Democratic People's Republic of Algeria
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*Faculty of Science and Technology*
Common Core Department of Science and Technology

# Course Handout

# Mathematics 3
# Courses and Exercises with Solutions

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This handout is intended for Second-year students Science and Technology bachelor's degree

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# PREFACE

This course is intended for second-year students in the LMD system (ST, SM), includes the Mathematics 3 subject. It contains the essentials of the course with many simple, illustrative examples without any tedious demonstrations to help students assimilate the course quickly. Various exercises with solutions are offered at the end of each chapter to allow students to test themselves.

I taught this subject for six years, sufficient time to figure out student's difficulties regarding certain concepts, hence my desire to make this course accessible to most students whatever their level and to create my own exercises.

This handout which contains all the fundamental concepts related to this subject is divided into three essentials parts. The first part contains chapter of simple and multiple integrals starting by an interesting and consistent reminder about definite integrals. Multiple integrals generalize the concept of integration to two and more dimensions. Double and triple integrals are natural extensions of single integral, useful in calculating volumes, surfaces and in various physical and technical applications. The second part contains chapter of series (infinite series, sequences and series of functions, power series and Fourier series) which offer a powerful tool for approximation, calculating and interpreting infinite sums. The third part includes two chapters of transforms (Laplace transform and Fourier transform) invaluable key techniques in several fields especially in engineering, physics and chemistry. Two other supplementary chapters were added, namely chapter of improper integrals to enable us to introduce transforms and chapter of differential equations containing ordinary differential equations and partial differential equations which will be solved later by both transforms.

Finally, we hope that this work will provide useful support for second- year students from various specialties, will assist them effectively throughout their learning journey and contribute to their academic success. It is our duty to accept all objective criticism, suggestion, opinions from fellow teacher and students to improve the quality of this handout.

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# Chapter I
# Simple and Multiple Integrals

# I. Definite integral:

## I .1 Definitions:

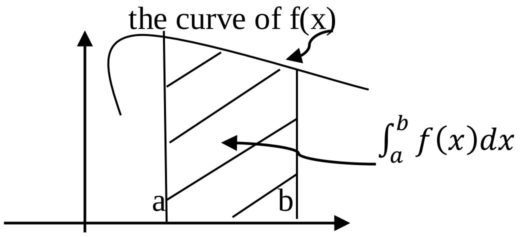
$Let\ f$: X $\longrightarrow$ ℝ be a continuous function on X

x $\longrightarrow f(x)$

Let a, b be $real\ numbers \in X$ :

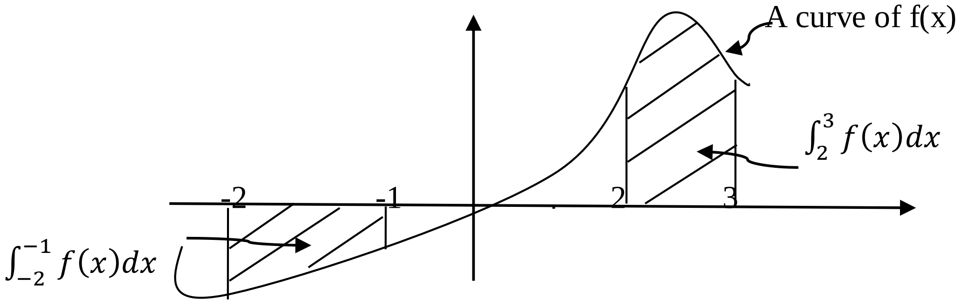
$\int_a^b f(x)dx$ is called definite integral of f(x) on [a, b] (or from a to b)=Area of the region bounded by the curve y= f(x), x-axis, straight line x=a and the straight line x=b.

Graphically:




A definite integral is just an area under a curve.

Examples



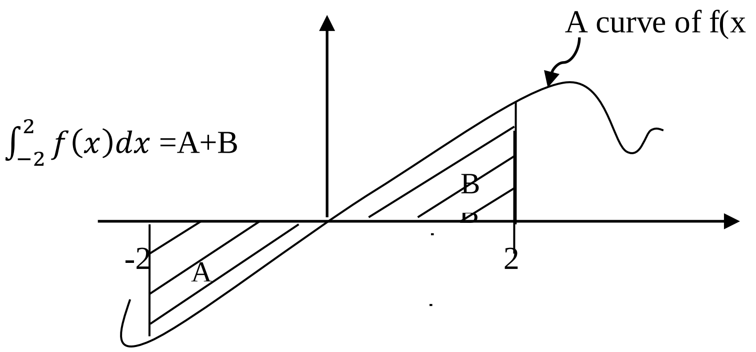




**Figure I.1** Meaning of a definite integral

## I.2 Evaluation of Definite Integral:

How to compute the area $\int_a^b f(x)dx$ ?

### I .2.1 Evaluation of definite integral using anti-derivative:

$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$ where F is an anti-derivative of f(x).

F is called an anti-derivative of f if F'(x)=f(x). That is, it is a function whose derivative is f(x).

Example1:

$f(x)= x^2$

$F(x)= \frac{x^3}{3}$ , $F(x)= \frac{x^3}{3} + 2$, $F(x)= \frac{x^3}{3} + C$ (C a constant) are all anti-derivatives of f.

Note:

-All anti-derivatives of f are represented by: $\int f(x)dx$ called indefinite integral of f.

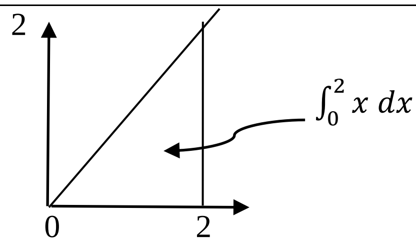
So $\int x^2 dx = \frac{x^3}{3} + C$

**Table I.1** indefinite integral of some usual functions

| f(x) | $\int f(x)dx$ =F(x)+C |
|---|---|
| 1 | $x$+c |
| $x^r$ $r \neq 1$ | $\frac{x^{r+1}}{r+1}$ +c |
| $e^x$ | $e^x$ +c |
| $\frac{1}{x}$ | lnx+ c |
| sin(x) | -cos(x)+c |
| cos(x) | sin(x)+c |
| $\frac{1}{1+x^2}$ | arctg(x)+c |
| $\frac{1}{\sqrt{1-x^2}}$ | arcsin(x)+c |
| $\frac{1}{cos^2(x)}$ | tan(x)+c |
| $\frac{1}{sin^2(x)}$ | -cot(x)+c |

Examples :

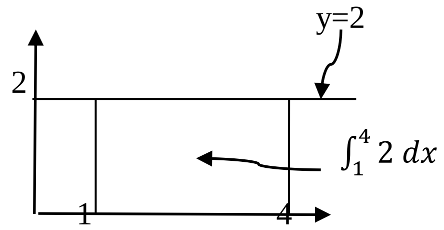
1) $\int_0^2 x\,dx = \left[\frac{x^2}{2}\right]_0^2 = \frac{2^2}{2} - \frac{0^2}{2} = \frac{4}{2} = 2$ $\int_0^2 x\,dx$




Note:

This definition coincides with triangle area= $\frac{1}{2}$ base. height=$\frac{1}{2}.2.2 = 2$

2) $\int_1^4 2\,dx = [2x]_0^4 = 8 - 2 = 6$




It is the rectangle area of length (4-1)=3 and width 2 ; 3.2=6

3) $\int\limits_0^3 e^x\,dx = [e^x]_0^3 = e^3 - e^0 = e^3 - 1$

1.2.1.1 Properties of definite integrals:

1) $\int_a^b cf(x)dx = c\int_a^b f(x)dx \ \ \forall c$ a constant.

2) $\int\limits_a^b (f(x) + g(x))dx = \int\limits_a^b f(x)dx + \int\limits_a^b g(x)dx$

3) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \ \ c \in$]a, b[.

4) $\int_a^b (f(x).g(x))dx \neq \int_a^b f(x)dx \,.\int_a^b g(x)dx$

Example1 :

$\int_0^2 2xdx = 2\int_0^2 xdx = 2\left[\frac{x^2}{2}\right]_0^2 = 4$

1)

Example2:

$\int_0^{\frac{\pi}{2}}(4cosx + x^3)dx = 4\int_0^{\frac{\pi}{2}} cosxdx + \int_0^{\frac{\pi}{2}}(x^3)dx = 4[sinx]_0^{\frac{\pi}{2}} + \left[\frac{x^4}{4}\right]_0^{\frac{\pi}{2}} = 4 + \frac{\pi^4}{64}$

1) and 2)

Example3:

$$\int\limits_0^2 (x^2 + x)(x^3 - 1)dx = \int\limits_0^2 (x^5 + x^4 - x^2 - x)dx = \left[\frac{x^6}{6} + \frac{x^5}{5} - \frac{x^3}{3} - \frac{x^2}{2}\right]_0^2$$

$$= \frac{32}{3} + \frac{32}{5} - \frac{8}{3} - \frac{1}{2} = \frac{417}{30}$$

**Note:**

-We can notice, it is easy to calculate $\int_a^b f(x)dx$ if $f(x)$ is one of usual functions of the table or combinations of them (addition, subtraction, multiplication by a real).

-The question that now arises how to calculate $\int_a^b (f(x).g(x))dx$ since it is different from

$\int_a^b f(x)dx \,.\int_a^b g(x)dx.$

If both f and g are polynomial functions, in general, we have just to develop and integrate such as in example 3.

If one of them is not a polynomial function, we will use one of the following methods

$$\begin{cases} U - Subtitution\ method \\ or \\ Integration\ by\ parts\ method \end{cases}$$

**I .2.2 Evaluation of definite integral by U-Substitution method:**

Most of the time, this method is used when we have a function and its derivative i.e our definite integral is of form$\int_a^b f(g(x)).\,g'(x)dx$.

Note that we have g(x) and its derivative$g'(x)$.

This integral can be transformed into another form ( easy to integrate) by doing the following substitution.

$g(x)$=U ⇒ dU= $g'(x)dx$So $\int_a^b f(g(x)).\,g'(x)dx = \int_{g(a)}^{g(b)} f(U).\,dU$

Then we can integrate f(U).

Example1:

$\int_0^2 (x^2+1)^a 2xdx$ a ∈ N

$if a = \begin{cases} 1 \\ ou \\ 2 \\ ou \\ 3 \end{cases}$ one can calculate$(x^2+1)^a$ and then $(x^2+1)^a 2x$ is a polynomial function that we can integrate easily.

If $a > 4$ then it takes time to compute$(x^2+1)^a$ that is why we use U-Substitution method (since we have a function and its derivatives)

Let us pick a=50

$$\int_0^2 (x^2+1)^{50} 2xdx \quad (1)$$

U-Substitution :

$x^2 + 1 = U \Rightarrow 2\text{xdx} = \text{dU}$

x=0⇒U=1

x=2⇒U=5

(1) $becomes \int_1^5 U^{50} dU = \left[\frac{U^{51}}{51}\right]_1^5 = \frac{5^{51}}{51} - \frac{1}{51} = \frac{5^{51}-1}{51}$

Example2:

$$\int_0^3 x^2(x^3-4)^5 dx \ \ (2)$$

U-Substitution :

$U = (x^3-4) \Rightarrow 3x^2dx = dU$

x=0⇒U=-4

$x = 3 \Rightarrow U = 3^3 - 4 = 23$

$$(2) = \frac{1}{3}\int_0^3 3x^2(x^3-4)^5 dx = \frac{1}{3}\int_{-4}^{23} U^5 dU = \left[\frac{U^6}{6}\right]_{-4}^{23} = \frac{23^6}{6} - \frac{(-4)^6}{6}$$

Example 3:

$\int_0^{\pi} 2x\cos(x^2)dx$ (3)

U-Substitution :

$U = x^2 \Rightarrow 2xdx = dU$

x=0⇒U=0

$x = \pi \Rightarrow U = \pi^2$

$(3) = \int_0^{\pi^2} \cos(U)\, dU = [\sin(U)]_0^{\pi^2} = \sin(\pi^2) - \sin(0) = \sin(\pi^2)$

### I .2.3 Evaluation of definite integral by integral by parts method:

$$\int_a^b U(x)V'(x)dx = [U(x)V(x)]_a^b - \int_a^b U'(x)V(x)dx$$

Example1 :

$$\int_0^1 xe^{-x}dx$$

U=x ⇒ dU=dx

$dV = e^{-x}dx \Rightarrow V = -e^{-x}$

$\int_0^1 xe^{-x}dx = [-xe^{-x}]_0^1 - \int_0^1 -e^{-x}dx = [-xe^{-x}]_0^1 + \int_0^1 e^{-x}dx$

U dV $= [-xe^{-x}]_0^1 + [-e^{-x}]_0^1 = -e^{-1} - e^{-1} + 1 = -2e^{-1} + 1$

Example 2:

$$\int_0^{\frac{\pi}{2}}(xsinxdx = [-xcosx]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} cosxdx = [-xcosx]_0^{\frac{\pi}{2}} + [sinx]_0^{\frac{\pi}{2}} = 1$$

U dV

# II. Double Integral:

## II.1 Definition of double integral:

$Let f$: AxB $\longrightarrow \mathbb{R}$ be a continuous function on AxB

(x,y)$\longrightarrow f(x,y)$

A, B are intervals of $\mathbb{R}$.

Let a, b, c, d be real constants such that a, b $are\ elements\ of A$ and c,d $of\ B$

$\int_a^b \int_c^d f(x,y)dydx$ is the Volume of a solid bounded by the area z=f(x,y), xy- plane , planes x=a, x=b, y=c and y=d.

Notes:

-Intersection between xy-plane and the four vertical planes x=a, x=b, y=c et y=d gives (a,b,c,d) rectangle.

-$\int_a^b \int_c^d f(x,y)dydx$ is the Volume of a solid, its ceiling is a part of the surface z= f(x,y) and its base is the full rectangle R=(a,b,c,d) ( see figure I.2).

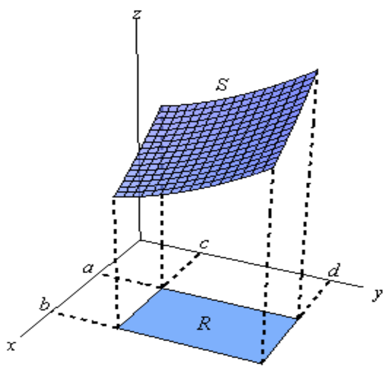



**Figure I.2** Meaning of a double integral

-Double integral provides a volume under a surface (z=f(x,y)) while a definite integral provides an area under a curve (y=f(x)).

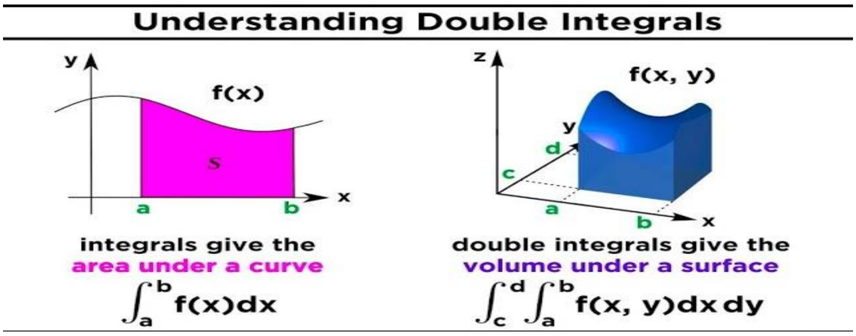



**Figure I.3** Indefinite integral vs double integral

## II.2 Evaluation of double integral:

$$\int_a^b \int_c^d f(x,y)dydx = \int_a^b [\int_c^d f(x,y)dy]dx$$

(1)

(2)

To compute $\int_a^b \int_c^d f(x,y)dydx$, first we calculate the definite integral (1) (where y plays the role of the variable and x a constant); the result of (1) is a function g of x then we compute the definite integral (2).

$$\int_a^b \int_c^d f(x,y)dydx = \int_a^b [\int_c^d f(x,y)dy]dx = \int_a^b [g(x)]dx$$

Example1:

$$\int_0^1 \int_1^2 x^2ydydx = \int_0^1 [\int_1^2 x^2ydy]dx = \int_0^1 x^2 \left[\frac{y^2}{2}\right]_{y=1}^{y=2} dx = \int_0^1 x^2[\frac{4}{2} - \frac{1}{2}]dx =$$

$$\int_0^1 \frac{3}{2}x^2dx = \frac{3}{2}\left[\frac{x^3}{3}\right]_{x=0}^{x=1} = \frac{3}{2}[\frac{1}{3} - 0] = \frac{1}{2}$$

Example2:

$$\int_0^2 \int_0^3 xe^ydydx = \int_0^2 [\int_0^3 xe^ydy]dx = \int_0^2 x[e^y]_{y=0}^{y=3}dx = \int_0^2 x[e^3 - 1]dx$$

$$= [e^3 - 1]\int_0^2 xdx = [e^3 - 1]\left[\frac{x^2}{2}\right]_{x=0}^{x=2} = [e^3 - 1]\,\frac{4}{2} = 2[e^3 - 1]$$

Note:

One can generalize this notion to multiple integrals for example for function of three variables f(x,y,z) (triple integral):

$$\int_a^b \int_c^d \int_e^f f(x,y,z)\,dzdydx = \int_a^b [\int_c^d [\int_e^f f(x,y,z)\,dz]dy]dx$$

(1)

(2)

(3)

To compute $\int_a^b \int_c^d \int_e^f f(x,y,z)\, dzdydx,$ first we calculate the definite integral (1) (where z plays the role of the variable and x,y constants); the result of (1) is a function g of x and y then we compute the definite integral (2) (where y plays the role of the variable and x a constant).Finally we compute the definite integral (3).

## II.3 Fubini's theorem:

$Let f$: AxB $\longrightarrow \mathbb{R}$ be a continuous function on AxB

(x,y) $\longrightarrow f(x,y)$

A, B are intervals of $\mathbb{R}$

Let a, b, c, d be real constants such that a,b are $elements\ of A$ and c,d $of\ B$

$$then \int_{x=a}^{x=b}\int_{y=c}^{y=d} f(x,y)dydx = \int_{y=c}^{y=d}\int_{x=a}^{x=b} f(x,y)dxdy$$

Fubini's theorem states that the order of integration (for a continuous function) does not matter ; if we integrate first with respect to x and then with respect to y or vice versa.

Example1 :

$$\int_0^1\int_1^2 x^2ydydx$$

$f(x,y) = x^2\, y$ is a continuous function on $\mathbb{R}$x$\mathbb{R}$ thus one can apply Fubini's theorem

We have already found : $\int_{x=0}^{x=1}\int_{y=1}^{y=2} x^2ydydx = \frac{1}{2}$

Let us calculate $\int_{y=1}^{y=2}\int_{x=0}^{x=1} x^2ydxdy$:

$$\int_{y=1}^{y=2}\int_{x=0}^{x=1} x^2ydxdy = \int_{y=1}^{y=2}[\int_{x=0}^{x=1} x^2ydx]dy = \int_{y=1}^{y=2} y[\frac{x^3}{3}]_{x=0}^{x=1}dy$$

$$= \int_{y=1}^{y=2} y\,[\frac{1}{3}]dy = \frac{1}{3}\int_{y=1}^{y=2} ydy = \frac{1}{3}[\frac{y^2}{2}]_{y=1}^{y=2} = \frac{1}{3}\left[\frac{4}{2}-\frac{1}{2}\right] = \frac{1}{3}\left[\frac{3}{2}\right] = \frac{1}{2}$$

So Fubini's theorem is well satisfied.

$$\int_{x=0}^{x=1}\int_{y=1}^{y=2} x^2ydydx = \int_{y=1}^{y=2}\int_{x=0}^{x=1} x^2ydxdy = \frac{1}{2}$$

Example 2:

$$\int_0^2 \int_0^3 xe^y dydx$$

$f(x,y) = xe^y$ is a continuous function on ℝxℝ thus one can apply Fubini's theorem

We have already found: $\int_{x=0}^{x=2} \int_{y=0}^{y=3} xe^y dydx = 2[e^3 - 1]$

Let us compute $\int_{y=0}^{y=3} \int_{x=0}^{x=2} xe^y dx\, dy$:

$$\int_{y=0}^{y=3} \int_{x=0}^{x=2} xe^y dx\, dy = \int_{y=0}^{y=3} e^y \left[\frac{x^2}{2}\right]_{x=0}^{x=2} dy$$

$$= \int_{y=0}^{y=3} e^y [\tfrac{4}{2}]\, dy = \tfrac{4}{2} \int_{y=0}^{y=3} e^y\, dy = \tfrac{4}{2} [e^y]_{y=0}^{y=3} = 2[e^3 - 1]$$

Consequently Fubini's theorem is verified.

$$\int_{x=0}^{x=2} \int_{y=0}^{y=3} xe^y dydx = \int_{y=0}^{y=3} \int_{x=0}^{x=2} xe^y dx\, dy = 2[e^3 - 1]$$

**Note:**

-The choice of the integral to calculate first depends on which integral is easy to compute.

**Example:**

Do 1) $\int_{x=0}^{x=1} [\underbrace{\int_{y=1}^{y=2} \frac{x}{1+xy} dy}_{A}]dx \ \ or \ \ 2) \int_{y=1}^{y=2} [\underbrace{\int_{x=0}^{x=1} \frac{x}{1+xy} dx}_{B}]dy$

We choose 1) since integral A is easier to calculate than integral B.

NB: in A, we use a U-substitution method since we have a function (1+xy) and its derivatives (x)

-Let us assume that $f(x,y) = k$ (k a constant).

$\int_a^b \int_c^d kdydx$ =volume of a parallelepiped of height k.

= [b-a].[d-c].k

By analogy, one can deduce:

$\int_a^b \int_c^d f(x,y)dydx$ =Volume of a parallelepiped whose ceiling is a surface

z=$f(x,y)$ (not a plane)

## II.4 Integral over a region D (calculation of volums):

Let f: D $\longrightarrow$ ℝ be a continuous function on D (D a surface of $\mathbb{R}^2$)

(x,y) $\longrightarrow f(x,y)$

$\iint_D f(x,y)ds$ =Volume of a solid bounded by the region D and the surface z= f(x,y).

Example1:

D= (full) rectangle $=\{(x,y) \in \mathbb{R}^2 \ / \ a \leq x \leq b \ and \ c \leq y \leq d\}$

$\iint_D f(x,y)ds = \int_a^b \int_c^d f(x,y)dydx$ we get back to the first case seen above.

= Volume of a parallelepiped whose ceiling is a surface S and its base is the region R ( full rectangle (a,b,c,d))

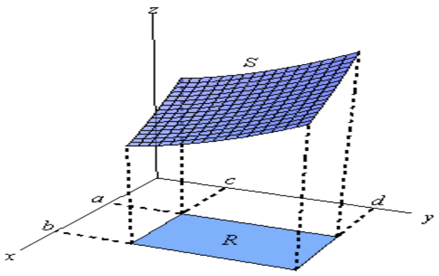



**Figure I.4** Integral over a full Rectangle R

Example2:

D= a disc R.

$\iint_D f(x,y)ds$ = volume of a cylinder whose ceiling is a surface z=f(x,y).

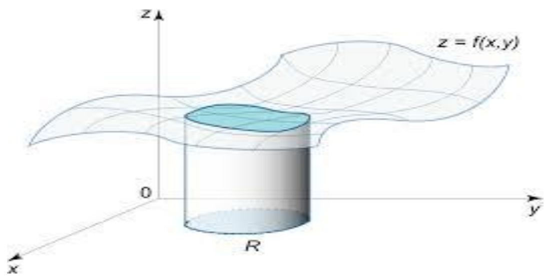



**Figure I.5** Integral over a disk R

Example3:

D= any surface on xy-plane.

$\iint_D f(x,y)ds$ = volume of a solid of form D.

## II.4.1 Integral over a rectangular region D:

D= $\{(x,y) \in \mathbb{R}^2 \ / \ a \leq x \leq b \ and \ c \leq y \leq d\}$= full rectangle.

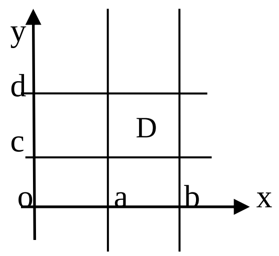



**Figure I.6** D a full rectangle in xy-plane

$$\iint_D f(x,y)ds = \int_a^b \int_c^d f(x,y)dydx$$

We know how to calculate this double integral

$$\int_a^b \int_c^d f(x,y)dydx = \int_a^b [\int_c^d f(x,y)dy]dx = \int_c^d [\int_a^b f(x,y)dx]dy$$

if one apply Fubini's theorem

## II.4.2 Integral over a non rectangular and non circular region D:

We have two cases:

### II.4.2.1 Integral over a region bounded by two curves and two vertical lines:

D= {(x,y) ∈ $\mathbb{R}^2$/ $\{a \leq x \leq b \ \ and \ \ \Psi_1(x) \leq y \leq \Psi_2(x) \ \ where \ \Psi_1(x), \Psi_2(x) \ are \ cuves\}$

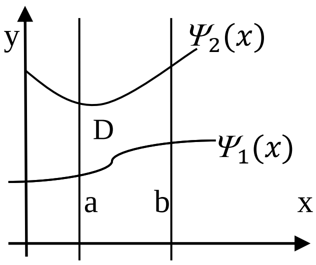



**Figure I.7** D bounded by two vertical lines in xy-plane

**Note :**

$\Psi_1(x)$, $\Psi_2(x)$ can be oblique lines so D will be for example of form:

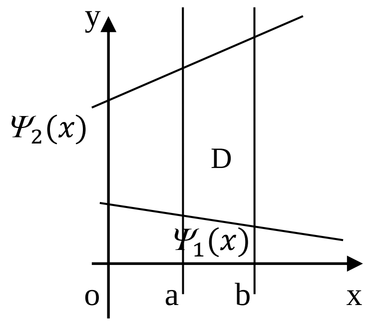



**Figure I.8** D bounded by two vertical and two oblic lines in xy-plane

$$\iint_D f(x,y)ds = \int_{x=a}^{x=b} \int_{y=\Psi_1(x)}^{y=\Psi_2(x)} f(x,y)dydx = \int_{x=a}^{x=b} [\int_{y=\Psi_1(x)}^{y=\Psi_2(x)} f(x,y)dy]dx$$

**Example1 :**

D= {(x,y)∈ $\mathbb{R}^2$ /$0 \leq x \leq 2 \ \ and \ 0 \leq y \leq \frac{x}{2}\}$

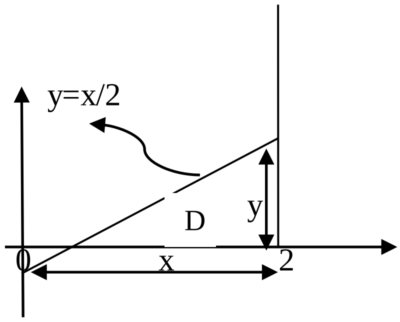



**Figure I.9** D a triangle in xy-plane

D is a triangle.

$$\iint_D f(x,y)ds = \int_{x=0}^{x=2}\int_{y=0}^{y=\frac{x}{2}} f(x,y)dydx = \int_{x=0}^{x=2}[\int_{y=0}^{y=\frac{x}{2}} f(x,y)dy]dx \text{ (1)}$$

Let us take for example$f(x,y)$=(x+1)y

$$(1) = \int_{x=0}^{x=2}[\int_{y=0}^{y=\frac{x}{2}}(x+1)ydy]dx = \int_{x=0}^{x=2}(x+1)\left[\frac{y^2}{2}\right]_{y=0}^{y=\frac{x}{2}}dx = \int_{x=0}^{x=2}(x+1)\,[\frac{\left(\frac{x}{2}\right)^2}{2} - 0]dx$$

$$= \int_{x=0}^{x=2}(x+1)\left[\frac{x^2}{8}\right]dx = \int_{x=0}^{x=2}(\frac{x^3}{8}+\frac{x^2}{8})\,dx = \left[\frac{x^4}{32}+\frac{x^3}{24}\right]_{x=0}^{x=2} = \frac{16}{32}+\frac{8}{24} = \frac{5}{6}$$

Example2:

D= {(x,y) ∈ $\mathbb{R}^2$ / $0 \le x \le 1$ $\;and\; x \le y \le 1$}

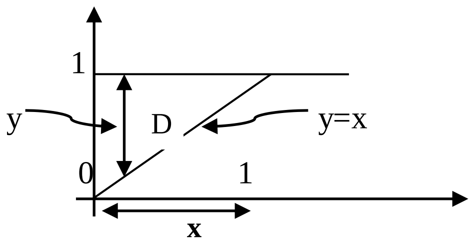



**Figure I.10** D a triangle in xy-plane

$$\iint_D e^{y^2}ds = \int_{x=0}^{x=1}\int_{y=x}^{y=1} e^{y^2}dydx = \int_{x=0}^{x=1}[\underbrace{\int_{y=x}^{y=1} e^{y^2}dy}_{(1)}]dx$$

Integral (1) is difficult to calculate, let us apply Fubini's theorem:

$$\int_{x=0}^{x=1}[\int_{y=0}^{y=x} e^{y^2}dy]dx = \int_{y=0}^{y=1}[\underbrace{\int_{x=0}^{x=y} e^{y^2}dx}_{(2)}]dy$$

Integral (2) is easy to calculate because x is the variable and y plays the role of a constant.

$$= \int_{y=0}^{y=1} [e^{y^2} \int_{x=0}^{x=y} dx]dy = \int_{y=0}^{y=1} e^{y^2} [x]_{x=0}^{x=y} dy = \int_{y=0}^{y=1} e^{y^2}\, ydy \quad (I)$$

We do an U-substitution:

$U = e^{y^2} \Rightarrow dU = 2ye^{y^2}dy$

$y = 0 \Rightarrow U = e^0 = 1$

$y = 1 \Rightarrow U = e$

One replace in (I) :

$$(I) = \int_{U=1}^{U=e} \frac{1}{2} dU = \frac{1}{2}[U]_{U=1}^{U=e} = \frac{1}{2}[e-1]$$

Example3:

D= $\{(x,y) \in \mathbb{R}^2 \ / \ 0 \le x \le 2 \ \ and\ x^2-1 \le y \le x^2+2\}$

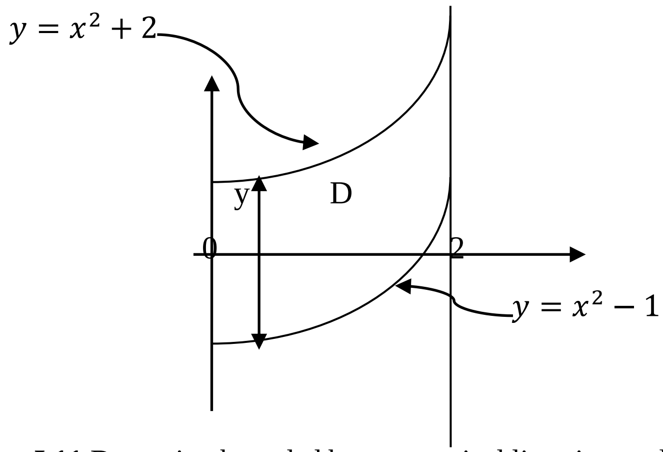



**Figure I.11** D a region bounded by two vertical lines in xy-plane

$$\iint_D xyds = \int_{x=0}^{x=2} \int_{y=x^2-1}^{y=x^2+2} xydydx = \int_{x=0}^{x=2} [\int_{y=x^2-1}^{y=x^2+2} xydy]dx = \int_{x=0}^{x=2} \left[x\frac{y^2}{2}\right]_{y=x^2-1}^{y=x^2+2} dx$$

$$= \int_{x=0}^{x=2} x[\frac{(x^2+2)^2}{2} - \frac{(x^2-1)^2}{2}] \ \ dx = \int_{x=0}^{x=2} x[\frac{6x^2+3}{2}] \ \ dx$$

$$= \frac{1}{2}\int_{x=0}^{x=2} 6x^3 + 3x \ \ dx = \frac{1}{2}\left[\frac{6x^4}{4} + \frac{3x^2}{2}\right]_{x=0}^{x=2} = \frac{1}{2}\left[\frac{6\ (16)}{4} + \frac{3(4)}{2}\right]$$

$$= \frac{1}{2}[64+6] = 35$$

**Example4 :**

D= {(x,y) ∈ $\mathbb{R}^2$/ $-2 \le x \le 2$ *and* $2 \le y \le -\frac{1}{4}x^2+3$}

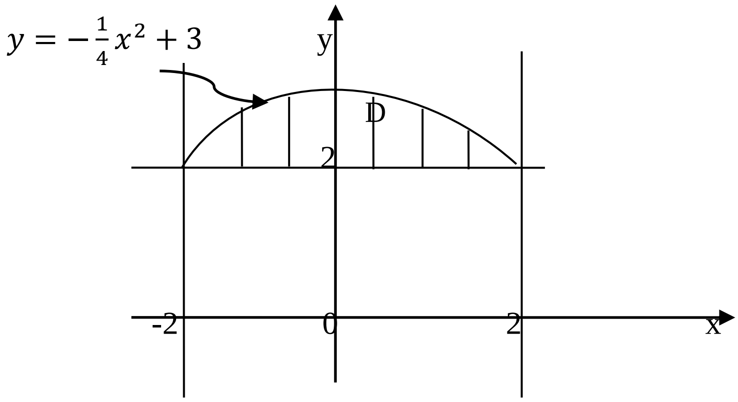



**Figure I.12** D a region bounded by two vertical lines in xy-plane

$$\iint_D (x-y)ds = \int_{x=-2}^{x=2}\int_{y=2}^{y=-\frac{1}{4}x^2+3}(x-y)dydx = \int_{x=-2}^{x=2}[\int_{y=2}^{y=-\frac{1}{4}x^2+3}(x-y)dy]dx$$

$$= \int_{x=-2}^{x=2}\left[xy-\frac{y^2}{2}\right]_{y=2}^{y=-\frac{1}{4}x^2+3}dx = \int_{x=-2}^{x=2} x(-\frac{1}{4}x^2+3)-2x-\frac{1}{2}[(-\frac{1}{4}x^2+3)^2-4]dx$$

$$= \int_{x=-2}^{x=2} -\frac{1}{4}x^3+x-\frac{1}{2}[\frac{x^4}{16}-\frac{3}{2}x^2+9-4]dx \quad = \int_{x=-2}^{x=2}[\frac{x^4}{32}-\frac{1}{4}x^3+\frac{3}{4}x^2+x-\frac{5}{2}]\,dx$$

$$= \left[\frac{x^5}{160}-\frac{1}{16}x^4+\frac{1}{4}x^3+\frac{1}{2}x^2-\frac{5}{2}\,x\right]_{x=-2}^{x=2}$$

$$= \left[\frac{32}{160}-1+\frac{8}{4}+2-5+\frac{32}{160}+1+\frac{8}{4}-2-5\right] = \frac{64}{160}+\frac{16}{4}-10 = \frac{-28}{5}$$

II.4.2.2 Integral over a region bounded by two curves and two horizontal lines:

D= {(x,y) / $\Psi_1(x) \le x \le \Psi_2(x)$ *and* c $\le y \le d$ *where* $\Psi_1(y)$, $\Psi_2(y)$ *are curbes*}

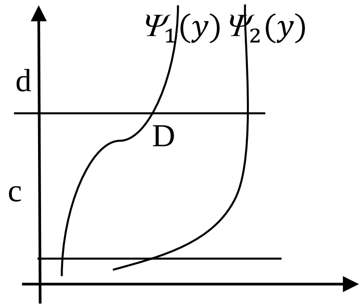



**Figure I.13** D bounded by two horizontal lines in xy-plane

$\iint_D f(x,y)ds = \int_{y=c}^{y=d}\int_{x=\Psi_1(y)}^{x=\Psi_2(y)} f(x,y)dxdy = \int_{y=c}^{y=d}[\int_{x=\Psi_1(y)}^{x=\Psi_2(y)} f(x,y)dx]dy.$

**Note:**

External integral is always the integral whose the limits of integration are constant.

Example1 :

D= {(x,y) ∈ $\mathbb{R}^2$ / $1 \leq x \leq e^y$ $\quad and\ 0 \leq y \leq 1$ }

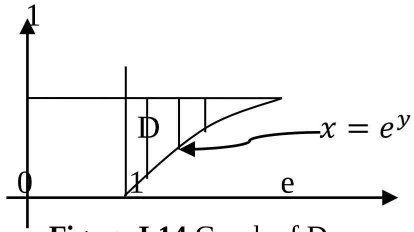



**Figure I.14** Graph of D

$$\iint_D xds = \int_{y=0}^{y=1}\int_{x=1}^{x=e^y} xdxdy = \int_{y=0}^{y=1}[\frac{x^2}{2}]_{x=1}^{x=e^y} dy = \int_{y=0}^{y=1}(\frac{(e^y)^2}{2} - \frac{1}{2})dy$$

This integral is difficult to calculate, let us apply Fubini's theorem:

$$\iint_D xds = \int_{y=0}^{y=1}\int_{x=0}^{x=e^y} xdxdy = \int_{x=1}^{x=e}\int_{y=lnx}^{y=1} xdydx = \int_{x=1}^{x=e}[\int_{y=lnx}^{y=1} xdy]dx$$

$$= \int_{x=1}^{x=e}[xy]_{y=lnx}^{y=1}\ dx = \int_{x=1}^{x=e} x[1-lnx]\ \ dx = \underbrace{\int_{x=1}^{x=e} x\ \ dx}_{(1)} - \underbrace{\int_{x=1}^{x=e} xlnx\ \ dx}_{(2)}$$

$$(1) = \left[\frac{x^2}{2}\right]_{x=1}^{x=e} = \frac{e^2}{2} - \frac{1}{2}$$

Let us use integration by parts method to compute (2):

Reminder: $\int_a^b UdV = [UV]_a^b - \int_a^b VdU$

$$\int_{x=1}^{x=e}\underbrace{(lnx)}_{U}\underbrace{x\,dx}_{dV} = \left[\frac{x^2}{2}lnx\right]_1^e - \int_1^e \frac{x^2}{2x}dx = \frac{e^2}{2}lne - \frac{1}{2}\left[\frac{x^2}{2}\right]_1^e = \frac{e^2}{2} - \frac{1}{4}[e^2-1] = \frac{1}{4}[e^2+1]$$

$U = lnx \Rightarrow dU = \frac{1}{x}dx,$ $\quad$ dV=x⇒ $V = \frac{x^2}{2} + C$

Therefore

$$\iint_D xds = (1) + (2) = \frac{e^2}{2} - \frac{1}{2} + \frac{1}{4}[e^2+1] = \frac{3e^2}{4} - \frac{1}{4}$$

**Example2:**

D= {(x,y) / $y^2 \le x \le y^2+3 \quad and\ 1 \le y \le 2$ }

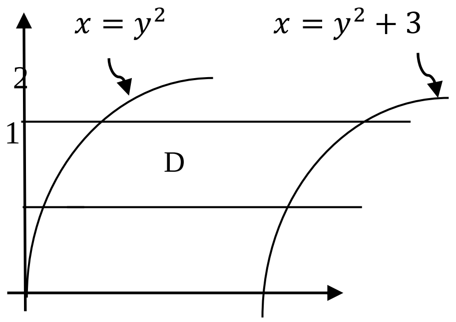



**Figure I.15** Graph of D

$$\iint_D xyds = \int_{y=1}^{y=2}\int_{x=y^2}^{x=y^2+3} xydxdy = \int_{y=1}^{y=2}[\int_{x=y^2}^{x=y^2+3} xydx]dy \int_{y=1}^{y=2} y[\frac{x^2}{2}]_{x=y^2}^{x=y^2+3}dy$$

$$= \int_{y=1}^{y=2} y(\frac{(y^2+3)^2}{2} - \frac{y^4}{2})dy = \frac{1}{2}\int_{y=1}^{y=2} y(6y^2+9)\,dy = \frac{1}{2}[\frac{6y^4}{4} + \frac{9y^2}{2}]_{x=1}^{x=2}$$

$$= \frac{1}{2}\left[24 + 18 - \frac{6}{4} - \frac{9}{2}\right] = \frac{1}{2}[42-6] = \frac{36}{2} = 18$$

## II.5.Integral over a disk D:

When D is a disk or a part of a disk, the double integral in Cartesian coordinates (x,y) becomes difficult to manipulate, we have to change to polar coordinates.

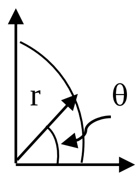
D is $\begin{cases} triangular \\ or \\ square \\ ou \\ rectangular \end{cases}$ region

D is a $\begin{cases} disk \\ or \\ a\ part\ of a\ disk \end{cases}$

Cartesian coordinates (x,y)

polar coordinates(r,θ)

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ x^2 + y^2 = r^2 \\ \frac{y}{x} = tg\theta \end{cases}$$




$ds = rdrd\theta$

$\iint_D f(x,y)ds$ will be converted into $\int_{\theta=}^{\theta=} \int_{r=}^{r=} f(r\cos\theta, r\sin\theta) r dr d\theta$

**Example 1:**

$\iint_D xds$

D={(x,y) ∈ $\mathbb{R}^2$ / $x^2 + y^2 \leq 4$ $and$ $0 < x < y$ } is a part of a disk.

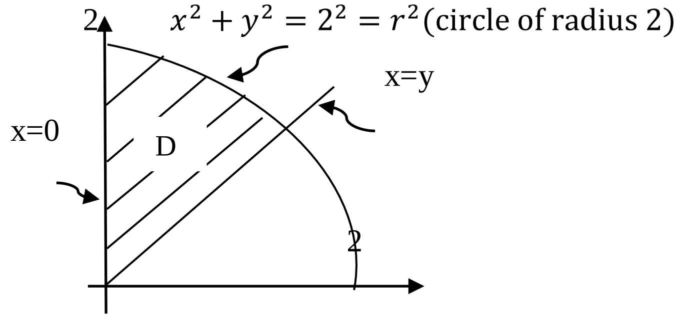



**Figure I.16** D a part of a disk

$$\iint_D xds = \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=2} rcos\theta r dr d\theta = \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} [\int_{r=0}^{r=2} r^2 cos\theta dr] d\theta = \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} [\frac{r^3}{3} cos\theta]_{r=0}^{r=2} d\theta$$

$$= \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} \frac{8}{3} cos\theta d\theta = \frac{8}{3}[sin\theta]_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} = \frac{8}{3}[1 - \frac{\sqrt{2}}{2}] = \frac{8}{3} - \frac{4\sqrt{2}}{3}$$

Example2:

$\iint_D (9 - x^2 - y^2)ds$

D={(x,y)∈ $\mathbb{R}^2$/ $x^2 + y^2 \leq 1$} is a disk of radius1.

