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COMPARTMENTAL MODELS ON TIME SCALES

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Presented by

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Dedication

To those who are not matched by anyone in the universe, to those who have made a great deal, and have given what cannot be returned, to my dear ***parents****.*

To my adorable brothers and my adorable sisters .

To my best friends

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Abstract

This thesis focuses on studying linear and nonlinear dynamic systems on time scales or both. The aim of the study is, on one hand, to find the exact solution for the non-population conserving SIR model on time scales, and on the other hand, to investigate the uniform stability of the $SICA$ model also on time scales. We introduce a fractional order SIR model and SICA model and we prove the existence and the positivity of solution.

Key words: Time Scales, dynamic equations on time scales, deterministic epidemic model, model SIR ,... existence of solution, SICA model for HIV transmission, permanence, almost periodic solution, uniform asymptotic stability, numerical simulations, fractional order model, existence of solution.

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Acronyms

Acronyms	Definitions
IVP	Initial Value Problem.
SIR	Susceptible, Infected, Removed.
SICA	Susceptible, Infected, Chronic, AIDS.
COVID-19	Coronavirus disease 2019.

Notations

$\mathbb{T}$	Time scales.
$\mathbb{R}$	Real numbers.
$\mathbb{Z}$	Integers.
$\mathbb{N}$	Natural numbers.
$\mathbb{N}_0$	Nonnegative integers.
$h\mathbb{Z}$	$hz; z \in \mathbb{Z}$, where h is a fixed real number.
$\mathbb{P}_{a,b}$	$\cup_{k=0}^{\infty} [k(a+b), k(a+b)+a]$.
$\mathbb{Q}$	Rational numbers.
$\mathbb{R} \setminus \mathbb{Q}$	Irrational numbers.
$\mathbb{C}$	Complex numbers.
$\sigma(.)$	Forward jump operator.
$\rho(.)$	Backward jump operator.
$\mu(.)$	Graininess function.
$f^{\Delta}(.)$	Delta derivative of f at t on $\mathbb{T}$.
Δ	Usual forward difference operator.
$C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$	Set of rd-continuous function.
$C_{rd}^1 = C_{rd}^1(\mathbb{T}) = C_{rd}^1(\mathbb{T}, \mathbb{R})$	Set of differentiable functions whose derivative is rd-continuous.
$\oplus$	Circle plus (addition) on $\mathbb{T}$.
$\ominus$	Circle minus (subtraction on $\mathbb{T}$).
$\xi_h(.)$	Cylinder transformation.
$\mathcal{R} = \mathcal{R}(\mathbb{R}) = \mathcal{R}(\mathbb{R}, \mathbb{T})$	Set of all regrissive and rd-continuous functions.
$\mathcal{R}^+ = \mathcal{R}^+(\mathbb{R}) = \mathcal{R}^+(\mathbb{R}, \mathbb{T})$	Set of all regrissive and rd-continuous functions such that

$1 + \mu(t)f(t) > 0$, for all $t \in \mathbb{T}$.

$(p \oplus q)(.)$	Addition on $\mathcal{R}$.
$(p \ominus q)(.)$	Subtraction on $\mathcal{R}$.
$(p \odot q)(.)$	Product on $\mathcal{R}$.
$e_p(., s)$	Exponential function on time scales.

Introduction

The historical trajectory of employing mathematical models to understand and predict the dynamics of disease transmission can be traced back to the pivotal year of 1766 and the work of Daniel Bernoulli. Indeed, Bernoulli's work laid a foundational understanding of the application of mathematical principles in describing the intricate patterns of disease spread [20].

Fast forward to the year 1927, a crucial milestone emerged with the seminal publication of Kermack and McKendrick [27]. In their groundbreaking work, they introduced the SIR (Susceptible–Infected–Recovered/Removed) model for epidemics. This pioneering model not only provided a conceptual framework for comprehending the transmission dynamics of infectious diseases, but also set the stage for the development of numerous subsequent models. In fact, SIR type models, with their compartmental classification of individuals into different classes, became a cornerstone in the field of epidemiology, offering a versatile and widely adopted template for modeling various contagious diseases [2].

Since then, the field has witnessed a burgeoning array of mathematical models, each tailored to address specific nuances and challenges associated with different diseases. The continuous evolution of these models reflects the ongoing commitment of researchers to refine and enhance our understanding of epidemic dynamics, ultimately contributing to more effective strategies for disease control and prevention [1].

Let $N(t) = S(t) + I(t) + R(t)$ denote the total population at time t. The classical and more standard SIR epidemic model assumes that there are no births or deaths during the period under study, based on the assumption that these are on a much slower time scales and can therefore be ignored. The combined dynamics is then given mathematically by

$$\begin{cases} \dot{S}(t) = -\lambda \frac{S(t)I(t)}{N(t)}, \\ \dot{I}(t) = \lambda \frac{S(t)I(t)}{N(t)} - \gamma I(t), \\ \dot{R}(t) = \gamma I(t), \end{cases}$$

where $\lambda > 0$ is the infection rate and $\gamma > 0$ is the rate for which infected individuals recover. Clearly, model SIR assumes the total population under study to be constant: $(S(t) + I(t) + R(t))' = 0$. Some of its limitations stand out immediately: for diseases such as Ebola [3, 40] or COVID-19 [12, 46], where death rates are not negligible, then we do not have a constant population and model SIR ceases to be valid.

In [12], Borkar and Manjunath propose a variant of SIR, called the SIR-NC model, that, unlike the standard SIR model, does not assume the conservation of the population. Surprisingly, by incorporating a nonzero death rate into the model, thus being more suitable to diseases like Ebola or COVID-19, the new SIR-NC model is analytically tractable [12, 17].

Calculus on time scales is a mathematical area that generalizes the traditional calculus by unifying continuous and discrete analysis on an arbitrary time scales. By combining both continuous and discrete elements, time scales allow for a more flexible and inclusive approach to modeling systems that exhibit both continuous and discrete behaviors (hybrid systems). The theory was introduced by Stefan Hilger from 1988 to 1990 as a special case of a general analysis on measure chains [21] (see also [4]).

The analysis on time scales allows one to generalize differential and difference equations, incorporating both continuous and discrete dynamics in a unified setting. Moreover, the new analysis holds in any nonempty closed set, such as the set of integers, rationals, or more complex structures like the Cantor set, offering a more comprehensive mathematical framework for analyzing and modeling systems with mixed continuous and discrete dynamics. This permits to extend the applicability of traditional calculus to a broader. In particular, this is true in epidemic modeling, where analysis on time scales have allowed the modeling of noncontinuous disease dynamics, e.g. diseases where a virus remains unnoticed within the host for several years before continuing to spread [10, 11].

Here we investigate the SIR-NC model on an arbitrary time scales. Our results are then given on chapter 2 Section 2.1: we formulate our SIR-NC model on time scales (cf. the dynamic system (2.1), proving that there exists a unique solution (cf. Theorem 2.3). More than that, an explicit analytical formula for the solution is obtained (cf. 2.4). We end with Section 2.2, discussing the SIR-NC model on time scales with imported infections. Under some conditions, a closed formula for the solution is also obtained (cf. Theorem 2.4). Along the text, two examples are given to illustrate the obtained results and showing how our results generalize those available in the literature (cf. Example 2.1.1 and Example 2.2.1).

Infectious diseases pose a massive threat to humans and the economy of states. Proper understanding of disease dynamics plays an important role in curtailing infections in a community. The implementation of suitable strategies against disease transmission is another challenge. A mathematical modeling approach is one key tool for addressing these challenges. A number of disease models have been developed in the existing literature, which enable us to explore and control the spread of infectious diseases more effectively.

Most of these models are based on integer-order differential equations (IDEs). However, in recent years, it has been noted that fractional-order equations (FDEs) can provide additional insights [28]. Let us consider the SIR model (Susceptibles, Infectives, Removed) as described in [19] :

$$\begin{cases} \frac{dx(t)}{dt} = -\alpha y(t)x(t) \\ \frac{dy(t)}{dt} = \alpha y(t)x(t) - \beta y(t) \\ \frac{dz(t)}{dt} = \beta y(t), \end{cases}$$

where $N(t) = x(t) + y(t) + z(t)$ the total population at time t, $x(t)$ represents susceptible, $y(t)$ represents number of infected and $z(t)$ represents the number of recovered individuals. The initial conditions are given by

$$x(0) = x_0,\ y(0) = y_0,\ z(0) = z_0.$$

The complete model that describes the system of fractional differential equations is presented as follows:

$$\begin{cases} \frac{^c d^\alpha x(t)}{dt^\alpha} = -\alpha y(t)x(t) \\ \frac{^c d^\alpha y(t)}{dt^\alpha} = \alpha y(t)x(t) - \beta y(t) \\ \frac{^c d^\alpha z(t)}{dt^\alpha} = \beta y(t). \end{cases}$$

In 2015, the deterministic SICA model was first presented as a sub-system of a TB-HIV/AIDS co-infection model [42]. One of the primary objectives of SICA models is to demonstrate how some of the fundamental relationships between epidemiological variables and the general pattern of the AIDS epidemic can be clarified using a straightforward mathematical model [36]. The celebrated SICA mathematical model [13, 14, 34, 45, 49] divides the total human population into four compartments, namely

- $S(t)$: susceptible individuals at time t;
- $I(t)$: HIV-infected individuals with no clinical symptoms of AIDS but able to transmit HIV to other individuals at time t;
- $C(t)$: HIV-infected individuals under antiretroviral therapy (ART), the so called chronic stage with a viral load remaining low at time t;
- $A(t)$: HIV-infected individuals with AIDS clinical symptoms at time t.

