parameterized paths, introduction of tensors, among others. However, certain notions such as the volume of a region or the length of a path necessitate an additional structural element, specifically the introduction of a metric, as discussed earlier in Part II. It might seem intuitive to associate the concept of **curvature**, previously employed informally, solely with the metric. However, this presumption is either inaccurate or incomplete. Indeed, an additional structural element, a **connection** is required. We will elucidate how the presence of a metric gives rise to a particular connection, the curvature of which can be likened to that of the metric.

The introduction of a connection becomes imperative when addressing the inadequacy of the partial derivative as a tensor operator. What is desired is a covariant derivative, an operator that behaves as a partial derivative in a flat space with Cartesian coordinates, yet transforms appropriately as a tensor on any given manifold.

**Definition 3** An affine connection $\nabla$ on a manifold $M$, is a map

$$\begin{aligned}\nabla : \Gamma(TM) \times \Gamma(TM) &\longrightarrow \Gamma(TM)\\ (X, Y) &\longmapsto \nabla_X Y\end{aligned}$$

Satisfying

$$\begin{cases} 1. \quad \nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z,\\ 2. \quad \nabla_X(fY) = X(f)Y + f\nabla_X Y,\\ 3. \quad \nabla_{fX+Y} Z = f\nabla_X Z + \nabla_Y Z,\end{cases}$$

herein, $X, Y, Z$ denote vector fields defined on the manifold $M$, while $f$ represents a smooth function defined on $M$.

**Figure 9**. **Hermann Klaus Hugo Weyl 1885-1955.**

The roots of the affine connection lie in 19th-century geometry and tensor calculus. **Élie Cartan** Figure 8 and **Hermann Weyl** Figure 9 [17–21, 77] played pivotal roles in its early 1920s development.

Cartan introduced the terminology, emphasizing the connection between tangent spaces via translation. The key idea is that an affine connection makes a manifold look infinitesimally like Euclidean space, not just smoothly, but as an affine space. Importantly, there are infinitely many affine connections on any positive-dimensional manifold. in the case of the sphere, an affine connection facilitates the transition of the affine tangent plane from one location to another. This transition engenders the movement of a point of contact along a trajectory within the plane, thereby delineating a curve.

A vector field $V$ on $M$ is deemed parallel with respect to the connection $\nabla$ if, and only if, for all vector fields $X$ on $M$, it satisfies the condition

$$\boxed{\nabla_X V = 0.} \tag{6.1}$$

In a more refined formulation, we write $\nabla V = 0$. Similarly, a metric $g$ is considered compatible with $\nabla$ (or parallel), if the condition

$$\nabla g = 0,$$

holds, that is, where for any vector fields $X, Y$ and $Z$ on $M$, we have

$$X\big(g(Y,Z)\big) = (\nabla_X g)(Y,Z) + g(\nabla_X Y, Z) + g(Y, \nabla_X Z),$$

which turns out to

$$\boxed{X\big(g(Y,Z)\big) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).} \tag{6.2}$$

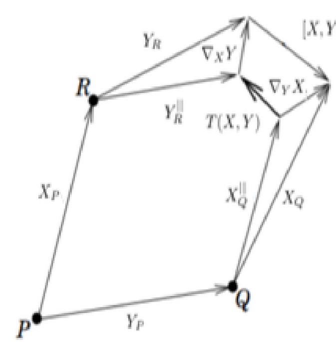


**Figure 10**. **Geometric Ineterpretation of Lie Brackaets and Torsion as Closure Failure.**

**Definition 4** The torsion $T$ associated with linear connection $\nabla$, on a manifolds $M$, is a tensor of type (1.2), is defined by

$$\begin{aligned} T : \Gamma(TM) \times \Gamma(TM) &\longrightarrow \Gamma(TM)\\ (X, Y) &\longmapsto T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]\end{aligned}$$

for all vector fields $X, Y$ on $M$

A refined exploration into the geometric interpretation of torsion Figure 10 entails the consideration of two vector fields denoted as $X$ and $Y$. At a given point $P$, the parallel transport of these fields along $Y$ and $X$, respectively, results in transformed fields $X_Q^{||}$ and $Y_R^{||}$. Should a torsion exist within the geometry, the closure of these transported

fields fails to occur, manifesting as a discernible **closure failure** $T(X,Y)$. A comprehensive elucidation of this phenomenon is expounded and specifically detailed in [67].
If $T$ vanishes identically then the linear connection $\nabla$ is torsion free, that is

$$\boxed{[X,Y] = \nabla_X Y - \nabla_Y X} \tag{6.3}$$

where

$$[\bullet,\bullet] : \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM)$$

is skew-symmetric tensor of type $(1,2)$ called the **Lie Bracket**, which verify the following properties

$$\begin{cases} 1. \quad \text{Antisymmetry} : [X,Y] = -[Y,X], \\ 2. \quad \mathbb{R}-\text{Linearity} : [aX+bY,Z] = a[X,Z]+b[Y,Z], \quad a,b \in \mathbb{R}, \\ 3. \quad \text{Jacobi Identity} : [X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]] = 0. \end{cases} \tag{6.4}$$

Subsection 6.1

## Levi-Cevita Connection

One of the cornerstone theorems in Riemannian geometry is the Levi-Civita theorem, unveiled by the Italian mathematician **Tullio Levi-Civita** Figure 11, which establishes the following fundamental principle:

**Figure 11. Tullio Levi-Civita 1873-1941.**

**Theorem 1** Let $(M,g)$ be a Riemannian manifold. Then there exists a unique torsion-free, affine connection $\nabla$, known as the **Levi-Civita** connection, which is compatible with the metric $g$.

The theorem above produces **Kozsul's formula**, derived by the French mathematician **Jean-Louis Koszul** Figure 12, which is used to compute explicitly the Levi-Cevita connection

$$\boxed{\begin{aligned} 2g(\nabla_X Y,Z) &= X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y)) \\ &\quad + g([X,Y],Z) + g([Z,X],Y) - g([Y,Z],X). \end{aligned}} \tag{6.5}$$

**Figure 12. Jean-Louis Koszul 1921-2018.**

Certainly, by leveraging the characteristics of the metric tensor $g$ in conjunction with the torsion-free and compatibility properties of $\nabla$, we establish the following

$$\begin{cases} X(g(Y,Z)) = g(\nabla_X Y,Z) + g(Y,\nabla_X Z), \\ Y(g(X,Z)) = g(\nabla_Y X,Z) + g(X,\nabla_Y Z), \\ Z(g(X,Y)) = g(\nabla_Z X,Y) + g(X,\nabla_Z Y). \end{cases}$$

On the other hand

$$\begin{cases} g([X,Y],Z) = g(\nabla_X Y,Z) - g(\nabla_Y X,Z), \\ g([Z,Y],X) = g(\nabla_Z Y,X) - g(\nabla_Y Z,X), \\ g([Y,Z],X) = g(\nabla_Y Z,X) - g(\nabla_Z Y,X). \end{cases}$$

Formula (6.5) is verified by direct substitution.
Locally, if one takes $X = X^i\partial_i$ and $Y = Y^j\partial_j$, then

$$\nabla_X Y = \nabla_{X^i\partial_i} Y^j\partial_j = X^i\big(\partial_i Y^k + \Gamma^k_{ij} Y^j\big)\partial_k,$$

**Figure 13**. **Elwin Bruno Christoffel 1829-1900.**

where

$$\boxed{\Gamma^k_{ij}\partial_k = \nabla_{\partial_i}\partial_j,} \tag{6.6}$$

are the **Christoffel symbols** introduced by the German mathematician **Elwin Christoffel** Figure 13. Since the connection is torsion free, the coefficients are symmetric

$$\Gamma^k_{ij} = \Gamma^k_{ji}.$$

For this reason, a torsion-free connection is often called **symmetric**. The **Christoffel symbols** can be derived from the metric tensor $g$ using the following formula

$$\boxed{\Gamma^k_{ij} = \frac{1}{2}g^{kl}\big(\partial_i g_{lj} + \partial_j g_{il} - \partial_l g_{ij}\big).} \tag{6.7}$$

Geometrically, Christoffel symbols can be interpreted as describing how basis vectors change throughout a given coordinate system (6.6). The basis vectors may change due to the coordinate system being curvilinear or due to the geometry of the space itself being curved, and the Christoffel symbols describe both of these. Physically, Christoffel symbols can be interpreted as describing fictitious forces arising from a non-inertial reference frame. In general relativity, Christoffel symbols represent gravitational forces as they describe how the gravitational potential (metric) varies throughout spacetime causing objects to accelerate. See [53].

**Example 6.2.** For the 2-Sphere Example 5.1, we compute the inverse metric $g^{-1}$

$$g = \begin{pmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi\theta} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{pmatrix} \implies g^{-1} = \begin{pmatrix} g^{\theta\theta} & g^{\theta\phi} \\ g^{\phi\theta} & g^{\phi\phi} \end{pmatrix} = \begin{pmatrix} \frac{1}{R^2} & 0 \\ 0 & \frac{1}{R^2\sin^2\theta} \end{pmatrix}. \tag{6.8}$$

According to (6.7), we know that we have to find the eight following symbols

$$\Gamma^\theta_{ij} = \begin{pmatrix} \Gamma^\theta_{\theta\theta} & \Gamma^\theta_{\theta\phi} \\ \Gamma^\theta_{\phi\theta} & \Gamma^\theta_{\phi\phi} \end{pmatrix} \quad \text{and} \quad \Gamma^\phi_{ij} = \begin{pmatrix} \Gamma^\phi_{\theta\theta} & \Gamma^\phi_{\theta\phi} \\ \Gamma^\phi_{\phi\theta} & \Gamma^\phi_{\phi\phi} \end{pmatrix}.$$

Let's start by calculating $\Gamma^\theta_{ij}$. By (6.7), one can write

$$\Gamma^\theta_{ij} = \frac{1}{2}g^{\theta l}\big(\partial_i g_{lj} + \partial_j g_{il} - \partial_l g_{ij}\big). \tag{6.9}$$

Since $g^{\theta\phi} = 0$, $l = \theta$ and formula (6.9) becomes

$$\Gamma^\theta_{ij} = \frac{1}{2}g^{\theta\theta}\big(\partial_i g_{\theta j} + \partial_j g_{i\theta} - \partial_\theta g_{ij}\big). \tag{6.10}$$

Hence, the four first symbols are now easy to deduce

$$\Gamma^\theta_{\theta\theta} = \Gamma^\theta_{\theta\phi} = \Gamma^\theta_{\phi\theta} = 0 \quad \text{and} \quad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta.$$

Simularly for $k = \phi$ in (6.7), we have

$$\Gamma^\phi_{ij} = \frac{1}{2}g^{\phi l}\big(\partial_i g_{lj} + \partial_j g_{il} - \partial_l g_{ij}\big). \tag{6.11}$$

From (6.8) $g^{\phi\theta} = 0$ then $l = \phi$ and (6.11) becomes

$$\Gamma^{\phi}_{ij} = \frac{1}{2} g^{\phi\phi}\big(\partial_i g_{\phi j} + \partial_j g_{i\phi} - \partial_\phi g_{ij}\big). \tag{6.12}$$

From there, we can easily deduce the last four connection coefficients

$$\Gamma^{\phi}_{\theta\theta} = \Gamma^{\phi}_{\phi\phi} = 0 \quad \text{and} \quad \Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \cot\theta.$$

So finally for a surface of a sphere, the eight Christoffel symbols are equal to

$$\Gamma^{\theta}_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & \sin\theta\cos\theta \end{pmatrix} \quad \text{and} \quad \Gamma^{\phi}_{ij} = \begin{pmatrix} 0 & \cot\theta \\ \cot\theta & 0 \end{pmatrix}.$$

**Example 6.3.** For the 2-Torus Example 5.2, we compute the inverse metric $g^{-1}$

$$g = \begin{pmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi\theta} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} (R + r\cos\phi)^2 & 0 \\ 0 & r^2 \end{pmatrix} \implies g^{-1} = \begin{pmatrix} g^{\theta\theta} & g^{\theta\phi} \\ g^{\phi\theta} & g^{\phi\phi} \end{pmatrix} = \begin{pmatrix} \frac{1}{(R+r\cos\phi)^2} & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix}. \tag{6.13}$$

For $k = \theta$, notice that from (6.13) we have $g^{\theta\phi} = g^{\phi\theta} = 0$, hence $l = \theta$, in this case we write (6.7)

$$\Gamma^{\theta}_{ij} = \frac{1}{2} g^{\theta\theta}\big(\partial_i g_{\theta j} + \partial_j g_{i\theta} - \partial_\theta g_{ij}\big). \tag{6.14}$$

Thus, the four symboles with upper index $\theta$ are

$$\Gamma^{\theta}_{\theta\theta} = \Gamma^{\theta}_{\phi\phi} = 0 \quad \text{and} \quad \Gamma^{\theta}_{\theta\phi} = \Gamma^{\theta}_{\phi\theta} = -\frac{r\sin\varphi}{R + r\cos\varphi}.$$

On the other hand, if $k = \phi$, again according to (6.13) $l = \phi$ and

$$\Gamma^{\phi}_{ij} = \frac{1}{2} g^{\phi\phi}\big(\partial_i g_{\phi j} + \partial_j g_{i\phi} - \partial_\phi g_{ij}\big). \tag{6.15}$$

Therefor, the symbols $\Gamma^{\phi}_{ij}$ are

$$\Gamma^{\phi}_{\theta\theta} = \frac{R}{r}\sin\theta + \sin\theta\cos\theta \quad \text{and} \quad \Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \Gamma^{\phi}_{\phi\phi} = 0.$$

In conclusion, the Christoffel symboles on a torus are

$$\Gamma^{\theta}_{ij} = \begin{pmatrix} 0 & -\frac{r\sin\varphi}{R+r\cos\varphi} \\ -\frac{r\sin\varphi}{R+r\cos\varphi} & 0 \end{pmatrix} \quad \text{and} \quad \Gamma^{\phi}_{ij} = \begin{pmatrix} \frac{R}{r}\sin\theta + \sin\theta\cos\theta & 0 \\ 0 & 0 \end{pmatrix}.$$

Subsection 6.4

## The exterior derivative

On a differentiable manifold, the exterior derivative generalizes the notion of the differential of a function to encompass differential forms of greater degree. This concept, articulated in its present form by **Élie Cartan** Figure 8 in 1899, constitutes the foundation of exterior calculus.

**Definition 5** The exterior derivative is defined to be the unique $\mathbb{R}$-linear mapping from $k$-forms to $(k+1)$-forms

$$\mathrm{d} : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$$

that has the following properties:

1. $\mathrm{d}f$ is the differential of $f$ for a 0-form $f$,
2. $\mathrm{d}(\mathrm{d}f) = 0$ for a 0-form $f$,
3. $\mathrm{d}(\omega_1 \wedge \omega_2) = \mathrm{d}\omega_1 \wedge \omega_2 + (-1)^p(\omega_1 \wedge \mathrm{d}\omega_2)$,

where $\omega_1$ is a $p$-form and $\omega_2$ is a $q$-form. A 1-form $\omega$ is closed if $\mathrm{d}\omega$=0, and is said to be exact if $\omega = \mathrm{d}f$. Consequently, every exact 1-form is closed.

The second defining characteristic holds with broader applicability: specifically, it asserts that for any $k$-form $\omega_1$, the exterior derivative of the exterior derivative of $\omega_1$ yields zero, succinctly expressed as $d^2 = 0$. Moreover, the third defining property entails, as a specific instance, that when $f$ represents a function and $\omega_1$ denotes a $k$-form, the exterior derivative of the product of $f$ and $\omega_1$, denoted as $d(f\omega_1)$, equals $df \wedge \omega_1 + f \wedge d\omega_1$, owing to the fundamental properties of scalar multiplication and the exterior product, especially notable when one of the operands reduces to a scalar. Alternatively, an explicit formula can be given [70] for the exterior derivative of a $k$-form $\omega$, when paired with $k+1$ arbitrary smooth vector fields $X_0, X_1, ..., X_k$

$$\begin{aligned}\mathrm{d}\omega(X_0, ..., X_k) = &\sum_{i=0}^{i=k}(-1)^i X_i\big(\omega(X_0, ..., \hat{X}_i, ..., X_k)\big)\\ &+ \sum_{i<j}(-1)^{(i+j)}\omega([X_i, X_j], X_0, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_k),\end{aligned}$$

where

$$\omega(X_0, ..., \hat{X}_i, ..., X_k) = \omega(X_0, ..., X_{i-1}, X_{i+1}, ..., X_k).$$

Taking into account the convention of **Kobayashi–Nomizu** and **Helgason**, the formula differs by a factor of $\frac{1}{k+1}$

$$\begin{aligned}\mathrm{d}\omega(X_0, ..., X_k) = &\frac{1}{k+1}\sum_{i=0}^{i=k}(-1)^i X_i\big(\omega(X_0, ..., \hat{X}_i, ..., X_k)\big)\\ &+ \frac{1}{k+1}\sum_{i<j}(-1)^{(i+j)}\omega([X_i, X_j], X_0, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_k).\end{aligned} \tag{6.16}$$

Applying formula (6.4) to a 1-form $\omega$, we have

$$\begin{aligned}\mathrm{d}\omega(X, Y) &= \frac{1}{2}\Big(X\big(\omega(Y)\big) - Y\big(\omega(X)\big) - \omega([X, Y])\Big)\\ &= \frac{1}{2}\Big((\nabla_X\omega)Y + \omega(\nabla_X Y) - (\nabla_Y\omega)X - \omega(\nabla_Y X) - \omega(\nabla_X Y) + \omega(\nabla_Y X)\Big).\end{aligned}$$

Thus

$$\mathrm{d}\omega(X, Y) = \frac{1}{2}\Big((\nabla_X\omega)Y - (\nabla_Y\omega)X\Big), \tag{6.17}$$

where, If $\omega = V^\flat$, then we have

$$(\nabla_X \omega)Y = g(\nabla_X V, Y), \tag{6.18}$$

and for a 2-form $\Phi$

$$3\mathrm{d}\Phi(X,Y,Z) = \Big( X\big(\Phi(Y,Z)\big) - Y\big(\Phi(X,Z)\big) + Z\big(\Phi(X,Y)\big) - \Phi([X,Y],Z) + \Phi([X,Z],Y) - \Phi([Y,Z],X) \Big). \tag{6.19}$$

Subsection 6.5

## Lie Derivative

The Lie derivative, attributed to **Sophus Lie** and further expounded by **Władysław Ślebodziński**, assesses the alteration undergone by a tensor field comprising scalar functions, vector fields, and one-forms along the trajectory defined by another vector field. This transformation remains invariant under changes of coordinates, rendering the Lie derivative applicable across any differentiable manifold. An in depth review of Lie derivative can be found in [74, 81]. If $T$ represents a tensor field and $X$ a vector field, the Lie derivative of $T$ with respect to $X$ is symbolized as $\mathcal{L}_X T$. However, we shall abstain from delving into specific intricacies, opting instead to provide a general methodology for Lie derivative computation. The algebraic definition for the Lie derivative of a tensor field arises from the following four axioms:

**Axiom 1**: The Lie derivative of a function equals the directional derivative of the function, often represented by the formula

**Figure 14. Marius Sophus Lie 1842-1899.**

$$\mathcal{L}_Y f = Y(f). \tag{6.20}$$

**Axiom 2**: The Lie derivative adheres to a version of Leibniz's rule, stating that for any tensor fields $S$ and $T$, we have

**Figure 15. Władysław Ślebodziński 1884-1972.**

$$\mathcal{L}_Y(S \otimes T) = (\mathcal{L}_Y S) \otimes T + S \otimes (\mathcal{L}_Y T). \tag{6.21}$$

**Axiom 3**: The Lie derivative follows the Leibniz rule with respect to contraction, expressed as

$$\mathcal{L}_X(T(Y_1,\ldots,Y_n)) = (\mathcal{L}_X T)(Y_1,\ldots,Y_n) + T((\mathcal{L}_X Y_1),\ldots,Y_n) + \cdots + T(Y_1,\ldots,(\mathcal{L}_X Y_n)). \tag{6.22}$$

**Axiom 4**: The Lie derivative commutes with the exterior derivative on functions, denoted as

$$[\mathcal{L}_X, d] = 0. \tag{6.23}$$

If these axioms are satisfied, then one can demonstrates that

$$(\mathcal{L}_X Y)(f) = X(Y(f)) - Y(X(f)) = [X,Y](f), \tag{6.24}$$

which constitutes one of the conventional definitions for the Lie bracket. Indeed, using **Axiom 3** and **Axiom 1**, we have

$$\mathcal{L}_X(Y(f)) = (\mathcal{L}_X Y)f + Y(\mathcal{L}_X f) \implies (\mathcal{L}_X Y)f = \mathcal{L}_X(Y(f)) - Y(\mathcal{L}_X f) = X(Y(f)) - Y(X(f)) = [X,Y](f).$$

Concretely, considering $T$ as a tensor field of type $(p,q)$, we envision $T$ as a differentiable multilinear mapping of smooth sections $\alpha_1, \alpha_2, ..., \alpha_p$ from the cotangent bundle $\Gamma(T^*M)$, and sections $X_1, X_2, ..., X_q$ from the tangent bundle $\Gamma(TM)$, yielding real numbers $\mathbb{R}$. The Lie derivative of $T$ along $Y$ is defined by the formula

$$\begin{aligned}(\mathcal{L}_Y T)(\alpha_1, \alpha_2, \dots, X_1, X_2, \dots) &= Y(T(\alpha_1, \alpha_2, \dots, X_1, X_2, \dots)) \\ &- T(\mathcal{L}_Y \alpha_1, \alpha_2, \dots, X_1, X_2, \dots) - T(\alpha_1, \mathcal{L}_Y \alpha_2, \dots, X_1, X_2, \dots) - \dots \\ &- T(\alpha_1, \alpha_2, \dots, \mathcal{L}_Y X_1, X_2, \dots) - T(\alpha_1, \alpha_2, \dots, X_1, \mathcal{L}_Y X_2, \dots) - \dots .\end{aligned} \tag{6.25}$$

Hence, employing the aforementioned axioms in conjunction with the expression provided in (6.5), we deduce the Lie derivative applicable to a one-form $\omega$

$$(\mathcal{L}_X \omega)(Y) = X\big(\omega(Y)\big) - \omega(\mathcal{L}_X Y) = X(\omega(Y)) - \omega([X,Y]) = (\nabla_X \omega)Y + \omega(\nabla_X Y) - \omega(\nabla_X Y) + \omega(\nabla_Y X).$$

Hence

$$(\mathcal{L}_X \omega)(Y) = (\nabla_X \omega)Y + \omega(\nabla_Y X), \tag{6.26}$$

or a 2-form, specifically the metric tensor $g$, we ascertain

$$\begin{aligned}(\mathcal{L}_X g)(Y,Z) &= X\big(g(Y,Z)\big) - g(\mathcal{L}_X Y, Z) - g(Y, \mathcal{L}_X Z) \\ &= (\nabla_X g)(Y,Z) + g(\nabla_X Y, Z) + g(Y, \nabla_X Z) - g([X,Y],Z) - g(Y,[X,Z]).\end{aligned}$$

Thus

$$(\mathcal{L}_V g)(X,Y) = g(\nabla_X V, Y) + g(\nabla_Y V, X). \tag{6.27}$$

A vector field $V$ is said to be conformal killing, if

$$(\mathcal{L}_V g)(X,Y) = f g(X,Y). \tag{6.28}$$

If the function $f$ equals zero, then the vector field $V$ is referred to as a Killing vector field.

Section 7

# Special Differential Operators

In the subsequent subsection, we shall furnish precise definitions elucidating the properties and functionalities of the differential operators such as **gradient**, **divergence**, **Hessian** and **Laplacian**.

Subsection 7.1

## Gradient

**Definition 6** We define the gradient operator grad on a Riemannian manifold $(M, g)$ by

$$\begin{aligned} \mathrm{grad} : C^\infty(M) &\longrightarrow \Gamma(TM) \\ f &\longmapsto \mathrm{grad} f = \sharp \mathrm{d}f, \end{aligned}$$

such that, for all $X \in \Gamma(TM)$, we have

$$\mathrm{d}f(X) = X(f) = g(X, \mathrm{grad} f). \tag{7.1}$$

On a local chart $U$, one has

$$\mathrm{d}f = \sum_{i=1}^{i=n} \partial_i f \mathrm{d}x^i.$$

Then

$$\sharp \mathrm{d}f = \sum_{i=1}^{i=n} g^{ij} (\mathrm{d}f)_i \partial_j = \sum_{i=1}^{i=n} g^{ij} \partial_i f \partial_j = \mathrm{grad} f_{|U}.$$

**Lemma 1** The gradient operator satisfies the following properties, for any $f, h \in \mathcal{C}^\infty(M)$

$$\begin{cases} 1. & \mathrm{grad}(f + h) = \mathrm{grad} f + \mathrm{grad} h, \\ 2. & \mathrm{grad}(fh) = f \mathrm{grad} h + h \mathrm{grad} f, \\ 3. & (\mathrm{grad} f) h = (\mathrm{grad} h)(f). \end{cases}$$

Subsection 7.2

## Divergeance

**Definition 7** We define the divergeance of a vector field $X \in \Gamma(TM)$, denoted by $\mathrm{div}(X)$, on a Riemannian manifold $(M, g)$ by

$$\mathrm{div}(X) = \mathrm{Tr_g}(\nabla X) = \sum_{i=1}^{i=n} g(\nabla_\bullet X, \bullet). \tag{7.2}$$

If $\{e_i\}_{\{i=1,\ldots,n\}}$ is an orthonormal basis, then

$$\boxed{\mathrm{div}(X) = \sum_{i=1}^{i=n} g(\nabla_{e_i} X, e_i).} \tag{7.3}$$

Consequently, the divergence of the 1-form $\omega$ is expressed as

$$\operatorname{div}(\omega) = \operatorname{Tr}_{\mathrm{g}}(\nabla\omega) = \sum_{i=1}^{i=n} \big(\nabla_{e_i}\omega\big)(e_i). \tag{7.4}$$

**Lemma 2** The divergence obeys the following properties, where $X$ and $Y$ are smooth vector fields and $f$ is a smooth function on $M$

$$\begin{cases} 1. & \operatorname{div}(X+Y) = \operatorname{div}(X) + \operatorname{div}(Y), \\ 2. & \operatorname{div}(fX) = f\operatorname{div}(X) + X(f). \end{cases}$$

SUBSECTION 7.3
## Hessian

**Definition 8** We define the $\mathcal{H}$essian operator of a function $f$ denoted $\mathcal{H}\mathbf{ess}_f$ on a Riemannian manifold $(M, g)$ by

$$\begin{aligned} \mathcal{H}ess_f : \Gamma(TM) \otimes \Gamma(TM) &\longrightarrow C^\infty(M) \\ (X,Y) &\longmapsto \mathcal{H}ess_f(X,Y) = g(\nabla_X \operatorname{grad} f, Y), \end{aligned} \tag{7.5}$$

for all $X, Y \in \Gamma(TM)$.

**Figure 16**. **Ludwig Otto Hesse 1811-1874.**

**Lemma 3** The $\mathcal{H}$essian operator is symetric.

SUBSECTION 7.4
## Laplacian

**Definition 9** We define the Laplacian operator of a function $f$ denoted $\Delta(f)$ on a Riemannian manifold $(M, g)$ by

$$\begin{aligned} \Delta : C^\infty(M) &\longrightarrow C^\infty(M) \\ f &\longmapsto \Delta(f) = \operatorname{div}(\operatorname{grad} f) = \operatorname{Tr}_g(\mathcal{H}ess_f), \end{aligned} \tag{7.6}$$

such that, if $e_{i\{1<i<n\}}$ is an orthonormal basis associated with $g$ then

$$\operatorname{Tr}_g(\mathcal{H}ess_f) = \sum_{i=1}^{n} \mathcal{H}ess_f(e_i, e_i).$$

**Figure 17**. **Pierre-Simon, Marquis de Laplace 1749-1827.**

**Lemma 4** For every pair of smooth functions $f$ and $h$ belonging to $C^\infty(M)$, the Laplacian operator adheres to the following properties

$$\begin{cases} 1. & \Delta(f+h) = \Delta(f) + \Delta(h), \\ 2. & \Delta(fh) = f\Delta(h) + 2g(\operatorname{grad} f, \operatorname{grad} h) + h\Delta(f). \end{cases}$$

SECTION 8

# Riemann Curvature Tensor

In this exposition, we introduce the Riemann curvature tensor alongside the sectional curvatures inherent to a Riemannian manifold. These conceptual constructs extend beyond the scope of Gaussian curvature, which traditionally holds prominence within the scope of classical differential geometry, particularly in the study of surfaces. Through our discourse, we elucidate the Gauss equation, a pivotal result facilitating the comparison between sectional curvatures of a given submanifold and those of its encompassing ambient space. Furthermore, we establish that Euclidean spaces, standard spheres, and hyperbolic spaces uniformly exhibit constant sectional curvature. Subsequently, we embark on a systematic elucidation of the Riemannian curvature tensor pertaining to manifolds characterized by constant sectional curvature.

**Definition 10** Let $(M, g)$ be a Riemannian manifold with Levi-Cevita connection $\nabla$. Then, the Riemann curvatur operator is a tensor field on $M$ of type $(1, 3)$

$$R : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM),$$

defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{8.1}$$

for all $X, Y$ and $Z$ vector fields on $M$.

**Lemma 5** The Riemann curvatur tensor satisfies the following properties:

1. Skew-Symmetry and Symmetry

$$\begin{cases} 1. & R(X, Y)Z = -R(Y, X)Z, \\ 2. & g(R(X, Y)Z, W) = -g(R(X, Y)W, Z), \\ 3. & g(R(X, Y)Z, W) = g(R(Z, W)X, Y). \end{cases}$$

2. First Bianchi Identity

$$R(X, Y)Z + R(Z, X)Y + R(Y, Z)X = 0.$$

3. Second Bianchi Identity

$$\nabla_X R(Y, Z)W + \nabla_Z R(X, Y)W + \nabla_Y R(Z, X)W = 0.$$

The Riemann curvature tensor, a rank-4 tensor, can be expressed in terms of the Christoffel symbols as follows

$$R^i_{jkl} = \partial_k \Gamma^i_{jl} - \partial_l \Gamma^i_{jk} + \Gamma^i_{mk} \Gamma^m_{jl} - \Gamma^i_{ml} \Gamma^m_{jk}. \tag{8.2}$$

However, the form $R^i_{jkl}$ does not exhibit all of the tensor's symmetries. It is often more convenient to use the fully covariant form $R_{ijkl}$, defined as

$$\boxed{R_{ijkl} = g_{ip} R^p_{jkl} = g_{ip} \left( \partial_k \Gamma^p_{jl} - \partial_l \Gamma^p_{jk} + \Gamma^p_{mk} \Gamma^m_{jl} - \Gamma^p_{ml} \Gamma^m_{jk} \right).} \tag{8.3}$$

On an $n$-dimensional manifold, the Riemann tensor initially has $4^n$ components. However, due to its inherent symmetries,

$$R_{ijkl} = R_{jilk} = -R_{jikl} = -R_{ijlk},$$

and the fact that components with three equal indices vanish, the number of independent components is reduced to

$$\boxed{\frac{n^2(n^2-1)}{12}.}$$

For a two-dimensional surface ($n = 2$), this reduction leads to only one independent component. In what follows, we will apply expression in (8.3) to compute the Riemann curvature tensor for the **2-sphere** and **2-torus**.

**Example 8.1.** For the 2-sphere, using results obtained in Example 5.1 and Example 6.2, we compute $R_{\theta\phi\theta\phi}$

$$\begin{aligned} R_{\theta\phi\theta\phi} &= g_{\theta p} R^p_{\phi\theta\phi} = g_{\theta\theta} R^\theta_{\phi\theta\phi} \\ &= R^2 \big(\partial_\theta \Gamma^\theta_{\phi\phi} - \partial_\phi \Gamma^\theta_{\phi\theta} + \Gamma^\theta_{\theta k}\Gamma^k_{\phi\phi} - \Gamma^\theta_{k\phi}\Gamma^k_{\theta\phi}\big) \\ &= R^2 \big(\sin^2\theta - \cos^2\theta - 0 + 0 + \cos^2\theta\big). \end{aligned}$$

Hence

$$R_{\theta\phi\theta\phi} = R_{\phi\theta\phi\theta} = R^2 \sin^2\theta \quad \text{and} \quad R_{\theta\phi\phi\theta} = R_{\phi\theta\theta\phi} = -R^2\sin^2\theta.$$

Using formula (8.3), we derive the Riemann curvature tensor components of the form $R^i_{jkl}$

$$R^\theta_{\phi\theta\phi} = -R^\theta_{\phi\phi\theta} = \sin^2\theta \quad \text{and} \quad R^\phi_{\theta\phi\theta} = -R^\phi_{\theta\theta\phi} = 1.$$

**Example 8.2.** For the 2-torus using results obtained in Example 5.2 and Example 6.3, we compute $R_{\theta\phi\theta\phi}$

$$\begin{aligned} R_{\theta\phi\theta\phi} &= g_{\theta p} R^p_{\phi\theta\phi} = g_{\theta\theta} R^\theta_{\phi\theta\phi} \\ &= (R + r\cos\phi)^2 \big(\partial_\theta \Gamma^\theta_{\phi\phi} - \partial_\phi \Gamma^\theta_{\phi\theta} + \Gamma^\theta_{\theta k}\Gamma^k_{\phi\phi} - \Gamma^\theta_{k\phi}\Gamma^k_{\theta\phi}\big) \\ &= (R + r\cos\phi)^2 \left(0 - \frac{r\cos\phi(R + r\cos\phi) + r^2\sin^2\phi}{(R + r\cos\phi)^2} + 0 - \frac{r^2\sin^2\phi}{(R + r\cos\phi)^2}\right). \end{aligned}$$

Thus

$$R_{\theta\phi\theta\phi} = R_{\phi\theta\phi\theta} = r\cos\phi(R + r\cos\phi) \quad \text{and} \quad R_{\theta\phi\phi\theta} = R_{\phi\theta\theta\phi} = -r\cos\phi(R + r\cos\phi).$$

Again with the help of formula (8.3), we compute the Riemann curvature tensor components of the form $R^i_{jkl}$

$$R^\theta_{\phi\theta\phi} = -R^\theta_{\phi\phi\theta} = \frac{r\cos\phi}{R + r\cos\phi} \quad \text{and} \quad R^\phi_{\theta\phi\theta} = -R^\phi_{\theta\theta\phi} = \frac{1}{r}\cos\phi\big(R + r\cos\phi\big).$$

SUBSECTION 8.3

## Sectional Curvature

**Sectional curvature** is a fundamental concept in differential geometry, particularly within the framework of Riemannian geometry. It provides a measure of the curvature of a surface in a given direction, encapsulating how much the surface deviates from being flat when viewed locally. More formally, the sectional curvature at a point on a manifold describes how the geometry of the manifold bends or curves in the plane defined by two tangent vectors at that point.

Figure 18. Hermann Günther Grassmann 1809-1877.

**Definition 11** Let $(M, g)$ be a Riemannian manifold and $p \in M$. Then a section $V$ at $p$ is a 2-dimensional subspace of the tangent space $T_pM$. The set

$$G_2(T_pM) = \{V \mid V \text{ is a section of } T_pM\}$$

of sections is called the Grassmannian of 2-planes at $p$.

The span of two vectors $X$ and $Y$, denoted as Span$\{X, Y\}$, represents all possible linear combinations of these vectors. In other words, it is the set of all vectors that can be obtained by scaling $X$ and $Y$ by any scalar and adding them together. Mathematically, it can be expressed as

$$Span_{\mathbb{K}}\{X, Y\} = \{aX + bY \mid a, b \text{ are scalar in } \mathbb{K}\}.$$

**Definition 12** Let $(M, g)$ be a Riemannian manifold and $p \in M$. Then the function

$$K_p : Span_{\mathbb{K}}\{X, Y\} \longrightarrow \frac{g(R(X,Y)Y, X)}{g(X,X)g(Y,Y) - g(X,Y)^2}, \tag{8.4}$$

is called the **sectional curvature** at $p$, usually denoted $K(X, Y)$.

As an immediate implication, we derive the subsequent valuable outcome:

**Theorem 2** Let $(M, g)$ be a Riemannian manifold of constant sectional curvature $\kappa$. Then, it's curvature tensor $R$ satisifies:

$$R(X,Y)Z = \kappa\big(g(Y,Z)X - g(X,Z)Y\big). \tag{8.5}$$

Section 9

# Ricci Tensor

The culmination of this chapter involves the establishment of the Ricci and scalar curvatures for a Riemannian manifold. These quantities are derived through the process of tracing over the curvature tensor and hold significant prominence within Riemannian geometry.