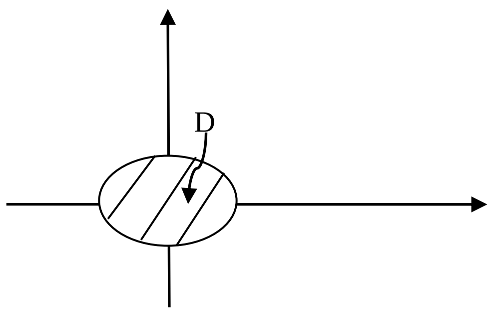



**Figure I.17** D an unit disk

We have : $9 - x^2 - y^2 = 9 - (x^2 + y^2) = 9 - r^2$

$$\iint_D (9-x^2-y^2)ds = \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1}(9-r^2)\mathrm{r}\mathrm{dr}\mathrm{d}\theta = \int_{\theta=0}^{\theta=2\pi}[\int_{r=0}^{r=1}(9r-r^3)\mathrm{dr}]\mathrm{d}\theta$$

$$= \int_{\theta=0}^{\theta=2\pi}[\int_{r=0}^{r=1}(9r-r^3)\mathrm{dr}]\mathrm{d}\theta = \int_{\theta=0}^{\theta=2\pi}\left[\frac{9r^2}{2}-\frac{r^4}{4}\right]_0^1 d\theta = \int_{\theta=0}^{\theta=2\pi}[\frac{9}{2}-\frac{1}{4}]d\theta = \frac{17}{4}\int_{\theta=0}^{\theta=2\pi} d\theta$$

$$= \frac{17}{4}[\theta]_0^{2\pi} = \frac{17\pi}{2}$$

## Exercises

**Exercise 1 :**

Evaluate the following indefinite integrals:

1)$\int x^6dx$ 2)$\int 25x^4dx$ 3) $\int \frac{dt}{t^3}$ 4) $\int \frac{10dx}{x}$

5) $\int(2x^2+x)dx$ 6) $\int(3\sin(t)+t)dt$

7) $\int xydy$ 8)$\int(ycos(t)+\ y^2)\,dt$

**Exercise 2 :**

Evaluate the given definite integrals:

1) $\int_0^2 x^6dx$ 2) $\int_{-3}^1 25x^4dx$ 3) $\int_{-1}^2 \frac{dt}{t^3}$ 4) $\int_1^3 \frac{10dx}{x}$

5) $\int_1^2(2x^2+x)dx$ 6) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(3\sin(t)+t)dt$

**Exercise 3 :**

A) Which method we are going to use to compute each of the following definite integrals:

1)$\int_0^3(2x^3+x)(x+1)dx$ 2) $\int_{-1}^0 3e^{3t}dt$ 3) $\int_0^{\frac{\pi}{2}}(\cos(y)+\sin(y))dy$

4) $\int_1^3 xlnx\,\mathrm{dx}$ 5) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\sin(x)dx}{(\cos(x))^3}$ 6) $\int_0^{\frac{\pi}{6}}xcos(x)dx$

B) Evaluate the following definite integrals: 2) ; 4) ; 5) ; 6).

**Exercise 4:**

Evaluate the following double integrals:

1)$\int_{x=1}^{x=3}\int_{y=0}^{y=2}x^2y^3dydx$ 2) $\int_{\theta=0}^{\theta=\frac{\pi}{6}}\int_{r=1}^{r=2}r\theta^2dr\,d\theta$

3)$\int_{y=0}^{y=3}\int_{x=1}^{x=2}xe^ydx\,dy$ 4) $\int_{t=0}^{t=2}\int_{y=-1}^{y=1}(t^2+ty+1)dy\,dt$

5) $\int_{y=0}^{y=2}\int_{x=1}^{x=3}(y^{\frac{1}{2}}+x^3+4)dxdy$

**Exercise 5 :**

1) Let D={(x,y) ∈ $\mathbb{R}^2$ / 0≤ $x \leq 2y$ $and$ $0 \leq y \leq 1$}be a region in xy-plane.

   Compute $\iint_D (4+2x-y^2)ds$

2) Let D={(x,y) ∈ $\mathbb{R}^2$ /0≤ $x \leq 2$ $and$ $1 \leq y \leq e^x$}be a region in xy-plane.

   Evaluate $\iint_D yds$

**Exercice6:**

1) Determine region D in the following integrals and draw D.
2) Invert order of integrals.
3) Compute.

   a. $\int_{y=0}^{y=1}\int_{x=0}^{x=1-y}(x^2+y^2)dx\,dy$

   b) $\int_{x=0}^{x=1}\int_{y=x^2}^{y=x}x^2y^3dydx$

## Solutions:

**Solution of exercise 1:**

$\int f(x)dx$= F(x)+C(C is a constant)

1)$\int x^6dx = \frac{x^{6+1}}{6+1}$ +C =$\frac{x^7}{7}$ +C (C is a constant)

2)$\int 25x^4dx$ =25 $\int x^4dx$=25$\frac{x^{4+1}}{4+1}$+C =25$\frac{x^5}{5}$+C =5$x^5$+C

3) $\int \frac{dt}{t^3}$ =$\int t^{-3}dt$= $\frac{x^{-3+1}}{-3+1}$ +C= $\frac{x^{-2}}{-2}$+C= - $\frac{1}{2x^2}$+C

4) $\int \frac{10dx}{x}$ =10 $\int \frac{dx}{x}$ =10lnx +C

5) $\int(2x^2+x)dx$ =2 $\int x^2dx+\int xdx$=2$\frac{x^3}{3}+\frac{x^2}{2}+C$

6) $\int(3\sin(t)+t)dt$= 3 $\int sintdt+\int tdt$=-3cos(t )+ $\frac{t^2}{2}$ +C

7) $\int xydy$=x$\int ydy$=x$\frac{y^2}{2}$+C (y is the variable, x plays the role of a constant)

8)$\int(ycos(t)+\ y^2)\,dt$=$y\int cost\,dt+y^2\int dt$ = ysin(t)+$y^2$t+C (t is the variable, y plays the role of a constant)

**Solution of exercise 2 :**

$\int_a^b f(x)dx$= $[F(x)]_a^b$=$F(b)-F(a)$where F is an anti-derivative of f(x).

1)$\int_0^2 x^6dx$=$[\frac{x^7}{7}]_1^2$ = $\frac{1}{7}[2^7-1^7]$=$\frac{127}{7}$

2) $\int_{-3}^{1}25x^4dx$= =$[5x^5]_{-3}^{1}$ =5$[1^5-(-3)^5]$ =5(1+243)= 1220

3) $\int_{-1}^{2}\frac{dt}{t^3}$ ==$[\frac{1}{-2x^2}]_{-1}^{1}$=$-\frac{1}{2}[\frac{1}{2^2}-\frac{1}{(-1)^2}]$= =$-\frac{1}{2}$ $[-\frac{3}{4}]$= =$\frac{3}{8}$

4) $\int_1^3\frac{10dx}{x}$ =10$[\ln x]_1^3$=10[ln3-ln1]=10ln3

5) $\int_1^2 (2x^2+x)dx$ =$[2\frac{x^3}{3}+\frac{x^2}{2}]_1^2$=$[2\frac{2^3}{3}+\frac{2^2}{2}-2\frac{1^3}{3}-\frac{1^2}{2}]$=$\frac{14}{3}-\frac{3}{2}$=$\frac{19}{6}$

6) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(3\sin(t)+t)dt$ ==$[-3\cos(\text{t})\ +\ \frac{t^2}{2}]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

=$[-3\cos\frac{\pi}{4}+\ \frac{(\frac{\pi}{4})^2}{2}-(-3\cos\frac{\pi}{2}+\ \frac{(\frac{\pi}{2})^2}{2}]$=[-3$\frac{\sqrt{2}}{2}$+$\frac{(\pi)^2}{32}+\frac{(\pi)^2}{8}$=-3$\frac{\sqrt{2}}{2}+\frac{5(\pi)^2}{32}$

**Solution of exercise 3:**

A)

1)$\int_0^3 (2x^3+x)(x+1)dx=\int_0^3 (2x^4+2x^3+x^2+x)dx$

Anti-derivative method (since we know anti-derivatives of al l functions above).

2)$\int_{-1}^0 3e^{3t}dt$ U-subtitution method because we have a function (3t) and its derivative(3)

3) $\int_0^{\frac{\pi}{2}}(\cos(y)+\sin(y))dy$ Anti-derivative method.

4) $\int_1^3 xlnx$ integral by parts method

5) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\sin(x)dx}{(\cos(x))^3}$ U-substitution method because we have a function (cos(x)) and its derivative (sin(x))

6) $\int_0^{\frac{\pi}{6}} xcos(x)dx$ integral by parts method

B)

2) $\int_{-1}^0 3e^{3t}dt$

U-substitution:

u=3t du= 3dt

t=-1 ⇒u=-3

t=0⇒u=0

$\int_{-1}^0 3e^{3t}dt=\int_{-3}^0 e^u du$==$[e^u]_{-3}^0$=$e^0-e^{-3}$=1-$\frac{1}{e^3}$=$\frac{e^3-1}{e^3}$

4) $\int_1^3 xlnx$ dx

Integral by parts method:

$\int_a^b UdV=[UV]_a^b-\int_a^b VdU$ ) (I)

U=lnx ⇒du=$\frac{dx}{x}$

dV= xdx⇒$V=\int dV$=$\int xdx$=$\frac{x^2}{2}$+C

$\int_1^3 lnxx$dx =$\left[\text{lnx}.\frac{x^2}{2}\right]_1^3$- $\int_1^3 \boldsymbol{\frac{x^2}{2}}.\frac{dx}{x}$ =[ln3.$\frac{3^2}{2}-ln1.\frac{1^2}{2}$]- $\boldsymbol{\frac{1}{2}}\int_1^3 \boldsymbol{x}\,dx$

U dV by (I) 0

$=\frac{9ln3}{2}$ - $\frac{1}{2}\left[\frac{x^2}{2}\right]_1^3$=$\frac{9ln3}{2}$ - $\frac{1}{2}\left[\frac{3^2}{2}-\frac{1^2}{2}\right]=\frac{9ln3}{2}-\frac{4}{2}$=$\frac{9ln3-4}{2}$

5) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\sin(x)dx}{(\cos(x))^3}$

U-substitution:

u=cos(x)$\Rightarrow$ du= -sin(x) dx

x=$\frac{\pi}{4}\Rightarrow$ u=cos$\frac{\pi}{4}=\frac{\sqrt{2}}{2}$

x=$\frac{\pi}{3}\Rightarrow$ u= cos$\frac{\pi}{3}$ =$\frac{1}{2}$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\sin(x)dx}{(\cos(x))^3}=-\int_{\frac{\sqrt{2}}{2}}^{\frac{1}{2}}\frac{du}{(U)^3}=-\int_{\frac{\sqrt{2}}{2}}^{\frac{1}{2}}(U)^{-3}du==-\left[\frac{(U)^{-3+1}}{-3+1}\right]_{\frac{\sqrt{2}}{2}}^{\frac{1}{2}}=-\left[\frac{1}{-2(U)^2}\right]_{\frac{\sqrt{2}}{2}}^{\frac{1}{2}}$$

$$=\frac{1}{2}\left[\frac{1}{(U)^2}\right]_{\frac{\sqrt{2}}{2}}^{\frac{1}{2}}=\frac{1}{2}\left[\frac{1}{\left(\frac{1}{2}\right)^2}-\frac{1}{\left(\frac{\sqrt{2}}{2}\right)^2}\right]=\frac{1}{2}[4-2]=1$$

6) $\int_0^{\frac{\pi}{6}}xcos(x)dx$

Integral by parts method:

$\int_a^b UdV=[UV]_a^b-\int_a^b VdU$ ) (I)

U=x $\Rightarrow$du=$dx$

dV= cos(x)dx$\Rightarrow V=\int dV$=$\int\cos(x)dx$= $\sin(x)$+C

$$\int_0^{\frac{\pi}{6}}xcos(x)dx=[x.\sin(x)]_0^{\frac{\pi}{6}}-\int_0^{\frac{\pi}{6}}\sin(x)\,dx=[\frac{\pi}{6}.sin\frac{\pi}{6}-0.sin0]-[-\cos(x)]_0^{\frac{\pi}{6}}$$

U dV by (I)

$=\frac{\pi}{6}\frac{1}{2}$ -[-cos$\frac{\pi}{6}$ +cos0]= $\frac{\pi}{12}$ -[-$\frac{\sqrt{3}}{2}$+1]=$\frac{\pi}{12}+\frac{2-\sqrt{3}}{2}$ =$\frac{\pi+6-6\sqrt{3}}{12}$

**Solution of exercise4:**

1)$\int_{x=1}^{x=3}\int_{y=0}^{y=2}x^2y^3dydx=\int_{x=1}^{x=3}\int_{y=0}^{y=2}x^2y^3dydx$

$=\int_{x=1}^{x=3}[\int_{y=0}^{y=2}x^2y^3dy]dx$=$\int_{x=1}^{x=3}x^2\left[\frac{y^4}{4}\right]_{y=0}^{y=2}dx$ =

x is the constant and y is the variable to integrate

$\int_{x=1}^{x=3}x^2[\frac{2^4}{4}-0]dx=\frac{16}{4}\int_{x=1}^{x=3}x^2dx=4\left[\frac{x^3}{3}\right]_{x=1}^{x=3}$=4 [$\frac{3^3}{3}-\frac{1^3}{3}$]=$\frac{104}{3}$

x is now the variable to ~~integrate~~

2) $\int_{\theta=0}^{\theta=\frac{\pi}{6}}\int_{r=1}^{r=2}r\theta^2dr\,d\theta$=$\int_{\theta=0}^{\theta=\frac{\pi}{6}}[\int_{r=1}^{r=2}r\theta^2dr]d\theta$

θ is the constant and $r$is the variable to integrate

$$=\int_{\theta=0}^{\theta=\frac{\pi}{6}}\left[\frac{r^2}{2}\theta^2\right]_{r=1}^{r=2}d\theta=\int_{\theta=0}^{\theta=\frac{\pi}{6}}\left[\frac{2^2}{2}\theta^2-\frac{1^2}{2}\theta^2\right]d\theta=\int_{\theta=0}^{\theta=\frac{\pi}{6}}\frac{3}{2}\theta^2\,d\theta$$

θ is now the variable to integrate.

$$=\left[\frac{3}{2}\frac{\theta^3}{3}\right]_{\theta=0}^{\theta=\frac{\pi}{6}}=\frac{1}{2}\left(\frac{\pi}{6}\right)^3-0=\frac{\pi^3}{432}$$

$$3)\int_{y=0}^{y=3}\int_{x=1}^{x=2}xe^y\,dx\,dy=\int_{y=0}^{y=3}\left[\int_{x=1}^{x=2}xe^y\,dx\right]dy=\int_{y=0}^{y=3}\left[\frac{x^2}{2}e^y\right]_{x=1}^{x=2}dy$$

y is the constant and $x$ is the variable to integrate

$$=\int_{y=0}^{y=3}\left[\frac{2^2}{2}e^y-\frac{1^2}{2}e^y\right]dy=\frac{3}{2}\int_{y=0}^{y=3}e^y\,dy=\frac{3}{2}\left[e^y\right]_{y=0}^{y=3}=\frac{3}{2}\left[e^3-e^0\right]=\frac{3}{2}\left[e^3-1\right]$$

$$4)\int_{t=0}^{t=2}\int_{y=-1}^{y=1}(t^2+ty+1)\,dy\,dt=\int_{t=0}^{t=2}\left[\int_{y=-1}^{y=1}(t^2+ty+1)\,dy\right]dt$$

t is the constant and y is the variable to integrate

$$=\int_{t=0}^{t=2}\left[t^2y+t\frac{y^2}{2}+y\right]_{y=-1}^{y=1}dt=\int_{t=0}^{t=2}\left[t^2.1+t\frac{1^2}{2}+1-(t^2(-1)+t\frac{(-1)^2}{2}+(-1))\right]dt=\int_{t=0}^{t=2}\left[t^2+\frac{t}{2}+1+t^2-\frac{t}{2}+1\right]dt=\int_{t=0}^{t=2}\left[2t^2+2\right]dt=\left[2\frac{t^3}{3}+2t\right]_{t=0}^{t=2}$$

t is now the variable to integrate

$$=\left[2\frac{2^3}{3}+2.2-0\right]=\frac{28}{3}$$

$$5)\int_{y=0}^{y=2}\int_{x=1}^{x=3}(y^{\frac{1}{2}}+x^3+4)\,dx\,dy=\int_{y=0}^{y=2}\left[\int_{x=1}^{x=3}(y^{\frac{1}{2}}+x^3+4)\,dx\right]dy$$

y is the constant and x is the variable to integrate

$$=\int_{y=0}^{y=2}\left[(y^{\frac{1}{2}}x+\frac{x^4}{4}+4x)\right]_{x=1}^{x=3}dy=\int_{y=0}^{y=2}\left[(y^{\frac{1}{2}}(3)+\frac{3^4}{4}+4(3)-((y^{\frac{1}{2}}1+\frac{1^4}{4}+4(1)\right]dy=\int_{y=0}^{y=2}\left[2y^{\frac{1}{2}}+\frac{80}{4}+8\right]dy=\int_{y=0}^{y=2}\left[2y^{\frac{1}{2}}+28\right]dy$$

y is now the variable to integrate

$$=\left[2\frac{y^{\frac{1}{2}+1}}{\frac{1}{2}+1}+28y\right]_{y=0}^{y=2}=2\frac{2^{\frac{3}{2}}}{\frac{3}{2}}+28(2)-0=\frac{4}{3}\sqrt{2^3}+56$$

**Solution of exercise 5 :**

$$1)\iint_D(4+2x-y^2)ds=$$

$\int_{y=0}^{y=1}\int_{x=0}^{x=2y}(4+2x-y^2)dx\,dy$=$\int_{y=0}^{y=1}[(\int_{x=0}^{x=2y}(4+2x-y^2)dx]\,dy$

$=\int_{y=0}^{y=1}\left[(4x+2\frac{x^2}{2}-y^2x)\right]_{x=0}^{x=2y}dy$=$\int_{y=0}^{y=1}[4(2y)+4y^2-2y^3)]dy$

$=\left[(8\frac{y^2}{2}+4\frac{y^3}{3}-2\frac{y^4}{4})\right]_{y=0}^{y=1}=4+\frac{4}{3}-\frac{1}{2}=\frac{29}{6}$

2)$\iint_D yds = \int_{x=0}^{x=2}\int_{y=1}^{y=e^x} ydydx = \int_{x=0}^{x=2}[\int_{y=1}^{y=e^x} ydy]dx$=$\int_{x=0}^{x=2}[\frac{y2}{2}]_{y=1}^{y=e^x}dx$

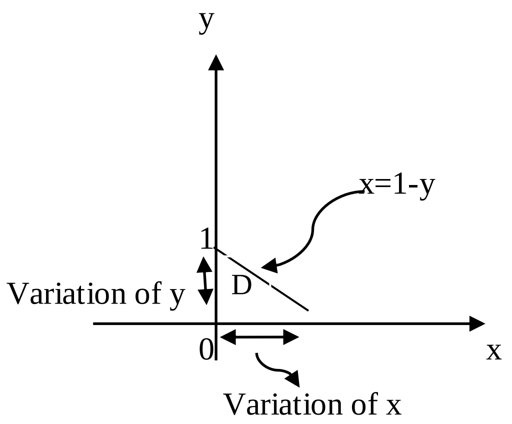
$=\frac{1}{2}\int_{x=0}^{x=2}[(e^x)^2-e^2]\,dx=\frac{1}{2}\int_{x=0}^{x=2}[e^{2x}-e^2]\,dx$=$\frac{1}{2}[\frac{1}{2}e^{2x}-e^2x]_{x=0}^{x=2}$

$=\frac{1}{2}[\frac{1}{2}e^4-2\,e^2-\frac{1}{2}]$=$\frac{1}{4}[e^4-4e^2-1]$

**Solution of exercise 6 :**

**a .** $\int_{y=0}^{y=1}\int_{x=0}^{x=1-y}(x^2+y^2)dx\,dy$

1) D={(x,y) ∈ $\mathbb{R}^2$ / 0≤ $y\leq 1$ *and* $0\leq x\leq 1-y$}




2)$\int_{y=0}^{y=1}\int_{x=0}^{x=1-y}(x^2+y^2)dx\,dy$=$\int_{x=0}^{x=1}\int_{y=0}^{y=1-x}(x^2+y^2)dydx$

3) Calcul:

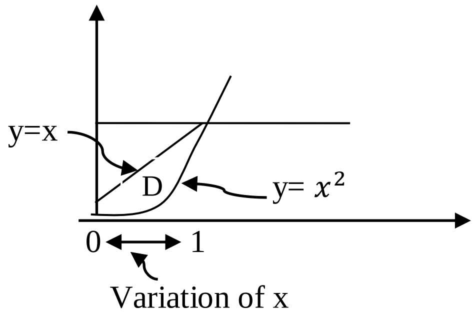
$\int_{x=0}^{x=1}\int_{y=0}^{y=1-x}(x^2+y^2)dydx=\int_{x=0}^{x=1}[\int_{y=0}^{y=1-x}(x^2+y^2)dy]dx$ $=\int_{x=0}^{x=1}[x^2y+\frac{y^3}{3}]_{y=0}^{y=1-x}dx$=$\int_{x=0}^{x=1}[x^2(1-x)+\frac{(1-x)^3}{3}]dx$=

$\int_{x=0}^{x=1}[\frac{-4x^3+6x^2-3x+1}{3}]dx$=$\frac{1}{3}[-4\frac{x^4}{4}+6\frac{x^3}{3}-3\frac{x^2}{2}+x]_{x=0}^{x=1}$

$=\frac{1}{3}\left[-1+2-\frac{3}{2}+1\right]=\frac{1}{6}$

b. $\int_{x=0}^{x=1}\int_{y=x^2}^{y=x}x^2y^3dydx$

1) D={(x,y) ∈ $\mathbb{R}^2$ / 0≤ $x \le 1$ $and$ $x^2 \le y \le x$}




2)$\int_{x=0}^{x=1} \int_{y=1}^{y=2x} x^2y^3 dydx = \int_{y=0}^{y=1} \int_{x=y}^{x=\sqrt{y}} x^2y^3 dxdy$

3) Calcul :

$$\int_{y=0}^{y=1} \int_{x=y}^{x=\sqrt{y}} x^2y^3 dxdy = \int_{y=0}^{y=1} [\frac{x^3}{3}\ y^3]_{x=y}^{x=\sqrt{y}}\ dy = \int_{y=0}^{y=1} [\frac{(\sqrt{y})^3}{3}\ y^3 - \frac{y^3}{3}\ y^3]\ dy$$

$$= \int_{y=0}^{y=1} [\frac{(y)^{\frac{9}{2}}}{3} - \frac{y^6}{3}]\ dy = [\frac{y^{\frac{11}{2}}}{3.\frac{11}{2}} - \frac{y^{7}}{3.7}]_{y=0}^{y=1} = \frac{1}{\frac{33}{2}} - \frac{1}{21} = \frac{2}{33} - \frac{1}{21} = \frac{3}{231}$$

# Chapter II
# Improper integrals

## II.1 Introduction

Improper integrals have numerous applications across various fields of science, engineering, and mathematics such as probability theory, quantum mechanics, fluid mechanics and especially in signal processing where they are essential to compute Fourier transform, which is used in analyzing signals and also to calculate Laplace Transform used in solving differential equations. Both Laplace transform Fourier Transform are introduced in this handout in chapter VIII and IX respectively.

## II.2 Definition of improper integrals:

### IV.2.1 Improper integral type I:

They are all integrals of form: $\int_a^{+\infty} f(x)dx$ or $\int_{-\infty}^{b} f(x)dx$ or $\int_{-\infty}^{+\infty} f(x)dx$

Where f is a continuous function on [a, $+\infty$[ , ] $-\infty, b$] and ] $-\infty, \infty$[ respectively.

Examples

$\int_{-1}^{+\infty}(x+1)dx$ $\int_{-\infty}^{3}\frac{2}{x-5}dx$ $\int_{-\infty}^{+\infty} e^x dx$

### II.2.2 Improper integral type II:

They are all integrals of form a) or b) or c) (see below):

a) $\int_a^b f(x)dx$ Where f is a continuous function on ]a, $b$] but discontinuous at a.

**Example:**

$\int_{-1}^{10}\frac{x^2+x+3}{x+1}dx$ f(x)= $\frac{x^2+x+3}{x+1}$ is a continuous on ]-1, 10] but not at -1.

$b$) $\int_a^b f(x)dx$ Where f is a continuous function on [a, $b$[ but discontinuous at b.

**Example :**

$\int_{-20}^{0}\frac{e^x+1}{x}dx$ f(x)= $\frac{e^x+1}{x}$ is a continuous function on [-20, 0[ but discontinuous at 0.

$c$) $\int_a^b f(x)dx$ Where f is a continuous function on [a, $b$] but ∃c∈[a, $b$] such that f is discontinuous at c.

**Example:**

$\int_{3}^{50}\frac{x\ln(x)+6}{(x+2)(x-7)}dx$ f(x)= $\frac{e^x+6}{(x+2)(x-7)}$ is a continuous function on [3, 50] except at 7∈ $[3, 50]$ .

## II.3 Convergence of improper integrals:

### II.3.1 Convergence of improper integral type I:

$\int_a^{+\infty} f(x)dx$ , $\int_{-\infty}^{b} f(x)dx$ , $\int_{-\infty}^{+\infty} f(x)dx$ converge if

$\lim_{t\to+\infty}\int_a^t f(x)dx$ exists and finite , $\lim_{t\to-\infty}\int_t^b f(x)dx$ exists and finite,

both ($\lim_{t\to+\infty}\int_{-t}^{0} f(x)dx$ and $\lim_{t\to+\infty}\int_0^t f(x)dx$ exist and finite) respectively.

Therefore, we write:

$\int_a^{+\infty} f(x)dx = \lim_{t\to+\infty} \int_a^t f(x)dx, \quad \int_{-\infty}^b f(x)dx = \lim_{t\to-\infty} \int_t^b f(x)dx,$

$\int_{-\infty}^{+\infty} f(x)dx = \lim_{t\to+\infty} \int_{-t}^0 f(x)dx + \lim_{t\to+\infty} \int_0^t f(x)dx$ respectively.

**Note:**

If $\lim_{t\to+\infty} \int_a^t f(x)dx = \begin{cases} +\infty \\ or \\ +\infty \\ or \\ does\ not\ exist \end{cases}$ then $\int_a^{+\infty} f(x)dx$ diverges

The same thing for $\lim_{t\to-\infty} \int_t^b f(x)dx$.

Examples:

1)$\int_{-2}^{+\infty} \sin(x)dx = \lim_{t\to+\infty} \int_2^t \sin(x)dx = \lim_{t\to+\infty} [-\cos(x)]_2^t = cos2 - \lim_{t\to+\infty} cost$

This limit does not exist so $\int_{-2}^{+\infty} \sin(x)dx$ diverges

2) $\int_{-\infty}^3 e^x dx = \lim_{t\to-\infty} \int_t^3 e^x dx = \lim_{t\to-\infty} [e^x]_t^3 = e^3 - \lim_{t\to-\infty} e^t = e^3 - 0 = e^3$

So $\int_{-\infty}^3 e^x dx$ converges and $\int_{-\infty}^3 e^x dx = e^3$

## II.3.2 Convergence of improper integral type II:

Let f be a continuous function on ]a, $b$] but discontinuous at a

$\int_a^b f(x)dx$ converges if $\lim_{t\to a^+} \int_a^b f(x)dx$ exists and finite

Therefore $\int_a^b f(x)dx = \lim_{t\to a^+} \int_a^b f(x)dx$

**Notes:**

In the case of f a continuous function on [a, $b$] but not at c, c ∈[a, $b$]

$\int_a^b f(x)dx$ converges if both ($\lim_{t\to c^-} \int_a^t f(x)dx$ and $\lim_{t\to c^+} \int_t^b f(x)dx$ exist and finite)

And $\int_a^b f(x)dx = \lim_{t\to c^-} \int_a^t f(x)dx + \lim_{t\to c^+} \int_t^b f(x)dx$

**Examples:**

1)$\int_0^4 \frac{1}{x}dx = \lim_{t\to0^+} \int_t^4 \frac{1}{x}dx = \lim_{t\to0^+} [\ln(x)]_t^4 = ln4 - \lim_{t\to0^+} \ln(t) = ln4 - (-\infty) = +\infty$

So $\int_0^4 \frac{1}{x}dx$ diverges

2) $\int_0^3 \frac{1}{(x-1)^{\frac{2}{3}}}dx$

$f(x) = \frac{1}{(x-1)^{\frac{2}{3}}}$ is continue on [0,3] but discontinue en x=1, so we have to split the integral into two integrals.

$\int_0^3 \frac{1}{(x-1)^{\frac{2}{3}}}dx = \lim_{t\to 1^-}\int_0^t \frac{1}{(x-1)^{\frac{2}{3}}}dx + \lim_{t\to 1^+}\int_t^3 \frac{1}{(x-1)^{\frac{2}{3}}}dx = \lim_{t\to 1^-}[3(x-1)^{\frac{1}{3}}]_0^t + \lim_{t\to 1^+}[3(x-1)^{\frac{1}{3}}]_t^3$

$=3+3\ 2^{\frac{1}{3}}$

So $\int_0^3 \frac{1}{(x-1)^{\frac{2}{3}}}dx$ converges and $\int_0^3 \frac{1}{(x-1)^{\frac{2}{3}}}dx = 3[1+ 2^{\frac{1}{3}}]$

**Notes:**

-All examples seen above, we were able to compute the improper integral because we knew the anti-derivative of the function f(x).

-Most of the time it is hard to calculate an improper integral (especially when the integrand f(x) is complex thus it is difficult to find its anti-derivative) so we can just determine whether it converges or diverges.

## II.4 Properties of convergent improper integrals

Let f and g be continuous functions on ]a, $b$] but discontinuous at a.

1) If both $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ converge then $\int_a^b [f(x)\pm g(x)]dx$ converges

2) If $\int_a^b f(x)dx$ converges then $\int_a^b kf(x)dx$ converges, $\forall k \in \mathbb{R}$

3)If either $\int_a^b f(x)dx$ or $\int_a^b g(x)dx$ diverges then $\int_a^b [f(x)\pm g(x)]dx$ diverges

**Note:**

All these properties work with all improper integral types.

## II.5 P-TEST

$$1)\int_1^{+\infty}\frac{1}{x^p}dx \begin{cases} \text{converges if } p > 1 \text{, and } \int_1^{+\infty}\frac{1}{x^p}dx = \frac{1}{p-1} \\ \text{diverges if } p \leq 1 \text{, and } \int_1^{+\infty}\frac{1}{x^p}dx = +\infty \end{cases}$$

$$2)\int_0^{1}\frac{1}{x^p}dx \begin{cases} \text{converges if } p < 1 \text{, and } \int_0^{1}\frac{1}{x^p}dx = 1 \\ \text{diverges if } p \geq 1 \text{, and } \int_0^{1}\frac{1}{x^p}dx = -\infty \end{cases}$$

**Examples :**

$\int_1^{+\infty}\frac{1}{x^6}dx$ converges since $p = 6 > 1$ and $\int_1^{+\infty}\frac{1}{x^6}dx = \frac{1}{6-1} = \frac{1}{5}$

$\int_1^{+\infty}\frac{1}{\sqrt{x}}dx$ diverges since $p = \frac{1}{2} \leq 1$

## II.6 Convergence tests

Convergence tests enable us just to determine whether an improper integral converges or diverges without calculating its value.

### II.6.1 Direct comparison test:

Let f and g be continuous function on [a,+ ∞[ such that 0≤ $f(x) \leq$ g(x) $\forall x \geq a$.

1) If $\int_a^{+\infty} g(x)dx$ converges then $\int_a^{+\infty} f(x)dx$ converges.
2) If $\int_a^{+\infty} f(x)dx$ diverges then $\int_a^{+\infty} g(x)dx$ diverges.

**Example**:

1)$\int_1^{+\infty} \frac{cos^4(x)}{x^{\frac{3}{2}}}dx$

We have: $\forall x$, 0≤ $cos^4(x) \leq 1 \Rightarrow \frac{cos^4(x)}{x^{\frac{3}{2}}} \leq \frac{1}{x^{\frac{3}{2}}} \forall x \geq 1$

Since $\int_1^{+\infty} \frac{1}{x^{\frac{3}{2}}}dx$ converges (according to P-test , p=$\frac{3}{2} > 1$), so does $\int_1^{+\infty} \frac{cos^4(x)}{x^{\frac{3}{2}}}dx$

2) $\int_2^{+\infty} \frac{1}{\ln(x)}dx$

Notice that both functions ln(x) and x are positive on [2, +∞[ and

$\ln(x) \leq x \; \forall x > 2 \Rightarrow \frac{1}{\ln(x)} \geq \frac{1}{x} \forall x > 2$

Since $\int_1^{+\infty} \frac{1}{x}dx$ diverges (by P-test ), so $\int_2^{+\infty} \frac{1}{\ln(x)}dx$ also diverges.

### II.6.2 Limit comparison test:

Let f and g be positive and continuous function on [a,+ ∞[.

If $\lim_{x\to+\infty} \frac{f(x)}{g(x)} = L \neq \begin{cases} 0 \\ +\infty \end{cases}$ then both $\int_a^{+\infty} f(x)dx$ and $\int_a^{+\infty} g(x)dx$ converges or diverges.

Example

$$\int_3^{+\infty} \frac{x^4+2}{x^6+x^3-x}dx$$

We notice that f(x)= $\frac{x^4+2}{x^6+x^3-x} \sim \frac{1}{x^2}$ as $x \to +\infty$

Let us choose then g(x)= $\frac{1}{x^2}$

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \lim_{x\to+\infty} \frac{\frac{x^4+2}{x^6+x^3-x}}{\frac{1}{x^2}} = \lim_{x\to+\infty} \frac{x^2(x^4+2)}{x^6+x^3-x} = \lim_{x\to+\infty} \frac{x^6}{x^6} = 1$$

Since $\int_1^{+\infty} \frac{1}{x^2}dx$ converges (by P-test ), so does $\int_3^{+\infty} \frac{x^4+2}{x^6+x^3-x}dx$

**Notes:**

We can notice that convergence tests are applied for positive functions but we can generalize for negative functions as well knowing that -$\int -f(x)$=$\int f(x)$

So we can say that that convergence tests can be applied to functions that keeps a constant sign( either positive or negative)

## II.7 Absolute Convergence of improper integrals:

Let f be a continuous function on [a,b[ except at b.

The improper integral $\int_a^b f(x)dx$ is said to be absolutely convergent if $\int_a^b |f(x)|dx$ converges

Notes:

-This works for all types of improper integrals.

- If an improper integral converges absolutely then it converges.

**Example**:

Let us check the absolute convergence of $\int_1^{+\infty} \frac{sinx}{x^2} dx$

$$\left|\frac{sinx}{x^2}\right| \leq \frac{1}{x^2}$$

Since $\int_1^{+\infty} \frac{1}{x^2} dx$ converges (by P-test ), so does $\int_1^{+\infty} \left|\frac{sinx}{x^2}\right| dx$ (by comparison test)

Therefore $\int_1^{+\infty} \frac{sinx}{x^2} dx$ is absolutely convergent.

**Note:**

Absolute Convergence can be applied to any functions specially to those changing sign( oscilate)

# Exercises

Exercise1

Find the values of the following improper integrals, if they converge.

1) $\int_0^1 \frac{1}{\sqrt{x}} dx$ 2) $\int_0^{+\infty} \frac{1}{x^2+1} dx$ 3) $\int_{-\infty}^{+\infty} xe^{-x^2} dx$

**Exercise2:**

State whether the following improper integrals converge or diverge.

1) $\int_1^{+\infty} \frac{3}{x^5+1} dx$ 2) $\int_1^{+\infty} \frac{2x^5+2x-1}{\sqrt{x^7+3}} dx$ 3) $\int_1^{+\infty} \frac{1}{x+\sin^2(x)} dx$

Exercise3:

Determine if the following improper integrals converge absolutely.

1) $\int_1^{+\infty} \frac{cos(x)}{x^4+3} dx$ 2) $\int_1^{+\infty} \sin(\frac{1}{x^2}) dx$

**Solutions**

**Solution of exercise1:**

**1)** $\int_0^1 \frac{1}{\sqrt{x}} dx = \int_0^1 x^{-\frac{1}{2}} dx = \lim_{t\to 0^-} [2x^{\frac{1}{2}}]_t^1 = 2 - 0 = 2$. Thus $\int_0^1 \frac{1}{\sqrt{x}} dx$ converges and $\int_0^1 \frac{1}{\sqrt{x}} dx = 2$

**2)** $\int_0^{+\infty} \frac{1}{x^2+1} dx = \lim_{t\to+\infty} [arctang(x)]_0^t = \lim_{t\to+\infty} arctang(t) - 0 = \frac{\pi}{2}$, so $\int_0^{+\infty} \frac{1}{x^2+1} dx$ converges and $\int_0^{+\infty} \frac{1}{x^2+1} dx = \frac{\pi}{2}$

3) $\int_{-\infty}^{+\infty} xe^{-x^2} dx$

Start by splitting up the integral:

$$\int_{-\infty}^{+\infty} xe^{-x^2} dx = \lim_{t\to+\infty} \int_{-t}^{0} xe^{-x^2} dx + \lim_{t\to+\infty} \int_{0}^{t} xe^{-x^2} dx$$

If either $\int_{-\infty}^{0} xe^{-x^2} dx$ or $\int_0^{+\infty} xe^{-x^2} dx$ diverges, then $\int_{-\infty}^{+\infty} xe^{-x^2} dx$ diverges.

Let us compute each integral separately.

For the first integral:

$$\lim_{t\to+\infty} \int_{-t}^{0} xe^{-x^2} dx$$

**By substitution:**

u=$-x^2 \Rightarrow du = -2x$dx

$\lim_{t\to+\infty} \frac{-1}{2} \int_{-t^2}^{0} e^u du = \frac{-1}{2} \lim_{t\to+\infty} [e^u]_{-t^2}^{0} = \frac{-1}{2} [e^0 - \lim_{t\to+\infty} e^{-t^2}] = \frac{-1}{2}$ [1-0]$= \frac{-1}{2}$

For the second integral, similarly:

$\lim_{t\to+\infty} \int_0^t xe^{-x^2} dx = \lim_{t\to+\infty} \frac{-1}{2} \int_0^{-t^2} e^u du = \frac{-1}{2} \lim_{t\to+\infty} [e^u]_0^{-t^2} = \frac{-1}{2}$ [0-1]$= \frac{1}{2}$

Since both integrals converges, so does $\int_{-\infty}^{+\infty} xe^{-x^2} dx$ and $\int_{-\infty}^{+\infty} xe^{-x^2} dx = \frac{-1}{2} + \frac{1}{2} = 0$

**Solution of exercise2:**

1)$\int_1^{+\infty} \frac{3}{x^5+1} dx$

$\forall x \in [1, +\infty[\,, \frac{3}{x^5+1} > 0.$

$\forall x \in [1, +\infty[\,, x^5 + 1 > x^5 \Rightarrow \frac{1}{x^5+1} < \frac{1}{x^5}$

Since $\int_1^{+\infty} \frac{3}{x^5+1} dx$ converges (by P-test), so does $\int_1^{+\infty} \frac{1}{x^5+1} dx$ (by comparison test).

Therefore $3\int_1^{+\infty} \frac{1}{x^5+1} dx = \int_1^{+\infty} \frac{3}{x^5+1} dx$ also converges.

2) $\int_1^{+\infty} \frac{2x^5+2x-1}{\sqrt{x^7+3}} dx$

$\forall x \in [1, +\infty[, \frac{2x^5+2x-1}{\sqrt{x^7+3}} > 0.$

$\frac{2x^5+2x-1}{\sqrt{x^7+3}} \sim \frac{2x^5}{\sqrt{x^7}} = \frac{2}{x^{-\frac{3}{2}}}$ as $x \to +\infty$

Let us choose g(x)= $\frac{1}{x^{-\frac{3}{2}}}$.

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \lim_{x\to+\infty} \frac{\frac{2x^5 + 2x - 1}{\sqrt{x^7 + 3}}}{\frac{1}{x^{-\frac{3}{2}}}} \lim_{x\to+\infty} \frac{2x^5 x^{-\frac{3}{2}}}{\sqrt{x^7}} = 2$$

Since$\int_1^{+\infty} g(x)dx = \int_1^{+\infty} \frac{1}{x^{-\frac{3}{2}}} dx$ diverges (by P-test), so does $\int_1^{+\infty} \frac{2x^5+2x-1}{\sqrt{x^7+3}} dx$ (by limit comparison test)

3) $\int_1^{+\infty} \frac{1}{x+\sin^2(x)} dx$

$\forall x \in [1, +\infty[\,, \frac{1}{x+\sin^2(x)}$ is a positive function.

We have sinx$\leq 1 \Rightarrow \sin^2(x) \leq 1 \Rightarrow \sin^2(x) + x \leq 1 + x \Rightarrow \frac{1}{\sin^2(x)+x} \geq \frac{1}{1+x}$

Let us check the convergence or divergence of $\int_1^{+\infty} \frac{1}{1+x} dx$ :

$\int_1^{+\infty} \frac{1}{1+x} dx = \lim_{t\to+\infty} [ln|1 + x|]_1^t = -ln2 + \lim_{t\to+\infty} ln|1 + t| = +\infty$ thus diverges, so does $\int_1^{+\infty} \frac{1}{x+\sin^2(x)} dx$ (by comparison test)

**Note:**

We could have determine the divergence of $\int_1^{+\infty} \frac{1}{1+x} dx$ using the limit comparison test, knowing that $\frac{1}{1+x} \sim \frac{1}{x}$ as $x \to +\infty$

**Solution exercise3:**

1) $\int_1^{+\infty} \frac{cos(x)}{x^4+3} dx$

$\int_1^{+\infty} \left|\frac{cos(x)}{x^4+3}\right| dx$ converges ?

$$|cos(x)| \leq 1 \Rightarrow \frac{|cos(x)|}{x^4 + 3} \leq \frac{1}{x^4 + 3} \leq \frac{1}{x^4}$$

Since$\int_1^{+\infty} \frac{1}{x^4} dx$ converges (according to P-test), so does $\int_1^{+\infty} \left|\frac{cos(x)}{x^4+3}\right| dx$ ( by comparison test).

2) $\int_1^{+\infty} \sin(\frac{1}{x^2}) dx$

As $x \to +\infty, \frac{1}{x^2} \to 0$

We know that $\lim_{t \to 0} \frac{sint}{t}$=1, let us call then f(x)= $\left|\sin(\frac{1}{x^2})\right|$ and g(x)= $\frac{1}{x^2}$ .

$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = \lim_{t \to 0} \frac{\left|\sin(\frac{1}{x^2})\right|}{\frac{1}{x^2}}$=1

Due to P-test, $\int_1^{+\infty} g(x)dx = \int_1^{+\infty} \frac{1}{x^2} dx$ converges, so does $\int_1^{+\infty} \left|\sin(\frac{1}{x^2})\right| dx$ (by Limit comparison test).

Therefore $\int_1^{+\infty} \sin(\frac{1}{x^2}) dx$ converges absolutely

# Chapter III
# Differential Equations

## III.1 Ordinary differential equations (ODE's)

### III.1.1 Definition

An Ordinary differential equation (ODE) is an equations that contain a function (usually denoted by y(x) or simply by y ) and some of its derivatives (the first derivative $y^{'}$ or higher-order derivatives $y'', y^{(3)} \dots..$).

**Examples:**

1) $y^{'} + 4y = 0$
2) $y^{'} = 5 + 4e^{2x}$
3) $y'' - y = 0$
4) $y^{(5)} - 2y''y = \ln(x)$

**Notes:**

- $y^{'}$ is sometimes written as :$\frac{dy}{dx}$ . We keep the first notation
- The word "ordinary" refers to the one variable x of the function y. If there are more than one ( ie y($x_1, x_2, x_3, \dots$) the equation become a partial differential equation see after sectionVI.2.
- The function y(x) is the unknown of the equation.
- The highest order of the derivative in the equation is the order of the differential equation.