Under some realistic assumptions, the dynamics of the disease proliferation in a community is then translated into a mathematical model given by the following system of four ordinary differential equations [42, 43, 44]:

$$\begin{cases} \dot{S}(t) = \Lambda - \beta\lambda(t)S(t) - \nu S(t), \\ \dot{I}(t) = \beta\lambda(t)S(t) - (\rho + \phi + \nu)I(t) + \gamma A(t) + \omega C(t), \\ \dot{C}(t) = \phi I(t) - (\omega + \nu)C(t), \\ \dot{A}(t) = \rho I(t) - (\gamma + \nu + d)A(t), \end{cases} \tag{1}$$

where Λ, β, ν, ρ, ϕ, γ, ω and d are real positive rates,

- Λ is the rate of new susceptible;
- β is the HIV transmission rate;

- ν is the natural death rate;
- ρ is the default treatment rate for I individuals;
- ϕ is the HIV treatment rate for I individuals;
- γ is the AIDS treatment rate;
- ω is the default treatment rate for C individuals;
- d is the AIDS induced death rate;

and where the effective contact rate with people infected with HIV is given by

$$\lambda(t) = \frac{\beta}{N(t)}(I(t) + \eta_C C(t) + \eta_A A(t))$$

with

- $N(t)$ the total population at time t, that is,

$$N(t) = S(t) + I(t) + C(t) + A(t);$$

- $0 \leq \eta_C \leq 1$ the modification parameter;
- $\eta_A \geq 1$ the partial restoration parameter of immune function of individuals with HIV infection that use correctly the ART treatment.

The study of dynamical systems on time scales is now a very active area of research. The books of Bohner and Peterson [7, 8] offer a good introduction with applications to the time scales calculus along with some advanced topics. Applications of the time scales calculus can be found in many areas, including economics [9, 15, 48], ecology [37, 38, 52] and epidemics [11, 39, 24]. Here we generalize the SICA model by considering dynamic equations on time scales and study it using the time-scales theory. By doing it, we unify the continuous and discrete-time models [49], generalizing it also to other contexts like the quantum [25, 35] or mixed/hybrid settings [16, 50].

Recently, Prasad and Khuddush proved the existence and uniform asymptotic stability of positive and almost periodic solutions for a 3-species Lotka–Volterra competitive system on time scales

[38]. Moreover, they also studied the permanence and positive almost periodic solutions of a n-species Lotka–Volterra system on time scales [37]. Motivated by these works, here we investigate the permanence and uniform asymptotic stability of the unique positive almost periodic solution of the following SICA model on time scales:

$$\begin{cases} x_1^{\Delta}(t) = \Lambda - \beta\lambda(t)x_1^{\sigma}(t) - \nu x_1^{\sigma}(t), \\ x_2^{\Delta}(t) = \beta\lambda(t)x_1(t) - (\rho + \phi + \nu)x_2^{\sigma}(t) + \gamma x_4(t) + \omega x_3(t), \\ x_3^{\Delta}(t) = \phi x_2(t) - (\omega + \nu)x_3^{\sigma}(t), \\ x_4^{\Delta}(t) = \rho x_2(t) - (\gamma + \nu + d)x_4^{\sigma}(t), \end{cases}$$

where $t \in \mathbb{T}^+$, with $\mathbb{T}^+$ a nonempty closed subset of $\mathbb{R}^+ =]0, +\infty[$.

Note that, in our notation, $(x_1(t), x_2(t), x_3(t), x_4(t))$ is interpreted as $(S(t), I(t), C(t), A(t))$.

The main objective of the thesis

We consider the time scales $\mathbb{T}$ and the SIR-NC model defined on $\mathbb{T}$ as follows:

$$\begin{cases} S^{\Delta}(t) = -\lambda \dfrac{S(t)I^{\sigma}(t)}{S(t) + I(t)} - \nu S^{\sigma}(t), \\ I^{\Delta}(t) = \lambda \dfrac{S(t)I^{\sigma}(t)}{S(t) + I(t)} + \nu S^{\sigma}(t) - \gamma I^{\sigma}(t), \\ R^{\Delta}(t) = \gamma I^{\sigma}(t), \end{cases}$$

where $S, I, R : \mathbb{T} \longrightarrow \mathbb{R}^+$ and $\lambda, \gamma, \nu > 0$, and we set $N(t) := S(t) + I(t)$.
We seek the exact and unique solution, namely finding $S(t), I(t), R(t)$ that satisfy the initial conditions $S(0) > 0, I(0) > 0, R(0) > 0$.

We will prove the existence and uniqueness of solutions of the following fractional SIR model:

$$\begin{cases} \frac{^c d^{\alpha} x}{dt^{\alpha}} = -\alpha y(t)x(t) \\ \frac{^c d^{\alpha} y}{dt^{\alpha}} = \alpha y(t)x(t) - \beta y(t) \\ \frac{^c d^{\alpha} z}{dt^{\alpha}} = \beta y(t) \end{cases}$$

Note that, in our notation, $(x(t), y(t), z(t))$ is interpreted as $(S(t), I(t), R(t))$. The initial conditions are given by

$$x(0) = x_0 \geq 0,\ y(0) = y_0 \geq 0,\ z(0) = z_0 \geq 0.$$

Based on the works by Prasad and Khuddush [37],[39] on studying positive and uniformly asymptotically stable almost periodic solutions for systems on time scales, including the Lotka-Volterra models [38], we investigate the permanence and uniform asymptotic stability of the unique positive almost periodic solution of the following SICA model on time scales:

$$\begin{cases} S^{\Delta}(t) = \Lambda - \beta\lambda(t)S^{\sigma}(t) - \nu S^{\sigma}(t), \\ I^{\Delta}(t) = \beta\lambda(t)S(t) - (\rho+\phi+\nu)I^{\sigma}(t) + \gamma A(t) + \omega C(t), \\ C^{\Delta}(t) = \phi I(t) - (\omega+\nu)C^{\sigma}(t), \\ A^{\Delta}(t) = \rho I(t) - (\gamma+\nu+d)A^{\sigma}(t), \end{cases}$$

where $t \in \mathbb{T}^+$, with $\mathbb{T}^+$ a nonempty closed subset of $\mathbb{R}^+ =]0, +\infty[$.

Finally we prove the existence and uniqueness of solutions of the following Caputo fractional SICA model:

$$\begin{cases} {}^{C}_{t_0}D^{\alpha}_{t} w(t) = \Lambda - \beta(x(t) + \eta_y y(t) + \eta_z z(t))w(t) - \mu w(t), \\ {}^{C}_{t_0}D^{\alpha}_{t} x(t) = \beta(x(t) + \eta_y y(t) + \eta_z z(t))w(t) - \xi_3 x(t) + \omega y(t) + \gamma z(t), \\ {}^{C}_{t_0}D^{\alpha}_{t} y(t) = \phi x(t) - \xi_2 y(t), \\ {}^{C}_{t_0}D^{\alpha}_{t} z(t) = \rho x(t) - \xi_1 z(t). \end{cases}$$

Note that, in our notation, $(w(t), x(t), y(t), z(t))$ is interpreted as $(S(t), I(t), C(t), A(t))$.

Tools and Techniques for Work

This thesis employs a variety of tools in time-scales calculus, encompassing differentiation and integration, [7, 8]. It builds upon extensive studies within the field, focusing on the precise analysis of solutions for epidemiological systems like the SIR model [10, 11], along with investigations into asymptotic stability [47, 22], Also fractional operator [28, 29, 30, 31, 41]

Description of the chapters and main results

In Chapter One, we include basic concepts in time-scale calculus, which are essential tools for the different chapters. This chapter covers the necessary calculus and the fundamental existence theory for dynamic systems on time scales, and we develop Lyapunov's second method.

In chapter Two, we propose the following SIR-NC models on time scales $\mathbb{T}$:

$$\begin{cases} S^{\Delta}(t) = -\lambda \dfrac{S(t)I^{\sigma}(t)}{S(t)+I(t)}, \\ I^{\Delta}(t) = \lambda \dfrac{S(t)I^{\sigma}(t)}{S(t)+I(t)} - \gamma I^{\sigma}(t), \\ R^{\Delta}(t) = \gamma I^{\sigma}(t), \end{cases} \tag{2}$$

and

$$\begin{cases} S^{\Delta}(t) = -\lambda \dfrac{S(t)I^{\sigma}(t)}{S(t)+I(t)} - \nu S^{\sigma}(t), \\ I^{\Delta}(t) = \lambda \dfrac{S(t)I^{\sigma}(t)}{S(t)+I(t)} + \nu S^{\sigma}(t) - \gamma I^{\sigma}(t), \\ R^{\Delta}(t) = \gamma I^{\sigma}(t), \end{cases} \tag{3}$$

where $S, I, R : \mathbb{T} \longrightarrow \mathbb{R}^{+}$ and $\lambda, \gamma, \nu > 0$, subject to given initial conditions

$$S(0) = S_0, \quad I(0) = I_0, \quad R(0) = R_0 \tag{4}$$

with $S_0 > 0$, $I_0 > 0$, and $R_0 \geq 0$. In the particular case $\mathbb{T} = \mathbb{R}$, problem (2)–(4) is studied in [12]. More precisely, we prove the following two theorems:

Theorem 0.1. *[6] Let $C = \dfrac{S(0)}{I(0)}$. If $\gamma - \lambda, p(t) \in \mathcal{R}$, then the unique solution to (3)–(4) is given by*

$$\begin{cases} S(t) = e_{(\ominus p(t)) \oplus (\gamma - \lambda)}(t, 0) S(0), \\ I(t) = e_{\ominus p(t)}(t, 0) I(0), \end{cases}$$

where $p(t) = \gamma - \dfrac{\lambda C}{e_{\ominus(\gamma-\lambda)}(t,0) + C}$.