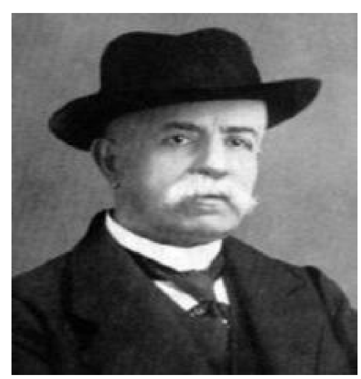

Figure 19. Gregorio Ricci-Curbastro 1853-1925.

**Definition 13** Let $(M, g)$ be a Riemannian manifold, then we define:

1. The **Ricci-Operator**, denoted Q by

$$Q : \Gamma(TM) \longrightarrow \Gamma(TM)$$
$$X \longmapsto Q(X) = \sum_{i=1}^{i=n} R(X, e_i)e_i, \tag{9.1}$$

2. The **Ricci-Tensor**, denoted S by

$$S : \Gamma(TM) \times \Gamma(TM) \longrightarrow C^\infty(M)$$
$$(X, Y) \longmapsto S(X, Y) = g(QX, Y). \tag{9.2}$$

3. The **scalar curvature**, denoted r by

$$r = \sum_{j=1}^{j=n} S(e_j, e_j) = \sum_{i=1}^{i=n}\sum_{j=1}^{j=n} g(R(e_j, e_i)e_i, e_j), \tag{9.3}$$

where $\{e_1, ..., e_n\}$ is any local orthonormal frame for $\Gamma(TM)$.

Clearly, we have

$$S(X,Y) = \sum_{i=1}^{i=n} g(R(X,e_i)e_i, Y) = \sum_{i=1}^{i=n} g(R(e_i,Y)X, e_i) = \sum_{i=1}^{i=n} g(R(Y,e_i)e_i, X) = S(Y,X). \tag{9.4}$$

Using (8.2) and (9.2), we derive the Ricci curvature tensor in terms of the Christoffel symbols

$$S_{ij} = R^k_{ikj} = \partial_k \Gamma^k_{ij} - \partial_j \Gamma^k_{ik} + \Gamma^k_{mk}\Gamma^m_{ij} - \Gamma^k_{mj}\Gamma^m_{ik}, \tag{9.5}$$

and the Ricci scalar curvature $r = g^{ij}S_{ij}$. Hence, we can write the Ricci tensor of both the 2-sphere and 2-torus using results from Example 8.1 and Example 8.2 in matrix form and the Ricci scalar curvature:

1. **For the 2-Sphere:**

$$S_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} \quad \text{and} \quad r = \frac{2}{R^2}. \tag{9.6}$$

2. **For the 2-Torus:**

$$S_{ij} = \begin{pmatrix} \frac{1}{r}\cos\phi\big(R + r\cos\phi\big) & 0 \\ 0 & \frac{r\cos\phi}{R + r\cos\phi} \end{pmatrix} \quad \text{and} \quad r = \frac{2\cos\phi}{r(R + r\cos\phi)}. \tag{9.7}$$

The geometric interpretation of the Ricci tensor is that it describes how much a volume element would differ in curved space compared to Euclidean or flat space. Different components of the Ricci tensor describe how the volume element evolves as one moves along a geodesic in any given direction. For more details we refer to [34]. The physical meaning of the Ricci tensor is that it describes how much the spacetime volume of an object changes due to gravitational tides in general relativity. This is because geometrically, the Ricci tensor describes volume changes due to curvature and spacetime curvature is equated to tidal forces. (See for example [53]). In scenarios characterized by constant sectional curvature, the ensuing result emerges:

**Corollary 1** Let $(M, g)$ be a Riemannian manifold of constant sectional curvature $\kappa$. Then, it's scalar curvature $r$ satisifies:

$$r = n(n-1)\kappa, \tag{9.8}$$

Section A

# Appendix A

**Proof to Lemma 1:** Consider a smooth vector field $X$ on $M$. Then, for any smooth functions $f, h \in C^{\infty}(M)$, according to (7.1), we have:

1. For the initial attribute

$$\begin{aligned} g(X, \text{grad}(f+h)) &= X(f+h) = X(f) + X(h) = g(X, \text{grad}f) + g(X, \text{grad}h) \\ &= g(X, \text{grad}f + \text{grad}h) \implies \text{grad}(f+h) = \text{grad}f + \text{grad}h. \end{aligned}$$

2. For the second

$$\begin{aligned} g(X, \text{grad}(fh)) &= X(fh) = hX(f) + fX(h) = hg(X, \text{grad}f) + fg(X, \text{grad}h) \\ &= g(X, h\text{grad}f + f\text{grad}h) \implies \text{grad}(fh) = h\text{grad}f + f\text{grad}h. \end{aligned}$$

3. Finally

$$(\text{grad}f)h = g(\text{grad}f, \text{grad}h) = (\text{grad}h)f.$$

---

**Proof to Lemma 2:** Consider $X$ and $Y$ as smooth vector fields, and let $\{e_i\}_{i=1,\ldots,n}$ denote an orthonormal basis on $M$. Then, for any smooth function $f$ defined on $M$, one can deduce from (5.2), (7.1) and (7.2) the following:

1. For the introductory characteristic, it suffices to utilize the metric tensor linearity

$$\begin{aligned} \text{div}(X+Y) &= \sum_{i=1}^{i=n} g(\nabla_{e_i}(X+Y), e_i) = \sum_{i=1}^{i=n} g(\nabla_{e_i}X, e_i) + \sum_{i=1}^{i=n} g(\nabla_{e_i}Y, e_i) \\ &= \text{div}X + \text{div}Y. \end{aligned}$$

2. Lastly

$$\begin{aligned} \text{div}(fX) &= \sum_{i=1}^{i=n} g(\nabla_{e_i}(fX), e_i) = f\sum_{i=1}^{i=n} g(\nabla_{e_i}X, e_i) + \sum_{i=1}^{i=n} g(e_i(f)X, e_i) \\ &= f\text{div}X + \sum_{i=1}^{i=n} g(X, e_i)e_i(f) = f\text{div}X + \sum_{i=1}^{i=n} g(X, e_i)g(\text{grad}f, e_i) \\ &= f\text{div}X + \sum_{i=1}^{i=n} g(\text{grad}f, g(X, e_i)e_i) = f\text{div}X + g(\text{grad}f, X) \\ &= f\text{div}X + X(f). \end{aligned}$$

---

**Proof to Lemma 3:** Consider $X$ and $Y$ as smooth vector fields and $f$ as a smooth function on M. Referring to equations (6.2), (7.1) and (7.5), we deduce the following

$$
\begin{aligned}
\mathcal{H}ess_f(X,Y) &= g(\nabla_X \mathrm{grad} f, Y) = X\big(g(\mathrm{grad} f, Y)\big) - g(\mathrm{grad} f, \nabla_X Y) \\
&= X\big(Y(f)\big) - (\nabla_X Y)(f) = [X,Y]5F + Y(X(f)) - (\nabla_X Y)(f) \\
&= (\nabla_X Y)(f) - (\nabla_Y X)(f) + Y\big(g(\mathrm{grad} f, X)\big) - (\nabla_X Y)(f) \\
&= g(\nabla_Y \mathrm{grad} f, X) + g(\mathrm{grad} f, \nabla_Y X) - (\nabla_Y X)(f) = g(\nabla_Y \mathrm{grad} f, X) \\
&= \mathcal{H}ess_f(Y,X).
\end{aligned}
$$

---

**Proof to Lemma 4:** Given $f$ and $h$ smooth functions on $M$, using results from Lemma 1 and Lemma 2 along with (7.6), we get:

1. We commence by establishing the initial property

$$
\begin{aligned}
\Delta(f+h) &= \mathrm{div}\big(\mathrm{grad}(f+h)\big) = \mathrm{div}(\mathrm{grad} f + \mathrm{grad} h) = \mathrm{div}(\mathrm{grad} f) + \mathrm{div}(\mathrm{grad} h) \\
&= \Delta(f) + \Delta(h).
\end{aligned}
$$

2. Lastly, we address the final property

$$
\begin{aligned}
\Delta(fh) &= \mathrm{div}\big(\mathrm{grad}(fh)\big) = \mathrm{div}\big(h\mathrm{grad} f + f\mathrm{grad} h\big) = h\mathrm{div}(\mathrm{grad} f) + \mathrm{grad} f(h) + \mathrm{grad} h(f) + f\mathrm{div}(\mathrm{grad} h) \\
&= h\Delta(f) + 2g(\mathrm{grad} f, \mathrm{grad} h) + f\Delta(h).
\end{aligned}
$$

---

**Proof to Lemma 5:**

1. The initial skew-symmetry readily emerges from Definition 10 delineated in (8.1)

$$
\begin{aligned}
R(X,Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \\
&= -\big(\nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[Y,X]} Z\big) \\
&= -R(Y,X)Z.
\end{aligned}
$$

The secondary skew-symmetry arises from the compatibility of the metric tensor $g$ with the Levi-Cevita connection $\nabla$

$$
\begin{aligned}
[X,Y]\big(g(Z,W)\big) &= X\big(Y(g(Z,W))\big) - Y\big(X(g(Z,W))\big) \\
&= X\big(g(\nabla_Y Z, W) + g(Z, \nabla_Y W)\big) - Y\big(g(\nabla_X Z, W) + g(Z, \nabla_X W)\big) \\
&= g(\nabla_X \nabla_Y Z, W) + g(\nabla_Y Z, \nabla_X W) + g(\nabla_X Z, \nabla_Y W) + g(Z, \nabla_X \nabla_Y W) \\
&- g(\nabla_Y \nabla_X Z, W) - g(\nabla_X Z, \nabla_Y W) - g(\nabla_Y Z, \nabla_X W) - g(Z, \nabla_Y \nabla_X W) \\
&= g(R(X,Y)Z, W) + g(\nabla_{[X,Y]} Z, W) + g(R(X,Y)W, Z) + g(\nabla_{[X,Y]} W, Z) \\
&= g(R(X,Y)Z, W) + g(\nabla_{[X,Y]} Z, W) + g(R(X,Y)W, Z) + [X,Y]\big(g(Z,W)\big) \\
&- g(\nabla_{[X,Y)} Z, W) \implies g(R(X,Y)Z, W) + g(R(X,Y)W, Z) = 0.
\end{aligned}
$$

To establish the final symmetry, we are compelled to demonstrate the first Bianchi identity.

2. Through direct computation employing equation (8.1)

$$\begin{aligned} R(X,Y)Z + R(Z,X)Y + R(Y,Z)X &= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z + \nabla_Z\nabla_X Y - \nabla_X\nabla_Z Y - \nabla_{[Z,X]}Y \\ &+ \nabla_Y\nabla_Z X - \nabla_Z\nabla_Y X - \nabla_{[Y,Z]}X \\ &= \nabla_X(\nabla_Y Z - \nabla_Z Y) - \nabla_{[Y,Z]}X + \nabla_Y(\nabla_Z X - \nabla_X Z) - \nabla_{[Z,X]}Y \\ &+ \nabla_Z(\nabla_X Y - \nabla_Y X) - \nabla_{[X,Y]}Z = [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]]. \end{aligned}$$

resulting from the application of Jacobi's identity as articulated in equation (6.4). Returning to the third symmetry property, we ascertain

$$\begin{aligned} g(R(X,Y)Z,W) &= -g(R(Z,X)Y,W) - g(R(Y,Z)X,W) \\ &= g(R(Z,X)W,Y) + g(R(Y,Z)W,X) \\ &= -g(R(X,W)Z,Y) - g(R(W,Z)X,Y) - g(R(W,Y)Z,X) - g(R(Z,W)Y,X) \\ &= 2g(R(Z,W)X,Y) + g(R(X,W)Y,Z) + g(R(W,Y)X,Z) \\ &= 2g(R(Z,W)X,Y) - g(R(Y,X)W,Z) \\ &= 2g(R(Z,W)X,Y) - g(R(X,Y)Z,W) \implies g(R(X,Y)Z,W) = g(R(Z,W)X,Y). \end{aligned}$$

3. For the second Bianchi identity, we derive

$$\begin{aligned} &\nabla_X R(Y,Z)W + \nabla_Z R(X,Y)W + \nabla_Y R(Z,X)W \\ &= \nabla_X(R(Y,Z)W) - R(\nabla_X Y,Z)W - R(Y,\nabla_X Z)W - R(Y,Z)\nabla_X W \\ &+ \nabla_Y(R(Z,X)W) - R(\nabla_Y Z,X)W - R(Z,\nabla_Y X)W - R(Z,X)\nabla_Y W \\ &+ \nabla_Z(R(X,Y)W) - R(\nabla_Z X,Y)W - R(X,\nabla_Z Y)W - R(X,Y)\nabla_Z W. \end{aligned}$$

Leveraging (6.4) in conjunction with the preceding findings, we deduce

$$\begin{aligned} R([X,Z],Y)W + R([Y,X],Z)W + R([Y,Z],X)W &= R(\nabla_X Z - \nabla_Z X,Y)W - R(\nabla_Z X,Y)W + R(\nabla_Y X,Z)W \\ &- R(\nabla_X Y,Z)W + R(\nabla_Z Y,X)W - R(\nabla_Y Z,X)W \\ &= -R(Y,\nabla_X Z)W - R(\nabla_Z X,Y)W - R(Z,\nabla_Y X)W \\ &- R(\nabla_X Y,Z)W - R(X,\nabla_Y Z)W - R(\nabla_Z Y,X)W. \end{aligned}$$

Subsequently, employing direct substitution, we arrive at

$$\begin{aligned} &\nabla_X R(Y,Z)W + \nabla_Z R(X,Y)W + \nabla_Y R(Z,X)W = R([X,Z],Y)W + R([Y,X],Z)W + R([Y,Z],X)W \\ &+ \nabla_X(R(Y,Z)W) + \nabla_Y(R(Z,X)W) + \nabla_Z(R(X,Y)W) - R(Y,Z)\nabla_X W - R(Z,X)\nabla_Y W - R(X,Y)\nabla_Z W = \star. \end{aligned}$$

Expanding $\star$ utilizing equation (8.1)

$$\begin{aligned} \star &= \nabla_{[X,Z]}\nabla_Y W - \nabla_Y\nabla_{[X,Z]}W - \nabla_{[[X,Z],Y]}W + \nabla_{[Y,X]}\nabla_Z W - \nabla_Z\nabla_{[Y,X]}W \\ &- \nabla_{[[Y,X],Z]}W + \nabla_{[Y,Z]}\nabla_X W - \nabla_X\nabla_{[Y,Z]}W - \nabla_{[[Y,Z],X]}W + \nabla_X\nabla_Y\nabla_Z W \\ &- \nabla_X\nabla_Z\nabla_Y W - \nabla_X\nabla_{[Y,Z]}W + \nabla_Y\nabla_Z\nabla_X W - \nabla_Y\nabla_X\nabla_Z W - \nabla_Y\nabla_{[Z,X]}W \\ &+ \nabla_Z\nabla_X\nabla_Y W - \nabla_Z\nabla_Y\nabla_X W - \nabla_Z\nabla_{[X,Y]}W - \nabla_Y\nabla_Z\nabla_X W + \nabla_Z\nabla_Y\nabla_X W \\ &+ \nabla_{[Y,Z]}\nabla_X W - \nabla_Z\nabla_X\nabla_Y W + \nabla_X\nabla_Z\nabla_Y W + \nabla_{[Z,X]}\nabla_Y W - \nabla_X\nabla_Y\nabla_Z W \\ &+ \nabla_Y\nabla_X\nabla_Z W + \nabla_{[X,Y]}\nabla_Z W. \end{aligned}$$

After simplification, we arrive at the following equality

$$\star = -\nabla_{[[X,Z],Y]+[[Y,X],Z]+[[Y,Z],X]}W = 0 \implies \nabla_X R(Y,Z)W + \nabla_Z R(X,Y)W + \nabla_Y R(Z,X)W = 0,$$

resulting from the application of Jacobi's identity as articulated in equation (6.4).

**Proof to Theorem 2:** From formula (8.4), we have

$$g(R(X,Y)Y,X) = \kappa\big(g(X,X)g(Y,Y) - g(X,Y)^2\big). \tag{A.1}$$

We shall compute the quantity $g(R(X+Z,Y)Y,X+Z)$. From one hand, using (A.1)

$$\begin{aligned} g(R(X+Z,Y)Y,X+Z) &= \kappa\big(g(X+Z,X+Z)g(Y,Y) - g(X+Z,Y)^2\big) \\ &= \kappa\Big(\big(g(X,X) + 2g(X,Z) + g(Z,Z)\big)g(Y,Y) - g(X,Y)^2 - 2g(Z,Y)g(X,Y) - g(Z,Y)^2\Big) \\ &= g(R(X,Y)Y,X) - g(Z,Y)g(X,Y) + g(R(Z,Y)Y,Z) + 2\kappa\big(g(X,Z)g(Y,Y) - g(Z,Y)g(X,Y)\big). \end{aligned} \tag{A.2}$$

on the other hand

$$g(R(X+Z,Y)Y,X+Z) = g(R(X,Y)Y,X) + g(R(Z,Y)Y,X) + g(R(X,Y)Y,Z) + g(R(Z,Y)Y,Z),$$

observe that g(R(Z,Y)Y,X)=g(R(X,Y)Y,X), thus

$$g(R(X+Z,Y)Y,X+Z) = g(R(X,Y)Y,X) + 2g(R(X,Y)Y,Z) + g(R(Z,Y)Y,Z). \tag{A.3}$$

Setting equations (A.2) and (A.3) equal, yields

$$g(R(X,Y)Y,Z) = \kappa\big(g(X,Z)g(Y,Y) - g(Z,Y)g(X,Y)\big) = \kappa\big(g(g(Y,Y)X,Z) - g(g(X,Y)Y,Z)\big).$$

Since $Z$ is an arbitrary vector field, we get

$$R(X,Y)Y = \kappa\big(g(Y,Y)X - g(X,Y)Y\big). \tag{A.4}$$

Using (A.4), we compute $R(X,Y+Z)(Y+Z)$

$$\begin{aligned} R(X,Y+Z)(Y+Z) &= \kappa\big(g(Y+Z,Y+Z)X - g(X,Y+Z)(Y+Z)\big) \\ &= \kappa\Big[\big(g(Y,Y) + 2g(Y,Z) + g(Z,Z)\big)X - g(X,Y)Y - g(X,Y)Z - g(X,Z)Y - g(X,Z)Z\Big] \\ &= \kappa\big(g(Y,Y)X - g(X,Y)Y\big) + \kappa\big(g(Z,Z)X - g(X,Z)Z\big) + \kappa\big(2g(Y,Z)X - g(X,Y)Z - g(X,Z)Y\big). \end{aligned}$$

Hence

$$R(X,Y+Z)(Y+Z) = R(X,Y)Y + R(X,Z)Z + \kappa\big(2g(Y,Z)X - g(X,Y)Z - g(X,Z)Y\big). \tag{A.5}$$

On the other hand, we have

$$R(X,Y+Z)(Y+Z) = R(X,Y)Y + R(X,Y)Z + R(X,Z)Y + R(X,Z)Z. \tag{A.6}$$

Setting equation (A.5) and (A.6) equal we obtain

$$R(X,Y)Z + R(X,Z)Y = \kappa(2g(Y,Z)X - g(X,Y)Z - g(X,Z)Y). \tag{A.7}$$

Using Bianchi's first identity along with equation (A.7), one can obtain

$$2R(X,Y)Z + R(Y,Z)X = \kappa(2g(Y,Z)X - g(X,Y)Z - g(X,Z)Y),$$

where after swapping $X$ and $Y$ we get

$$2R(X,Y)Z + R(Y,Z)X = \kappa(2g(Y,Z)X - g(X,Y)Z - g(X,Z)Y). \tag{A.8}$$

Substracting formulas (A.7) and (A.8) results in

$$3R(X,Y)Z = 3\kappa\big(g(Y,Z)X - g(X,Z)Y\big).$$

**Proof to Corollary 1:** Let $\{e_1, ..., e_n\}$ be an orthonormal basis of $(M, g)$. Using the fact (5.2) and (8.5) along with (9.1), we have

$$QX = \sum_{i=1}^{n} R(X, e_i)e_i = \kappa \sum_{i=1}^{n} (g(e_i, e_i)X - g(X, e_i)e_i)$$
$$= \kappa \left( nX - \sum_{i=1}^{n} g(X, e_i)e_i \right) = \kappa(n-1)X. \quad \text{(A.9)}$$

Substituting (A.9) in (9.2), gives

$$S(X, Y) = \kappa(n-1)g(X, Y), \quad \text{(A.10)}$$

and by direct computation of the trace of (A.10) we obtain

$$r = \kappa n(n-1).$$

# *Almost Contact Structures*

PART

III

This chapter delves into almost contact metric manifolds, revisiting their fundamental definition and properties. It proceeds to categorize these structures into distinct classes, providing illustrative examples along the way. A particular focus will be given to the comprehensive examination of Trans-Sasakian manifolds and $C_{12}$ manifolds within this broader framework.



**Table 4**. Contents for Part III

Section 10

## Generalities

Contact geometry originated from the pioneering work of the Norwegian mathematician **Sophus Lie** during the late $19^{th}$ century [44]. Lie, renowned for his advancements in differential equations and the formulation of Lie groups, introduced the fundamental concept of contact transformations as part of his investigations into partial differential equations. Subsequently, luminaries such as **Élie Cartan** and **Vladimir Arnold** Figure 20 enriched contact geometry, extending its applications and forging connections with other mathematical domains, notably symplectic geometry and classical mechanics [2, 3]. In a distinct application, **Hoffman** utilized contact geometry to elucidate phenomena in the visual cortex [80]. From [15, 81], we shall extract what is necessary for Part IV and Part VI.

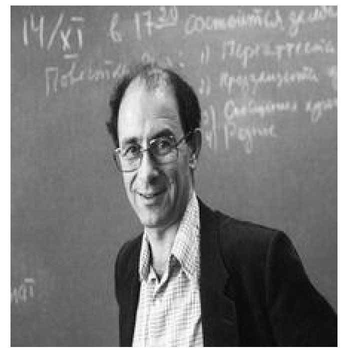

**Figure 20**. **Vladimir Igorevich Arnold 1937-2010.**

**Definition 14** An almost contact structure on a smooth manifold $M$ is denoted by the triple $(\varphi, \xi, \eta)$, where $\varphi$ is a tensor field of type $(1,1)$, $\xi$ is a vector field, and $\eta$ is a differential one-form, where the endomorphism

$$\varphi : \Gamma(TM) \longrightarrow \Gamma(TM)$$

satisfies

$$\varphi^2 = -\mathrm{Id} + \eta \otimes \xi \quad \text{and} \quad \eta(\xi) = 1. \tag{10.1}$$

In this case, $(\varphi, \xi, \eta)$ is referred to as an almost contact structure on $M$.

*Remark* From Definition 14, we can conclude the following:

1. $\varphi$ induces a complex structure on the orthogonal distribution $\mathcal{D}$ to $\xi$.
2. The vector field $\xi$ belongs to the kernel of $\varphi$, signifying that it is tangent to the distribution $\mathcal{D}$ defined by $\eta$

$$\mathcal{D} = \{X \in \Gamma(TM) \quad | \quad \eta(X) = 0\}. \tag{10.2}$$

In essence, an almost contact structure delineates a geometric configuration wherein $\varphi$ governs a complex-like structure on a specific subspace of tangent vectors, while $\xi$ and $\eta$ dictate the characteristics of a complementary subspace, forming a distinctive interplay between different geometric objects on the manifold $M$. Conditions (2) and (**??**) provides a splitting of the tangeant bundle $\Gamma(TM)$

$$\boxed{\Gamma(TM) = \mathcal{D} \oplus \{\xi\}.} \tag{10.3}$$

An almost contact metric manifold is typically defined on a manifold of odd dimension, specifically of dimension $(2n+1)$. This is because the structure involves a tangent hyperplane field, which is naturally defined in odd dimensions (see, [**?** ]). Thus, it satisfies the following

$$\varphi\xi = 0, \qquad \eta \circ \varphi = 0, \qquad rank\varphi = 2n. \tag{10.4}$$

**Theorem 3** On any smooth almost contact manifold $(M, \varphi, \xi, \eta)$, there exist a metric $g$ compatible with the almost contact structure $(\varphi, \xi, \eta)$, such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{10.5}$$

for any vector field $X, Y$ on $M$ and $(M, \varphi, \eta, \xi, g)$ is called an almost contact metric manifold.

Immediately, from (10.1) and (10.5), we obtain

$$g(\varphi X, Y) = -g(X, \varphi Y), \quad \eta(X) = g(X, \xi), \tag{10.6}$$

**Definition 15** We associates to every almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ a fundamental 2-form $\Phi$ expressed by

$$\Phi(X, Y) = g(X, \varphi Y), \tag{10.7}$$

for any vector fields $X, Y$ on $M$.

**Lemma 6** The fundamental 2-form $\Phi$ satisfy the following for any arbitraty vector fields $X$ and $Y$ on $M$

$$\begin{cases} 1. \quad \Phi(X, Y) = -\Phi(Y, X), \\ 2. \quad \Phi(\varphi X, \varphi Y) = \Phi(Y, X). \end{cases}$$

**Definition 16** We say that $(M, \varphi, \xi, \eta, g)$ is **normal** if and only if

$$N^{(1)} = N_\varphi + 2\mathrm{d}\eta \otimes \xi = 0, \tag{10.8}$$

and is **integrable** if and only if

$$N_\varphi = 0, \tag{10.9}$$

where $N_\varphi$ is the **Nijenhuis** tensor, expressed by

$$N_\varphi(X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[X, \varphi Y] - \varphi[\varphi X, Y]. \tag{10.10}$$

When referring to the concept of integrability, we specifically denote the integrability of the distribution $\mathcal{D}$ in accordance with the **Frobenius sense** [38], elucidating the following

$$\forall X, Y \in \mathcal{D} : [X, Y] \in \mathcal{D}$$

We now state the following important result:

**Theorem 4** Let $(M, \varphi, \xi, \eta, g)$ be an almost contact metric manifold such that $N_\varphi \neq 0$. Then, any smooth function $f$ on $M$ that depends only on the direction of $\xi$ is constant.

PROOF Using the fact that $\mathcal{D}$ is not integrable along with (10.3) then, there exists $X, Y \in \mathcal{D}$ such that

$$[X, Y] \propto \xi.$$

**Figure 21. Ferdinand Georg Frobenius 1849-1917.**

Thus, for a function $f$ that depends only on the direction of $\xi$

$$[X,Y](f) = X(Y(f)) - Y(X(f)) = 0 \propto \xi(f) \implies \xi(f) = 0,$$

which yields the result. □

Section 11

# Some Classes of Almost Contact Metric Structures

In 1990, [25] Chinea and González introduced a systematic classification framework for almost contact metric structures, leveraging the properties of the second fundamental form $\Phi$ to delineate 12 distinct classes:

| Class | Defining equation |
|---|---|
| $C_1$ | $(\nabla_X\Phi)(Y,Z) = 0$, $\nabla\eta = 0$. |
| $C_2$ | $\mathrm{d}\Phi = \nabla\eta = 0$. |
| $C_3$ | $(\nabla_X\Phi)(Y,Z) - (\nabla_{\varphi X}\Phi)(\varphi Y, Z) = 0$, $\delta\Phi = 0$. |
| $C_4$ | $(\nabla_X\Phi)(Y,Z) = -\frac{1}{2(n-1)}\Big(g(\varphi X,\varphi Y)\delta\Phi(Z) - g(\varphi X,\varphi Z)\delta\Phi(Y) - \Phi(X,Y)\delta\Phi(\varphi Z) + \Phi(X,Z)\delta\Phi(\varphi Y)\Big)$, $\delta\Phi(\xi) = 0$. |
| $C_5$ | $(\nabla_X\Phi)(Y,Z) = \frac{1}{2n}\Big(\Phi(X,Z)\eta(Y) - \Phi(X,Y)\eta(Z)\Big)\delta\eta$. |
| $C_6$ | $(\nabla_X\Phi)(Y,Z) = \frac{1}{2n}\Big(g(X,Z)\eta(Y) - g(X,Y)\eta(Z)\Big)\delta\Phi(\xi)$. |
| $C_7$ | $(\nabla_X\Phi)(Y,Z) = \eta(Z)(\nabla_Y\eta)\varphi X + \eta(Y)(\nabla_{\varphi X}\eta)Z$, $\delta\Phi = 0$. |
| $C_8$ | $(\nabla_X\Phi)(Y,Z) = -\eta(Z)(\nabla_Y\eta)\varphi X + \eta(Y)(\nabla_{\varphi X}\eta)Z$, $\delta\eta = 0$. |
| $C_9$ | $(\nabla_X\Phi)(Y,Z) = \eta(Z)(\nabla_Y\eta)\varphi X - \eta(Y)(\nabla_{\varphi X}\eta)Z$. |
| $C_{10}$ | $(\nabla_X\Phi)(Y,Z) = -\eta(Z)(\nabla_Y\eta)\varphi X - \eta(Y)(\nabla_{\varphi X}\eta)Z$. |
| $C_{11}$ | $(\nabla_X\Phi)(Y,Z) = -\eta(X)(\nabla_\xi\Phi)(\varphi Y,\varphi Z)$. |
| $C_{12}$ | $(\nabla_X\Phi)(Y,Z) = \eta(X)\eta(Z)(\nabla_\xi\eta)\varphi Y - \eta(X)\eta(Y)(\nabla_\xi\eta)\varphi Z$. |

**Table 5**. Defining relations for each of the twelve classes

In the context of Table 5, several notable classes can be distinguished. For instance, the trivial class, commonly denoted by $C$ or $|C|$, represents the category of **cosymplectic** manifolds (sometimes referred to as **co-Kähler** by certain authors). The class $C_1$ corresponds to **nearly K-cosymplectic** manifolds, $C_5$ to $\beta$-**Kenmotsu** manifolds, and $C_6$ to $\alpha$-**Sasakian** manifolds. Additionally, the class $C_6 \oplus C_7$ encompasses **quasi-Sasakian** manifolds, while $C_{12}$ pertains to **corner** manifolds.

Specifically, for a 3-dimensional manifolds, we have

$$C = C_5 \oplus C_6 \oplus C_9 \oplus C_{12}.$$

Subsection 11.1

## Trans-Sasakian Manifolds

In the work of [59], the author establishes a clear definition when the structure $(\varphi, \xi, \eta, g)$ attains Trans-Sasakian status:

**Definition 17** An almost contact metric structure $(\varphi, \xi, \eta, g)$ is Trans-Sasakian if and only if

$$d\eta = \alpha\Phi \quad \text{and} \quad d\Phi = 2\beta\eta \wedge \Phi, \tag{11.1}$$

where $\alpha$ and $\beta$ are smooth functions on $M$, expressed as follow:

$$2\alpha = Tr_g(\varphi\nabla\xi), \quad \text{and} \qquad 2\beta = \text{div}\xi.$$

The subsequent delineation of a Trans-Sasakian manifold is provided in [57]:

**Theorem 5** An almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is an $(\alpha, \beta)$ Trans-Sasakian manifold if and only if

$$\big(\nabla_X\varphi\big)Y = \alpha\big(g(X,Y)\xi - \eta(Y)X\big) + \beta\big(g(\varphi X, Y)\xi - \eta(Y)\varphi X\big). \tag{11.2}$$

In their work [29], **De** and **Tripathi** established that for an $(\alpha, \beta)$ Trans-Sasakian manifold, the relationship between $\alpha$ and $\beta$:

**Proposition 1** In an $(\alpha, \beta)$ Trans-Sasakian manifold, the following is satisfied

$$\xi(\alpha) + 2\alpha\beta = 0. \tag{11.3}$$

Subsection 11.2

## $\alpha$-Sasakian Manifolds

Sasakian manifolds, and more generally $\alpha$-Sasakian manifolds, are named in honor of the mathematician **Shigeo Sasaki** Figure 22.

**Definition 18** Let $(M, \varphi, \xi, \eta, g)$ be a $(2n+1)$-dimensional almost contact metric manifold and $\Phi$ it's fundamental 2-form. We say that $M$ is an $\alpha$-Sasakian manifold if it is normal and

$$d\eta = \alpha\Phi \quad \text{and} \quad d\Phi = 0. \tag{11.4}$$

From formula (11.2), we deduce the following result for $\alpha$-Sasakian manifold:

**Corollary 2** $(M, \varphi, \xi, \eta, g)$ is an $\alpha$-Sasakian manifold if and only if

$$\big(\nabla_X\varphi\big)Y = \alpha\big(g(X,Y)\xi - \eta(Y)X\big), \tag{11.5}$$

for all $X, Y$ vector field on $M$.

**Corollary 3** $\alpha$-Sasakian non-Cosymplectic manifolds are non-integrable.

**Proposition 2** In an $\alpha$-Sasakian manifold, the following is satisfied for any vector fields $X, Y$ on $M$

$$\nabla_X \xi = -\alpha \varphi X, \tag{11.6}$$
$$(\nabla_X \eta)Y = -\alpha g(\varphi X, Y), \tag{11.7}$$
$$S(X, \xi) = 2n\alpha^2 \eta(X) - \varphi X(\alpha). \tag{11.8}$$

for all $X, Y$ vector field on $M$.

**Figure 22**. **Shigeo Sasaki 1912-1987.**

Subsection 11.3

## $\beta$-Kenmotsu Manifolds

Kenmotsu manifolds, or more broadly, $\beta$-Kenmotsu manifolds, are named in honor of the Japanese mathematician **Katsuei Kenmotsu**.

**Definition 19** Let $(M, \varphi, \xi, \eta, g)$ be a $(2n+1)$-dimensional almost contact metric manifold and $\Phi$ it's fundamental 2-form. We say that $M$ is an $\beta$-Kenmotsu manifold if it is normal, integrable and

$$\mathrm{d}\eta = 0 \quad \text{and} \quad \mathrm{d}\Phi = 2\beta\eta \wedge \Phi. \tag{11.9}$$

The following results can be derived with ease:

**Theorem 6** $(M, \varphi, \xi, \eta, g)$ is an $\beta$-Kenmotsu manifold if and only if

$$(\nabla_X \varphi)Y = \beta\big(g(\varphi X, Y)\xi - \eta(Y)\varphi X\big), \tag{11.10}$$

for all $X, Y$ vector field on $M$.

**Proposition 3** In an $\beta$ Kenmotsu manifold, the following is satisfied for any vector fields $X, Y$ on $M$

$$\nabla_X \xi = \beta\big(X - \eta(X)\xi\big), \tag{11.11}$$
$$(\nabla_X \eta)Y = \beta\big(g(X, Y) - \eta(X)\eta(Y)\big), \tag{11.12}$$
$$S(X, \xi) = (-2n\beta^2 - \xi(\beta))\eta(X) - (2n-1)X(\beta), \tag{11.13}$$

for all $X, Y$ vector field on $M$.

**Corollary 4** $\beta$-Kenmotsu manifolds are integrable.

Consequently, we obtain:

$$\begin{cases} (1) : \alpha - \mathit{Sasakian} \Leftrightarrow \text{Trans-Sasaki of type } (\alpha, 0) \\ (2) : \beta - \mathit{Kenmotsu} \Leftrightarrow \text{Trans-Sasaki of type } (0, \beta) \\ (3) : \mathit{Cosymplectic} \Leftrightarrow \text{Trans-Sasaki of type } (0, 0) \end{cases}$$

Additionally, **Marrero** noted in [50] that for dimensions greater than or equal to 5, every $(\alpha, \beta)$ trans-Sasakian manifold must either be an $\alpha$-Sasakian or a $\beta$-Kenmotsu manifold, indicating that either $\alpha$ or $\beta$ must be zero. Thus, based on the equation (11.3), in an $\alpha$-Sasakian manifold, we observe that $\xi(\alpha) = 0$.

Subsection 11.4

## $C_{12}$ Manifolds

In the framework of the classification of almost contact metric manifolds by **Chinea** and **Gonzalez** [25], Table 5, a specific category of manifolds, termed $C_{12}$-manifolds, is identified. These manifolds, though potentially integrable, inherently lack normalcy. The characterization of $C_{12}$-manifolds is expressed as

$$(\nabla_X \phi)(Y, Z) = \eta(X)\eta(Z)(\nabla_\xi \eta)\varphi Y - \eta(X)\eta(Y)(\nabla_\xi \eta)\varphi Z.$$

The works of **Bouzir, H.; Beldjilali, G.; Bayour, B.** [16] and **de Candia, S.; Falcitelli, M.** [27] further elaborate on the characterization of $(2n+1)$-dimensional $C_{12}$-manifolds

$$(\nabla_X \varphi)Y = \eta(X)\big(\omega(\varphi Y)\xi + \eta(Y)\varphi\psi\big), \tag{11.14}$$

for all $X$ and $Y$ vector fields on $M$ where $\omega = -\nabla_\xi \eta$ is a closed 1-form and $\psi = -\nabla_\xi \xi$ is a vector field such that $\omega(X) = g(X, \psi)$. Firstly, notice that the vector field $\psi$ is perpendicular to $\xi$ because $g(\psi, \xi) = -g(\nabla_\xi \xi, \xi) = 0$.