**Examples:**

1) and 2) are first-order differential equations since we have only first derivative.
3) is a second -order differential equation, it contains only second derivatives.
4) Is a fifth -order differential equation because the fifth derivative $y^{(5)}$ is the highest order.

- The general solution of a differential equation is given by the set of all functions that satisfy the equation.

**Examples (above):**

1) $y^{'} + 4y = 0$ solutions: y(x)=$ke^{-4x}$, k∈ ℝ
5) $y^{'} = 5 + 4e^{2x}$ solutions: y(x)=5x+$2e^{2x} + c$, c∈ ℝ
6) $y'' - y = 0$ solutions: y(x)= a$e^{x} + be^{-x}$, a, b∈ ℝ

### III.1.2 Linear Ordinary differential equation

We are not able to solve all differential equation because of the complexity of most of them the reason why we are going to focus in this section on two types of equations: first-order linear ODE's and Second-Order Linear ODE's with Constant Coefficients

´ • An nth order differential equation is linear if it is of form:

$a_0(x)\, y + a_1(x)\, y' + \cdot\cdot\cdot + a_n(x) y^{(n)} = f(x)$ (1)

where $a_i(x)$ are functions

**Notes:**

•If coefficients $a_i$ are constants ie (1) is written as $a_0y + a_1y' + \cdot\cdot\cdot + a_n y^{(n)} = f(x)$ the latter one is a Linear ODE's with constant Coefficients.

•if f(x)=0 then the equation $a_0(x)\, y + a_1(x)\, y' + \cdot\cdot\cdot + a_n(x) y^{(n)} = 0$ is said to be homogenous.

•Linear means: y and its derivatives ($y'$, $y''$ , ...) occurs at most to the first power and there is no product between them.

**Examples:**

1) $2x\, y'' + y\, y' + 2 = 0$ (not linear because of the product $y\, y'$)
2) $\text{Sin}(x)\, y' + y^2 = x^2$ (not linear because of $y^2$, y appears to the second powers)
3) $x^4\, y'' + \text{Ln}(x)\, y' + 4\, y = e^x$ ( linear because satisfies all requirements cited above)

## III.1.3. First-order linear ODE's

### III.1.3.1 Definition

First-order non-homogeneous linear ODE is of form:

$y' + b(x)y = f(x)$ (I)

b(x) and f(x) are functions of x

**Note:**

-If we have an equation of form: $a(x)y' + b(x)y = f(x)$. It is still a first-order non-homogeneous linear ODE, we have just to divide by $a(x)$.

### III.1.3.2 Solving a first-order linear ODE's

Steps to follow:

1)Find the homogenous solution (denoted $y_H$) of the homogenous equation $y' + b(x)y = 0$.

2)Find a particular solution (denoted $y_p$) of the equation (I).

3) General solution of our equation (I) is $y_G = y_H + y_p$

Trough an example, we are going to apply 1), 2) and 3)

Example: Solve $xy' + 2xy = x^3$

We can bring this equation to the standard form (I): $y' + 2y = x^2$ (II)

1)Find the homogenous solution $y_H$ of equation $y' + 2y = 0$.

Separate variables:

We rearrange the equation so that dy and all other expression containing y are on the left and dx and all expressions containing x are on the right.

$y' + 2y = 0 \Rightarrow \frac{dy}{y} = -2dx$ for $y \neq 0$ (y=0 is an obvious solution)

since $dy = y'\,dx$

Now, we can integrate: $\int \frac{dy}{y} = \int 2dx \Rightarrow ln|y| = -2x + C$

$\Rightarrow e^{ln|y|} = e^{2x+C} \Rightarrow |y| = e^{C}e^{-2x} \Rightarrow y = \pm e^{C}e^{-2x} \Rightarrow y_H = Ke^{-2x}$, K∈ ℝ

**Note:**

- For any first-order non-homogeneous linear ODE ( $y' + b(x)y = 0$), the set of solutions is

$y_H = Ke^{-\int b(x)dx}$, K∈ ℝ

2)Find a particular solution $y_p$ of the original equation (II) $y' + 2y = x^2$

Sometimes the search for a particular solution is done by noticing an 'obvious' solution. In most cases, it's difficult, so we use variation of parameters method to find this particular solution.

Variation of Parameters:

Consider the function $K(x)e^{-2x}$ (as particular solution $y_p$) , in which we have replaced the constant parameter K with the function K(x). This technique is called *variation of parameters*.

Let us determine $K(x)$:

$y_p = K(x)e^{-2x}$ (III)

We differentiate (III):

$$y_p' = (K(x))'e^{-2x} - 2\,K(x)e^{2x}$$

Replace $y_p'$ and $y_p$ in the original equation (II):

$(K(x))'e^{-2x} - \cancel{2\,K(x)e^{2x}} + \cancel{2K(x)e^{-2x}} = x^2 \Rightarrow (K(x))' = x^2e^{2x} \Rightarrow K(x) = \int x^2e^{2x}dx$

Using twice integration by parts method, we get expression of K(x):

K(x)=$\frac{1}{2}x^2e^{2x} - 2\left(\frac{1}{2}xe^{2x} - \frac{1}{2}e^{2x}\right) = \frac{1}{2}x^2e^{2x} - xe^{2x} + e^{2x}$

$\Rightarrow y_p = K(x)e^{-2x} = [\frac{1}{2}x^2e^{2x} - xe^{2x} + e^{2x}]e^{-2x}$

Thus our particular solution $y_p = \frac{1}{2}x^2 - x + 1$

3) General solution of our equation is $y_G = y_H + y_p$

So $y_G(x) = Ke^{-2x} + \frac{1}{2}x^2 - x + 1$, K∈ ℝ

## III.1.4. Second-Order Linear ODE's with Constant Coefficients

## III.1.4.1 Definition

Second-order ODE's with Constant Coefficients are those containing the second derivative $y''$, in which coefficients of $y''$, $y'$and y are all constant. These equations are of form of:

$$ay''+by' + cy = f(x) \quad (IV)$$

where a, b, c are constant and f a function of x

VI.1.4.1.2 Solving a Second-Order Linear ODE with Constant Coefficients(my method)

We apply the same steps as for first-order linear ODE's:

1)Find the homogenous solution $y_H$ of the homogenous equation $ay''+by' + cy=0$ $(HE)$.

2)Find a particular solution $y_p$ of the equation ( IV)

3) General solution of our equation ( IV)is $y_G=y_H + y_p$

First step:

Let us determine $y_H$, solution of $ay''+by' + cy=0$ $(HE)$.

We define a characteristic equation (CE) associated to homogenous equation $(HE) by$:

$ar^2+br + c=0$ (CE)

If $\Delta= b^2 - 4ac > 0 \Rightarrow$ we have two real roots $r_1 = \frac{-b-\sqrt{\Delta}}{2a}$ , $r_2 = \frac{-b+\sqrt{\Delta}}{2a}$

and our homogenous solution $y_H = k_1e^{r_1x} + k_2e^{r_2x}$ , $k_1, k_2 \in \mathbb{R}$

If $\Delta= b^2 - 4ac = 0 \Rightarrow$ we have a double real root $r = \frac{-b}{2a}$ and our homogenous solution $y_H = (k_1x + k_2)e^{rx}$ , $k_1, k_2 \in \mathbb{R}$

If $\Delta= b^2 - 4ac < 0$ $\Rightarrow$ we have two complex roots $r_1 = \frac{-b-i\sqrt{\Delta}}{2a}=\alpha - i\beta$, $r_2 = \frac{-b+i\sqrt{\Delta}}{2a}= \alpha + i\beta$ and our homogenous solution $y_H = e^{\alpha x}(k_1\cos(\beta x) + k_2 sin(\beta x))$ , $k_1, k_2 \in \mathbb{R}$

Second step:

Let us determine a particular solution $y_p$ of the equation $ay''+by' + cy = f(x)$ ( IV):

It depends on the form of f(x).

1) If $f(x)=P_n(x)e^{kx}, k \in \mathbb{R}$ ($P_n(x)$ is $n^{th}$ order polynomial function)

   Then we have three possibilities:

   a- If k is not a root of the characteristic equation (see above) then $y_p = Q_n(x)e^{kx}$
      Where $Q_n(x)e^{kx}$ a polynomial function with the same order as $P_n(x)$.

   b- If k is a real (simple) root of the characteristic equation then $y_p = Q_n(x)e^{kx}x$.

c- If k is a real double root of the characteristic equation then $y_p = Q_n(x)e^{kx}x^2$.

In the three possibilities, we calculate $y_p{}'', y_p{}'$ $and$ $y_p$ , put their expressions in the equation a$y''$+b$y' + cy$ =f(x) ( IV) and by identification with f(x) we deduce $Q_n(x)$.

2) If $f$(x)=$e^{kx}(P_n(x)\sin(\theta x) + P_m(x)cos(\theta x))$ , $k \in \mathbb{R}$

($P_n(x), P_m(x)$ are n^th^, m^th^ order polynomial functions respectively)

Then we have two possibilities (for each case):

a- If (k+i$\theta$) is not a root of the characteristic equation (seen above) then

$y_p = e^{kx}(Q_l(x)sin(\theta x) + Q_l(x)cos(\theta x))$

Where $Q_l(x)$ a $l$ th order polynomial function such that $l$ =max(n,m).

b- If (k+i$\theta$) is a root of the characteristic equation then

$y_p = xe^{kx}(Q_l(x)sin\,(\theta x)$+ $Q_l(x)cos(\theta x))$

In all these possibilities, we calculate $y_p{}'', y_p{}'$ $and$ $y_p$ , put their expressions in the equation a$y''$+b$y' + cy$ =f(x) ( IV) and by identification with f(x) we deduce $Q_n(x)$.

3) General solution of our equation ( IV)is $y_G$=$y_H + y_p$

**Example:**

$y''$+$y' - 6y$ =(2x+1) $e^{-2x}$ ($P_1(x)e^{kx}$)

1) Determine $y_H$, solution of $y''$+$y' - 6y$ =0 $(HE)$.

Characteristic equation: $r^2$+$r - 6$=0 (CE)

$\Delta = 1^2 - 4(-6) = 25 > 0 \Rightarrow$ we have two real roots $r_1 = \frac{-1-5}{2} = -3$ , $r_2 = \frac{-1+5}{2} = 2$

and our homogenous solution $y_H = k_1e^{-3x} + k_2e^{2x}$ , $k_1, k_2 \in \mathbb{R}$

2) Determine a particular solution $y_p$ of the equation $y''$+$y' - 6y$ =(2x+1) $e^{-2x}$

f(x)= (2x+1) $e^{-2x}$ of form$P_1(x)e^{kx}$ ($P_1(x) = (2x + 1)$ is first order polynomial function)

k=-2 is not a root of the characteristic equation then $y_p = Q_1(x)e^{-2x}$

where $Q_1(x) = (ax + b)$ ($Q_1(x)$ is a polynomial function with the same order than $P_1(x)$)

To calculate constants a and b we have to compute $y_p{}'', y_p{}'$ $and$ $y_p$.

$y_p = (ax + b)\ e^{-2x}$

$y_p{}' = e^{-2x}(-2ax - 2b + a)$

$y_p{}'' = e^{-2x}(4ax + 4b - 4a)$

Put expressions of $y_p{}'', y_p{}'$ $and$ $y_p$in the original equation:

$\cancel{e^{-2x}}(4ax + 4b - 4a) + \cancel{e^{-2x}}(-2ax - 2b + a) - 6(ax + b)\ \cancel{e^{-2x}}$=(2x+1) $\cancel{e^{-2x}}$

After developing the left side, we obtain:

(-4a)x+(-4b-3a)= (2x+1)

By identification, we have:

$a = -\frac{1}{2}$ and $b = \frac{1}{8}$

Thus our particular solution $y_p(x) = (-\frac{1}{2}x + \frac{1}{8})e^{-2x}$

Therefore our general solution

$y_{G(x)}=y_H(x) + y_p(x) = k_1e^{-3x} + k_2e^{2x}(-\frac{1}{2}x + \frac{1}{8})e^{-2x}$ , $k_1, k_2 \in \mathbb{R}$

## III.2 Partial differential equations (PDE's)

Many of PDEs are coming from different domains of physics (acoustics, optics, elasticity, hydro and aerodynamics, electromagnetism, quantum mechanics, seismology etc). However PDEs appear in other fields of science as well (like quantum chemistry, chemical kinetics); some PDEs are coming from economics and financial mathematics, or computer science.

### III.2.1 Definition of a partial derivative

If we consider a function that depends on several variables, we can differentiate with respect to either variable while keeping the other variable constant. For example, if we have a function depending on two real variables u(x,y) taking its values in ℝ.

We can compute the derivative with respect to x while keeping y fixed. This leads to $\frac{\partial u}{\partial x}$ which is called partial derivative of u with respect to x. Similarly, we can hold x fixed and differentiate with respect to y $\frac{\partial u}{\partial y}$ (it is the partial derivative of u with respect to y).

**Examples:**

1) u(x,y)= $x^2 + 3y^3$

   $\frac{\partial u}{\partial x} = 2xy$ (here y is considered as a constant)

   $\frac{\partial u}{\partial y} = x^2 + 9y^2$ (here x is held fixed)

2) u(x,y,z)= $x^4 lny + e^{5x}\sin(y) + zy + 10$

   $\frac{\partial u}{\partial x} = 4x^3\ lny + 5e^{5x}\sin(y)$(here y and are considered as a constant)

   $\frac{\partial u}{\partial y} = \frac{x^4}{y} + e^{5x}\cos(y) + z$ (here x and z are held fixed)

   $\frac{\partial u}{\partial z} = y$ (here x and y are held fixed)

**Notes:**

- $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ can be denoted also by $u_x$ , $u_y$ respectively. We keep the first notation.

- All partial derivatives computed above are first-order-partial derivatives (since we differentiate the function u one time with respect to a certain variable. If we do it twice we obtain second-order partial derivatives denoted by $\frac{\partial^2 u}{(\partial x)^2}$, $\frac{\partial^2 u}{\partial x \partial y}$, $\frac{\partial^2 u}{\partial y \partial x}$, $\frac{\partial^2 u}{(\partial y)^2}$. More generally, we can differentiate u more than twice such as $\frac{\partial^i}{(\partial x)^i}\frac{\partial^j}{(\partial y)^j}\frac{\partial^k u}{(\partial x)^k}$ which is an (i+j+k)-order partial derivative.

**Examples:**

$$u(x,y)= ye^{xy}$$

$$\frac{\partial u}{\partial x} = y^2 e^{xy} \qquad \frac{\partial u}{\partial y} = e^{xy} + yxe^{xy}$$

$$\frac{\partial}{\partial x}\frac{\partial u}{\partial x} = \frac{\partial^2 u}{(\partial x)^2} = \frac{\partial}{\partial x}(y^2 e^{xy}) = y^3 e^{xy}$$

$$\frac{\partial}{\partial x}\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}(e^{xy} + yxe^{xy}) = ye^{xy} + ye^{xy} + y^2 xe^{xy} = 2ye^{xy} + y^2 xe^{xy}$$

$$\frac{\partial}{\partial y}\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}(y^2 e^{xy}) = 2\, ye^{xy} + y^2 xe^{xy}$$

$$\frac{\partial}{\partial y}\frac{\partial u}{\partial y} = \frac{\partial^2 u}{(\partial y)^2} = \frac{\partial}{\partial y}(e^{xy} + yxe^{xy}) = xe^{xy} + xe^{xy} + yx^2 e^{xy} = 2xe^{xy} + yx^2 e^{xy}$$

$$\frac{\partial}{\partial y}\frac{\partial}{\partial x}\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(2ye^{xy} + y^2 xe^{xy}) = 2e^{xy} + 2yxe^{xy} + y^2 x^2 e^{xy}$$

We can notice that $\frac{\partial}{\partial x}\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\frac{\partial u}{\partial x}$ but in general it is not always true..

## III.2.2 Definition of a partial differential equation (PDE)

A partial differential equation is an equation which involves a function depending on more than one variable, and partial derivatives of the function.

**Examples**

1) $\frac{\partial u}{\partial x} = 0$
2) $\frac{\partial^2 u}{(\partial x)^2} = 0$
3) $\frac{\partial^2 u}{\partial x \partial y} = 0$
4) $\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$ is the transport equation.
5) $\frac{\partial^2 u}{(\partial x)^2} + \frac{\partial^2 u}{(\partial y)^2} = 0$ is the Laplace’s equation
6) $\frac{\partial u}{\partial t} - c^2\left(\frac{\partial^2 u}{(\partial x)^2} + \frac{\partial^2 u}{(\partial y)^2}\right) = 0$ is the two-dimensional Heat equation
7) $\frac{\partial^2 u}{(\partial t)^2} - c^2\frac{\partial^2 u}{(\partial x)^2} = 0$ is the One-dimensional wave equation (or string equation)

**Notes:**

-All PDE’s above are of second order (except for 1 and 4, they are of a first order) since they involve second-order partial derivatives.

**-** If a PDE contains partial derivatives such as $\frac{\partial^i}{(\partial x)^i}\frac{\partial^j}{(\partial y)^j}\frac{\partial^k u}{(\partial x)^k}$, its order is then the largest order of partial derivatives i+ j+k that appears in the equation.

## III.2.2 Linear partial differential equation:

A PDE is said to be linear if:

i)The function u and its partial derivatives occurs at first power

ii)There is no product between the function u and its partial derivatives.

Otherwise it's non-linear.

**Note:**

- if u depends on two variables x and y, i) means there is no $u^m$, $(\frac{\partial u}{\partial x})^m$, $\left(\frac{\partial u}{\partial x}\right)^m$ $\left(\frac{\partial^2 u}{\partial x\partial y}\right)^m$ ....., $m \geq 2$

**Examples:**

1) All PDE's above are linear since they satisfy both requirements.
2) $\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}+$ u(x,t)=4x+t is linear
2) $2xy\frac{\partial^2 u}{\partial x\partial y}+5\frac{\partial u}{\partial x}=2x\sin(y)$ is linear
3) $(\frac{\partial u}{\partial x})^2+\frac{\partial^3 u}{(\partial y)^3}+6x\frac{\partial u}{\partial x}=0$ is non- linear because of $(\frac{\partial u}{\partial x})^2$ (second power. The first condition is not satisfied).
4) $u(x,y)\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=x$ is non- linear because of the product $u(x,y)\frac{\partial u}{\partial x}$. (the second condition is not satisfied).

## III.2.3 Solving linear second-order PDE's

PDE's are often hard to solve because of their complexity indeed they involve functions of multiple variables, different partial derivatives and most of them are non linear. The reason why we are going to restrain our study to linear second-order PDE's using a method called separable of variables to solve them and in chapter V we will introduce a tool to solve PDE's called Fourier transform.

A general linear second –order PDE is of form:

$$A\frac{\partial^2 u}{(\partial x)^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{(\partial y)^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=G$$

Where all coefficients A, B,…G are functions of x and y.

**Notes:**

-If G=0⇒ PDE is homogeneous.

- A solution of a linear second–order PDE is a function u(x, y) that possesses partial derivatives in LPDE and satisfies the equation.

- if $u_1, u_2, \dots . u_3$ are solution of homogeneous LPDE then $u = c_1u_1 + c_2u_2, \dots . + c_ku_3$is also a solution (superposition principle)

## III.2.3 Separable variables method:

This method is about looking for a solution of form u(x,y)=f(x).g(y)

So $\frac{\partial u}{\partial x} = f'(x).g(y)$, $\frac{\partial u}{\partial y} = f(x).g'(y)$, $\frac{\partial^2 u}{(\partial y)^2} = f(x)g''(y)$, ……etc

At the end we have to solve two different equations.

Example1:

$\frac{\partial^2 u}{(\partial x)^2} = 4\frac{\partial u}{\partial y}$ (I)

Let us put u(x,y)=f(x).g(y)

(I)Becomes $f''(x)g(y) = 4\, f(x).g'(y) \Rightarrow \frac{f''(x)}{4f(x)} = \frac{g'(y)}{g(y)}$

independent of y independent of x

From this equality we can deduce both of them are independent of x and y therefore they are constant.

$\frac{f''(x)}{4f(x)} = \frac{g'(y)}{g(y)} = -k$ ( -k because it's more convenient for next calculations)

$\Rightarrow f''(x) + 4kf(x) = 0$ and $g'(y) + kg(y) = 0$

We have three cases: $k = 0, k < 0, k > 0$

1) $k = 0 \Rightarrow \begin{cases} f''(x) = 0 \Rightarrow \frac{\partial^2 f(x)}{(\partial x)^2} = 0 \Rightarrow \frac{\partial f(x)}{\partial x} = c_1 \Rightarrow f(x) = c_1x + c_2 \\ g'(y) = 0 \Rightarrow \frac{\partial g(x)}{\partial y} = 0 \Rightarrow g(y) = c_3 \end{cases}$

So u(x,y)=f(x).g(y)= $(c_1x + c_2)\, c_3 \Rightarrow$ u(x,y)= $a_1x + b_1$, $a_1, b_1 \in \mathbb{R}$

2) $k < 0 \Rightarrow k = -\alpha^2$

$f''(x) - 4\alpha^2 f(x) = 0$

A second-order homogeneous EDO

$r^2 - 4\alpha^2 = 0$ (CE)

Two real roots $r_1 = 2\alpha$ , $r_2 = -2\alpha$

solution $f(x) = k_1e^{2\alpha x} + k_2e^{-2\alpha x}$ ,

$k_1, k_2 \in \mathbb{R}$

$g'(y) - \alpha^2 g(y) = 0$

A first-order homogeneous EDO

Solution g(y)= $k_3e^{\alpha^2 y}$ , $k_3 \in \mathbb{R}$

So u(x,y)=f(x).g(y)= $(k_1e^{2\alpha x}+k_2e^{-2\alpha x})(k_3e^{\alpha^2 y}) \Rightarrow$ u(x,y)= $a_2e^{-2\alpha x+\alpha^2 y}+b_2e^{2\alpha x+\alpha^2 y}$, $a_2, b_2 \in \mathbb{R}$

3) $k > 0 \Rightarrow k = \alpha^2$

| | |
|---|---|
| $f''(x)+4\alpha^2\text{f(x)}=0$ | $\text{g}'(\text{y})+\alpha^2\text{g(y)}=0$ |
| A second-order homogeneous EDO | A first-order homogeneous EDO |
| $r^2+4\alpha^2=0$ (CE) | Solution g(y)= $k_6e^{-\alpha^2 y}$, $k_6 \in \mathbb{R}$ |
| Two complex roots $r_1=-2\alpha i$, $r_2=2\alpha i$ | |
| solution $f(x)=k_4cos2\alpha x+k_5sin2\alpha x$, $k_4, k_5 \in \mathbb{R}$ | |

So u(x,y) =f(x).g(y)= $a_3e^{-\alpha^2 y}cos2\alpha x+b_3e^{-\alpha^2 y}sin2\alpha x$, $a_3, b_3 \in \mathbb{R}$

**Example2:**

$x\frac{\partial u}{\partial x}=t\frac{\partial u}{\partial t}$ (II)

u(x,t)=f(x).g(t)

(II) Becomes $\text{x}f'(x)\text{g(t)}=\text{t f(x).g}'(\text{t}) \Rightarrow \frac{xf'(x)}{\text{f(x)}}=\frac{\text{tg}'(\text{t})}{\text{g(t)}}=-k$

$\Rightarrow xf'(x)+k\text{f(x)}=0$ and $\text{tg}'(\text{t})+\text{kg(t)}=0$

1) $k=0 \Rightarrow \begin{cases} f'(x)=0 \Rightarrow f(x)=c_1 \\ \text{g}'(\text{y})=0 \Rightarrow g(y)=c_2 \end{cases}$ so u(x,y)= $c_1+c_2=a_1$, $a_1 \in \mathbb{R}$

2) $k \neq 0$

| | |
|---|---|
| $xf'(x)+k\text{f(x)}=0$<br>A first-order homogeneous EDO<br>$xf'(x)=-kf(x)$<br>$\frac{f'(x)}{f(x)}=\frac{-k}{x}$<br>Integrate: ln$\lvert f(x)\rvert$=-k$ln\lvert x\rvert$+$c_3$<br>$\Rightarrow f(x)=c_4x^{-k}$<br>$\Rightarrow$solution $f(x)=c_4x^{-k}$, $c_4 \in \mathbb{R}$ | $\text{tg}'(\text{t})+\text{kg(t)}=0$<br>A first-order homogeneous EDO<br>$xg'(x)=-kg(x)$<br>$\frac{g'(t)}{g(t)}=\frac{-k}{t}$<br>Integrate: ln$\lvert g(t)\rvert$=-k$ln\lvert x\rvert$+$c_3$<br>$\Rightarrow g(t)=c_5t^{-k}$<br>Solution g(t)= $c_5t^{-k}$, $c_5 \in \mathbb{R}$ |

So u(x,t) =f(x).g(y)= $a_2(xt)^{-k}$, $a_2 \in \mathbb{R}$

**Notes :**

We have considered just two cases of k because of first-order homogeneous EDO (

three cases when we have second-order homogeneous EDO).

## Exercises

**Exercice1**:

Solve the following First-order ODE:

$xy' - y = x^2 lnx$

**Exercice2:**

Solve the following second-order ODE:

$$y'' + y' - 6y = sinx$$

**Exercice3:**

Which of the following PDE's are linear?

1) $\frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial u}{\partial x} = 2xe^y$

2) $\frac{\partial^2 u}{(\partial y)^2} + (\frac{\partial u}{\partial x})^3 + 6x\frac{\partial u}{\partial x} = 0$

3) $\frac{\partial^2 u}{(\partial t)^2} + \frac{\partial u}{\partial x} + u(x,t) + 2xt = 0$

4 ) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u(x,y)\frac{\partial^2 u}{\partial x \partial y}$

**Exercice4:**

Solve the following simple PDE's.

1) $\frac{\partial u(x,y)}{\partial x} = 0$ 2) $\frac{\partial^2 u}{(\partial x)^2} = 0$ 3) $\frac{\partial^2 u}{\partial x \partial y} = 0$

**Exercice5:**

Solve the following linear second-order PDE's using separable of variables method:

1) $\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$ (the transport equation).

2) $\frac{\partial^2 u}{(\partial x)^2} + \frac{\partial^2 u}{(\partial y)^2} = 0$ ( the Laplace's equation)

**Solution of exercise 1**

We can bring this equation to the standard form (I): $y' - \frac{1}{x}y = xlnx$

1)Find the homogenous solution $y_H$ of equation $y' - \frac{1}{x}y = 0$:

**Separate variables:**

$y' - \frac{1}{x}y = 0 \Rightarrow \frac{dy}{y} = \frac{dx}{x}$ (since $dy = y'\, dx$)

Integrate: $\int \frac{dy}{y} = \int \frac{dx}{x} \Rightarrow ln|y| = ln|x| + C$

$\Rightarrow e^{ln|y|} = e^{ln|x|+C} \Rightarrow |y| = e^C|x| \Rightarrow y_H = e^C x = Kx$, $K \in \mathbb{R}$

2) Find a particular solution $y_p$ of $y' - \frac{1}{x}y = xlnx$ (I) using variation of parameters:

$y_p = K(x)x \Rightarrow y_p' = K'^{(x)}x + K(x)$

Put expressions of $y_p$ and $y_p'$ into (I):

$K'^{(x)}x + K(x) - \frac{1}{x}K(x)x = \ xlnx \Rightarrow K'^{(x)} = lnx \Rightarrow \ K(x) = \int lnx\, dx$

Using integration by parts method, we get $K(x) = xlnx - x + c, \ \ c \in \mathbb{R}$

$$\Rightarrow y_p = (xlnx - x + c)x$$

Therefore, our general solution: $y_G = \ Kx + (xlnx - x + c)x$

Finally, $y_G = \ Lx + (xlnx - x)x$, $L \in \mathbb{R}$

**Solution of exercise2**:

$y'' + y' - 6y = \sin x$

1) Determine $y_H$ of homogeneous equation $y'' + y' - 6y = 0$

Characteristic equation (CE): $r^2 + r - 6 = 0 = 0$

$\Delta = 1 + 24 = 25 \Rightarrow$ two complex roots $r_1 = \frac{-1-5}{2} = -3$

$r_2 = \frac{-1+5}{2} = 2$

So $y_H = y_H = k_1e^{-3x} + k_2e^{2x}$ , $k_1, k_2 \in \mathbb{R}$

3) Determine a particular solution $y_p$ of the equation $y'' + y' - 6y = \sin x$ (II)

f(x)= sin(x) of form $e^{0x}(P_0(x)\sin x + R_0(x)\cos x)$ , here $P_0(x) = 1$ and $R_0(x) = 0$

(0+i) is not a root of the characteristic equation (seen above) then

$y_p = e^{0x}(Q_0(x)\sin x + S_0(x)\cos x) = \mathrm{A}\sin x + B\cos x$ A, B constants

Let us determine A and B:

$y_p = \mathrm{A}\sin x + B\cos x$

$y_p' = Acosx - Bsinx$

$y_p'' = -Asinx - Bcosx$

Put expressions of $y_p, y_p', y_p''$ into equation (II).

$-Asinx - Bcosx + Acosx - Bsinx - 6\,\mathrm{A}\sin x - 6B\cos x = sinx \Leftrightarrow$

$sinx\,[-A - B - 6A] + cosx\,[-B + A - 6B] = sinx + 0cosx \ \ \Leftrightarrow$

$$\begin{cases} -7\mathrm{A} - \mathrm{B} = 1 \\ A - 7B = 0 \end{cases} \Leftrightarrow \begin{cases} \mathrm{A} = \frac{-7}{50} \\ B = \frac{-1}{50} \end{cases}$$

Thus $y_p = \frac{-7}{50}\sin x + \frac{-1}{50}\cos x$

Therefore our general solution $y_G = k_1e^{-3x} + k_2e^{2x} + \frac{-7}{50}\sin x + \frac{-1}{50}\cos x$ , $k_1, k_2 \in \mathbb{R}$

**Solution of exercise3**

1) $\frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial u}{\partial x} = 2xe^y$ is linear

2) $\frac{\partial^2 u}{(\partial y)^2} + (\frac{\partial u}{\partial x})^3 + 6x\frac{\partial u}{\partial x} = 0$ is non- linear because of $(\frac{\partial u}{\partial x})^3$ (Third power. The first condition is not satisfied).

3) $\frac{\partial^2 u}{(\partial t)^2} + \frac{\partial u}{\partial x}$+ u(x,t)+2xt=0 is linear

4 ) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u(x,y)\frac{\partial^2 u}{\partial x \partial y}$ is non- linear because of the product $u(x,y)\frac{\partial^2 u}{\partial x \partial y}$ (the second condition is not satisfied).

**Solution of exercice4:**

1) $\frac{\partial u(x,y)}{\partial x} = 0$

That means the function u does not depend on the variable x, but only on the Variable y

$\Rightarrow$ by integrating with respect to x, u(x,y)= f(y) where f(y) is any arbitrary function of y.

2) $\frac{\partial^2 u}{(\partial x)^2} = 0 \Leftrightarrow \frac{\partial}{\partial x}\left[\frac{\partial u(x,y)}{\partial x}\right] = 0$

That means the function $\frac{\partial u(x,y)}{\partial x}$ does not depend on the variable x, but only on the variable y $\Rightarrow \frac{\partial u(x,y)}{\partial x}=$ f(y) $\Rightarrow$by integrating with respect to x, we get u(x,y)=f(y)x+g(x), where f(y), g(x) are respectively any arbitrary function of y, x.

3) $\frac{\partial^2 u}{\partial x \partial y} = 0 \Leftrightarrow \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial y}\right] = 0 \Rightarrow$by integrating with respect to x, we obtain $\frac{\partial u}{\partial y} =$ f(y)

$\Rightarrow$by integrating with respect to y, we get u(x,y)= $\int$ f(y)dy +g(x)

$\Rightarrow$ u(x,y)= $F(y)$ +g(x) where F(y) is an anti-derivative of f.

**Solution of exercice5:**

1) $\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$ (1).

u(x,t)=f(x).g(t)

(1)$\Leftrightarrow$ f(x).g'(t) +c$f'(x)$g(t) $= 0 \Rightarrow \frac{cf'(x)}{f(x)} = \frac{g'(t)}{g(t)} = -k$ (k cste)

$\Rightarrow cf'(x) + k f(x) = 0$ and $g'(t) + k g(t) = 0$

1)$k = 0 \Rightarrow \begin{cases} f'(x) = 0 \Rightarrow f(x) = c_1 \\ g'(y) = 0 \Rightarrow g(y) = c_2 \end{cases}$ so u(x,y)= $c_1 + c_2 = a_1$, $a_1 \in \mathbb{R}$

2)$k \neq 0$

$cf'(x) + k\text{f}(\text{x}) = 0$ $\text{g}'(\text{t}) + \text{kg}(\text{t}) = 0$

A first-order homogeneous EDO A first-order homogeneous EDO

$\Rightarrow$solution $f(x) = a_1 e^{-\frac{k}{c}x}$ , $a_1 \in \mathbb{R}$

Solution g(t)= $a_2 e^{-kt}$ , $a_2 \in \mathbb{R}$

So u(x,t) =f(x).g(y)= $a_3 e^{-\frac{k}{c}x} e^{-kt}$ , $a_3 \in \mathbb{R}$

2) $\frac{\partial^2 u}{(\partial x)^2} + \frac{\partial^2 u}{(\partial y)^2} = 0$ (II)

u(x,y)=f(x).g(y)

(II) $\Leftrightarrow f''(x)\text{g}(\text{y}) + \text{f}(\text{x}).\text{g}''(\text{y}) = 0 \Rightarrow -\frac{f''(x)}{\text{f}(\text{x})} = \frac{\text{g}''(\text{y})}{\text{g}(\text{y})} = -k$ (k cste)

$\Rightarrow -f''(x) + k\text{f}(\text{x}) = 0$ and $\text{g}''(\text{y}) + \text{kg}(\text{y}) = 0$

1) $k = 0 \Rightarrow \begin{cases} f''(x) = 0 \Rightarrow \frac{\partial^2 f(x)}{(\partial x)^2} = 0 \Rightarrow \frac{\partial f(x)}{\partial x} = c_1 \Rightarrow f(x) = c_1 x + c_2 \\ \text{g}''(\text{y}) = 0 \Rightarrow \frac{\partial^2 g(y)}{(\partial y)^2} = 0 \Rightarrow \frac{\partial g(y)}{\partial y} = c_3 \Rightarrow g(y) = c_3 y + c_4 \end{cases}$

So u(x,y)=f(x).g(y)= $(c_1 x + c_2)(c_3 y + c_4) \Rightarrow$, $c_i \in \mathbb{R}$

2) $k < 0 \Rightarrow k = -\alpha^2$

-$f''(x) - \alpha^2\text{f}(\text{x}) = 0$ $\text{g}''(\text{y}) - \alpha^2\text{g}(\text{y}) = 0$

A second-order homogeneous EDO A second-order homogeneous EDO

$-r^2 - \alpha^2$=0 (CE) $r^2 - \alpha^2$=0 (CE)

Two complex roots $r_1 = i\alpha$ , $r_2 = -i\alpha$ Two real roots $r_1 = \alpha$ , $r_2 = -\alpha$

solution $f(x) = k_1\cos(\alpha x) + k_2 sin(\alpha x)$ solution $g(y) = k_3 e^{\alpha y} + k_4 e^{-\alpha y}$ ,

$k_1, k_2 \in \mathbb{R}$ $k_3, k_4 \in \mathbb{R}$

So u(x,y)=f(x).g(y)= ($k_1\cos(\alpha x) + k_2 sin(\alpha x)$ )( $k_3 e^{\alpha y} + k_3 e^{-\alpha y}$), $k_i \in \mathbb{R}$

3) $k > 0 \Rightarrow k = \alpha^2$

$-f''(x) + \alpha^2\text{f}(\text{x}) = 0$ $\text{g}''(\text{y}) + \alpha^2\text{g}(\text{y}) = 0$

A second-order homogeneous EDO A second-order homogeneous EDO

$-r^2 + \alpha^2$=0 (CE) $r^2 + \alpha^2$=0 (CE)

Two real roots $r_1 = \alpha$ , $r_2 = -\alpha$ Two complex roots $r_1 = -\alpha i$ , $r_2 = \alpha i$

solution $f(x) = k_4 e^{\alpha x} + k_5 e^{-\alpha x}$ , solution $g(y) = k_6 cos\alpha y + k_7 sin\alpha y$ ,

$k_4, k_5 \in \mathbb{R}$ $k_6, k_7 \in \mathbb{R}$

So u(x,y) =f(x).g(y)= $(k_4 e^{\alpha x} + k_5 e^{-\alpha x})(k_6 cos\alpha y + k_7 sin\alpha y)$, $k_i \in \mathbb{R}$

# Chapter IV
# Series

## IV.1. Infinite series:

### IV.1.1 Definition of an infinite series

Let $(U_n)_{n\geq 1}$be a sequence of real numbers so

$U_1+ U_2+ U_3+$……………..=$\sum_{n=1}^{\infty} U_n$is called an infinite series or more simply just a series.

In other words an infinite series is an infinite sum of elements of a sequence or roughly speaking it is an infinite sum of real numbers.

$U_1, U_2, U_3,$………………..are called terms of the series$\sum_{n=1}^{\infty} U_n$

**Notes:**

-If $(U_n)_{n\geq 0}$ then $\sum_{n=0}^{\infty} U_n$ is the associate series to this sequence.

If $(U_n)_{n\geq 3}$ then $\sum_{n=3}^{\infty} U_n$ is the associate series to this sequence.

-The goal of chapter is to understand the meaning of such an infinite sum and to develop methods to calculate it.

Examples:

1 .Let (1, 2, 3……….) be a sequence, one can write I tas :

$U_n$= n $\forall n \geq 1$ so the associate series is$\sum_{n=1}^{\infty} n$

2. Let (0,-1,-2,-3……….) be a sequence which can be defined also as follows:

$U_n$= -n $\forall n \geq 0$ so the associate series is $\sum_{n=0}^{\infty} -n$

3. The general term of the following sequence (-6,-9,-12,-15………) is

$U_n$= -3n $\forall n \geq 2$ so the associate series is $\sum_{n=2}^{\infty} -3n$

**Note:**

The general term is usually given, so it is easy to write the series:

Example :

$U_n = \frac{1}{n}$ $\forall n \geq 2$ so the associate series is $\sum_{n=1}^{\infty} \frac{1}{n}$

### IV.1.2 Convergence of an infinite series:

Let$\sum_{n=1}^{\infty} U_n$be a series.

We build a sequence$(S_n)_{n\geq 1}$ as follows:

$S_1 = U_1$

$S_2 = U_1 + U_2$

$S_3 = U_1 + U_2 + U_3$

$S_n = U_1 + U_2 + \ldots\ldots U_n$

This sequence$(S_n)_{n\geq 1}$is called the sequence of partial sums of the series $\sum_{n=1}^{\infty} U_n$.

## IV.1.2.1 Definition:

The series $\sum_{n=1}^{\infty} U_n$ converges if and only if the sequence $(S_n)_{n\geq 1}$ converges, and otherwise the series diverges.

That is, $\sum_{n=1}^{\infty} U_n$ converges $\Leftrightarrow \lim\limits_{n\to+\infty} S_n = S$  S is a finite number

In this case we write: $\sum_{n=1}^{\infty} U_n = S$ and S is called sum of the series $\sum_{n=1}^{\infty} U_n$

**Note:**

If $\lim\limits_{n\to+\infty} S_n = \begin{cases} +\infty \\ or \\ -\infty \\ or \\ 2\ limits \end{cases}$ then series $\sum_{n=1}^{\infty} U_n$ diverges

and $\sum_{n=1}^{\infty} U_n = \begin{cases} +\infty \\ or \\ -\infty \\ or \\ does\ not\ exist \end{cases}$

Examples:

1) $\sum_{n=1}^{\infty}(1)^n$

Let $(S_n)_{n\geq 1}$ be its partial sums.

$S_1 = U_1 = 1$

$S_2 = U_1 + U_2 = 1+1$

$S_3 = U_1 + U_2 + U_3 = 1+1+1$

$S_n = U_1 + U_2 + \ldots\ldots U_n = 1+1+1\ldots\ldots\ldots\ldots+1 = n \Rightarrow \lim\limits_{n\to+\infty} S_n = \lim\limits_{n\to+\infty} n = +\infty$

Then series $\sum_{n=1}^{\infty}(1)^n$ diverges and $\sum_{n=1}^{\infty}(1)^n = +\infty$

2) $\sum_{n=1}^{\infty}(-1)^n$

$S_1 = U_1 = -1$

$S_2 = U_1 + U_2 = -1+1=0$

$S_3 = U_1 + U_2 + U_3 = -1+1-1 = -1$

$\Rightarrow \lim\limits_{n\to+\infty} S_n = \begin{cases} -1\ if\ n\ is\ odd \\ or \\ 0\ if\ n\ is\ even \end{cases}$ hence series $\sum_{n=1}^{\infty}(-1)^n$ diverges.

3) $\sum_{n=0}^{\infty}(\frac{1}{8})^n$

Note:

$U_n = (\frac{1}{8})^n$ is a geometric sequence with a common ratio $r = \frac{1}{8}$

so Sn=$U_0 + U_1 + U_2 + \ldots + U_n = 1 + \frac{1}{8} + (\frac{1}{8})^2 + \ldots + (\frac{1}{8})^n$ is the finite sum of first terms of geometric sequencethat we know how to compute it:

Sn=$U_0(\frac{1-r^{n+1}}{1-r})$

$=1(\frac{1-(\frac{1}{8})^{n+1}}{1-\frac{1}{8}})$

$$\lim_{n\to+\infty} S_n = \lim_{n\to+\infty} (\frac{1-(\frac{1}{8})^{n+1}}{1-\frac{1}{8}})$$

Since $\lim_{n\to+\infty} (\frac{1}{8})^{n+1} = 0 \Rightarrow \lim_{n\to+\infty} S_n = \frac{1}{1-\frac{1}{8}} = \frac{8}{7}$

So the geometric series$\sum_{n=0}^{\infty}(\frac{1}{8})^n$ converges and$\sum_{n=0}^{\infty}(\frac{1}{8})^n=\frac{8}{7}$

## IV.1.2.2 Generalization about geometric series:

$\sum_{n=0}^{\infty}(q)^n$is ageometric series with a common ratio q ( q a real constant)

$\sum_{n=0}^{\infty}(q)^n$=1+q+$q^2$+……....