Theorem 0.2. *[6] If $\nu, \gamma - \lambda \in \mathcal{R}$, then the solution to the SIR-NC system (2) with imported* *infections is given as follows:*

$$\begin{cases} S(t) = \dfrac{x(t)}{1-x(t)} e_{\ominus g(t)}(t,0) I(0), \\ \\ I(t) = e_{\ominus g(t)}(t,0) I(0), \end{cases}$$

$t \in \mathbb{T}$, *where* $S, I : \mathbb{T} \longrightarrow \mathbb{R}^+$, $\lambda, \gamma, \nu > 0$, *and* $g(t) = -\lambda x(t) + \gamma - \dfrac{\nu x^\sigma(t)}{1 - x^\sigma(t)}$,
$x(t) = \dfrac{x(0)}{e_{\nu \ominus (\gamma - \lambda)}(t,0)\left(1 + \frac{x(0)(\gamma-\lambda)}{\nu - \gamma + \lambda}\right) - \frac{x(0)(\gamma - \lambda)}{\nu - \gamma + \lambda}}.$

In Chapter Three, we examine a fractional-order SIR model:

$$\begin{cases} \frac{{}^c d^\alpha x}{dt^\alpha} = -\alpha y(t) x(t) \\ \frac{{}^c d^\alpha y}{dt^\alpha} = \alpha y(t) x(t) - \beta y(t) \\ \frac{{}^c d^\alpha z}{dt^\alpha} = \beta y(t) \end{cases} \tag{5}$$

Note that, in our notation, $(x(t), y(t), z(t))$ is interpreted as $(S(t), I(t), R(t))$, and ${}^c d^\alpha$ fractional Caputo derivative having $0 < \alpha \leq 1$. The initial conditions are given by

$$x(0) = x_0 \geq 0,\ y(0) = y_0 \geq 0,\ z(0) = z_0 \geq 0,$$

and we take the conditions of growth non-linear vector operator $\phi : \mathbb{R}^3_+ \times [0, T] \longrightarrow \mathbb{R}_+$ as:
$(A1)$ There is a constant $L_\phi > 0$; $\forall (W(t), W'(t)) \in \mathbb{R} \times \mathbb{R}$;

$$|\phi(W(t), t) - \phi(W'(t), t)| \leq L_\phi \, |W(t) - W'(t)| \, .$$

$(A2)$ there is a constants $C_\phi > 0, M_\phi > 0$;

$$|\phi(W(t), t)| \leq C_\phi \, |W| + M_\phi.$$

More precisely, we prove the following two theorems:

Theorem 0.3. *Under the continuity of* ϕ *together with assumption* $(A2)$*, system* (5) *has at least* *one solution.*

Theorem 0.4. *Using (A1), system (5) has unique or one solution if* $\frac{T^\alpha}{\Gamma(\alpha+1)} L_\phi < 1$.

In Chapter Four, we investigate the permanence and existence of solutions, and provide sufficient conditions to indicate the existence of a unique almost periodic uniformly asymptotically stable solution of the following SICA model on time scales.

$$\begin{cases} S^\Delta(t) = \Lambda - \beta\lambda(t)S^\sigma(t) - \nu S^\sigma(t), \\ I^\Delta(t) = \beta\lambda(t)S(t) - (\rho+\phi+\nu)I^\sigma(t) + \gamma A(t) + \omega C(t), \\ C^\Delta(t) = \phi I(t) - (\omega+\nu)C^\sigma(t), \\ A^\Delta(t) = \rho I(t) - (\gamma+\nu+d)A^\sigma(t), \end{cases} \tag{6}$$

where $t \in \mathbb{T}^+$, with $\mathbb{T}^+$ a nonempty closed subset of $\mathbb{R}^+ =]0, +\infty[$.

Under certain assumptions:

(H_1) $\lambda(t)$ is a bounded and almost periodic function satisfying

$$0 < \lambda^L \le \lambda(t) \le \lambda^U.$$

(H_2) $\Gamma_2 < \Gamma_1$ with $\Gamma_1, \Gamma_2 \in \mathcal{R}^+$.

More precisely, we prove the following results

Lemma 0.1. *[5] Suppose hypothesis (H_1) holds. Then, for any positive solution $(x_1(t), x_2(t), x_3(t), x_4(t))$ of system (6), there exists positive constants M and T such that $x_i(t) \le M$, $i = 1, 2, 3, 4$, for $t \ge T$. Note that, in our notation, $(x_1(t), x_2(t), x_3(t), x_4(t))$ is interpreted as $(S(t), I(t), C(t), A(t))$.*

Lemma 0.2. *[5] Suppose that (H_1) holds. Then, system (6) is permanent.*

Theorem 0.5. *[5] Suppose that (H_1) and (H_2) hold. Then the dynamic system (6) has a unique almost periodic solution $Z(t) = (x_1(t), x_2(t), x_3(t), x_4(t)) \in \Omega$ that is uniformly asymptotically stable.*

In chapitre Five, we examine the Caputo fractional order SICA model:

$$\begin{cases} {}_{t_0}^{C}D_t^{\alpha}w(t) = \Lambda - \beta(x(t) + \eta_y y(t) + \eta_z z(t))w(t) - \mu w(t), \\ {}_{t_0}^{C}D_t^{\alpha}x(t) = \beta(x(t) + \eta_y y(t) + \eta_z z(t))w(t) - \xi_3 x(t) + \omega y(t) + \gamma z(t), \\ {}_{t_0}^{C}D_t^{\alpha}y(t) = \phi x(t) - \xi_2 y(t), \\ {}_{t_0}^{C}D_t^{\alpha}z(t) = \rho x(t) - \xi_1 z(t). \end{cases} \tag{7}$$

And we take the conditions of growth non-linear vector operator $\psi : [0, T] \times \mathbb{R}_+^4 \longrightarrow \mathbb{R}_+$ as:

(A1) $\exists$ a constants $L_\psi > 0$; $\forall (U(t), U'(t)) \in \mathbb{R} \times \mathbb{R}$;

$$|\psi(U(t), t) - \psi(U'(t), t)| \leq L_\psi |U(t) - U'(t)|$$

.

(A2) $\exists$ a constants $C_\psi > 0, M_\psi > 0$;

$$|\ \psi(U(t), t)\ | \leq C_\psi \mid U \mid + M_\psi$$

. More precisely, we prove the following results

Theorem 0.6. *Under the continuity of ψ together with assumption (A2), system(7) has at least one solution.*

Theorem 0.7. *Using (A1), system (7) has unique or one solution if* $\dfrac{T^\alpha}{\Gamma(\alpha+1)} L_\psi < 1$.

Contributions

(1) Z. Belarbi and B. Bayour, D. F. M. Torres, Uniform stability of dynamic SICA HIV transmission models on time scales, Applicationes Mathematicae, vol 51,2 (2024), pp. 163-177, Doi: 10.4064/am2521-6-2024

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Chapter 1

Preliminaries

In this chapter, we introduce some basic concepts related to calculus on time scales, provide exact solutions with given examples, discuss fractional operators, and explore stability. The definitions and results presented in this chapter can be found in [7, 8, 10, 11, 47, 22, 18].

1.1 The Time Scales calculus

Definition 1.1.1. *A time scales is an arbitrary nonempty closed subset of the real numbers. Thus*

$$\mathbb{R},\ \ \mathbb{N}_0,\ \ \mathbb{N},\ \ \mathbb{Z}.$$

i.e., the real numbers, the integers, the natural numbers, and the nonnegative integers are examples of time scales, as are

$$[0,1]\cup[2,3]\,,[0,1]\cup\mathbb{N},\ \textit{and the Cantor set},$$

while

$$\mathbb{Q},\ \mathbb{R}\setminus\mathbb{Q},\ \mathbb{C},\ (0,1),$$

are not time scales. Throughout this thesis we will denote a time scales by the symbol $\mathbb{T}$. We assume throughout that a time scales $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology.

We will introduce the delta derivative f^{Δ} for a function f defined on $\mathbb{T}$, and it turns out that

(i) $f^{\Delta} = f'$ is the usual derivative if $\mathbb{T} = \mathbb{R}$ and

(ii) $f^{\Delta} = \Delta f$ is the usual forward difference operator if $\mathbb{T} = \mathbb{Z}$.

In this section we introduce the basic notions connected to time scales. We start by defining the forward and backward jump opertors.