Therefore, we have the following definition from [16] :

**Definition 20** Let $(M, \varphi, \xi, \eta, g)$ be an almost contact manifold. $M$ is called almost $C_{12}$-manifold if there exists a closed one-form $\omega$ which satisfies

$$\mathrm{d}\eta = \omega \wedge \eta \quad \text{and} \quad \mathrm{d}\phi = 0.$$

In addition, if $N_\varphi = 0$ we say that $M$ is a $C_{12}$-manifold and we denote it by $(M, \varphi, \xi, \psi, \eta, \omega, g)$.

By direct substitution of $Y = \xi$ in (11.14) and derivating $\varphi\xi = 0$ along $X$ an arbitrary vector field on $M$

$$\begin{aligned} 0 = X(\varphi\xi) = (\nabla_X \varphi)\xi + \varphi(\nabla_X \xi) = \eta(X)\varphi\psi + \varphi\nabla_X \xi \\ \implies \varphi\nabla_X \xi = -\eta(X)\varphi\psi. \end{aligned}$$

Applying $\varphi$ to both sides of the equation and using the fact $\eta(\nabla_X \xi) = 0$ and $\eta(\psi) = 0$ yields

$$-\nabla_X \xi = \eta(X)\psi \implies \nabla_X \xi = -\eta(X)\psi.$$

Taking the inner product of $\nabla_X \xi$ with $Y$ gives

$$(\nabla_X \eta)Y = g(\nabla_X \xi, Y) = -\eta(X)g(\psi, Y) = -\eta(X)\omega(Y).$$

Using (8.1), we have

$$\begin{aligned} R(X,Y)\xi &= \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X,Y]}\xi = -X(\eta(Y)\psi) + Y(\eta(X)\psi) + \eta([X,Y])\psi \\ &= -\big((\nabla_X \eta)Y + \eta(\nabla_X Y)\big)\psi - \eta(Y)\nabla_X \psi + \big((\nabla_Y \eta)X + \eta(\nabla_Y X)\big)\psi + \eta(X)\nabla_Y \psi + \big(\eta(\nabla_X Y) - \eta(\nabla_Y X)\big)\psi \\ &= \big(-(\nabla_X \eta)Y + (\nabla_Y \eta)X\big)\psi + \eta(X)\nabla_Y \psi - \eta(Y)\nabla_X \psi. \end{aligned}$$

Using formula (6.17), we get an important result

$$\boxed{R(X, Y)\xi = -2\mathrm{d}\eta(X, Y)\psi + \eta(X)\nabla_Y \psi - \eta(Y)\nabla_X \psi.} \tag{11.15}$$

Taking the inner product of $R(X, Y)\xi$ with $Z$ and using symmetries we obtain

$$g(R(X, Y)\xi, Z) = -2\mathrm{d}\eta(X, Y)\omega(Z) + \eta(X)g(\nabla_Y \psi, Z) - \eta(Y)g(\nabla_X \psi, Z) = g(R(\xi, Z)X, Y).$$

Considering an orthonormal basis $\{e_i\}_{\{i=1\dots 2n\}}$ and using formula (9.2)

$$S(\xi,Y)=\sum_{i=1}^{i=2n} g(R(\xi,e_i)e_i,Y)=g(\nabla_Y\psi,\xi)-\sum_{i=1}^{i=2n}+2\mathrm{d}\eta(e_i,Y)\omega(e_i)+\eta(Y)g(\nabla_{e_i}\psi,e_i).$$

First, we have

$$\begin{aligned}\sum_{i=1}^{i=2n}-2\mathrm{d}\eta(e_i,Y)\omega(e_i)&=\sum_{i=1}^{i=2n}-2\mathrm{d}\eta(e_i,Y)g(\psi,e_i)=\sum_{i=1}^{i=2n}-2\mathrm{d}\eta(g(\psi,e_i)e_i,Y)\\&=-2\mathrm{d}\eta(\psi,Y)=-(\nabla_\psi\eta)Y+(\nabla_Y\eta)\psi=\eta(\psi)\omega(Y)-\eta(Y)\omega(\psi)=-\eta(Y).\end{aligned}$$

On the other hand, differentiating $\omega(\xi)=0$ along $X$, we get

$$0=X(\omega(\xi))=X(g(\psi,\xi))=g(\nabla_X\psi,\xi)+g(\psi,\nabla_X\xi)=g(\nabla_X\psi,\xi)-\eta(X)\implies g(\nabla_X\psi,\xi)=\eta(X).$$

With direct substitutions and using (7.2), the result obtained in [8] is concluded:

**Proposition 4** The following is satisfied in a $C_{12}$-manifold

$$\nabla_X\xi=-\eta(X)\psi, \tag{11.16}$$
$$(\nabla_X\eta)Y=-\eta(X)\omega(Y), \tag{11.17}$$
$$S(X,\xi)=-\eta(X)\mathrm{div}\psi. \tag{11.18}$$

for all $X,Y$ vector field on $M$.

In 3-dimensional configuration, the theorem articulated in [16] stands affirmed:

**Theorem 7** Let $(M,\varphi,\xi,\eta,g)$ be a 3-dimensional almost contact metric manifold. $M$ is $C_{12}$-manifold if and only if

$$\nabla_X\xi=-\eta(X)\psi,$$

for all vector field on $M$.

Herein, we present an alternative delineation of a $C_{12}$-manifold possessing a dimensionality of $2n+1$. We show that

$$(\nabla_X\varphi)Y=\eta(X)\big(\omega(\varphi Y)\xi+\eta(Y)\varphi\psi\big),$$

is equivalent to

$$(\nabla_{\varphi X}\varphi)Y=0.$$

Suppose that $(\nabla_X\varphi)Y=\eta(X)\big(\omega(\varphi Y)\xi+\eta(Y)\varphi\psi\big)$. So due to (10.4), replacing $X$ by $\varphi X$ we get $(\nabla_{\varphi X}\varphi)Y=0$. Conversely, assume that

$$(\nabla_{\varphi X}\varphi)Y=0 \tag{11.19}$$

and replacing $X$ by $\varphi X$ we get

$$(\nabla_X\varphi)Y=\eta(X)(\nabla_\xi\varphi)Y. \tag{11.20}$$

Also, in (11.19) replacing $Y$ by $\xi$ then apply $\varphi$ and then replacing $X$ by $\varphi X$, we obtain

$$\begin{aligned} 0 &= (\nabla_{\varphi X}\varphi)\xi \\ &= \varphi\nabla_{\varphi X}\xi \\ &= \nabla_{\varphi X}\xi \\ &= \nabla_{\varphi^2 X}\xi \\ &= -\nabla_X\xi + \eta(X)\nabla_\xi\xi. \end{aligned}$$

Hence

$$\nabla_X\xi = -\eta(X)\psi, \qquad \text{where} \quad \psi = -\nabla_\xi\xi. \tag{11.21}$$

On the other hand, knowing that $\mathrm{d}\phi = 0$ that is

$$g\big(X, (\nabla_Z\varphi)Y\big) + g\big(Z, (\nabla_Y\varphi)X\big) + g\big(Y, (\nabla_X\varphi)Z\big) = 0,$$

putting $Z = \xi$, we obtain

$$\begin{aligned} 0 &= g\big(X, (\nabla_\xi\varphi)Y\big) + g\big(\xi, (\nabla_Y\varphi)X\big) + g\big(Y, (\nabla_X\varphi)\xi\big) \\ &= g\big(X, (\nabla_\xi\varphi)Y\big) + g\big(\xi, \nabla_Y\varphi X\big) - g\big(Y, \varphi\nabla_X\xi\big) \\ &= g\big(X, (\nabla_\xi\varphi)Y\big) - g\big(\nabla_Y\xi, \varphi X\big) + g\big(\varphi Y, \nabla_X\xi\big), \end{aligned}$$

now, using (11.21), we get

$$\begin{aligned} 0 &= g\big(X, (\nabla_\xi\varphi)Y\big) + \eta(Y)g\big(\psi, \varphi X\big) - \eta(X)g\big(\varphi Y, \psi\big) \\ &= g\big(X, (\nabla_\xi\varphi)Y\big) - g\big(\eta(Y)\varphi\psi, X\big) - g\big(\omega(\varphi Y)\xi, X\big), \end{aligned}$$

where $\omega(X) = g(X, \psi)$, which gives

$$(\nabla_\xi\varphi)Y = \eta(Y)\varphi\psi + \omega(\varphi Y)\xi,$$

replacing this relation in (11.20), we obtain

$$(\nabla_X\varphi)Y = \eta(X)\big(\omega(\varphi Y)\xi + \eta(Y)\varphi\psi\big).$$

This yields the following deep result:

**Theorem 8** An almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is of class $C_{12}$ if and only if for all $X$ and $Y$ vector fields on $M$

$$(\nabla_{\varphi X}\varphi)Y = 0. \tag{11.22}$$

A 3-dimensional $C_{12}$ manifold exhibits complete controllability, signifying the existence of a naturally derived global orthonormal basis denoted by $\{\xi, \psi, \varphi\psi\}$.

Section 12

## Examples

In the concluding section of this chapter, we present concrete examples illustrating each of the previously introduced classes:

**Example 12.1.** [10]$(\alpha,\beta)$ **Trans-Sasakian:**

We denote the Cartesian coordinates in a 3-dimensional Euclidean space $\mathbb{R}^3$ by $(x,y,z)$ and define a symmetric tensor field $g$ by

$$g = \begin{pmatrix} \rho^2+\tau^2 & 0 & -\tau \\ 0 & \rho^2 & 0 \\ -\tau & 0 & 1 \end{pmatrix},$$

where $\rho$ and $\tau$ are functions on $\mathbb{R}^3$ such that $\rho \neq 0$ everywhere. Further, we define an almost contact metric $(\varphi,\xi,\eta)$ on $\mathbb{R}^3$ by

$$\varphi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \xi = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \eta = (-\tau, 0, 1).$$

The fundamental 1-form $\eta$ and the 2-form $\varphi$ can be expressed as

$$\eta = dz - \tau dx, \quad \varphi = -2\rho^2 dx \wedge dy,$$

and hence

$$d\eta = \tau_2 dx \wedge dy + \tau_3 dx \wedge dz, \quad d\varphi = -4\rho^3 dx \wedge dy \wedge dz,$$

where $\rho_i = \frac{\partial \rho}{\partial x_i}$ and $\tau_i = \frac{\partial \tau}{\partial x_i}$.
We know that the components of the Nijenhuis tensor $N_\varphi$ can be written as

$$N_{ij} = \varphi_k(\partial_i \varphi_j^k - \partial_j \varphi_i^k) - \varphi_i^k(\partial_k \varphi_j^l - \partial_l \varphi_i^l) + \eta_k(\partial_j \xi^k) - \eta_j(\partial_k \xi^k),$$

where the indices $i,j,k$ and $l$ run over the range $\{1,2,3\}$, then by a direct computation we can verify that

$$N_{ij} = 0, \quad \forall i,j,k,$$

implying that the structure $(\varphi,\xi,\eta,g)$ is normal and:

$$\begin{cases} (1): \text{Sasakian when} \quad \tau_2 = -2\rho^2 \quad \text{and} \quad \tau_3 = \tau = 0, \\ (2): \text{Cosymplectic when} \quad \rho_3 = 0 \quad \text{and} \quad \tau_2 = \tau_3 = 0, \\ (3): \text{Kenmotsu when} \quad \rho_3 = \rho \quad \text{and} \quad \tau_2 = \tau_3 = 0. \end{cases}$$

**Example 12.2.** $\alpha$**-Sasakian:**

Let $M = \{(x,y,z) \in \mathbb{R}^3 \;/\; z > 0\}$ and $\{e_1,e_2,e_3\}$ be the frame of vector fields on $M$ given by

$$e_1 = \frac{\partial}{\partial x}, \qquad e_2 = -\frac{2\sigma}{z}\frac{\partial}{\partial y}, \qquad e_3 = y\frac{\partial}{\partial x} + z\frac{\partial}{\partial z},$$

where $\sigma = \sigma(y)$ is a non zero function on $M$. We define a Riemannian metric $g$ by

$$g = \begin{pmatrix} 1 & 0 & -\frac{y}{z} \\ 0 & \frac{1}{4\sigma^2} & 0 \\ -\frac{y}{z} & 0 & \frac{1+y^2}{z^2} \end{pmatrix},$$

Let $\nabla$ be the Riemannian connection of $g$, then we have

$$[e_2, e_3] = -2\sigma e_1.$$

By using the Koszul's formula (6.5) for the Riemannian metric $g$, the non zero components of the Levi-Civita connection corresponding to $g$ are given by:

$$\nabla_{e_1}e_2 = \nabla_{e_2}e_1 = \sigma e_3, \qquad \nabla_{e_1}e_3 = \nabla_{e_3}e_1 = -\sigma e_2 \quad \text{and} \quad \nabla_{e_3}e_2 = -\nabla_{e_2}e_3 = \sigma e_1.$$

For $\xi = e_1$ and setting

$$\varphi e_1 = 0, \qquad \varphi e_2 = \epsilon e_3 \qquad \text{and} \qquad \varphi e_3 = -\epsilon e_2,$$

which gives

$$\varphi = \epsilon \begin{pmatrix} 0 & -\frac{y}{2\sigma} & 0 \\ 0 & 0 & \frac{2\sigma}{z} \\ 0 & -\frac{z}{2\sigma} & 0 \end{pmatrix},$$

then, $(\varphi, \xi, \eta, g)$ is an almost contact metric structure on $M$ with $\eta = dx - \frac{y}{z}dz$. One easily can get

$$\alpha = -\sum_{i=1}^{3} g(\nabla_{e_i}\xi, \varphi e_i) = -2\sigma, \quad \text{and} \quad \beta = 0,$$

which implies that $(\varphi, \xi, \eta, g)$ is an $\alpha$-Sasakian structure where $\alpha = -\epsilon\sigma$.

**Example 12.3.** ***β*-Kenmotsu:**

Let $M = \{(x, y, z) \in \mathbb{R}^3 \ / \ z > 0\}$ and $\{e_1, e_2, e_3\}$ be the frame of vector fields on $M$ given by

$$e_1 = \frac{1}{z}\frac{\partial}{\partial x}, \qquad e_2 = \frac{1}{z}\frac{\partial}{\partial y}, \qquad e_3 = \frac{\partial}{\partial z}.$$

We define a Riemannian metric $g$ by

$$g = z^2(dx^2 + dy^2) + dz^2.$$

Let $\nabla$ be the Riemannian connection of $g$, then we have

$$[e_1, e_3] = \frac{1}{z}e_1, \qquad [e_2, e_3] = \frac{1}{z}e_2.$$

By using Koszul formula (6.5) for the Riemannian metric $g$, the non zero components of the Levi-Civita connection corresponding to $g$ are given by:

$$\nabla_{e_1}e_1 = -\frac{1}{z}e_3, \qquad \nabla_{e_1}e_3 = \frac{1}{z}e_1, \qquad \nabla_{e_2}e_2 = -\frac{1}{z}e_3 \quad \text{and} \quad \nabla_{e_2}e_3 = \frac{1}{z}e_2.$$

For $U = ze_3$ and define

$$\varphi = \epsilon \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

then, $(\varphi, \xi, \eta, g)$ is an almost contact metric structure on $M$ with $\eta = dz$. One easily can get

$$\alpha = -\sum_{i=1}^{3} g(\nabla_{e_i}\xi, \varphi e_i) = 0 \qquad \text{and} \qquad \beta = \frac{1}{2}\sum_{i=1}^{3} g(\nabla_{e_i}\xi, e_i) = \frac{1}{z},$$

which allows us to conclude that $(\varphi, \xi, \eta, g)$ is a Trans-Sasakian structure of type $(0, \frac{1}{z})$ i.e., $\beta$-Kenmotsu structure.

**Example 12.4.** $C_{12}$**:**

We denote the Cartesian coordinates in a 3-dimensional Euclidean space $\mathbb{R}^3$ by $(x_1, x_2, x_3)$ and define a symmetric tensor field $g$ by

$$g = \begin{pmatrix} \rho(x_1,x_2,x_3)^2 & 0 & 0 \\ 0 & \tau(x_1,x_2,x_3)^2 & 0 \\ 0 & 0 & \sigma(x_1,x_2,x_3)^2 \end{pmatrix},$$

where $\rho$, $\tau$ and $\sigma$ are functions on $\mathbb{R}^3$ and $\rho\tau\sigma \neq 0$ everywhere. Further, we define an almost contact metric $(\varphi, \xi, \eta)$ on $\mathbb{R}^3$ by

$$\varphi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \xi = \frac{1}{\rho}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \eta = \big(0, 0, \rho\big).$$

Notice that $\mathrm{d}\eta = \rho_2 dx_2 \wedge dx_1 + \rho_3 dx_3 \wedge dx_1 = \omega \wedge \eta$ with $\omega = \frac{\rho_2}{\rho}dx_2 + \frac{\rho_3}{\rho}dx_3$, where $\rho_i = \frac{\partial \rho}{\partial x_i}$. Therefore

$$\psi = \frac{\rho_2}{\tau\rho}e_1 + \frac{\rho_3}{\sigma\rho}e_2.$$

We give the following orthonormal basis

$$\xi = \frac{1}{\rho}\frac{\partial}{\partial x_1}, \qquad e_1 = \frac{1}{\tau}\frac{\partial}{\partial x_2}, \qquad e_2 = \frac{1}{\sigma}\frac{\partial}{\partial x_3}.$$

So, the components of the Levi-Civita connection corresponding to $g$ are written

$$\begin{array}{ccc} \nabla_\xi \xi = -\frac{\rho_2}{\tau\rho}e_1 - \frac{\rho_3}{\rho\sigma}e_2, & \nabla_\xi e_1 = \frac{\rho_2}{\tau\rho}\xi, & \nabla_\xi e_2 = \frac{\rho_3}{\rho\sigma}\xi, \\ \nabla_{e_1}\xi = \frac{\tau_1}{\tau\rho}e_1, & \nabla_{e_1} e_1 = -\frac{\tau_1}{\tau\rho}\xi - \frac{\tau_3}{\tau\sigma}e_2, & \nabla_{e_1} e_2 = \frac{\tau_3}{\tau\sigma}e_1, \\ \nabla_{e_2}\xi = \frac{\sigma_1}{\rho\sigma}e_2, & \nabla_{e_2} e_1 = \frac{\sigma_2}{\tau\sigma}e_2, & \nabla_{e_2} e_2 = -\frac{\sigma_1}{\rho\sigma}\xi - \frac{\sigma_2}{\tau\sigma}e_1. \end{array}$$

Using Theorem 7, one can check that $(\mathbb{R}^3, \varphi, \xi, \eta, g)$ is a 3-parameter family of $C_{12}$-manifolds if and only if

$$\nabla_{e_i}\xi = -\eta(e_i)\psi = -\eta(e_i)\Big(\frac{\rho_2}{\tau\rho}e_1 + \frac{\rho_3}{\sigma\rho}e_2\Big),$$

where $i \in \{0, 1, 2\}$ with $e_0 = \xi$, i.e.

$$\nabla_\xi \xi = -\frac{\rho_2}{\tau\rho}e_1 - \frac{\rho_3}{\rho\sigma}e_2, \qquad \nabla_{e_1}\xi = \nabla_{e_2}\xi = 0.$$

From the above components of the Levi-Civita connection, we get

$$\tau_1 = \sigma_1 = 0.$$

**Example 12.5.** $C_{12}$**:**

We denote the Cartesian coordinates in a 3-dimensional Euclidean space $\mathbb{R}^3$ by $(x_1, x_2, x_3)$ and define a metric tensor $g$ by

$$g = e^{2f}\begin{pmatrix} \rho^2 + \tau^2 & 0 & -\tau \\ 0 & \rho^2 & 0 \\ -\tau & 0 & 1 \end{pmatrix},$$

where $f = f(y)$, $\tau = \tau(x)$ and $\rho = \rho(x, y)$ are functions on $\mathbb{R}^3$ with $f' = \frac{\partial f}{\partial y}$.

Further, we define an almost contact structure $(\varphi, \xi, \eta)$ on $\mathbb{R}^3$ by

$$\varphi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & -\tau & 0 \end{pmatrix}, \qquad \xi = e^{-f} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \eta = e^f \big( -\tau, 0, 1 \big).$$

Thus

$$\mathrm{d}\eta = f' e^f \bigg( \tau \mathrm{d}x \wedge \mathrm{d}y + \mathrm{d}y \wedge \mathrm{d}z \bigg) \quad \text{and} \quad \mathrm{d}\Phi = 0.$$

By direct computation, the non-zero components of $N_k^{(1)i} j$ are

$$N_{12}^{(1)3} = \tau f' \quad \text{and} \quad N_{23}^{(1)3} = f'.$$

On the other hand

$$(N_\varphi)^i_{kj} = 0, \qquad \forall i, j, k \in \{1, 2, 3\},$$

implies that the structure $(\varphi, \xi, \eta, g)$ is integrable. To ensure that the defined structure is not normal, it suffices to take $f' \neq 0$. Also, taking $\omega = f' dy$, we can see that

$$\mathrm{d}\eta = \omega \wedge \eta, \quad \omega(\xi) = 0 \quad \text{and} \quad \mathrm{d}\omega = 0.$$

We denote $\psi$ the $g$-dual of $\omega$

$$\psi = \frac{f'}{\rho^2} e^{2f} \frac{\partial}{\partial y}.$$

Thus, $(M, \varphi, \xi, \psi, \eta, \omega, g)$ is a $C_{12}$-structure on $\mathbb{R}^3$.

Section B

# Appendix B

**Proof to Proposition 1:** Using (11.1) from Definition 17, we have

$$\begin{cases} \mathrm{d}\eta = \alpha\Phi, \\ \mathrm{d}\Phi = 2\beta\eta\wedge\Phi, \end{cases} \implies \begin{cases} 0 = \mathrm{d}^2\eta = \mathrm{d}\alpha\wedge\Phi + \alpha\mathrm{d}\Phi, \\ \mathrm{d}\Phi = 2\beta\eta\wedge\Phi, \end{cases}$$

this leads to

$$(\mathrm{d}\alpha + 2\alpha\beta\eta)\wedge\Phi = 0 \implies X(\alpha) + 2\alpha\beta\eta(X) = 0, \quad \forall X\in\Gamma(TM).$$

The proof is concluded by setting $X = \xi$.

---

**Proof to Lemma 6:** Consider any vector fields $X$ and $Y$ on $M$. Then, by virtue of (10.1), (10.5) and (10.6):

1. For the first equality, we have

$$\Phi(X,Y) = g(X,\varphi Y) = -g(\varphi X, Y) = -\Phi(Y,X).$$

2. For the second one

$$\Phi(\varphi X,\varphi Y) = g(\varphi X,\varphi^2 Y) = g(\varphi X, -Y+\eta(X)\xi) = -g(\varphi X, Y) = \Phi(Y,X).$$

# *Metric Deformations*

PART

IV

In this chapter, we will explore different deformations of the metric tensor and analyze the resulting Levi-Civita connection and curvature tensors. These deformations will be applied to both Riemannian manifolds and certain classes of almost contact metric manifolds. Additionally, we will derive conditions of rigidity for the latter.

The concept of metric deformation serves as a fundamental tool in differential geometry, offering a versatile framework for exploring the intrinsic properties of geometric structures. Metric deformation involves systematically altering the metric tensor while preserving certain geometric properties, such as curvature or volume.



**Table 6**. Contents for Part IV

By allowing for controlled modifications to the metric, this approach enables mathematicians and physicists to investigate the behavior of geometric objects under perturbations, shedding light on the underlying symmetries and geometric relationships.Metric deformation assumes a fundamental role in investigating novel geometric structures. It serves a dual purpose by facilitating the examination of their rigidity while also enabling the generation of new examples from pre-existing structures through deformation. In essence, the notion of metric deformation embodies a powerful analytical technique, empowering researchers to uncover deeper insights into the intricate geometry of mathematical and physical systems. This chapter endeavors to delve into conformal metric deformation and its diverse manifestations. Through a meticulous examination of different scenarios, we aim to unravel the nuanced intricacies inherent in such deformations. As a practical application, we shall scrutinize these deformations within the context of select classes of almost contact metric manifolds, thereby elucidating their implications and ramifications in the broader domain of differential geometry.

Section 13

## Conformal Deformation

In the context of an $n$-dimensional Riemannian manifold $(M, g)$, a conformal deformation refers to a modification of the metric $g$ in the following manner

$$\tilde{g} = f^2 g, \tag{13.1}$$

where $f$ represents a smooth non-zero function defined on $M$. Utilizing Kozsul's formula (6.5), we obtain

$$\begin{aligned}
2\tilde{g}(\nabla_X Y, Z) &= X(\tilde{g}(Y, Z)) + Y(\tilde{g}(X, Z)) - Z(\tilde{g}(X, Y)) + \tilde{g}([X, Y], Z) + \tilde{g}([Z, Y], X) - \tilde{g}([Y, Z], X) \\
&= f^2\Big(Xg(Y, Z) + Y(g(X, Z) - Zg(X, Y) + g([X, Y], Z) + g([Z, Y], X) - g([Y, Z], X)\Big) \\
&\quad + X(f^2)g(Y, Z) + Y(f^2)g(X, Z) - Z(f^2)g(X, Y).
\end{aligned}$$

By employing Definition 1 and formula (7.1), we obtain

$$\begin{cases}
1. \quad X(f^2)g(Y, Z) = g\big(X(f^2)Y, Z\big), \\
2. \quad Y(f^2)g(X, Z) = g\big(Y(f^2)X, Z\big), \\
3. \quad Z(f^2)g(X, Y) = g(\mathrm{grad} f^2, Z)g(X, Y) = g\big(g(X, Y)\mathrm{grad} f^2, Z\big).
\end{cases}$$

Furthermore, derived from equation (13.1), we have

$$2\tilde{g}(\tilde{\nabla}_X Y, Z) = 2f^2 g(\tilde{\nabla}_X Y, Z).$$

This leads to the following conclusion

$$2f^2 g(\tilde{\nabla}_X Y, Z) = 2f^2 g(\nabla_X Y, Z) + g\big(Y(f^2)X, Z\big) + g\big(Y(f^2)X, Z\big) - g\big(g(X,Y)\mathrm{grad} f^2, Z\big),$$

which gives

$$\tilde{\nabla}_X Y = \nabla_X Y + \frac{Y(f^2)}{f^2}X + \frac{X(f^2)}{f^2}Y - \frac{\mathrm{grad} f^2}{f^2}g(X,Y).$$

Utilizing the fact that $X(\ln f^2) = \frac{X(f^2)}{f^2}$ and setting $h = \ln f^2$, the following result is attained

$$\tilde{\nabla}_X Y = \nabla_X Y + Y(h)X + X(h)Y - g(X,Y)\mathrm{grad}(h). \tag{13.2}$$

**Proposition 5** By conformal deformation, the Levi-Cevita connection $\tilde{\nabla}$ is provided by formula (13.2)

In light of equation (8.1), we have

$$\tilde{R}(X,Y)Z = \tilde{\nabla}_X\tilde{\nabla}_Y Z - \tilde{\nabla}_Y\tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]}Z. \tag{13.3}$$

Through lengthy direct computations employing equation (13.2), we obtain

$$\begin{aligned}\tilde{\nabla}_X\tilde{\nabla}_Y Z = {} & \nabla_X\nabla_Y Z - g(Y,Z)\big(\nabla_X \mathrm{grad}h + |\mathrm{grad}h|^2 X\big) + \big((\nabla_X Y)(h) + \mathcal{H}ess_h(X,Z)\big)Z \\ & + Y(h)\big(\nabla_X Z + Z(h)X + X(h)Z - g(X,Z)\mathrm{grad}h\big) + \big((\nabla_X Z)(h) + \mathcal{H}ess_h(X,Z)\big)Y \\ & + Z(h)\big(\nabla_X Y + Y(h)X + X(h)Y - g(X,Y)\mathrm{grad}h\big) + X(h)\nabla_Y Z + (\nabla_Y Z)(h)X \\ & - \big(g(\nabla_X Y, Z) + g(Y, \nabla_X Z)\big)\mathrm{grad}h.\end{aligned} \tag{13.4}$$

Likewise, in a similar manner

$$\begin{aligned}\tilde{\nabla}_Y\tilde{\nabla}_X Z = {} & \nabla_Y\nabla_X Z - g(X,Z)\big(\nabla_Y \mathrm{grad}h + |\mathrm{grad}h|^2 Y\big) + \big((\nabla_Y X)(h) + \mathcal{H}ess_h(Y,Z)\big)Z \\ & + X(h)\big(\nabla_Y Z + Z(h)Y + Y(h)Z - g(Y,Z)\mathrm{grad}h\big) + \big((\nabla_Y Z)(h) + \mathcal{H}ess_h(Y,Z)\big)X \\ & + Z(h)\big(\nabla_Y X + X(h)Y + Y(h)X - g(Y,X)\mathrm{grad}h\big) + Y(h)\nabla_X Z + (\nabla_X Z)(h)Y \\ & - \big(g(\nabla_Y X, Z) + g(X, \nabla_Y Z)\big)\mathrm{grad}h.\end{aligned} \tag{13.5}$$

Similarly

$$\tilde{\nabla}_{[X,Y]}Z = \nabla_{[X,Y]}Z + Z(h)\big(\nabla_X Y - \nabla_Y X\big) + (\nabla_X Y)(h)Z - (\nabla_Y X)(h)Z - \big(g(\nabla_X Y, Z) - g(\nabla_Y X, Z)\big)\mathrm{grad}h. \tag{13.6}$$

Hence, $\tilde{R}$ arises through the substitution of equations (13.4), (13.5), and (13.6) into equation (13.3)

$$\begin{aligned}\tilde{R}(X,Y)Z = {} & R(X,Y)Z - g(Y,Z)\big(\nabla_X \mathrm{grad}h + |\mathrm{grad}h|^2 X\big) + g(X,Z)\big(\nabla_Y \mathrm{grad}h + |\mathrm{grad}h|^2 Y\big) \\ & + \big(X(h)g(Y,Z) - Y(h)g(X,Z)\big)\mathrm{grad}h + \mathcal{H}ess_h(X,Z)Y - \mathcal{H}ess_h(Y,Z)X \\ & + \big(g(X, \nabla_Y Z) - g(Y, \nabla_X Z)\big)\mathrm{grad}h + Z(h)\big(Y(h)X - X(h)Y\big).\end{aligned} \tag{13.7}$$

**Proposition 6** By conformal deformation, the Riemann curvature tensor $\tilde{R}$ is provided by formula (13.7).

Let us examine the orthonormal basis $\{\tilde{e}_i\}_{\{1<i<n\}}$ such that

$$\tilde{e}_i = e^{-h}e_i.$$

Drawing from equations (9.1) and (13.7), we discern

$$
\begin{aligned}
\tilde{Q}X &= \sum_{i=1}^{2n} \tilde{R}(X,\tilde{e}_i)\tilde{e}_i = e^{-h}\sum_{i=1}^{n} \tilde{R}(X,e_i)e_i \\
&= e^{-h}\Big( R(X,e_i)e_i - g(e_i,e_i)\big(\nabla_X \mathrm{grad}h + |\mathrm{grad}h|^2 X\big) + g(X,e_i)\big(\nabla_{e_i}\mathrm{grad}h + |\mathrm{grad}h|^2 e_i\big) \\
&+ \big(X(h)g(e_i,e_i) - e_i(h)g(X,e_i)\big)\mathrm{grad}h + \mathcal{H}ess_f(X,e_i)e_i - \mathcal{H}ess_f(e_i,e_i)X \\
&+ \big(g(X,\nabla_{e_i}e_i) - g(e_i,\nabla_X e_i)\big)\mathrm{grad}h + e_i(h)\big(e_i(h)X - X(h)e_i\big).\Big)
\end{aligned} \tag{13.8}
$$

With the understanding that $g(X,\nabla_{e_i}e_i) = g(e_i,\nabla_X e_i) = 0$, in conjunction with formulas (5.2) and (7.1), we deduce the following facts

$$
\begin{cases}
\sum_{i=1}^n g(X,e_i)\nabla_{e_i}\mathrm{grad}h = \nabla_{\sum_{i=1}^n g(X,e_i)e_i}\mathrm{grad}h = \nabla_X\mathrm{grad}h, \\
\sum_{i=1}^n g(X,e_i)|\mathrm{grad}h|^2 e_i = |\mathrm{grad}h|^2 \sum_{i=1}^n g(X,e_i)e_i = |\mathrm{grad}h|^2 X, \\
\sum_{i=1}^n e_i(h)g(X,e_i) = \sum_{i=1}^n g(X,e_i(h)e_i) = g(X,\mathrm{grad}h) = X(h), \\
\sum_{i=1}^n \mathcal{H}_h(X,e_i)e_i = \sum_{i=1}^n g(\nabla_X\mathrm{grad}h,e_i)e_i = \nabla_X\mathrm{grad}h,
\end{cases}
$$

and lastly

$$
\sum_{i=1}^n e_i(h)e_i(h) = \sum_{i=1}^n g(\mathrm{grad}h,e_i)g(\mathrm{grad}h,e_i) = \sum_{i=1}^n g(\mathrm{grad}h, g(\mathrm{grad}h,e_i)e_i) = \sum_{i=1}^n g(\mathrm{grad}h,\mathrm{grad}h) = |\mathrm{grad}h|^2.
$$

Therefore, upon substitution into formula (13.8), the following outcome is obtained

$$
\tilde{Q}X = e^{-h}\Big( QX - \Delta(h)X - (n-2)\big(X(h)\mathrm{grad}h + \nabla_X\mathrm{grad}h + |\mathrm{grad}h|^2 X\big)\Big). \tag{13.9}
$$

**Proposition 7** By conformal deformation, the Ricci operator $\tilde{Q}$ is provided by formula (13.9).

By considering the inner product $\tilde{g}$ of equation (13.9) with $Y$, and recognizing the relationship $e^{-h} = \frac{1}{f^2}$, we obtain

$$
\tilde{S}(X,Y) = \tilde{g}(\tilde{Q}X,Y) = S(X,Y) - \Delta(h)g(X,Y) - (n-2)\big(X(h)g(\mathrm{grad}h,Y) + g(\nabla_X\mathrm{grad}h,Y) + |\mathrm{grad}h|^2 g(X,Y)\big).
$$

The conclusion follows promptly from equations (7.1) and (7.5)

$$
\tilde{S}(X,Y) = S(X,Y) - (n-2)\big(\mathcal{H}_h(X,Y) + X(h)Y(h)\big) - \big(\Delta(h) + (n-2)|\mathrm{grad}h|^2\big)g(X,Y). \tag{13.10}
$$

**Proposition 8** By conformal deformation, the Ricci curvature tensor $\tilde{S}$ is provided by formula (13.10).

Employing equations (9.3) and (13.10) the following is obtained

$$
\begin{aligned}
\tilde{r} &= \sum_{i=1}^n \tilde{S}(\tilde{e}_i,\tilde{e}_i) = e^{-h}\sum_{i=1}^n \tilde{S}(e_i,e_i) = e^{-h}\sum_{i=1}^n \Big( S(e_i,e_i) - (n-2)\big(\mathcal{H}_h(e_i,e_i) + e_i(h)e_i(h)\big) - \big(\Delta(h) + (n-2)|\mathrm{grad}h|^2\big)\Big) \\
&= e^{-h}\Big( r - (n-2)\big(\Delta(h) + |\mathrm{grad}h|^2\big) - n\big(\Delta(h) + (n-2)|\mathrm{grad}h|^2\big)\Big).
\end{aligned}
$$

Through the straightforward rearrangement of analogous terms

$$
\tilde{r} = e^{-h}\big(r - 2(n-1)\Delta(h) - (n-1)(n-2)|\mathrm{grad}h|^2\big). \tag{13.11}
$$

**Proposition 9** Under conformal deformation the scalar curvature $\tilde{r}$ associated with $\tilde{g}$ is provided by formula (13.11).