Its partial sums $S_n$ =1+q+$q^2$+……....+$q^n$=$U_0(\frac{1-q^{n+1}}{1-q})$ si q$\neq 1$

where $U_0 = 1$

$$\lim_{n\to+\infty} S_n = \lim_{n\to+\infty} (\frac{1-q^{n+1}}{1-q}) \text{ si } q\neq 1$$

a) if $-1< q <1$

$$\lim_{n\to+\infty} q^{n+1}=0 \Rightarrow \lim_{n\to+\infty} S_n = \lim_{n\to+\infty} \left(\frac{1-q^{n+1}}{1-q}\right) = \frac{1}{1-q}$$

b) if q$> 1$

$$\lim_{n\to+\infty} q^{n+1}=+\infty \Rightarrow \lim_{n\to \lim_{n\to+\infty} +\infty} S_n = \lim_{n\to+\infty} \left(\frac{1-q^{n+1}}{1-q}\right)=-\infty$$

c) if q$< -1$

$$\lim_{n\to+\infty} q^{n+1} = \begin{cases} +\infty & \text{if n is even} \\ -\infty & if \text{ n is odd} \end{cases} \Rightarrow \lim_{n\to \lim_{n\to+\infty} +\infty} S_n = \lim_{n\to+\infty} \left(\frac{1-q^{n+1}}{1-q}\right)$$

$$= \begin{cases} +\infty & \text{if n is even} \\ -\infty & if \text{ n is odd} \end{cases}$$

d) if q=1

$$\lim_{n\to+\infty} S_n = \lim_{n\to+\infty} 1+q+q^2+\ldots+q^n = \lim_{n\to+\infty} 1+1+1+\cdots\ldots+1$$

$$= \lim_{n\to+\infty} n+1 = +\infty$$

e) Si q=-1

$\lim_{n\to+\infty} S_n = \lim_{n\to+\infty} 1+q+q^2+\dots\dots+q^n$

$$= \lim_{n\to+\infty} 1-1+1+\cdots\ldots+(-1)^n = \begin{cases} 1 & \text{if n is even} \\ 0 & \text{if n is odd} \end{cases}$$

To summarize:

Geometric series $\sum_{n=0}^{\infty}(q)^n$ converges if $-1< q <1$ and its sum $S=\sum_{n=0}^{\infty}(q)^n = \frac{1}{1-q}$

Geometric series $\sum_{n=0}^{\infty}(q)^n$ diverges if $q\leq -1$ or $q \geq 1$.

The above Example:

$\sum_{n=0}^{\infty}(\frac{1}{8})^n$ is a geometric series with a common ratio $q=\frac{1}{8}$ ; $-1< \frac{1}{8} <1$ converges and its sum $S=\sum_{n=0}^{\infty}(\frac{1}{8})^n = \frac{1}{1-\frac{1}{8}}=\frac{8}{7}$

Example:

$\sum_{n=0}^{\infty}(\frac{3}{2})^n$ is a geometric series with a common ratio $q=\frac{3}{2}$; $\frac{3}{2} \geq 1$ so this series diverges and $\sum_{n=0}^{\infty}(\frac{3}{2})^n=+\infty$

### IV.1.2.3 Properties of convergent series

Proposition 1:

if $\sum_{n=1}^{\infty} U_n$ converges $\Rightarrow \lim_{n\to+\infty} U_n = 0$

Proof :

$S_n= U_1+U_2+\dots\dots U_n$

$S_{n-1}= U_1+U_2+\dots\dots U_{n-1}$

$S_n- S_{n-1}= U_n$

If $\sum_{n=1}^{\infty} U_n$ converges $\Rightarrow \begin{cases} \lim_{n\to+\infty} S_n = S \\ \lim_{n\to+\infty} S_{n-1} = S \end{cases}$

$$\Rightarrow \lim_{n\to+\infty}(S_n - S_{n-1}) = \lim_{n\to+\infty} S_n - \lim_{n\to+\infty} S_{n-1} = S - S = 0$$

The contrapositive of this theorem is called the divergence test:

If $\lim_{n\to+\infty} U_n \neq 0 \Rightarrow \sum_{n=1}^{\infty} U_n$ diverges

In other words, if $\lim_{n\to+\infty} U_n$ of a given series does not exist or has a value other than zero, the series diverges.

This test (the divergence test) is easy to apply, can save us a lot of time and guesswork that is why it is used a lot.

Note well that the converse is not true: if $\lim_{n\to+\infty} U_n = 0$ then the series does not necessarily converge (It may converge or it may diverge)

Example1 :

$$\sum_{n=0}^{\infty}(1)^n$$

$\lim_{n\to+\infty} (1)^n = 1 \neq 0 \Rightarrow \sum_{n=1}^{\infty}(1)^n$ is divergent.

Example2 :

$$\sum_{n=0}^{\infty}(-1)^n$$

$\lim_{n\to+\infty} (-1)^n = \begin{cases} 1 & \text{if n is even} \\ -1 & \text{if n is odd} \end{cases} \Rightarrow \sum_{n=1}^{\infty}(-1)^n$ is divergent.

Example3 :

$$\sum_{n=1}^{\infty}\left(2+\frac{1}{n}\right)$$

$\lim_{n\to+\infty} \left(2+\frac{1}{n}\right) = 2 \neq 0 \Rightarrow \sum_{n=1}^{\infty}\left(2+\frac{1}{n}\right)$ is divergent.

Example4 :

$$\sum_{n=0}^{\infty} -n$$

$\lim_{n\to+\infty} -n = -\infty \neq 0 \Rightarrow \sum_{n=1}^{\infty} -n$ diverges.

Proposition 2 :

Let $\sum_{n=0}^{\infty} U_n$ , $\sum_{n=p}^{\infty} U_n$ be two series that are different just by a finite number of terms.

If $\sum_{n=0}^{\infty} U_n$ is convergent and its sum is S $\Rightarrow \sum_{n=p}^{\infty} U_n$ is convergent

and $\sum_{n=p}^{\infty} U_n = S - \left(U_0 + U_1 + \cdots . + U_{p-1}\right)$

Example:

$\sum_{n=0}^{\infty}(\frac{1}{2})^n$ is a geometric series with a common ratio q=$\frac{1}{2}$ ; -1< $\frac{1}{2}$ <1 hence it is convergent and its sum S=$\sum_{n=0}^{\infty}(\frac{1}{2})^n = \frac{1}{1-\frac{1}{2}}$=2

So $\sum_{n=3}^{\infty}\left(\frac{1}{2}\right)^n$is also convergentand its sumS=$\sum_{n=3}^{\infty}\left(\frac{1}{2}\right)^n = 2-\left(1+\frac{1}{2}+\frac{1}{4}\right)$

=$2-\left(\frac{7}{4}\right) = \frac{1}{4}$

**Proposition 3:**

If $\sum_{n=0}^{\infty} U_n$ $and$ $\sum_{n=0}^{\infty} V_n$are convergent then$\sum_{n=0}^{\infty}(U_n + V_n)$is convergent.

If $\sum_{n=0}^{\infty} U_n$is convergentthen$\sum_{n=0}^{\infty} k.U_n$ is convergent$\forall\ k \in R$

And $\sum_{n=0}^{\infty} k.U_n = k\sum_{n=0}^{\infty} U_n$

Example:

We know that $\sum_{n=0}^{\infty}(\frac{1}{2})^n$is a convergent series$\Rightarrow \sum_{n=0}^{\infty} 3(\frac{1}{2})^n$ converges

And $\sum_{n=0}^{\infty} 3\left(\frac{1}{2}\right)^n = 3\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n$

=3x2

=6

## IV.1.3 P-Series:

### IV.1.3.1 Definition:

Are series of the form $\sum_{n=1}^{\infty}\frac{1}{n^p}$, p is a real constant.

**Examples:**

$\sum_{n=1}^{\infty}\frac{1}{n}$ (p=1), $\sum_{n=1}^{\infty}\frac{1}{n^2}$ (p=2) ,$\sum_{n=1}^{\infty}\frac{1}{n^3}$ $(p = 3)$, $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ (p=$\frac{1}{2}$), $\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ (p=$\frac{3}{2}$) are P-séries.

II.1.3 .2 Convergence of P-series:

$\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges if p$> 1$.

$\sum_{n=1}^{\infty}\frac{1}{n^p}$ diverges if p $\leq 1$.

Example1 :

$\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent because p=1$\leq 1$

Example2:

$\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges since p=2$> 1$.

**Example3 :**

$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges because p=$\frac{1}{2} \leq 1$

## IV.1.4 Abel's Test:

Let $\sum_{n=1}^{\infty} a_n . b_n$an infinite series such as:

1) $(a_n)_n$is an increasing sequence that converges to 0 ;$a_n \in \mathbb{R}^+ \ \forall n.$
2) $\exists M \in \mathbb{R}^+ , \forall p \in N \left|\sum_{n=1}^{p} b_n\right| < M$

Then $\sum_{n=1}^{\infty} a_n . b_n$ is convergent.

Example :

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

Let us choose $a_n = \frac{1}{n}$and $b_n = (-1)^n$ and check if Abel's test is satisfied:

We have n+1≥ $n \Rightarrow \frac{1}{n+1} \leq \frac{1}{n} \Rightarrow a_{n+1} \leq a_n$thus $(a_n)_n$is an increasing sequence.

$$\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} \frac{1}{n} = 0$$

Condition 1) is satisfied.

Let us find M such as$\forall p \in N \left|\sum_{n=1}^{p} (-1)^n\right| < M$ :

P=1 $|\sum_{n=1}^{1} (-1)^n| = 1$

P=2 $|\sum_{n=1}^{2} (-1)^n|=|-1+1| = 0$

P=3 $\sum_{n=1}^{3} (-1)^n=|-1+1-1| = 1$

P=4 $\sum_{n=1}^{4} (-1)^n=|-1+1-1+1| = 0$

So we have found M=2 such as$\forall p \in N \left|\sum_{n=1}^{p} (-1)^n\right| < 2$ Condition 2) is satisfied.

**Conclusion :**

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$is a convergent series according to Abel's Test.

## IV.1.5 Series with positive terms:

are series $\sum_{n=1}^{\infty} a_n$such that $a_n \geq 0 \ \forall n.$

Examples:

$\sum_{n=1}^{\infty} \frac{1}{n}$ , $\sum_{n=1}^{\infty} n$ , $\sum_{n=1}^{\infty} e^n$

### IV.1.5.1 Comparison Test:

Let$\sum_{n=1}^{\infty} a_n$be a series with positive terms.

1) If $\exists \sum_{n=1}^{\infty} b_n$a series with positive terms that is convergent such that :

$$a_n \leq b_n \forall n \geq n_0$$

Then $\sum_{n=1}^{\infty} a_n$ is convergent.

2) If $\exists \sum_{n=1}^{\infty} c_n$a series with positive terms that is divergent such that:

$$a_n \geq c_n \forall n \geq n_0$$

then $\sum_{n=1}^{\infty} a_n$is divergent.

**Example1:**

Let be the following series $\sum_{n=1}^{\infty} \frac{1}{2+n^2}$

We are going to determine whether this series converges or diverges:

We have $2 + n^2 \geq n^2 \quad \forall n \Rightarrow \frac{1}{2+n^2} \leq \frac{1}{n^2} \quad \forall n$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$is a P-Series ( p=2 $>$ 1) that converges so $\sum_{n=1}^{\infty} \frac{1}{2+n^2}$ converges according to comparison test 1).

**Example2:**

$$\sum_{n=1}^{\infty} \frac{5^n + 4}{2^n}$$

$5^n + 4 > 0 \; and 2^n > 0 \; \forall n > 1 \Rightarrow \frac{5^n+4}{2^n} > 0 \; \forall n > 1$ so

$\sum_{n=1}^{\infty} \frac{5^n+4}{2^n}$ is a series with positive terms.

We have $5^n + 4 \geq 5^n \quad \forall n \Rightarrow \frac{5^n+4}{2^n} \geq \frac{5^n}{2^n} \quad \forall n$

$\sum_{n=1}^{\infty} \frac{5^n}{2^n} = \sum_{n=1}^{\infty} (\frac{5}{2})^n$ is a geometric series with a common ratio q= $\frac{5}{2}$ >1 that diverges hence

$\sum_{n=1}^{\infty} \frac{5^n+4}{2^n}$ is divergent according to comparison test 2).

So the general approach is this: If you believe that a new series is convergent, attempt to find a convergent series whose terms are larger than the terms of the new series; if you believe that a new series is divergent, attempt to find a divergent series whose terms are smaller than the terms of the new series.

## IV.1.5.2 Cauchy Root Test:

Let $\sum_{n=1}^{\infty} a_n$be a series with positive terms.

We suppose that $\lim\limits_{n \to +\infty} \sqrt[n]{a_n} = L$

1) if $L < 1$then series $\sum_{n=1}^{\infty} a_n$is convergent.
2) If $L > 1$ then series $\sum_{n=1}^{\infty} a_n$is divergent.

**Note :**

If L=1, use another test.

**Example1 :**

$$\sum_{n=1}^{\infty} \frac{1}{n^n}$$

$n^n > 0\ \forall n > 1 \Rightarrow \frac{1}{n^n} > 0\ \forall n > 1 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^n}$is a series with positive terms.

$\lim_{n\to+\infty} \sqrt[n]{a_n} = \lim_{n\to+\infty} \sqrt[n]{\frac{1}{n^n}} = \lim_{n\to+\infty} \frac{1}{n} = 0 = L < 1 \quad \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^n}$ is a convergent series according to Cauchy Root test 1).

**Example2 :**

$$\sum_{n=0}^{\infty} 2^n$$

$2^n > 0\ \forall n > 0 \Rightarrow \sum_{n=0}^{\infty} 2^n$is a series with positive terms.

$\lim_{n\to+\infty} \sqrt[n]{a_n} = \lim_{n\to+\infty} \sqrt[n]{2^n} = \lim_{n\to+\infty} 2 = 2 = L > 1 \quad \Rightarrow \sum_{n=0}^{\infty} 2^n$ is a divergent series according to Root test 2)

When an contains power of n, as in the above examples, the root test is often useful.

**IV.1.5.3 D'Alembert's Ratio Test:**

Let $\sum_{n=1}^{\infty} a_n$be a series with positive terms.

We suppose that $\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = L$

1) if $L < 1$ then series $\sum_{n=1}^{\infty} a_n$is convergent.
2) If $L > 1$ then series $\sum_{n=1}^{\infty} a_n$is divergent.

**Note:**

If L=1, use another test.

**Example1 :**

$$\sum_{n=1}^{\infty} \frac{n}{n!}$$

$\frac{n}{n!} > 0\ \forall n > 1 \Rightarrow \sum_{n=1}^{\infty} \frac{n}{n!}$ is a series with positive terms.

$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to+\infty} \frac{n+1}{(n+1)!} \cdot \frac{n!}{n} = \lim_{n\to+\infty} \frac{1}{n} = 0 = L < 1 \quad \Rightarrow \sum_{n=1}^{\infty} \frac{n}{n!}$ is a convergent series according to Ratio Test1).

**Example2 :**

$$\sum_{n=1}^{\infty} n(\frac{5}{4})^n$$

$n(\frac{5}{4})^n > 0 \ \forall n > 1 \Rightarrow \sum_{n=1}^{\infty} n(\frac{5}{4})^n$ is a series with positive terms.

$\frac{a_{n+1}}{a_n} = \lim\limits_{n\to+\infty} \frac{(n+1)\left(\frac{5}{4}\right)^{n+1}}{n(\frac{5}{4})^n} \lim\limits_{n\to+\infty} \frac{(n+1)\left(\frac{5}{4}\right)}{n} = \frac{5}{4} = L > 1 \sum_{n=1}^{\infty} n(\frac{5}{4})^n$is a divergent series according to Ratio test 2).

When ($a_n$) contains factorials, as in the above examples, the ratio test is often useful.

### IV.1.5.4 Limit comparison Test:

Let$\sum_{n=1}^{\infty} a_n$and $\sum_{n=1}^{\infty} b_n$be series with positive terms.

If $\lim\limits_{n\to+\infty} \frac{a_n}{b_n} = L \neq \begin{cases} 0 \\ +\infty \end{cases}$ then either both series converge or both diverge.

Tip:

We can compare a series (with Positive terms) to a well known series to determine if it converges or diverges such as P-series, geometric series.

Determine if the following series converge or diverge:

Example1 :

$$\sum_{n=1}^{\infty} \frac{1}{3n^3+1}$$

$a_n = \frac{1}{3n^3+1}$, pick $b_n = \frac{1}{n^3}$

$$\lim_{n\to+\infty} \frac{a_n}{b_n} = \lim_{n\to+\infty} \frac{\frac{1}{3n^3+1}}{\frac{1}{n^3}} = \lim_{n\to+\infty} \frac{1}{3n^3+1}.\ n^3 = \lim_{n\to+\infty} \frac{n^3}{3n^3} = \frac{1}{3} = L \neq \begin{cases} 0 \\ +\infty \end{cases}$$

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is a P-series with P=3, converges $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{3n^3+1}$ converges according to limit comparison test.

Example2 :

$$\sum_{n=1}^{\infty} \frac{1}{e^n+2}$$

$a_n = \frac{1}{e^n+2}$, pick $b_n = \frac{1}{e^n}$

$$\lim_{n\to+\infty} \frac{a_n}{b_n} = \lim_{n\to+\infty} \frac{\frac{1}{e^n+2}}{\frac{1}{e^n}} = \lim_{n\to+\infty} \frac{1}{e^n+2}.\ e^n = \lim_{n\to+\infty} \frac{1}{1+\frac{2}{e^n}} = 1 \neq \begin{cases} 0 \\ +\infty \end{cases}$$

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (\frac{1}{e})^n$ is a geometric series with a common ratio q==$\frac{1}{2}$ ; -1< $\frac{1}{e}$ <1 hence it is convergent $\Rightarrow \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{e^n+2}$ converges according to limit comparison test.

## IV.1.6 Alternating series:

### IV.1.6.1 Definition:

An alternating series is of form $\sum_{n=1}^{\infty}(-1)^n a_n,\ a_n \in \mathbb{R}^+$.

$\sum_{n=1}^{\infty}(-1)^n a_n = -a_1 + a_2 - a_3 + \cdots \ldots \ldots$

That is, series with both positive and negative terms, but in a regular pattern: they alternate.

(i.e infinite series in which the signs alternate).

**Example1:**

$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\quad a_n = \frac{1}{n} \in \mathbb{R}^+$.

**Example2**:

$\sum_{n=1}^{\infty}(-1)^n n^2 \quad a_n = n^2 \in \mathbb{R}^+$.

**Example3:**

$\sum_{n=1}^{\infty}(\frac{-1}{2})^n = \sum_{n=1}^{\infty}(-1)^n.\frac{1}{2^n} \quad a_n = \frac{1}{2^n} \in \mathbb{R}^{++}$.

### IV.1.6.2 Leibnitz Test ( Alternating series Test):

Alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$ is convergent if

1) $(a_n)_n$ is a decreasing sequence.

2) $\lim\limits_{n\to+\infty} a_n = 0$.

**Example1:**

$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\ ,\ a_n = \frac{1}{n}$

We have $\frac{1}{n+1} \le \frac{1}{n}$ ie $a_{n+1} \le a_n$ ie $(a_n)_n$ is a decreasing sequence so condition 1) is satisfied.

$\lim\limits_{n\to+\infty} a_n = \lim\limits_{n\to+\infty}\frac{1}{n} = 0$ so condition 2) is satisfied.

Conclusion: the alternating series $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ is convergent.

**Example2:**

$\sum_{n=1}^{\infty}(-1)^n n^2\ ;\ a_n = n^2$

We have $(n+1)^2 \ge n^2$ ie $a_{n+1} \ge a_n$ ie $(a_n)_n$ is not a decreasing sequence so condition 1) is not satisfied.

Conclusion: The alternating series $\sum_{n=1}^{\infty}(-1)^n n^2$ is divergent since one of the condition is not satisfied.

**Example3:**

$\sum_{n=1}^{\infty}(\frac{-1}{2})^n = \sum_{n=1}^{\infty}(-1)^n\frac{1}{2^n}\ ,\ a_n = \frac{1}{2^n}$

$2^{n+1} \geq 2^n \Rightarrow \frac{1}{2^{n+1}} \leq \frac{1}{2^n} ie a_{n+1} \leq a_n (a_n)_n$is a decreasing sequence so condition 1) is satisfied.

$\lim\limits_{n\to+\infty} a_n = \lim\limits_{n\to+\infty} \frac{1}{2^n} = 0$ so condition 2) is satisfied.

Conclusion: the alternating series $\sum_{n=1}^{\infty}(\frac{-1}{2})^n$ is convergent.

## IV.1.6.7 Absolute Convergence of a series:

### IV.1.6.7. 1 Definition:

An infinite series is absolutely convergent if the absolute values of its terms form a convergent series.

That is, $\sum_{n=1}^{\infty} a_n$ converges absolutely if $\sum_{n=1}^{\infty}|a_n|$ converges.

**Example 1:**

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

We have $\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n^2}\right| = \sum_{n=1}^{\infty}\frac{1}{n^2}$ and we know that $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a P-series (P= 2> 1) converges $\Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$ converges absolutely.

**Example 2 :**

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

$\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n}$ and since $\sum_{n=1}^{\infty}\frac{1}{n}$is a P-series (P= =1$\leq$ 1so it diverges $\Rightarrow$ $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$ does not converge absolutely.

### IV.1.6.7.2 Theorem 4:

If a series converges absolutely then its converges (see example1).

The contrapositive is not true (see example 2)

If it converges, but not absolutely, it is termed conditionally convergent (such as example 2).

## Exercises of infinites series

**Exercise 1:**

Identify geometric series, P- series among the following infinite series and State whether these series converge or diverge and evaluate their sum**:**

1- $\sum_{n=0}^{\infty}(-5)^n$ 2-$\sum_{n=1}^{\infty}(n^4-2)$ 3- $\sum_{n=1}^{\infty} n^{-7}$

4-$\sum_{n=0}^{\infty}(-\frac{2}{15})^n$ $5-\sum_{n=1}^{\infty}(\frac{3}{n})^n$ 6-$\sum_{0}^{\infty}\frac{5n^5+n}{25n^5+2}$

**Exercise 2:**

Suppose that:$\sum_{n=1}^{\infty} a_n = 3$,$\sum_{n=1}^{\infty} b_n = -3$,$a_1 = 4$ and $b_1 = -5$. Compute the sum of the following series:

$$\sum_{n=1}^{\infty}(a_n + 2b_n)\sum_{n=2}^{\infty}(a_n - b_n)\sum_{n=1}^{\infty}(a_{n+1} + b_{n+1})$$

**Exercise 3:**

Use the sequence of partial sums $(\boldsymbol{S_n})$ to determine whether the following series converge or diverges, find the exact value of their sum S:

$\sum_{n=1}^{\infty}(1-(-1)^n)\sum_{n=1}^{\infty}(\frac{1}{n}-\frac{1}{n+1})$ 2) $\sum_{n=1}^{\infty}\ln(\frac{n}{n+1})$

**Exercise 4:**

**1-** State whether these series converge or diverge and evaluate their sum**:**

2- $\sum_{n=1}^{\infty} 2n^{-10}$ 2-$\sum_{n=1}^{\infty}(n^4+3n)^n$ 3 –$\sum_{0}^{\infty}\frac{9n^3-4}{n^3+1}$

4 – $\sum_{n=0}^{\infty}(n+4)!$ 5 – $\sum_{n=1}^{\infty}\frac{1}{n^5+6n^3}$

6- $\sum_{n=0}^{\infty}(\frac{5n^5+n}{25n^5+2})^n$ 7-$\sum_{n=1}^{\infty}\frac{n^2+1}{n^2 lnn}$

**Exercise 5:**

Use the Limit Comparison Test to determine whether the following series converge or diverges and evaluate their sum S.

1-$\sum_{n=1}^{\infty}\frac{n+5}{n^4+2n}$ 2- $\sum_{n=1}^{\infty}\frac{11}{2^n+6}$

**Exercise 6 :**

State whether the following series converge or converge absolutely:

1- $\sum_{n=1}^{\infty}(-1)^n e^{-2n}$ 2- $\sum_{n=1}^{\infty}(-1)^n n^5$ 3- $\sum_{n=1}^{\infty}(-1)^n\ln(\frac{3n^2}{n^2+7})$

## Solutions of exercises of infinite series:

**Solution of exercise 1:**

1-$\sum_{n=0}^{\infty}(-5)^n$is a geometric series because of form$\sum_{n=0}^{\infty} q^n$(q=cste) where q=-5$\leq -1$thus$\sum_{n=0}^{\infty}(-5)^n$ is divergente and its sum

S= $\sum_{n=0}^{\infty}(-5)^n$=$\begin{cases}+\infty & \text{if n is even}\\ -\infty & \text{if n is odd}\end{cases}$

2- $\sum_{n=1}^{\infty}(n^4-2)$it is neither a geometric series nora P-series.

3- $\sum_{n=1}^{\infty} n^{-7}\sum_{n=1}^{\infty}\frac{1}{n^7}$ is a P-series (of form$\sum_{n=1}^{\infty}\frac{1}{n^p}$) where p=7 $>1$ thus $\sum_{n=1}^{\infty} n^{-7}$ is convergent and its sum S=$\sum_{n=1}^{\infty} n^{-7}$ =A a finite number that exists but we cannot compute it.

4-$\sum_{n=0}^{\infty}(-\frac{2}{15})^n$ is a geometric series with common ratio q=$-\frac{2}{15}$ -1$< q = -\frac{2}{15}$<1 so this series converges and its sum S=$\sum_{n=0}^{\infty}(-\frac{2}{15})^n = \frac{1}{1-q} = \frac{1}{1-(-\frac{2}{15})} = \frac{15}{17}$

5- $\sum_{n=1}^{\infty}(\frac{3}{n})^n$it is not a geometric series because ($\frac{3}{n}$ depends on n, it is not constant).

6-$\sum_{0}^{\infty}\frac{5n^5+n}{25n^5+2}$) it is neither a geometric series nor a P-series.

Solution of exercise2:

We have $\sum_{n=1}^{\infty} a_n = 3, \sum_{n=1}^{\infty} b_n = -3 \Rightarrow both\ series\ are$ convergent.

And we know

If $\sum_{n=0}^{\infty} U_n\ and \sum_{n=0}^{\infty} V_n$ are convergent then $\sum_{n=0}^{\infty}(U_n + V_n)$ is convergent.

If $\sum_{n=0}^{\infty} U_n$ is convergent then $\sum_{n=0}^{\infty} k.U_n$ is convergent $\forall\ k \in R$

And $\sum_{n=0}^{\infty} k.U_n = k\sum_{n=0}^{\infty} U_n$

So

1)$\sum_{n=1}^{\infty}(a_n + 2b_n)$ is convergent and $\sum_{n=1}^{\infty}(a_n + 2b_n) = \sum_{n=1}^{\infty} a_n + 2\sum_{n=1}^{\infty} b_n$

=3 +2(-3)=-3

2) $\sum_{n=2}^{\infty}(a_n - b_n)$is convergent and$\sum_{n=2}^{\infty}(a_n - b_n) = \sum_{n=2}^{\infty} a_n - \sum_{n=2}^{\infty} b_n$

= $(\sum_{n=1}^{\infty} a_n - a_1) - (\sum_{n=1}^{\infty} b_n - b_1)$

= (3-4) - (-3-(-5))

=-3

2)$\sum_{n=1}^{\infty}(a_{n+1} + b_{n+1})$ is also convergent and

$\sum_{n=1}^{\infty}(a_{n+1} + b_{n+1})$=$\sum_{n=2}^{\infty}(a_n + b_n)$=$(\sum_{n=1}^{\infty} a_n - a_1)$ +$(\sum_{n=1}^{\infty} b_n - b_1)$

= (3-4) + (-3-(-5))

=1

**Solution of exercise3:**

1) $\sum_{n=1}^{\infty}(\frac{1}{n} - \frac{1}{n+1}) = \sum_{n=1}^{\infty} U_n$

Let ($S_n$) be the sequence of partial sums of this series such that:

$$S_1 = U_1 = \frac{1}{1} - \frac{1}{1+1} = \frac{1}{1} - \frac{1}{2}$$

$$S_2 = U_1 + U_2 = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3}$$

$$S_2 = U_1 + U_2 + U_3 = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4}$$

$$S_n = U_1 + U_2 + U_3 + \cdots . + U_n = \frac{1}{1} \cancel{- \frac{1}{2}} \cancel{+ \frac{1}{2}} \cancel{- \frac{1}{3}} \cancel{+ \frac{1}{3}} \cancel{- \frac{1}{4}} \cancel{+ \cdots} \cancel{..} \cancel{+} \frac{1}{n} - \frac{1}{n+1}$$

$=1 - \frac{1}{n+1}$

We know that :

$\sum_{n=1}^{\infty} U_n$ converges $\Leftrightarrow$ the sequence$(S_n)$ converge$s \Leftrightarrow \lim\limits_{n\to+\infty} S_{n=}A(\ a\ finite\ number)$.

$\sum_{n=1}^{\infty} U_n$ diverges $\Leftrightarrow$the sequence$(S_n)$ diverge$s \Leftrightarrow \lim\limits_{n\to+\infty} S_{n=} \begin{cases} +\infty \\ ou \\ -\infty \\ ou \\ 2\ limits \end{cases}$

In this case:

$\lim\limits_{n\to+\infty} S_{n=} \lim\limits_{n\to+\infty} 1 - \frac{1}{n+1}$=1 thus $\sum_{n=1}^{\infty}(\frac{1}{n} - \frac{1}{n+1})$ is convergent.

And $\sum_{n=1}^{\infty}(\frac{1}{n} - \frac{1}{n+1}) = 1$

2) $\sum_{n=1}^{\infty} \ln(\frac{n}{n+1}) = \sum_{n=1}^{\infty} \ln n - \ln(n+1) = \sum_{n=1}^{\infty} U_n$

Let $(S_n)$ be the sequence of partial sums of this series:

$S_{1=}U_1 = ln1 - ln2$

$S_{2=}U_1 + U_2 = ln1 - ln2 + ln2 - ln3$

$S_{n=}U_1 + U_2 + \cdots. + U_n = ln1 - ln2 + ln2 - ln3 + \cdots \ldots.. + \ln n - \ln(n+1)$

$= ln1 - \ln(n+1)$

$\lim\limits_{n\to+\infty} S_{n=} \lim\limits_{n\to+\infty} -\ln(n+1) = -\infty$

Thus $\sum_{n=1}^{\infty} \ln(\frac{n}{n+1})$ is divergent and $\sum_{n=1}^{\infty} \ln(\frac{n}{n+1}) = -\infty$

**Solution of exercise 3:**

1-$\sum_{n=1}^{\infty} 2n^{-10}$=2$\sum_{n=1}^{\infty} \frac{1}{n^{10}}$

$\sum_{n=1}^{\infty} \frac{1}{n^{10}}$ is a P-series with p=10> 1 so converges$\Rightarrow$ 2$\sum_{n=1}^{\infty} \frac{1}{n^{10}}$ stays convergent (see above)

Its sum S=$\sum_{n=1}^{\infty} \frac{2}{n^{10}}$= A a finite number that exists but we can not compute it.

2-$\sum_{n=1}^{\infty}(n^4 + 3n)^n$is a series with positive terms whose general term is a power of n consequently we apply Cauchy Root Test.

$\lim\limits_{n\to+\infty} \sqrt[n]{a_n} = \lim\limits_{n\to+\infty} \sqrt[n]{(n^4 + 3n)^n} = \lim\limits_{n\to+\infty} (n^4 + 3n) = +\infty > 1$so $\sum_{n=1}^{\infty}(n^4 + 3n)^n$ is divergent and S=$+\infty$

3 –$\sum_{0}^{\infty} \frac{9n^3-4}{n^3+1}$it is neither a geometric series nor a P-series.

Let us apply the divergence test:

$\lim\limits_{n\to+\infty} \frac{9n^3-4}{n^3+1} = \lim\limits_{n\to+\infty} \frac{9n^3}{n^3} = 9 \neq 0$ then $\sum_{0}^{\infty} \frac{n^2+2}{4n^2+1}$ is divergent.

and S=$\sum_0^{\infty}\frac{9n^3-4}{n^3+1}=+\infty$

4 – $\sum_{n=0}^{\infty}(n+4)!$ General term contains a factorial, let us use then D'Alembert ratio Test:

$$\lim_{n\to+\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to+\infty}\frac{(n+1+4)!}{(n+4)!}=\lim_{n\to+\infty}\frac{(n+5)!}{(n+4)!}=\lim_{n\to+\infty}\frac{(n+5)\cancel{(n+4)}\cancel{(n+3)}\cancel{(n+2)}......\cancel{2.1}}{\cancel{(n+4)}\cancel{(n+3)}\cancel{(n+2)}.........\cancel{2.1}}$$

$=\lim_{n\to+\infty}(n+5)$=$+\infty$ >1 thus$\sum_{n=0}^{\infty}(n+4)!$ is divergent according to D'Alembert ratio Test2) and S=$\sum_{n=0}^{\infty}(n+4)!$ =$+\infty$

5 – $\sum_{n=1}^{\infty}\frac{1}{n^5+6n^3}$

-This series is neither a geometric series nor a P-series.

- If we apply the divergence Test:

$\lim_{n\to+\infty}\frac{1}{n^5+6n^3}$ =0 then we can not say anything.

- We can not apply Cauchy Root Test because the general term $\frac{1}{n^5+6n^3}$ is not a power of n.

- If we apply d'Alembert ratio test:

$\lim_{n\to+\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to+\infty}\frac{n^5+6n^3}{(n+1)^5+6(n+1)^3}=\lim_{n\to+\infty}\frac{n^5}{n^5}=1$ then we can not say anything.

-Let us use ComparisonTest :

$n^5+6n^3>n^5\forall n\Rightarrow\frac{1}{n^5+6n^3}<\frac{1}{n^5}$

$\sum_{n=1}^{\infty}\frac{1}{n^5}$is a P-series with p=5>1 ⇒converges⇒ $\sum_{n=1}^{\infty}\frac{1}{n^5+6n^3}$is convergent by Comparison Test 1).

and S=$\sum_{n=1}^{\infty}\frac{1}{n^5+6n^3}$= k a finite number.

6- $\sum_{n=0}^{\infty}(\frac{5n^5+n}{25n^5+2})^n$is not a geometric series because q=$\frac{5n^5+n}{25n^5+2}$is not constant, q depends on n.

We notice that $(\frac{5n^5+n}{25n^5+2})^n$ is positive and is a power of n consequently we apply Cauchy Root Test.

$\lim_{n\to+\infty}\sqrt[n]{a_n}=\lim_{n\to+\infty}\sqrt[n]{(\frac{5n^5+n}{25n^5+2})^n}=\lim_{n\to+\infty}\frac{5n^5+n}{25n^5+2}=\lim_{n\to+\infty}\frac{5n^5}{25n^5}=\frac{1}{5}<1$then$\sum_{n=0}^{\infty}(\frac{5n^5+n}{25n^5+2})^n$is convergent and S=$\sum_{n=0}^{\infty}(\frac{5n^5+n}{25n^5+2})^n$=k a finite number.

7-$\sum_{n=1}^{\infty}\frac{n^2+1}{n^2 lnn}$

We do the same work as 5)

-This series is neither a geometric series nor a P-series.

- if we apply the divergence Test:

$\lim\limits_{n\to+\infty} \frac{n^2+1}{n^2 lnn} = \lim\limits_{n\to+\infty} \frac{n^2}{n^2 lnn} = \lim\limits_{n\to+\infty} \frac{1}{lnn} = 0$ , use another test

-We can not apply Cauchy Root Test because the general term $\frac{n^2+1}{n^2 lnn}$ is not a power of n.

- If we apply d'Alembert ratio test:

$\lim\limits_{n\to+\infty} \frac{\frac{(n+1)^2+1}{(n+1)^2\ln(n+1)}}{\frac{n^2+1}{n^2 lnn}} = \lim\limits_{n\to+\infty} \frac{n^2((n+1)^2+1)lnn}{(n+1)^2(n^2+1)\ln(n+1)} = \lim\limits_{n\to+\infty} \frac{n^4 lnn}{n^4\ln(n+1)} = \lim\limits_{n\to+\infty} \frac{lnn}{lnn} = 1$then use another test.

-Let us use Comparison Test :

$lnn < n\forall n \Rightarrow n^2 lnn < n^2 n \forall n \Rightarrow \frac{1}{n^2 lnn} > \frac{1}{n^3} \forall n \Rightarrow \frac{n^2+1}{n^2 lnn} > \frac{n^2+1}{n^3} > \frac{n^2}{n^3} = \frac{1}{n}$

$\sum_{n=1}^{\infty} \frac{1}{n}$ is a P-series that diverges(since p=1≤ 1) $\Rightarrow \sum_{n=1}^{\infty} \frac{n^2+1}{n^2 lnn}$ is divergent by Comparison 2).

And S=$\sum_{n=1}^{\infty} \frac{n^2+1}{n^2 lnn} = +\infty$

**Solution of exercise4*:***

1-$\sum_{n=1}^{\infty} \frac{n+5}{n^4+2n}$

$a_n = \frac{n+5}{n^4+2n}$,pick$b_n = \frac{1}{n^3}$

**Note:**

$\sum_{n=1}^{\infty} a_n$ is the series which we want to know whether it converges or diverges.

$\sum_{n=1}^{\infty} b_n$ is the series that we choose and we know whether it converges or diverges.

We know that:$\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a convergent P-series (p=3>1).

$\lim\limits_{n\to+\infty} \frac{a_n}{b_n} = \lim\limits_{n\to+\infty} \frac{\frac{n+5}{n^4+2n}}{\frac{1}{n^3}} = \lim\limits_{n\to+\infty} \frac{(n+5)n^3}{n^4+2n} = \lim\limits_{n\to+\infty} \frac{n^4}{n^4} = 1 \neq \begin{cases} 0 \\ +\infty \end{cases} \Rightarrow \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n+5}{n^4+2n}$is convergent.

2- $\sum_{n=1}^{\infty} \frac{11}{2^n+6}$

$a_n = \frac{11}{2^n+6}$, pick $b_n = \frac{1}{2^n}$

We know that:$\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} (\frac{1}{2})^n$ is a convergent geometric series (-1<q=$\frac{1}{2}$<1).

$\lim\limits_{n\to+\infty} \frac{a_n}{b_n} = \lim\limits_{n\to+\infty} \frac{\frac{11}{2^n+6}}{\frac{1}{2^n}} = \lim\limits_{n\to+\infty} \frac{11.2^n}{2^n+6} = \lim\limits_{n\to+\infty} \frac{11.2^n}{2^n} = 11 \neq \begin{cases} 0 \\ +\infty \end{cases} \Rightarrow \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{11}{2^n+6}$ is convergent.

**Solution of exercise5:**

1-$\sum_{n=1}^{\infty} (-1)^n e^{-2n}$

Absolute convergence?

$\sum_{n=1}^{\infty}|(-1)^n e^{-2n}|$ =$\sum_{n=1}^{\infty} e^{-2n} = \sum_{n=1}^{\infty}\frac{1}{e^{2n}} = \sum_{n=1}^{\infty}\left(\frac{1}{e^2}\right)^n$ is a geometric series with a common ratio q=$\frac{1}{e^2}$ -1< $q = \frac{1}{e^2}$ <1 that converges thus $\sum_{n=1}^{\infty}(-1)^n e^{-2n}$ converges absolutely ⇒$\sum_{n=1}^{\infty}(-1)^n e^{-2n}$ converges.

Note:

If we have started first by studying convergence we have two ways to do that:

a) $\sum_{n=1}^{\infty}(-1)^n e^{-2n}$ It is an alternating serieswith $a_n = e^{-2n} = \frac{1}{e^{2n}} \in \mathbb{R}^+$.

Let us check Leibnitz Test:

1) $e^{2(n+1)} \geq e^{2n} \Rightarrow \frac{1}{e^{2(n+1)}} \leq \frac{1}{e^{2n}} \Rightarrow (a_n)_n$ is a decreasing sequence.

2) $\lim\limits_{n\to+\infty} a_n = \lim\limits_{n\to+\infty} \frac{1}{e^{2n}}$ =0 ⇒conditions 1) and 2) are satisfied⇒ $\sum_{n=1}^{\infty}(-1)^n e^{-2n}$is a convergent alternating series and its sum S=$\sum_{n=1}^{\infty}(-1)^n e^{-2n}$=k a finite number.

b) $\sum_{n=1}^{\infty}(-1)^n e^{-2n}$=$\sum_{n=1}^{\infty}\left(\frac{-1}{e^2}\right)^n$is a geometric series with a common ratio q=$\frac{-1}{e^2}$ -1< $\frac{-1}{e^2}$ <1 hence it is convergent and its sum S=$\sum_{n=0}^{\infty}(\frac{-1}{e^2})^n = \frac{1}{1-(\frac{-1}{e^2})}$=$\frac{e^2}{e^2+1}$

2) $\sum_{n=1}^{\infty}(-1)^n n^5$is not a series with positive terms so we can not apply (Comparison test, Cauchy Root test, d'Alembert ratio Test orlimit comparison test). It left only divergence test or Leibnitz Test.

Divergence test:

$$\lim_{n\to+\infty} (-1)^n n^5 = \begin{cases} +\infty & if\ n\ is\ even \\ & or \\ -\infty & if\ n\ is\ odd \end{cases} \Rightarrow \exists \text{ 2 limits } \neq 0 \Rightarrow$$

$\sum_{n=1}^{\infty}(-1)^n n^5$does not converge ⇒$\sum_{n=1}^{\infty}(-1)^n n^5$does not converge absolutely.

$3 - \sum_{n=1}^{\infty}(-1)^n \ln(\frac{3n^2}{n^2+7})$ It is an alternating series with $a_n = \ln(\frac{3n^2}{n^2+7}) \in \mathbb{R}^+$.

Let us check Leibnitz Test:

$\lim\limits_{n\to+\infty} a_n = \lim\limits_{n\to+\infty} \ln(\frac{3n^2}{n^2+7}) = \lim\limits_{n\to+\infty} \ln(\frac{3n^2}{n^2})$ =ln3≠0 ⇒condition 2) is not satisfied ⇒$\sum_{n=1}^{\infty}(-1)^n \ln\left(\frac{3n^2}{n^2+7}\right)$ does not converge ⇒$\sum_{n=1}^{\infty}(-1)^n \ln\left(\frac{3n^2}{n^2+7}\right)$ does not converge absolutely.

## IV.2. Sequences and Series of functions:

### IV.2. 1 Definition of series of functions:

Let$(f_n)_{n\geq 1}$be a sequence of real functions defined as:

$f_1$:I $\longrightarrow \mathbb{R}$, $f_2$:I$\longrightarrow \mathbb{R}$ , $f_3$:I $\longrightarrow \mathbb{R}$ ……………….

x$\longrightarrow f_1(x)$ x$\longrightarrow f_2(x)$ x$\longrightarrow f_3(x)$

I is a real interval.

So$\sum_{n=1}^{\infty} f_n$is a series of functions.

In other words, a series of functions is an infinite sum of elements of a sequence of functions or simply it is an infinite sum of functions.

Examples:

1)$f_n$: [0,1] $\longrightarrow \mathbb{R}$ i.e $(f_n)_{n\geq 0}$ is a sequence of functions

x $\longrightarrow f_n(x) = x^n$

For n=0 $f_0(x) = 1$

For n=1 $f_1(x) = x$

For n=2 $f_2(x) = x^2$

$\sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} x^n$ $=1 + x+x^2 + x^3 + x^4 + \cdots$ is a series of functions.

2) $g_n: \mathbb{R} \longrightarrow \mathbb{R}$ i.e $(g_n)_{n\geq 1}$ is a sequence of functions

x$\longrightarrow g_n(x) = \frac{x}{n}$

For n=1 $g_1(x) = x$

For n=2 $g_2(x) = \frac{x}{2}$

For n=3 $g(x) = \frac{x}{3}$

$\sum_{n=1}^{\infty} g_n(x) = \sum_{n=1}^{\infty} \frac{x}{n}$ $=x + \frac{x}{2} + \frac{x}{3} + \frac{x}{4} + \cdots$ is a series of functions.

3) $h_n: \mathbb{R} \longrightarrow \mathbb{R}$ i.e $(h_n)_{n\geq 1}$ is a sequence of functions

x$\longrightarrow h_n(x) = \frac{x^2}{n}$

For n=1 $h_1(x) = x^2$

For n=2 $h_2(x) = \frac{x^2}{2}$

For n=3 $h_3(x) = \frac{x^2}{3}$

$\sum_{n=1}^{\infty} h_n(x) = \sum_{n=1}^{\infty} \frac{x^2}{n}$ $=x^2 + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$ is a series of functions.