Definition 1.1.2. *Let $\mathbb{T}$ be a time scales . For $t \in \mathbb{T}$ we define the forward jump operator $\sigma : \mathbb{T} \longrightarrow \mathbb{T}$ by*

$$\sigma(t) := inf\{s \in \mathbb{T} : \quad s > t\} \quad for\ all \quad t \in \mathbb{T},$$

while the backward jump operator $\rho : \mathbb{T} \longrightarrow \mathbb{T}$ is defined by

$$\rho(t) := sup\{s \in \mathbb{T} : \quad s < t\} \quad for\ allt \in \mathbb{T},$$

and the graininess function $\mu :\ \mathbb{T} \longrightarrow [0, \infty)$ is defined by

$$\mu := \sigma(t) - t \ \ for\ all \ \ t \in \mathbb{T}.$$

Using these operators, any $t \in \mathbb{T}$ can be classified as

- *right-scattered (left-scatterd), if $\sigma(t) > t$ $(\rho(t) < t)$, and*
- *right-dense (left-dense), if $\sigma(t) = t$ $(\rho(t) = t)$.*

We say that a point $t \in \mathbb{T}$ is isolated, if it is right and left-scattered. we say that a point $t \in \mathbb{T}$ is dense, if it is right-and left-dense.

Example 1.1.1. *Let us consider different time scales*

- *If $\mathbb{T} = \mathbb{R}$, we have $\sigma(t) = \rho(t) = t$, then t is dense.*
- *If $\mathbb{T} = \mathbb{Z}$, we have $\sigma(t) = t + 1 > t$, and $\rho(t) = t - 1 < t$, then t is isolated.*
- *If $\mathbb{T} = \mathbb{N}_0^2 = \{n^2, \ \ n \in N_0\}$, we have $\sigma(t) = (\sqrt{t} + 1)^2 > t$, and $\rho(t) = (\sqrt{t} - 1)^2 < t$, then t is isolated.*
- *If $\mathbb{T} = [0, 1[\cup \{1, 2, 5, 6\}$, we have $\sigma(1) = 2 > 1$, $\rho(1) = 1 = t$ then 1 is right-scatterd and left-dense. $\{2, 5, 6\}$ are isolated.*

Definition 1.1.3. *Let $\mathbb{T}$ be a time scales . For $t \in \mathbb{T}$ we define the set $\mathbb{T}^k$ by*

$$\mathbb{T}^k = \begin{cases} \mathbb{T} -]\rho(\sup \mathbb{T}), \sup \mathbb{T}] & if \quad \sup \mathbb{T} < +\infty \\ \mathbb{T} & if \quad \sup \mathbb{T} = +\infty. \end{cases} \tag{1.1}$$

If $\mathbb{T}$ has a left-scattered maximum M, then we define $\mathbb{T}^k = \mathbb{T} - \{M\}$, otherwise $\mathbb{T}^k = \mathbb{T}$.

Example 1.1.2. • *Let $\mathbb{T} = \{2, 4, 6, 8, ...\}$. We have* $\sup \mathbb{T} = +\infty$ *then $\mathbb{T}^k = \mathbb{T}$.*

• *Let $\mathbb{T} =]-\infty, 0] \cup \{1, 2, 5, 7\}$, we have* $\sup \mathbb{T} = 7$, *and* $\rho(\sup \mathbb{T}) = 5$, *then*

$$\mathbb{T}^k = \mathbb{T} -]5, 7] =]-\infty, 0] \cup \{1, 2, 5\} .$$

Definition 1.1.4. *Assume $f : \mathbb{T} \longrightarrow \mathbb{R}$ is a function and let $t \in \mathbb{T}$. The function is defined $f^\sigma : \mathbb{T} \longrightarrow \mathbb{R}$ par*

$$f^\sigma(t) := (f \circ \sigma)(t) = f(\sigma(t)), \quad pour\ tout\ t \in \mathbb{T}.$$

1.1.1 Differentiation

Definition 1.1.5. *Let $f : \mathbb{T} \longrightarrow \mathbb{R}$ and $t \in \mathbb{T}^k$. If there exists $f^\Delta(t) \in \mathbb{R}$ such that for all $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\left| f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s) \right| \leq \varepsilon \left| \sigma(t) - s \right| \quad for\ all \quad s \in (t - \delta, t + \delta) \cap \mathbb{T},$$

then we call $f^\Delta(t)$ the delta derivative of f at $t \in \mathbb{T}^k$.
If $f^\Delta(t)$ exists for all $t \in \mathbb{T}^k$, we say that f is delta differentiable (differentiable) and the function $f^\Delta : \mathbb{T} \longrightarrow \mathbb{R}$ is called delta derivative of f on $\mathbb{T}^k$.
If f is differentiable at $t \in \mathbb{T}^k$, then

$$f(\sigma(t)) = f(t) + \mu(t) f^\Delta(t).$$

The definition of a delta derivative can be extended to consider higher order derivatives. We say that f is twice delta differentiable with the second (delta) derivative $f^{\Delta\Delta}$, if f^Δ is (delta) differentiable on $\mathbb{T}^{k^2} = (\mathbb{T}^k)^k$.

Theorem 1.1. *Let $f : \mathbb{T} \longrightarrow \mathbb{R}$ and $t \in \mathbb{T}^k$. Then, the following holds:*

i. If t is right-dense, then

$$f^{\Delta}(t) = \lim_{s \longrightarrow t} \frac{f(t) - f(s)}{t - s},$$

provided that the limit exists (as a finite number).

ii. If f is continuous at the right-scattered point t, then

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

Applying Theorem 1.1 for the case of $\mathbb{T} = \mathbb{R}$, that is, $f^{\Delta}(t) = f'(t)$ for $t \in \mathbb{R}$. For $\mathbb{T} = \mathbb{Z}$, that is, $f^{\Delta}(t) = f(t+1) - f(t) = \Delta f(t)$ for $t \in \mathbb{Z}$, where Δ is the usual forward difference operator defined by the last equation above.

Example 1.1.3. *Let $f : \mathbb{T} \longrightarrow \mathbb{R}$ a function defined by $f(t) = logt$, If $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$ we have $\sigma(t) = qt$ and $\mu(t) = t(q-1)$, then*

$$f^{\Delta}(t) = \lim_{s \longrightarrow t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} = \lim_{s \longrightarrow t} \frac{\log qt - \log s}{qt - s} = \frac{\log q}{t(q-1)}.$$

Example 1.1.4. *(σ is in general not differentiable). Here we present an example of a time scales $\mathbb{T}$ whose jump function $\sigma : \mathbb{T} \longrightarrow \mathbb{T}$ is continuous but not differentiable at a right-dense point $t \in \mathbb{T}$. Let*

$$\mathbb{T} = \left\{ t_n = (1/2)^{2^n} \ : n \in \mathbb{N}_0 \right\} \cup \{0, 1\}.$$

Then

$$\sigma(t_n) = t_{n-1} \longrightarrow 0 = \sigma(0), \ \ n \longrightarrow \infty,$$

and hence $\lim_{s \longrightarrow 0} \sigma(s) = \sigma(0)$ *so* σ *is continuous at* 0*. But*

$$\begin{aligned}
\lim_{s \longrightarrow 0} \frac{\sigma(\sigma(0)) - \sigma(s)}{\sigma(0) - s} &= \lim_{s \longrightarrow 0} \frac{\sigma(s)}{s} \\
&= \lim_{s \longrightarrow 0} \frac{\sqrt{s}}{s} \\
\Delta &= \lim_{s \longrightarrow 0} \frac{1}{\sqrt{S}} \\
&= \infty
\end{aligned} \tag{1.2}$$

so that σ is not differentiable at 0.

Theorem 1.2. *Assume* $f, g : \mathbb{T} \longrightarrow \mathbb{R}$ *are differentiable at* $t \in \mathbb{T}^k$*. Then:*

(i) The sum $f + g : \mathbb{T} \longrightarrow \mathbb{R}$ *is differentiable at* t *with*

$$(f + g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t).$$

(ii) For any constant α*,* $\alpha f : \mathbb{T} \longrightarrow \mathbb{R}$ *is differentiable at* t *with*

$$(\alpha f)^{\Delta}(t) = \alpha f^{\Delta}(t).$$

(iii) The product $fg : \mathbb{T} \longrightarrow \mathbb{R}$ *is differentiable at* t *with*

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma)(t).$$

(iv) If $f(t)f(\sigma(t)) \neq 0$*, then* $\frac{1}{f}$ *is differentiable at* t *with*

$$\left(\frac{1}{f}\right)^{\Delta}(t) = -\frac{f^{\Delta}(t)}{f(t)f(\sigma(t))}.$$

(v) If $g(t)g(\sigma(t)) \neq 0$*, then* $\frac{f}{g}$ *is differentiable at* t *and*

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}.$$

For $\mathbb{T}=\mathbb{R}$, we have $f^\sigma = f$ and $g^\sigma = g$ so that the classical product and quotient rule are retrieved. In the case of $\mathbb{T}=\mathbb{Z}$, we have

$$(fg)^\Delta(t) = \Delta(fg)(t) = (\Delta f(t))g(t+1) + f(t)(\Delta g(t)) = (\Delta f(t))g(t) + f(t+1)(\Delta g(t)).$$

If $g(t), g(t+1) \neq 0$, then

$$\left(\frac{f}{g}\right)^\Delta(t) = \Delta\left(\frac{f(t)}{g(t)}\right) = \frac{(\Delta f(t))g(t) - (\Delta g(t))f(t)}{g(t)g(t+1)}$$