Section 14

# $\mathcal{D}$-Isometric Deformation

The findings delineated within this section are documented in the scholarly [31]. This form of deformation becomes feasible exclusively when the tangent bundle $\Gamma(TM)$ of the manifold admits a split structure represented as

$$\Gamma(TM) = \mathcal{D} \oplus \{\xi\},$$

where $\xi$ signifies a non-trivial vector field on $M$. Consequently, employing such a deformation method proves advantageous for delving into the properties of almost contact metric manifolds, given the inherent existence of $\xi$ within this structural context. However, as our initial published findings encompassed a broader scope, encompassing what is known as **compact gradient manifolds**, we shall defer the exploration of almost contact metric manifolds to a subsequent section, where a more comprehensive deformation shall be investigated.
A Riemannian manifold $(M, g)$ of dimension $n$ is said **compact gradient**, if there exists a unit closed 1-form $\eta$ (i.e. $\mathrm{d}\eta = 0$), such that

$$g(\nabla_X \xi, Y) = g(\nabla_Y \xi, X), \qquad g(\nabla_X \xi, \xi) = 0 \qquad \text{and} \qquad \nabla_\xi \xi = 0. \tag{14.1}$$

where $\eta$ be the $g$-dual of $\xi$ which means $\eta(X) = g(X, \xi)$ for all vector field $X$ on $M$. On the other hand, $\xi$ is said a Jacobi-Type vector field if and only if [23]

$$\nabla_X \nabla_Y \xi - \nabla_{\nabla_X Y} \xi - R(X, \xi)Y = 0. \tag{14.2}$$

Substituting $Y = \xi$ in (14.2) and using (14.1), one can obtain

$$\nabla_{\nabla_X \xi} \xi + R(X, \xi)\xi = 0. \tag{14.3}$$

From [24], we have the following deep result

**Theorem 9** Every Jacobi-type vector field on a compact Riemannian manifold is a Killing vector field.

We define on $M$ a Riemannian metric, denoted $\tilde{g}$, by

$$\tilde{g}(X, Y) = g(X, Y) + \eta(X)\eta(Y) \quad \text{where} \quad \eta(\xi) = 1. \tag{14.4}$$

The equation $\eta = 0$ defines a $(n-1)$-dimensional distribution $\mathcal{D}$ on $M$. Then, we have

$$\begin{cases} \tilde{g}(\xi, \xi) = 2, \\ \tilde{g}(X, X) = g(X, X), \qquad \forall X \in \mathcal{D}. \end{cases}$$

That is why, we refer to this construction as $\mathcal{D}$-isometric deformation. Note that the simplest case for this deformation is for $\eta = \mathrm{d}f$ where $f \in \mathcal{C}^\infty(M)$. In [46], **Innami** proved that $M$ admits a non constant affine function if and only if $M$ splits as a Riemannian product $M = N \times \mathbb{R}$. In our situation, since $\mathrm{d}\eta = 0$ then $\xi$ is locally of type gradient, what means that at each point $p$ on $M$ there exists a function $f$ such that $\xi = \nabla f$ on a neighborhood at $p$, where $\nabla f$ denotes the gradient vector field of $f$. Using Koszul's formula (6.5) for the metric $\tilde{g}$ (14.4)

$$\begin{aligned} 2\tilde{g}(\tilde{\nabla}_X Y, Z) &= X\tilde{g}(Y, Z) + Y\tilde{g}(Z, X) - Z\tilde{g}(X, Y) - \tilde{g}(X, [Y, Z]) + \tilde{g}(Y, [Z, X]) + \tilde{g}(Z, [X, Y] \\ &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]) \\ &+ X\big(\eta(Y)\eta(Z)\big) + Y\big(\eta(Z)\eta(X)\big) - Z\big(\eta(X)\eta(Y)\big) - \eta(X)\eta([Y, Z]) + \eta(Y)\eta([Z, X]) + \eta(Z)\eta([X, Y]), \end{aligned}$$

one can obtain

$$\tilde{g}(\tilde{\nabla}_X Y, Z) = \tilde{g}(\nabla_X Y, Z) + \frac{1}{2}\big((\nabla_X \eta)Y + (\nabla_Y \eta)X\big)\eta(Z).$$

Knowing that $\mathrm{d}\eta = 0$, by virtue of (6.17)

$$2\mathrm{d}\eta(X,Y) = (\nabla_X \eta)Y - (\nabla_Y \eta)X = 0 \implies (\nabla_X \eta)Y = (\nabla_Y \eta)X.$$

Then, we get

$$\tilde{g}(\tilde{\nabla}_X Y, Z) = \tilde{g}(\nabla_X Y, Z) + (\nabla_X \eta)(Y)\eta(Z),$$

with

$$(\nabla_X \eta)(Y) = g(\nabla_X \xi, Y).$$

On the other hand, we have

$$\eta(Z) = g(\xi, Z) = \tilde{g}(\xi, Z) - \eta(Z),$$

which gives

$$\eta(Z) = \frac{1}{2}\tilde{g}(\xi, Z).$$

Therefore

$$\tilde{g}(\tilde{\nabla}_X Y, Z) = \tilde{g}(\nabla_X Y, Z) + \frac{1}{2}(\nabla_X \eta)(Y)\tilde{g}(Z,\xi) = \tilde{g}(\nabla_X Y, Z) + \frac{1}{2}g(\nabla_X \xi, Y)\tilde{g}(Z,\xi) = \tilde{g}(\nabla_X Y, Z) + \frac{1}{2}\tilde{g}(Z, g(\nabla_X \xi, Y)\xi).$$

Hence, employing the non-degeneracy property inherent to the metric tensor denoted as $\tilde{g}$ yields the final result

$$\tilde{\nabla}_X Y = \nabla_X Y + \frac{1}{2}g(\nabla_X \xi, Y)\xi, \tag{14.5}$$

**Proposition 10** Under $\mathcal{D}$-isometric deformation the Levi-Cevita connection $\tilde{\nabla}$ is provided by formula (14.5).

By utilizing formula (14.5), the first term of (13.3) is rendered as

$$\begin{aligned}\tilde{\nabla}_X \tilde{\nabla}_Y Z &= \tilde{\nabla}_X\Big(\nabla_Y Z + \frac{1}{2}g(\nabla_Y \xi, Z)\xi\Big) = \tilde{\nabla}_X \nabla_Y Z + \frac{1}{2}g(\nabla_X \nabla_Y \xi, Z)\xi + \frac{1}{2}g(\nabla_Y \xi, \nabla_X Z)\xi + \frac{1}{2}g(\nabla_Y \xi, Z)\tilde{\nabla}_X \xi \\ &= \nabla_X \nabla_Y Z + \frac{1}{2}g(\nabla_X \xi, \nabla_Y Z)\xi + \frac{1}{2}g(\nabla_X \nabla_Y \xi, Z)\xi + \frac{1}{2}g(\nabla_Y \xi, \nabla_X Z)\xi + \frac{1}{2}g(\nabla_Y \xi, Z)\nabla_X \xi. \end{aligned}\tag{14.6}$$

Employing a similar methodology, the second term of (14.6) is determined as

$$\tilde{\nabla}_Y \tilde{\nabla}_X Z = \nabla_Y \nabla_X Z + \frac{1}{2}g(\nabla_Y \xi, \nabla_X Z)\xi + \frac{1}{2}g(\nabla_Y \nabla_X \xi, Z)\xi + \frac{1}{2}g(\nabla_X \xi, \nabla_Y Z)\xi + \frac{1}{2}g(\nabla_X \xi, Z)\nabla_Y \xi. \tag{14.7}$$

Given that $\mathrm{d}\eta = 0$, the final term of (14.9) simplifies to

$$\begin{aligned}\tilde{\nabla}_{[X,Y]} Z &= \nabla_{[X,Y]} + \frac{1}{2}g(\nabla_{[X,Y]}\xi, Z)\xi = \nabla_{[X,Y]} + \frac{1}{2}g(\nabla_{\nabla_X Y}\xi, Z)\xi - \frac{1}{2}g(\nabla_{\nabla_Y X}\xi, Z)\xi \\ &= \nabla_{[X,Y]} + \frac{1}{2}g(\nabla_Z \xi, \nabla_X Y)\xi - \frac{1}{2}g(\nabla_Z \xi, \nabla_Y X)\xi. \end{aligned}\tag{14.8}$$

A formula for $\tilde{R}$ ensues from the substitution of equations (14.6), (14.7), and (14.8) into (13.3)

$$2\tilde{R}(X,Y)Z = 2R(X,Y)Z - g\big(R(X,Y)Z, \xi\big)\xi + g(\nabla_Y \xi, Z)\nabla_X \xi - g(\nabla_X \xi, Z)\nabla_Y \xi. \tag{14.9}$$

**Proposition 11** Under $\mathcal{D}$-isometric deformation the Riemann curvature tensor $\tilde{R}$ is provided by formula (14.9).

let $\{\xi, e_i\}_{2\le i\le n}$ denote the orthonormal basis on $M$ with respect to the metric $g$. It can be readily shown that $\{\frac{1}{\sqrt{2}}\xi, e_i\}_{2\le i\le n}$ forms an orthonormal basis on $M$ with respect to the metric $\tilde{g}$. Utilizing formulas (9.1) and (14.9), we obtain

$$\begin{aligned}\tilde{Q}X &= \frac{1}{2}\tilde{R}(X,\xi)\xi + \sum_{i=2}^{n}\tilde{R}(X,e_i)e_i = -\frac{1}{2}\tilde{R}(X,\xi)\xi + \sum_{i=1}^{n}\tilde{R}(X,e_i)e_i \\ &= -\frac{1}{2}R(X,\xi)\xi + \sum_{i=1}^{n}\Big(R(X,e_i)e_i - \frac{1}{2}g\big(R(X,e_i)e_i,\xi\big)\xi + \frac{1}{2}g(\nabla_{e_i}\xi,e_i)\nabla_X\xi - \frac{1}{2}g(\nabla_X\xi,e_i)\nabla_{e_i}\xi\Big). \end{aligned} \tag{14.10}$$

Utilizing equations (5.2), (7.2), (14.1), and the fact that

$$\sum_{i=1}^{n} g(\nabla_X\xi,e_i)\nabla_{e_i}\xi = \nabla_{\sum_{i=1}^{n} g(\nabla_X\xi,e_i)e_i}\xi = \nabla_{\nabla_X\xi}\xi = -R(X,\xi)\xi,$$

we drive the following

$$2\tilde{Q}X = 2QX - S(X,\xi)\xi + (\mathrm{div}\xi)\nabla_X\xi. \tag{14.11}$$

**Proposition 12** Under $\mathcal{D}$-isometric deformation the Ricci operator $\tilde{Q}$ is provided by formula (14.11).

Employing formulas (9.2) and (14.4)

$$\begin{aligned}\tilde{S}(X,Y) &= \tilde{g}(\tilde{Q}X,Y) = g(\tilde{Q}X,Y) + \eta(\tilde{Q}X)\eta(Y) \\ &= g(QX,Y) - \frac{1}{2}S(X,\xi)g(Y,\xi) + \frac{1}{2}(\mathrm{div}\xi)g(\nabla_X\xi,Y) + \Big(\eta(QX) - \frac{1}{2}S(X,\xi) + \frac{1}{2}(\mathrm{div}\xi)\eta(\nabla_X\xi)\Big)\eta(Y),\end{aligned}$$

suffice to recall that $\eta(QX) = g(QX,\xi) = S(X,\xi)$ and $\eta(\nabla_X\xi) = 0$. Hence, we obtain

$$2\tilde{S}(X,Y) = 2S(X,Y) + \mathrm{div}\xi g(\nabla_X\xi,Y). \tag{14.12}$$

**Proposition 13** Under $\mathcal{D}$-isometric deformation the Ricci curvature tensor $\tilde{S}$ is provided by formula (14.12).

**Corollary 5** Let $\tilde{r}$ (resp. $r$) be the Ricci operator assoaciated with $\tilde{g}$ (resp.$g$). Then:

$$2\tilde{r} = 2r - \frac{1}{2}S(\xi,\xi) + (\mathrm{div}\xi)^2. \tag{14.13}$$

Proof Through direct computations utilizing equations (7.2), (9.3), and (14.12). □

Section 15

# Generalized $\mathcal{D}$-Conformal Deformation

This section is a compilation of results from [1, 61]. Consider an almost contact metric manifold $(M,\varphi,\eta,\xi,g)$, if one takes

$$\tilde{g} = b^2 g + (a^2 - b^2)\eta\otimes\eta, \quad \tilde{\varphi} = \varphi, \quad \tilde{\eta} = a\eta, \quad \tilde{\xi} = \frac{1}{a}\xi, \tag{15.1}$$

where $a$ and $b$ are smooth, non-zero functions on $M$. The above transformation is called **generalized $\mathcal{D}$-conformal deformation**. This deformation generalizes the previous ones:

$$\begin{cases} 1. & \text{If } a = b \text{ we get conformal deformation.} \\ 2. & \text{If } b = 1 \text{ and } a = \sqrt{2} \text{ we get the } \mathcal{D}\text{-isometric deformation (14.4).} \\ 3. & \text{If } a = \pm 1 \text{ we get the deformation of Olzsak.} \\ 4. & \text{If } a = b^2 \text{ we get the case of Tanno, } \mathcal{D}\text{-homothetic deformation.} \end{cases}$$

**Theorem 10** $(M, \varphi, \tilde{\eta}, \tilde{\xi}, \tilde{g})$ is an almost contact metric manifold.

Proof One must make sure that the data $(M, \varphi, \tilde{\eta}, \tilde{\xi}, \tilde{g})$ satisfies (10.1),(**??**) and (10.5):

1. For (10.1), we have
$$\tilde{\varphi}^2 = \varphi^2 = -Id + \eta \otimes \xi = -Id + a\eta \otimes \frac{1}{a}\xi = -Id + \tilde{\eta} \otimes \tilde{\xi}.$$
2. On the other hand, by direct substitution, we have
$$\tilde{\eta}(\tilde{\xi}) = a\eta\left(\frac{1}{a}\xi\right) = a.\frac{1}{a}\eta(\xi) = 1.$$
3. Finally
$$\tilde{g}(\tilde{\varphi}X, \tilde{\varphi}Y) = b^2 g(\varphi X, \varphi Y) + (a^2 - b^2)\eta(\varphi X)\eta(\varphi Y) = b^2 g(X, Y) - b^2\eta(X)\eta(Y).$$
Using (15.1), we substitute
$$b^2 g(X, Y) = \tilde{g}(X, Y) - (a^2 - b^2)\eta(X)\eta(Y),$$
and we get
$$\tilde{g}(\tilde{\varphi}X, \tilde{\varphi}Y) = \tilde{g}(X, Y) - a^2\eta(X)\eta(Y) = \tilde{g}(X, Y) - \tilde{\eta}(X)\tilde{\eta}(Y),$$

of which the proof is concluded. □

On the other hand, using (10.7) we have

$$\tilde{\Phi}(X, Y) = \tilde{g}(X, \varphi Y) = b^2 g(X, \varphi Y).$$

Hence

$$\tilde{\Phi}(X, Y) = b^2\Phi(X, Y). \tag{15.2}$$

where $\tilde{\Phi}$ is the fundamental 2-form associated with $(M, \tilde{g})$. Using formulas (10.7) and (15.1) we get

$$\begin{cases} \mathrm{d}\tilde{\eta} = \mathrm{d}a \wedge \eta + a\mathrm{d}\eta, \\ \mathrm{d}\tilde{\Phi} = 2b.\mathrm{d}b \wedge \Phi + b\mathrm{d}\Phi. \end{cases} \tag{15.3}$$

With the aid of Kozsul's formula (6.5)

$$2\tilde{g}(\tilde{\nabla}_X Y, Z) = X(\tilde{g}(Y, Z)) + Y(\tilde{g}(Z, X)) - Z(\tilde{g}(X, Y)) - \tilde{g}(X, [Y, Z]) + \tilde{g}(Y, [Z, X]) + \tilde{g}(Z, [X, Y]).$$

It is sufficient to compute $X(\tilde{g}(Y, Z))$

$$\begin{aligned} X(\tilde{g}(Y, Z)) &= X\Big(b^2 g(Y, Z) + (a^2 - b^2)\eta(Y)\eta(Z)\Big) = 2bX(b)g(Y, Z) + b^2 X\big(g(Y, Z)\big) + 2\Big(aX(a) - bX(b)\Big)\eta(Y)\eta(Z) \\ &+ (a^2 - b^2)\Big[\big\{(\nabla_X\eta)Y + \eta(\nabla_X Y)\big\}\eta(Z) + \eta(Y)\big\{(\nabla_X\eta)Z + \eta(\nabla_X Z)\big\}\Big] \\ &= 2bX(b)g(\varphi Y, \varphi Z) + b^2 X\big(g(Y, Z)\big) + 2aX(a)\eta(Y)\eta(Z) \\ &+ (a^2 - b^2)\Big[\big\{(\nabla_X\eta)Y + \eta(\nabla_X Y)\big\}\eta(Z) + \eta(Y)\big\{(\nabla_X\eta)Z + \eta(\nabla_X Z)\big\}\Big], \end{aligned}$$

subsequently permute the variables $X$, $Y$ and $Z$ to derive $Y(\tilde{g}(Z,X))$ and $Z(\tilde{g}(X,Y))$

$$\begin{aligned}Y(\tilde{g}(Z,X)) &= 2bY(b)g(\varphi Z,\varphi X) + b^2 Y\big(g(Z,X)\big) + 2aY(a)\eta(Z)\eta(X)\\ &\quad + (a^2-b^2)\Big[\big\{(\nabla_Y\eta)Z + \eta(\nabla_Y Z)\big\}\eta(X) + \eta(Z)\big\{(\nabla_Y\eta)X + \eta(\nabla_Y X)\big\}\Big],\\ Z(\tilde{g}(X,Y)) &= 2bZ(b)g(\varphi X,\varphi Y) + b^2 Z\big(g(X,Y)\big) + 2aZ(a)\eta(X)\eta(Y)\\ &\quad + (a^2-b^2)\Big[\big\{(\nabla_Z\eta)X + \eta(\nabla_Z X)\big\}\eta(Y) + \eta(X)\big\{(\nabla_Z\eta)Y + \eta(\nabla_Z Y)\big\}\Big].\end{aligned}$$

For the terms enclosed in brackets, we have

$$\begin{aligned}\tilde{g}(X,[Y,Z]) &= b^2 g(X,[Y,Z]) + (a^2-b^2)\eta(X)\eta([Y,Z]) = b^2 g(X,[Y,Z]) + (a^2-b^2)\eta(X)\Big(\eta(\nabla_Y Z) - \eta(\nabla_Z Y)\Big),\\ \tilde{g}(Y,[Z,X]) &= b^2 g(Y,[Z,X]) + (a^2-b^2)\eta(Y)\eta([Z,X]) = b^2 g(Y,[Z,X]) + (a^2-b^2)\eta(Y)\Big(\eta(\nabla_Z X) - \eta(\nabla_X Z)\Big),\\ \tilde{g}(Z,[X,Y]) &= b^2 g(Z,[X,Y]) + (a^2-b^2)\eta(Z)\eta([X,Y]) = b^2 g(Z,[X,Y]) + (a^2-b^2)\eta(Z)\Big(\eta(\nabla_X Y) - \eta(\nabla_Y X)\Big).\end{aligned}$$

Therefore, after regrouping and rearranging appropriately, we obtain

$$\begin{aligned}2\tilde{g}(\tilde{\nabla}_X Y, Z) = b^2\Big(&X\big(g(Y,Z) + Y\big(g(Z,X)\big) - Z\big(g(X,Y) - g(X,[Y,Z]) - g(Y,[Z,X]) + g(Z,[X,Y])\big)\Big)\\ &+ 2b\Big(X(b)g(\varphi Y,\varphi Z) + Y(b)g(\varphi Z,\varphi X) - Z(b)g(\varphi X,\varphi Y)\Big) + 2a\Big(X(a)\eta(Y)\eta(Z) + Y(a)\eta(X)\eta(Z)\\ &- Z(a)\eta(X)\eta(Y)\Big) + (a^2-b^2)\Big(\big\{(\nabla_X\eta)Y + \eta(\nabla_X Y) + (\nabla_Y\eta)X + \eta(\nabla_Y X) + \eta(\nabla_X Y)\\ &- \eta(\nabla_Y X)\big\}\eta(Z) + \big\{(\nabla_X\eta)Z + \eta(\nabla_X Z) - (\nabla_Z\eta)X - \eta(\nabla_Z X) + \eta(\nabla_Z X) - \eta(\nabla_X Z)\big\}\eta(Y)\\ &+ \big\{(\nabla_Y\eta)Z + \eta(\nabla_Y Z) - (\nabla_Z\eta)Y - \eta(\nabla_Z Y) - \eta(\nabla_Y Z) + \eta(\nabla_Z Y)\big\}\eta(X)\Big).\end{aligned}$$

By utilizing (6.27), we can identify that

$$(\mathcal{L}_\xi g)(X,Y) = (\nabla_X\eta)Y + (\nabla_Y\eta)X,$$

and recalling from (6.17)

$$\begin{cases} 2\mathrm{d}\eta(X,Z) = (\nabla_X\eta)Z - (\nabla_Z\eta)X,\\ 2\mathrm{d}\eta(Y,Z) = (\nabla_Y\eta)Z - (\nabla_Z\eta)Y.\end{cases}$$

The outcome arises upon identification of the initial term as Koszul's formula enacted on $g$

$$\begin{aligned}2\tilde{g}(\tilde{\nabla}_X Y, Z) = 2b^2 g(\nabla_X Y, Z) &+ 2(a^2-b^2)\eta(\nabla_X Y)\eta(Z) + 2b\Big(X(b)g(\varphi Y,\varphi Z) + Y(b)g(\varphi Z,\varphi X) - Z(b)g(\varphi X,\varphi Y)\Big)\\ &+ 2a\Big(X(a)\eta(Y)\eta(Z) + Y(a)\eta(X)\eta(Z) - Z(a)\eta(X)\eta(Y)\Big) + (a^2-b^2)\Big((\mathcal{L}_\xi g)(X,Y)\eta(Z)\\ &+ 2\mathrm{d}\eta(X,Z)\eta(Y) + 2\mathrm{d}\eta(Y,Z)\eta(X)\Big).\end{aligned} \tag{15.4}$$

As an illustrative example, we will employ a generalized $\mathcal{D}$-conformal deformation on cosymplectic, $\alpha$-Sasakian, $\beta$-Kenmotsu and $C_{12}$ manifolds.

Subsection 15.1
## Cosymplectic manifolds

Let $(M, \varphi, \xi, \eta, g)$ denote a cosymplectic manifold. Utilizing equation Definition 17, the differentiation of $\tilde{\eta}$ and $\tilde{\Phi}$ as expressed in (15.1) yields

$$\begin{cases} \mathrm{d}\tilde{\eta} = \mathrm{d}a \wedge \eta, \\ \mathrm{d}\tilde{\Phi} = 2b\mathrm{d}b \wedge \Phi. \end{cases} \tag{15.5}$$

**Case 1:** When $(M, \varphi, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is $\beta$-Kenmotsu, the ensuing conditions must be satisfied

$$\begin{cases} \mathrm{d}\tilde{\eta} = 0, \\ \mathrm{d}\tilde{\Phi} = 2\beta\tilde{\eta} \wedge \tilde{\Phi} = 2ab^2\beta\eta \wedge \Phi. \end{cases} \tag{15.6}$$

Therefore, we derive

$$2b\mathrm{d}b \wedge \Phi = 2\beta\tilde{\eta} \wedge \tilde{\Phi} = 2ab^2\beta\eta \implies \big(ab\beta\eta - \mathrm{d}b\big) \wedge \Phi = 0. \tag{15.7}$$

The aforementioned relations lead to the following conclusions for every vector field $X$ on $M$

$$ab\beta\eta(X) = \mathrm{d}b(X). \tag{15.8}$$

By evaluating both sides for $X = \xi$ and $X = V \perp \xi$, we obtain

$$V(b) = 0 \implies \mathrm{grad}b = \xi(b)\xi \quad \text{and} \quad \beta = \frac{\xi(b)}{ab}. \tag{15.9}$$

**Case 2:** When $(M, \varphi, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is a $C_{12}$ manifold, the following conditions must hold

$$\begin{cases} \mathrm{d}\tilde{\eta} = \omega \wedge \tilde{\eta}, \quad \omega(\tilde{\xi}) = 0, \\ \mathrm{d}\omega = 0, \\ \mathrm{d}\tilde{\Phi} = 0, \end{cases} \tag{15.10}$$

where $\psi$ represents the $\tilde{g}$-dual of $\omega$. From equations (11.1) $(\alpha = \beta = 0)$, we obtain

$$\omega = \mathrm{d}\ln(a) \quad \text{with} \quad \xi(a) = 0 \quad \text{and} \quad b \in \mathbb{R}^*_+.$$

The resultant $C_{12}$ structure is defined by $(\Phi, \tilde{\eta}, \omega, \tilde{\xi}, \psi)$ where

$$\omega = \mathrm{d}\ln(a), \quad \psi = \mathrm{grad}\ln(a), \quad \tilde{\eta}(\psi) = 0 \quad \text{and} \quad \omega(\tilde{\xi}) = 0.$$

**Case 3:** When $(M, \varphi, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is a cosymplectic manifold, this case has been previously investigated in [60]. Thus, summarizing all the preceding analyses, we arrive at the following conclusions:

**Theorem 11** Let $(M, \varphi, \xi, \eta, g)$ be a cosymplectic manifold. Then under generalized $\mathcal{D}$-conformal deformation, we have:

- $(M, \varphi, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is cosymplectic if $\mathrm{grad}a = \xi(a)\xi$ and $b \in \mathbb{R}^*$ [**?** ].
- $(M, \varphi, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is $\beta$-Kenmotsu if $a \in \mathbb{R}^*$ and $\mathrm{grad}b = \xi(b)\xi$.
- $(M, \varphi, \tilde{\eta}, \omega, \tilde{\xi}, \psi)$ is $C_{12}$-manifold if $\xi(a) = 0$ and $b = 1$.

The rigid scenario holds significant relevance for our concluding section. We examine a cosymplectic manifold $(M,\varphi,\xi,\eta,g)$ which remains invariant under generalized $\mathcal{D}$-conformal deformation, characterized by the conditions grad$a=\xi(a)\xi$ and $b\in\mathbb{R}^*$. Under the assumptions grad$a=\xi(a)\xi$ and $b\in\mathbb{R}^*$, formula from (15.4) becomes

$$2\tilde{g}(\tilde{\nabla}_X Y,Z)=2\tilde{g}(\nabla_X Y,Z)+2\xi(a)\eta(X)\eta(Y)\tilde{\eta}(Z),$$

which yields

$$\tilde{\nabla}_X Y=\nabla_X Y+\xi(\ln a)\eta(X)\eta(Y)\xi. \tag{15.11}$$

**Proposition 14** Under generalized $\mathcal{D}$-conformal deformation on a cosymplectic manifold the Levi-Cevita connection $\tilde{\nabla}$ is provided by formula (15.11).

Utilizing formulas (11.6), (11.7) for example with $\alpha=0$, (15.11) and substitute $A=\xi(\ln a)$ a smooth function that depends only on the direction of $\xi$, we find that the first term of (13.3) can be expressed as

$$\begin{aligned}\tilde{\nabla}_X\tilde{\nabla}_Y Z=&\nabla_X\big(\nabla_Y Z+A\eta(Y)\eta(Z)\xi\big)+A\eta(X)\big(\eta(\nabla_Y Z)+A\eta(Y)\eta(Z)\xi\big)\\ =&\nabla_X\nabla_Y Z+\xi(A)\eta(X)\eta(Y)\eta(Z)\xi+A\big(\eta(\nabla_X Y)\eta(Z)\xi\\ &+\eta(Y)\eta(\nabla_Y Z)\xi\big)+A\eta(X)\eta(\nabla_Y Z)+A^2\eta(X)\eta(Y)\eta(Z)\xi.\end{aligned} \tag{15.12}$$

Considering the fact that $X(A)=\xi(A)\eta(X)$, where $A$ solely depends on the direction of $\xi$, we can further simplify the expression. Using a similar approach, the second term of equation (13.3) is determined as

$$\begin{aligned}\tilde{\nabla}_Y\tilde{\nabla}_X Z=&\nabla_Y\nabla_X Z+\xi(A)\eta(Y)\eta(X)\eta(Z)\xi+A\big(\eta(\nabla_Y X)\eta(Z)\xi\\ &+\eta(X)\eta(\nabla_X Z)\xi\big)+A\eta(Y)\eta(\nabla_X Z)+A^2\eta(Y)\eta(X)\eta(Z)\xi.\end{aligned} \tag{15.13}$$

The last term of equation (13.3) can be represented as

$$\tilde{\nabla}_{[X,Y]}Z=\nabla_{[X,Y]}+A\eta(\nabla_X Y)\eta(Z)\xi-A\eta(\nabla_Y X)\eta(Z)\xi. \tag{15.14}$$

Substituting equations (15.12), (15.13), and (15.14) into (13.3), we derive

$$\tilde{R}(X,Y)Z=R(X,Y)Z. \tag{15.15}$$

**Proposition 15** Under generalized $\mathcal{D}$-conformal deformation on a cosymplectic manifold the Riemann curvature tensor is invariant.

Consider $\{\xi,e_i\}_{\{2\leq i\leq 2n+1\}}$ as the orthonormal basis on $M$ with respect to the metric $g$. It is straightforward to establish that $\{\frac{1}{\sqrt{a}}\xi,\frac{1}{b}e_i\}_{\{2\leq i\leq 2n+1\}}$ forms an orthonormal basis on $M$ with respect to the metric $\tilde{g}$. By employing formulas (9.1) and (15.15), we obtain:

$$\tilde{Q}X\;=\;\frac{1}{a^2}\tilde{R}(X,\xi)\xi+\frac{1}{b^2}\sum_{i=2}^{n}\tilde{R}(X,e_i)e_i. \tag{15.16}$$

Since $R(X,\xi)\xi=0$ holds true for any cosymplectic manifold, the result follows immediately

$$\tilde{Q}X=\frac{1}{b^2}QX. \tag{15.17}$$

**Proposition 16** Under generalized $\mathcal{D}$-conformal deformation on a cosymplectic manifold the Riemann curvature tensor is the Ricci operator is provided by formula (15.17).

By employing the formulas (9.2), (15.17), and (15.1), we can proceed

$$\begin{aligned}\tilde{S}(X,Y) &= \tilde{g}(\tilde{Q}X,Y) = b^2 g(\tilde{Q}X,Y) + (a^2-b^2)\eta(\tilde{Q}X)\eta(Y)\\ &= g(QX,Y) + \frac{(a^2-b^2)}{b^2}\eta(QX)\eta(Y)\\ &= S(X,Y) + \frac{(a^2-b^2)}{b^2}S(X,\xi)\eta(Y).\end{aligned}$$

Recall that $S(X,\xi)=0$ for any cosymplectic manifold, hence

$$\tilde{S}(X,Y) = S(X,Y). \tag{15.18}$$

**Proposition 17** Under generalized $\mathcal{D}$-conformal deformation on a cosymplectic manifold the Ricci curvature tensor $\tilde{S}$ is invariant.

Using (9.3) along with (15.18), we have

$$\tilde{r} = \sum_{i=1}^{2n+1}\tilde{S}(\tilde{e}_i,\tilde{e}_i) = \frac{1}{b^2}\sum_{i=1}^{2n+1}S(e_i,e_i) \implies \tilde{r} = \frac{1}{b^2}r. \tag{15.19}$$

Clearly, for $b=\pm1$ we have $\tilde{r}=r$ and the manifolds $(M,g)$ and $(M,\tilde{g})$ are locally isometric.

**Corollary 6** Cosymplectic manifolds invariant under generalized $\mathcal{D}$-conformal deformation are locally isometric if $b=\pm1$.

Subsection 15.2
## $\alpha$-Sasakian manifolds

Let $(M,\varphi,\xi,\eta,g)$ represent an $\alpha$-Sasakian manifold. Employing equation (11.4), the derivative of $\tilde{\eta}$ and $\tilde{\Phi}$ in (15.1) is

$$\begin{cases} \mathrm{d}\tilde{\eta} = \mathrm{d}a\wedge\eta + a\mathrm{d}\eta = \mathrm{d}a\wedge\eta + a\alpha\Phi,\\ \mathrm{d}\tilde{\Phi} = 2b\mathrm{d}b\wedge\Phi.\end{cases} \tag{15.20}$$

Utilizing equations (10.7) and (15.20) and considering $Y=\xi$, we obtain

$$\Big(\mathrm{d}a\wedge\eta + a\alpha\Phi\Big)(X,\xi) = \big(\mathrm{d}a\wedge\eta\big)(X,\xi). \tag{15.21}$$

Upon setting equation (15.21) to zero, we obtain

$$\big(\mathrm{d}a\wedge\eta\big)(X,\xi)=0 \implies X(a)=\xi(a)\eta(X) \implies \mathrm{grad}a = \xi(a)\xi.$$

As a consequence of the non-integrability characteristic of $\alpha$-Sasakian manifolds, according to Theorem 4, $a$ remains constant. Moreover, given that

$$\mathrm{d}\tilde{\eta} = a\alpha\Phi = \frac{a\alpha}{b^2}\tilde{\Phi} = \tilde{\alpha}\tilde{\Phi},$$

never vanishes identically (except when $\alpha=0$) then, $(M,\varphi,\tilde{\xi},\tilde{\eta},\tilde{g})$ cannot be classified as a cosymplectic, $\beta$-Kenmotsu, or $C_{12}$ manifold. Therefore, $(M,\varphi,\tilde{\xi},\tilde{\eta},\tilde{g})$ is an $\tilde{\alpha}$-Sasakian manifold with:

$$\tilde{\alpha} = \frac{a\alpha}{b^2} \quad \text{where} \quad a,b\in\mathbb{R}^*. \tag{15.22}$$

**Theorem 12** Let $(M,\varphi,\xi,\eta,g)$ be a $\alpha$-Sasakian manifold. Then under generalized $\mathcal{D}$-conformal deformation, $(M,\varphi,\tilde{\xi},\tilde{\eta},\tilde{g})$ is $\tilde{\alpha}$-Sasakian manifold and $\tilde{\alpha}=\frac{a\alpha}{b^2}$.

Hence, for an $\alpha$-Sasakian manifold under generalized $\mathcal{D}$-conformal deformation, formula (15.4) becomes

$$\begin{aligned}2\tilde{g}(\tilde{\nabla}_XY,Z)&=2b^2g(\nabla_XY,Z)+2(a^2-b^2)\eta(\nabla_XY)\eta(Z)\\&+(a^2-b^2)\Big((\mathcal{L}_\xi g)(X,Y)\eta(Z)+2\mathrm{d}\eta(X,Z)\eta(Y)+2\mathrm{d}\eta(Y,Z)\eta(X)\Big).\end{aligned}\tag{15.23}$$

Recall that $\xi$ is Killing in an $\alpha$-Sasakian manifold and from (11.4), we have

$$\begin{cases}\mathrm{d}\eta(X,Z)=\alpha\Phi(X,Z)=\alpha g(X,\varphi Z)=-\alpha g(\varphi X,Z)=-\frac{\alpha}{b^2}\tilde{g}(\varphi X,Z)\\ \mathrm{d}\eta(Y,Z)=\alpha\Phi(Y,Z)=\alpha g(Y,\varphi Z)=-\alpha g(\varphi Y,Z)=-\frac{\alpha}{b^2}\tilde{g}(\varphi Y,Z).\end{cases}$$

Substituting in formula from (15.23)

$$2\tilde{g}(\tilde{\nabla}_XY,Z)=2\tilde{g}(\nabla_XY,Z)-2\alpha\frac{a^2-b^2}{b^2}\Big(\tilde{g}(\varphi X,Z)\eta(Y)+\tilde{g}(\varphi Y,Z)\eta(X)\Big).$$

Hence

$$\tilde{\nabla}_XY=\nabla_XY-\alpha\frac{a^2-b^2}{b^2}\Big(\eta(Y)\varphi X+\eta(X)\varphi Y\Big).\tag{15.24}$$

**Proposition 18** Under generalized $\mathcal{D}$-conformal deformation on an $\alpha$-Sasakian manifold the Levi-Cevita connection $\tilde{\nabla}$ is provided by formula (15.24).