**Note:**

For a given value $x_0$ of x, $\sum_{n=1}^{\infty} f_n(x_0)$ becomes a series of numbers or just what we usually call an infinite series (see chapter 2) .

**Examples:**

1) $\sum_{n=0}^{\infty} x^n$

For a given value $x_0$ of x: $x_0=\frac{1}{2}$ $\sum_{n=0}^{\infty} f_n(\frac{1}{2})=\sum_{n=0}^{\infty}(\frac{1}{2})^n$ is a series of numbers.

$x_0=4$ $\sum_{n=0}^{\infty} f_n(4)=\sum_{n=0}^{\infty}(4)^n$ is a series of numbers.

2) $\sum_{n=1}^{\infty} \frac{x}{n}$

For a given value $x_0$ of x: $x_0=1$ $\sum_{n=1}^{\infty} f_n(1)=\sum_{n=1}^{\infty} \frac{1}{n}$ is a series of numbers.

$x_0=2$ $\sum_{n=1}^{\infty} f_n(2)==\sum_{n=1}^{\infty} \frac{2}{n}$ is a series of numbers.

- One way to study convergence of a series of functions is to give values to x and study convergence of the corresponding series of numbers.

## IV.2. 2. Convergence of a series of functions:

We have two types of convergence for series of functions:

-Pointwise convergence.

- Uniform convergence.

## IV.2. 2.1 Pointwise convergence of a series of functions

### IV.2. 2.1 .1 Definition

Let $f_n$: I $\longrightarrow \mathbb{R}$ be a sequence of functions.

$\sum_{n=1}^{\infty} f_n(x)$ is said to converge pointwise to f at $x_0 \in$ I if the infinite series $\sum_{n=1}^{\infty} f_n(x_0)$ converges to $f(x_0)$.

Example1:

$\sum_{n=0}^{\infty} x^n$, $x \in I=[0,1]$

$x_0=\frac{1}{2}$, $\sum_{n=0}^{\infty}(\frac{1}{2})^n$ is a geometric series with a common ratio $q=\frac{1}{2}$ that converges

$\Rightarrow \sum_{n=0}^{\infty} x^n$ converges pointwise at $x_0=\frac{1}{2}$ to f ($x_0$), f a function to find.

$x_0=\frac{1}{3}$, $\sum_{n=1}^{\infty}(\frac{1}{3})^n$ is a geometric series with a common ratio $q=\frac{1}{3}$ that converges

$\Rightarrow \sum_{n=0}^{\infty} x^n$ converges pointwise at $x_0=\frac{1}{3}$ to f ($x_0$), f a function to find.

$x_0=1$, $\sum_{n=1}^{\infty}(1)^n$ is a geometric series with a common ratio $q=1$ that diverges $\Rightarrow \sum_{n=0}^{\infty} x^n$ does not converge pointwise at $x_0=1$.

Example2:

$$\sum_{n=1}^{\infty} \frac{x}{n}$$

$x_0=1$, $\sum_{n=1}^{\infty} \frac{1}{n}$ is a P-series (p=1) that diverges $\Rightarrow \sum_{n=1}^{\infty} \frac{x}{n}$ does not converge pointwise at

$x_0=1$.

$x_0=2$, $\sum_{n=1}^{\infty}\frac{2}{n}=2.\sum_{n=1}^{\infty}\frac{1}{n}$ is a series in form of k. $\sum_{n=1}^{\infty}\frac{1}{n}$(k a constant) that diverges$\Rightarrow$ $\sum_{n=1}^{\infty}\frac{x}{n}$ does not converge pointwise at $x_0=2$.

$\forall x\in\mathbb{R}^*$ $\sum_{n=1}^{\infty}\frac{x}{n}=x\sum_{n=1}^{\infty}\frac{1}{n}$ is a divergent series of function.

**Example3:**

$$\sum_{n=1}^{\infty}\frac{x^2}{n}$$

For $x_0=0$ $\sum_{n=1}^{\infty}0=0+0+0+\ldots\ldots=0\Rightarrow\sum_{n=1}^{\infty}\frac{x^2}{n}$ converges pointwise at $x_0=0$.

$x_0\neq 0$ $\sum_{n=1}^{\infty}\frac{x^2}{n}=x_0^2\sum_{n=1}^{\infty}\frac{1}{n}$ is a series in form of k.$\sum_{n=1}^{\infty}\frac{1}{n}$(k a constant) that diverges $\Rightarrow$ $\sum_{n=1}^{\infty}\frac{x^2}{n}$ does not converge pointwise at $x_0\neq 0$.

**Example4:**

$\sum_{n=1}^{\infty}\frac{n}{x}=\frac{1}{x}+\frac{2}{x}+\frac{3}{x}+\cdots\ldots.$ $x\in\mathbb{R}^*$

For $x>0$ $\lim\limits_{n\to+\infty}\frac{n}{x}=\frac{1}{x}\lim\limits_{n\to+\infty}n=+\infty\neq 0\Rightarrow\sum_{n=1}^{\infty}\frac{n}{x}$ is divergent according to divergence test.

For $x<0$ $\lim\limits_{n\to+\infty}\frac{n}{x}=\frac{1}{x}\lim\limits_{n\to+\infty}n=-\infty\neq 0\Rightarrow\sum_{n=1}^{\infty}\frac{n}{x}$ is divergent according to divergence test.

Hence, $\sum_{n=1}^{\infty}\frac{n}{x}$does not converge pointwise at x, $\forall x\in\mathbb{R}^*$.

**Example5 :**

$\sum_{n=1}^{\infty}\frac{x}{n^2}$ $x\in\mathbb{R}$.

$x_0=0$ $\sum_{n=1}^{\infty}0=0+0+0+\ldots\ldots\Rightarrow\sum_{n=0}^{\infty}\frac{x}{n^2}$ converges pointwise at $x_0=0$.

$x_0\neq 0$ $\sum_{n=1}^{\infty}\frac{x}{n^2}=x\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a series in form of k.$\sum_{n=1}^{\infty}\frac{1}{n^2}$ (k a constant)

Since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a P-series ( p=2) that converges $\Rightarrow$ k. $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges also

$\Rightarrow\sum_{n=1}^{\infty}\frac{x}{n^2}$ converges pointwise at $x_0\neq 0$.

Consequently $\sum_{n=1}^{\infty}\frac{x}{n^2}$ converges pointwise at x, $\forall x\in\mathbb{R}$ to a function f (to find).

**Example6:**

$\sum_{n=1}^{\infty}\left(\frac{x}{n}-\frac{x}{n+1}\right)$, $x\in\mathbb{R}$.

First method:

$x_0=0$ $\sum_{n=1}^{\infty}0=0+0+0+\ldots\ldots\Rightarrow\sum_{n=1}^{\infty}\left(\frac{x}{n}-\frac{x}{n+1}\right)$ converges pointwise at $x_0=0$.

For $x_0\neq 0$:

$\frac{x_0}{n} - \frac{x_0}{n+1} = \frac{(n+1)x_0 - nx_0}{n(n+1)} = \frac{nx_0 + x_0 - nx_0}{n(n+1)} = \frac{x_0}{n(n+1)}$

$\sum_{n=1}^{\infty} \frac{x_0}{n(n+1)} = x_0 . \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

We study whether $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges or diverges:

**Comparison Test:**

$$n(n+1) = n^2 + n \geq n^2 \Rightarrow \frac{1}{n^2+n} \leq \frac{1}{n^2}$$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$is a P-series ( p=2) that converges $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges according to comparison

test1) $\Rightarrow x_0 . \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges pointwise at $x_0$ ($x_0$a constant $\in \mathbb{R}^*$)

Therefore $\sum_{n=1}^{\infty} \frac{x}{n(n+1)}$ converges pointwise at $x, \forall x \in \mathbb{R}$ to f a function f to find.

Second method:

$\sum_{n=1}^{\infty} \left(\frac{x}{n} - \frac{x}{n+1}\right), x \in \mathbb{R}$.

**Sequence of partial sums:**

$S_1(x) = x - \frac{x}{2}$

$S_2(x) = x - \frac{x}{2} + \frac{x}{2} - \frac{x}{3}$

$S_3(x) = x - \frac{x}{2} + \frac{x}{2} - \frac{x}{3} + \frac{x}{3} - \frac{x}{4}$

$S_n(x) = x \cancel{- \frac{x}{2}} \cancel{+ \frac{x}{2}} \cancel{- \frac{x}{3}} \cancel{+ \frac{x}{3}} \cancel{- \frac{x}{4}} + \cdots \ldots \ldots \ldots . \cancel{- \frac{x}{n-1}} + \cancel{\frac{x}{n-1}} - \frac{x}{n} \Rightarrow S_n(x) = x - \frac{x}{n}$

Therefore $\lim_{n \to +\infty} S_n(x) = \lim_{n \to +\infty} x - \frac{x}{n} = x$

We can conclude $\sum_{n=1}^{\infty} (\frac{x}{n} - \frac{x}{n+1})$converges pointwise at $x, \forall x \in \mathbb{R}$ to f(x)= x.

**Notes:**

-In the last example, we were able to find the function f comparing to the example 5 where we know just there exists.

-A convergent infinite series is equal to a number while a convergent series of function is equal to a function

## IV.2. 2.1.2 Domain of convergence D:

Let $\sum_{n=1}^{\infty} f_n(x)$ be a series of function defined on I:

$$f_n : I \longrightarrow \mathbb{R}$$

D=$\{x_0 \in I \text{ so that} \sum_{n=1}^{\infty} f_n(x_0)$ converges pointwise$\}$is called the domain of convergence.

i.e D is the set of those values $x_0$ for which the series $\sum_{n=1}^{\infty} f_n(x_0)$ is convergent.

We are going to determine D for all the examples above.

**Example1 :**

1) $\sum_{n=0}^{\infty}(x_0)^n$, $x_0 \in I=[0,1]$

We have seen for every $x_0 \in I=[0,1[$ $\sum_{n=0}^{\infty}(x_0)^n$ is a geometric series with a common ratio $x_0$, $0 \leq x_0<1 \Rightarrow \sum_{n=0}^{\infty}(x_0)^n$ converges$\Rightarrow$D =[0,1[=I.

**Note:**

$\sum_{n=0}^{\infty}(x_0)^n$is a geometric series that converges to $f(x_0)=\frac{1}{1-x_0}$ (see chapter 2)

Therefore $\sum_{n=0}^{\infty}(x)^n$converges pointwise to $f(x)=\frac{1}{1-x}$, $x \in I=[0,1[$

This is another example where the function f is determined.

Example2:

$\sum_{n=1}^{\infty}\frac{x}{n}$, $x \in \mathbb{R}$.

**We have found that:**

$\forall x \in \mathbb{R}^*$ $\sum_{n=1}^{\infty}\frac{x}{n}$ is a divergent series of function.

For $x_0 =0$ $\sum_{n=1}^{\infty}\frac{x}{n} = \sum_{n=1}^{\infty} 0 =0$ is convergent

Thus D={0}

**Example3:**

$$\sum_{n=1}^{\infty}\frac{x^2}{n}$$

We have found:

For $x_0 \neq 0$ $\sum_{n=1}^{\infty}\frac{x^2}{n}$ is divergent.

For $x_0 =0$ $\sum_{n=1}^{\infty}\frac{x^2}{n}$ is convergent.

$\Rightarrow$ D={0}

**Example4:**

$\sum_{n=1}^{\infty}\frac{n}{x}$, $x \in \mathbb{R}^*$.

We have seen $\sum_{n=1}^{\infty}\frac{n}{x}$does not converge pointwise at x, $\forall x \in \mathbb{R}^*$.

$\Rightarrow$ D=$\emptyset$

**Example5:**

$\sum_{n=1}^{\infty}\frac{x}{n^2} x \in \mathbb{R}$.

We have found $\sum_{n=0}^{\infty}\frac{x}{n^2}$converges pointwise at x , $\forall x \in \mathbb{R}$.

$\Rightarrow$ D=$\mathbb{R}$

**Examples 6:**

We have found $\sum_{n=1}^{\infty}\left(\frac{x}{n}-\frac{x}{n+1}\right)$ converges pointwise at $x, \forall x \in \mathbb{R}$.

$\Rightarrow$ D=$\mathbb{R}$

## IV.2. 2.2 Uniform convergences of a series of functions

$\sum_{n=1}^{\infty} f_n$ is said to converge uniformly to f on I if:

$$\lim_{n\to+\infty} sup_{x\in I}\,|S_n(x)-f(x)|=0$$

Sup is the maximal value of $|S_n(x)-f(x)|$ determined on I.

$(S_n(x))_n$ is the sequence of partial sums of $\sum_{n=1}^{\infty} f_n$ i.e

$S_n(x)$=$f_1(x)+f_2(x)+f(x)+\cdots \ldots .+f_n(x)$

**Note:**

The definition of the uniform convergence requires first to compute $S_n(x)$ and to have expression of $f(x)$.

**Example1:**

We have seen $\sum_{n=0}^{\infty}(x)^n$ converges pointwise to f(x)=$\frac{1}{1-x}$ $x$ ∈I=[0,1[

Let us compute $S_n(x)$:

$S_n(x)$=1+x+x$^2$+………+x$^n$ = 1. $\left(\frac{1-x^{n+1}}{1-x}\right)$

$\lim_{n\to+\infty} sup_{x\in[0,1[}\,|S_n(x)-f(x)| = \lim_{n\to+\infty} sup_{x\in[0,1[}\left|\frac{1-x^{n+1}}{1-x}-\frac{1}{1-x}\right| = \lim_{n\to+\infty} sup_{x\in[0,1[}\left|\frac{1-x^{n+1}-1}{1-x}\right| = \lim_{n\to+\infty} sup\left|\frac{x^{n+1}}{1-x}\right|$

Put $g_n(x)$= $\left|\frac{x^{n+1}}{1-x}\right|$

We look for the upper value of $g_n(x)$ on interval I=[0,1[ :

We can get rid of the absolute value because $\frac{x^{n+1}}{1-x}>0$ on interval I=[0,1[ .

$g_n'(x)= \frac{(n+1)x^n(1-x)+x^{n+1}}{(1-x)^2} = \frac{nx^n+x^n-nx^{n+1}-x^{n+1}+x^{n+1}}{(1-x)^2} = \frac{x^n(-nx+n+1)}{(1-x)^2}$

$g_n'(x)=0$ if $\begin{cases} x=0 \\ or \\ x=\frac{n+1}{n} \end{cases}$

Table of variation of $g_n(x)$ :

0 1 n+1

| | | | |
|---|---|---|---|
| $x^n$ | + | + | + |
| $-nx+n+1$ | + | + | - |
| $x^n(-nx+n+1)$ | + | + | - |
| $g_n(x)$ | ↗ | ↗ | ↘ |

Since our work is on interval I=[0,1[ , $g_n(x)$reaches its maximal value on x=1 hence :

$\lim\limits_{n\to+\infty} sup_{x\in[0,1[}\left|\frac{x^{n+1}}{1-x}\right| = \lim\limits_{n\to+\infty}\left|\frac{1^{n+1}}{1-1}\right|==+\infty \neq 0 \Rightarrow \sum_{n=0}^{\infty}(x)^n$does not converge uniformly to f on I.

Let us try to study uniform convergence on [0,A], A< 1:

The previous table is simplified to:

0 A 1

| | |
|---|---|
| $x^n$ | + |
| $-nx+n+1$ | + |
| $x^n(-nx+n+1)$ | + |
| $g_n(x)$ | ↗ |

$g_n(x)$ reaches its maximal value at x=A thus :

$\lim\limits_{n\to+\infty} sup_{x\in[0,A]}\left|\frac{x^{n+1}}{1-x}\right|= \lim\limits_{n\to+\infty}\left|\frac{A^{n+1}}{1-A}\right|== \lim\limits_{n\to+\infty} A^{n+1} = 0 \Rightarrow \sum_{n=0}^{\infty}(x)^n$converges uniformly to f on

[0,A] .

**Example6 (see above):**

$\sum_{n=1}^{\infty}\left(\frac{x}{n}-\frac{x}{n+1}\right), \quad x\in\mathbb{R}.$

We have found:

$S_n(x) = x-\frac{x}{n}$and f(x) = x.

$\lim\limits_{n\to+\infty} sup_{x\in]-\infty,+\infty[}|S_n(x)-f(x)| = \lim\limits_{n\to+\infty} sup_{x\in]-\infty,+\infty[}\left|x-\frac{x}{n}-x\right|= \lim\limits_{n\to+\infty} sup_{x\in]-\infty,+\infty[}\left|\frac{x}{n}\right|$

$f_n(x)=\left|\frac{x}{n}\right|$

Since x$\in]-\infty,+\infty[$, $f_n(x)$does not have a maximal value on$]-\infty,+\infty[$

consequently $\sum_{n=1}^{\infty}(\frac{x}{n}-\frac{x}{n+1})$does not converge uniformly to fon$]-\infty,+\infty[$.

Let us check uniform convergence on$[-1,+1]$:

$f_n(x)$ reaches its maximal value on x=$-1$ $and\ x = 1 \Rightarrow \lim_{n\to+\infty} sup_{x\in[-1,1]} \left|\frac{x}{n}\right| = \lim_{n\to+\infty}\left|\frac{\pm1}{n}\right| =$
$\lim_{n\to+\infty}\frac{1}{n}$=0 $\Rightarrow \sum_{n=1}^{\infty}(\frac{x}{n} - \frac{x}{n+1})$converge uniformly to f in$[-1, +1]$.

Note:

In general, it is difficult to study uniform convergence because to find
$\lim_{n\to+\infty} sup_{x\in I}\,|S_n(x) - f(x)|$ we have to :

- Compute $S_n(x)$, which is difficult to calculate in general.
- Have $f(x)$which is hard to find most of the time. In the majority of situation, f exists but we are not able to calculate it (example 5).

$\sum_{n=1}^{\infty}\frac{x}{n^2}$ where we have found this series converges pointwise to a function f(x) ( unknown) but we know there exists and the sequence of partial sums$S_n(x)$ difficult to calculate $S_n(x)$,
$S_n(x)$= x+$\frac{x}{2^2} + \frac{x}{3} + \frac{x}{4^2} + \cdots \ldots + \frac{x}{n^2}$

That is why we usually use other methods to study uniform convergence for example normal convergence.

## IV.2. 2.3 Normal Convergence of a series of functions:

### IV.2. 2.3.1 Definition:

A series of function $\sum_{n=1}^{\infty} f_n$ converges normally on I if:

The infinite series $\sum_{n=1}^{\infty} sup_{x\in I}|f_n(x)|$ converges.

Where the sup means the maximal value on the interval I.

Note:

If $\sum_{n=1}^{\infty} f_n$ converges normally on I then $\sum_{n=1}^{\infty} f_n$converges uniformly on I.

Example1:

Let us study normal convergence of $\sum_{n=0}^{\infty} x^n$on [0,1].

For that we have to study convergence of the infinite series $\sum_{n=0}^{\infty} sup_{x\in[0,1]}|x^n|$

$f_n(x) = |x^n|$

It is obvious that $f_n(x)$ reaches its maximum value at x=1$\Rightarrow \sum_{n=0}^{\infty} sup_{x\in[0,1]}|x^n|$=$\sum_{n=0}^{\infty} 1^n$and this series is divergent $\Rightarrow \sum_{n=0}^{\infty} x^n$does not converge normally on [0,1].

Let us study normal convergence of $\sum_{n=0}^{\infty} x^n$on [0,a] a$<$ 1:

The same work as before, the only difference is the boundaries of the interval.

$f_n(x)$ reaches its maximal value at x=$a \Rightarrow \sum_{n=0}^{\infty} sup_{x\in[0,a]}|x^n|$=$\sum_{n=0}^{\infty} a^n$ is a geometric series with a common ratio a$<$ 1, is convergent $\Rightarrow \sum_{n=0}^{\infty} x^n$converges normally on [0,a].

**Example5 (see above):**

$\sum_{n=1}^{\infty} \frac{x}{n^2} x \in \mathbb{R}$.

We have found that$\sum_{n=0}^{\infty} \frac{x}{n^2}$ converges pointwise on $\mathbb{R}$.

Normal convergence on [-a, a] :

$\sum_{n=0}^{\infty} sup_{x\in\,[-a,a]} \left|\frac{x}{n^2}\right|$ converges ?

$f_n(x) = \left|\frac{x}{n^2}\right|$

It is clear that $f_n(x)$reaches its maximal value at x=$-a \; and \; x = a$

$\Rightarrow \sum_{n=1}^{\infty} sup_{x\in\,[-a,a]} \left|\frac{x}{n^2}\right|$=$\sum_{n=1}^{\infty} \left|\frac{\pm a}{n^2}\right| =$

$\sum_{n=1}^{\infty} \frac{a}{n^2}$ = a.$\sum_{n=0}^{\infty} \frac{1}{n^2}$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a P-Series (p=2) so converges ⇒ a.$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges also

$\Rightarrow \sum_{n=1}^{\infty} \frac{x}{n^2}$ converges normallyon [-a, a].

**Example 6 (see above):**

$$\sum_{n=1}^{\infty} (\frac{x}{n} - \frac{x}{n+1})$$

Normal convergence on [-a, a] :

$\sum_{n=1}^{\infty} sup_{x\in\,[-a,a]} \left|\frac{x}{n} - \frac{x}{n+1}\right|$ converges ?

$\sum_{n=0}^{\infty} sup_{x\in\,[-a,a]} \left|\frac{x}{n} - \frac{x}{n+1}\right| = \sum_{n=1}^{\infty} sup_{x\in\,[-a,a]} \left|\frac{x}{n(n+1)}\right|$

$f_n(x) = \left|\frac{x}{n(n+1)}\right|$

$f_n(x)$ reaches its maximal value at x=$-a \; and \; x = a \Rightarrow \sum_{n=1}^{\infty} sup_{x\in\,[-a,a]} \left|\frac{x}{n(n+1)}\right|$=$\sum_{n=1}^{\infty} \frac{a}{n(n+1)}$

= a.$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

$n(n+1) \quad > n^2 \Rightarrow \frac{1}{n(n+1)} < \frac{1}{n^2}$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$is a P-Series ( p=2) so converges ⇒$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges (thanks to comparison test 1)) ⇒a.$\sum_{n=0}^{\infty} \frac{1}{n(n+1)}$ converges also ⇒$\sum_{n=1}^{\infty} (\frac{x}{n} - \frac{x}{n+1})$ converges normally On [-a, a].

**Example7:**

$$\sum_{n=1}^{\infty} \frac{nx^2}{(n+1)^3}$$

Normal convergence on [-a, a]:

$$\sum_{n=1}^{\infty} sup_{x\in [-a,a]} \left|\frac{nx^2}{(n+1)^3}\right|$$

$$f_n(x) = \left|\frac{nx^2}{(n+1)^3}\right|$$

$f_n(x)$ reaches its maximal value at x=$-a$ $and$ $x = a$

$\Rightarrow \sum_{n=1}^{\infty} sup \left|\frac{nx^2}{(n+1)^3}\right|=\sum_{n=1}^{\infty} \frac{n(\pm a)^2}{(n+1)^3}=a^2 \sum_{n=1}^{\infty} \frac{n}{(n+1)^3}$

$$(n+1)^3 > n^3 \Rightarrow \frac{1}{(n+1)^3} < \frac{1}{n^3} \Rightarrow \frac{n}{(n+1)^3} < \frac{n}{n^3} = \frac{1}{n^2}$$

$\sum_{n=1}^{\infty} \frac{n}{(n+1)^3}$ converges according to comparison test 1$\Rightarrow$ $a^2 \sum_{n=1}^{\infty} \frac{n}{(n+1)^3}$ converges also

$\Rightarrow \sum_{n=1}^{\infty} \frac{nx^2}{(n+1)^3}$ converges normally on [-a, a].

### IV.2. 2.3.2 Weierstrass M-Test:

Let $\sum_{n=1}^{\infty} f_n$(x) be a seriesof functions.

If $|f_n(x)| \le M_n, \ \forall x \in I$

And if $\sum_{n=1}^{\infty} M_n$ is an infinite series that converges then $\sum_{n=1}^{\infty} f_n$(x) converges normally on I.

Example1 (see above):

$\sum_{n=0}^{\infty} (x)^n$ $x$ ∈[0,a] a< 1.

We have$|(x)^n| \le a^n, \ \forall x$ ∈[0,a].

$\sum_{n=0}^{\infty} a^n$ is a geometric series with a common ratio a< 1 that converges $\Rightarrow \sum_{n=0}^{\infty}(x)^n$converges normally on [0,a].

Example5 (see above) :

$\sum_{n=1}^{\infty} \frac{x}{n^2},$ $x$ ∈[-a,a]

We have: $\left|\frac{x}{n^2}\right| \le \frac{a}{n^2} \quad \forall x$ ∈[-a,a].

$\sum_{n=0}^{\infty} \frac{a}{n^2}$= a$\sum_{n=0}^{\infty} \frac{1}{n^2}$is a P-Series ( p =2) that converges $\Rightarrow \sum_{n=1}^{\infty} \frac{x}{n^2}$converges normally on [-a,a].

## Exercises of series of functions:

## Exercise 1:

Study the pointwise convergence and **f**ind domain of convergence of the following series of functions:

a- $\sum_{n=1}^{\infty} \cos(x) n^3$ b- $\sum_{n=0}^{\infty} x^2 (\frac{2}{9})^n$ c-$\sum_{n=1}^{\infty} \frac{n^4 x}{(n^7+10)}$

d-$\sum_{n=1}^{\infty} x(x^2+1)n!$ e-$\sum_{n=1}^{\infty} (\frac{x^4}{n^2+5})^n$

## Exercise 2 :

Study the pointwise convergence and determine domain of convergence of this series of function: $\sum_{n=0}^{\infty} x^n(4)^n$.

1-Deduce its function sum f(x).

2-Show that this series of function converges uniformly on [0,a] (a<$\frac{1}{4}$).

## Exercise 3:

1- Study the pointwise convergence on]1,+∞[ of this series of function:$\sum_{n=1}^{\infty}\frac{(-1)^n}{(lnx)^n}$.

2- Deduce its domain of convergence.

3-State whether this series converges or diverges uniformly on [a, +∞] (a>1).

## Solutions of series of functions

## Solutions of exercise1:

a- $\sum_{n=1}^{\infty}\cos(x)n^3$

For a given value $x_0$ of x $\sum_{n=1}^{\infty} cosx_0 n^3$ becomes an infinite series(of numbers)

$\sum_{n=1}^{\infty} cosx_0 n^3 = cosx_0 \sum_{n=1}^{\infty} n^3 = cosx_0 \sum_{n=1}^{\infty}\frac{1}{n^{-3}}$

a constant

$\sum_{n=0}^{\infty}\frac{1}{n^{-3}}$is a P-series that diverges⇒$cosx_0\sum_{n=1}^{\infty}\frac{1}{n^{-3}}$ diverges also thus$\sum_{n=1}^{\infty} cosxn^3$does not converge pointwise at x,∀x

Domain of convergence of this series is D= $\emptyset$

b- $\sum_{n=0}^{\infty} x^2(\frac{2}{9})^n$

For a given value $x_0$ of x, $\sum_{n=0}^{\infty} {x_0}^2(\frac{2}{9})^n = {x_0}^2\sum_{n=0}^{\infty}(\frac{2}{9})^n$ becomes an infinite series.

a constant

$\sum_{n=0}^{\infty}(\frac{2}{9})^n$is ageometric series with a common ratio$q = \frac{2}{9}$that converges (since -1<$\frac{2}{9}$<1)

⇒${x_0}^2\sum_{n=0}^{\infty}(\frac{2}{9})^n$ converges∀$x_0$ ∈R ⇒$\sum_{n=0}^{\infty} x^2(\frac{2}{9})^n$converges pointwise at $x$, ∀$x \in \mathbb{R}$.

Domain of convergence of this series is D= ℝ=]-∞, +∞[

c-$\sum_{n=1}^{\infty}\frac{n^4x}{(n^7+10)}$

For a given value $x_0$ of x, $\sum_{n=0}^{\infty}\frac{n^4x_0}{(n^7+10)}$ becomes an infinite series.

$\sum_{n=1}^{\infty}\frac{n^4x_0}{(n^7+10)} = x_0\sum_{n=1}^{\infty}\frac{n^4}{(n^7+10)}$

Let us study convergence of the infinite series $\sum_{n=1}^{\infty}\frac{n^4}{(n^7+10)}$ :

- It is neither a geometric series nor a P-series.

- If we apply Divergence Test; $\lim_{n\to+\infty} \frac{n^4}{(n^7+10)} = \lim_{n\to+\infty} \frac{n^4}{n^7} = \lim_{n\to+\infty} \frac{1}{n^3} = 0$ thus we can not say anything.

- Let us use Comparison Test :

$n^7 + 10 > n^7 \Rightarrow \frac{1}{(n^7+10)} < \frac{1}{n^7} \Rightarrow \frac{n^4}{(n^7+10)} < \frac{n^4}{n^7} = \frac{1}{n^3}$

$\sum_{n=0}^{\infty} \frac{1}{n^3}$ is a P-series where p=3>1 that converges$\Rightarrow \sum_{n=1}^{\infty} \frac{n^4}{(n^7+10)}$ is convergent by Comparison Test 1) $\Rightarrow x_0 \sum_{n=1}^{\infty} \frac{n^4}{(n^7+10)}$ converges$\forall x_0 \in$R

So$\sum_{n=1}^{\infty} \frac{n^4 x}{(n^7+10)}$ converges pointwise at $x$, $\forall x \in \mathbb{R}$

Domain of convergence of this series is D= $\mathbb{R}$=]-$\infty, +\infty$[

d-$\sum_{n=1}^{\infty} x(x^2 + 1)n!$

For a given value $x_0$ of x, $\sum_{n=1}^{\infty} x_0({x_0}^2 + 1)n!$ is an infinite series.

$$\sum_{n=1}^{\infty} x_0({x_0}^2 + 1)n! = x_0({x_0}^2 + 1)\sum_{n=1}^{\infty} n!$$

Convergence of $\sum_{n=1}^{\infty} n!$ ?

Let us use D'Alembert ratio test:

$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to+\infty} \frac{(n+1)!}{(n)!} = \lim_{n\to+\infty} n + 1 = +\infty > 1 \Rightarrow \sum_{n=1}^{\infty} n!$ is divergent

$\Rightarrow x_0({x_0}^2 + 1)\sum_{n=1}^{\infty} n!$ stays divergent$\Rightarrow \sum_{n=1}^{\infty} x(x^2 + 1)n!$ does not converge pointwise at x,$\forall$x

Domain of convergence of this series is D= $\emptyset$

e-$\sum_{n=1}^{\infty} (\frac{x^4}{n^2+5})^n$

For a given value $x_0$ of x, $\sum_{n=1}^{\infty} (\frac{{x_0}^4}{n^2+5})^n$ is an infinite series with positive terms.

$(\frac{{x_0}^4}{n^2+5})^n$ is a power of n consequently we apply Cauchy Root Test.

$\lim_{n\to+\infty} \sqrt[n]{a_n} = \lim_{n\to+\infty} \sqrt[n]{(\frac{{x_0}^4}{n^2+5})^n} = \lim_{n\to+\infty} \left(\frac{{x_0}^4}{n^2+5}\right) = 0 < 1 \quad \Rightarrow \sum_{n=1}^{\infty} (\frac{{x_0}^4}{n^2+5})^n$converges $\forall x_0 \in \mathbb{R}$

thus $\sum_{n=1}^{\infty} (\frac{{x_0}^4}{n^2+5})^n$ converges pointwise at $x$, $\forall x \in \mathbb{R}$

Domain of convergence of this series is D=$\mathbb{R}$=]- $\infty, +\infty$[

## Solutions of exercice 2 :

1- For a given value $x_0$ of x, $\sum_{n=0}^{\infty} (x_0)^n (4)^n$ becomes an infinite series

$\sum_{n=0}^{\infty} (x_0)^n (4)^n = \sum_{n=0}^{\infty} (x_0 4)^n$

$\sum_{n=0}^{\infty} (4x_0)^n$ is a geometric series with a common ratio $q = 4x_0$that converges if $-1<4x_0<1 \Rightarrow \sum_{n=0}^{\infty}(4x_0)^n$ converges if $-\frac{1}{4}<x_0<\frac{1}{4}$

thus $\sum_{n=0}^{\infty} x^n(4)^n$ converges pointwise at $x$ such that $-\frac{1}{4}<x<\frac{1}{4}$

Domain of convergence of this series is the interval $]-\frac{1}{4},\frac{1}{4}[=D$

2- We know for a convergent geometric series $\sum_{n=0}^{\infty}(q)^n$, its sum $S=\sum_{n=0}^{\infty}(q)^n = \frac{1}{1-q}$

So for a given value $x_0 \in ]-\frac{1}{4},\frac{1}{4}[$ $\quad \sum_{n=0}^{\infty}(4x_0)^n=\frac{1}{1-4x0}$

Thus $\sum_{n=0}^{\infty} x^n(4)^n = \frac{1}{1-4x} = f(x) \quad \forall x \in ]-\frac{1}{4},\frac{1}{4}[$

3-Let us prove normal convergence on [0,a] ($a<\frac{1}{4}$).

For that let us apply Weierstrass M-Test:

For a given value $x_0$ on [0,a] $f_n(x_0)=(4x_0)^n$,we are going to look for $U_n$ such that

$$|f_n(x_0)| \leq U_n$$

We have: $x0 \in [0,a] \Rightarrow 4x_0<4a \Rightarrow (4x_0)^n<(4a)^n$

$\sum_{n=0}^{\infty}(4a)^n$ is a geometric series with a common ratio $q=4a<1$ (since $a<\frac{1}{4}$)thus converges.

$(U_n=(4a)^n) \Rightarrow \sum_{n=0}^{\infty} x^n(4)^n$ converges normally on [0,a] ($a<\frac{1}{4}$)

Therefore $\sum_{n=0}^{\infty} x^n(4)^n$ converges uniformly on [0,a] ($a<\frac{1}{4}$)

## Solutions of exercice3:

1- $\sum_{n=1}^{\infty}\frac{(-1)^n}{(lnx)^n}$

- For all fixed $x_0 \in ]1,+\infty[$ ; $lnx_0>0$ $then$ $\sum_{n=1}^{\infty}\frac{(-1)^n}{(lnx)^n}$ is an alternating series

  with $a_n = \frac{1}{(lnx)^n}$ ($a_n>0$)

  $\forall x_0 \in ]1,+\infty[$

  - $\frac{1}{(lnx_0)^n}$ is a decreasing sequence (since $(lnx_0)^{n+1}>(lnx_0)^n \Rightarrow \frac{1}{(lnx_0)^{n+1}}<\frac{1}{(lnx_0{}^n}$)
  - $\lim\limits_{n\to+\infty}\frac{1}{(lnx_0)^n}=0$

$\Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n}{(lnx_0)^n}$ converges (by Leibnitz Test)

So $\sum_{n=1}^{\infty}\frac{(-1)^n}{(lnx_0)^n}$ converges pointwise at $x \in ]1,+\infty[$

2- we have found that $\sum_{n=1}^{\infty}\frac{(-1)^n}{(lnx_0)^n}$converges pointwise at $x \in ]1,+\infty[$ thus convergence domain of this series of function is $]1,+\infty[$.

3- To study the normal convergence, it is sufficient to study convergence of the infinite series

$\sum_{n=1}^{\infty} sup\left|\frac{(-1)^n}{(lnx)^n}\right|$ :

$f_n(x)=\left|\frac{(-1)^n}{(lnx)^n}\right|=\frac{1}{(lnx)^n}$ (since $lnx>0$ $on$ ]a , +∞[)

$f_n(x)=(lnx)^{-n} \Rightarrow f'_n(x)=-n(lnx)^{-n-1}=\frac{-n}{(lnx)^{n+1}}<0$

Table of variation of $f_n(x)$

| | a $\qquad\qquad$ $+\infty$ |
|---|---|
| $f_n{}'(x)=\dfrac{-n}{(lnx)^{n+1}}$ | - |
| $f_n(x)$ | $\searrow$ |

$f_n(x)$ reaches its maximal value at x=$a$, $\sum_{n=1}^{\infty} sup\left|\frac{(-1)^n}{(lnx)^n}\right|=\sum_{n=1}^{\infty}\frac{1}{(lna)^n}$

Convergence of the series $\sum_{n=1}^{\infty}\frac{1}{(lna)^n}$ ?:

Let us apply Cauchy's Root Test :

$$\lim_{n\to+\infty}\sqrt[n]{a_n}=\lim_{n\to+\infty}\sqrt[n]{\frac{1}{(lna)^n}}=\lim_{n\to+\infty}\left(\frac{1}{lna}\right)=\frac{1}{lna}$$

$\frac{1}{lna}<1$ if $lna>1 \Leftrightarrow e^{lna}>e^1=e \Leftrightarrow a>e$

thus $\sum_{n=1}^{\infty}\frac{1}{(lna)^n}$ converges if $a>e$

Conclusion:

$\sum_{n=1}^{\infty}\frac{(-1)^n}{(lnx)^n}$ does not converge normally on $[a,+\infty]$ (a>1) but converges normally

on $[a,+\infty]$ (a>e) $\Rightarrow\sum_{n=1}^{\infty}\frac{(-1)^n}{(lnx)^n}$ converges uniformly on $[a,+\infty]$ (a>e)

## IV.3 Power Series

### IV.3.1 Definition:

A power series (centered at 0) is a series of functions of the form$\sum_{n=1}^{\infty} a_n x^n$ where $(a_n)_{n\geq 1}$is a real sequences and x$\in \mathbb{R}$.

Examples:

1) $\sum_{n=0}^{\infty} x^n \, a_n = 1 \quad \forall \, n \geq 0$

2) $\sum_{n=1}^{\infty} \frac{x^n}{n} a_n = \frac{x}{n} \quad \forall \, n \geq 1$

3) $\sum_{n=0}^{\infty} \frac{x^n}{(n+1)(n+2)} a_n = \frac{1}{(n+1)(n+2)} \quad \forall \, n \geq 0$

**Notes:**

-$S_n$(x)=$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$is an n^th degree polynomial function therefore power series $\sum_{n=0}^{\infty} a_n x^n$is a generalization of a polynomial function.

-A power series, being a series of functions, we can then study its convergence like that of series of functions ( see II.2)

### IV.3.2 Radius of convergence of a power series :

Let $\sum_{n=1}^{\infty} a_n x^n$ be apower series then there exits R$\geq$ 0( can be equal to$+\infty$) such that:

a)$\sum_{n=1}^{\infty} a_n x^n$ converges absolutely for all reals x so that $|x|$<R.

b)$\sum_{n=1}^{\infty} a_n x^n$ diverges for all real x so that $|x|$>R.

c)For$|x|$ =R the power series may converge or diverge.

R is called radius of convergence of $\sum_{n=1}^{\infty} a_n x^n$.

Note:

-Radius of convergence R enable us to determine the domain of absolute convergence of the power series $\sum_{n=1}^{\infty} a_n x^n$, which is the open interval ]-R, R[. To close the interval, we have to study the absolute convergence at the boundaries (i.e x=$\pm R$)

### IV.3.3 Cauchy-Hadamard formula:

Let $\sum_{n=1}^{\infty} a_n x^n$ be a power series. Radius of convergence R is given by :

$$1)\frac{1}{R} = \lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$$or$$

$$2)\frac{1}{R} = \lim_{n\to+\infty} \sqrt[n]{|a_n|}$$

**Example1:**

$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^n}$ ; $a_n = \frac{(-1)^n}{n^n}$

$\frac{1}{R} = \lim_{n\to+\infty} \sqrt[n]{|a_n|} = \lim_{n\to+\infty} \sqrt[n]{\left|\frac{(-1)^n}{n^n}\right|} = \lim_{n\to+\infty} \frac{1}{n} = 0$

$\Rightarrow$ R= +∞

Therefore $\forall\ x \epsilon ]-\infty, +\infty[\ \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^n}$ converges absolutely.

**Example 2:**

$\sum_{n=1}^{\infty}(2n)!\,x^n;\ a_n = (2n)!$

$\frac{1}{R} = \lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to+\infty} \left|\frac{(2(n+1))!}{(2n)!}\right| = \lim_{n\to+\infty} \left|\frac{(2n+2)!}{(2n)!}\right| \lim_{n\to+\infty} \frac{(2n+2)(2n+1)(2n)!}{(2n)!}$

$= \lim_{n\to+\infty} (2n+2)(2n+1) = \lim_{n\to+\infty} 4n^2 = +\infty$

$\Rightarrow$ R= 0

thus $\sum_{n=0}^{\infty} \frac{x^n}{n!}$converges absolutely only at x=0.

**Example 3:**

$\sum_{n=1}^{\infty} \frac{x^n}{n^2};\ a_n = \frac{1}{n^2}$

$\frac{1}{R} = \lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to+\infty} \left|\frac{n^2}{(n+1)^2}\right| = \lim_{n\to+\infty} \frac{n^2}{n^2} = 1$

$\Rightarrow$ R= 1

So $\forall\ x \epsilon ]-1,1[$ ,$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^n}$ converges absolutely.

Let us check absolute convergence at x=-1 and x=1:

x=-1:

$\sum_{n=1}^{\infty} \left|\frac{x^n}{n^2}\right| = \sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is a P-Series ( p=2) that converges

Therefore $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$converges absolutely at x=-1.

x =1:

$\sum_{n=1}^{\infty} \left|\frac{x^n}{n^2}\right| = \sum_{n=1}^{\infty} \left|\frac{1^n}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is a P-Series ( p=2) that converges

So $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ converges absolutely at x=1

**Conclusion :**

$\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ converges absolutely on [-1, 1].

## IV.3.4 Addition of power series:

Let $\sum_{n=1}^{\infty} a_n x^n$ and $\sum_{n=1}^{\infty} b_n x^n$ be two power series of radius of convergence R ,R' respectively.

1- If R≠R' then radius of convergence R''of the power series $\sum_{n=1}^{\infty}(a_n+b_n)x^n$ is R''=min{R, R'}.

2- If R=R' then radius of convergence R''$> R$ .

**Examples:**

1- Let $\sum_{n=1}^{\infty} a_n x^n$ be the power series $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$,we have already calculated its radius of convergence R=1

And $\sum_{n=1}^{\infty} b_n x^n = \sum_{n=1}^{\infty} \frac{x^n}{2^n}$

$$\frac{1}{R'} = \lim_{n\to+\infty} \sqrt[n]{|b_n|} = \lim_{n\to+\infty} \sqrt[n]{\left|\frac{1}{2^n}\right|} \lim_{n\to+\infty} \frac{1}{2} = \frac{1}{2} \Rightarrow R' = 2.$$

$n^2$

Let us determine R'' of $\sum_{n=1}^{\infty}(a_n+b_n)x^n = \sum_{n=1}^{\infty}(\frac{1}{n^2}+\frac{1}{2^n})x^n$

$$= \sum_{n=1}^{\infty}\left(\frac{2^n+n^2}{n^2 2^n}\right)x^n$$

$$\frac{1}{R''} = \lim_{n\to+\infty}\left|\frac{a_{n+1}+b_{n+1}}{a_n+b_n}\right| = \lim_{n\to+\infty}\left|\frac{2^{n+1}+(n+1)^2}{(n+1)^2 2^{n+1}}.\frac{n^2 2^n}{2^n+n^2}\right| = \lim_{n\to+\infty}\left|\frac{n^2 2^n}{(n+1)^2 2^{n+1}}.\frac{(2^{n+1}+(n+1)^2)}{(2^n+n^2)}\right|$$

$$= \lim_{n\to+\infty}\left|\frac{n^2 2^n}{(n+1)^2 2^{n+1}}.\frac{(2^{n+1}+(n+1)^2)}{(2^n+n^2)}\right| = \lim_{n\to+\infty}\left|\frac{n^2}{(n+1)^2 2}.\frac{(2^{n+1}+(n+1)^2)}{(2^n+n^2)}\right| = \lim_{n\to+\infty}\left|\frac{2^{n+1}}{2.2^n}\right|$$

$= \frac{1}{2}.2 = 1 \Rightarrow$ R''=1.