Example 1.1.5. *Let $\mathbb{T} = 2^{\mathbb{N}_0} = \{2, 2^2, 2^3, ...\}$, then $\sigma(t) = 2t$, $\mu(t) = t$ and $u, v : \mathbb{T} \longrightarrow \mathbb{R}$, $u(t) = log(t)$, $v(t) = e^t$.*
We have

$$u^\Delta(t) = \frac{u(\sigma(t)) - u(t)}{\mu(t)} = \frac{log(2t) - log(t)}{t} = \frac{log2}{t},$$

and

$$v^\Delta(t) = \frac{v(\sigma(t)) - v(t)}{\mu(t)} = \frac{e^{2t} - e^t}{t},$$

then

$$\begin{aligned}(u(t)v(t))^\Delta &= u(t)v^\Delta(t) + u^\Delta(t)v^\sigma(t)\\ &= log(t)\frac{e^{2t}-e^t}{t} + \frac{log(2)}{t}e^{2t}\\ &= \frac{e^t}{t}\left((e^t-1)log(t) + log(2)e^t\right),\end{aligned}$$

and

$$\begin{aligned}\left(\frac{u(t)}{v(t)}\right)^\Delta &= \frac{u^\Delta(t)v(t) - u(t)v^\Delta(t)}{v(t)v^\sigma(t)}\\ &= \frac{\frac{log(2)}{t}e^t - log(t)\frac{e^{2t}-e^t}{t}}{e^{3t}}\\ &= \frac{log(2) - (e^t-1)log(t)}{te^{2t}}.\end{aligned}$$

Theorem 1.3. *([8], Theorem 1.90). Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be continuously differentiable and suppose $g : \mathbb{T} \longrightarrow \mathbb{R}$ is (delta) differentiable. Then $f \circ g : \mathbb{T} \longrightarrow \mathbb{R}$ is (delta) differentiable and*

$$(f \circ g)^{\Delta}(t) = \left\{ \int_0^1 f'(g(t) + h\mu(t)g^{\Delta}(t))dh \right\} g^{\Delta}(t).$$

An interesting observation is that the operators, Δ and σ, do generally not commute, that is, $(f^{\Delta})^{\sigma} \neq (f^{\sigma})^{\Delta}$, Take for example $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$, then

$$(f^{\Delta})^{\sigma}(t) = \frac{f(q^2t) - f(qt)}{\mu(qt)} \neq \frac{f(q^2t) - f(qt)}{\mu(t)} = (f^{\sigma})^{\Delta}(t),$$

since $\mu(qt) = qt(q-1) \neq t(q-1) = \mu(t)$.

Example 1.1.6. *If $x, y,$ and z are delta differentiable at t, then*

$$\begin{aligned} (xyz)^{\Delta} = (x.(yz))^{\Delta} &= x^{\Delta}(yz) + x^{\sigma}(yz)^{\Delta} \\ &= x^{\Delta}yz + x^{\sigma}(y^{\Delta}z + y^{\sigma}z^{\Delta}) \\ &= x^{\Delta}yz + x^{\sigma}y^{\Delta}z + x^{\sigma}y^{\sigma}z^{\Delta}. \end{aligned} \tag{1.3}$$

Mean Value Results

Definition 1.1.6. *We say that a function $f : \mathbb{T} \longrightarrow \mathbb{R}$ is right-increasing at a point $t_0 \in \mathbb{T} \setminus \{max\mathbb{T}\}$ provided*

(i) if t_0 is right-scatterd, then $f(\sigma(t_0)) > f(t_0)$;

(ii) if t_0 is right-dense, then there is a neighborhood U of t_0 such that

$$f(t) > f(t_0) \;\; for \;\; all \;\; t_0 \in U \;\; with \;\; t > t_0$$

Similarly, we say that f is a right-decreasing if above in (i), $f(\sigma(t_0)) < f(t_0)$ and in (ii), $f(t) < f(t_0)$.

Example 1.1.7. *Let $f : \mathbb{T} \longrightarrow \mathbb{R}$ a function define by $f(t) = 2t, \mathbb{T} = \{t \in h\mathbb{Z} \cup [2,3], \;\; h > 0\}$.*

- *If* $t_0 \in h\mathbb{Z}$ *we have* $\sigma(t_0) = t_0 + h$ *then* t_0 *is right scatterd and*

$$\forall t_0 \in \mathbb{T},\ f(\sigma(t_0)) = 2t_0 + 2h > f(t_0)$$

then f *is right-increasing at* t_0*.*

- *If* $t_0 \in \,]2,3[\,, \sigma(t_0) = t_0$ *then* t_0 *is dense and*

$$f(t) > f(t_0),\ \ \forall t \in \,]2,3[\ \ with\ \ t > t_0.$$

then f *is increasing at* t_0*.*

Theorem 1.4. *Suppose* $f : \mathbb{T} \longrightarrow \mathbb{R}$ *is differentiable at* $t_0 \in \mathbb{T} \setminus \{max\mathbb{T}\}$*.*
(i) If $f^{\Delta}(t_0) > 0$*, then* f *is right-increasing.*
(i) If $f^{\Delta}(t_0) < 0$*, then* f *is right-decreasing.*

Example 1.1.8. *Let* $f : \mathbb{T} \longrightarrow \mathbb{R}$ *a function define by* $f(t) = t^2,\ \ \mathbb{T} = \{q^n,\ \ q > 1,\ \ n \in \mathbb{N}\}$*, If* $t_0 \in \mathbb{T} \setminus \{\infty\}\,,\ \ \sigma(t) = qt > t$ *and* $f^{\Delta}(t) = (q+1)t > 0$*, then* f *is right-increasing.*

Definition 1.1.7. *We say that a function* $f : \mathbb{T} \longrightarrow \mathbb{R}$ *is right-maximum at* $t_0 \in \mathbb{T} \setminus \{max\mathbb{T}\}$ *provided.*
(i) If t_0 *is right-scatterd, then* $f(\sigma(t_0)) \leq f(t_0)$*;*
(ii) if t_0 *is right-dense, then there is a neighborhood* U *of* t_0 *such that*

$$f(t) \leq f(t_0)\ \ for\ \ all\ \ t \in U\ \ with\ \ t > t_0$$

Similarly, we say that f its local right-minimum if in (i), $f(\sigma(t_0)) \geq f(t_0)$ *and in (ii),* $f(t) \geq f(t_0)$*.*

Example 1.1.9. *Let* $f : \mathbb{T} \longrightarrow \mathbb{R}$ *a function define by* $f(t) = 2t,\ \ \mathbb{T} = [0,2] \cup \{3,4,5,6\}$

- *If* $t_0 \in [0,2]$ *is dense and*

$$\exists D = \,]0,2]\,,\ f(t) \geq f(t_0),\ \ for\ \ all\ \ t \in D\ \ with\ \ t > t_0.$$

since f *assume its local minimum at* t_0*.*

- *If* $t_0 = 4$, $\sigma(t_0) = 5$, t_0 *is right-scatterd, we have* $f(\sigma(4)) = 10 \geq f(4) = 8$ *since* f *assume its local right-minimum at* $t_0 = 4$.

Theorem 1.5. *Suppose* $f : \mathbb{T} \longrightarrow \mathbb{R}$ *is differentiable at* $t_0 \in \mathbb{T} \setminus \{max\mathbb{T}\}$.

(i) If $f^{\Delta}(t_0) > 0$, *then* f *assumes its local right-minimum at* t_0.

(ii) If $f^{\Delta}(t_0) < 0$, *then* f *assumes its local right-maximum at* t_0.

Theorem 1.6. *Suppose* $f : \mathbb{T} \longrightarrow \mathbb{R}$ *is differentiable at* $t_0 \in \mathbb{T} \setminus \{max\mathbb{T}\}$.

(i) If f *assumes its local right-minimum at* t_0 , *then* $f^{\Delta}(t_0) \geq 0$.

(ii) If f *assumes its local right-maximum at* t_0, *then* $f^{\Delta}(t_0) \leq 0$.

Example 1.1.10. *Let* $f : \mathbb{T} \longrightarrow \mathbb{R}$ *a function define by* $f(t) = log(t)$, $\mathbb{T} = \sqrt{n}$, $n \in \mathbb{N}_0$, $t_0 \in \mathbb{T} \setminus \{\infty\}$, $\sigma(t_0) = \sqrt{t_0^2 + 1}$ *we have*

$$f^{\Delta}(t_0) = \frac{f(\sigma(t_0)) - f(t_0))}{\sigma(t_0) - t_0} = \frac{log(\sqrt{t_0^2 + 1}) - log(t_0)}{\sqrt{t_0^2 + 1} - t_0}$$

since $log(\sqrt{t_0^2 + 1}) > log(t_0)$, *and* $\sqrt{t_0^2 + 1} > t_0$ *then* $f^{\Delta}(t_0) > 0$ *therefore* f *assumes its right-minimum at* t_0..

Example 1.1.11. *Let* $f : \mathbb{T} \longrightarrow \mathbb{R}$ *a function define by* $f(t) = \frac{1}{t}$, $\mathbb{T} = \mathbb{N} \setminus \{0\}$, $\sigma(t_0) = t_0 + 1$. *If* $t_0 \in \mathbb{T} \setminus \{\infty\}$ *we possess*

$$f^{\Delta}(t_0) = \frac{f(\sigma(t_0)) - f(t_0)}{\sigma(t_0) - t_0} = -\frac{1}{t_0(t_0 + 1)} < 0$$

then f *assumes its right-maximum at* t_0.