Henceforth, we shall regard $\alpha$ as a constant, as demonstrated in [28], wherein it was proven that the manifold $(M,g)$ remains invariant solely under such an assumption. From formula (13.3), we find that the first term can be expressed as

$$\begin{aligned}\tilde{\nabla}_X\tilde{\nabla}_YZ&=\nabla_X\nabla_YZ-\alpha\frac{a^2-b^2}{b^2}\Big(\big(\eta(\nabla_XY)-\alpha g(\varphi X,Y)\big)\varphi Z\\&+\eta(Y)\big(\alpha(g(X,Z)\xi-\eta(Z)X)+\varphi\nabla_XZ\big)+\big(\eta(\nabla_XZ)-\alpha g(\varphi X,Z)\big)\varphi Y\\&+\eta(Z)\big(\alpha(g(X,Y)\xi-\eta(Y)X)+\varphi\nabla_XY\big)+\eta(\nabla_YZ)\varphi X+\eta(X)\varphi\nabla_YZ\Big)\\&-X(\alpha)\frac{a^2-b^2}{b^2}\Big(\eta(Y)\varphi Z+\eta(Z)\varphi Y\Big)+\alpha^2\frac{(a^2-b^2)^2}{b^4}\Big(\eta(Y)\varphi^Z+\eta(Z)-\varphi^2Y\Big)\eta(X).\end{aligned}\tag{15.25}$$

With the similar method, the second term of (13.3) is given by

$$\begin{aligned}\tilde{\nabla}_Y\tilde{\nabla}_XZ&=\nabla_Y\nabla_XZ-\alpha\frac{a^2-b^2}{b^2}\Big(\big(\eta(\nabla_YX)-\alpha g(\varphi Y,X)\big)\varphi Z\\&+\eta(X)\big(\alpha(g(Y,Z)\xi-\eta(Z)Y)+\varphi\nabla_YZ\big)+\big(\eta(\nabla_YZ)-\alpha g(\varphi Y,Z)\big)\varphi X\\&+\eta(Z)\big(\alpha(g(Y,X)\xi-\eta(X)Y)+\varphi\nabla_YX\big)+\eta(\nabla_XZ)\varphi Y+\eta(Y)\varphi\nabla_XZ\Big)\\&-Y(\alpha)\frac{a^2-b^2}{b^2}\Big(\eta(X)\varphi Z+\eta(Z)\varphi X\Big)+\alpha^2\frac{(a^2-b^2)^2}{b^4}\Big(\eta(X)\varphi^Z+\eta(Z)\varphi^2X\Big)\eta(Y).\end{aligned}\tag{15.26}$$

The final term of (15.15) can be expressed as

$$\tilde{\nabla}_{[X,Y]}Z = \nabla_{[X,Y]}Z - \alpha\frac{a^2-b^2}{b^2}\Big(\big(\eta(\nabla_X Y) - \eta(\nabla_Y X)\big)\varphi Z + \eta(Z)\big(\varphi\nabla_X Y - \varphi\nabla_Y X\big)\Big). \tag{15.27}$$

Substituting equations (15.25), (15.26) and (15.27) into (13.3), we derive

$$\begin{aligned}\tilde{R}(X,Y)Z = R(X,Y)Z + \alpha^2\frac{a^2-b^2}{b^2}\Big(&g(\varphi X,Z)\varphi Y - g(\varphi Y,Z)\varphi X - 2g(X,\varphi Y)\varphi Z + 2\eta(Y)\eta(Z)X \\ &- 2\eta(X)\eta(Z)Y - \eta(Y)g(X,Z)\xi + \eta(X)g(Y,Z)\xi\Big) + \alpha^2\frac{(a^2-b^2)^2}{b^4}\Big(\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\Big).\end{aligned} \tag{15.28}$$

**Proposition 19** Under generalized $\mathcal{D}$-conformal deformation on an $\alpha$-Sasakian manifold the Riemann curvature tensor $\tilde{R}$ is provided by formula (15.28).

Knowing that in an $\alpha$-Sasakian manifold the following holds

$$R(X,\xi)\xi = \alpha^2(X - \eta(X)\xi)$$

Then, utilizing formula (3.51), we have

$$\tilde{R}(X,\tilde{\xi})\tilde{\xi} = \frac{1}{b^2}R(X,\xi)\xi + \alpha^2\frac{a^2-b^2}{b^4}\Big(X - \eta(X)\xi\Big), \tag{15.29}$$

and

$$\tilde{R}(X,\tilde{e}_i)\tilde{e}_i = \frac{1}{b^2}R(X,e_i)e_i - 3\alpha^2\frac{a^2-b^2}{b^4}g(X,\varphi e_i)\varphi e_i + \alpha^2\frac{a^2-b^2}{b^4}\eta(X)\xi. \tag{15.30}$$

Hence, the outcome is derived through the direct substitution of equations (15.29) and (15.30) into equation (9.1) with the additional utilization of equation (5.2) and (7.1)

$$\tilde{Q}X = \frac{1}{b^2}QX + \frac{a^2-b^2}{b^4}\Big((2n+2)\alpha^2\eta(X)\xi - 2\alpha^2 X\Big). \tag{15.31}$$

**Proposition 20** Under generalized $\mathcal{D}$-conformal deformation on an $\alpha$-Sasakian manifold the Ricci operator $\tilde{Q}$ is provided by formula (15.31).

We can re-write formula (15.31) as follow

$$\tilde{Q}X = \frac{1}{b^2}QX + 2n\alpha^2\frac{a^2-b^2}{b^4}\eta(X)\xi + 2\alpha^2\frac{a^2-b^2}{b^4}\varphi^2 X.$$

From one hand we have

$$g(\tilde{Q}X,Y) = \frac{1}{b^2}S(X,Y) - 2\alpha^2\frac{a^2-b^2}{b^4}g(X,Y) + (2n+2)\alpha^2\frac{a^2-b^2}{b^4}\eta(X)\eta(Y). \tag{15.32}$$

Taking $Y = \xi$, yields

$$\eta(\tilde{Q}X) = \frac{1}{b^2}S(X,\xi) + 2n\alpha^2\frac{a^2-b^2}{b^4}\eta(X) = 2n\alpha^2\frac{a^2}{b^4}\eta(X). \tag{15.33}$$

The result is obtained using equations (15.1), (15.32) and (15.33)

$$\tilde{S}(X,Y) = S(X,Y) - 2\alpha^2\frac{a^2-b^2}{b^2}g(X,Y) + 2n\alpha^2\frac{a^2}{b^4}\eta(X)\eta(Y) + (2n+2)\alpha^2\frac{a^2-b^2}{b^2}\eta(X)\eta(Y). \tag{15.34}$$

**Proposition 21** Under generalized $\mathcal{D}$-conformal deformation on an $\alpha$-Sasakian manifold the Ricci curvature tensor $\tilde{S}$ is provided by formula (15.34).

By formulas (9.3) and formula from (15.34), we have

$$\tilde{r} = \tilde{S}(\tilde{\xi},\tilde{\xi}) + \sum_{i=2}^{2n+1} \tilde{S}(\tilde{e}_i,\tilde{e}_i) = \frac{1}{a^2}S(\xi,\xi) + \frac{2n\alpha^2}{b^4} + 2n\alpha^2\frac{a^2-b^2}{a^2b^2} - 4n\alpha^2\frac{a^2-b^2}{b^4} + \frac{1}{b^2}\sum_{i=2}^{2n+1} S(e_i,e_i).$$

Using equation (11.8) and

$$\sum_{i=2}^{2n+1} S(e_i,e_i) = r - S(\xi,\xi).$$

We conclude the following

$$\tilde{r} = \frac{1}{b^2}r - 2n\alpha^2\frac{a^2-b^2}{b^4}. \tag{15.35}$$

**Corollary 7** Under generalized $\mathcal{D}$-conformal deformation on an $\alpha$-Sasakian manifold the scalar curvature $\tilde{r}$ is provided by formula (15.35).

Subsection 15.3

## $\beta$-Kenmotsu manifolds

Let $(M,\varphi,\xi,\eta,g)$ denote a $\beta$-Kenmotsu manifold. Utilizing equation (11.9), the derivative of $\tilde{\eta}$ and $\tilde{\Phi}$ in (15.1) is

$$\begin{cases} \mathrm{d}\tilde{\eta} = \mathrm{d}a \wedge \eta \\ \mathrm{d}\tilde{\Phi} = 2b\mathrm{d}b \wedge \Phi + b^2\mathrm{d}\Phi = \big(2b\mathrm{d}b + 2b^2\beta\eta\big) \wedge \Phi. \end{cases} \tag{15.36}$$

**Case 1:** $(M,\varphi,\tilde{\xi},\tilde{\eta},\tilde{g})$ is $\tilde{\beta}$-Kenmotsu, the following conditions must be satisfied

$$\begin{cases} \mathrm{d}\tilde{\eta} = 0, \\ \mathrm{d}\tilde{\Phi} = 2\tilde{\beta}\tilde{\eta} \wedge \tilde{\Phi} = 2ab^2\tilde{\beta}\eta \wedge \Phi. \end{cases} \tag{15.37}$$

In a manner akin to the previous case

$$\mathrm{d}a \wedge \eta = 0 \implies \mathrm{grad}a = \xi(a)\xi.$$

Upon selecting $X = \xi$ followed by $X = V \perp \xi$ in equation

$$\big(2b\mathrm{d}b - 2b^2\beta\eta\big)(X) = 2ab^2\tilde{\beta}\eta(X),$$

yields the following outcomes

$$\tilde{\beta} = \frac{\beta}{a} + \frac{\xi(b)}{ab} \quad \text{and} \quad V(b) = 0 \implies \mathrm{grad}b = \xi(b)\xi. \tag{15.38}$$

**Case 2:** $(M,\varphi,\tilde{\xi},\tilde{\eta},\tilde{g})$ constitutes a $C_{12}$ manifold, wherein the following conditions must be validated

$$\begin{cases} \mathrm{d}\tilde{\eta} = \omega \wedge \tilde{\eta} = a\omega \wedge \eta, \quad \omega(\tilde{\xi}) = 0, \\ \mathrm{d}\omega = 0, \\ \mathrm{d}\tilde{\Phi} = 0, \end{cases} \tag{15.39}$$

where $\psi$ represents the $\tilde{g}$-dual of $\omega$. By equating equations (15.36) and (15.39), we deduce

$$\mathrm{d}a \wedge \eta = a\omega \wedge \eta \implies \mathrm{dln}a = \omega \quad \text{and} \quad \mathrm{gradln}a = \psi \implies \xi(a) = 0.$$

Alternatively, we have

$$\big(2b\mathrm{d}b + 2b^2\beta\eta\big)(X) = 0.$$

Choosing $X = \xi$ and subsequently $X = V \perp \xi$ will yield

$$\xi(\mathrm{ln}b) = -\beta \quad \text{and} \quad V(b) = 0 \implies \mathrm{gradln}b = -\beta\xi. \tag{15.40}$$

The resultant $C_{12}$ structure is defined by $(\varphi, \tilde{\eta}, \omega, \tilde{\xi}, \psi)$, where

$$\omega = \mathrm{dln}(a), \quad \psi = \mathrm{gradln}(a), \quad \tilde{\eta}(\psi) = 0 \quad \text{and} \quad \omega(\tilde{\xi}) = 0.$$

**Case 3:** $(M, \varphi, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is a cosymplectic manifold. This conclusion can be readily inferred by setting $\tilde{\beta} = 0$, resulting in

$$\frac{\beta}{a} + \frac{\xi(b)}{2ab} = 0 \implies \mathrm{gradln}b = -2\beta\xi. \tag{15.41}$$

**Theorem 13** Let $(M, \varphi, \xi, \eta, g)$ be a $\beta$-Kenmotsu manifold. Then under generalized $\mathcal{D}$-conformal deformation, we have:

- $(M, \varphi, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is cosymplectic if $\mathrm{grad}a = \xi(a)\xi$ and $\mathrm{gradln}b = -2\beta\xi$ .
- $(M, \varphi, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is $\tilde{\beta}$-Kenmotsu if $\mathrm{grad}a = \xi(a)\xi$ and $\mathrm{grad}b = \xi(b)\xi$.
- $(M, \varphi, \tilde{\eta}, \omega, \tilde{\xi}, \psi)$ is $C_{12}$-manifold if $\xi(a) = 0$ and $\mathrm{gradln}b = -\beta\xi$.

Similar to the previous paragraphs, we shall investigate in depth the rigid case assuming $\mathrm{grad}a = \xi(a)\xi$ and $\mathrm{grad}b = \xi(b)\xi$ then, formula (15.4) becomes

$$\begin{aligned} 2\tilde{g}(\tilde{\nabla}_X Y, Z) &= 2b^2 g(\nabla_X Y, Z) + 2(a^2 - b^2)\eta(\nabla_X Y)\eta(Z) \\ &+ 2b\Big(\xi(b)\eta(X)g(\varphi Y, \varphi Z) + \xi(b)\eta(Y)g(\varphi Z, \varphi X) - \xi(b)\eta(Z)g(\varphi X, \varphi Y)\Big) \\ &+ 2a\xi(a)\eta(X)\eta(Y)\eta(Z) + (a^2 - b^2)\Big((\mathcal{L}_\xi g)(X, Y)\eta(Z) + 2\mathrm{d}\eta(X, Z)\eta(Y) + 2\mathrm{d}\eta(Y, Z)\eta(X)\Big). \end{aligned} \tag{15.42}$$

Recall that in a $\beta$-Kenmotsu manifold and from (11.9), we have

$$\mathrm{d}\eta = 0 \quad \text{and} \quad \big(\mathcal{L}_\xi g\big)(X, Y) = 2\beta(g(\varphi X, \varphi Y)).$$

Substituting in (15.42)

$$\begin{aligned} 2\tilde{g}(\tilde{\nabla}_X Y, Z) &= 2b^2 g(\nabla_X Y, Z) + 2(a^2 - b^2)\eta(\nabla_X Y)\eta(Z) + \xi(a^2)\eta(X)\eta(Y)\eta(Z) \\ &+ \xi(b^2)\big(\eta(X)g(Y, Z) + \eta(Y)g(X, Z) - 2\eta(X)\eta(Y)\eta(Z) - g(\varphi X, \varphi Y)\eta(Z)\big) + 2\beta(a^2 - b^2)g(\varphi X, \varphi Y)\eta(Z). \end{aligned} \tag{15.43}$$

Taking $Z = \xi$ in (15.43), we get

$$\eta(\tilde{\nabla}_X Y) = \eta(\nabla_X Y) - \frac{\xi(b^2)}{2a^2}g(\varphi X, \varphi Y) + \frac{\xi(a^2)}{2a^2}\eta(X)\eta(Y) + \beta\frac{a^2 - b^2}{a^2}g(\varphi X, \varphi Y), \tag{15.44}$$

substituting (15.44) in (15.43), we get

$$\begin{aligned} g(\tilde{\nabla}_X Y, Z) &= g(\nabla_X Y, Z) + \beta\frac{a^2 - b^2}{a^2}g(\varphi X, \varphi Y)\eta(Z) + \frac{\xi(b^2)}{2a^2}g(\varphi X, \varphi Y)\eta(Z) \\ &+ \frac{\xi(b^2)}{2b^2}\big(\eta(X)g(Y, Z) + \eta(Y)g(X, Z) - 2\eta(X)\eta(Y)\eta(Z)\big) + \frac{\xi(a^2)}{2a^2}\eta(X)\eta(Y)\eta(Z). \end{aligned}$$

Hence

$$\tilde{\nabla}_X Y = \nabla_X Y + \beta\frac{a^2-b^2}{a^2}g(\varphi X,\varphi Y)\xi + \frac{\xi(\ln a^2)}{2}\eta(X)\eta(Y)\xi + \frac{\xi(\ln b^2)}{2}\Big(\eta(X)Y+\eta(Y)X-2\eta(X)\eta(Y)\xi\Big) + \frac{\xi(b^2)}{2a^2}g(\varphi X,\varphi Y)\xi. \tag{15.45}$$

**Proposition 22** Under generalized $\mathcal{D}$-conformal deformation on an $\beta$-Kenmotsu manifold the Levi-Cevita connection $\tilde{\nabla}$ is provided by formula (15.45).

In the context of this specific instance, we confine our examination to scenarios in which the variables $a$ and $b$ retain fixed values, thereby simplifying formula (15.45) to

$$\tilde{\nabla}_X Y = \nabla_X Y + \beta\frac{a^2-b^2}{a^2}g(\varphi X,\varphi Y)\xi. \tag{15.46}$$

The motivation behind this elucidation stems from an oversight identified within the context of [54]. The authors therein employed formula (15.45), which was derived under the premise that the functions $a$ and $b$ solely depend on the direction of $\xi$. Subsequently, they proceeded to calculate the Riemann tensor $\tilde{R}$ and the Ricci tensor $\tilde{S}$ under the assumption that $a$ and $b$ remain constant in the direction of $\xi$. However, this approach yielded conflicting outcomes since the combination of the aforementioned assumptions implies that $a$ and $b$ are reduced to constants.
Through extensive direct computations, employing equations (13.3) and (15.46). Firstly

$$\begin{aligned}\tilde{\nabla}_X\tilde{\nabla}_Y Z &= \nabla_X\Big(\nabla_Y Z + \beta\frac{a^2-b^2}{a^2}g(\varphi Y,\varphi Z)\xi\Big) + \beta\frac{a^2-b^2}{a^2}g(\varphi X,\varphi\nabla_Y Z)\xi\\ &= \nabla_X\nabla_Y Z + X(\beta)\frac{a^2-b^2}{a^2}g(\varphi Y,\varphi Z)\xi - \beta^2\frac{a^2-b^2}{a^2}g(\varphi Y,\varphi Z)\varphi^2 X\\ &\quad+\beta\frac{a^2-b^2}{a^2}\Big(g(\varphi\nabla_X Y,\varphi Z) - g(\varphi X,\varphi Z)\eta(Y) + g(\varphi Y,\varphi\nabla_X Z) - g(\varphi Y,\varphi X)\Big)\xi + \beta\frac{a^2-b^2}{a^2}g(\varphi X,\varphi\nabla_Y Z)\xi.\end{aligned}$$

Similarly

$$\begin{aligned}\tilde{\nabla}_Y\tilde{\nabla}_X Z &= \nabla_Y\nabla_X Z + Y(\beta)\frac{a^2-b^2}{a^2}g(\varphi X,\varphi Z)\xi - \beta^2\frac{a^2-b^2}{a^2}g(\varphi X,\varphi Z)\varphi^2 Y\\ &\quad+\beta\frac{a^2-b^2}{a^2}\Big(g(\varphi\nabla_Y X,\varphi Z) - g(\varphi Y,\varphi Z)\eta(X) + g(\varphi X,\varphi\nabla_Y Z) - g(\varphi X,\varphi Y)\Big)\xi + \beta\frac{a^2-b^2}{a^2}g(\varphi Y,\varphi\nabla_X Z)\xi.\end{aligned}$$

Finally

$$\tilde{\nabla}_{[X,Y]}Z = \nabla_{[X,Y]}Z + \beta\frac{a^2-b^2}{a^2}\Big(g(\varphi\nabla_X Y,\varphi Z) - g(\varphi\nabla_Y X,\varphi Z)\Big)\xi,$$

substituting in (13.3), we obtain

$$\begin{aligned}\tilde{R}(X,Y)Z &= R(X,Y)Z + \frac{a^2-b^2}{a^2}\Big(X(\beta)g(\varphi Y,\varphi Z) - Y(\beta)g(\varphi X,\varphi Z)\Big)\xi\\ &\quad-\beta^2\frac{a^2-b^2}{a^2}\Big(g(\varphi Y,\varphi Z)X - g(\varphi X,\varphi Z)Y\Big).\end{aligned} \tag{15.47}$$

**Proposition 23** Under generalized $\mathcal{D}$-conformal deformation on an $\beta$-Kenmotsu manifold with deformation functions $a$ and $b$ are constants, the Riemann curvature tensor $\tilde{R}$ is provided by formula (15.47).

Using $\{\tilde{\xi},\tilde{e_i}\}=\{\frac{1}{a}\xi,\frac{1}{b}e_i\}$, where $\{i=2,...,2n+1\}$ and formula (15.47), we have

$$\begin{cases} \tilde{R}(X,\tilde{\xi})\tilde{\xi}=\frac{1}{b^2}R(X,\xi)\xi-\frac{a^2-b2}{a^2b^2}R(X,\xi)\xi, \\ \tilde{R}(X,\tilde{e_i})\tilde{e_i}=\frac{1}{b^2}R(X,e_i)e_i+\frac{a^2-b2}{a^2b^2}\big(X(\beta)-g(X,e_i(\beta)e_i)\big)\xi-\beta^2\frac{a^2-b2}{a^2b^2}\big(X-g(X,e_i)e_i\big). \end{cases} \tag{15.48}$$

Substituting (15.48) into (9.1), we get

$$\tilde{Q}X=\frac{1}{b^2}QX+(2n-1)\frac{a^2-b2}{a^2b^2}X(\beta)\xi-(2n-1)\beta^2\frac{a^2-b2}{a^2b^2}X-\frac{a^2-b2}{a^2b^2}R(X,\xi)\xi. \tag{15.49}$$

**Proposition 24** Under generalized $\mathcal{D}$-conformal deformation on an $\beta$-Kenmotsu manifold with deformation functions $a$ and $b$ are constants, the Ricci operator $\tilde{Q}$ is provided by formula (15.49).

According to (9.2), (15.1) and (15.49), we have

$$\tilde{S}(X,Y)=\tilde{g}(\tilde{Q}X,Y)=b^2g(\tilde{Q}X,Y)+(a^2-b^2)\eta(\tilde{Q}X)\eta(Y) \tag{15.50}$$

$$=S(X,Y)+(2n-1)\frac{(a^2-b^2)}{a^2}X(\beta)\eta(Y)-(2n-1)\beta^2\frac{a^2-b^2}{a^2}g(X,Y)-\frac{a^2-b^2}{a^2}g(R(X,\xi)\xi,Y)$$

$$+(2n-1)\frac{a^2-b^2}{b^2}S(X,\xi)\eta(Y)+(2n-1)\frac{(a^2-b^2)^2}{a^2b^2}X(\beta)\eta(Y)-(2n-1)\beta^2\frac{(a^2-b^2)^2}{a^2b^2}\eta(X)\eta(Y), \tag{15.51}$$

hence

$$\tilde{S}(X,Y)=S(X,Y)+(2n-1)\frac{a^2-b^2}{b^2}S(X,\xi)\eta(Y)-(2n-1)\beta^2\frac{a^2-b^2}{a^2}g(\varphi X,\varphi Y)-\frac{a^2-b^2}{a^2}\eta(R(\xi,X)Y)$$

$$+(2n-1)\frac{a^2-b^2}{b^2}X(\beta)\eta(Y)-(2n-1)\beta^2\frac{a^2-b^2}{b^2}\eta(X)\eta(Y). \tag{15.52}$$

One could further simplify formula (15.50) by substituting in (11.13) and

$$\eta(R(\xi,X)Y)=(\beta^2+\xi(\beta))\big(\eta(X)\eta(Y)-g(X,Y)\big)=-(\beta^2+\xi(\beta))g(\varphi X,\varphi Y),$$

but it is deemed unecessary for all purposes in this book.

**Proposition 25** Under generalized $\mathcal{D}$-conformal deformation on an $\beta$-Kenmotsu manifold with deformation functions $a$ and $b$ are constants, the Ricci curvature tensor $\tilde{S}$ is provided by formula (15.52).

By virtue of formula (9.3) and (15.52)

$$\tilde{r}=\frac{1}{b^2}r-\frac{a^2-b^2}{a^2b^2}S(\xi,\xi)-(4n^2-1)\beta^2\frac{a^2-b^2}{a^2b^2}+(2n-1)\frac{a^2-b^2}{a^2b^2}\xi(\beta). \tag{15.53}$$

**Corollary 8** Under generalized $\mathcal{D}$-conformal deformation on a $\beta$-Kebmotsu manifold with deformation functions a and b are constant, the scalar curvature $\tilde{r}$ is provided by formula (15.53).

Subsection 15.4
## $C_{12}$ manifolds

An intriguing feature of $C_{12}$ manifolds is the coexistence of both $\eta$ and $\omega$, either of which can be substituted into equation (15.1). In the case of $\omega$

$$\bar{\varphi}X = \frac{a}{b}\omega(X)\varphi\psi + \frac{b}{a}\omega(\varphi X)\psi, \quad \bar{\eta} = b\eta, \quad \bar{\omega} = a\omega, \quad \bar{\xi} = \frac{1}{b}\xi, \quad \bar{\psi} = \frac{1}{a}\psi, \quad \bar{g} = b^2 g + (a^2 - b^2)\omega\otimes\omega. \tag{15.54}$$

where a suitable choice for $\bar{\varphi}$ has been selected to fulfill the conditions expressed in (10.5) and (10.6). From now on, we shall call (15.54) $\omega$-conformal deformation.

**Proposition 26** $(M, \bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is an almost contact metric manifold.

### 15.4.1 $C_{12}$ manifolds under $\eta$-conformal deformation

We will explore the $\eta$-conformal deformation (15.1) on a $C_{12}$ manifold. Through direct computations utilizing Definition 20, we obtain

$$\begin{cases} \mathrm{d}\tilde{\eta} = \mathrm{d}a\wedge\eta + a\mathrm{d}\eta = \big(\mathrm{d}a + a\omega\big)\wedge\eta, \\ \mathrm{d}\tilde{\Phi} = 2b.\mathrm{d}b\wedge\Phi + b\mathrm{d}\Phi = 2b\mathrm{d}b\wedge\Phi. \end{cases} \tag{15.55}$$

**Case 1:** $(M, \varphi, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ being $\tilde{\beta}$-Kenmotsu, the following conditions must be satisfied

$$\begin{cases} \mathrm{d}\tilde{\eta} = 0, \\ \mathrm{d}\tilde{\Phi} = 2\tilde{\beta}\tilde{\eta}\wedge\tilde{\Phi} = 2ab^2\tilde{\beta}\eta\wedge\Phi. \end{cases} \tag{15.56}$$

On one hand

$$2b\mathrm{d}b(X) = 2ab^2\tilde{\beta}\eta(X).$$

Taking $X = \xi$ then $X = V \perp \xi$ in this latter yields

$$\tilde{\beta} = \frac{\xi(b)}{ab} \quad \text{and} \quad \mathrm{grad}b = \xi(b)\xi. \tag{15.57}$$

On the other hand

$$\Big(\big(\mathrm{d}a + a\omega\big)\wedge\eta\Big)(X,Y) = \big(\mathrm{d}a + a\omega\big)(X)\eta(Y) - \big(\mathrm{d}a + a\omega\big)(Y)\eta(X) = 0.$$

Selecting $X = \xi$ yields

$$\xi(a)\eta(Y) = Y(a) - a\omega(Y) \implies \mathrm{grad}a = \xi(a)\xi + a\psi.$$

**Case 2:** $(M, \varphi, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ being $C_{12}$, the following conditions must be validated

$$\begin{cases} \mathrm{d}\tilde{\eta} = \tilde{\omega}\wedge\tilde{\eta} = a\tilde{\omega}\wedge\eta, \quad \omega(\tilde{\xi}) = 0, \\ \mathrm{d}\tilde{\omega} = 0, \\ \mathrm{d}\tilde{\Phi} = 0. \end{cases} \tag{15.58}$$

In order to attain $\mathrm{d}\tilde{\eta} = \tilde{\omega}\wedge\tilde{\eta}$, it is necessary for the functions $a$ and $b$ to be constant. Consequently, under these conditions, $(M, \varphi, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is a $C_{12}$ manifold.

**Case 3:** If $(M, \varphi, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is cosymplectic, then setting $\beta = 0$ in equation (15.57) yields

$$\beta = \frac{\xi(b)}{ab} = 0 \implies \xi(b) = 0 \quad \text{and} \quad \text{grad} b = \xi(b)\xi \implies b \in \mathbb{R}^*.$$

**Theorem 14** Let $(M, \varphi, \xi, \eta, g)$ be a $C_{12}$ manifold. Then under generalized $\mathcal{D}$-conformal deformation, we have:

- $(M, \varphi, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is cosymplectic if $\text{grad} a = \xi(a)\xi + a\psi$ and $b \in \mathbb{R}^*$ .
- $(M, \varphi, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is $\beta$-Kenmotsu if $\text{grad} a = \xi(a)\xi + a\psi$ and $\text{grad} b = \xi(b)\xi$.
- $(M, \varphi, \tilde{\eta}, \omega, \tilde{\xi}, \psi)$ is $C_{12}$-manifold if $a, b \in \mathbb{R}^*$.

We can encapsulate the preceding findings succinctly within the ensuing table:

| | Cosymplectic | $\tilde{\alpha}$-Sasakian | $\tilde{\beta}$-Kenmotsu | $C_{12}$ |
|---|---|---|---|---|
| Cosymplectic | $\text{grad} a = \xi(a)\xi$<br>$b \in \mathbb{R}^*$ | ✗ | $a \in \mathbb{R}^*$<br>$\text{grad} b = \xi(b)\xi$<br>$\beta = \frac{\xi(b)}{ab}$ | $\xi(a) = 0$, $b = 1$ |
| $\alpha$-Sasakian | ✗ | $a, b \in \mathbb{R}^*$<br>$\tilde{\alpha} = \frac{a\alpha}{b^2}$ | ✗ | ✗ |
| $\beta$-Kenmotsu | $\text{grad} a = \xi(a)\xi$<br>$\text{grad} \ln b = -\beta\xi$ | ✗ | $\text{grad} a = \xi(a)\xi$<br>$\text{grad} b = \xi(b)\xi$<br>$\tilde{\beta} = \frac{\beta}{a} + \frac{\xi(b)}{ab}$ | $\text{grad} \ln b = -\beta\xi$<br>$\xi(a) = 0$ |
| $C_{12}$ | $\text{grad} a = \xi(a)\xi + a\psi$<br>$b \in \mathbb{R}^*$ | ✗ | $\text{grad} a = \xi(a)\xi + a\psi$<br>$\text{grad} b = \xi(b)\xi$<br>$\tilde{\beta} = \frac{\xi(b)}{ab}$ | $a, b \in \mathbb{R}^*$ |

**Table 7**. **Generalized $\mathcal{D}$-Conformal Deformation Of Certain Almost Contact Metric Structures**

The transition from a $\beta$-Kenmotsu manifold to an $\alpha$-Sasakian manifold ($\alpha \neq 0$) via generalized $\mathcal{D}$-conformal deformation is precluded. This restriction stems from the foundational principle that integrability is contingent upon the nullification of the Nijenhuis tensor $N_\varphi = 0$, as delineated in (10.10). Notably, this tensor is solely dependent upon the tensor $\varphi$, and the inherent invariance of $\varphi$ under generalized $\mathcal{D}$-conformal deformation, as explicated in (15.1), underscores the inherent impossibility of effecting such a transition.
Using Kozsul's formula (6.5) for the metric $\tilde{g}$ (15.1) along with (11.16) one can obtain

$$2g(\tilde{\nabla}_X Y, Z) + 2(a^2 - 1)\eta(\tilde{\nabla}_X Y)\eta(Z) = 2g(\nabla_X Y, Z) + 2(a^2 - 1)\eta(\nabla_X Y)\eta(Z) - 2(a^2 - 1)\eta(X)\eta(Y)\omega(Z). \quad (15.59)$$

Taking $Z = \xi$ in (15.1), we get

$$\tilde{\eta}(\tilde{\nabla}_X Y) = \eta(\nabla_X Y). \quad (15.60)$$

Thus, substituting (15.60) in (15.59), we get

$$g(\tilde{\nabla}_X Y, Z) = g(\nabla_X Y, Z) + (1 - a^2)\eta(X)\eta(Y)\omega(Z),$$

which yields

$$\tilde{\nabla}_X Y = \nabla_X Y + (1 - a^2)\eta(X)\eta(Y)\psi. \quad (15.61)$$

**Proposition 27** Under $\eta$-conformal deformation on a $C_{12}$ manifold the Levi-Cevita connection $\tilde{\nabla}$ is provided by formula (15.61).

Now using formula (13.3) and (15.61), we compute

### 15.4.2 $C_{12}$ manifolds under $\omega$ conformal deformation

We will explore the $\omega$-conformal deformation introduced in equation (15.54) within the context of 3-dimensional $C_{12}$ manifolds. By employing explicit calculations based on the definitions provided in Definition 20, we obtain

$$\begin{cases} \mathrm{d}\bar{\omega} = \mathrm{d}a \wedge \omega + a\mathrm{d}\omega = \mathrm{d}a \wedge \omega, \\ \mathrm{d}\bar{\eta} = \mathrm{d}b \wedge \eta + b\mathrm{d}\eta = \big(\mathrm{dln}b + \omega\big) \wedge \bar{\eta}, \\ \mathrm{d}\bar{\Phi} = 2b\mathrm{d}b \wedge \Phi + b\mathrm{d}\Phi = 2\mathrm{d}b \wedge \tilde{\Phi}. \end{cases} \tag{15.62}$$

From equations (15.62), the structure $(\varphi, \bar{\xi}, \bar{\eta}, \bar{g})$ is $C_{12}$ if $a = 1$ and $b$ is a non-zero constant. To reach a conclusion, we must ensure that $N_{\bar{\varphi}} = 0$. First, recall the following results from [14]:

**Corollary 9** For any 3-dimensional unit $C_{12}$ manifold, we have

$$\nabla_\xi \xi = \psi, \quad \nabla_\xi \psi = \xi, \quad \nabla_{\varphi\psi}\psi = (\mathrm{div}\psi - 1)\varphi\psi, \quad \nabla_{\varphi\psi}\varphi\psi = (1 - \mathrm{div}\psi)\psi,$$
$$\nabla_\xi \varphi\psi = \nabla_\psi \xi = \nabla_\psi \psi = \nabla_\psi \varphi\psi = \nabla_{\varphi\psi}\xi = 0.$$

Using a $\varphi$-basis $\{\xi, \psi, \varphi\psi\}$ (notice that $a = 1$ and $b$ is a chosen constant that does not affect the derivations, thus the choice of a $\varphi$-basis instead of a $\bar{\varphi}$-basis), it suffices to verify that $N_{\bar{\varphi}}(\xi, \psi) = N_{\bar{\varphi}}(\xi, \varphi\psi) = N_{\bar{\varphi}}(\psi, \varphi\psi) = 0$. Employing Corollary 9, we have

$$[\xi, \psi] = \xi, \quad [\xi, \varphi\psi] = 0, \quad \text{and} \quad [\psi, \varphi\psi] = (1 - \mathrm{div}\psi)\varphi\psi,$$

now, using (10.10) and (15.54), we obtain

$$\begin{cases} N_{\bar{\varphi}}(\xi, \psi) = \bar{\varphi}^2[\xi, \psi] - \frac{1}{b}\bar{\varphi}[\xi, \varphi\psi] = 0, \\ N_{\bar{\varphi}}(\xi, \varphi\psi) = \bar{\varphi}^2[\xi, \varphi\psi] + b\bar{\varphi}[\xi, \psi] = 0, \\ N_{\bar{\varphi}}(\psi, \varpi\psi) = \bar{\varphi}^2[\psi, \varphi\psi] - [\varphi\psi, \psi] = -[\psi, \varphi\psi] - [\varphi\psi, \psi] = 0. \end{cases}$$

Summarizing all of the above, we have:

**Proposition 28** A $C_{12}$ structure is invariant under the following $\omega$-conformal deformation

$$\bar{\varphi}X = \frac{1}{b}\omega(X)\varphi\psi + b\omega(\varphi X)\psi, \quad \bar{\eta} = b\eta, \quad \bar{\omega} = \omega, \quad \bar{\xi} = \frac{1}{b}\xi, \quad \bar{\psi} = \psi, \quad \bar{g} = b^2 g + (1 - b^2)\omega \otimes \omega.$$

From Kozsul's formula, one can get

$$\begin{aligned} 2b^2 g(\bar{\nabla}_X Y, Z) + 2(1 - b^2)\omega(\bar{\nabla}_X Y)\omega(Z) &= 2b^2 g(\nabla_X Y, Z) + 2(1 - b^2)\omega(\nabla_X Y)\omega(Z) \\ &+ (1 - b^2)\Big((\mathcal{L}_\psi g)(X, Y)\omega(Z) + 2\mathrm{d}\omega(X, Z)\omega(Y) + 2\mathrm{d}\omega(Y, Z)\omega(X)\Big). \end{aligned}$$

Since $\mathrm{d}\omega = 0$, then $g(\nabla_X \psi, Y) = g(\nabla_Y \psi, X)$, yields

$$(\mathcal{L}_\psi g)(X, Y) = g(\nabla_X \psi, Y) + g(\nabla_Y \psi, X) = 2g(\nabla_X \psi). \tag{15.63}$$

Thus, we have

$$\begin{aligned} 2b^2 g(\bar{\nabla}_X Y, Z) + 2(1 - b^2)\omega(\bar{\nabla}_X Y)\omega(Z) &= 2b^2 g(\nabla_X Y, Z) + 2(1 - b^2)\omega(\nabla_X Y)\omega(Z) \\ &+ 2(1 - b^2)g(\nabla_X \psi, Y)\omega(Z). \end{aligned} \tag{15.64}$$

Taking $Z = \psi$ in (3.15), we get

$$\omega(\bar{\nabla}_X Y) = \omega(\nabla_X Y) + (1 - b^2)g(\nabla_X Y, Z).$$

Finally

$$g(\bar{\nabla}_X Y, Z) = g(\nabla_X Y, Z) + (1 - b^2)g(\nabla_X \psi, Y)\omega(Z).$$

Hence

$$\bar{\nabla}_X Y = \nabla_X Y + (1 - b^2)g(\nabla_X \psi, Y)\psi. \tag{15.65}$$

**Proposition 29** Under $\omega$-conformal deformation on an $C_{12}$ manifold the Levi-Cevita connection $\bar{\nabla}$ is provided by formula (15.65).