According to 1) R''=min{R, R'}=min R''=min{1, 2}=1.

2-Let $\sum_{n=1}^{\infty} a_n x^n$ be the series $\sum_{n=1}^{\infty} x^n$ whose radius of convergence R=1

And $\sum_{n=1}^{\infty} b_n x^n = \sum_{n=1}^{\infty} \frac{(1-3^n)x^n}{3^n}$

$$\frac{1}{R'} = \lim_{n\to+\infty}\left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to+\infty}\left|\frac{(1-3^{n+1})}{3^{n+1}}.\frac{3^n}{(1-3^n)}\right| = \lim_{n\to+\infty}\left|\frac{3^n}{3^{n+1}}.\frac{(1-3^{n+1})}{(1-3^n)}\right|$$

$= \frac{1}{3}\lim_{n\to+\infty}\left|\frac{(1-3^{n+1})}{(1-3^n)}\right| = \frac{1}{3}.3 = 1 \Rightarrow$ R'=1.

Let us determine R'' of $\sum_{n=1}^{\infty}(a_n+b_n)x^n = \sum_{n=1}^{\infty}(1+\frac{(1-3^n)}{3^n})x^n$

$= \sum_{n=1}^{\infty}(\frac{3^n+1-3^n}{3^n})x^n = \sum_{n=1}^{\infty}(\frac{1}{3^n})x^n$

$$\frac{1}{R''} = \lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to+\infty}\left|\frac{3^n}{3^{n+1}}\right| = \frac{1}{3} \Rightarrow R''=3$$

According to 2) R''$> R$ .

### IV.3.5. Power series properties:

Let $\sum_{n=1}^{\infty} a_n x^n$ be a power series of radius of convergence R and f its sum f(x)=$\sum_{n=1}^{\infty} a_n x^n$on ]-R, R[ then :

a- f is continue on ]-R, R[.

b- f is differentiable on]-R, R[ and its derivative is f '(x)=$\sum_{n=1}^{\infty} n a_n x^{n-1}$.

c- f is integrable and its primitive (or anti-derivative) F(x)=$\sum_{n=1}^{\infty} \frac{a_n}{n+1} x^{n+1}$.

**Notes:**

-A power series and its derivative have the same radius of convergence.

$$\frac{1}{R'} = \lim_{n\to+\infty} \left|\frac{(n+1)a_{n+1}}{n a_n}\right| = \lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{R}$$

-A power series and its primitive have the same radius of convergence.

$$\frac{1}{R''} = \lim_{n\to+\infty} \left|\frac{a_{n+1}}{(n+2)}.\frac{n+1}{a_n}\right| = \lim_{n\to+\infty} \left|\frac{n+1}{(n+2)}\frac{a_{n+1}}{a_n}\right| = \lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{R}$$

**Example 1:**

Let $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$ be a power series

Let us determine its radius of convergence R:

$$\frac{1}{R} = \lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to+\infty} \left|\frac{(-1)^n}{(n+1)}.\frac{n}{(-1)^n}\right| = \lim_{n\to+\infty} \left|\frac{n}{(n+1)}\right| = 1$$

$\forall x \in$]-1, 1[, $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$ converges absolutely to f ; f(x)=$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$

(f exists but we do not know its expression).

f is differentiable on ]-1, 1[

$\forall x \in$]-1, 1[ f '(x)=$\sum_{n=1}^{\infty} (-1)^n \frac{n}{n} x^{n-1} = \sum_{n=1}^{\infty} (-1)^n x^{n-1}$=-1+x-$x^2 + x^3 - \cdots$ ...

=-(1-x+$x^2 - x^3 + \cdots \dots \dots \dots$)

=- $\sum_{n=0}^{\infty} (-1)^n x^n$

=-$\sum_{n=0}^{\infty} (-x)^n$ wich is a geometric series of functions with a common ratio –x

⇒f '(x)= - $\frac{1}{1+x}$ (1)

By integrating (1), we find expression of f(x):

f(x)= $\int_0^x \frac{-1}{1+t} dt = |-\ln|(1+t)||_0^x = -\ln|(1+x)|$

**Conclusion :**

$\forall x \in$]-1, 1[, $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$ converges absolutely to f ; f(x)=- ln$|(1+x)|$

$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n} =$ - $\ln(1+x)$

$\forall x \in ]-1, 1[, \sum_{n=1}^{\infty}(-1)^n x^{n-1}$ converges absolutely to f'; f'(x)= - $\frac{1}{1+x}$

$\sum_{n=1}^{\infty}(-1)^n x^{n-1} = -\frac{1}{1+x}$

**Notes:**

-Let $\sum_{n=1}^{\infty} a_n x^n$ be a power series of radius of convergence R and f its sum f(x)=$\sum_{n=1}^{\infty} a_n x^n$ on ]-R, R[ .

If we differentiate it f '(x)=$\sum_{n=1}^{\infty} n a_n x^{n-1}$ will be a new power series of radius of convergence R.

If we differentiate the new one f''(x)=$\sum_{n=1}^{\infty} n(n-1) a_n x^{n-2}$ will be a new power series of radius of convergence R.

And so on.

Thus the sum of a power series $\sum_{n=1}^{\infty} a_n x^n$ is infinitely differentiable ($\in C^{\infty}$) on its interval of convergence ]-R, R[and its derivatives are given term-by-term differentiation of the power series.

- We can deduce the same thing for integration.

## IV.3.6. Function representable by power series (RPS Function):

### IV.3.6.1 Definition:

Let f be a real function defined in a neighborhood of 0, we say that f is representable by a power series if there exists A> 0 and a power series $\sum_{n=1}^{\infty} a_n x^n$ with radius of convergence A such that: f(x)= $\sum_{n=1}^{\infty} a_n x^n \quad \forall x \in ]-A, A[$.

**Example1:**

Let f defined by f: :$\mathbb{R}$-{1}$\longrightarrow \mathbb{R}$

$x \longrightarrow f(x)= \frac{1}{1-x}$

f is representable by a power series in a neighborhood of 0 on ]-1, 1[ because we know (see II.2) :

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ on ]-1, 1[

**Example2:**

f :$\mathbb{R}$-{2}$\longrightarrow \mathbb{R}$

$x \longrightarrow f(x)= \frac{1}{2-x}$

f(x)= $\frac{1}{2-x} = \frac{1}{2(1-\frac{x}{2})} = \frac{1}{2}\frac{1}{(1-\frac{x}{2})} = \frac{1}{2}\sum_{n=0}^{\infty}(\frac{x}{2})^n = \frac{1}{2}\sum_{n=0}^{\infty}\frac{x^n}{2^n} = \sum_{n=0}^{\infty}\frac{x^n}{2^{n+1}}$

Hence f is representable by a power series in a neighborhood of 0 on ]-R, R[ :

$\frac{1}{2-x} = \sum_{n=0}^{\infty}\frac{x^n}{2^{n+1}}$ where $a_n = \frac{1}{2^{n+1}}$

$$\frac{1}{R} = \lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to+\infty}\left|\frac{1}{2^{n+2}}.2^{n+1}\right| = \lim_{n\to+\infty}\left|\frac{1}{2}\right| = \frac{1}{2} \Rightarrow R = 2$$

$\frac{1}{2-x} = \sum_{n=0}^{\infty}\frac{x^n}{2^{n+1}}$on]-2, 2[ .

Proposition (necessary condition):

Let f be a real function defined in a neighborhood of 0.

If f is infinitely differentiable ($\in C^{\infty}$) then f is representable by a power series in this neighborhood.

Thus f $\in C^{\infty}$is a necessary condition.

## IV.3.6.2 Taylor's Series:

Let f be a real function infinitely differentiable ($\in C^{\infty}$) . We call Taylor's series of f, the power series $\sum_{n=0}^{\infty}\frac{f^n(0)}{n!}x^n$

Proposition (sufficient condition)

Let f be a real function defined in a neighborhood of 0 and infinitely differentiable ($\in C^{\infty}$).

if ∃M> 0 such that $\forall n$ ∈N and $\forall\ x\ \in$]-R, R[ $|f^n(x)| \le M$ then Taylor's series of f $\sum_{n=0}^{\infty}\frac{f^n(0)}{n!}x^n$ converges pointwise to f on ]-R, R[ i.e f(x)= $\sum_{n=0}^{\infty}\frac{f^n(0)}{n!}x^n$on ]-R, R[ .

Explanation:

If f is infinitely differentiable and all its derivatives are bounded on ]-R, R[ then f is equal to its Taylor's series (i.e f is representable by a power series on ]-R, R[ ).

**Example1:**

f(x)=sin(x) is infinitely differentiable and we have :

$f^n(x)$=sin(x+n$\frac{\pi}{2}$) n≥ 1 ⇒all derivatives of f are bounded on]-1, 1[ $\forall\ x \in \mathbb{R}$ (i.e M=1)

Hence f is representable by a power series (f is equal to its Taylor's series)

$\forall\ x \in \mathbb{R}$ sin(x)= $\sum_{n=0}^{\infty}\frac{f^n(0)}{n!}x^n = \sum_{n=0}^{\infty}\frac{\sin\left(n\frac{\pi}{2}\right)}{n!}x^n$

$$= \sin(0)+\frac{1}{1!}\sin\left(\frac{\pi}{2}\right)x + \frac{1}{2!}\sin(\pi)\,x^2 + \frac{1}{3!}\sin\left(\frac{3\pi}{2}\right)x^3 + \frac{1}{4!}\sin(2\pi)\,x^4 + \frac{1}{5!}\sin\left(\frac{5\pi}{2}\right)x^5 + \cdots$$

$$=0 \quad + \ x \quad + 0 \quad - \frac{1}{3!}x^3 \quad + 0 \quad + \frac{1}{5!}x^5 + \cdots..$$

$$=\sum_{p=0}^{\infty}\frac{(-1)^P}{2p+1!}x^{2p+1}$$

So $\forall\ x \in \mathbb{R}$, sin(x)= $\sum_{p=0}^{\infty}\frac{(-1)^P}{2p+1!}x^{2p+1}$

**Example2:**

f(x)=cos(x) is infinitely differentiable and all its derivatives are bounded on ]-1, 1[ $\forall\ x \in \mathbb{R}$ (i.e M=1)

Thus cos(x) is representable by a power series ( is equal to its Taylor's series) f(x)=cos(x)= (sinx)'=$\left(\sum_{p=0}^{\infty}\frac{(-1)^P}{2p+1!}x^{2p+1}\right)' = \sum_{p=0}^{\infty}\left(\frac{(-1)^P}{2p+1!}x^{2p+1}\right)'$=$\sum_{p=0}^{\infty}\frac{(-1)^P 2p+1}{2p+1!}x^{2p}$

=$\sum_{p=0}^{\infty}\frac{(-1)^P}{2p!}x^{2p}$

$\forall\ x \in \mathbb{R}$, cos(x) =$\sum_{p=0}^{\infty}\frac{(-1)^P}{2p!}x^{2p}$

**Example3:**

f(x)=$e^x$is infinitely differentiable.

Proposition:

Let $\sum_{n=0}^{\infty}\frac{f^n(0)}{n!}x^n$ be Taylor's series of a function f infinitely differentiable.

$\sum_{n=0}^{\infty}\frac{f^n(0)}{n!}x^n$ = f(0)+$f'(0)x + \cdots. + f^n(0)x^n + R_n(x)$

Where $R_n(x) = \frac{f^{n+1}(z)}{n!}x^{n+1}$ is the reminder of Taylor's series ( 0$< z < x$ ou x$< z < 0$)

If $\lim\limits_{n\to+\infty} R_n(x) = 0$ then Taylor's series of f converges pointwise to f.

Let us check this condition for the function f(x)=$e^x$.

$R_n(x) = \frac{f^{n+1}(z)}{n!}x^{n+1} = \frac{e^z}{n!}x^{n+1}$

If 0$< z < x$ $\quad 0 < |R_n(x)| \le \frac{e^x}{n!}|x^{n+1}|$

$$\lim_{n\to+\infty}\frac{e^x}{n!}|x^{n+1}| = 0 \Rightarrow \lim_{n\to+\infty} R_n(x) = 0$$

If x$< z < 0$ $\quad 0 < |R_n(x)| \le \frac{|x^{n+1}|}{n!}$

$$\lim_{n\to+\infty}\frac{|x^{n+1}|}{n!} = 0 \Rightarrow \lim_{n\to+\infty} R_n(x) = 0$$

So according to the proposition:

f(x)=$e^x$, f is representable by its Taylor series at 0 i.e $e^x = \sum_{n=0}^{\infty}\frac{f^n(0)}{n!}x^n$

Since $f^n(0)$=$e^0 = 1$, therefore $e^x$ =$\sum_{n=0}^{\infty}\frac{x^n}{n!}\forall\ x \in \mathbb{R}$

**Note**:

Now, knowing that some functions are representable by power series such as

($e^x, \sin(x), \cos(x), \frac{1}{1-x}$ ... ) we can deduce representation by power series of other functions obtained by addition, subtraction,(or in terms of finite combinations), differentiation, integration of these familiar functions.

## IV.3.7. Using Power series to Solve Differential Equations

We call linear m$^{th}$ order differential equation an equation of form:

A(x) $Y^m$+ B(x) $Y^{m-1}$……….+ G(x) $Y^{'}$+H(x)Y=f(x)

Where:

Y: a function of x.

$Y^m$, $Y^{m-1}$………., $Y^{'}$ derivatives of Y of order (m, m-1,….).

A(x), B(x) ,……., G(x),f(x) are functions of x.

**Examples:**

$xY^{''}-(2x+1)Y^{'}+x^2Y=3x$

$x^2Y^{'}+4xY=0$.

Let us solve the following differential equation:

$Y^{'}-Y=0$ (1)

The method :

Let us look for Y of form a power series:

$Y=\sum_{n=0}^{\infty} a_n x^n$ (2)

We can differentiate power series term by term, so

$Y^{'}=\sum_{n=1}^{\infty} n a_n x^{n-1}$ (3)

In order to compare the expressions for y and y' more easily, we rewrite as follows:

$Y^{'}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$ (4)

Substituting the expressions in Equations 2 and 4 into the differential equation (1), we obtain

$\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n-\sum_{n=0}^{\infty} a_n x^n=0$ (5)

or $\sum_{n=1}^{\infty}[(n+1)a_{n+1}-a_n]x^n=0$ (6)

If two power series are equal, then the corresponding coefficients must be equal. Therefore, the coefficients of series in Equation 6 must be 0:

$(n+1)a_{n+1}-a_n=0$ (7)

$\Rightarrow a_{n+1}=\frac{a_n}{n+1}$ (8)

Equation 8 is called a recursion relation. If $a_0$ is known, this equation allows us to determine the remaining coefficients recursively by putting n= 0,1, 2, 3, ….in succession.

Put n=0: $a_1=\frac{a_0}{1}$

Put n=1: $a_2=\frac{a_1}{2}=\frac{a_0}{2.1}$

Put n=2: $a_3=\frac{a_2}{3}=\frac{a_0}{3.2.1}$

By now we see the pattern:

$$a_{n+1}=\frac{a_0}{(n+1)n(n-1)\ldots..1}$$

$$a_{n+1} = \frac{a_0}{(n+1)!} \Rightarrow a_n = \frac{a_0}{(n)!}$$

Putting this value back into Equation 2, we write the solution as

Y=$\sum_{n=0}^{\infty} a_n x^n$=$\sum_{n=0}^{\infty} \frac{a_0}{(n)!} x^n$=$a_0 \sum_{n=0}^{\infty} \frac{x^n}{(n)!}$= $a_0 e^x$

So then the solution is Y(x)=$a_0 e^x$

## Exercises of Power series

### Exercise 1:

Identify power series among the following series of functions and calculate their radius of convergence:

a)$\sum_{n=0}^{\infty}\frac{7^n x^n}{3^n}$ b)$\sum_{n=1}^{\infty}\frac{x^2}{nx^n}$ c)$\sum_{n=1}^{\infty}\frac{x^n e^x}{n!}$

d)$\sum_{n=1}^{\infty}\frac{(2n!)x^n}{(n!)^2}$ e)$\sum_{n=1}^{\infty} n^{\alpha}x^n \ \alpha\in R$

### Exercise 2 :

Determine radius of convergence R, domain of pointwise convergence(P) and domain of absolute convergence (A) of the following power series:

a)$\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{\ln n}$ b)$\sum_{n=1}^{\infty}\frac{x^n 3^n}{(n+1)^2}$

### Exercise 3:

Determine representation by power series of the following functions:

1) f(x)=$\frac{5}{x-1}$ 2) g(x)= $e^{\frac{x}{2}}-1$ 3) h(x)=$\frac{6x}{3-2x}$ 4) I(x)=$\frac{x+1}{(2-x)(x-4)}$

### Exercise 4:

Solve the following differential Equations using power series :

1) $x^2y''+4xy'+2y=e^x$
2) $y''+y=0$withy(0)=$\frac{1}{2}$ et $y'(0)=0$

## Solutions of exercises of Power series

### Solutions of exercise 1 :

a) $\sum_{n=0}^{\infty}\frac{7^n x^n}{3^n}$ is a power series where $a_n$=$\frac{7^n}{3^n}$

Radius of convergence R :

$\frac{1}{R}=\lim\limits_{n\to+\infty}\sqrt[n]{|a_n|}=\lim\limits_{n\to+\infty}\sqrt[n]{\left|\frac{7^n}{3^n}\right|}=\lim\limits_{n\to+\infty}\sqrt[n]{(\frac{7}{3})^n}=\lim\limits_{n\to+\infty}\frac{7}{3}=\frac{7}{3}\Rightarrow R=\frac{3}{7}$thus$\sum_{n=0}^{\infty}\frac{7^n x^n}{3^n}$ converges absolutely on ]- $\frac{7}{3}$,$\frac{7}{3}$[.

b) $\sum_{n=1}^{\infty}\frac{x^2}{nx^n}=\sum_{n=1}^{\infty}\frac{x^{2-n}}{n}$ is not a power series because of negative powerof x ($x^{2-n}$)

c)$\sum_{n=1}^{\infty}\frac{x^n e^x}{n!}$ is not a power series because $a_n$=$\frac{e^x}{n!}$ depends on x ($e^x$)

d) $\sum_{n=1}^{\infty}\frac{(2n!)x^n}{(n!)^2}$

$\sum_{n=1}^{\infty}\frac{(2n!)x^n}{(n!)^2}$is a power series where $a_n=\frac{(2n)!}{(n\,!)^2}$

Radius of convergence R :

$$\frac{1}{R}=\lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to+\infty}\left|\frac{\frac{(2(n+1))!}{((n+1)!)^2}}{\frac{(2n)!}{(n!)^2}}\right|=\lim_{n\to+\infty}\left|\frac{(2n+2))!}{((n+1)!)^2}\frac{(n!)^2}{(2n)!}\right|=\lim_{n\to+\infty}\left|\frac{(2n+2))!}{(n+1)!(n+1)!}\frac{(n!)^2}{(2n)!}\right|$$

$$=\lim_{n\to+\infty}\left|\frac{(2n+2))(2n+1)(\cancel{2n})!}{(n+1)\cancel{n!}(n+1)\cancel{n!}}\frac{(\cancel{n!})^2}{(\cancel{2n})!}\right|=\lim_{n\to+\infty}\frac{(2n+2))(2n+1)}{(n+1)(n+1)}=\lim_{n\to+\infty}\frac{4n^2}{n^2}=4\Rightarrow R=\frac{1}{4}$$

$\sum_{n=1}^{\infty}\frac{(2n!)x^n}{(n!)^2}$ converges absolutely on $]-\frac{1}{4},\frac{1}{4}[$.

e) $\sum_{n=1}^{\infty} n^{\alpha}x^{n}$ $\alpha\in R$ is a power series where $a_n=n^{\alpha}$

$$\frac{1}{R}=\lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to+\infty}\left|\frac{(n+1)^{\alpha}}{n^{\alpha}}\right|=\lim_{n\to+\infty}\left(\frac{n+1}{n}\right)^{\alpha}=\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{\alpha}=1$$

$\sum_{n=1}^{\infty} n^{\alpha}x^{n}$ converges absolutely on $]-1,1[$.

## Solutions of exercise 2 :

a) $\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{\ln n}$

$\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{\ln n}$ is a power series where $a_n=\frac{(-1)^n}{\ln n}$

Radius of convergence R:

$$\frac{1}{R}=\lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to+\infty}\left|\frac{\frac{(-1)^{n+1}}{\ln(1+n)}}{\frac{(-1)^n}{\ln n}}\right|=\lim_{n\to+\infty}\frac{\ln n}{\ln(1+n)}=\lim_{n\to+\infty}\frac{\ln n}{\ln n}=1$$

$\Rightarrow R=1$

<u>Domain of pointwise convergence P</u>:

$R=1\Rightarrow\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{\ln n}$ converges absolutely on $]-1,1[=A\Rightarrow\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{\ln n}$ converges <u>pointwise</u> on $]-1,1[\Rightarrow P=]-1,1[$

Let us study the pointwise convergence at x=1 :

$\sum_{n=1}^{\infty}\frac{(-1)^n 1^n}{\ln n}=\sum_{n=1}^{\infty}\frac{(-1)^n}{\ln n}$

It is an alternating series of form $\sum_{n=1}^{\infty}(-1)^n b_n$ where $b_n=\frac{1}{\ln n}$

According to Leibnitz Test, both conditions are satisfied :

1) $(b_n)_n$ is a decreasing sequence $\left(\frac{1}{\ln n+1}<\frac{1}{\ln n}\right)$

2) $\lim_{n\to+\infty} b_n=\lim_{n\to+\infty}\frac{1}{\ln n}=0$

thus $\sum_{n=1}^{\infty}\frac{(-1)^n}{\ln n}$ is convergent $\Rightarrow\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{\ln n}$ converges pointwise at x=1 $\Rightarrow P=[-1,1[$

Pointwise convergence at x= -1:

$\sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n}{\ln n}=\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{\ln n}=\sum_{n=1}^{\infty}\frac{1}{\ln n}$

Comparison Test:

$\forall n\geq 1 \quad \ln n<n \Rightarrow \frac{1}{\ln n}>\frac{1}{n}$

since $\sum_{n=1}^{\infty}\frac{1}{n}$ is a P-series that diverges$\Rightarrow\sum_{n=1}^{\infty}\frac{1}{\ln n}$ diverges

thus$\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{\ln n}$ does not converge pointwise at x= -1$\Rightarrow$S=]-1,1]

Domain of absolute convergence A:

R=1 $\Rightarrow$this series converges absolutelyon ]-1,1 [=A.

Note :

We know that absolute convergence $\Rightarrow$ pointwise convergence

(or the contrapositive) non-pointwise convergence $\Rightarrow$ non-absolute convergence

We have found this series does not converge pointwise at x=1$\Rightarrow$this series does not converge absolutely at x=1.

It left just to study absolute convergence at x= -1

Absolute convergence at x= -1 :

$\sum_{n=1}^{\infty}\left|\frac{(-1)^n(-1)^n}{\ln n}\right|=\sum_{n=1}^{\infty}\frac{1}{\ln n}$ that diverges ( see above)

thus $\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{\ln n}$ does not converge absolutely at x=-1

So domain of absolute convergence does not change A=]-1,1 [.

b) $\sum_{n=1}^{\infty}\frac{x^n 3^n}{(n+1)^2}$is a power series where$a_n=\frac{3^n}{(n+1)^2}$

Radius of convergence R :

$$\frac{1}{R}=\lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to+\infty}\left|\frac{\frac{3^{n+1}}{(n+2)^2}}{\frac{3^n}{(n+1)^2}}\right|=\lim_{n\to+\infty}\frac{\cancel{3^{n}}^{+1}}{\cancel{3^n}}\frac{(n+1)^2}{(n+2)^2}=3\lim_{n\to+\infty}\frac{\cancel{n^2}}{\cancel{n^2}}=3\Rightarrow R=\frac{1}{3}$$

Domain of absolute convergence:

R=$\frac{1}{3}\Rightarrow\sum_{n=1}^{\infty}\frac{x^n 3^n}{(n+1)^2}$ converges absolutely on ]-$\frac{1}{3}$,$\frac{1}{3}$ [.

Absolute convergence at x=-$\frac{1}{3}$ :

$\sum_{n=1}^{\infty}\left|\frac{(-\frac{1}{3})^n 3^n}{(n+1)^2}\right|=\sum_{n=1}^{\infty}\frac{1}{(n+1)^2}$

Comparaison Test:

$(n+1)^2\geq n^2\Rightarrow\frac{1}{(n+1)^2}\leq\frac{1}{n^2}$

$\sum\frac{1}{n^2}$ is a P-series with a=2 that converges$\Rightarrow\sum_{n=1}^{\infty}\frac{1}{(n+1)^2}$ is convergent by comparison test1).

Thus $\sum_{n=1}^{\infty}\frac{x^n 3^n}{(n+1)^2}$ converges absolutely at x=-$\frac{1}{3}$

Absolute convergence at x=$\frac{1}{3}$ :

$\sum_{n=1}^{\infty}\left|\frac{(\frac{1}{3})^n 3^n}{(n+1)^2}\right| = \sum_{n=1}^{\infty}\frac{1}{(n+1)^2}$ (same reasoning)

$\sum_{n=1}^{\infty}\frac{x^n 3^n}{(n+1)^2}$ converges absolutely at x=$\frac{1}{3}$

Thus domain of absolute convergence A=[ $-\frac{1}{3},\frac{1}{3}$ ].

<u>Domain of pointwise convergence P</u>:

We have found that $\sum_{n=1}^{\infty}\frac{x^n 3^n}{(n+1)^2}$ converges absolutely on [$-\frac{1}{3},\frac{1}{3}$ ]$\Rightarrow\sum_{n=1}^{\infty}\frac{x^n 3^n}{(n+1)^2}$ converges pointwise on [ $-\frac{1}{3},\frac{1}{3}$ ] so domain of pointwise convergence P=[$-\frac{1}{3},\frac{1}{3}$ ]

## Solutions of exercise 3:

1) $f(x)=\frac{5}{x-1} = -\frac{5}{1-x} = -5\sum x^n$ (since $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \ \forall x\in]-1,1[$)

2) $g(x)= e^{\frac{x}{2}} - 1$

$\forall\, x \in \mathbb{R},\ e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$

$e^{\frac{x}{2}} = \sum_{n=0}^{\infty}\frac{(\frac{x}{2})^n}{n!} = 1+\sum_{n=1}^{\infty}\frac{(\frac{x}{2})^n}{n!} \Rightarrow e^{\frac{x}{2}} - 1 = 1+\sum_{n=1}^{\infty}\frac{(\frac{x}{2})^n}{n!} - 1 = \sum_{n=1}^{\infty}\frac{(x)^n}{(2)^n n!}$

$so\ \forall\, x \in \mathbb{R},\quad e^x - 1 = \sum_{n=1}^{\infty}\frac{(x)^n}{(2)^n n!}$

2) $h(x)=\frac{6x}{3-2x} = \frac{6x}{3(1-\frac{2}{3}x)} = \frac{6}{3}x.\frac{1}{(1-\frac{2}{3}x)} = 2x\sum_{n=0}^{\infty}(\frac{2}{3}x)^n = 2x\sum_{n=0}^{\infty}\frac{2^n x^n}{3^n} \quad = \sum_{n=0}^{\infty}\frac{2^{n+1}x^{n+1}}{3^n}$

$h(x)=\sum_{n=0}^{\infty}\frac{2^{n+1}x^{n+1}}{3^n}$

4) $I(x)=\frac{x+1}{(2-x)(x-4)} = \frac{a}{(2-x)} + \frac{b}{(x-4)}$ a, b real constant to determine after

$= a\frac{1}{2\left(1-\frac{x}{2}\right)} + b\frac{1}{-(4-x)} = \frac{a}{2}\sum_{n=0}^{\infty}(\frac{x}{2})^n - \frac{b}{4}\frac{1}{(1-\frac{x}{4})} = \frac{a}{2}\sum_{n=0}^{\infty}(\frac{x}{2})^n - \frac{b}{4}\sum_{n=0}^{\infty}(\frac{x}{4})^n$

$= \sum_{n=0}^{\infty}[\frac{a}{2^{n+1}} - \frac{b}{4^{n+1}}](x)^n$

Calcul of a and b:

We have $\frac{x+1}{(2-x)(x-4)} = \frac{a}{(2-x)} + \frac{b}{(x-4)}$ (I)

To determine a :

1) Multiply both sides of equation (I) by (2-x) :

$\frac{x+1}{\cancel{(2-x)}(x-4)}\cancel{(2-x)} = \frac{a}{\cancel{(2-x)}}\cancel{(2-x)} + \frac{b}{(x-4)}(2-x)$

2) Put x=2 :

$\frac{2+1}{(2-4)} = a + \frac{b}{(x-4)}(2-2)$

So $-\frac{3}{2} = \mathrm{a}$

To determine b :

1) Multiply both sides of equation (I) by(x-4) :

$$\frac{\mathrm{x}+1}{(2-\mathrm{x})\cancel{(\mathrm{x}-4)}}\cancel{(\mathrm{x}-4)} = \frac{\mathrm{a}}{(2-\mathrm{x})}(\mathrm{x}-4) + \frac{\mathrm{b}}{\cancel{(\mathrm{x}-4)}}\cancel{(\mathrm{x}-4)}$$

2) Put x=4:

$$\frac{4+1}{(2-4)} = a\ \frac{\mathrm{a}}{(2-\mathrm{x})}\underbrace{(4-4)}_{0} + b$$

So $-\frac{5}{2} = \mathrm{b}$

Consequently $\mathrm{I(x)} = \sum_{n=0}^{\infty}[\frac{a}{2^{\mathrm{n}+1}} - \frac{b}{4^{\mathrm{n}+1}}](\mathrm{x})^n = \sum_{n=0}^{\infty}[\frac{-3}{2^{\mathrm{n}+2}} + \frac{5}{2.\ 4^{\mathrm{n}+1}}](\mathrm{x})^n$

## Solutions of exercise 4 :

1) $\mathrm{x}^2\mathrm{y}'' + 4\mathrm{xy}' + 2\mathrm{y} = e^x$

$\mathrm{y(x)} = \sum_{n=0}^{\infty} a_n \mathrm{x}^n$

$\mathrm{y'(x)} = \sum_{n=1}^{\infty} na_n\mathrm{x}^{n-1} \Rightarrow \mathrm{x\ y'(x)} = \sum_{n=1}^{\infty} na_n\mathrm{x}^n = \sum_{n=0}^{\infty} na_n\mathrm{x}^n$

because for n=0 term($na_n = 0$) we can start this sum at n=0

$\mathrm{y''(x)} = \sum_{n=2}^{\infty} n(n-1)a_n\mathrm{x}^{n-2} \Rightarrow \mathrm{x}^2\mathrm{y''(x)} = \sum_{n=2}^{\infty} n(n-1)a_n\mathrm{x}^n = \sum_{n=0}^{\infty} n(n-1)a_n\mathrm{x}^n$

because for n=0 term ($na_n = 0$) and for n=1 term [$n(n-1)a_n = 0$)] we can start this sum at n=0

Put expressions of y'', y', y in equation 1)

$\mathrm{x}^2\mathrm{y}'' + 4\mathrm{xy}' + 2\mathrm{y} = e^x \Leftrightarrow \sum_{n=0}^{\infty} n(n-1)a_n\mathrm{x}^n + 4\sum_{n=0}^{\infty} na_n\mathrm{x}^n + 2\sum_{n=0}^{\infty} a_n\mathrm{x}^n = e^x$

$\Leftrightarrow \sum_{n=0}^{\infty}[n^2 + 3n \quad + 2]a_n\mathrm{x}^n = e^x$

$\Leftrightarrow \sum_{n=0}^{\infty}(n+1)(n+2)a_n\mathrm{x}^n = e^x$

since $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$

$\Leftrightarrow \sum_{n=0}^{\infty}(n+1)(n+2)a_n\mathrm{x}^n = \sum_{n=0}^{\infty}\frac{x^n}{n!}$

By identification of both series (their coefficients are equals) :

$(n+1)(n+2)a_n = \frac{1}{n!} \forall \mathrm{n} \geq 0 \ \Leftrightarrow a_n = \frac{1}{(n+1)(n+2)n!} = \frac{1}{(n+2)!} \ \forall \mathrm{n} \geq 0$

Put $a_n$ *in our solution* $y(x) = \sum_{n=0}^{\infty}\frac{1}{(n+2)!}\mathrm{x}^n$

Let us look for the function corresponding to this power series:

$y(x) = \sum_{n=0}^{\infty}\frac{1}{(n+2)!}\mathrm{x}^n = \frac{1}{x^2}\sum_{n=0}^{\infty}\frac{1}{(n+2)!}\mathrm{x}^{n+2} = \frac{1}{x^2}[\frac{x^2}{(2)!} + \frac{x^2}{(3)!} + \frac{x^4}{(4)!} + \cdots ...]$

since $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{(2)!} + \frac{x^2}{(3)!} + \frac{x^4}{(4)!} + \cdots ...]$

thus y(x)= $\frac{1}{x^2}[e^x - 1 - x]$

2)$y''$+y $= 0$ (withy(0)=$\frac{1}{2}$ and $y'(0) = 0$)

y(x)=$\sum_{n=0}^{\infty} a_n x^n$

y'(x)=$\sum_{n=1}^{\infty} na_n x^{n-1}$

y''(x)=$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$

Put expressions of $y''$and y in equation 2)

$y''+y = 0 \Leftrightarrow \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$

Let us unify powers (ie both expressions have the same power$x^n$)

For that, in the first expression put k=n-2 (⇔n=k+2):

$y''+y = 0 \Leftrightarrow \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2} x^k + \sum_{n=0}^{\infty} a_n x^n = 0$

And now rename k by n :

$\sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2} x^k + \sum_{n=0}^{\infty} a_n x^n = 0$

$\Leftrightarrow \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$

$\Leftrightarrow \sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + a_n] x^n = 0$

So coefficients are equal to zero$\Leftrightarrow (n+2)(n+1)a_{n+2} + a_n = 0 \quad \forall n \geq 0$

$\Leftrightarrow \qquad a_{n+2} = \frac{-a_n}{(n+1)(n+2)} \forall n \geq 0$

We have :

y(0)=$\frac{1}{2} \Leftrightarrow$ y(0)=$\sum_{n=0}^{\infty} a_n (0)^n = a_0 + a_1(0) + a_2(0)^2 + \cdots \ldots . = \frac{1}{2} \Leftrightarrow a_0 = \frac{1}{2}$

$y'(0) = 0 \Leftrightarrow$y'(0)=$\sum_{n=1}^{\infty} na_n 0^{n-1} = a_1 + 2a_2(0) + 3a_3(0)^2 + \cdots . = 0 \Leftrightarrow a_1 = 0$

We have found $a_{n+2} = \frac{-a_n}{(n+1)(n+2)} \quad \forall n \geq 0$

n=0 $\quad a_2 = \frac{-a_0}{(0+1)(0+2)} = \frac{-1}{2.1.2.} = \frac{-1}{2.2!}$ ( since $a_0 = \frac{1}{2}$)

n=1 $\quad a_3 = \frac{-a_1}{(n+1)(n+2)} = 0 \quad (since\ a_1 = 0)$

n=2 $\quad a_4 = \frac{-a_2}{(2+1)(2+2)} = \frac{1}{2.1.2.3.4} = \frac{1}{2.4!}$

since $a_2 = \frac{-1}{2.2!}$

n=3 $\quad a_5 = \frac{-a_3}{(n+1)(n+2)} = 0 \quad (since\ a_3 = 0)$

if we keep going we will find :

$a_{2k+1} = 0 \quad \forall k \geq 0$

$a_{2k} = \frac{(-1)^k}{2.(2k)!} \quad \forall k \geq 0$

It left only even powers in the solution $y(x)=\sum_{n=0}^{\infty} a_n x^n$

$\Rightarrow y(x)=\sum_{n=0}^{\infty} a_{2n}x^{2n} \Leftrightarrow y(x)=\sum_{n=0}^{\infty} \frac{(-1)^n}{2.(2n)!}x^{2n}=\frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}$

Let us look for the function corresponding to this power series:

We know that $\cos(x)=\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}$ so $y(x)=\frac{1}{2}\cos(x)$.

## IV.4 Fourier Series

### IV.4.1 Introduction:

Studying Fourier series is important in several fields particularly in physics and engineering. In physics, many natural phenomena (such as sound waves, light waves and vibrations) are periodic. Fourier series allow to represent complex periodic function as a sum of simpler sinusoidal ones, they are therefore essential for modeling and analyzing phenomena cited above.

### IV.4.2 Basics:

#### IV.4.2.1 Definition of a periodic function:

f:ℝ→ℝ is called a T-periodic function if f(x+T)=f(x)

Examples:

1)Functions sin and cos are 2π periodic functions.

2)Functions tang and cot are π periodic functions.

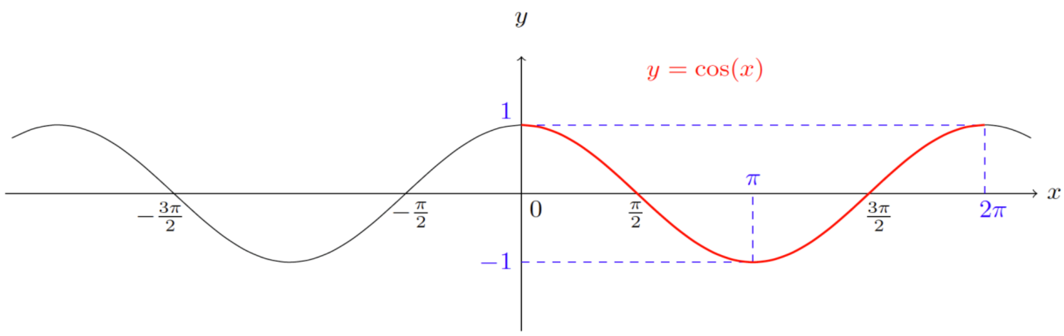



**Figure IV.4.1** Graph of cos(x)

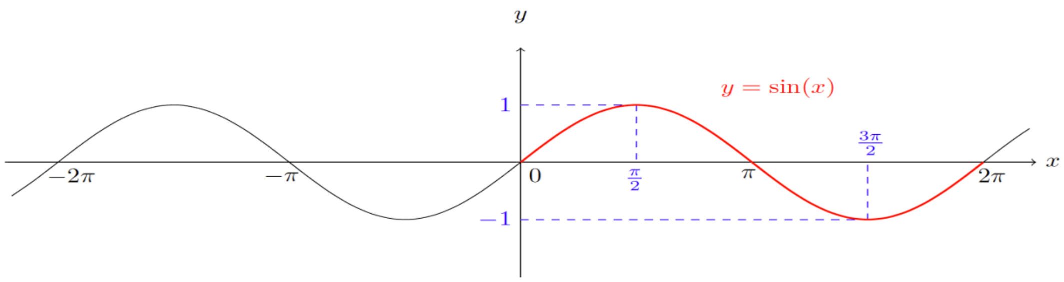



**Figure IV.4.2** Graph of sin(x)

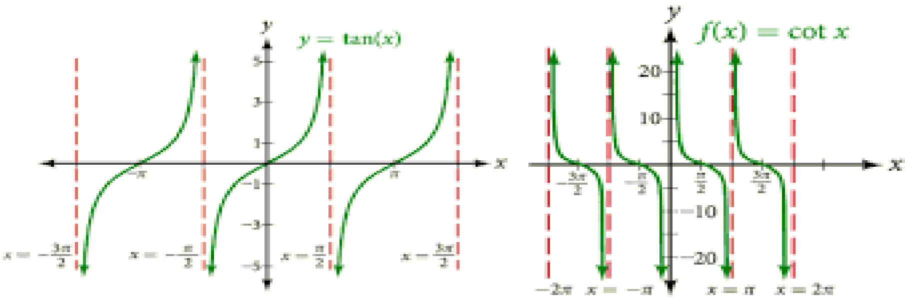



**Figure IV.4.3** Graph of tan(x) and cot(x)

### IV.4.2.2 Definition of an odd function:

f:ℝ→ℝ is said to be an odd function if f(-x)=-f(x)

**Examples:**

1) Functions sin, tang and cot are odd functions.

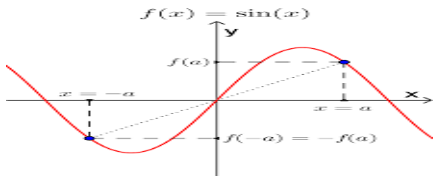



**Figure IV.4.4** sin(x) is an odd function

Notes:

If f is an odd function, then the graph of f is symmetric with respect to the origin.

If f is an odd function, then $\int_{-\pi}^{\pi} f(x)dx = 0$.

### IV.4.2.3 Definition of an even function:

f:ℝ→ℝ is said to be an even function if and f(-x) =f(x).

Example:

Function cos is an even function.

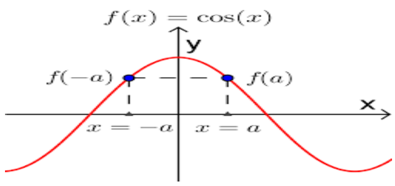



**Figure IV.4.5** cos(x) is an even function

**Notes:**

If f is an even function, then the graph of f is symmetric with respect to y-axis.

if f is an even function, then $\int_{-\pi}^{\pi} f(x)dx = 2\int_{0}^{\pi} f(x)dx$.

### IV.4.2.4 Definition of a piecewise continuous function:

A function f : ℝ → ℝ is said to be piecewise continuous on an interval [a,b], if there exist a finite number of points $x_1 < x_2 < \cdots < x_p$ of [a,b] such as on each open subinterval ]a, $x_1$[ , ]$x_i$, $x_{i+1}$[ , ]$x_p$, b[ the function is continuous, and the four following limits exist ; $\lim_{\substack{x \to x_i \\ <}} f(x) = f(x^-), \lim_{\substack{x \to x_i \\ >}} f(x) = f(x^+)$ , $f(a^+)$ et $f(b^-)$.

**Example:**

Graph of a piecewise continuous function.

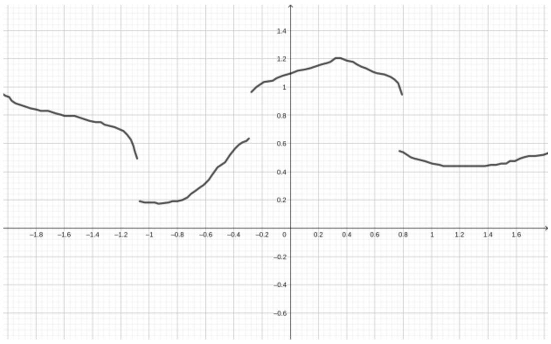


**Figure IV.4.6** A piecewise continuous function

## IV.4.3Definition of Fourier series:

Let f(x) be a T-periodic function, integrable in any closed interval of ℝ.