Theorem 1.7. *Let* f *be a continuous function on* $[a, b]$ *that is differentiable on* $[a, b)$ *(the differentiability at* a *is understood as right-sided) and satisfies*

$$f(a) = f(b).$$

Then there exist $\xi, \tau \in [a, b)$ *such that*

$$f^{\Delta}(\tau) \leq 0 \leq f^{\Delta}(\xi).$$

Theorem 1.8. *Let f be a continuous function on $[a, b]$ that is differentiable on $[a, b)$. Then there exist $\xi, \tau \in [a, b)$ such that*

$$f^{\Delta}(\tau) \leq \frac{f(b) - f(a)}{b - a} \leq f^{\Delta}(\xi).$$

Corollary 1.1. *Let f be a continuous function on $[a, b]$ that is differentiable on $[a, b)$. If $f^{\Delta}(t) = 0$ for all $t \in [a, b)$, then f is a constant function on $[a, b]$.*

Corollary 1.2. *Let f be a continuous function on $[a, b]$ that is differentiable on $[a, b)$. Then f is increasing, decreasing, nondecreasing, and nonincreasing on $[a, b]$ if $f^{\Delta}(t) > 0$, $f^{\Delta}(t) < 0$, $f^{\Delta}(t) \geq 0$, and $f^{\Delta}(t) \leq 0$ for all $t \in [a, b)$, respectively.*

Definition 1.1.8. *A function $f : \mathbb{T} \longrightarrow \mathbb{R}$ is called pre-differentiable (with region of differentiation D) provided that the following conditions hold:*

(i) f is continuous on $\mathbb{T}$;

(ii) $D \subset \mathbb{T}^k$;

(iii) $\mathbb{T}^k \setminus D$ countable and contains no right-scattered elements of $\mathbb{T}$;

(iv) f is differentiable at each $t \in D$.

Theorem 1.9. *Let f and g be real-valued functions defined on $\mathbb{T}$. Suppose both f and g are pre-differentiable with region of differentiation D. Then*

$$\left|f^{\Delta}(t)\right| \leq g^{\Delta}(t) \quad for \ \ all \ \ t \in D,$$

implies

$$|f(r) - f(s)| \leq g(r) - g(s) \quad for \ \ r, s \in \mathbb{T} \ \ with \ \ r \leq s.$$

Proposition 1.1.1. *Let $\gamma : \mathbb{T} \longrightarrow \mathbb{R}$ be a strictly increasing function. Then $\gamma(\mathbb{T})$ is a time scales if and only if*

(i) γ is continuous

and

(ii) γ is bounded above (respectively below) only when $\mathbb{T}$ is bounded above (respectively below).

1.1.2 Integration

Definition 1.1.9. *A function $f : \mathbb{T} \longrightarrow \mathbb{R}$ is called regulated provided its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$.*

Example 1.1.12. *Let $\mathbb{T} = \mathbb{R}$ and*

$$f(t) = \begin{cases} \dfrac{2}{t-1} & for \ \ t \in \mathbb{R} \setminus 1 \\ 3 & for \ \ t = 1 \end{cases} \tag{1.4}$$

All points of $\mathbb{T}$ are denses and $\lim\limits_{\substack{t \leq 1 \\ \to}} f(t) = -\infty$, $\lim\limits_{\substack{t \geq 1 \\ \to}} f(t) = +\infty$. Therefore, the function f isn't regulated on $\mathbb{R}$.

Example 1.1.13. *Let $\mathbb{T} = \mathbb{N} \cup [0, 1]$ and*

$$f(t) = \frac{1}{t}, \quad g(t) = \frac{t}{t+1}$$

We have 0 is left dense, and we obtain $\lim\limits_{t \longrightarrow 0} f(t) = \infty$ then the function f isn't regulated.
On the other hand, we have $\lim\limits_{t \longrightarrow 0} g(t) = 0$ (exist and finite) then the function g is regulated.

Definition 1.1.10. *A function $f : \mathbb{T} \longrightarrow \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \longrightarrow \mathbb{R}$ will be denoted in this thesis by*

$$C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R}).$$

The set of functions $f : \mathbb{T} \longrightarrow \mathbb{R}$ that are differentiable and whose derivative is rd-continuous is denoted by

$$C^1_{rd} = C^1_{rd}(\mathbb{T}) = C^1_{rd}(\mathbb{T}, \mathbb{R}).$$

The main existence theorem for pre-antiderivatives now reads as follows.

Theorem 1.10. *(Existence of Pre-Antiderivatives). Let f be regulated. Then there exists a func-*

tion F which is pre-differentiable with region of differentiation D such that

$$F^{\Delta}(t) = f(t) \ \ holds\ for\ all \ \ t \in D.$$

Example 1.1.14. *Let $\mathbb{T} = \mathbb{P}_{2,1} = \bigcup_{k=0}^{\infty} [3k, 3k+2]$ and let $f : \mathbb{T} \longrightarrow \mathbb{R}$ be defined by*

$$f(t) = \begin{cases} 0 & if \ \ t \in \bigcup_{k=0}^{\infty} [3k, 3k+1] \\ t - 3k - 1 & if \ \ t \in \bigcup_{k=0}^{\infty} [3k+1, 3k+2]\,, \ \ k \in \mathbb{N}_0. \end{cases}$$

Then f is pre-differentiable with

$$D := \mathbb{T} \setminus \bigcup_{k=0}^{\infty} \{3k+1\}$$

Example 1.1.15. *Let $\mathbb{T} = \mathbb{R}$ and let $f : \mathbb{T} \longrightarrow \mathbb{R}$ be defined by*

$$f(t) = \begin{cases} 0 & if \ \ t = 0 \\ \dfrac{1}{t} & if \ \ t \in \mathbb{R} \setminus \{0\}\,. \end{cases}$$

Then f is pre-differentiable with

$$D = \mathbb{R} \setminus \{0\}\,.$$

Definition 1.1.11. *Assume $f : \mathbb{T} \longrightarrow \mathbb{R}$ is a regulated function. Any function F as in Theorem 1.10 is called a pre-antiderivative of f. We define the indefinite integral of a regulated function f by*

$$\int f(t)\Delta t = F(t) + C,$$

where C is an arbitrary constant and F is a pre-antiderivative of f. We define the Cauchy integral by

$$\int_{\tau}^{s} f(t)\Delta t = F(s) - F(\tau) \ \ for\ all \ \ \tau, s \in \mathbb{T}.$$

A function $F : \mathbb{T} \longrightarrow \mathbb{R}$ *is called an antiderivative of* $f : \mathbb{T} \longrightarrow \mathbb{R}$ *provided*

$$F^{\Delta}(t) = f(t) \;\; \textit{holds for all} \;\; t \in \mathbb{T}^{k}.$$

Example 1.1.16. *If* $\mathbb{T} = \mathbb{Z}$*, evaluate the indefinite integral*

$$\int a^{t} \Delta t,$$

where $a \neq 1$ *is a constant. Since*

$$\left(\frac{a^{t}}{a-1}\right)^{\Delta} = \frac{a^{t+1} - a^{t}}{a-1} = a^{t},$$

we get that

$$\int a^{t} \Delta t = \frac{a^{t}}{a-1} + C,$$

where C *is an arbitrary constant.*

Example 1.1.17. *If* $\mathbb{T} = \mathbb{Z}$*, evaluate the indefinite integral*

$$\int \left(2t + 1 + e^{2t}(e^{2} - 1)\right) \Delta t,$$

Since

$$\left(t^{2} + e^{2t}\right)^{\Delta} = \frac{\left((\sigma(t))^{2} + e^{2\sigma(t)}\right) - (t^{2} - e^{2t})}{\mu(t)} = 2t + 1 + e^{2t}(e^{2} - 1),$$

where $\sigma(t) = t + 1, \;\; \mu(t) = 1$ *we get that*

$$\int \left(2t + 1 + e^{2t}(e^{2} - 1)\right) \Delta t = t^{2} + e^{2t} + C,$$

where C *is an arbitrary constant*

Theorem 1.11. *(Existence of Antiderivatives). Every rd-continuous function has an antiderivative. In particular if* $t_0 \in \mathbb{T}$*, then* F *defined by*

$$F(t) := \int_{t_0}^{t} f(\tau) \Delta \tau \quad \textit{for} \;\; t \in \mathbb{T},$$

is an antiderivative of f.

Theorem 1.12. *If $f \in C_{rd}$ and $t \in \mathbb{T}^k$, then*

$$\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t)f(t).$$

The following theorem gives several elementary properties of the delta integral.

Theorem 1.13. *If $a, b, c \in \mathbb{T}$, $a \in \mathbb{R}$, and $f, g \in C_{rd}$, then*

$$\begin{aligned}
&(i) \int_a^b [f(t) + g(t)]\, \Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t;\\
&(ii) \int_a^b (\alpha f)(t)\Delta t = \alpha \int_a^b f(t)\Delta t;\\
&(iii) \int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t;\\
&(iv) \int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t;\\
&(v) \int_a^b f(\sigma(t))g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(t)\Delta t;\\
&(vi) \int_a^b f(t)g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(\sigma(t))\Delta t;\\
&(vii) \int_a^a f(t)\Delta t = 0;\\
&(viii)\ if\ f(t) \geq 0\ for\ all\ a \leq t < b,\ then\ \int_a^b f(t)\Delta t \geq 0;\\
&(ix)\ if\ \mid f(t) \mid \leq g(t)\ on\ [a, b),\ then\ \left|\int_a^b f(t)\Delta t\right| \leq \int_a^b g(t)\Delta t.
\end{aligned} \tag{1.5}$$

Theorem 1.14. *Let $a, b \in \mathbb{T}$ and $f \in C_{rd}$.*

(i) If $\mathbb{T} = \mathbb{R}$, then

$$\int_a^b f(t)\Delta t = \int_a^b f(t)dt,$$

where the integral on the right is the usual Riemann integral from calculus.