Let us denote the Riemannian curvature tensor $\bar{R}$ associated with the metric $\bar{g}$ defined by

$$\bar{R}(X,Y)Z = \bar{\nabla}_X\bar{\nabla}_Y Z - \bar{\nabla}_Y\bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]}Z. \tag{15.66}$$

Using (15.65), we have

$$\bar{\nabla}_X\bar{\nabla}_Y Z = \nabla_X\nabla_Y Z + (1-b^2)\big(g(\nabla_X\nabla_Y\psi, Z)\psi + g(\nabla_Y\psi, \nabla_X Z)\psi + g(\nabla_X\psi, \nabla_Y Z)\psi + g(\nabla_Y\psi, Z)\nabla_X\psi\big). \tag{15.67}$$

In the same manner

$$\bar{\nabla}_Y\bar{\nabla}_X Z = \nabla_Y\nabla_X Z + (1-b^2)\big(g(\nabla_Y\nabla_X\psi, Z)\psi + g(\nabla_X\psi, \nabla_Y Z)\psi + g(\nabla_Y\psi, \nabla_X Z)\psi + g(\nabla_X\psi, Z)\nabla_Y\psi\big). \tag{15.68}$$

Also, we have

$$\bar{\nabla}_{[X,Y]}Z = \nabla_{[X,Y]} + (1-b^2)g(\nabla_{[X,Y]}\psi, Z)\psi. \tag{15.69}$$

Hence, substituting (15.67), (15.68) and (15.69) in (15.66) yields

$$\bar{R}(X,Y)Z = R(X,Y)Z + (1-b^2)\Big(g\big(\nabla_Y\psi, Z\big)\nabla_X\psi - g\big(\nabla_X\psi, Z\big)\nabla_Y\psi - \omega(R(X,Y)Z)\psi\Big). \tag{15.70}$$

**Proposition 30** Under $\omega$-conformal deformation on an $C_{12}$ manifold the Riemann curvature tensor $\bar{R}$ is provided by formula (15.70).

Let us consider the orthonormal basis $\{\psi, e_i\}_{1\le i\le 2n}$ on the manifold $M$, relative to the metric $g$. It can be readily demonstrated that the set $\{\psi, \frac{1}{b}e_i\}_{1\le i\le 2n}$ forms an orthonormal frame on $M$ in relation to the metric $\bar{g}$. By contracting equation (15.70) with respect to the vectors $Y$ and $Z$ , we obtain

$$\begin{aligned}\bar{Q}X &= \sum_{i=1}^{i=2n}\bar{R}(X,\bar{e_i})\bar{e_i} + \bar{R}(X,\psi)\psi\\ &= \frac{1}{b^2}\sum_{i=1}^{i=2n}\Big(R(X,e_i)e_i + (1-b^2)\big(g(\nabla_{e_i}\psi, e_i)\nabla_X\psi - g(\nabla_X\psi, e_i)\nabla_{e_i}\psi - \omega(R(X,e_i)e_i)\psi\big)\Big) + R(X,\psi)\psi.\end{aligned}$$

Upon substituting expressions (5.2) and (7.2), the following outcomes are derived

$$\bar{Q}X = \frac{1}{b^2}QX + \frac{1-b^2}{b^2}\Big(\mathrm{div}\psi\nabla_X\psi - S(X,\psi)\psi - R(X,\psi)\psi - \sum_{i=1}^{i=2n} g(\nabla_X\psi, e_i)\nabla_{e_i}\psi\Big). \tag{15.71}$$

**Proposition 31** Under $\omega$-conformal deformation on an $C_{12}$ manifold the Ricci operator $\bar{Q}$ is provided by formula (15.71).

Performing the inner product of (15.71) with any arbitrary vector field $Y$ defined on the manifold $M$ results in

$$\begin{aligned}\bar{S}(X,Y) &= \bar{g}(\bar{Q}X,Y) = b^2 g(X,Y) + (1-b^2)\omega(\bar{Q}X)\omega(Y)\\ &= S(X,Y) + (1-b^2)\Big(\mathrm{div}\psi g(\nabla_X\psi, Y) - g(R(X,\psi)\psi, Y) - \sum_{i=1}^{i=2n} g(\nabla_X\psi, e_i)g(\nabla_{e_i}\psi, Y)\Big).\end{aligned}$$

Using (5.2), one can observe that

$$\sum_{i=1}^{i=2n} g(\nabla_X\psi, e_i)g(\nabla_{e_i}\psi, Y) = g\Big(\nabla_{\sum_{i=1}^{i=2n} g(\nabla_X\psi, e_i)e_i}\psi, Y\Big) = g\big(\nabla_{\nabla_X\psi}\psi, Y\big).$$

Hence

$$\bar{S}(X,Y) = S(X,Y) + (1-b^2)\Big(\mathrm{div}\psi g(\nabla_X\psi, Y) - g(R(X,\psi)\psi, Y) - g(\nabla_{\nabla_X\psi}\psi, Y)\Big) \tag{15.72}$$

**Proposition 32** Under $\omega$-conformal deformation on an $C_{12}$ manifold the Ricci curvature tensor $\bar{S}$ is provided by formula (15.72).

The outcomes derived from (15.72) manifest a discernible level of intricacy. Nevertheless, it is conceivable to distill and articulate more accessible formulations specifically tailored to the 3-dimensional case. These simplified expressions not only hold considerable significance for our present inquiry but also foreshadow their relevance in subsequent sections. Using (5.2) and Corollary 9, we have

$$\nabla_X \psi = \eta(X)\xi + (\mathrm{div}\psi - 1)\omega \circ \varphi(X)\varphi\psi.$$

With direct computations we have

$$\nabla_\psi \nabla_X \psi = \eta(\nabla_\psi X)\xi + (\mathrm{div}\psi - 1)\omega \circ \varphi(\nabla_\psi X)\varphi\psi = \nabla_{\nabla_\psi X}\psi. \tag{15.73}$$

Substituting (15.73) in (8.1), we get

$$R(X,\psi)\psi = -\nabla_{\nabla_X \psi}\psi. \tag{15.74}$$

Thus, formula (15.72) becomes

$$\bar{S}(X,Y) = S(X,Y) + (1-b^2)\mathrm{div}\psi g(\nabla_X \psi, Y). \tag{15.75}$$

**Proposition 33** Under $\omega$-conformal deformation on a 3-dimensional $C_{12}$ manifold the Ricci curvature tensor $\bar{S}$ is provided by formula (15.75).

Using formulas (9.3) and (15.75), one can obtain

$$\bar{r} = \frac{1}{b^2} r - \frac{1-b^2}{b^2}\big(S(\psi,\psi) - (\mathrm{div}\psi)^2\big). \tag{15.76}$$

**Corollary 10** Under $\omega$-conformal deformation on a 3-dimensional $C_{12}$ manifold the scalar curvature $\bar{r}$ is provided by formula (15.76).

### 15.4.3 Lorentz $C_{12}$ manifolds through metric deformation

Contrary to the Riemann case, any smooth manifold cannot admit a Lorentz structure, in fact, this is possible if and only if there exists a global vector field ([58], page 149). Additionally, the following well-known result shows how to obtain such a metric:

**Figure 23. Hendrik Antoon Lorentz 1853-1928.**

**Theorem 15** Let $(M,g)$ be a Riemann manifold, $V$ a unit global vector field and $V^\flat$ its dual 1-form. Then $\tilde{g} = g - 2V^\flat \otimes V^\flat$ is a Lorentz metric on $M$. Furthermore, $V$ becomes time-like so the resulting Lorentz manifold is time-orient-able.

**Definition 21** Let $(M,\varphi,\xi,\eta,g)$ be a Lorentz almost contact manifold. $M$ is said to be a Lorentz almost contact $C_{12}$-manifold if

$$\mathrm{d}\eta = \omega \wedge \eta \quad \text{and} \quad \mathrm{d}\Phi = 0$$

If, in addition $N_\varphi = 0$ then $(M,\varphi,\xi,\eta,g)$ is called a Lorentz $C_{12}$-manifold.

Now, we consider a $C_{12}$-manifold $(M^{2n+1},\varphi,\xi,\eta,g)$ and we obtain a Lorentz metric putting

$$g^* = g - 2\eta \otimes \eta. \tag{15.77}$$

**Theorem 16** The manifold $(M^{2n+1},\varphi,\xi,\eta,g^*)$ is a Lorentz-$C_{12}$-manifold.

PROOF Since $(\varphi,\xi,\eta)$ is an almost contact structure, it is easily to see that $\xi$ is time-like with respect to the metric $g^*$

$$g^*(\xi,\xi) = g(\xi,\xi) - 2\eta(\xi)\eta(\xi) = -1. \tag{15.78}$$

We check the compatibility of $g^*$ with the structure, for any vector fields $X$ and $Y$ on $M$ we have

$$\begin{aligned} g^*(\varphi X,\varphi Y) &= g(\varphi X,\varphi Y) \\ &= g(X,Y) - \eta(X)\eta(Y) \\ &= g^*(X,Y) + \eta(X)\eta(Y). \end{aligned}$$

Moreover, we have

$$\begin{aligned} \Phi^*(X,Y) &= g^*(X,\varphi Y) \\ &= \Phi(X,Y), \end{aligned}$$

for any vector fields $X$ and $Y$ on $M$. So, we obtain $\mathrm{d}\Phi^* = \mathrm{d}\Phi = 0$.
Finally, the integrability condition (i.e $N_\varphi = 0$) holds since it does not depend on the metric. Then, we obtain a Lorentz-$C_{12}$-manifold on $M$. □

From now on, such a Lorentz-$C_{12}$-manifold is said to be the associated Lorentz-$C_{12}$-manifold.
To compare the Levi-Civita connections $\nabla$ and $\nabla^*$ with respect to the Riemannian metric $g$ and the Lorentz one $g^*$, by Koszul's formula for $\nabla^*$ and applying the definition of $g^*$ we have

$$g^*\big(\nabla^*_X Y, Z\big) = g^*\big(\nabla_X Y, Z\big) - \big(g(\nabla_X\xi, Y) + g(\nabla_Y\xi, X)\big)\eta(Z) - \mathrm{d}\eta(X,Z)\eta(Y) - \mathrm{d}\eta(Y,Z)\eta(X).$$

With the help of (11.17) and Definition 20, one can get

$$g^*\big(\nabla^*_X Y, Z\big) = g^*\big(\nabla_X Y, Z\big) + \eta(X)\eta(Y)\omega(Z),$$

since

$$\omega(Z) = g(\psi, Z) = \tilde{g}(\psi, Z),$$

then

$$\nabla^*_X Y = \nabla_X Y + \eta(X)\eta(Y)\psi. \tag{15.79}$$

**Proposition 34** The Levi-Cevita connection $\nabla^*$ associated with the metric $g^*$ on a $C_{12}$ manifold is provided by formula (15.79).

As a consequence of the relation between the Levi-Civita connections, we have the following theorem:

**Theorem 17** An almost contact Lorentzian manifold $(M^{2n+1},\varphi,\xi,\eta,\tilde{g})$ is a Lorentz-$C_{12}$-manifold if and only if

$$(\nabla^*_X\varphi)Y = \eta(X)\omega(\varphi Y)\xi. \tag{15.80}$$

PROOF Using (15.79), we have

$$\begin{aligned} (\nabla^*_X\varphi)Y &= \nabla^*_X\varphi Y - \varphi\nabla^*_X Y \\ &= (\nabla_X\varphi)Y - \eta(X)\eta(Y)\varphi\psi, \end{aligned}$$

from (11.14), we get our formula. □

From formula (15.79), firstly we have

$$\nabla^*_X\nabla^*_Y Z = \nabla_X\nabla_Y Z + \Big[\big(\eta(\nabla_X Y) - \eta(X)\omega(Y)\big)\eta(Z) + \big(\eta(\nabla_X Z) - \eta(X)\omega(Z)\big)\eta(Y) + \eta(X)\eta(\nabla_Y Z)\Big]\psi + \eta(Y)\eta(Z)\nabla_X\psi. \quad (15.81)$$

Secondly

$$\nabla^*_Y\nabla^*_X Z = \nabla_Y\nabla_X Z + \Big[\big(\eta(\nabla_Y X) - \eta(Y)\omega(X)\big)\eta(Z) + \big(\eta(\nabla_Y Z) - \eta(Y)\omega(Z)\big)\eta(X) + \eta(Y)\eta(\nabla_X Z)\Big]\psi + \eta(X)\eta(Z)\nabla_Y\psi. \quad (15.82)$$

Finally

$$\nabla^*_{[X,Y]}Z = \nabla_{[X,Y]}Z + \eta(\nabla_X Y)\eta(Z)\psi - \eta(\nabla_Y X)\eta(Z)\psi. \quad (15.83)$$

Substituting (15.81), (15.82) and (15.83) in (8.1), we evaluate

$$R^*(X,Y)Z = R(X,Y)Z + \eta(Z)\Big[\big(\eta(Y)\omega(X) - \eta(X)\omega(Y)\big)\psi + \eta(Y)\nabla_X\psi - \eta(X)\nabla_Y\psi\Big], \quad (15.84)$$

hence, substituting (11.15) into (15.84) yields

$$R^*(X,Y)Z = R(X,Y)Z - \eta(Z)R(X,Y)\xi. \quad (15.85)$$

**Proposition 35** The Riemann curvature tensor $R^*$ associated with the metric $g^*$ on a $C_{12}$ manifold is provided by (15.85).

A standard orthonormalization process shows that if $\{\xi, e_i\}_{1\leq i\leq 2n}$ is a local orthonormal basis with respect to $g$ then it is a local pseudo-orthonormal basis with respect to $g^*$ . Using formulas (9.1) and (15.85)

$$Q^*X = R^*(X,\xi)\xi + \sum_{i=1}^{i=2n} R^*(X,e_i)e_i = \sum_{i=1}^{i=2n} R(X,e_i)e_i,$$

hence

$$Q^*X = QX - R(X,\xi)\xi. \quad (15.86)$$

**Proposition 36** The Ricci operator $Q^*$ associated with the metric $g^*$ on a $C_{12}$ manifold is provided by (15.86).

By a simple computation using (9.2), (15.77) and (15.86), one can get

$$S^*(X,Y) = S(X,Y) - g(R(X,\xi)\xi,Y) - 2\eta(QX)\eta(Y),$$

again using formula (11.15), we obtain

$$S^*(X,Y) = S(X,Y) + \omega(X)\omega(Y) + g(\nabla_X\psi,Y) - \eta(Y)g(\nabla_X\psi,\xi) + 2\eta(X)\eta(Y)\mathrm{div}\psi,$$

using the fact $g(\nabla_X\psi,\xi) = -g(\psi,\nabla_X\xi) = |\psi|^2\eta(X)$, we conclude

$$S^*(X,Y) = S(X,Y) + (2\mathrm{div}\psi - |\psi|^2)\eta(X)\eta(Y) + \omega(X)\omega(Y) + g(\nabla_X\psi,Y). \quad (15.87)$$

**Proposition 37** The Ricci curvature tensor $S^*$ associated with the metric $g^*$ on a $C_{12}$ manifold is provided by (15.87).

Contacting formula (15.87) yields

$$r^* = S^*(\xi, \xi) + S^*(\psi, \psi) + \sum_{i=1}^{i=2n-1} S^*(e_i, e_i). \tag{15.88}$$

By virtue of (7.2) and (9.3), we get

$$r^* = r + 3\text{div}\psi. \tag{15.89}$$

**Corollary 11** The scalar curvature $r^*$ associated with the metric $g^*$ on a $C_{12}$ manifold is provided by (15.89).

PART

V



**Table 8**. Contents for Part V

# *Ricci Flow and Ricci-Solitons*

In the present chapter, we initiate discussion on the Ricci flow equation, originally formulated by **R. Hamilton**, as delineated in [43]. Essential foundational insights into the Ricci flow equation are further expounded upon in [26].

SECTION 16

## RicciFlow

**Definition 22** Consider a Riemannian manifold $(M, g)$. The Ricci flow equation describes the evolution of the Riemannian metric $g$ over time

$$\partial_t g = -2S, \tag{16.1}$$

where $S$ refers to the Ricci tensor.

**Figure 24**. **Richard Streit Hamilton 1943-2024.**

In a set of harmonic coordinates $\{x_1, x_2, ..., x_n\}$

$$\Delta x_i = 0, \quad \forall i \in \{1, ..., n\},$$

where $\Delta$ is the Laplace operator, and the Ricci flow equation is actually a heat equation for the Riemannian metric [43].

SUBSECTION 16.1

### Geometric Interpretation of Ricci-flow

Ricci-flow is a way of changing the metric tensor $g$ of a Riemannian manifold $(M, g)$ over time $t$ so that the manifold becomes round (**Approaching geometric characteristics akin to those exhibited by a sphere**). On the figure on the right, the Ricci flow will inflate the region pointed out by the blue arrow. By convention we say that it has negative Ricci curvature (i.e. $S < 0$). Hence, we can observe that the lenght increases (i.e. $g \uparrow$)

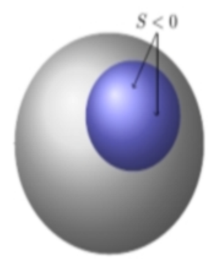


**Figure 25**. **Ricci-flow on an inflated sphere.**

Therefor, one can deduce that

$$\partial_t g = -2S,$$

where $K$ is a positive constant. One must inquire about the case $K = 2$ from Hamilton, as it remains an unresolved puzzle for myself. As an example, we shall solve the Ricci-flow equation (16.1) for a 2-Sphere.

**Example 16.2.** Recall from Example 5.1

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2 \sin^2\theta \end{pmatrix},$$

and its Ricci tensor from (9.6)

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}.$$

Assuming that over time $0 \le t < \infty$, only $R$ changes and $\theta$ is time independant. Thus

$$\partial_t g = \begin{pmatrix} \partial_t g_{11} & \partial_t g_{12} \\ \partial_t g_{21} & \partial_t g_{22} \end{pmatrix} = \begin{pmatrix} \partial_t R^2 & 0 \\ 0 & \partial_t R^2 \sin^2\theta \end{pmatrix} = \begin{pmatrix} 2R\partial_t R & 0 \\ 0 & 2R\partial_t R \sin^2\theta \end{pmatrix}.$$

Then substituting in (16.1), gives

$$\partial_t g_t = -2S \implies \begin{pmatrix} 2R\partial_t R & 0 \\ 0 & 2R\partial_t R \sin^2\theta \end{pmatrix} = -2 \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix},$$

Wich leads to the simple to solve differential equation

$$2R\partial_t R = -2 \implies R(t) = \sqrt{-2t + R_0^2}.$$

where $R_0$ is the initial radius of the sphere at time $t = 0$. We say that a 2-Sphere of radius $R$ has an extinction time of $t = \frac{R_0^2}{2}$.

Section 17

# Some Exact Solutions To Ricci Flow

Subsection 17.1

## Einstein Manifolds

An Einstein manifold, named in honor of **Albert Einstein**, is a Riemannian or pseudo-Riemannian manifold, whose metric is a special solution to Einstein's field equation with cosmological constant

$$S_{ij} - \big(\Lambda - \frac{1}{2}r\big)g_{ij} = \frac{8\pi G}{c^4} T_{ij},$$

where $\Lambda$ is the cosmological constant, $G$ the gravitational constant, $c$ the speed of light and $T$ is the stress-energy tensor. In vaccum, $T = 0$ the equation reduces to

$$S_{ij} = \big(\Lambda - \frac{1}{2}r\big)g_{ij}.$$

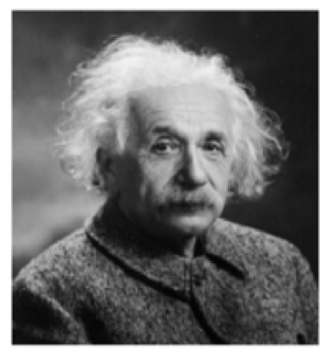

**Figure 26. Albert Einstein 1879-1955.**

Thus, an Einstein manifold is a Riemannian or pseudo-Riemannian whose Ricci tensor is proportional to the metric

$$S = \lambda g. \tag{17.1}$$

where $\lambda$ is a constant. For any constant $c$, if $g = cg_0$, from (13.10) we have

$$S(g) = S(g_0) = \lambda g_0 = \frac{\lambda}{c} g.$$

Using this, we can construct a family of solution to the Ricci-flow equation as follows. Consider $g(t) = u(t)g_0$. If this one parameter family is a solution to the Ricci-flow equation, then:

$$\partial_t g_t = u'(t)g_0 = -2S(u(t)g_0) = -2S(g_0) = -2\lambda g_0,$$

where $\lambda$ is a constant. Hence, we obtain

$$u'(t) = -2\lambda \implies u(t) = 1 - 2\lambda t.$$

The cases $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$ correspond to **shrinking**, **steady** and **expanding** solutions. Notice that the solution exists for $0 \leq t < \frac{1}{2\lambda}$ and vanishes (or **goes singular**) at $t = \frac{1}{2\lambda}$.
Over the course of the last decade, there were many generalization of Einstein manifolds, for example, we say that $(M, g)$ is an $\eta$-Einstein manifold if $S$ satisfies:

$$S = \mu g + \nu \eta \otimes \eta, \tag{17.2}$$

and it is said **almost quasi-Einstein** and **nearly quasi-Einstein**, respectively if

$$S = cg + d\big(\omega_1 \otimes \omega_2 + \omega_2 \otimes \omega_1\big), \tag{17.3}$$

and

$$S = ag + bE, \tag{17.4}$$

where $a, b, c, d$ are functions, $\omega_1$, $\omega_2$ are 1-forms and $E$ is a non-vanishing symmetric $(0, 2)$-tensor on $M$.
In physics, manifolds whose metric satisfies equation (17.2) are called **perfect fluid spaces**.

Subsection 17.2
## Ricci-Soliton

A Ricci soliton denotes a solution to the Ricci flow equation $(M, g_t)$, where $0 \leq t < T \leq \infty$, possessing the property that for each $t \in [0, T)$, there exists a diffeomorphism

$$\phi_t : M \to M$$

generated by a vector field $V$ and a constant $\sigma_t$ such that

$$\sigma_t \phi_t^* g_0 = g_t.$$

In essence, a Ricci soliton ensures that all Riemannian manifolds $(M, g_t)$ are isometric up to a varying scale factor, which may depend on $t$. A method to construct Ricci solitons involves the following steps: Assume the existence of a vector field $X$ on $M$, a constant $\lambda$, and an initial metric $g_0$ such that

$$-S(g_0) = \frac{1}{2}(\mathcal{L}_X g_0) - \lambda g_0$$

Hence, a Ricci soliton is a triplet $(g, X, \lambda)$ satisfying equation:

$$(\mathcal{L}_X g)(Y, Z) + 2S(Y, Z) = 2\lambda g(Y, Z). \tag{17.5}$$

The generalized Ricci-soliton equation in Riemann manifold $(M, g)$ is defined by (see [56])

$$\mathcal{L}_V g = -2c_1 V^\flat \otimes V^\flat + 2c_2 S + 2\lambda g, \tag{17.6}$$

where $V^\flat(X) = g(V, X)$ and $c_1, c_2, \lambda \in \mathbb{R}$. Equation (17.6) is a generalization of:

- Killing's equation $c_1 = c_2 = \lambda = 0$.
- Equation for homotheties $c_1 = c_2 = 0$.
- Ricci soliton $c_1 = 0$, $c_2 = -1$.
- Cases of Einstein-Weyl $c_1 = 1$, $c_2 = \frac{-1}{n-2}$.

- Metric projective structures with skew-symmetric Ricci tensor in projective class $c_1 = 1$, $c_2 = -\frac{1}{n-1}$, $\lambda = 0$.
- Vacuum near-horzion geometry equation $c_1 = 1$, $c_2 = \frac{1}{2}$.

Moreover, in the event that $V$ is a Killing vector field (i.e., $\mathcal{L}_V g = 0$), equation (17.6) characterizes $(M, g)$ as $V^\flat$-Einstein, given that $c_1$ is non-zero, and it is termed Einstein if $c_1 = 0$.
A further generalization of the Ricci-soliton equation in the Riemann manifold $(M, g)$, given in [24] by the following equation

$$\mathcal{L}_{V_1} g = -2c_1 V_2^\flat \otimes V_2^\flat + 2c_2 S + 2\lambda g, \tag{17.7}$$

where $V_1, V_2$ are two vector fields on $M$.
Recently, in [6], the authors introduced the generalized $\eta$-Ricci soliton equation in Riemann manifold $(M, g)$ by

$$\mathcal{L}_V g = -2c_1 V^\flat \otimes V^\flat + 2c_2 S + 2\lambda g + 2\mu\eta \otimes \eta, \tag{17.8}$$

where $c_1, c_2, \lambda, \mu \in \mathbb{R}$ and $\eta$ is a 1-form on $M$.
Inspired by equations (17.7) and (17.8), we can guess the existence of a generalization that includes all previous cases, which we define by the following equation

$$\mathcal{L}_{V_1} g = -2c_1 V_2^\flat \otimes V_2^\flat + 2c_2 S + 2\lambda g + 2\mu\eta \otimes \eta. \tag{17.9}$$

We refer to this generalization as **generalized $\eta$-Ricci bi-soliton** and the confirmation of the existence of this generalization will be in the last two theorems.
Conversely, a Ricci-Yamabe soliton (briefly, RYS) is defined as a semi-Riemannian manifold $(M^n, g)$ equipped with a vector field $V$ on $M$ that satisfies

$$\mathcal{L}_V g = 2\alpha S + 2(\lambda + r\rho)g, \tag{17.10}$$

where $\rho \in \mathbb{R}$ is constant and $r$ denotes the scalar curvature, defined as the trace of the Ricci tensor $S$ with respect to the metric $g$

$$r = Tr_g S. \tag{17.11}$$

Likewise, equation (17.10) is a natural generalization of:

- Ricci Soliton (briefly, **RS**) $\alpha = 1$, $\rho = 0$.
- Ricci-Bourguignon Soliton (briefly, **GBS**) $\alpha = 1$, $\rho \in \mathbb{R}$.
- Yamabe Soliton (brielfly, **YS**) $\alpha = 0$, $\rho = -1$.

Ricci-Yamabe solitons has been an active area of investigation in differential geometry. These solitons generalize both Ricci solitons and Yamabe solitons, serving as self-similar solutions to the Ricci flow and the Yamabe flow, respectively. Recent studies have explored various aspects of Ricci-Yamabe solitons, including their existence, uniqueness, and classification under different geometric conditions. Notable contributions include the work of Deshmukh and Alodan [33], which examined the geometric properties of Ricci-Yamabe solitons on warped product manifolds, and Blaga [14], who studied $\eta$-Ricci-Yamabe solitons in the context of almost contact metric manifolds. These investigations provide valuable insights into the interplay between curvature and the underlying geometry of the manifolds.
Motivated by the work of [57], we define a generalized Ricci-Yamabe soliton (briefly, **GRYS**) as follows

$$\mathcal{L}_V g = -2c_1 V^\flat \otimes V^\flat + 2c_2 S + 2(\lambda + r\rho)g. \tag{17.12}$$

Equation (17.12) is an immediate generalization of the following:

- **GRS** equation (17.6) for $\rho = 0$.
- **RYS** equation (17.10) for $c_1 = 0$.

Section 18

# Generalized Ricci-Yamabe Soliton On 3-Dimensional Lie Groups

This segment is dedicated to the computation of the generalized Ricci-Yamabe soliton (briefly, **GRYS**) on 3-dimensional Lie groups.

Subsection 18.1

## Left-Invariant 3-Dimensional Lie Groups

**Figure 27. Leopold Kronecker 1823-1891.**

**Figure 28. Luigi Bianchi 1856-1928.**

A 3-dimensional left-invariant Lie group $G$ is a smooth manifold of dimension 3 equipped with a group structure such that left translations $L_a$ defined by

$$\begin{aligned} L_a : G &\longrightarrow G \\ x &\longrightarrow L_a(x) = a.x, \end{aligned}$$

for $a, x \in G$ are diffeomorphisms. This implies that the tangent space $T_eG$ at the identity element $e \in G$, equipped with the Lie bracket operation derived from the group multiplication, forms a 3-dimensional Lie algebra.

A Riemannian frame on a 3-dimensional left-invariant Lie group $(G, g)$ consists of three smooth vector fields $\{e_1, e_2, e_3\}$ on $G$, which are left-invariant and form an orthonormal basis with respect to the Riemannian metric $g$. Specifically, at each point $p \in G$,

$$g(e_i, e_j)_{|p} = \delta_{ij},$$

where $\delta_{ij}$ is the Kronecker delta. In Jantzen's work [47], L. Bianchi compiled a catalog of 3-dimensional real Lie algebras, accompanied by a demonstration that each 3-dimensional Lie algebra finds isomorphism with a singular entry on his list. Given our focus on left-invariant structures, our analysis is confined to the Lie algebras associated with their respective Lie groups. The ensuing outcome elucidates the various classes of 3-dimensional Lie algebras [60].

**Proposition 38** Let $\mathfrak{g}$ be a 3-dimensional real Lie algebra. Then if $\mathfrak{g}$ is not abelian, it is isomorphic to one and only one of the Lie algebras listed below:

| Algebra | Structure equations |
|---|---|
| $\mathcal{A}_{3,1}$ | $[e_2, e_3] = e_1$ |
| $\mathcal{A}_{3,2}$ | $[e_1, e_3] = e_1$, $[e_2, e_3] = e_1 + e_2$ |
| $\mathcal{A}_{3,3}$ | $[e_1, e_3] = e_1$, $[e_2, e_3] = e_2$ |
| $\mathcal{A}_{3,4}$ | $[e_1, e_3] = e_1$, $[e_2, e_3] = -e_2$ |
| $\mathcal{A}_{3,5}^{\delta}$ | $[e_1, e_3] = e_1$, $[e_2, e_3] = \delta e_2$, $(0 < \|\delta\| < 1)$ |
| $\mathcal{A}_{3,6}$ | $[e_1, e_3] = -e_2$, $[e_2, e_3] = e_1$ |
| $\mathcal{A}_{3,7}^{\delta}$ | $[e_1, e_3] = -\delta e_1 - e_2$, $[e_2, e_3] = e_1 + \delta e_2$, $(\delta > 0)$ |
| $\mathcal{A}_{3,8}$ | $[e_1, e_2] = e_1$, $[e_1, e_3] = -2e_2$, $[e_2, e_3] = e_3$ |
| $\mathcal{A}_{3,9}$ | $[e_1, e_2] = e_3$, $[e_1, e_3] = -e_2$, $[e_2, e_3] = e_1$ |

**Table 9**. Classification of 3-Dimensional Real Lie Algebras and Their Structure Equations.

Subsection 18.2
## GRYS On 3-Dimensional Lie Groups

Our inquiry will delve into the presence of a **GRYS** (17.12) within 3-dimensional left-invariant Lie groups, on each algebra $\mathcal{A}_{3,k}, \quad k \in \{1, ..., 9\}$, with the potential vector field $V$

$$V = ae_1 + be_2 + ce_3, \quad \text{and} \quad V^\flat \otimes V^\flat = \begin{pmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{pmatrix}.$$

Obviously, formula (17.12) is symmetric and we are lead to solve a system of 6 equations

$$(\mathcal{L}_V g)_{ij} = -2c_1 V_i V_j + 2c_2 S_{ij} + 2(\lambda + r\rho)\delta_{ij}. \tag{18.1}$$

### 18.2.1 The algebra $\mathcal{A}_{3,1}$

The covariant derivatives of the basis elements are given by the following expressions

$$\begin{aligned} &\nabla_{e_1} e_1 = 0, && \nabla_{e_1} e_2 = -\frac{1}{2} e_3, && \nabla_{e_1} e_3 = \frac{1}{2} e_2, \\ &\nabla_{e_2} e_1 = -\frac{1}{2} e_3, && \nabla_{e_2} e_2 = 0, && \nabla_{e_2} e_3 = \frac{1}{2} e_1, \\ &\nabla_{e_3} e_1 = \frac{1}{2} e_2, && \nabla_{e_3} e_2 = -\frac{1}{2} e_1, && \nabla_{e_3} e_3 = 0. \end{aligned}$$

Using formulas (9.2), (6.27), and (17.11), we obtain

$$(\mathcal{L}_V g)_{ij} = \begin{pmatrix} 0 & c & -b \\ c & 0 & 0 \\ -b & 0 & 0 \end{pmatrix}, \quad S_{ij} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} \tag{18.2}$$

and the scalar curvature is

$$r = -\frac{1}{2}. \tag{18.3}$$

By directly substituting (18.2) into (18.1), we must address the challenge of solving

$$c_1 a^2 - \frac{c_2}{2} - \left(\lambda - \frac{\rho}{2}\right) = 0, \tag{18.4}$$

$$c_1 b^2 + \frac{c_2}{2} - \left(\lambda - \frac{\rho}{2}\right) = 0, \tag{18.5}$$

$$c_1 c^2 + \frac{c_2}{2} - \left(\lambda - \frac{\rho}{2}\right) = 0, \tag{18.6}$$

$$c + 2c_1 ab = 0, \tag{18.7}$$

$$b - 2c_1 ac = 0, \tag{18.8}$$

$$c_1 bc = 0. \tag{18.9}$$

By analyzing various cases related to equation (18.9), we obtain the following results:

- If $c_1 = 0$, then $b = c = 0$ is obtained from (18.7) and (18.8). The system becomes

$$-\frac{c_2}{2} - \left(\lambda - \frac{\rho}{2}\right) = 0, \tag{18.10}$$
$$\frac{c_2}{2} - \left(\lambda - \frac{\rho}{2}\right) = 0, \quad . \tag{18.11}$$

  From one hand, summing equations (18.10) and (18.11) yields $\lambda = \frac{\rho}{2}$. On the other hand, substracting equations (18.10) and (18.11) results in $c_2 = 0$.

- If $c_1 \neq 0$, then $bc = 0$. Assume $b = 0$, then $c = 0$ by virtue of (18.7). Substituting in (18.5) yields $\lambda = \frac{c_2}{2} + \frac{\rho}{2}$, this along with (18.4) yields $a = 0$. Similar result to this latter is obtained in the case where $c = 0$.