The following trigonometric series: $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nwx) + b_n \sin(nwx)$

is called the Fourier series associated to the function f(x).

We denote that by:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nwx) + b_n \sin(nwx)$$

where: $w = \frac{2\pi}{T}$

$$a_0 = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)dx = \frac{2}{T}\int_0^T f(x)dx$$

$$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cos(nwx)\,dx = \frac{2}{T}\int_0^T f(x)\cos(nwx)\,dx$$

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\sin(nwx)\,dx = \frac{2}{T}\int_0^T f(x)\sin(nwx)\,dx$$

Notes:

-If f is $2\pi$ periodic, then

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

where: $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nwx)\,dx = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nwx)\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nwx)\,dx = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nwx)\,dx$$

-If f is $2\pi$ periodic and even, then $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\sin(nx)\,dx = 0$

Even. Odd

Odd

and

f(x) ~ $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$

where: $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx$

Even. Odd

Even

-If f is $2\pi$ periodic and odd then $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\cos(nx)\,dx = 0$

odd. even

Odd

and

f(x) ~ $\sum_{n=1}^{\infty} b_n \sin(nx)$

where: $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$

Examples

1)Let S be an odd $2\pi$ periodic function such as S(x) = 1 for 0 <x<πand f(nπ) = 0 for n ∈ℤ.

S(x) ~ $\sum_{n=1}^{\infty} b_n \sin(nx)$

where

$$b_n = \frac{2}{\pi}\int_0^{\pi} S(x)\sin(nx)\,dx = \frac{2}{\pi}\left[\frac{-cosnx}{n}\right]_0^{\pi} = \frac{2}{\pi}\left\{\frac{2}{1},\frac{0}{2},\frac{2}{3},\frac{0}{4},\frac{2}{5},\frac{0}{6},\ldots\ldots\right\}$$

Thus,

$$f(x) \sim \frac{2}{\pi}\left[\frac{2\sin(x)}{1} + \frac{2\sin(3x)}{3} + \frac{2\sin(5x)}{5} + \cdots\right] = \frac{4}{\pi}\left[\frac{\sin(x)}{1} + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \cdots\right]$$

$$= \sum_{n=1}^{\infty} \frac{4sin((2n+1)x)}{\pi(2n+1)}$$

## IV.4.4 Dirichlet's Theorem:

Let f be a T ($T=\frac{2\pi}{\omega}$) periodic function such that*:*

1. f is continuous on any interval I = [a, a + T ] except for finite numbers of points in which f has a hand-right limit f (x + 0)($f(x^+)$)and a hand-left limit f (x − 0)($f(x^-)$)
2. f is differentiable on any interval I =]a, a + T [ except for finite numbers of points in which f has a right derivative and a left one.

then, the Fourier series associated to the function f is such that :

$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nwx) + b_n \sin(nwx)$= f(x) , ∀x f continuous at x

$\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos(nwx)+b_n\sin(nwx)=\frac{f(x^+)+f(x^-)}{2}$, $\forall$x f is discontinuous at x

**Note** :

Fourier series decomposition consists in representing a periodic function as the sum of the most elementary possible periodic functions (namely sines and cosines).

**Examples**

1)Let f *be a* 2π periodic function defined as *:*

f(x)=x ∀ x,−π≤x<π.

Let us check up if Direchlet's requirements are satisfied :

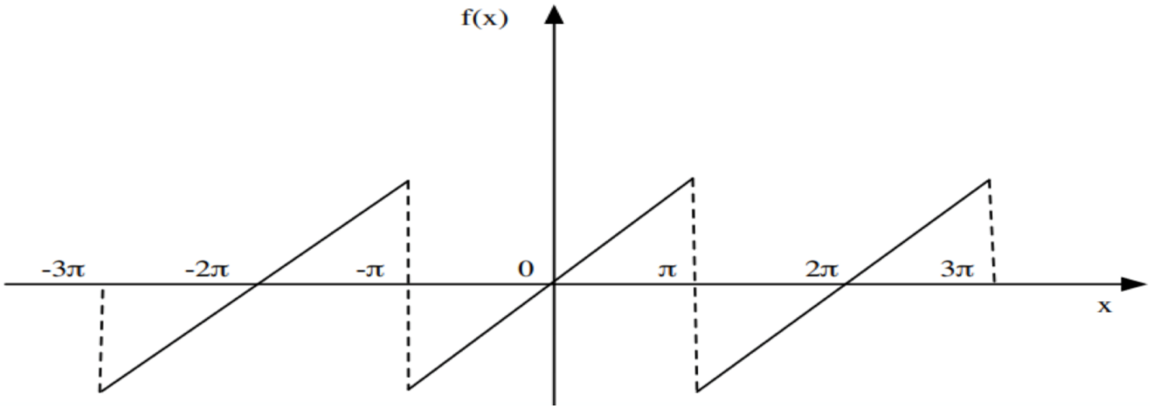



**Figure IV.4.7 Graph of f(x)**

The graph of f is drawn in 3 periods. We can notice that:

- f is continuous on $]-\pi,\pi[$ and it is not continuous at $-\pi$ and at $\pi$

(because also $\lim_{x\to-\pi^+} f(x)=-\pi \neq \lim_{x\to-\pi^-} f(x)=\pi$

and $\lim_{x\to\pi^+} f(x)=-\pi\neq \lim_{x\to\pi^-} f(x)=\pi$)

- f is differentiable on$]-\pi,\pi[$ and it is not differentiable at $-\pi$ and at $\pi$, since it is not continuous at these points .

So, Direchlet's requirements are satisfied and thus;

f(x)=$\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos(nx)+b_n\sin(nx)$on $]-\pi,\pi[$

Since f is odd then $a_n$=0 ∀ n and

$b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx=\frac{2}{\pi}\int_0^{\pi} x\sin(nx)\,dx$

To compute $b_n$, we use integration by parts method:

$$\int_a^b U(x)V'(x)dx=[U(x)V'(x)]_a^b-\int_a^b U'(x)V(x)dx$$

Where U(x)=x, V'(x)=sin(nx)

$b_n=\frac{2}{\pi}\int_0^{\pi} x\sin(nx)\,dx=\frac{2}{\pi}\left[-x\frac{1}{n}\cos(nx)\right]_0^{\pi}-\frac{2}{\pi}\int_0^{\pi}-\frac{1}{n}\cos(nx)dx$

$=\frac{2}{\pi}\left[-\pi\frac{1}{n}\cos(n\pi)-0\right]+\frac{2}{n\pi}\int_0^{\pi}\cos(nx)dx$

$= -\frac{2}{n}\cos(n\pi)+\frac{2}{n\pi}\frac{1}{n}\sin(n\pi)$

We know that: $\cos(n\pi)=(-1)^n$ and $\sin(n\pi)=0$

So $b_n=(-1)^{n+1}\frac{2}{n}$

and consequently $f(x)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\sin(nx)\forall\ x\in]-\pi,\pi[$

2) Let f *be an even* π periodic function defined as:

$g(x)=1-\frac{2x}{\pi}\forall\ x, 0\leq x\leq\pi.$

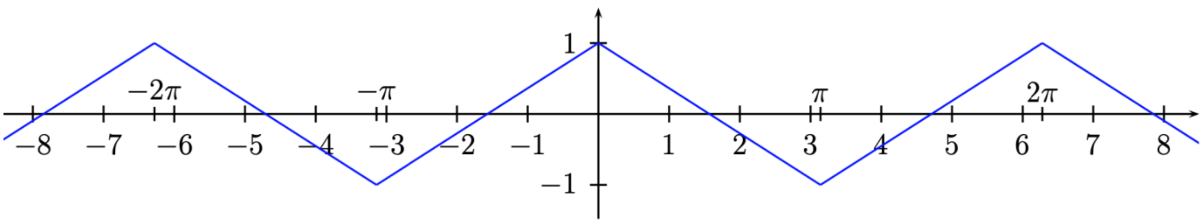



**Figure IV.4.8 Graph of g(x)**

The graph of f is drawn in 3 periods. From the graph, we can see that g is continuous $\forall\ x\in\mathbb{R}$ g is differentiable on any interval I =]0, π [ *so* Direchlet's requirements are satisfied and thus ;

$g(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos(nx)+b_n\sin(nx)\quad\forall\ x\in\mathbb{R}$

Since g is even then $b_n=0\ \ \forall$ n.

$a_0=\frac{2}{\pi}\int_0^{\pi}g(x)dx=\frac{2}{\pi}\int_0^{\pi}(1-\frac{2x}{\pi})dx=\frac{2}{\pi}\left[(x-\frac{x^2}{\pi})\right]_0^{\pi}=0$

$a_n=\frac{2}{\pi}\int_0^{\pi}g(x)\cos(nx)\,dx=\frac{2}{\pi}\int_0^{\pi}(1-\frac{2x}{\pi})\cos(nx)\,dx$

We use integration by parts method:

Where $U(x)=(1-\frac{2x}{\pi})$, $V'(x)=\cos(nx)$

$a_n=\frac{2}{\pi}\left[(1-\frac{2x}{\pi})\frac{\sin(nx)}{n}\right]_0^{\pi}+\frac{4}{\pi^2n}\int_0^{\pi}\sin(nx)\,dx=\frac{4}{\pi^2n}\left[-\frac{\cos(nx)}{n}\right]_0^{\pi}=\frac{4(1-(-1)^n)}{\pi^2n^2}$

Thus $f(x)=\sum_{n=1}^{\infty}\frac{4(1-(-1)^n}{\pi^2n^2}\cos(nx)\forall\ x\in\mathbb{R}$

Since$1-(-1)^n=\begin{cases}0\ if\ n\ is\ even\\2\ if\ n\ is\ odd\end{cases}$, then $g(x)=\frac{8}{\pi^2}\sum_{p=0}^{\infty}\frac{\cos((2p+1)x)}{(2p+1)^2}\forall\ x\in\mathbb{R}$

**Note:**

$g(x)=\frac{8}{\pi^2}\sum_{p=0}^{\infty}\frac{\cos((2p+1)x)}{(2p+1)^2}\forall\ x\in\mathbb{R}$

For x=0, we have $g(0)=1=\frac{8}{\pi^2}\sum_{p=0}^{\infty}\frac{1}{(2p+1)^2}\Rightarrow\sum_{p=0}^{\infty}\frac{1}{(2p+1)^2}=\frac{\pi^2}{8}$

We recall that In chapter 2 (infinite series), we knew that $\sum_{p=0}^{\infty}\frac{1}{(2p+1)^2}$ converges but we were not able to evaluate its sum, now using Fourier series we can do it.

We can also deduce from this example the sum of $\sum_{p=0}^{\infty}\frac{1}{p^2}$

Let S be the sum of $\sum_{p=0}^{\infty}\frac{1}{p^2}$i.e S=$\sum_{p=0}^{\infty}\frac{1}{p^2}$

So S=$\sum_{p=0}^{\infty}\frac{1}{(2p+1)^2}+\sum_{p=1}^{\infty}\frac{1}{(2p)^2}=\frac{\pi^2}{8}+\frac{S}{4}\Rightarrow\frac{3S}{4}=\frac{\pi^2}{8}\Rightarrow S=\frac{4\,\pi^2}{8x3}=\frac{\pi^2}{6}$

Thus, $\sum_{p=0}^{\infty}\frac{1}{p^2}=\frac{\pi^2}{6}$and $\sum_{p=1}^{\infty}\frac{1}{(2p)^2}=\frac{\pi^2}{24}$

## IV.4.5 Parseval's identity:

If f is T periodic and piecewise continuous

Then

$$\frac{2}{T}\int_0^T (f(x))^2\,dx=\frac{(a_0)^2}{2}+\sum_{n=1}^{\infty}(a_n)^2+(b_n)^2$$

$(or\ \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}(f(x))^2\,dx=\frac{(a_0)^2}{2}+\sum_{n=1}^{\infty}(a_n)^2+(b_n)^2)$

**Example 1 :**

Let f *be an even* $\pi$ periodic function defined as:

g(x)=1-$\frac{2x}{\pi}\forall$ x,0≤x≤π.

We have already seen this example. The function f satisfies Parseval's identity requirement.

We have found :

$a_0=0\ ,a_n=\frac{8}{\pi^2(2n+1)^2}$ n≥ 1

$$\frac{2}{\pi}\int_0^{\pi}(g(x))^2\,dx=\sum_{n=1}^{\infty}(\frac{8}{\pi^2(2n+1)^2})^2\Leftrightarrow\frac{2}{\pi}\int_0^{\pi}(1-\frac{2x}{\pi})^2\,dx=\sum_{n=1}^{\infty}\frac{64}{\pi^4(2n+1)^4}$$

$$\Leftrightarrow\frac{2}{\pi}\int_0^{\pi}(1-\frac{4x}{\pi}+\frac{4x^2}{\pi^2})\,dx=\sum_{n=1}^{\infty}\frac{64}{\pi^4(2n+1)^4}\Leftrightarrow\frac{2}{\pi}\left[x-\frac{4x^2}{2\pi}+\frac{4x^3}{3\pi^2}\right]_0^{\pi}=\sum_{n=1}^{\infty}\frac{64}{\pi^4(2n+1)^4}$$

$$\Leftrightarrow\frac{2}{\pi}\left[\pi-\frac{4\pi^2}{2\pi}+\frac{4\pi^3}{3\pi^2}\right]_0^{\pi}=\sum_{n=1}^{\infty}\frac{64}{\pi^4(2n+1)^4}\Leftrightarrow\frac{2}{\pi}\left[\frac{6\pi^3-12\pi^3+8\pi^3}{6\pi^2}\right]=\sum_{n=1}^{\infty}\frac{64}{\pi^4(2n+1)^4}$$

$\Leftrightarrow\frac{2}{3}=\sum_{n=1}^{\infty}\frac{64}{\pi^4(2n+1)^4}\Leftrightarrow\sum_{n=1}^{\infty}\frac{1}{(2n+1)^4}=\frac{\pi^4}{96}$

**Example2**

Given the Fourier series $x^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx)\forall$ x, -π $< x < \pi$(see exercise1a)below)

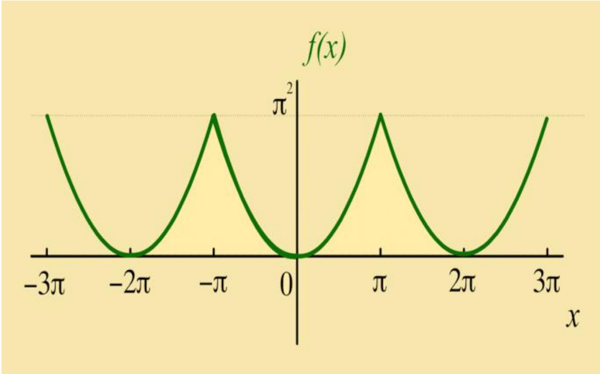



**Figure IV.4.9** Graph of f(x)= $x^2 \forall x \in ]-\pi, \pi[$, 2π periodic

We can deduce the value of $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

Indeed :

Here $a_0 = \frac{2\pi^2}{3}$, $a_n = \frac{4(-1)^n}{n^2}$, $b_n = 0$.

Using Parseval's identity :

We have $\frac{2}{2\pi} \int_{-\pi}^{\pi} x^4 dx = \frac{2\pi^4}{2x9} + \sum_{n=1}^{\infty} \frac{16}{n^4}$

Given us $\frac{2\pi^4}{5} = \frac{2\pi^4}{9} + \sum_{n=1}^{\infty} \frac{16}{n^4} \Rightarrow \frac{8\pi^4}{45} = \sum_{n=1}^{\infty} \frac{16}{n^4}$

Therefore $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$.

## Exercises of Fourier Series

**Exercise1:**

Let f be a 2π periodic function defined as:

a)f(x)=$x^2 \forall x \in ]-\pi, \pi[$

b)f(x)=$x^2 \forall x \in [0,2\pi[$

Determine Fourier series of f in both cases 1) and 2).

**Exercise2:**

Determine Fourier series of a 10 periodic function f, defined as

F(x)=$\begin{cases} 0 & if\ x \in ]-5,0[ \\ 3 & if\ x \in ]0,5[ \end{cases}$

Exercise3 :

Consider a 2π periodic function defined as

F(x)=$\begin{cases} -x & if\ x \in [-\pi, 0] \\ x & if\ x \in ]0, \pi] \end{cases}$

1) Determine Fourier series of f.

2) Evaluate the sum of the following infinite series :$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$ ,$\sum_{n=1}^{\infty} \frac{1}{n^2}$and $\sum_{n=1}^{\infty} \frac{1}{(2n)^2}$

## Solution of exercises of Fourier Series

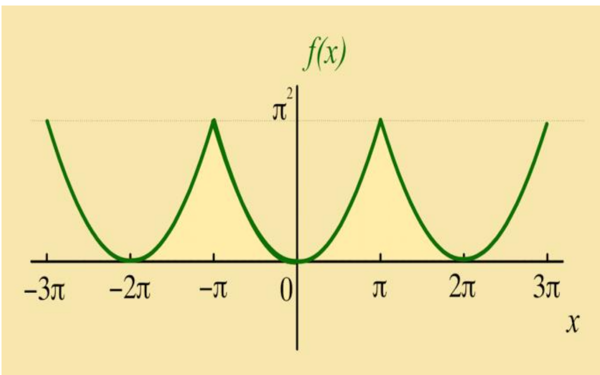
Usually, what to do in a Fourier series exercise ?

- Draw the graph of f in several periods.
- Check parity, classe (continuity, differentiability) of f.
- Compute Fourier's coefficients of f ($a_n$, $b_n$ according to the context).
- Apply Direchlet and/or Parseval's identity.

**Solution of exercise1**

a)f(x)=$x^2 \forall x \in ]-\pi, \pi[$

Graph of f




We have drawn the graph of f in 3 periods. We can see that our function is even so

$b_n = 0$

Let us compute $a_{0,}\ a_n$

$a_0=\frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi} x^2\, dx= \frac{2(\pi)^3}{3\pi} = \frac{2\pi^2}{3}$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2\cos(nx)\,dx$$

We use integration by parts method twice:

Where in the first one: U(x)=$x^2$, V'(x)=cos(nx)

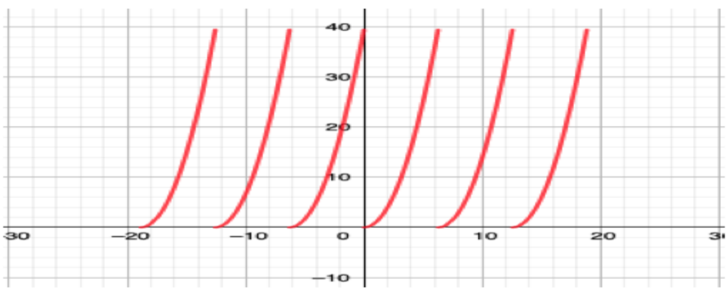
$a_n = \frac{2}{\pi}\left[x^2\frac{\sin(nx)}{n}\right]_0^{\pi} - \frac{4}{\pi n}\int_0^{\pi} x\sin(nx)\,dx = =0 - \frac{4}{\pi n}\int_0^{\pi} x\sin(nx)\,dx = -\frac{4}{\pi n}\left[-\frac{x\cos(nx)}{n}\right]_0^{\pi} - \frac{4}{\pi n^2}\int_0^{\pi}\cos(nx)\,dx$

$= -\frac{4}{\pi n}\left[-\frac{x\cos(nx)}{n}\right]_0^{\pi} - \frac{4}{\pi n^2}\int_0^{\pi}\cos(nx)\,dx = -\frac{4}{\pi n}\left[-\frac{x\cos(nx)}{n}\right]_0^{\pi} - \frac{4}{\pi n^2}\left[-\frac{\sin(nx)}{n}\right]_0^{\pi} = -\frac{4}{\pi n}\left[-\frac{x\cos(nx)}{n}\right]_0^{\pi} - 0 = -\frac{4}{\pi n}\left[-\frac{\pi\cos(n\pi)}{n}\right] = \frac{4(-1)^n}{n^2}$

Since f is continuous ∀ x∈ ℝ

Thus f(x)=$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nwx) = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos(nx)$ ∀ x∈ℝ

b) Graph of f




We have drawn the graph of f in 6 periods. From the graph we can see that our function is neither even nor odd.

Let us evaluate $a_{0,}\ a_n, b_n$ :

$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \int_0^{2\pi} x^2\, dx = \frac{1}{\pi}\frac{(2\pi)^3}{3} = \frac{4\pi^2}{3}$

$a_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)dx=\frac{1}{\pi}\int_0^{2\pi} x^2\cos(nx)dx$

Using integration by parts methods twice, we obtain :

$a_n=\frac{8}{n^2}$ (see case1 above, same thing just the interval has changed)

$b_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)dx=\frac{1}{\pi}\int_0^{2\pi} x^2\sin(nx)dx$

As above, we use integration by parts method twice:

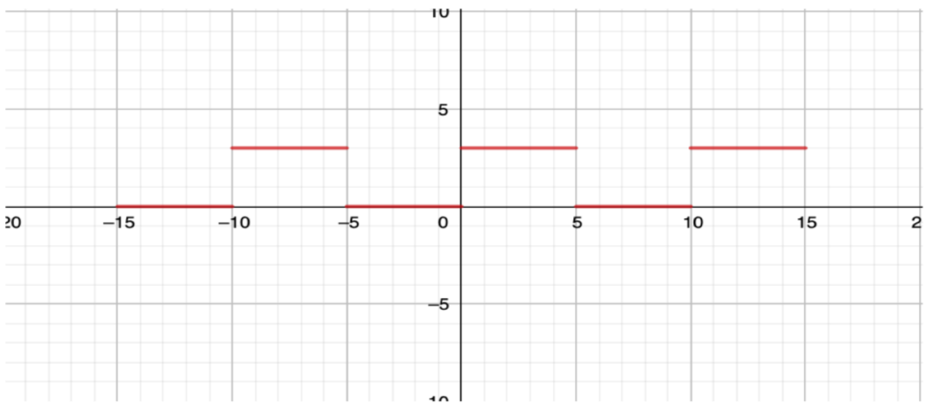
$b_n = \frac{1}{\pi}\left[-x^2\frac{\cos(nx)}{n}\right]_0^{2\pi} + \frac{1}{\pi n}\int_0^{2\pi} x\cos(nx)\,dx = \frac{1}{\pi n}\left[-\frac{(2\pi)^2\cos(2n\pi)}{n}\right] + \frac{1}{\pi n}\int_0^{2\pi} x\cos(nx)\,dx = -\frac{4\pi}{n^2} + \frac{1}{\pi n}\left[\frac{x\sin(nx)}{n}\right]_0^{2\pi} - \frac{1}{\pi n^2}\int_0^{2\pi}\sin(nx)\,dx = -\frac{4\pi}{n^2} + 0 - \frac{1}{\pi n^2}\int_0^{2\pi}\sin(nx)\,dx = -\frac{4\pi}{n^2} - \frac{1}{\pi n^2}\left[-\frac{\cos(nx)}{n}\right]_0^{2\pi} = -\frac{4\pi}{n^2} + 0$

$b_n = -\frac{4\pi}{n^2}$

Fourier series of f is $\frac{2\pi^2}{3} + \sum_{n=1}^{\infty}\frac{8}{n^2}\cos(nx) - \frac{4\pi}{n^2}\sin(nx)$

**Solution of exercise2**

Graph of f




The graph of f is drawn in 6 periods. We can notice that f is neither even nor odd function.

F is piecewise continuous on $\mathbb{R}$, its discontinuity points are 5k (k$\in\mathbb{R}$)

Let us compute $a_0,\ a_n, b_n$ :

$a_0 = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)dx = \frac{2}{T}\int_0^T f(x)dx$

$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cos(nwx)\,dx = \frac{2}{T}\int_0^T f(x)\cos(nwx)\,dx$

$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\sin(nwx)\,dx = \frac{2}{T}\int_0^T f(x)\sin(nwx)\,dx$

$a_0 = \frac{2}{10}\int_{-5}^{5} f(x)\,dx = \frac{2}{10}\left(\int_{-5}^{0} f(x)\,dx + \frac{1}{10}\int_0^5 f(x)\,dx\right) = \frac{2}{10}\int_0^5 3\,dx = \frac{2x5(3)}{10} = 3$

$a_n = \frac{2}{10}\int_{-5}^{5} f(x)\cos(\frac{2n\pi x}{10})dx = \frac{2}{10}\int_0^5 3\cos(\frac{n\pi x}{5})dx = \frac{3}{5}\left[\frac{5}{n\pi}sin\,\frac{n\pi x}{5}\right]_0^5 = 0$ if $n \neq 0$

$b_n = \frac{2}{10}\int_{-5}^{5} f(x)\sin(\frac{2n\pi x}{10})dx = \frac{2}{10}\int_0^5 3\sin(\frac{n\pi x}{5})dx =$

$\frac{3}{5}\left[-\frac{5}{n\pi}\cos(\frac{n\pi x}{5})\right]_0^5 = 3\left[-\frac{1}{n\pi}\cos(\frac{n\pi x}{5})\right]_0^5 = 3(\frac{-\cos(n\pi)}{n\pi} + \frac{1}{n\pi}) = \frac{3(1-\cos(n\pi))}{n\pi}$

So the Fourier series of f is $\frac{3}{2} + \sum_{n=1}^{\infty}\frac{3(1-\cos(n\pi))}{n\pi}\sin(\frac{n\pi x}{5})$

**Notes:**

According to Direchlet's theorem :

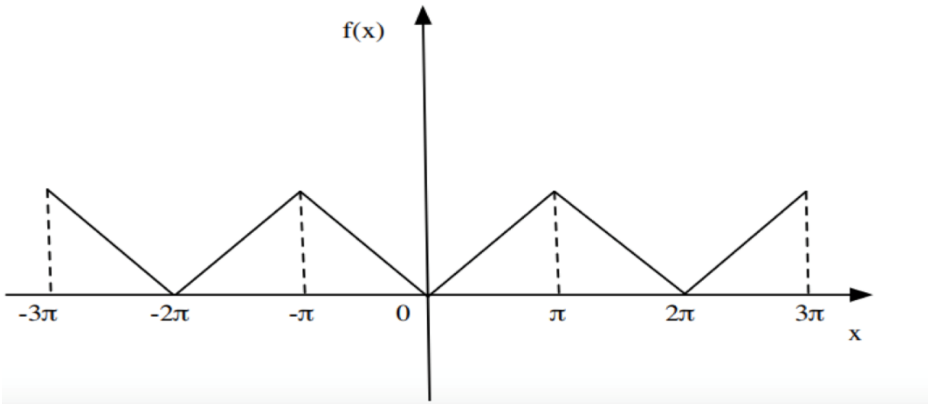
$\frac{3}{2}+\sum_{n=1}^{\infty}\frac{3(1-\cos(n\pi))}{n\pi}\sin\left(\frac{n\pi x}{5}\right)$ =f(x) in any point of continuity ( in particularly on $]-5,0[$ $and$ $]0,5[$)

$\frac{3}{2}+\sum_{n=1}^{\infty}\frac{3(1-\cos(n\pi))}{n\pi}\sin\left(\frac{n\pi x}{5}\right)=\frac{f(x^+)+f(x^-)}{2}$ in any point of discontinuity in particulary at -5, 0, 5

Thus

$$\frac{3}{2}+\sum_{n=1}^{\infty}\frac{3(1-\cos(n\pi))}{n\pi}\sin\left(\frac{n\pi x}{5}\right)=\begin{cases}\frac{3}{2} & if\ x=-5\\ 0 & if\ \ x\in\,]-5,0[\\ \frac{3}{2} & if\ x=0\\ 3 & if\ \ x\in\,]0,5[\\ \frac{3}{2} & if\ x=5\end{cases}$$

**Solution of exercise3**




1) We have drawn the graph of f in 3 periods. We can see that our function is even so $b_n=0$

Let us compute $a_{0,}$ $a_n$

$a_0$=$\frac{2}{\pi}\int_0^{\pi}f(x)dx=\frac{2}{\pi}\int_0^{\pi}x\,dx$= $\frac{2(\pi)^2}{2\pi}=\pi$

$$a_n=\frac{2}{\pi}\int_0^{\pi}x\cos(nx)\,dx$$

We use integration by parts method:

Where : U(x)=$x$, V'(x)=cos(nx)

$a_n=\frac{2}{\pi}\left[x\frac{\sin(nx)}{n}\right]_0^{\pi}-\frac{2}{\pi n}\int_0^{\pi}\sin(nx)\,dx$==0$-\frac{2}{\pi n}\int_0^{\pi}\sin(nx)\,dx=$

$-\frac{2}{\pi n}\left[-\frac{\cos(nx)}{n}\right]_0^{\pi}$=$\frac{-2(1-(-1)^n)}{\pi n^2}$=$\frac{2((-1)^n-1)}{\pi n^2}$

From the graph, we can see that f is continuous $\forall$ x$\in\mathbb{R}$

f is differentiable on any interval I =]0, π [ *so* Direchlet's requirements are satisfied and thus ;

f(x)=$\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos(nwx)$=$\frac{\pi}{2}+\sum_{n=1}^{\infty}\frac{2((-1)^n-1)}{\pi n^2}\cos(nx)$ $\forall$ x$\in\mathbb{R}$

Since$(-1)^n-1=\begin{cases}0 \; if \; n \; is \; even\\ -2 \; if \; n \; is \; odd\end{cases}$, then f(x)= $\frac{\pi}{2}-\frac{4}{\pi}\sum_{p=0}^{\infty}\frac{\cos((2p+1)x)}{(2p+1)^2}$ $\forall$ x$\in\mathbb{R}$

2) We have f(x)=$\frac{\pi}{2}-\frac{4}{\pi}\sum_{p=0}^{\infty}\frac{\cos((2p+1)x)}{(2p+1)^2}$ $\forall$ x$\in\mathbb{R}$

For x=0 f(0)=0=$\frac{\pi}{2}-\frac{4}{\pi}\sum_{p=0}^{\infty}\frac{1}{(2p+1)^2} \Rightarrow \sum_{p=0}^{\infty}\frac{1}{(2p+1)^2}=\frac{\pi^2}{8}$.

We have:

$\sum_{p=1}^{\infty}\frac{1}{p^2}=\sum_{p=0}^{\infty}\frac{1}{(2p+1)^2}+\sum_{p=1}^{\infty}\frac{1}{(2p)^2}=\sum_{p=0}^{\infty}\frac{1}{(2p+1)^2}+\frac{1}{4}\sum_{p=1}^{\infty}\frac{1}{p^2}=\frac{\pi^2}{8}+\frac{1}{4}\sum_{p=1}^{\infty}\frac{1}{p^2}$

$\Rightarrow \frac{3}{4}\sum_{p=1}^{\infty}\frac{1}{p^2}=\frac{\pi^2}{8}$

$\Rightarrow \sum_{p=1}^{\infty}\frac{1}{p^2}=\frac{4\,\pi^2}{8x3}=\frac{\pi^2}{6}$

Thus, $\sum_{p=0}^{\infty}\frac{1}{p^2}=\frac{\pi^2}{6}$and $\sum_{p=1}^{\infty}\frac{1}{(2p)^2}=\frac{\pi^2}{24}$

# Chapter V
# Fourier Transform

## V.1. Introduction:

Fourier transform is one of the important concepts used in analyzing signals because it allows us to break down complex signals into their frequency components. This makes it easier to filter, compress, and interpret the signals, which has applications in many fields including telecommunication, audio, and control system. By working in the frequency domain we can gain insights into periodicities and anomalies that are not easily visible in the time domain.

## V.2 Definition:

Let f be a function f: ]- ∞, +∞[ $\longrightarrow$ $\mathbb{R}/\ C$ such that $\int_{-\infty}^{+\infty}|f(t)|dt$ is finite i.e. convergent.

t $\longrightarrow$ f(t)

Fourier transform of f(t) is the application $\hat{f}$: $]-\infty, +\infty[\ \longrightarrow C$

s $\longrightarrow$ $\hat{f}$ (s)=$\int_{-\infty}^{+\infty} f(t)e^{-2\pi ist}dt$

where $e^{-2\pi ist}$ is the exponential complex: $e^{i(-2\pi st)} = \cos(-2\pi st) + i\ sin(-2\pi st)$ (Euler's Formula)

**Notes:**

- $\hat{f}$ (s) is a complex number therefore it has a real part and an imaginary part ($\hat{f}$ (s) =a+ib) as well as a magnitude ( $|\hat{f}\ (s)| = \sqrt{(a)^2 + (b)^2}$) and an argument (or phase).

- We can also denote Fourier transform of f(t) by $F$ (f(t))= $\hat{f}$ (s) =$\int_{-\infty}^{+\infty} f(t)e^{-2\pi ist}dt$

-Fourier transform is an improper integral since limits of integration are infinity, so the condition required above ($\int_{-\infty}^{+\infty}|f(t)|dt$ convergent) insure the existence of $\hat{f}(s)$(just use the comparison test see chapter 2)

-Fourier transform is a generalization of Fourier series (which is applied only for periodic functions) to non-periodic functions.

-It is important to mention that they are various definitions of Fourier transform and the difference is about the factor ($2\pi$) where to put it and the sign in the exponential (+) or (-). The consequences of that are only multiplicative factors in future formulas. All the conventions are in use in practice. We will summarize these different variations below.

$F$ (f(t))= $\hat{f}$ (s) =$\frac{1}{L}\int_{-\infty}^{+\infty} f(t)e^{Mist}dt$

L = $\sqrt{2\pi}$ M = ±1

L=2π M = ±1

L = 1 M = ±2π

L = 1 M = ±1

Our definition of the Fourier transform (L = 1 M = -2π) is a standard one, used in mechanics, electronics, quantum physics, etc.

- The variable t can be time , in this case, the variable s has the dimension of a frequency (Hz). Then $\hat{f}$ (s) represents the frequency spectrum of the signal f(t).

- The variable t can be sometimes a position x (m); in this case, $\hat{f}$ (k)=$\int_{-\infty}^{+\infty} f(x)e^{ikt}dx$) where the variable k=$\frac{2\pi}{\lambda}$ (λ wavelength) has the dimension of the inverse of a length so k is the so-called wave number.

When the function f(t) represents a signal e.g. an image, sound wave, electromagnetic wave (t designating time or space variable), its Fourier transform $\hat{f}$(s) is its spectrum, with s represents the frequency or pulsation. Therefore the Fourier transform converts the time-domain signal into the frequency-domain representation $\hat{f}$ (s), which tells us what frequencies are present in the signal and their corresponding amplitudes ($|\hat{f}$ (s)$|$), phases ( arg$\widehat{(f}$ (s))), and their power spectral density ( $|\hat{f}$ (s)$|$) $^2$).

Examples:

1) f(t) =$\delta(t)$

   $\delta$ is Dirac Delta Function or Unit Impulse Function is defined as $\delta(t)$ = 0, t≠ 0 such that

$$\int_{-\infty}^{+\infty} \delta(t)dt = 1,$$

It is zero everywhere except one point '0'. Delta function in sometimes thought of having infinite value at t=0.

We are going to use one property of Dirac function to compute its Fourier Transform, namely

$$\int_{-\infty}^{+\infty} f(t)\delta(t)dt = \mathrm{f}(0),$$

Consequently,

F($\delta(t)$)=$\hat{\delta}$ (s) = $\int_{-\infty}^{+\infty} \delta(t)e^{-2\pi ist}dt$=$e^{-2\pi is0}$ =1

2) $f(t) = \begin{cases} b & -a \leq t \leq a \\ 0 & |t| \quad > a \end{cases}$

   f is called the box (or gate) function. We are going to denote it by (rect)(t) for further use (rect: referring to rectangle).

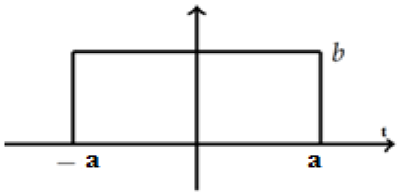



**Figure V.1** graph of the box functions

F$(f(t)) = \hat{f}$ (s)=$\int_{-\infty}^{+\infty} f(t)e^{-2\pi ist}dt$=$\int_{-a}^{+a} be^{-2\pi ist}dt$=$b\left[\frac{e^{-2\pi ist}}{-2\pi is}\right]_{-a}^{a}$= $b[\frac{e^{-2\pi isa}}{-2\pi is} - \frac{e^{-2\pi is(-a)}}{-2\pi is}]$

Knowing that sin(x)=$\frac{e^{ix}-e^{-ix}}{2i}$ so

$\hat{f}$ (s)= $\widehat{rect}$ (s)= $\frac{b}{\pi s}\left[\frac{e^{i(2a\pi s)}-e^{-i(2a\pi s)}}{2i}\right] = \frac{b}{\pi s}sin(2a\pi s) = \frac{2ab}{2a\pi s}sin(2a\pi s) = 2ab\frac{\sin(2a\pi s)}{2a\pi s}$

$\widehat{rect}$ (s)= $2ab\frac{\sin(2a\pi s)}{2a\pi s}$ for s$\neq$ 0

For s=0, $\widehat{rect}$ (s)= $\int_{-a}^{+a} bdt$ =2ab

We can notice that the Fourier Transform of the function box is the cardinal sine function by a factor of 2ab (this function is denoted sinc, sinc(x)= $\frac{\sin(x)}{x}$)

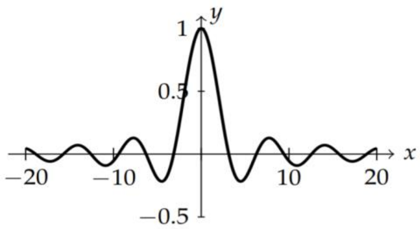



**Figure V.2** General shape of $\frac{\sin(x)}{x}$.

## V.3 Fourier transform properties:

### V.3.1 time shift

F[f(t − c)] = $e^{-i2\pi sc}\ \hat{f}$ (s) (time- shift)

Proof:

F[f(t − c)] =$\int_{-\infty}^{+\infty} f(t-c)e^{-2\pi ist}dt$

Substitute: u=t-c

F[f(t − c)] =$\int_{-\infty}^{+\infty} f(u)e^{-2\pi is(u+c)}du$=$e^{-2\pi isc}\int_{-\infty}^{+\infty} f(u)e^{-2\pi isu}du$=$e^{-2\pi isc}$F(f(t))

= $e^{-i2\pi sc}\ \hat{f}$ (s)

**Notes:**

-If f(t) is a signal, a shift in time does not affect the magnitude of the spectrum

$(|F[f(t-c)]| = |e^{-i2\pi sc}\ \hat{f}(s)\ | = |\hat{f}(s)|)$ but alters its phase;

$\arg(e^{-i2\pi sc}\,\hat{f}\,(s) \;= \arg(\hat{f}(s)) - 2\pi sc$

$e^{-i2\pi sc}$ is called phase factor.

**Example:**

$F[\delta\,(t-c)] = e^{-i2\pi sc}\,\underbrace{\hat{\delta}\,(s)}_{1} = e^{-i2\pi sc}$

## V.3.2 Frequential shift

$F[f(t)\,e^{i2\pi ct}] = \hat{f}\,(s-c)$

Proof:

$F[f(t)\,e^{-ibt}] = \int_{-\infty}^{+\infty} f(t)e^{i2\pi ct}e^{-2\pi ist}dt \int_{-\infty}^{+\infty} f(t)e^{-i2\pi(s-c)t}dt = \hat{f}\,(s-c)$

## V.3.3 time- Scaling

$F[f(ct)] = \frac{1}{|c|}\hat{f}\,(\frac{s}{c})$ a∈R*

Proof:

$F[f(ct)] = \int_{-\infty}^{+\infty} f(ct)e^{-2\pi ist}dt$

Substitute: u=ct

If $c > 0$

$F[f(ct)] = \int_{-\infty}^{+\infty} f(u)e^{-2\pi is\frac{u}{c}}\frac{du}{c} = \frac{1}{c}\int_{-\infty}^{+\infty} f(u)e^{-2\pi i\frac{s}{c}u}du = \frac{1}{c}\hat{f}\,(\frac{s}{c})$

If $c < 0$

$F[f(ct)] = \int_{+\infty}^{-\infty} f(u)e^{-2\pi is\frac{u}{c}}\frac{du}{c} = -\int_{-\infty}^{+\infty} f(u)e^{-2\pi is\frac{u}{c}}\frac{du}{c} = \frac{1}{-c}\,\hat{f}\,(\frac{s}{c})$

If f(t) is a signal, the time-scaling states that if a signal is expended in time by (c), then its Fourier transform is compressed in frequency by the same amount.

## V.3.4 Duality

Suppose f(t) has the Fourier Transform $\hat{f}(s)$ and we would like to evaluate the Fourier transform of the new function $\hat{f}\,(t)$ (we have to change s by t to determine it).

So $F(\hat{f}\,(t)) = f(-s)$ (we get back to the initial function)

We use this property specially to compute the Fourier transform of some complex function that we cannot do it directly with the definition.

Examples:

1) We have found before, that the Fourier Transform of the function box (rect(t)) is the cardinal sine function ie $F(rect(t)) = \frac{\sin(s)}{s}$ =sinc(s)

So by applying duality property, we obtain:

$F(\frac{\sin(t)}{t})$= rect (-s) and knowing that the function box is even so

$F(\frac{\sin(t)}{t})$= rect (-s)=rect (s)= $\begin{cases} b & -a \leq s \leq a \\ 0 & |s| \ > a \end{cases}$

2) We have found before that F($\delta$ (t))= $\hat{\delta}$ (s)=1

By using the duality property, we get

F[$\hat{f}$ (t)] = f(−s) ⇒ F[$\hat{\delta}$ (t)] = f(−s) ⇒ F[1] = $\delta$ (−s)

3)We have seen before that F[$\delta$ (t − c)]=$e^{-i2\pi sac}$=$\hat{f}$ (s)

So F[$\hat{f}$ (t)] = f(−s) ⇒ F[$e^{-i2\pi tc}$] = $\delta$ (−s)

### V.3.5 Convolution and Modulation:

The convolution of two functions in time is defined by

f(t)*g(t)= $\int_{-\infty}^{+\infty} f(\mathrm{u})g(t-u)du$

then F(f(t)*g(t))=F$\big($f(t)$\big)$. F(g(t)) $=$ $\hat{f}$ (s) . $\hat{g}$ (s)

A function is modulated by another function if they are multiplied in time

Then F[f(t).g(t)]=F(f(t))*F(g(t)).

## V.4 linearity of Fourier transform:

$F$(f(t) $\pm$g(t)) = $F$(f (t)) $\pm F$(g(t))

$F(k$f(t))=kF (f (t)) k∈ $\mathbb{R}^*$

**Examples:**

1) $F$(4-7$\delta$ (t))=$F$(4)$-F$(7$\delta$ (t))

=4$F$(1)$-$7$F$($\delta$ (t))

=$\delta$ (−s) $-$7

2)$F(5\frac{\sin(t)}{t}+3\delta$ (t))=5$F(\frac{\sin(t)}{t})$+3$F$($\delta$ (t))

=5rect (s) +3 =$\begin{cases} b+3 & -a \leq t \leq a \\ 3 & |t| \ > a \end{cases}$

## V.5 Inverse of Fourier Transform

**Definition:**

Let f be a function: f: ]- ∞, +∞[ ⟶ ℝ and F(s) Fourier transform of f(t)

t ⟶ f(t)

ie $\hat{f}$ (s) = $F$(f(t)) then f(t)= $F^{-1}(\hat{f}(s))$ = $\int_{-\infty}^{+\infty} \hat{f}\ (\mathrm{s})e^{2\pi ist}ds$ where $F^{-1}$ is called the inverse of Fourier transform.