(ii) $[a,b]$ *consists of only isolated points, then*

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{t\in[a,b)} \mu(t)f(t) & if \quad a<b \\ 0 & if \;\; a=b \\ -\sum_{t\in[b,a)} \mu(t)f(t) & if \quad a>b. \end{cases}$$

(iii) $\mathbb{T} = h\mathbb{Z} = \{hk : \; k \in \mathbb{Z}\}$, *where* $h>0$, *then*

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} f(kh)h & if \quad a<b \\ 0 & if \;\; a=b \\ -\sum_{k=\frac{b}{h}}^{\frac{a}{h}-1} f(kh)h & if \quad a>b. \end{cases}$$

(iv) If $\mathbb{T} = \mathbb{Z}$, *then*

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{t=a}^{b-1} f(t) & if \quad a<b \\ 0 & if \;\; a=b \\ -\sum_{t=b}^{a-1} f(t) & if \quad a>b. \end{cases}$$

Example 1.1.18. *For* $\mathbb{T} = \mathbb{Z}$, *let's calculate* $\int_1^4 (t^2+2t+\frac{5}{6})\Delta t$, *we have*

$$\int_1^4 (t^2+2t+\frac{5}{6})\Delta t = \sum_{t=1}^{3} (t^2+2t+\frac{5}{6}) = \frac{57}{2}.$$

Example 1.1.19. *Let $\mathbb{T}=\mathbb{Z}$, $f(t)=t^2+t$, we have*

$$\begin{aligned}\int_0^t f(t)\Delta t &= \int_0^t (s^2+s)\Delta s = \sum_{k=0}^{\frac{t}{h}-1} f(kh)h\\ &= h^3\sum_{k=0}^{\frac{t}{h}-1}k^2 + h^2\sum_{k=0}^{\frac{t}{h}-1}k\\ &= h^3\frac{(\frac{t}{h}-1)\frac{t}{h}(\frac{2t}{h}-1)}{6} + h^2\frac{(\frac{t}{h}-1)\frac{t}{h}}{2}\\ &= \frac{ht}{2}(\frac{t}{h}-1)\frac{2t-h+3}{3}.\end{aligned}$$

Theorem 1.15. *(Change of Variable)*

Let $\gamma:\mathbb{T}\longrightarrow\mathbb{R}$ be a strictly increasing function such that $\tilde{\mathbb{T}}=\gamma(\mathbb{T})$ is a time scales.

Let $\tilde{\Delta}$ denote the Δ-derivative on $\tilde{\mathbb{T}}$. Suppose $f:\mathbb{T}\longrightarrow\mathbb{R}$ is Δ-integrable on each finite interval of $\mathbb{T}$. Suppose also that γ is Δ-differentiable and γ^Δ is Δ-integrable, we have that

$$\int_a^b f(t)\gamma^\Delta(t)\Delta t = \int_{\gamma(a)}^{\gamma(b)} (f\circ\gamma^{-1})(s)\tilde{\Delta}s,$$

for $a,b\in\mathbb{T}$.

Example 1.1.20. *Let $\mathbb{T}:=\mathbb{N}_0^{\frac{1}{2}}=\{\sqrt{n}:\ n\in\mathbb{N}_0\}$. Let's calculate*

$$\int_0^t \left(\sqrt{\tau^2+1}+\tau\right)3^{\tau^2}\Delta\tau.$$

We take $\gamma(t)=t^2$,

for $t\in\mathbb{N}_0^{\frac{1}{2}}$. Then $\gamma:\mathbb{N}_0^{\frac{1}{2}}\longrightarrow\mathbb{R}$ is strictly increasing and $\gamma(\mathbb{N}_0^{\frac{1}{2}})=\mathbb{N}_0$ is a time scales, and

$$\gamma^\Delta(t)=\sqrt{t^2+1}+t.$$

Hence if $f(t) = 3^{t^2}$ we get from Theorem 1.15 that

$$\begin{aligned}
\int_0^t \left(\sqrt{\tau^2+1}+\tau\right) 3^{\tau^2}\Delta\tau &= \int_0^t f(\tau)\gamma^{\Delta}(\tau)\Delta\tau \\
&= \int_0^{t^2} f(\sqrt{s})\tilde{\Delta}s \\
&= \int_0^{t^2} 3^s\tilde{\Delta}s \\
&= \left[\frac{1}{2}3^s\right]_{s=0}^{s=t^2} \\
&= \frac{1}{2}\left(3^{t^2}-1\right).
\end{aligned}$$

1.1.3 The Regressive Group

Definition 1.1.12. *We say that a function $\mathbb{T} \longrightarrow \mathbb{R}$ is regressive provided*

$$1+\mu(t)p(t) \neq 0 \;\; for\ all\;\; t \in \mathbb{T}^k, \tag{1.6}$$

holds. The set of all regressive and rd-continuous functions $f : \mathbb{T} \longrightarrow \mathbb{R}$ will be denoted in this thesis by

$$\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R}).$$

We define the set

$$\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}) = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{f \in \mathcal{R} : 1+\mu(t)f(t) > 0,\;\; for\;\; all\;\; t \in \mathbb{T}\}.$$

Definition 1.1.13. *Let $p, q \in \mathcal{R}$, we define the circle plus addition $\oplus$ and the circle minus subtraction $\ominus$ on $\mathcal{R}$ by*

$$\begin{aligned}
(p \oplus q)(t) &:= p(t)+q(t)+\mu(t)p(t)q(t) \quad for\ all\;\; t \in \mathbb{T}, \\
(p \ominus q)(t) &:= \frac{p(t)-q(t)}{1+\mu(t)q(t)} \quad for\ all\;\; t \in \mathbb{T}.
\end{aligned}$$

Let introduce the notation

$$\mathcal{R}(\alpha) = \begin{cases} \mathcal{R} & if \;\; \alpha \in \mathbb{N} \\ \mathcal{R}^{+} & if \;\; \alpha \in \mathbb{R} \setminus \mathbb{N}. \end{cases}$$

Not that $p \in \mathcal{R}^{+}$ *implies that*

$$1 + \mu(t)p(t)\tau > 0 \;\; for \; all \;\; \tau \in [0, 1].$$

Definition 1.1.14. *For* $h > 0$*, define* $\mathbb{Z}_h$ *to be the strip*

$$\mathbb{Z}_h := \left\{ z \in \mathbb{C} : -\frac{\pi}{h} < Im(z) \leq \frac{\pi}{h} \right\}$$

Definition 1.1.15. *If* $p \in \mathcal{R}$*, then one defines the exponential function on time scales* $\mathbb{T}$ *by*

$$e_p(t, s) = exp\left(\int_s^t \xi_{\mu(\tau))}(p(\tau))\Delta\tau \right), \;\; for \; t, s \in \mathbb{T},$$

where the cylinder transformation

$$\xi_h(z) = \begin{cases} \dfrac{Log(1 + hz)}{h} & if \;\; h \neq 0 \\ z & if \;\; h = 0. \end{cases}$$

where Log is the principal logarithm function.

Example 1.1.21. *Let* $\alpha \in \mathcal{R}$ *be a constant and* $p : \mathbb{Z} \longrightarrow \mathbb{R}$*.*

- *If* $\mathbb{T} = \mathbb{Z}$*, then* $e_\alpha(t, t_0) = (1 + \alpha)^{t - t_0}, \;\; for \; all \;\; t \in \mathbb{T}.$

- *If* $\mathbb{T} = \mathbb{Z}$*, then* $e_{p(t)}(t, t_0) = \prod_{i=t_0}^{i=t} (1 + p(i)),$

- *If* $\mathbb{T} = \mathbb{R}$*, then* $e_\alpha(t, t_0) = e^{\alpha(t - t_0)}, \;\; for \; all \;\; t \in \mathbb{T}.$

Example 1.1.22. *Let $\mathbb{T} = \{h\mathbb{Z}, h > 0\}$, and $\alpha \in \mathcal{R}$ be a constant then*

$$\begin{aligned} e_\alpha(t,0) &= exp\left(\int_0^t \frac{log(1+\mu(\tau)\alpha)}{\mu(\tau)}\Delta\tau\right) \\ &= exp\left(\int_0^t \frac{log(1+h\alpha)}{h}\Delta\tau\right) = exp\left(\sum_{\tau=0}^{\frac{t}{h}-1} log(1+h\alpha)\right) \\ &= exp\left(\frac{t}{h}log(1+h\alpha)\right) = (1+h\alpha)^{\frac{t}{h}} \end{aligned}$$

Definition 1.1.16. *For $\alpha \in \mathbb{R}$ and $p \in \mathcal{R}$, we define*

$$(\alpha \odot p)(t) := \alpha p(t) \int_0^1 (1+\mu(t)p(t)\tau)^{\alpha-1} d\tau.$$

Example 1.1.23. *Let $\mathbb{T} = \{h\mathbb{Z}, h > 0\}$, $\alpha = 2$, $p(t) = t^2 \in \mathcal{R}$, we have*

$$\begin{aligned} \alpha \odot p(t) = 2 \odot t^2 = 2t^2 &\int_0^1 (1+ht^2\tau)d\tau \\ &= 2t^2\left[\tau + ht^2\frac{\tau^2}{2}\right]_0^1 = 2t^2(1+\frac{ht^2}{2}) \\ &= 2t^2 + ht^4. \end{aligned}$$

Theorem 1.16. *Suppose $p \in \mathcal{R}$ and fix $t_0 \in \mathbb{T}$. Then the initial value problem*

$$y^\Delta = p(t)y, \quad y(t_0) = 1, \tag{1.7}$$

has a unique solution on $\mathbb{T}$, which is the exponential function.