In summary, the solutions of equation (18.1) within the algebra $\mathcal{A}_{3,1}$ is as follows

$$V = ae_1, \quad c_1 = \frac{c_2}{a^2}, \quad \lambda = \frac{\rho}{2} + \frac{c_2}{2}, \quad a \in \mathbb{R}^* \quad \text{and} \quad c_2, \rho \in \mathbb{R}.$$

### 18.2.2 The algebra $\mathcal{A}_{3,2}$

The derivatives with respect to covariant bases are delineated as follows

$$\begin{aligned}
&\nabla_{e_1} e_1 = -e_3, && \nabla_{e_1} e_2 = -\frac{1}{2}e_3, && \nabla_{e_1} e_3 = e_1 + \frac{1}{2}e_2,\\
&\nabla_{e_2} e_1 = -\frac{1}{2}e_3, && \nabla_{e_2} e_2 = -e_3, && \nabla_{e_2} e_3 = \frac{1}{2}e_1 + e_2,\\
&\nabla_{e_3} e_1 = \frac{1}{2}e_2, && \nabla_{e_3} e_2 = -\frac{1}{2}e_1, && \nabla_{e_3} e_3 = 0.
\end{aligned}$$

Thus, from equations (9.2), (6.27), and (17.11), we derive

$$(\mathcal{L}_V g)_{ij} = \begin{pmatrix} 2c & c & -a-b \\ c & 2c & -b \\ -a-b & -b & 0 \end{pmatrix}, \quad S_{ij} = \begin{pmatrix} -\frac{3}{2} & -1 & 0 \\ -1 & -\frac{5}{2} & 0 \\ 0 & 0 & -\frac{5}{2} \end{pmatrix} \tag{18.12}$$

and the scalar curvature is

$$r = -\frac{13}{2}. \tag{18.13}$$

By employing equation (18.12) and (18.13) within (17.12), we are prompted to undertake the challenge of resolving

$$c_1 a^2 + \frac{3}{2}c_2 - \left(\lambda - \frac{13}{2}\rho\right) + c = 0, \tag{18.14}$$
$$c_1 b^2 + \frac{5}{2}c_2 - \left(\lambda - \frac{13}{2}\rho\right) + c = 0, \tag{18.15}$$
$$c_1 c^2 + \frac{5}{2}c_2 - \left(\lambda - \frac{13}{2}\rho\right) = 0, \tag{18.16}$$
$$c_1 ab + c_2 + \frac{c}{2} = 0, \tag{18.17}$$
$$c_1 ac - \frac{a+b}{2} = 0, \tag{18.18}$$
$$c_1 bc - \frac{b}{2} = 0. \tag{18.19}$$

Exploring different scenarios informed by equation (18.19), we discover:

- If $b = 0$, then either $a = 0$ or $c_1 c = \frac{1}{2}$, as indicated by (18.18):
  - If $a = 0$, the system of equations simplifies to

$$\frac{3}{2}c_2 - \left(\lambda - \frac{13}{2}\rho\right) + c = 0, \tag{18.20}$$

$$\frac{5}{2}c_2 - \left(\lambda - \frac{13}{2}\rho\right) + c = 0, \tag{18.21}$$

$$c_1 c^2 + \frac{5}{2}c_2 - \left(\lambda - \frac{13}{2}\rho\right) = 0, \tag{18.22}$$

$$c_2 + \frac{c}{2} = 0. \tag{18.23}$$

Subtracting (18.20) from (18.21) yields $c_2 = 0$. Substituting this into (18.23) gives $c = 0$, thus $\lambda = \frac{13}{2}\rho$.

  - If $c_1 c = \frac{1}{2}$, then substituting $c = \frac{1}{2c_1}$ results in

$$c_1 a^2 + \frac{3}{2}c_2 - \left(\lambda - \frac{13}{2}\rho\right) + \frac{1}{2c_1} = 0, \tag{18.24}$$

$$\frac{5}{2}c_2 - \left(\lambda - \frac{13}{2}\rho\right) + \frac{1}{2c_1} = 0, \tag{18.25}$$

$$\frac{1}{4c_1} + \frac{5}{2}c_2 - \left(\lambda - \frac{13}{2}\rho\right) = 0, \tag{18.26}$$

$$c_2 + \frac{1}{4c_1} = 0. \tag{18.27}$$

Subtracting (18.25) from (18.26), we obtain $\frac{1}{c_1} = 0$, which has no solution.

- If $c_1 c = \frac{1}{2}$, then from (18.18) necessarily $b = 0$, and from (18.17) $c_2 = -\frac{1}{4c_1}$. Direct substitution gives

$$\frac{1}{8c_1} - \left(\lambda - \frac{13}{2}\rho\right) = 0, \tag{18.28}$$

$$c_1 b^2 - \left(\lambda - \frac{13}{2}\rho\right) - \frac{1}{8c_1} = 0, \tag{18.29}$$

$$\frac{3}{8c_1} + \left(\lambda - \frac{13}{2}\rho\right) = 0. \tag{18.30}$$

Summing equation (18.28) and (18.30) gives $\frac{1}{c_1} = 0$, which has no solution.

Summarizing the above, equation (18.1) on the algebra $\mathcal{A}_{3,2}$ has no solution.

### 18.2.3 The algebra $\mathcal{A}_{3,3}$

The covariant derivatives of the basis elements are as follows:

$$\begin{aligned}
&\nabla_{e_1} e_1 = -e_3, && \nabla_{e_1} e_2 = 0, && \nabla_{e_1} e_3 = e_1, \\
&\nabla_{e_2} e_1 = 0, && \nabla_{e_2} e_2 = -e_3, && \nabla_{e_2} e_3 = e_2, \\
&\nabla_{e_3} e_1 = 0, && \nabla_{e_3} e_2 = 0, && \nabla_{e_3} e_3 = 0.
\end{aligned}$$

Hence, from (9.2), (6.27), and (17.11), we have

$$(\mathcal{L}_V g)_{ij} = \begin{pmatrix} 2c & 0 & -a \\ 0 & 2c & -b \\ -a & -b & 0 \end{pmatrix}, \quad S_{ij} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{18.31}$$

and the scalar curvature (17.11) is

$$r = -6. \tag{18.32}$$

Upon substituting equations (18.31) and (18.32) into (17.12), we find it necessary to address the task of resolving

$$c_1 a^2 + 2c_2 - \lambda + 6\rho + c = 0, \tag{18.33}$$
$$c_1 b^2 + 2c_2 - \lambda + 6\rho + c = 0, \tag{18.34}$$
$$c_1 c^2 + 2c_2 - \lambda + 6\rho = 0, \tag{18.35}$$
$$c_1 ab = 0, \tag{18.36}$$
$$2c_1 ac - a = 0, \tag{18.37}$$
$$2c_1 bc - b = 0. \tag{18.38}$$

Examining various scenarios based on equation (18.36), the outcomes are as follows:

- If $c_1 = 0$, then $a = b = 0$, and from (18.35) we get $\lambda = 2c_2 + 6\rho$. Substituting in either (18.33) or (18.34) yields $c = 0$.
- If $ab = 0$, we distringuish two particular cases:
  - If $a = 0$ and $b \neq 0$, then from (18.38) we have $c_1 c = \frac{1}{2}$. By direct substitution, we get

$$2c_2 - \lambda + 6\rho + \frac{1}{2c_1} = 0, \tag{18.39}$$
$$c_1 b^2 + 2c_2 - \lambda + 6\rho + \frac{1}{2c_1} = 0, \tag{18.40}$$
$$\frac{1}{4c_1} + 2c_2 - \lambda + 6\rho = 0. \tag{18.41}$$

    Substituting (18.41) from (18.39) yields $\frac{1}{c_1} = 0$, which has no solution.
  - The case where $a \neq 0$, $b = 0$ and $c_1 c = \frac{1}{2}$ yields an identical result as previously discussed.
- If $a = b = 0$, we obtain

$$2c_2 - \lambda + 6\rho + c = 0, \tag{18.42}$$
$$c_1 c^2 + 2c_2 - \lambda + 6\rho = 0. \tag{18.43}$$

  Subtracting (18.41) from (18.43) provides $c(c_1 c - 1) = 0$. Therefore:
  - Either $c = 0$ and $\lambda = 2c_2 + 6\rho$.
  - Or $c = \frac{1}{c_1}$ and $\lambda = 2c_2 + 6\rho + \frac{1}{c_1}$.

In conclusion, the solution of equation (18.1) on the algebra $\mathcal{A}_{3,3}$ are given by

$$V = ce_3, \quad c_1 = \frac{1}{c}, \quad \lambda = \frac{1}{c_1} + 2c_2 + 6\rho, \quad \text{where} \quad c \in \mathbb{R}^* \quad \text{and} \quad c_2, \rho \in \mathbb{R}.$$

### 18.2.4 The algebra $\mathcal{A}_{3,4}$

The covariant derivatives of the basis elements are as follows

$$\begin{aligned} \nabla_{e_1} e_1 &= -e_3, & \nabla_{e_1} e_2 &= 0, & \nabla_{e_1} e_3 &= e_1, \\ \nabla_{e_2} e_1 &= 0, & \nabla_{e_2} e_2 &= e_3, & \nabla_{e_2} e_3 &= -e_2, \\ \nabla_{e_3} e_1 &= 0, & \nabla_{e_3} e_2 &= 0, & \nabla_{e_3} e_3 &= 0. \end{aligned}$$

With the help of (9.2), (6.27), and (17.11), we have

$$(\mathcal{L}_V g)_{ij} = \begin{pmatrix} 2c & 0 & -a \\ 0 & -2c & b \\ -a & b & 0 \end{pmatrix}, \quad S_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{18.44}$$

and the scalar curvature is

$$r = -2. \tag{18.45}$$

After substituting equations (18.44) and (18.45) into (18.1), we need to resolve

$$c_1 a^2 - \lambda + 2\rho + c = 0, \tag{18.46}$$
$$c_1 b^2 - \lambda + 2\rho - c = 0, \tag{18.47}$$
$$c_1 c^2 + 2c_2 - \lambda + 2\rho = 0, \tag{18.48}$$
$$c_1 ab = 0, \tag{18.49}$$
$$a - 2c_1 ac = 0, \tag{18.50}$$
$$b + 2c_1 bc = 0. \tag{18.51}$$

Considering different scenarios outlined in equation (18.49), the following observations arise:

- If $c_1 = 0$, then (18.50) and (18.51) yield $a = b = 0$. By direct substitution, we obtain

$$\lambda = 2\rho + c, \tag{18.52}$$
$$\lambda = 2\rho - c, \tag{18.53}$$
$$\lambda = 2c_2 + 2\rho. \tag{18.54}$$

  Subtracting equation (18.53) from (18.52) gives $c = 0$. Summing equations (18.52) and (18.53) yields $\lambda = 2\rho$. Putting all of the above in (18.54), we obtain $c_2 = 0$.

- If $ab = 0$ and $c_1 \neq 0$, we consider three cases:
  - If $a = b = 0$, again using equations (18.52), (18.53) and (18.54), we find $c = 0$, $\lambda = 2\rho$, and $c_2 = 0$.
  - If $a = 0$ and $b \neq 0$, then $c_1 c = -\frac{1}{2}$. Substituting gives

$$\lambda + 2\rho - \frac{1}{2c_1} = 0, \tag{18.55}$$
$$c_1 b^2 - \lambda + 2\rho + \frac{1}{2c_1} = 0, \tag{18.56}$$
$$\frac{1}{4c_1} + 2c_2 - \lambda + 2\rho = 0. \tag{18.57}$$

    From (18.55), we get $\lambda = 2\rho - \frac{1}{2c_1}$. Substituting in (18.56) yields $b^2 = -\frac{1}{2c_1^2}$, which has no real solutions.
  - The case where $b = 0$, $a \neq 0$, and $c_1 c = -\frac{1}{2}$ is similar to the previous one.

Combining all the results, the solutions to equation (18.1) within the algebra $\mathcal{A}_{3,4}$ exhibits no solutions.

### 18.2.5 The algebra $\mathcal{A}_{3,5}^{\delta}$

The covariant derivatives of the basis elements are as follows

$$\begin{aligned} &\nabla_{e_1} e_1 = -e_3, && \nabla_{e_1} e_2 = 0, && \nabla_{e_1} e_3 = e_1, \\ &\nabla_{e_2} e_1 = 0, && \nabla_{e_2} e_2 = -\delta e_3, && \nabla_{e_2} e_3 = \delta e_2, \\ &\nabla_{e_3} e_1 = 0, && \nabla_{e_3} e_2 = 0, && \nabla_{e_3} e_3 = 0, \end{aligned}$$

where $0 < |\delta| < 1$. With direct computations, we have

$$(\mathcal{L}_V g)_{ij} = \begin{pmatrix} 2c & 0 & -a \\ 0 & 2c\delta & -b\delta \\ -a & -b\delta & 0 \end{pmatrix}, \quad S_{ij} = \begin{pmatrix} -1-\delta & 0 & 0 \\ 0 & -\delta^2-\delta & 0 \\ 0 & 0 & -\delta^2-1 \end{pmatrix} \tag{18.58}$$

and the scalar curvature

$$r = -2\delta^2 - 2\delta - 2. \tag{18.59}$$

Substituting equations (18.58) into (18.1), we are compelled to engage in the process of resolving

$$c_1 a^2 + (1+\delta)c_2 - \lambda + 2(\delta^2+\delta+1)\rho + c = 0, \tag{18.60}$$

$$c_1 b^2 + \delta(\delta+1)c_2 - \lambda + 2(\delta^2+\delta+1)\rho + \delta c = 0, \tag{18.61}$$

$$c_1 c^2 + (\delta^2+1)c_2 - \lambda + 2(\delta^2+\delta+1)\rho = 0, \tag{18.62}$$

$$c_1 ab = 0, \tag{18.63}$$

$$c_1 ac - \frac{a}{2} = 0, \tag{18.64}$$

$$c_1 bc - \delta\frac{b}{2} = 0. \tag{18.65}$$

Evaluating different possibilities with respect to equation (18.63), we conclude:

- If $c_1 = 0$, then equations (18.64) and (18.65) give $a = b = 0$. Thus, we have

$$(1+\delta)c_2 - \lambda + 2(\delta^2+\delta+1)\rho + c = 0, \tag{18.66}$$

$$\delta(\delta+1)c_2 - \lambda + 2(\delta^2+\delta+1)\rho + \delta c = 0, \tag{18.67}$$

$$(\delta^2+1)c_2 - \lambda + 2(\delta^2+\delta+1)\rho = 0. \tag{18.68}$$

  Subtracting equations (18.66) from (18.67), we find $c = -(\delta+1)c_2$. From equation (18.68), $\lambda = (\delta^2+1)c_2 + 2(\delta^2+\delta+1)\rho$. By substituting these results into either equation (18.66) or (18.67), we get $(\delta^2+1)c_2 = 0$, hence $c_2 = 0$.

- If $a = b = 0$ and $c_1 \neq 0$, then equations (18.66) and (18.67) result in $c = -(\delta+1)c_2$ and $\lambda = 2(\delta^2+\delta+1)\rho$. Substituting into (18.62), we obtain $c_2 = -\frac{\delta^2+1}{(\delta^2+2\delta+1)c_1}$.

- If $a \neq 0$, $b = 0$, then from (18.64) we get $c = \frac{1}{2c_1}$. Thus

$$c_1 a^2 + (1+\delta)c_2 - \lambda + 2(\delta^2+\delta+1)\rho + \frac{1}{2c_1} = 0, \tag{18.69}$$

$$\delta(\delta+1)c_2 - \lambda + 2(\delta^2+\delta+1)\rho + \frac{\delta}{2c_1} = 0, \tag{18.70}$$

$$\frac{1}{4c_1} + (\delta^2+1)c_2 - \lambda + 2(\delta^2+\delta+1)\rho = 0. \tag{18.71}$$

Subtracting equation (18.70) from (18.71), we find $c_1 = \frac{-2\delta+1}{4(\delta-1)c_2}$. Using these results along with equation (18.69), we find $a^2 = \frac{-2\delta^2+\delta-1}{4c_1^2}$, which leads to an impossibility due to $-2\delta^2+\delta-1<0$.

- If $a=0$, $b\neq 0$, then from (18.65) we get $c=\frac{\delta}{2c_1}$. Using direct substitution, we have

$$(1+\delta)c_2-\lambda+2(\delta^2+\delta+1)\rho+\frac{\delta}{2c_1}=0, \tag{18.72}$$

$$c_1b^2+\delta(\delta+1)c_2-\lambda+2(\delta^2+\delta+1)\rho+\frac{\delta^2}{2c_1}=0, \tag{18.73}$$

$$\frac{\delta^2}{4c_1}+(\delta^2+1)c_2-\lambda+2(\delta^2+\delta+1)\rho=0. \tag{18.74}$$

Subtracting equation (18.73) from (18.74), we find $c_1=\frac{\delta-2}{4(1-\delta)c_2}$. From (18.72) we pull

$$\lambda=-\frac{\delta^2-\delta+2}{4(1-\delta)c_1}.$$

Finally, substituting in (18.73), we obtain

$$b^2=-\frac{\delta^2-\delta+2}{4c_1^2},$$

for which is absurd due to the fact $\delta^2-\delta+2>0$ for all $0<|\delta|<1$.

In summary, the equation (18.1) within the algebra $\mathcal{A}_{3,5}^{\delta}$ admits the following solution

$$V=ce_3,\quad c=-(\delta+1)c_2,\quad c_2=-\frac{\delta^2+1}{(\delta^2+2\delta+1)c_1},\quad \lambda=2(\delta^2+\delta+1)\rho, c_1\in\mathbb{R}^*\quad \text{and}\quad \rho\in\mathbb{R}$$

### 18.2.6 The algebra $\mathcal{A}_{3,6}$

The covariant derivatives of the basis elements are as follows

$$\begin{array}{lll}\nabla_{e_1}e_1=0, & \nabla_{e_1}e_2=0, & \nabla_{e_1}e_3=0,\\ \nabla_{e_2}e_1=0, & \nabla_{e_2}e_2=0, & \nabla_{e_2}e_3=0,\\ \nabla_{e_3}e_1=e_2, & \nabla_{e_3}e_2=-e_1, & \nabla_{e_3}e_3=0.\end{array}$$

With direct computations, we have

$$(\mathcal{L}_Vg)_{ij}=\begin{pmatrix}0&0&-b\\0&0&a\\-b&a&0\end{pmatrix},\quad S_{ij}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix} \tag{18.75}$$

and the scalar curvature

$$r=0. \tag{18.76}$$

Substituting these into (18.1), we tackle the endeavor of resolving

$$c_1a^2 - \lambda = 0, \tag{18.77}$$
$$c_1b^2 - \lambda = 0, \tag{18.78}$$
$$c_1c^2 - \lambda = 0, \tag{18.79}$$
$$c_1ab = 0, \tag{18.80}$$
$$c_1ac - \frac{b}{2} = 0, \tag{18.81}$$
$$c_1bc + \frac{a}{2} = 0. \tag{18.82}$$

Reviewing several scenarios outlined by equation (18.80), the analysis indicates:

- If $c_1 = 0$, then from (18.77), (18.81) and (18.82) we get $\lambda = a = b = 0$.
- Consider $c_1 \neq 0$:
  - If $a = 0$, then from (18.81), $b = 0$, leading to $\lambda = 0$ and $c = 0$.
  - Similarly, if $b = 0$, from (18.82), $a = 0$, resulting in $\lambda = 0$ and $c = 0$.

Combining all the results, the equation (18.1) within the algebra $\mathcal{A}_{3,6}$ satisfies only the Killing equation for $V = ce_3$ where $c \neq 0$.

### 18.2.7 The algebra $\mathcal{A}_{3,7}^{\delta}$

The covariant derivatives of the basis elements are as follows:

$$\begin{aligned}
&\nabla_{e_1}e_1 = \delta e_3, && \nabla_{e_1}e_2 = 0, && \nabla_{e_1}e_3 = -\delta e_1,\\
&\nabla_{e_2}e_1 = 0, && \nabla_{e_2}e_2 = -\delta e_3, && \nabla_{e_2}e_3 = \delta e_2,\\
&\nabla_{e_3}e_1 = e_2, && \nabla_{e_3}e_2 = -e_1, && \nabla_{e_3}e_3 = 0,
\end{aligned}$$

where $\delta > 0$.
With direct computations, we have

$$(\mathcal{L}_V g)_{ij} = \begin{pmatrix} -2c\delta & 0 & a\delta - b \\ 0 & 2c\delta & a - \delta b \\ a\delta - b & a - \delta b & 0 \end{pmatrix}, \quad S_{ij} = \begin{pmatrix} 0 & 2\delta & 0 \\ 2\delta & 0 & 0 \\ 0 & 0 & -2\delta^2 \end{pmatrix} \tag{18.83}$$

and the scalar curvature is

$$r = -2\delta^2. \tag{18.84}$$

We need to solve the following equations from (18.1)

$$c_1a^2 - \lambda + 2\delta^2\rho - c\delta = 0, \tag{18.85}$$
$$c_1b^2 - \lambda + 2\delta^2\rho + c\delta = 0, \tag{18.86}$$
$$c_1c^2 + 2c_2\delta^2 - \lambda + 2\delta^2\rho = 0, \tag{18.87}$$
$$c_1ab - 2c_2\delta = 0, \tag{18.88}$$
$$c_1ac + \frac{a}{2}\delta - \frac{b}{2} = 0, \tag{18.89}$$
$$c_1bc + \frac{a}{2} - \frac{b}{2}\delta = 0. \tag{18.90}$$

Analyzing equation (18.89), we obtain $b = 2c_1ac + a\delta$. Substituting into (18.90) yields

$$a\left(c_1^2c^2 + \frac{1-\delta}{4}\right) = 0. \tag{18.91}$$

Reviewing several scenarios outlined by equation (18.91), the analysis indicates:

- If $a = 0$, substituting in (18.89) and using (18.88) gives $b = 0$ and $c_2 = 0$. Substituting into (18.85) and (18.86) gives $c = 0$ and $\lambda = 2\delta^2\rho$.
- If $c_1c = \frac{\sqrt{\delta-1}}{2}$, which is valid only for $\delta \geq 1$, then by direct substitution we get:
  - If $\delta > 1$, then $a = 0$ and similar results are obtained as discussed previously.
  - If $\delta = 1$, then $c_1 = 0$. In the first case, from (18.88) we get $c_2 = 0$, and from (18.89) $a = b$. Hence, we are left with

    $$-\lambda + 2\rho - c = 0, \tag{18.92}$$
    $$-\lambda + 2\rho + c = 0, \tag{18.93}$$
    $$-\lambda + 2\rho = 0, \tag{18.94}$$

    which clearly gives $\lambda = 2\rho$ and $c = 0$.

    In the second case, where $c = 0$, again from (18.89) we have $a = b$, and using (18.88) we get $a = \sqrt{\frac{2c_2}{c_1}}$. Finally, from (18.87) we obtain $\lambda = 2c_2 + 2\rho$.

Combining all the results, the solutions to equation (18.1) within the algebra $\mathcal{A}_{3,7}^{\delta}$ are

$$V = a(e_1 + e_2), \quad c_1 = \frac{2c_2}{a^2}, \quad \lambda = 2c_2 + 2\rho, \quad a \in \mathbb{R}^* \quad \text{and} \quad c_2, \rho \in \mathbb{R}.$$

### 18.2.8 The algebra $\mathcal{A}_{3,8}$

The covariant derivatives of the basis elements are

$$\begin{aligned}
&\nabla_{e_1}e_1 = -e_2, && \nabla_{e_1}e_2 = e_1 + e_3, && \nabla_{e_1}e_3 = -e_2,\\
&\nabla_{e_2}e_1 = e_3, && \nabla_{e_2}e_2 = 0, && \nabla_{e_2}e_3 = -e_1,\\
&\nabla_{e_3}e_1 = e_2, && \nabla_{e_3}e_2 = -e_1 - e_3, && \nabla_{e_3}e_3 = e_2.
\end{aligned}$$

From direct computations, we obtain

$$(\mathcal{L}_V g)_{ij} = \begin{pmatrix} 2b & -a-2c & 0 \\ -a-2c & 0 & 2a+c \\ 0 & 2a+c & -2b \end{pmatrix}, \quad S_{ij} = \begin{pmatrix} -2 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & -2 \end{pmatrix}. \tag{18.95}$$

The scalar curvature is given by

$$r = -4. \tag{18.96}$$

Therefore, upon substituting (18.95) and (18.96) into (18.1) the following equations must be satisfied

$$c_1a^2 + 2c_2 - \lambda + 4\rho + b = 0, \tag{18.97}$$
$$c_1b^2 - \lambda + 4\rho = 0, \tag{18.98}$$
$$c_1c^2 + 2c_2 - \lambda + 4\rho - b = 0, \tag{18.99}$$
$$c_1ab - \frac{a}{2} - c = 0, \tag{18.100}$$
$$c_1ac + 2c_2 = 0, \tag{18.101}$$
$$c_1bc + a + \frac{c}{2} = 0. \tag{18.102}$$

From (18.100), we deduce $c = c_1ab - \frac{a}{2}$. Substituting this into (18.102) gives:

$$a\left(c_1^2b^2 + \frac{3}{4}\right) = 0.$$

Hence, $a = 0$ and $c = 0$. This implies $c_2 = 0$. Substituting $c_2 = 0$ into (18.98) yields $\lambda = 4\rho$. Finally, substituting $\lambda = 4\rho$ into either (18.97) or (18.99) provides $b = 0$.
In conclusion, equation (18.1) in the algebra $\mathcal{A}_{3,8}$ has no solution.

### 18.2.9 The algebra $\mathcal{A}_{3,9}$

The covariant derivatives of the basis elements are

$$\begin{aligned}
&\nabla_{e_1}e_1 = 0, && \nabla_{e_1}e_2 = \frac{1}{2}e_3, && \nabla_{e_1}e_3 = -\frac{1}{2}e_2,\\
&\nabla_{e_2}e_1 = -\frac{1}{2}e_3, && \nabla_{e_2}e_2 = 0, && \nabla_{e_2}e_3 = \frac{1}{2}e_1,\\
&\nabla_{e_3}e_1 = \frac{1}{2}e_2, && \nabla_{e_3}e_2 = -\frac{1}{2}e_1, && \nabla_{e_3}e_3 = 0.
\end{aligned}$$

From direct computations, we obtain

$$(\mathcal{L}_V g)_{ij} = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{pmatrix}, \quad S_{ij} = \begin{pmatrix} \frac{1}{2} & 0 & 0\\ 0 & \frac{1}{2} & 0\\ 0 & 0 & \frac{1}{2} \end{pmatrix} \tag{18.103}$$

and the scalar curvature

$$r = \frac{3}{2}. \tag{18.104}$$

Thus, the following equations must be satisfied

$$2c_1a^2 - c_2 - 2\lambda - 3\rho = 0, \tag{18.105}$$
$$2c_1b^2 - c_2 - 2\lambda - 3\rho = 0, \tag{18.106}$$
$$2c_1c^2 - c_2 - 2\lambda - 3\rho = 0, \tag{18.107}$$
$$c_1ab = 0, \tag{18.108}$$
$$c_1ac = 0, \tag{18.109}$$
$$c_1bc = 0. \tag{18.110}$$

Analyzing equations (18.108), (18.109), and (18.110) yields

- If $c_1 = 0$, then $\lambda = -\frac{1}{2}c_2 - \frac{3}{2}\rho$, and $c_2, \rho \in \mathbb{R}$.
- If $c_1 \neq 0$, then $a = b = c = 0$, $\lambda = -\frac{1}{2}c_2 - \frac{3}{2}\rho$, and $c_2, \rho \in \mathbb{R}$.

Therefore, the solution set is

$$V = ae_1 + be_2 + ce_3, \quad c_1 = 0, \quad \lambda = -\frac{1}{2}c_2 - \frac{3}{2}\rho, \quad \text{and} \quad a, b, c, c_2, \rho \in \mathbb{R}.$$

**Theorem 18** Generalized Ricci-Yamabe soliton equation

$$\mathcal{L}_V g = -2c_1 V^\flat \otimes V^\flat + 2c_2 S + 2(\lambda + r\rho)g,$$

admits the following solutions on 3-dimensional left invariant Lie algebras:

- **Algebra** $\mathcal{A}_{3,1}$:
$$V = ae_1, \quad c_1 = \frac{c_2}{a^2}, \quad \lambda = \frac{\rho}{2} + \frac{c_2}{2} \quad \text{where} \quad a \in \mathbb{R}^* \quad \text{and} \quad c_2, \rho \in \mathbb{R}.$$
- **Algebra** $\mathcal{A}_{3,3}$:
$$V = ce_3, \quad c_1 = \frac{1}{c}, \quad \lambda = \frac{1}{c_1} + 2c_2 + 6\rho, \quad \text{where} \quad c \in \mathbb{R}^* \quad \text{and} \quad c_2, \rho \in \mathbb{R}.$$
- **Algebra** $\mathcal{A}^\delta_{3,5}$:
$$V = ce_3, \quad c = -(\delta + 1)c_2, \quad c_2 = -\frac{\delta^2 + 1}{(\delta^2 + 2\delta + 1)c_1}, \quad \lambda = 2(\delta^2 + \delta + 1)\rho,$$
where $c_1 \in \mathbb{R}^*$ and $\rho \in \mathbb{R}$.
- **Algebra** $\mathcal{A}^\delta_{3,7}$:
$$V = a(e_1 + e_2), \quad c_1 = \frac{2c_2}{a^2}, \quad \lambda = 2c_2 + 2\rho, \quad a \in \mathbb{R}^* \quad \text{and} \quad c_2, \rho \in \mathbb{R}.$$
- **Algebra** $\mathcal{A}_{3,9}$:
$$V = ae_1 + be_2 + ce_3, \quad c_1 = 0, \quad \lambda = -\frac{1}{2}c_2 - \frac{3}{2}\rho, \quad \text{and} \quad a, b, c, c_2, \rho \in \mathbb{R}.$$

We have extended the concept of the Ricci-Yamabe soliton through equation (18.1) and explored the presence of this structure on left-invariant three-dimensional Lie groups. The findings provide concrete examples that substantiate the existence of this structure, thus demonstrating its viability. This work opens a wide range of possibilities for future research in this area. We can summarize the existence of various solitonic structure on left-invariant 3-dimensional Lie algebras in the following:

| Algebra | **GRYS** | **GRS** | **RBS** | **RS** | **YS** | **PFS** |
|---|---|---|---|---|---|---|
| $\mathcal{A}_{3,1}$ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |
| $\mathcal{A}_{3,2}$ | ✗ | ✗ | ✗ | ✗ | ✗ | ✗ |
| $\mathcal{A}_{3,3}$ | ✓ | ✓ | ✓ | ✓ | ✓ | ✗ |
| $\mathcal{A}_{3,4}$ | ✗ | ✗ | ✗ | ✗ | ✗ | ✗ |
| $\mathcal{A}^\delta_{3,5}$ | ✓ | ✓ | ✓ | ✓ | ✓ | ✗ |
| $\mathcal{A}_{3,6}$ | ✗ | ✗ | ✗ | ✗ | ✗ | ✗ |
| $\mathcal{A}^\delta_{3,7}$ | ✓ | ✓ | ✓ | ✓ | ✓ | ✗ |
| $\mathcal{A}_{3,8}$ | ✗ | ✗ | ✗ | ✗ | ✗ | ✗ |
| $\mathcal{A}_{3,9}$ | ✓ | ✓ | ✓ | ✓ | ✓ | ✗ |

**Table 10**. Possible Solitonic Structure On Left-Invariant 3-Dimensional Lie Algebras

SECTION C

# Appendix C

SUBSECTION C.1

## Tensor Product

For non-negative integers $r$ and $s$, a type $(r,s)$ tensor on a vector space $V$ is an element of

$$T^r_s(\mathcal{V}) = \underbrace{\mathcal{V}\otimes\cdots\otimes\mathcal{V}}_{r}\otimes\underbrace{\mathcal{V}^*\otimes\cdots\otimes\mathcal{V}^*}_{s} = \mathcal{V}^{\otimes r}\otimes(\mathcal{V}^*)^{\otimes s}.$$

Here $\mathcal{V}^*$ is the dual vector space (which consists of all linear maps $f$ from $V$ to the field $\mathcal{K}$). There is a product map, called the **tensor product** of tensors

$$T^r_s(\mathcal{V})\otimes_{\mathcal{K}} T^{r'}_{s'}(\mathcal{V}) \to T^{r+r'}_{s+s'}(\mathcal{V}).$$

If $\mathcal{V}$ is finite dimensional, then picking a basis of $\mathcal{V}$ and the corresponding dual basis of $\mathcal{V}^*$ naturally induces a basis of $T^r_s(\mathcal{V})$ (this basis is described in the article on Kronecker products). In terms of these bases, the components of a (tensor) product of two (or more) tensors can be computed. For example, if $F$ and $G$ are two covariant tensors of orders $m$ and $n$ respectively (i.e. $F\in T^0_m$ and $G\in T^0_n$), then the components of their tensor product are given by

$$(F\otimes G)_{i_1 i_2\cdots i_{m+n}} = F_{i_1 i_2\cdots i_m} G_{i_{m+1} i_{m+2}\cdots i_{m+n}}.$$

Thus, the components of the tensor product of two tensors are the ordinary product of the components of each tensor. Another example: let $U$ be a tensor of type $(1,1)$ with components $U^i_j$, and let $V$ be a tensor of type $(1,0)$ with components $V^k$. Then

$$(V\otimes U)^i_{kj} = V_k U^i_j.$$

In our case, consider the basis $\{e_1,e_2,e_3\}$ of the the vector space $T_eG$. Let $V\in T_eG$ a tensor of type $(1,0)$

$$V = ae_1 + be_2 + c_3.$$

Using (5.6) and for an arbitrary vector field $X$ in $T_eG$, we obtain a tensor of type $(0,1)$, the dual of $V$

$$V^\flat(X) = g(V,X) = ag(e_1,X) + bg(e_2,X) + cg(e_3,X).$$

Hence, withing the co-frame $\{e^1,e^2,e^3\}$ (such that $e^i(e_j)=\delta_{ij}$)

$$V^\flat = ae^1 + be^2 + ce^3.$$

Using previous definitions and remarks, $V^\flat\otimes V^\flat$ is a tensor of type $(0,2)$ with components

$$\left(V^\flat\otimes V^\flat\right)_{ij} = \left(V^\flat\otimes V^\flat\right)(e_i,e_j) = V^\flat(e_i)V^\flat(e_j) = V_iV_j.$$

Subsection C.2
## Covariant Derivative On Lie Algebras

Let $\{e_1, ...e_n\}$ be an orthonormal babis for the vector space $T_eG$, then by the bilinearity of the Lie bracket it will be completely determined by the elements $[e_i, e_j]$. The strucutre constans $C^k_{ij}$ are defined such that

$$[e_i, e_j] = \sum_{k=1}^{i=n} C^k_{ij} e_k. \tag{C.1}$$

Hence, using the the metric $g$ we can get

$$C^k_{ij} = g([e_i, e_j], e_k). \tag{C.2}$$

Using Kozsul's formula (6.5) and (C.2), one can get

$$\begin{aligned} 2g(\nabla_{e_i} e_j, e_k) &= g([e_i, e_j], e_k) + g([e_k, e_i], e_j) - g([e_j, e_k], e_i) \\ &= C^k_{ij} + C^j_{ki} - C^i_{jk}. \end{aligned} \tag{C.3}$$

On the other hand, using formula (5.2) for $\nabla_{e_i} e_j$

$$\nabla_{e_i} e_j = \sum_{k=1}^{k=n} g(\nabla_{e_i} e_j, e_k) e_k. \tag{C.4}$$

Thus, substituting (C.3) into (C.4) yields

$$\boxed{\nabla_{e_i} e_j = \frac{1}{2} \sum_{k=1}^{k=n} (C^k_{ij} + C^j_{ki} - C^i_{jk}) e_k.} \tag{C.5}$$

As an application, we shall use datas provided in Table 9 to compute the covariant derivatives of the basis elements in $\mathcal{A}_{3,1}$: The non-zero structure constants are $C^1_{23} = 1$. Using formula (C.5)

$$\begin{aligned}
\nabla_{e_1} e_1 &= \frac{1}{2}\Big( (C^1_{11} + C^1_{11} - C^1_{11}) e_1 + (C^2_{11} + C^1_{21} - C^1_{12}) e_2 + (C^3_{11} + C^1_{31} - C^1_{13}) e_3 \Big) = 0, \\
\nabla_{e_1} e_2 &= \frac{1}{2}\Big( (C^1_{12} + C^2_{11} - C^1_{21}) e_1 + (C^2_{12} + C^2_{21} - C^1_{22}) e_2 + (C^3_{12} + C^2_{31} - C^1_{23}) e_3 \Big) = -\frac{1}{2} e_3, \\
\nabla_{e_1} e_3 &= \frac{1}{2}\Big( (C^1_{13} + C^3_{11} - C^1_{31}) e_1 + (C^2_{13} + C^3_{21} - C^1_{32}) e_2 + (C^3_{13} + C^3_{31} - C^1_{33}) e_3 \Big) = \frac{1}{2} e_2, \\
\nabla_{e_2} e_1 &= \frac{1}{2}\Big( (C^1_{21} + C^1_{12} - C^2_{11}) e_1 + (C^2_{21} + C^1_{22} - C^2_{12}) e_2 + (C^3_{21} + C^1_{32} - C^2_{13}) e_3 \Big) = -\frac{1}{2} e_3, \\
\nabla_{e_2} e_2 &= \frac{1}{2}\Big( (C^1_{22} + C^2_{12} - C^2_{21}) e_1 + (C^2_{22} + C^2_{22} - C^2_{22}) e_2 + (C^3_{22} + C^2_{32} - C^2_{23}) e_3 \Big) = 0, \\
\nabla_{e_2} e_3 &= \frac{1}{2}\Big( (C^1_{23} + C^3_{12} - C^2_{31}) e_1 + (C^2_{23} + C^3_{22} - C^2_{32}) e_2 + (C^3_{23} + C^3_{32} - C^2_{33}) e_3 \Big) = -\frac{1}{2} e_1,
\end{aligned}$$

$$\nabla_{e_3} e_1 = \frac{1}{2}\Big( (C^1_{31} + C^1_{13} - C^3_{11}) e_1 + (C^2_{31} + C^1_{23} - C^3_{12}) e_2 + (C^3_{31} + C^1_{33} - C^3_{13}) e_3 \Big) = \frac{1}{2} e_2,$$
$$\nabla_{e_3} e_2 = \frac{1}{2}\Big( (C^1_{32} + C^2_{13} - C^3_{21}) e_1 + (C^2_{32} + C^2_{23} - C^3_{22}) e_2 + (C^3_{32} + C^2_{33} - C^3_{23}) e_3 \Big) = -\frac{1}{2} e_1,$$
$$\nabla_{e_3} e_3 = \frac{1}{2}\Big( (C^1_{33} + C^3_{13} - C^3_{31}) e_1 + (C^2_{33} + C^3_{23} - C^3_{32}) e_2 + (C^3_{33} + C^3_{33} - C^3_{33}) e_3 \Big) = 0.$$

# *Ricci-Soliton Under Deformations*

PART

VI



**Table 11**. Contents for Part VI

In the concluding section of our study, we will delve into the dynamic properties of Ricci solitons across various categories of manifolds subject to specific deformations. Additionally, we aim to establish the presence of a rudimentary Ricci soliton within select instances of $C_{12}$ manifolds.