-We notice that the expression of the inverse Fourier transform $F^{-1}$is very similar to that of Fourier transform F. Indeed, only the sign of the complex exponential changes:+i instead of –i

- This inverse transformation implies that from the frequency representation of the signal we can get its temporal representation.

**Examples:**

A) Using the definition

$F^{-1}(\delta(s)) = \int_{-\infty}^{+\infty} \delta(s)e^{2\pi ist}ds = e^{2\pi i0t} = 1$

$F^{-1}(rect(s)) = \int_{-a}^{+a} be^{2\pi ist}ds = b\left[\frac{e^{2\pi ist}}{2\pi it}\right]_{-a}^{a} = b\left[\frac{e^{2\pi ita}}{2\pi it} - \frac{e^{2\pi it(-a)}}{2\pi it}\right]$

$= \frac{b}{\pi t}\left[\frac{e^{i2a\pi t} - e^{-i2a\pi t}}{2i}\right] = 2ab\frac{\sin(2a\pi t)}{2\pi t}$

Therefore $F^{-1}(rect(s)) = 2absinc(2a\pi t)$

B) From the examples seen above in **VII.2** we can say:

1) $F(\delta(t)) = 1 \Rightarrow F^{-1}(1) = \delta(t)$

2) $F(rect(t)) = sinc(s) \Rightarrow F^{-1}(sinc(s)) = rect(t)$

**Notes:**

- the inverse of Fourier transform $F^{-1}$ is also linear ie

$F^{-1}(c_1\hat{f}_1(s) + c_2\hat{f}_2(s)) = c_1 F^{-1}(\hat{f}_1(s)) + c_1 F^{-1}(\hat{f}_2(s))$ $\forall c_1, c_2$ real constant

## V.6 Fourier Transform of some common functions.

Formulas are given both in frequency (s) and in angular frequency (ω, ω= 2πs) to enable students to do a variety of exercises.

$F(f(t)) = \hat{f}(s) = \int_{-\infty}^{+\infty} f(t)e^{-2\pi ist}dt \rightleftarrows f(t) = F^{-1}(\hat{f}(s)) = \int_{-\infty}^{+\infty} \hat{f}(s)e^{2\pi ist}ds$

$F(f(t)) = \hat{f}(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-i\omega t}dt \rightleftarrows f(t) = F^{-1}(\hat{f}(\omega)) = \int_{-\infty}^{+\infty} \hat{f}(\omega)e^{i\omega t}d\omega$

## Notes:

All Properties seen above are not affected (if we change 2πs by ω )

**Table V.6** Fourier Transform of some usuel functions

| f(t) | $F(f(t)) = \hat{f}(s)$ | $F(f(t)) = \hat{f}(\omega)$ |
|---|---|---|
| $\delta(t)$ | 1 | 1 |
| $e^{-a\|t\|}$ | $\frac{2a}{a^2 + 4\pi^2 s^2}$ | $\frac{2a}{a^2 + \omega^2}$ |
| $e^{-at^2}$ | $\sqrt{\frac{\pi}{a}}e^{-\frac{\pi^2 s^2}{a}}$ | $\sqrt{\frac{\pi}{a}}e^{-\frac{\omega^2}{4a}}$ |
| Cos$(2\pi at)$ | $\frac{1}{2}[\delta(s+a) + \delta(s-a)]$ | $\pi[\delta(\omega + 2\pi a) + \delta(\omega - 2\pi a)]$ |

| | | |
|---|---|---|
| $\sin(2\pi a t)$ | $\frac{i}{2}[\delta(s+a)-\delta(s-a)]$ | $i\pi[\delta(\omega+2\pi a)-\delta(\omega-2\pi a)]$ |
| rect(t) | sinc(s) | sinc($\frac{\omega}{2\pi}$) |
| sinc(t) | rect(s) | rect ($\frac{\omega}{2\pi}$) |

**Note:**

- To determine the Fourier transform of any function, we have either to use the table if this function is a combination of functions cited in the table and using properties given above (linearity, duality, time-shift…. ) or use the definition of the Fourier transform.

## Examples:

1) Find Fourier transform of the following f(t):

a- f(t)= $2+\frac{1}{7}e^{-6t^2}$ $\Rightarrow$F($2+\frac{1}{7}e^{-6t^2}$) $=2\mathrm{F}(1)+\frac{1}{7}F\left(e^{-6t^2}\right)=2\delta(s)+\frac{1}{7}\frac{\sqrt{\pi}}{6}e^{-\frac{\pi^2 s^2}{6}}$

b- g(t)=$t\ e^{-4|t|}$

we can write $e^{-4|t|}$=$e^{\frac{-2\pi 2|t|}{\pi}}$ in form of $e^{-2\pi s_0|t|}$ where $s_0=\frac{2}{\pi}$

So F($t\ e^{-4|t|}$)=i($\frac{1}{\pi}\frac{\frac{2}{\pi}}{s^2-(\frac{2}{\pi})^2}$)' = i($\frac{2}{\pi^2}\frac{1}{s^2-\frac{4}{\pi^2}}$)'= $-i(\frac{2}{\pi^2}\frac{2s}{(s^2-\frac{4}{\pi^2})^2})$

F[tf(t)] = i $\hat{f}$' (s)

Thus F($t\ e^{-4|t|}$)= $-i(\frac{4s}{\pi^2(s^2-\frac{4}{\pi^2})^2})$

c- f(t)=sin(4t) -5

we can write sin(4t) =sin($2\pi\frac{2}{\pi}t$) in form of sin($2\pi s_0 t$) where $s_0=\frac{2}{\pi}$

*So* $F$(sin(4t) -5) = $F$(sin(4t))-5 $F$(1)= $\frac{1}{2}\mathrm{i}\left[\delta\left(s+\frac{2}{\pi}\right)-\delta\left(s-\frac{2}{\pi}\right)\right]-5\ \delta(s)$

## V.7 Fourier transform of a derivative ( differentiation in time-domain)

Let f be a function f: [, +∞[ $\longrightarrow$ ℝ such that $\int_{-\infty}^{+\infty}|f(t)|dt$ is convergent

t $\longrightarrow$ f(t)

and $f'$ is its derivative then F[$f'$ (t)] =2πi s $\hat{f}$ (s)

A brief proof:

g(t)=$\frac{d}{dt}f(t)$

Let us determine $\hat{g}$ (s):

We have $F$(f(t))= $\hat{f}$ (s) =$\int_{-\infty}^{+\infty} f(t)e^{-2\pi i s t}dt$ (Fourier transform) (I)

And f(t)= $\int_{-\infty}^{+\infty}\hat{f}$ (s)$e^{2\pi i s t}ds$ (inverse of Fourier transform) (II) g(t)=$\frac{d}{dt}f(t)$ = $\frac{d}{dt}(2\pi\int_{-\infty}^{+\infty}\hat{f}\ (s)e^{2\pi ist}ds)$ =$\frac{d}{dt}(2\pi\int_{-\infty}^{+\infty}\frac{d}{dt}(\hat{f}\ (s)e^{2\pi ist})ds)$

=($\int_{-\infty}^{+\infty}2\pi is\hat{f}\ (s)e^{2\pi ist})ds)$=$\int_{-\infty}^{+\infty}2\pi i\ s\hat{f}\ (s)e^{2\pi ist}ds$

g(t)= $\int_{-\infty}^{+\infty}2\pi i\ s\hat{f}\ (s)e^{2\pi ist}ds$

$\hat{g}$ (s) (similarity to (I) and (II))

$\hat{g}$ (s)= $\widehat{\frac{d}{dt}f(t)}$ =$2\pi i\ s\hat{f}\ (s)$

Thus F[$f'$ (t)] =2πi s $\hat{f}$ (s)

-For higher-order derivatives:

F[ $f^{(n)}$ (t)]= $(2\pi is)^{n}$ $\hat{f}$ (s)

**Note:**

- if formula are given with angular frequency (ω) then

  F[ $f^{(n)}$ (t)]= $(i\omega)^{n}$ $\hat{f}$ (ω)

## V.8 Using Fourier transform and its inverse for solving Partial differential equation (PDE):

It is important to mention that in this section, we are going to work with the following formulas (with ω angular frequency and the variable position x ):

$F$ (f(x))=$\hat{f}$ (ω)= $\int_{-\infty}^{+\infty}f(x)e^{-i\omega x}dx \rightleftarrows$ f(x)= $F^{-1}(\hat{f}(\omega))$ = $\int_{-\infty}^{+\infty}\hat{f}\ (\omega)e^{i\omega x}d\omega$

Solve the following PDE:

$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}$ (1) $-\infty<x<+\infty$, t> 0

u(x,0)=f(x) (2) $-\infty<x<+\infty$ (initial condition)

$\frac{\partial u}{\partial t}(x,0)=0$ (3) $-\infty<x<+\infty$

$\lim\limits_{x\to+\infty}$ u(x,t) = 0 $\lim\limits_{x\to+\infty}\frac{\partial u}{\partial x}(x,t)=0$

-Apply Fourier Transform to both sides of equation (1)

$F\left[\frac{\partial^2 u}{\partial t^2}\right]=F[\frac{\partial^2 u}{\partial x^2}]$ (2)

-Use property of Fourier transform of a second derivative (with respect to the variable x] for the left side:

$$F\left[\frac{\partial^2 u}{\partial t^2}\right]=(i\omega)^2\hat{u}(\omega,t)$$

$F\left[\frac{\partial^2 u}{\partial t^2}\right]=\int_{-\infty}^{+\infty}\frac{\partial^2 u}{\partial t^2}e^{-i\omega x}dx$=$\frac{\partial^2}{\partial t^2}\int_{-\infty}^{+\infty}u(x,t)e^{-i\omega x}dx$=$\frac{\partial^2}{\partial t^2}\hat{u}(\omega,t)$

Since $e^{-i\omega x}$ is independent of the variable t

Therefore we obtain an ordinary differential equation (ODE) of second order (3) :

$\frac{\partial^2}{\partial t^2}\hat{u}(\omega,t) = (i\omega)^2\hat{u}(\omega,t) \Leftrightarrow \frac{\partial^2}{\partial t^2}\hat{u}(\omega,t) = -\omega^2\hat{u}(\omega,t) \quad \Leftrightarrow \hat{u}'' + \omega^2\hat{u}=0$ (3)

Here $\hat{u}(\omega,t)$ is the function to find

The characteristic equation of (2) is : $r^2+\omega^2 \Rightarrow r = \pm i\omega$

So the solution is: $\hat{u}(\omega,t)= A(\omega)\cos(\omega t) + B(\omega)\sin(\omega t)$ (4) (see chapter III)

Let us determine A and B from equations (2) and (3):

(2) leads to $\hat{u}(\omega,0)=\hat{f}(\omega)$ and (4) gives $\hat{u}(\omega,0)=A(\omega)\cos(0) + B(\omega)\sin(0) = A(\omega)$

Therefore $A(\omega) =\hat{f}(\omega)$

(3) gives $\frac{\partial\hat{u}}{\partial t}(\omega,0) = 0$ and so (4) yields to $\frac{\partial\hat{u}}{\partial t}(\omega,0) = -\omega A(\omega)\sin(0) + \omega B(\omega)\cos(0)$

$\Rightarrow B(\omega) = 0.$

Hence the solution for our (ODE) is $\hat{u}(\omega,t) = \hat{f}(\omega)\cos(\omega t)$

Now, we have just to invert $\hat{u}(\omega,t)$ (using inverse Fourier Transform) to get our solution $u(x,t)$ of our PDE (1):

$u(x,t) = F^{-1}\{\hat{f}(\omega)\cos(\omega t)\}=F^{-1}\left\{\hat{f}(\omega)[\frac{e^{\omega t}+e^{-\omega t}}{2}]\right\} = \frac{1}{2}\{F^{-1}[\hat{f}(\omega)e^{\omega t}] + F^{-1}[\hat{f}(\omega)e^{-\omega t}]\}$

by linearity of $F^{-1}$

$= \frac{1}{2}\{f(x+t) + f(x-t)\}$

Using the time-shift property

Finally $u(x,t) = \frac{1}{2}\{f(x+t) + f(x-t)\}$

# Exercises

### Exercise1

Let f be a function f: ]- ∞, +∞[ $\longrightarrow$ ℝ/ $C$ such that $\int_{-\infty}^{+\infty}|f(t)|dt$ convergent.

t $\longrightarrow$ f(t)

Compute the Fourier transform of f if f is an even, odd function respectively.

### Exercise 2

Determine the Fourier transform of the following function (called the triangle function):

$$f(t) = \begin{cases} 1 - \dfrac{|t|}{a} & -a \leq t \leq a \\ 0 & |t| \; > a \end{cases}$$

### Exercise3

Find the Fourier transform of the following function using the time-shifting property

$$f(t) = \begin{cases} 1, & 4 \leq t \leq 6 \\ 0, & otherwise \end{cases}$$

### Exercise 4

Solve the following partial differential equation (Heat equation) using Fourier transform: $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ (1) $-\infty < x < +\infty$

u(x,0)=$\delta(x)$ (2) $-\infty < x < +\infty$ (initial condition).

## Solutions

Solution of exercise1:

We are going to use the definition with angular frequency (ω):

a) if f is an even function then,

$$F\ (f(t)) = \hat{f}\ (\omega) = \int_{-\infty}^{+\infty} f(t)e^{-i\omega t}dt = \int_{-a}^{+a} f(t)[\cos(-wt) + i\,sin(-wt)]dt$$

$$= \int_{-\infty}^{+\infty} f(t)\cos(wt)\,dt - i\,\underbrace{\int_{-\infty}^{+\infty} f(t)sin(wt)]dt}_{0} \text{ (since cos is even and sin is odd)}$$

$$= 2\int_{0}^{+\infty} f(t)\cos(wt)\,dt$$

Knowing that:1) $f(t)\cos(wt)$=an even function g and $\int_{-a}^{+a} g(t)dt = 2\int_{0}^{+a} g(t)dt$

even.even

2) $f(t)\cos(wt)$=an odd function h and $\int_{-a}^{+a} h(t)dt = 0$.

even.odd

b) if f is an odd function then,

$$F\,(f(t)) = \hat{f}\,(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-i\omega t}dt = \int_{-a}^{+a}[f(t)[\cos(-wt) + i\,sin(-wt)]dt$$

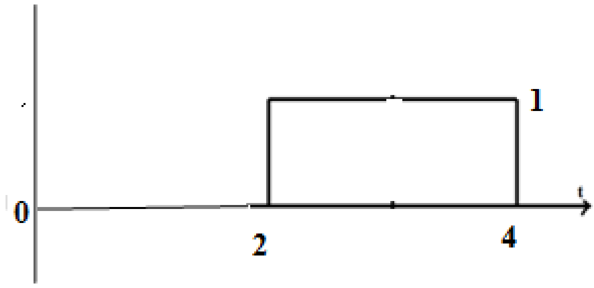
$$= \underbrace{\int_{-\infty}^{+\infty} f(t)\cos(wt)\,dt}_{0} - i\int_{-\infty}^{+\infty} f(t)sin(wt)]dt$$

$$= -2i\int_{0}^{+\infty} f(t)\sin(wt)\,dt$$

(Same explanation like above)

Solution exercise2 :

$$f(t) = \begin{cases} 1, & 4 \le t \le 6 \\ 0, & otherwise \end{cases}$$

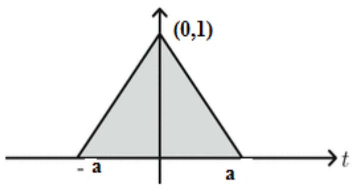



**Graph of f(t)**

We can notice that f(t) is the rectangular function (where b=1, a=1)but shifted to the right by 3 units . Thus by putting c = 3 in the time shift property, we get:

F(f(t))=F[rect(t – 3)] = $e^{-i2\pi s3}\,\widehat{rect}$ (s)= 2 $e^{-i6\pi s}\frac{\text{sinc}(2s)}{2s}$

**Solution exercise 3:**




Graph of triangular function

We are going to use the definition with angular frequency ($\omega$) :

$$F\,(f(t)) = \hat{f}\,(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-i\omega t}dt = \int_{-a}^{+a}(1 - \frac{|t|}{a})e^{-i\omega t}dt = 2\int_{0}^{+a}(1 - \frac{t}{a})\cos(\omega t)\quad dt$$

Since f(t) is an even function

Integration by parts:

u= $1 - \frac{t}{a} \Rightarrow du = \frac{-1}{a}dt$, dv=$\cos(\omega t)\quad dt \Rightarrow$v=$\frac{1}{\omega}\,\sin(\omega t)$

$$2\int_{0}^{+a}(1 - \frac{t}{a})\cos(\omega t)\quad dt = \frac{2}{\omega}\underbrace{\left[(1 - \frac{t}{a})\sin(\omega t)\right]_{0}^{a}}_{0} + \frac{2}{a\omega}\int_{0}^{+a} sin(\omega t)dt$$

$$=\frac{2}{a\omega^2}[-\cos(\omega t)]_0^a=\frac{2}{a\omega^2}[1-\cos(wa)]$$

Thus $F$ (f(t))= $\frac{2}{a\omega^2}[1-\cos(wa)]$

**Notes :**

-If we continue calculations using properties of trigonometric functions, we will have:

$$\frac{2}{a\omega^2}[1-\cos(wa)]=\frac{2}{a\omega^2}\left[1-\left(1-2\sin^2\left(\frac{wa}{2}\right)\right)\right]=\frac{4}{a\omega^2}\sin^2\left(\frac{wa}{2}\right)=a\frac{\sin^2\left(\frac{wa}{2}\right)}{(\frac{wa}{2})^2}=a\mathrm{sinc}^2(\frac{wa}{2})$$

So We can notice that the Fourier Transform of the triangle function is the cardinal sine function (by a factor of a), a similar result to that of rectangular function.

**Solution of exercise 4:**

$\frac{\partial u}{\partial t}=k\frac{\partial^2 u}{\partial x^2}$ (1) $-\infty<x<+\infty$

u(x,0)=$\delta(x)$ (2) $-\infty<x<+\infty$ (initial condition)

-Apply Fourier Transform to both sides of equation (1)

F$[\frac{\partial u}{\partial t}]=F[\,k\frac{\partial^2 u}{\partial x^2}]$

-Use linearity of Fourier transform

$\frac{\partial \hat{u}(x,t)}{\partial t}=kF\left[\frac{\partial^2 u}{\partial x^2}\right]$

-Use property of Fourier transform of a second derivative

$$\frac{\partial \hat{u}(x,t)}{\partial t}=k[(i\omega)^2\hat{u}(x,t)]\Leftrightarrow\frac{\partial \hat{u}(x,t)}{\partial t}=-k\omega^2\hat{u}(x,t)$$

Therefore we obtain an ordinary differential equation (ODE) of first order:

$\frac{\partial \hat{u}(\omega,t)}{\partial t}+k\omega^2\hat{u}(\omega,t)=0$ ($\hat{u}$ is the function to find)

The solution is given by $\hat{u}$(ω,t)=C(ω) $e^{-k\omega^2 t}$ (3)(see chapter III)

Where C(ω) is to determine from initial condition

Apply Fourier Transform to both sides of equation (2):

F [u(x,0)]=F[f(x)] $\Leftrightarrow \hat{u}(x,0)=\hat{\delta}(\omega)$

From (III) $\hat{u}$(ω,0)=C(ω) $e^{-0}$= C(ω)

Thus C(ω)= $\hat{\delta}(\omega)=1\Rightarrow\hat{u}$(ω,t)= $e^{-k\omega^2 t}$ (3)

See table above

Apply inverse Fourier Transform to both sides of equation (3)

$F^{-1}(\hat{u}$(ω,t))=$F^{-1}(\,e^{-k\omega^2 t})\Rightarrow$ u(x,t) $=\frac{1}{2\sqrt{kt\pi}}e^{-\frac{x^2}{4kt}}$

See table above ($F^{-1}[\sqrt{\frac{\pi}{a}}e^{-\frac{\omega^2}{4a}}]=e^{-at^2}$ , where here a=$\frac{1}{4kt}$

# Chapter VI
# Laplace transform

## VI.1. Introduction:

Laplace transform is a technique that involves taking a function (generally in terms of time) and transforming it into another function, i.e., moving from one space to another space. In this space, many things can be done, including solving differential equation more easily than doing so directly and afterward, returning to the initial space via the inverse of the Laplace transform. This technique is widely applied in engineering, physics, and control theory.

## VI.2 Definition:

Let f be a time function f: $[0, +\infty[ \longrightarrow \mathbb{R}$ such that $\int_0^{+\infty} |f(t)| dt$ is convergent.

$t \longrightarrow f(t)$

Laplace transform of f(t) is the application F: $[0, +\infty[ \longrightarrow \mathbb{R}$

$s \longrightarrow F(s)=\int_0^{+\infty} f(t)e^{-st}dt$

**Notes:**

- We can also denote Laplace transform of f(t) by $\mathcal{L}$ (f(t)); $\mathcal{L}$ (f(t))=F(s)=$\int_0^{+\infty} f(t)e^{-st}dt$

-Laplace transform is an improper integral since the upper limit of integration is infinity, so the condition cited above ($\int_0^{+\infty} |f(t)| dt$ convergent) insure the existence of $F(s)$(just use the comparison test see chapter 2)

- Laplace transform is an integral all over time t.

- Laplace transform convert a function of time ( f(t)) into a function of frequency (F(s))

**Examples:**

We will calculate the Laplace transform of some basic functions, which will then be used to find the Laplace transform of other more complex functions.

3) f(t) =1

$$\mathcal{L}(1) = F(s)=\int_0^{+\infty} 1e^{-st}dt = \lim_{h\to+\infty} \left(\int_{t=0}^{t=h} e^{-st}dt\right) = \lim_{h\to+\infty} \left[\frac{e^{-st}}{-s}\right]_0^h = \lim_{h\to+\infty} \left[-\frac{e^{-sh}}{s}+\frac{1}{s}\right]$$

$$= \underbrace{\lim_{h\to+\infty} \left(-\frac{e^{-sh}}{s}\right)}_{0} + \frac{1}{s} = \frac{1}{s}$$

So f(t)=1 $\longrightarrow$ $\mathcal{L}(1)=\frac{1}{s}$

4) f(t) = $e^t$

$$\mathcal{L}\ (e^t)=\ F(s)=\int_0^{+\infty} e^t e^{-st}dt\ = \lim_{h\to+\infty}\left(\int_{t=0}^{t=h} e^{-(s-1)t}dt\right)\ = \lim_{h\to+\infty}\left[\frac{e^{-(s-1)t}}{-(s-\lambda)}\right]_0^h=$$

$$\lim_{h\to+\infty}\left[-\frac{e^{-(s-1)h}}{(s-1)}+\frac{1}{s-1}\right]=\underbrace{\lim_{h\to+\infty}\left(-\frac{e^{-(s-1)h}}{s-1}\right)}_{0\ (\text{for } s>1)}+\frac{1}{s-1}\ =\frac{1}{s-1}$$

So f(t)= $e^t \longrightarrow \mathcal{L}\ (e^t)=\frac{1}{s-1}$  for s> 1

5) *f(t)*= $t^n$, n≥1

$$\mathcal{L}\ (e^t)= F(s)=\int_0^{+\infty} t^n e^{-st}dt = \lim_{h\to+\infty}\ \underbrace{\left(\int_{t=0}^{t=h} t^n e^{-st}dt\right)}_{(I)}$$

We integrate (I) by parts, i.e., use $\int_a^b U(t)V'(t)dt = [U(t)V(t)]_a^b - \int_a^b U'(t)V(t)dt$

Where :

U(t)=$t^n \Rightarrow U'(t)dt = nt^{n-1}dt$

$V'(t)dt = e^{-st}dt \Rightarrow V(t) = -\frac{1}{s}e^{-st}$

$$F(s)=\int_0^{+\infty} t^n e^{-st}dt = \underbrace{\lim_{h\to+\infty}\ [-t^n s e^{-st}]_0^h}_{0} - \lim_{h\to+\infty}\int_0^h -\frac{n}{s}t^{n-1}e^{-st}dt=\frac{n}{s}\int_0^{+\infty} t^{n-1}e^{-st}dt$$

Thus, F(s)= $\frac{n}{s}\ \mathcal{L}(t^{n-1})$

Applying the formula recursively, we obtain

F(s)=$\frac{n!}{s^{n+1}}$

So for f(t)= $t^n$, n≥1  $\mathcal{L}\ (t^n)=\frac{n!}{s^{n+1}}$

## VI.3 Laplace transform properties:

a-If f and g differ only at a finite number of points then F = G

**Examples:**

1) f(t) defined as f(t)=$\begin{cases} 3 & t=1 \\ 0 & t\neq 1\end{cases}$ has a Laplace transform F(s)=0

   f differs from the function zero just by one point (so we have found they have the same Laplace transform F(s)=0.

2) f(t) defined as f(t)=$\begin{cases} \frac{3}{4} & t=0 \\ 1 & t>0\end{cases}$ has a Laplace transform F(s)= $\frac{1}{s}$ (the same as that of f(t)=1)

b-If g(t) = f(at), a∈ ℝ; then G(s) = $\frac{1}{a}$ F ($\frac{s}{a}$)

**Example :**

g(t)=$e^{4t}$

We have $\mathcal{L}(e^{t}) = \frac{1}{s-1}$ so $\mathcal{L}(g(t)) = \mathcal{L}(e^{4t}) = \frac{1}{4}\frac{1}{(\frac{s}{4}-1)} = \frac{1}{s-4}$

Thus if g(t)=$e^{at}$ so $\mathcal{L}(e^{at}) = \frac{1}{s-a}$

c-If g(t) = $e^{at}$f(t); then G(s) = F (s − a)

**Example:**

g(t)=$t^3e^{-2t}$ =$e^{-2t}\,t^3$ of form of $e^{at}$f(t)

We have $\mathcal{L}(t^3) = \frac{3!}{s^{3+1}}$ ⇒ G(s) = F (s − a)⇒ $\mathcal{L}(e^{-2t}t^3) = \frac{3!}{(s+2)^{3+1}} = \frac{6}{(s+2)^4}$

d- if g(t) = tf(t) then G(s) = $-F'(s)$

**Example:**

g(t)= $te^{\frac{2}{3}t}$ of form of tf(t) ⇒ G(s) =$-F'(s) = -(\frac{1}{s-\frac{2}{3}})' = -(-\frac{1}{(s-\frac{2}{3})^2})$

so $\mathcal{L}\,(te^{\frac{2}{3}t}) = \frac{1}{(s-\frac{2}{3})^2}$

## VI.4 linearity of Laplace transform:

$\mathcal{L}$(f(t) ±g(t)) = $\mathcal{L}$(f (t)) ±$\mathcal{L}$(g(t))

$\mathcal{L}(k$f(t))=k$\mathcal{L}$ (f (t)) k∈ $\mathbb{R}^*$

**Examples:**

1) $\mathcal{L}(3-2e^{t}) = \mathcal{L}(3) - \mathcal{L}(2e^{t})$

$=3\mathcal{L}(1) - 2\mathcal{L}(e^{t})$

$=3\frac{1}{s} - \frac{2}{s-1} = \frac{3s-5}{s(s-1)}$

2) $\mathcal{L}(\frac{1}{4} + \frac{1}{2}t^4 + 8e^{5t}) = \frac{1}{4}\mathcal{L}(1) + \frac{1}{2}\mathcal{L}(t^4) + 8\mathcal{L}(e^{5t})$

$=\frac{1}{4}\frac{1}{s} + \frac{1}{2}\frac{4!}{s^{4+1}} + 8\frac{1}{s-5}$

$=\frac{1}{4s} + \frac{12}{s^5} + \frac{8}{s-5} = \frac{s^4(s-5)+48(s-5)+32s^5}{4(s-5)s^5}$

$=\frac{33s^5-5s^4+48s-240}{4(s-5)s^5}$

## VI.5 Inverse of Laplace Transform

**Definition:**

Let f be a time function: f: [0, +∞[ ⟶ ℝ and F(s) the Laplace transform of f(t)

$t \longrightarrow f(t)$

ie F(s) = $\mathcal{L}$ (f(t)) then f(t)= $\mathcal{L}^{-1}(F(s))$ where $\mathcal{L}^{-1}$ is called the inverse of Laplace transform.

**Examples :**

From the examples seen above we can say:

3) $\mathcal{L}(1)=\frac{1}{s} \Rightarrow \mathcal{L}^{-1}\left(\frac{1}{s}\right)=1$

4) $\mathcal{L}(e^{t})=\frac{1}{S-1} \Rightarrow \mathcal{L}^{-1}(\frac{1}{S-1})=e^{t}$

## Notes:

- the inverse of Laplace transform $\mathcal{L}^{-1}$ is also linear ie

  $\mathcal{L}^{-1}(c_1 F_1(s)+c_2F_2(s))=c_1\mathcal{L}^{-1}(F_1(s))+c_1\mathcal{L}^{-1}(F_2(s))$ $\forall c_1, c_2$ real constant

## VI.6 Laplace Transform and its inverse of some common functions.

**Table VI.1** LaplaceTransform of some usual functions and its inverse

| f(t) | $\mathcal{L}(f(t))$ | F(s) | $\mathcal{L}^{-1}$(F(s)) |
|---|---|---|---|
| 1 | $\frac{1}{s}$ | $\frac{1}{S}$ | 1 |
| $e^{at}$ | $\frac{1}{S-a}$ | $\frac{1}{S-a}$ ( a >0) | $e^{at}$ (a >0) |
| $t^n$ (n≥ 1) | $\frac{n!}{s^{n+1}}$ | $\frac{n!}{s^{n+1}}$ | $t^n$ |
| Cos($\beta t$) | $\frac{s}{s^2+\beta^2}$ | $\frac{s}{s^2+\beta^2}$ | Cos($\beta t$) |
| sin($\beta t$) | $\frac{\boldsymbol{\beta}}{s^2+\boldsymbol{\beta}^2}$ | $\frac{1}{s^2+\beta^2}$ | $\frac{1}{\beta}\sin(\beta t)$ |

## Note:

-This Table shows how the Laplace transform converts the time-domain exponential, …function into a rational function in the s- domain

- These functions, combined with the properties given above (linearity,…. ) enable us to evaluate Laplace transform( and its inverse) of most of functions.

**Examples:**

1)Let us find Laplace transform of the following f(t):

a- f(t)=sin(5t) +3 $\Rightarrow$ $\mathcal{L}$(sin(5t) +3) $=$ $\mathcal{L}$(sin(5t))+5 $\mathcal{L}$(1)= $\frac{\mathbf{5}}{s^2+\mathbf{5}^2}+3\frac{1}{s}=\frac{\mathbf{3}s^2+\mathbf{5}s++75}{s(s^2+25)}$

b- f(t)= $t^2-7+\cos(2t)$ $\Rightarrow$ $\mathcal{L}(t^2-7+\cos(2t))=\mathcal{L}(t^2)$-7 $\mathcal{L}$(1)+ $\mathcal{L}(\cos(2t))$

$=\frac{2!}{s^3}-7\frac{1}{S}+\frac{s}{s^2+2^2}$ $\Rightarrow$ $\mathcal{L}$ (f(t))= $\frac{-6s^4-26s^2+8}{s^3(s^2+4)}$

2)Let us find f(t) in the following cases:

a- F(s)= $\frac{1}{2S+1}=\frac{1}{2}\,\frac{1}{S+\frac{1}{2}} \Rightarrow \mathcal{L}^{-1}\left(\frac{1}{2}\,\frac{1}{S+\frac{1}{2}}\right)=\frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{S+\frac{1}{2}}\right)=\frac{1}{2}\,e^{-\frac{t}{2}}$=f(t)

b- F(s)= $\frac{1}{s^2+3}=\frac{1}{s^2+(\sqrt{3})^2} \Rightarrow \mathcal{L}^{-1}\left(\frac{1}{s^2+(\sqrt{3})^2}\right)=\frac{1}{\sqrt{3}}\sin(\sqrt{3}\ t)$=f(t)

## VI.7 Laplace transform of a derivative

Let f be a time function f: $[0,+\infty[\ \longrightarrow \mathbb{R}$ and $f'$ is its derivative

$$t \longrightarrow f(t)$$

then $\mathcal{L}(f'(t))$= s $\mathcal{L}\big(f(t)\big)-f(0)=sF(s)-f(0)$

**Note:**

-The above formula is obtained by integration by parts method.

**-** Time-domain differentiation becomes multiplication by frequency variable s plus a term that includes initial condition($-$f(0)).

**-**For second-order derivatives, we have just to apply derivative formula twice:

$$\mathcal{L}(f''(t)) = s\,\mathcal{L}\big(f'(t)\big)-f'(0) = s[\ s\,\mathcal{L}\big(f(t)\big)-f(0)\ ]-f'(0)$$
$$= s^2\,\mathcal{L}\big(f(t)\big)-s\,f(0)-f'(0)$$
$$= s^2\,F(s)-s\,f(0)-f'(0)$$

So for higher-order derivatives, similar formulas hold for $\mathcal{L}(f^n(t))$

$$\mathcal{L}(f^n(t)) = s^n\,F(s)-s^{n-1}\,f(0)-s^{n-2}\,f'(0)-s^{n-3}\,f''(0)\ldots\ldots\ldots\ldots-f^{(n-1)}(0)$$

## VI.8 Using Laplace transform and its inverse for solving differential equation:

**Example1:**

y'(t) + y(t) = 1 (y(0)=0) it is a first order differential equation with limit condition.

Method of resolution:

Apply Laplace transform both sides of equation

$$\mathcal{L}\big(y'(t)+y(t)\big)=\mathcal{L}(1) \Rightarrow \mathcal{L}\big(y'(t)\big)+\mathcal{L}\big(y(t)\big)=\mathcal{L}(1) \Rightarrow s\mathcal{L}\big(y(t)\big)-y(0)+\mathcal{L}\big(y(t)\big)=\frac{1}{s}$$

Linearity of $\mathcal{L}$ Laplace transform of a derivative formula $\rightarrow$ 0

$$\Rightarrow \mathcal{L}\big(y(t)\big)(S+1)=\frac{1}{s} \Rightarrow \mathcal{L}\big(y(t)\big)=\frac{1}{s(s+1)}=\frac{1}{s}-\frac{1}{(s+1)}$$

Now, we apply inverse of Laplace transform both sides of equation:

$y(t) = \mathcal{L}^{-1}(\frac{1}{s} - \frac{1}{(s+1)}) = \mathcal{L}^{-1}\left(\frac{1}{s}\right) - \mathcal{L}^{-1}\left(\frac{1}{(s+1)}\right) = 1 - e^{-t}$

Linearity of $\mathcal{L}^{-1}$

Thus, our solution y(t)= $1 - e^{-t}$

**Example2:**

$y'(t) = 5 - 2t$, $y(0) = 1$.

Apply Laplace transform both sides of equation

$\mathcal{L}(y'(t)) = \mathcal{L}(5 - 2t) \Rightarrow s\mathcal{L}(y(t)) - y(0) = 5\mathcal{L}(1) - 2\,\mathcal{L}(t)$

1

$\Rightarrow s\mathcal{L}(y(t)) - 1 = \frac{5}{s} - 2\frac{1!}{s^2} \Rightarrow \mathcal{L}(y(t)) = \frac{5}{s^2} - \frac{2}{s^3} + \frac{1}{s}$

Now, we apply inverse of Laplace transform both sides of equation:

$y(t) = \mathcal{L}^{-1}(\frac{5}{s^2} - \frac{2}{s^3} + \frac{1}{s}) \Rightarrow y(t) = 5\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) - \mathcal{L}^{-1}\left(\frac{2}{s^3}\right) + \mathcal{L}^{-1}(\frac{1}{s})$

Thus, our solution y(t)= $5t - t^2 + 1$

**Note:**

Laplace transform turned a differential equation into an algebraic equation.

## Exercises

**Exercise 1:**

Determine Laplace transform of the following functions:

a) f(t)= $t^2 + 3$  b)f(t)= $e^{5t} - 20t^5$  c) f(t)=6cos(4t) +9

d)$f(t) = \frac{5}{3}\sin(6t) + \frac{4}{3}\ e^{-2t} + 7$

**Exercice 2 :**

Find the inverse of Laplace transform of the following functions:

a) F(S)=$-\frac{1}{4-S}$  b) F(S)= $\frac{3}{2S^6}$  c) F(S)= $\frac{5}{S^2+4}$

d)F(S)= $\frac{7}{3S+2}$  e) F(S)= $\frac{S}{2S^2+5} + \frac{9}{S^4} - \frac{4}{s}$

**Exercise 3 :**

Using Laplace transform and its inverse, solve the following differential equations :

1) $x' = t^2 - 2$ avec $x(0) = 3$
2) $x' + x = 3\cos(t)$ avec $x(0) = 5$

**Solution of exercise 1 :**

a) f(t)= $t^2 + 3$

$\mathcal{L}(t^2 + 3) = \mathcal{L}(t^2) + 3\mathcal{L}(1) = \frac{2!}{s^{2+1}} + 3\frac{1}{s} = \frac{2+3s^2}{s^3}$

Linearity  see table VI.1

Thus $\mathcal{L}(t^2 + 3) = \frac{2+3s^2}{s^3}$

b)f(t)= $e^{5t} - 20t^5$

$\mathcal{L}(e^{5t} - 20t^5) = \mathcal{L}(e^{5t}) - 20\mathcal{L}(t^5) = \frac{1}{s-5} - 20\frac{5!}{s^{5+1}} = \frac{s^6 - 20x120(s-5)}{(s-5)s^6}$

So $\mathcal{L}(e^{5t} - 20t^5) = \frac{s^6 - 2400s - 12000}{(s-5)s^6}$

b) f(t)=6cos(4t) +9

$\mathcal{L}(6\cos(4t) + 9) = 6\mathcal{L}(\cos(4t)) + 9\,\mathcal{L}(1) = 6\frac{s}{s^2+4^2} + 9\frac{1}{s}$

Consequently $\mathcal{L}(6\cos(4t) + 9) = \frac{15s^2+144}{(s^2+4^2)s}$

d)$f(t) = \frac{5}{3}\sin(6t) + \frac{4}{3}\ e^{-2t} + 7$

$\mathcal{L}(\frac{5}{3}\sin(6t) + \frac{4}{3}\ e^{-2t} + 7) = \frac{5}{3}\mathcal{L}(\sin(6t)) + \frac{4}{3}\mathcal{L}(e^{-2t}) + 7\mathcal{L}(1) = \frac{5}{3}\frac{\mathbf{6}}{s^2+\mathbf{6}^2} + \frac{4}{3}\frac{1}{S+2} + 7\frac{1}{s}$

$$= \frac{\mathbf{30s}(S+2)+4s\left(s^2+\mathbf{6}^2\right)+21(s^2+\mathbf{6}^2)(S+2)}{3s(s^2+\mathbf{6}^2)(S+2)}$$

Thus $\mathcal{L}(\frac{5}{3}\sin(6t)+\frac{4}{3}\ e^{-2t}+7) = \frac{\mathbf{25}s^3+\mathbf{72}s^2+\mathbf{960}s+\ \mathbf{1512}}{3s(s^2+\mathbf{6}^2)(S+2)}$

**Solution of exercice 2 :**

a) $F(S)=-\frac{1}{4-S}$

$\mathcal{L}^{-1}\left(-\frac{1}{4-S}\right)=\mathcal{L}^{-1}\left(\frac{1}{s-4}\right)=e^{4t}$ (see table VI.1)

b) $F(s)=\frac{3}{2s^6}$

$\mathcal{L}^{-1}(\frac{5}{s^2+4}) = \mathcal{L}^{-1}(\frac{5}{2}\frac{2}{s^2+2^2}) = \frac{5}{2}\mathcal{L}^{-1}(\frac{2}{s^2+2^2}) = \frac{5}{2}\sin(2t)$

d) $F(s)=\frac{7}{3s+2}$

$\mathcal{L}^{-1}(\frac{7}{3s+2}) = \mathcal{L}^{-1}(\frac{7}{3s+2}) = \frac{7}{3}\ \mathcal{L}^{-1}(\frac{1}{s+\frac{2}{3}}) = \frac{7}{3}\ e^{-\frac{2}{3}t}$

e) $F(s)=\frac{s}{2s^2+5}+\frac{9}{s^4}-4$

$\mathcal{L}^{-1}\ (\frac{s}{2s^2+5}+\frac{9}{s^4}-\frac{4}{s})= \mathcal{L}^{-1}\ (\frac{s}{2s^2+5})+\mathcal{L}^{-1}\ (\frac{9}{s^4})-4\mathcal{L}^{-1}(\frac{1}{s})$

$=\frac{1}{2}\mathcal{L}^{-1}\ (\frac{s}{s^2+(\sqrt{\frac{5}{2}})^2})+\frac{3}{2}\ \mathcal{L}^{-1}\ (\frac{3x2}{s^{3+1}})\ )-4\mathcal{L}^{-1}(\frac{1}{s})$

$=\frac{1}{2}\cos(\sqrt{\frac{5}{2}}t)+\frac{3}{2}\ t^3-4$

**Solution of exercise 3:**

1) $x' = t^2-2$ avec $x(0)=3$

Apply Laplace transform both sides of equation

$\mathcal{L}(x'(t)) = \mathcal{L}(t^2-2) \Rightarrow s\mathcal{L}(x(t)) - x(0) = \mathcal{L}(t^2) - 2\ \mathcal{L}(1)$

3

$\Rightarrow s\mathcal{L}(x(t)) - 3 = \frac{2!}{s^{2+1}} - 2\frac{1}{s} \Rightarrow \mathcal{L}(x(t)) = \frac{2}{s^4} - \frac{2}{s^2} + \frac{3}{s}$

Now, we apply inverse of Laplace transform both sides of equation:

$x(t) = \mathcal{L}^{-1}(\frac{2}{s^4}-\frac{2}{s^2}+\frac{3}{s}) \Rightarrow y(t) = \frac{1}{3}\mathcal{L}^{-1}\left(\frac{2x3}{s^4}\right) - \mathcal{L}^{-1}\left(\frac{2}{s^3}\right) + 3\mathcal{L}^{-1}(\frac{1}{s})$

Thus, our solution $x(t)=\frac{1}{3}t^3-t^2+3$

2) $x'+x=3\cos(t)$ avec $x(0)=5$

Apply Laplace transform both sides of equation

$\mathcal{L}(x'(t)+x(t)) = \mathcal{L}(3\cos(t)) \Rightarrow s\mathcal{L}(x(t)) - x(0) + \mathcal{L}(x(t)) = 3\mathcal{L}(\cos(t))$

$\Rightarrow \mathcal{L}(x(t))[s+1] = 3\frac{s}{s^2+1} + 5 \Rightarrow \mathcal{L}(x(t)) = \frac{3s}{(s^2+1)(s+1)} = \frac{a}{(s+1)} + \frac{bs+c}{(s^2+1)}$

(According to partial fraction decomposition)

(a, b and c constants to determine by identification)

Now, we apply inverse of Laplace transform both sides of equation:

$x(t) = \mathcal{L}^{-1}(\frac{a}{(s+1)} + \frac{bs+c}{(s^2+1)}) \Rightarrow x(t) = a\mathcal{L}^{-1}\left(\frac{1}{(s+1)}\right) + b\mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) + c(\frac{1}{s^2+1})$

Thus, our solution x(t)=$ae^{-t} + b\cos(t) + c\sin(t)$ ( where a=$-\frac{3}{2}$, b=$\frac{3}{2}$, c=$-\frac{3}{2}$ )

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