SECTION 19

## Ricci-Soliton on a Class of Riemannian manifold under $\mathcal{D}$-Isometric Deformation

In this section, we will investigate Ricci-Soliton on compact gradient manifolds admitting a Jacobi-Type vector field [31]. First, we have the following immediate result:

**Theorem 19** Let $(M^n, g)$ be a Riemannian manifold endowed with a unit closed vector field $\xi$ admitting Ricci-Soliton $(g, \xi, \lambda)$. The following holds:

- $r = n\lambda - \mathrm{div}\xi$,
- $\lambda$ is eigenvalue of Ricci operator $Q$ with $\xi$ its associated eigenvector.

In addition, if $(M^n, g)$ is compact and $\xi$ is of Jacobi-type, then:

- $(M^n, g)$ is Einstein.
- $r = n\lambda$.

PROOF Since $(g, \xi, \lambda)$ is a Ricci-Soliton then equation (17.5) along with (14.1) gives

$$g(\nabla_X \xi, Y) + S(X, Y) = \lambda g(X, Y). \tag{19.1}$$

Computing the trace of $S$ from equation (19.1) yields

$$r = \sum_{i=1}^{n} S(e_i, e_i) = \sum_{i=1}^{n} \lambda g(e_i, e_i) - \sum_{i=1}^{n} g(\nabla_{e_i} \xi, e_i) = n\lambda - \mathrm{div}\xi.$$

Again taking $X = \xi$ in (19.1) and using equations (11.20) and (14.1), we can obtain

$$S(\xi, Y) = \lambda \eta(Y) \Rightarrow Q\xi = \lambda \xi.$$

The second part is immediately obtained using Theorem 9. □

**Proposition 39** Let $(M^n, g)$ be a Riemannian manifold endowed with a unit closed vector field $\xi$ admitting $(g, \xi, \lambda)$ Ricci-Soliton. Then $(g, f\xi, \lambda)$ is a Ricci-Soliton if and only if $f$ is constant.

Proof Using the definition of the Lie derivative, we have

$$(\mathcal{L}_V g)(X,Y) = f(\mathcal{L}_\xi g)(X,Y) + X(f)\eta(Y) + Y(f)\eta(X).$$

Hence, from equation (17.5) we obtain

$$\begin{aligned}(\mathcal{L}_{f\xi} g)(X,Y) + 2S(X,Y) - 2\lambda g(X,Y) &= f(\mathcal{L}_\xi g)(X,Y) + X(f)\eta(Y) \\ &+ Y(f)\eta(X) + 2S(X,Y) - 2\lambda g(X,Y) = 0.\end{aligned} \tag{19.2}$$

Consider $\{\xi, e_i\}_{2\leq i\leq n}$ an orthonormal frame with respect to $g$ and since $(g,\xi,\lambda)$ is a Ricci-Soliton $(S(e_i,\xi) = 0)$, substituting $X = Y = \xi$ in (19.2) gives

$$\xi(f) = 0.$$

Again, putting $X = \xi$ and $Y = e_i$ in (19.2) yields

$$e_i(f) = 0.$$

□

In the following, we will study Ricci-Soliton behaviour under $\mathcal{D}$-isometric deformation (14.4). First, we consider the case where the potential field $V$ is pointwise collinear with the vector field $\xi$ (i.e $V = f\xi$, $f$ is a smooth function on $M^n$). With direct computations we have

$$\begin{aligned}(\mathcal{L}_V \tilde{g})(X,Y) &= \tilde{g}(\tilde{\nabla}_X V, Y) + \tilde{g}\big(\tilde{\nabla}_Y V, X\big) \\ &= \tilde{g}\big(\tilde{\nabla}_X (f\xi), Y\big) + \tilde{g}\big(\tilde{\nabla}_Y (f\xi), X\big) \\ &= 2fg(\nabla_X \xi, Y) + 2X(f)\eta(Y) + 2Y(f)\eta(X) \\ &= 2f(\mathcal{L}_\xi g)(X,Y) + 2X(f)\eta(Y) + 2Y(f)\eta(X).\end{aligned}$$

Thus

$$(\mathcal{L}_V \tilde{g})(X,Y) = 2(\mathcal{L}_V g)(X,Y) = (\mathcal{L}_{2V} g)(X,Y). \tag{19.3}$$

Replacing (14.4), (14.12) and (19.3) in (17.5) we obtain

$$\begin{aligned}(\mathcal{L}_V \tilde{g})(X,Y) + 2\tilde{S}(X,Y) - 2\lambda\tilde{g}(X,Y) &= (\mathcal{L}_{2V} g)(X,Y) + 2S(X,Y) \\ &+ \operatorname{div}\xi g(\nabla_X \xi, Y) - 2\lambda g(X,Y) - 2\lambda\eta(X)\eta(Y).\end{aligned} \tag{19.4}$$

Thus $(M, \tilde{g}, f\xi, \lambda)$ is a Ricci soliton if and only if

$$\operatorname{div}\xi g(\nabla_X \xi, Y) - \lambda\eta(X)\eta(Y) = 0. \tag{19.5}$$

Therefore, summing up the arguments above, we have the following theorem:

**Theorem 20** Let $(M^n, g)$ be a Riemannian manifold endowed with a unit closed Jacobi-Type vector field $\xi$ admitting Ricci-Soliton $(g, 2f\xi, \lambda)$. Then, under
$\mathcal{D}$-isometric deformation $(\tilde{g}, f\xi, \lambda)$ is a Ricci-Soliton if and only if

$$\operatorname{div}\xi g(\nabla_X \xi, Y) - \lambda\eta(X)\eta(Y) = 0,$$

and $(g, 2f\xi, \lambda)$ , $(\tilde{g}, f\xi, \lambda)$ are steady.

**Corollary 12** Let $(M^n, g)$ be a compact Riemannian manifold endowed with a unit closed Jacobi-Type vector field $\xi$. Then, under $\mathcal{D}$-isometric deformation $\eta$-Einstein Ricci-Soliton $(g, f\xi, \lambda)$ deforms to an Einstein metric.

Next, we consider the case where $V$ is orthogonal to $\xi$ i.e. $\eta(V) = 0$. We compute

$$\tilde{g}(\tilde{\nabla}_X V, Y) = \tilde{g}\Big(\nabla_X V + \frac{1}{2}g(\nabla_X \xi, V)\xi, Y\Big) = \tilde{g}(\nabla_X V, Y) + \frac{1}{2}g(\nabla_X \xi, V)\tilde{g}(\xi, Y),$$

knowing that $\tilde{g} = g + \eta \otimes \eta$ and $g(\nabla_X \xi, V) = -\eta\big(\nabla_X V\big)$, we get

$$\tilde{g}(\tilde{\nabla}_X V, Y) = g(\nabla_X V, Y).$$

Then

$$(\mathcal{L}_V \tilde{g})(X, Y) = g(\nabla_X V, Y) + g(\nabla_Y V, X) = (\mathcal{L}_V g)(X, Y). \tag{19.6}$$

Replacing (14.4), (14.12) and (19.6) in (17.5), we obtain

$$\big(\mathcal{L}_V \tilde{g} + 2\tilde{S} - 2\lambda \tilde{g}\big)(X, Y) = \big(\mathcal{L}_V g + 2S - 2\lambda g\big)(X, Y) + g\Big(\mathrm{div}\xi \nabla_X \xi - 2\lambda \eta(X)\xi, Y\Big). \tag{19.7}$$

If $(M, g, V, \lambda)$ is a Ricci-Soliton, the above equation takes the form

$$\big(\mathcal{L}_V \tilde{g} + 2\tilde{S} - 2\lambda \tilde{g}\big)(X, Y) = +g\Big(\mathrm{div}\xi \nabla_X \xi - 2\lambda \eta(X)\xi, Y\Big).$$

Thus, $(M, \tilde{g}, V, \lambda)$ is a Ricci-Soliton if and only if

$$\mathrm{div}\xi \nabla_X \xi - 2\lambda \eta(X)\xi = 0. \tag{19.8}$$

By taking $X = \xi$ in (19.8), we obtain:

$$\lambda = 0, \tag{19.9}$$

substituting equations (19.8) and (19.9) in (14.12), we obtain

$$\tilde{S} = S. \tag{19.10}$$

Then we get:

$$\mathcal{L}_V \tilde{g} + 2\tilde{S} - 2\lambda \tilde{g} = \mathcal{L}_V g + 2S - 2\lambda g - 2\lambda\, \eta \otimes \eta.$$

Hence, we state the following:

**Theorem 21** Let $(M, g, V, \lambda)$ be a Ricci-Soliton with the potential vector field $V$ orthogonal to $\xi$. Then $(M, \tilde{g}, V, \lambda)$ is a steady Ricci-Soliton.

Subsection 19.1

## A Class of Examples

**Example 19.2. Hyperbolic Cylinder:**

Let $M = \mathbb{H}^n \times \mathbb{R} = \{(x_i, z) \in \mathbb{R}^{n+1}/x_n > 0\}$, where $(x_i, z)_{1 \le i \le n}$ are standard co-ordinate in $\mathbb{R}^{n+1}$. Let $\{e_i, \xi\}_{1 \le i \le n}$ be linearly independent vector fields given by

$$e_i = x_n \frac{\partial}{\partial x_i}, \qquad \xi = \frac{\partial}{\partial z}.$$

We define a Riemannian metric $g$ by

$$g = \frac{1}{x_n^2} \sum_{i=1}^{n} dx_i^2 + dz^2.$$

Let $\nabla$ be the Riemannian connection of $g$, then we have

$$[e_i, e_n] = -e_i, \qquad [e_i, \xi] = [e_i, e_j] = 0, \qquad \forall i \neq j \in \{1, .., n-1\}.$$

By using the Koszul formula for the Riemannian metric $g$

$$2g(\nabla_{e_i} e_j, e_k) = -g(e_i, [e_j, e_k]) + g(e_j, [e_k, e_i]) + g(e_k, [e_i, e_j]),$$

the non zero components of the Levi-Civita connection corresponding to $g$ are given by

$$\nabla_{e_i} e_i = e_n, \qquad \nabla_{e_i} e_n = -e_i, \qquad \forall i \in \{1, .., n-1\}.$$

The non-vanishing curvature tensor $R$ components are computed as

$$R(e_i, e_j)e_j = -e_i, \quad R(e_i, e_n)e_i = e_n, \quad R(e_i, e_n)e_n = -e_i, \qquad \forall i \neq j \in \{1, .., n-1\}.$$

The Ricci operator $Q$ and the Ricci curvature $S$ components are computed as

$$Qe_i = (1-n)e_i, \qquad S(e_i, e_j) = (1-n)\delta_{ij}, \qquad S(\xi, \xi) = 0, \qquad \forall i \in \{1, .., n\}.$$

One can easily check that for $f$ constant function on $M$, $(g, f\xi, \lambda)$ is a steady Ricci-Soliton. Since $\mathrm{div}\xi = 0$, using Theorem 20, we obtain that $(\tilde{g}, f\xi, \lambda)$ is also a steady Ricci-Soliton such that

$$\tilde{g} = \frac{1}{x_n^2} \sum_{i=1}^{n} dx_i^2 + 2dz^2.$$

Considering Theorem 20 , it is proved that there exists an infinite number of Ricci-Solitons $(M, g_m, f\xi, \lambda)$ where $g_m = g + m\eta \otimes \eta$.

**Example 19.3. 3-D Cigar-Soliton**:
Let $M = \mathbb{S}^1 \times \mathbb{R} \times \mathbb{R}$ and let $\{e_1, e_2, e_3\}$ be linearly independent vector fields given by

$$e_1 = (1 + x^2)\frac{\partial}{\partial x}, \qquad e_2 = \frac{\partial}{\partial r}, \qquad e_3 = \frac{1}{x}\frac{\partial}{\partial t}.$$

and $\{\theta^1, \theta^2, \theta^3\}$ be the dual frame of differential 1-forms such that

$$\theta^1 = \frac{1}{1 + x^2}dx, \qquad \theta^2 = dr, \qquad \theta^3 = xdt.$$

We define a Riemannian metric $g$ by $g = \sum_{i=1}^{3} \theta^i \otimes \theta^i$, That is the form

$$g = \begin{pmatrix} \frac{1}{(1+x^2)^2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x^2 \end{pmatrix}.$$

The potential vector field is given by $V = \mathrm{grad} f = 2x(1 + x^2)e_1$ where the potential function is $f = \ln(1 + x^2)$.

With simple computations one can find

$$S = -2(1+x^2)\begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix}, \qquad \mathcal{L}_V g = 4(1+x^2)\begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix}.$$

We can easily notice that $\mathcal{L}_V g + 2S = 0$ which implies that $(M, g, V)$ is a steady Ricci soliton.
We take $\xi = e_2$ to ensure the conditions $V \perp \xi$ and $\mathrm{d}\theta^2 = 0$ then we deform the metric as follows

$$\tilde{g} = g + \theta^2 \otimes \theta^2 = \begin{pmatrix}\frac{1}{(1+x^2)^2}&0&0\\0&2&0\\0&0&x^2\end{pmatrix}.$$

Using formulas (19.6) and (19.10) we conclude that $(M, \tilde{g}, V)$ is a steady Ricci soliton aswell.

Section 20

# Ricci-Soliton on Deformed $C_{12}$-Manifolds

The significance inherent in establishing conditions for the triviality of a Ricci soliton arises from its role as a broader conceptualization encompassing Einstein manifolds. In this context, our investigation commences by elucidating a specific equivalence between these two constructs within the framework of $C_{12}$-manifolds. The established results can be found in [32]

Subsection 20.1

## Under $\eta$-conformal deformation

Let us initially presume that $(M, g)$ constitutes an Einstein manifold, denoted by the relation

$$S(X,Y) = \mu g(X,Y) + \nu\eta(X)\otimes\eta(Y) \quad \text{where} \quad \mu+\nu = -\mathrm{div}\psi.$$

Subsequently, employing (15.87), we derive

$$\tilde{S}(X,Y) = \mu\tilde{g}(X,Y) + a^2\nu\eta(X)\otimes\eta(Y) = \mu\tilde{g}(X,Y) + \nu\tilde{\eta}(X)\otimes\tilde{\eta}(Y).$$

Consequently, our preliminary finding is articulated as:

**Theorem 22** An $\eta$-Einstein $C_{12}$ manifold is invariant under $\eta$-conformal deformation.

Continuing our analysis, through direct computations involving (15.1) and (15.61), we arrive at the following expression

$$\begin{aligned}\big(\mathcal{L}_V\tilde{g}\big)(X,Y) &= \tilde{g}(\tilde{\nabla}_X V, Y) + \tilde{g}(\tilde{\nabla}_Y V, X)\\ &= \big(\mathcal{L}_V g\big)(X,Y) + (1-a^2)\big(\eta(\nabla_X V)\eta(Y) + \eta(\nabla_Y V)\eta(X)\big).\end{aligned} \tag{20.1}$$

By substituting expressions (15.1), (15.61), and (20.1) into the Ricci soliton equation (17.5), we obtain the subsequent relation

$$\begin{aligned}\big(\mathcal{L}_V\tilde{g}\big)(X,Y) + 2\tilde{S}(X,Y) - 2\lambda\tilde{g}(X,Y) &= \big(\mathcal{L}_V g\big)(X,Y) + 2S(X,Y) - 2\lambda g(X,Y)\\ &+ (1-a^2)\Big(2(\mathrm{div}\psi+\lambda)\eta(X)\eta(Y) + \eta(\nabla_X V)\eta(Y) + \eta(\nabla_Y V)\eta(X)\Big).\end{aligned} \tag{20.2}$$

**Case 01:** Initially, we shall examine the vector field $V = f\xi$ to be pointwise colinear with $\xi$, where $f$ represents a smooth function on $M$

$$\eta(\nabla_X V) = X(f) + f\eta(\nabla_X \xi) = X(f).$$

Substituting this expression into (20.2), the resulting equation is

$$\begin{aligned}\big(\mathcal{L}_V\tilde{g}\big)(X,Y) + 2\tilde{S}(X,Y) - 2\lambda\tilde{g}(X,Y) &= \big(\mathcal{L}_V g\big)(X,Y) + 2S(X,Y) - 2\lambda g(X,Y)\\ &+ (1-a^2)\Big(2(\mathrm{div}\psi + \lambda)\eta(X)\eta(Y) + X[f]\eta(Y) + Y[f]\eta(X)\Big).\end{aligned}$$

If $(g, V, \lambda)$ constitutes a Ricci soliton, then $(\tilde{g}, V, \lambda)$ assumes the same role if and only if

$$2(\mathrm{div}\psi + \lambda)\eta(X)\eta(Y) + X[f]\eta(Y) + Y[f]\eta(X) = 0 \tag{20.3}$$

Setting $X = Y = \xi$ in (20.3), the ensuing expression is derived

$$\xi(f) = -\lambda + \mathrm{div}\psi.$$

Consequently, our initial result manifests as:

**Theorem 23** A Ricci soliton $(g, f\xi, \lambda)$ on a $C_{12}$ manifold is invariant under $\eta$-conformal deformation if and only if $\mathrm{grad}(f) = -(\lambda + \mathrm{div}\psi)\xi$.

**Corollary 13** A Ricci soliton $(g, f\xi, -\mathrm{div}\psi)$ on a $C_{12}$ manifold is invariant under $\eta$-conformal deformation.

**Case 02:** In this scenario, we consider the vector field $V$ to be orthogonal to $\xi$. Consequently, we obtain the relationship

$$\eta(\nabla_X V) = -g(V, \nabla_X \xi) \implies \begin{cases} \eta(\nabla_X V) = f\eta(X), \text{if } V = f\psi, \\ \eta(\nabla_X V) = 0, \text{if not.} \end{cases} \tag{20.4}$$

In the circumstance where $V$ is perpendicular to both $\xi$ and $\psi$ according to (20.2), the triplet $(\tilde{g}, V, \lambda)$ forms a Ricci soliton if and only if

$$\mathrm{div}\psi + \lambda = 0 \implies \lambda = -\mathrm{div}\psi.$$

Conversely, if $V$ is pointwise colinear with $\psi$, then

$$\mathrm{div}\psi + \lambda + f = 0 \implies f = -(\mathrm{div}\psi + \lambda).$$

Hence, we articulate the ensuing results:

**Theorem 24** A Ricci soliton $(g, V, -\mathrm{div}\psi)$ on $C_{12}$ manifold, where $V$ is perpendicular to both $\xi$ and $\psi$ is invariant under $\eta$-conformal deformation.

**Theorem 25** A Ricci soliton $(g, f\psi, \lambda)$ on $C_{12}$ manifold is invariant under $\eta$-conformal deformation if and only if $f = -(\mathrm{div}\psi + \lambda)$.

Subsection 20.2

## Under $\omega$-conformal deformation

In this transformation, our focus is exclusively on the 3-dimensional case. Analogous to the preceding analysis, through a direct examination of the formula (15.72), we observe the following relationship

$$\bar{S}(X,Y) = S(X,Y) + (1-b^2)\mathrm{div}\psi g(\nabla_X\psi, Y) = S(X,Y) + \frac{1}{2}(\mathcal{L}_{\alpha\psi} g)(X,Y),$$

where $\alpha = (1-b^2)\text{div}\psi$. If $(g, \alpha\psi, \lambda)$ constitutes a Ricci soliton on $(M, g)$, it is evident that $\lambda = -b^2\text{div}\psi$, thus

$$\begin{aligned}\bar{S}(X,Y) &= S(X,Y) + +\frac{1}{2}(\mathcal{L}_{\alpha\psi}g)(X,Y) = \lambda g(X,Y)\\ &= -\text{div}\psi\bar{g}(X,Y) - (1-b^2)\text{div}\psi\omega(X)\omega(Y).\end{aligned}$$

This leads to the subsequent result:

**Proposition 40** A Ricci soliton $(g, \alpha\psi, -b^2\text{div}\psi)$ on a $C_{12}$ manifold becomes $\omega$-Einstein under $\omega$-conformal deformation where $\alpha = (1-b^2)\text{div}\psi$ and $b$ is a constant.

Consider now an arbitrary vector field $V$ on $M$. Utilizing (6.27), (15.54) and (15.65), we obtain

$$\big(\mathcal{L}_V\bar{g}\big)(X,Y) = \bar{g}(\bar{\nabla}_X V, Y) + \bar{g}(\bar{\nabla}_Y V, X) \;= b^2\big(\mathcal{L}_V g\big)(X,Y), \tag{20.5}$$

Substituting (15.54), (15.75) and (20.5), we obtain

$$\begin{aligned}\big(\mathcal{L}_V\bar{g}\big)(X,Y) + 2\bar{S}(X,Y) - 2\lambda\bar{g}(X,Y) &= b^2\big(\mathcal{L}_V g\big)(X,Y) + 2S(X,Y)\\ &+ (1-b^2)\text{div}\psi\big(\mathcal{L}\psi g\big)(X,Y) - 2b^2\lambda g(X,Y) - 2\lambda(1-b^2)\omega(X)\omega(Y)\end{aligned} \tag{20.6}$$

Since

$$\big(\mathcal{L}\psi\bar{g}\big)(X,Y) = \big(\mathcal{L}\psi g\big)(X,Y),$$

one can rewrite equation (20.6) as follows

$$\big(\mathcal{L}_{\frac{1}{b^2}V_1}\bar{g}\big)(X,Y) + 2\bar{S}(X,Y) - 2\lambda\bar{g}(X,Y) = \big(\mathcal{L}_{V_1}g\big)(X,Y) + 2S(X,Y) - 2b^2\lambda g(X,Y) - 2\lambda(1-b^2)\omega(X)\omega(Y),$$

where $V_1 = b^2V - b^2(\text{div}\psi)$. We state the following:

**Theorem 26** A generalized Ricci soliton $(g, V_1, -b^2\lambda)$ on $C_{12}$ manifold:

$$\mathcal{L}_{V_1}g = -2c_1V_2^\flat \otimes V_2^\flat + 2c_2S + 2\lambda g$$

where $V_1 = b^2V - b^2(\text{div}\psi)\psi$, $V_2 = \psi$, $c_1 = \lambda(1-b^2)$ and $c_2 = 1$, becomes a Ricci soliton $(\bar{g}, \frac{1}{b^2}V_1, \lambda)$ under $\omega$-conformal deformation.

SUBSECTION 20.3

## Ricci Soliton on $C_{12}$ Lorentz-Manifolds

In [6], significant results concerning Ricci solitons on 3-dimensional $C_{12}$-manifolds are presented. Here, we extend these findings, demonstrating their validity for manifolds of arbitrary odd dimension. In this section, we explore the behavior of Ricci solitons and generalized Ricci solitons within the framework of Lorentzian $C_{12}$-manifolds. We begin by introducing the fundamental concepts required for our analysis.
Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a $C_{12}$-manifold and $(M^{2n+1}, \varphi, \xi, \eta, g^*)$ it's associated Lorentz-$C_{12}$-manifold.

**Proposition 41** Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a $C_{12}$-manifold and consider $(M^{2n+1}, \varphi, \xi, \eta, g^*)$ the associated Lorentz-$C_{12}$-manifold admitting $(g, \xi, \lambda)$ Ricci-soliton. The following holds:

- $(M, g^*)$ is Einstein.
- The scalar curvature is given by $r^* = (2n+1)\text{div}\psi$

Proof Since $(g^*, \xi, \lambda)$ is a Ricci-soliton, then

$$\mathcal{L}_\xi g^*(X,Y) + 2S^*(X,Y) - 2\lambda g^*(X,Y) = 0.$$

From (15.79), we have

$$\nabla^*_X \xi = \nabla_X \xi + \eta(X)\psi,$$

and in view of (11.16), we get $\nabla^*_X \xi = 0$. Then

$$\mathcal{L}_\xi g^*(X,Y) = 0.$$

Knowing that

$$r^* = \sum_{1}^{2n+1} S^*(e_i, e_i),$$

we obtain the desired result. □

For our first motivation, we consider the case where the potential field $V$ be point-wise co-linear with the vector field $\xi$ i.e. $V = f\xi$, where $f$ is a function on $M$. We compute

$$\begin{aligned}(\mathcal{L}_{f\xi}\tilde{g})(X,Y) &= \tilde{g}\big(\tilde{\nabla}_X(f\xi), Y\big) + \tilde{g}\big(\tilde{\nabla}_Y(f\xi), X\big)\\ &= -X(f)\eta(Y) - Y(f)\eta(X).\end{aligned} \tag{20.7}$$

Replacing (20.22), (15.87) and (15.77) in (17.5), we obtain

$$\begin{aligned}(\mathcal{L}_{f\xi}\tilde{g})(X,Y) + 2\tilde{S}(X,Y) - 2\lambda\tilde{g}(X,Y) =&\mathcal{L}_\psi g(X,Y) + 2\omega(X)\omega(Y) + 2S(X,Y)\\ &- 2\lambda g(X,Y) - X(f)\eta(Y) - Y(f)\eta(X)\\ &+ 2\big(2\lambda + 2\mathrm{div}\psi - |\psi|^2\big)\eta(X)\eta(Y).\end{aligned} \tag{20.8}$$

$(\tilde{g}, f\xi, \lambda)$ is a Ricci-soliton if and only if

$$\begin{aligned}\mathcal{L}_\psi g(X,Y) =& - 2\omega(X)\omega(Y) - 2S(X,Y) + 2\lambda g(X,Y)\\ &+ X(f)\eta(Y) + Y(f)\eta(X) - 2\big(2\lambda + 2\mathrm{div}\psi - |\psi|^2\big)\eta(X)\eta(Y).\end{aligned} \tag{20.9}$$

By setting $Y = \xi$ in (20.9), we obtain

$$X(f) = \big(2\lambda + 2\mathrm{div}\psi - \xi(f)\big)\eta(X). \tag{20.10}$$

Again replacing $X$ by $\xi$ in (20.10), we get

$$\xi(f) = \lambda + \mathrm{div}\psi. \tag{20.11}$$

Substituting this in (20.10), we have

$$X(f) = \big(\lambda + \mathrm{div}\psi\big)\eta(X), \tag{20.12}$$

witch implies

$$\mathrm{d}f = \big(\lambda + \mathrm{div}\psi\big)\eta. \tag{20.13}$$

Substituting (20.12) in (20.9) , we obtain

$$\mathcal{L}_\psi g(X,Y) = - 2\omega(X)\omega(Y) - 2S(X,Y) + 2\lambda g(X,Y) - 2\big(\lambda + \mathrm{div}\psi - |\psi|^2\big)\eta(X)\eta(Y). \tag{20.14}$$

Thus, we state the following:

**Theorem 27** Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a $C_{12}$-manifold and consider $(M^{2n+1}, \varphi, \xi, \eta, g^*)$ the associated Lorentz-$C_{12}$-manifold. If $(g^*, f\xi, \lambda)$ is a Ricci- soliton then $g$ satisfies the generalized $\eta$-Ricci- soliton equation (17.8) with

$$V = \psi, \quad c_1 = 1, \quad c_2 = -1 \quad \text{where} \quad \mu = -2\big(\lambda + \mathrm{div}\psi - |\psi|^2\big) \in \mathbb{R} \quad \text{and} \quad \mathrm{d}f = \big(\lambda + \mathrm{div}\psi\big)\eta.$$

In addition, if $\lambda = |\psi|^2 - \mathrm{div}\psi \in \mathbb{R}$, then $g$ satisfies the generalized Ricci-soliton equation (17.6) with $V = \psi$, $c_1 = 1$, $c_2 = -1$ where $\mathrm{d}f = |\psi|^2\eta$ .

Conversely, suppose that $g$ satisfies the generalized $\eta$-Ricci-soliton equation (17.8) with $V = \psi$, that is

$$\mathcal{L}_\psi g = -2c_1\omega \otimes \omega + 2c_2 S + 2\lambda g + \mu\eta \otimes \eta, \tag{20.15}$$

where $c_1, c_2, \lambda, \mu \in \mathbb{R}$.
Using (15.87) and (15.77) taking into account $\mathcal{L}_\psi g(X, Y) = 2g(\nabla_X\psi, Y)$, (20.15) reduces to

$$\begin{aligned} c_2\tilde{S}(X, Y) =& -\lambda\tilde{g}(X, Y) + \big(c_2(2\mathrm{div}\psi - |\psi|^2) - 2\lambda - \frac{\mu}{2}\big)\eta(X)\eta(Y) \\ &+ (1 + c_2)g(\nabla_X\psi, Y) + (c_1 + c_2)\omega(X)\omega(Y), \end{aligned} \tag{20.16}$$

i.e. $\tilde{g}$ is $\eta$-Einstein if and only if

$$c_2 = -1 \quad \text{and} \quad c_1 = 1.$$

So, (20.16) becomes

$$\tilde{S}(X, Y) = \lambda\tilde{g}(X, Y) + \big(2\lambda + 2\mathrm{div}\psi - |\psi|^2 + \frac{\mu}{2}\big)\eta(X)\eta(Y), \tag{20.17}$$

where $2\lambda + 2\mathrm{div}\psi - |\psi|^2 + \frac{\mu}{2} \in \mathbb{R}$. Further, setting $X = Y = \xi$ in (20.17), we obtain

$$\frac{\mu}{2} = -\lambda - \mathrm{div}\psi + |\psi|^2,$$

and (20.17) reduces to

$$\tilde{S}(X, Y) = \lambda\tilde{g}(X, Y) - (\lambda + \mathrm{div}\psi)\eta(X)\eta(Y). \tag{20.18}$$

on the other hand, if $g$ satisfies the generalized Ricci-soliton equation (17.6) with $V = \psi$, from (20.17) with $\mu = 0$ we get

$$\tilde{S}(X, Y) = \lambda\tilde{g}(X, Y) + \big(2\lambda + 2\mathrm{div}\psi - |\psi|^2\big)\eta(X)\eta(Y), \tag{20.19}$$

where $2\lambda + 2\mathrm{div}\psi - |\psi|^2 \in \mathbb{R}$. Again, setting $X = Y = \xi$ in (20.19), we obtain

$$\lambda = |\psi|^2 - \mathrm{div}\psi,$$

and (20.19) reduces to

$$\tilde{S}(X, Y) = (|\psi|^2 - \mathrm{div}\psi)\tilde{g}(X, Y) + |\psi|^2\eta(X)\eta(Y), \tag{20.20}$$

Therefore, we have the following:

**Theorem 28** Let $(M, \varphi, \xi, \eta, g)$ be a $C_{12}$-manifold with $\mathrm{div}\psi \in \mathbb{R}$ and consider $(M, \varphi, \xi, \eta, \tilde{g})$ the associated Lorentz-$C_{12}$-manifold.

- If $g$ satisfies the generalized $\eta$-Ricci-soliton equation (17.8) with

$$V = \psi, \quad c_1 = 1, \quad c_2 = -1, \quad \frac{\mu}{2} = -\lambda - \mathrm{div}\psi + |\psi|^2$$

then, $(M, g^*)$ is $\eta$-Einstein manifold. In addition, if $\lambda = -\mathrm{div}\psi$, then $(M, g^*)$ is Einstein manifold.

- If $g$ satisfies the generalized Ricci-soliton equation (17.6) with

$$V = \psi, \quad c_1 = 1, \quad c_2 = -1, \quad \textit{and} \quad \lambda = |\psi|^2 - \mathrm{div}\psi$$

then, $(M, g^*)$ is $\eta$-Einstein manifold.

For our second motivation, we consider the case where the potential field $V$ be orthogonal to Reeb vector field $\xi$:

**Theorem 29** Let $(M, \varphi, \xi, \eta, g)$ be a $C_{12}$-manifold and $(M, \varphi, \xi, \eta, g^*)$ the associated Lorentz $C_{12}$-manifold and $V$ a vector field on $M$ orthogonal to $\xi$.
If $(g^*, V, \lambda)$ is a Ricci-soliton then $g$ satisfies the generalized $\eta$-Ricci bi-soliton equation (17.9) with

$$V_1 = V + \psi, \quad V_2 = \psi, \quad c_1 = 1, \quad c_2 = -1, \quad \textit{and} \quad \mu = |\psi|^2.$$

PROOF Let $V$ be orthogonal to $\xi$ that is $\eta(V) = 0$, it provides

$$\begin{aligned} \eta(\nabla_X V) &= g(\nabla_X V, \xi) \\ &= -g(V, \nabla_X \xi) \\ &= \eta(X)\omega(V). \end{aligned} \tag{20.21}$$

So, using (15.79) and (15.77), one can get

$$\begin{aligned} \mathcal{L}_V g^*(X, Y) =& g^*\big(\tilde{\nabla}_X V, Y\big) + g^*\big(\tilde{\nabla}_Y V, X\big) \\ =& \mathcal{L}_V g(X, Y) - 4\omega(V)\eta(X)\eta(Y). \end{aligned} \tag{20.22}$$

Then, from (20.22), (15.87) and (15.77) we obtain

$$\begin{aligned} \mathcal{L}_V g^*(X, Y) + 2S^*(X, Y) - 2\lambda g^*(X, Y) =& \mathcal{L}_V g(X, Y) + 2S(X, Y) - 2\lambda g(X, Y) \\ &+ 2\omega(X)\omega(Y) + 2g(\nabla_X \psi, Y) \\ &+ 2\big(2\lambda + 2\mathrm{div}\psi - 2\omega(V) - |\psi|^2\big)\eta(X)\eta(Y). \end{aligned} \tag{20.23}$$

Suppose that $\mathcal{L}_V g^*(X, Y) + 2S^*(X, Y) - 2\lambda g^*(X, Y) = 0$. Setting $X = Y = \xi$ we get

$$\lambda = \omega(V) - \mathrm{div}\psi. \tag{20.24}$$

Knowing that $\mathcal{L}_\psi g(X, Y) = 2g(\nabla_X \psi, Y)$, the equation (20.23) becomes

$$\mathcal{L}_{(V+\psi)} g(X, Y) = -2\omega(X)\omega(Y) - 2S(X, Y) + 2\lambda g(X, Y) + 2|\psi|^2\eta(X)\eta(Y). \tag{20.25}$$

This completes the proof. □

Now, suppose that $(g, V, \lambda)$ be a Ricci soliton, that is $\mathcal{L}_V g + 2S - 2\lambda g = 0$ with $V$ orthogonal to $\xi$. Setting $X = Y = \xi$ we obtain

$$\lambda = \omega(V) - \mathrm{div}\psi. \tag{20.26}$$

Using (20.22), (15.87), (15.77) and (20.26), one can get

$$\mathcal{L}_{(V-\psi)}g^*(X,Y) = 2\omega(X)\omega(Y) - 2S^*(X,Y) + 2\lambda g^*(X,Y) + 2|\psi|^2\eta(X)\eta(Y). \tag{20.27}$$

Therefore, we have the following theorem:

**Theorem 30** Let $(M,\varphi,\xi,\eta,g)$ be a $C_{12}$-manifold and $(M,\varphi,\xi,\eta,g^*)$ the associated Lorentz-$C_{12}$-manifold and $V$ a vector field on $M$ orthogonal to $\xi$.
If $(g,V,\lambda)$ is a Ricci soliton then $g^*$ satisfies the generalized $\eta$-Ricci bi-soliton equation (17.9) with

$$V_1 = V - \psi, \quad V_2 = \psi, \quad c_1 = -1, \quad c_2 = 1, \quad \textit{and} \quad \mu = |\psi|^2.$$

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