Example 1.1.24. *Let $\mathbb{T} = \mathbb{N}^2 = \{n^2,\ n \in \mathbb{N}\}$. Proof that $e_1(t,0) = 2^{\sqrt{t}}(\sqrt{t})!$*

We have $\sigma(t) = (n+1)^2 = t + 2\sqrt{t} + 1$, then $\mu(t) = 2\sqrt{t} + 1$.

Let $y(t) = 2^{\sqrt{t}}(\sqrt{t})!$, we possess

$$\begin{aligned} y^\Delta(t) &= \frac{y(\sigma(t)) - y(t)}{\mu(t)} = \frac{2^{\sqrt{t}+1}(\sqrt{t}+1)! - 2^{\sqrt{t}}(\sqrt{t})!}{2\sqrt{t}+1} \\ &= \frac{2^{\sqrt{t}}(\sqrt{t})!(2(\sqrt{t}+1)-1)}{2\sqrt{t}+1} = 2^{\sqrt{t}}(\sqrt{t})! \\ &= e_1(t,0), \end{aligned}$$

then $e_1(t,0) = 2^{\sqrt{t}}(\sqrt{t})!$.

Som useful properties of the exponential function are the following.

Theorem 1.17. *If* $p \in \mathcal{R}$, *then*

- $e_0(t,s) = 1$ *and* $e_p(t,t) = 1$;
- $e_p(\sigma(t),s) = (1+\mu(t)p(t))e_p(t,s)$;
- $\dfrac{1}{e_p(t,s)} = e_{\ominus}(t,s)$;
- $e_p(t,s) = \frac{1}{e_p(s,t)}$;
- $e_p(t,r)e_p(r,s) = e_p(t,s)$;
- $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$;
- $\dfrac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t,s)$;
- $\left(\dfrac{1}{e_p(.,s)}\right)^{\Delta} = -\dfrac{p(t)}{e_p^{\sigma}(.,s)}$;
- $\left[e_p(c,.)\right]^{\Delta} = -p\left[e_p(c,.)\right]^{\sigma}$, *where* $c \in \mathbb{T}$.

Theorem 1.18. *If* $\alpha \in \mathbb{R}$ *and* $p \in \mathcal{R}(\alpha)$, *then*

$$e_{\alpha\odot p} = e_p^{\alpha}.$$

1.2 Point fixed theorems

Let X and Y be Two Banach spaces, S a family of functions from X to Y, and $A \subset X$.

Definition 1.2.1. *(Uniformly bounded) we call* S *uniformly bounded if there exists* $M > 0$ *such that*

$$\parallel T \parallel = \sup_{t\in A}|T(x)| \leq M \text{ on } X \text{ for } T \in S.$$

Definition 1.2.2. *(Equicontinuous) The family S is equicontinuous on A if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for every pair of elements $x, y \in A$ and every $T \in S$ we have*

$$\| y - x \|_X < \delta \Rightarrow \| T(y) - T(x) \|_Y < \varepsilon.$$

Theorem 1.19. *(Ascoli-Arzela theorem) Assume that A is a compact set in X. then a set $S \subset C(A)$ is relatively compact in $C(A)$ if and only if the functions in S are uniformly bounded and equicontinuous on A.*

Theorem 1.20. *(Schauder theorem) Let A be a closed convex set in Banach space X and assume that $T : A \longrightarrow A$ is a continuous mapping such that $T(A)$ is relatively compact subset of A. Then T has a fixed point.*

Theorem 1.21. *(Banach theorem) Let T be a contraction on a Banach space X. Then T has a unique fixed point.*

1.3 Fractional operators

We now recall the celebrated gamma function.

Definition 1.3.1. *(Gamma function). For complex numbers with a positive real part, the gamma function $\Gamma(t)$ is defined by the following convergent improper integral:*

$$\Gamma(t) := \int_0^\infty s^{t-1} e^{-s} ds.$$

Remark 1.3.1. *The gamma function satisfies the following useful property:*

$$\Gamma(t+1) = t\Gamma(t)$$

.

Definition 1.3.2. *Let we have any operator say $x(t)$, then we may define the arbitrary order integration w.r.t t as*

$$I_t^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \zeta)^{\alpha - 1} x(\zeta) d\zeta, \ \alpha > 0,$$

such that integral on right side converges.

Definition 1.3.3. *For any mapping $x(t)$, one may define the non-integer order derivative in Caputo sense w.r.t t as*

$$\frac{{}^c d^\alpha x(t)}{dt^\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\zeta)^{n-\alpha-1} \frac{d^n}{d\zeta^n}[x(\zeta)]d\zeta,\ \alpha > 0,$$

with right side is point wise continuous on $\mathbb{R}_+$ and $n = [\alpha] + 1$. If $\alpha \in (0, 1]$, then we have

$$\frac{{}^c d^\alpha x(t)}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\zeta)^{-\alpha} \frac{d}{d\zeta}[x(\zeta)]d\zeta,\ \alpha > 0.$$

Lemma 1.1. *The solution of*

$$\frac{{}^C d^\alpha x(t)}{dt^\alpha} = w(t),\ 0 < \alpha < 1.$$

is

$$x(t) = c_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha-1} w(\zeta)d\zeta.$$

1.4 Stability

In this section we discuss the stability of dynamics systems on time scales. Consider the dynamic system

$$x^\Delta = f(t,x),\ t \in \mathbb{T},\ x(t_0) = x_0,\ \ t_0 \geq 0. \tag{1.8}$$

Where $f \in C_{rd}\,[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n]$ and x^Δ denotes the deravative of x with respect to $t \in \mathbb{T}$.

Definition 1.4.1. *A function $\phi : [0,r] \longrightarrow [0,\infty)$ is of class $\mathcal{K}$ if it is well-defined, continuous, and strictly increasing on $[0,r]$ with $\phi(0) = 0$.*

Definition 1.4.2. *A continuous function $P : \mathbb{R}^n \longrightarrow \mathbb{R}$ with $P(0) = 0$ is called positive definite (negative definite) on $\mathcal{D}$ if there exists a function $\phi \in \mathcal{K}$, such that $\phi(|x|) \leq P(x)$ $(\phi(|x|) \leq -P(x))$ for $x \in \mathcal{D}$, where $\mathcal{D}$ is a compact set.*

Definition 1.4.3. *A continuous function $P : \mathbb{R}^n \longrightarrow \mathbb{R}$ with $P(0) = 0$ is called positive semidefinite (negative semidefinite) on $\mathcal{D}$ if $P(x) \geq 0$ $(P(x) \leq 0)$, for all $x \in \mathcal{D}$.*

Definition 1.4.4. *The trivial solution of (1.8) is said to be*

(i) stable if given an $\epsilon > 0$ *and* $t_0 \in \mathbb{T}$*, there exists a* $\delta > 0$ *such that* $|x_0| \leq \delta$ *implies*

$$|x(t)| \leq \epsilon, \ \ t \geq t_0;$$

(ii) asymptotically stable if it is stable and $\lim_{t \longrightarrow \infty} |x(t)| = 0$.

1.4.1 Comparison Theorems

Let $V \in C_{rd}\left[\mathbb{T}^k \times \mathbb{R}^n, \mathbb{R}_+\right]$ Then we define

$$D^+V^{\Delta}(t,x) = \begin{cases} \dfrac{V(\sigma(t), x(\sigma(t)) - V(t, x(t))}{\mu(t)}, & if \ \ \sigma(t) > t, \\ \limsup_{s \longrightarrow t^+} \dfrac{V(s, x(t) + (s-t)f(t, x(t))) - V(t, x(t))}{s-t}, & if \ \sigma(t) = t \end{cases} \tag{1.9}$$

If V is differentiable, then $D^+V^{\Delta}(t,x) = V^{\Delta}(t,x)$.

Definition 1.4.5. *Let* $V \in C_{rd}\left[\mathbb{T}^k \times \mathbb{R}^n, \mathbb{R}_+\right]$*. Then we define the generalized derivative of* $V(t,x)$ *relative to (1.8) as follows: given* $\epsilon > 0$*, there exists a neighbourhood* $N(\epsilon)$ *of* $t \in \mathbb{T}$ *such that*

$$\frac{1}{\mu(t,s)}\left[V(\sigma(t), x(\sigma(t))) - V(s, x(\sigma(t)) - \mu(t,s)f(t, x(t)))\right] < D^+V^{\Delta}(t, x(t)) + \epsilon$$

for each $s \in N(\epsilon)$ *and* $s > t$*, where* $\mu(t,s) = \sigma(t) - s$ *and* $x(t)$ *is any solution of (1.8).*

Theorem 1.22 ([47])**.** *Let* $V \in C_{rd}\left[\mathbb{T}^k \times \mathbb{R}^n, \mathbb{R}_+\right]$*,* $V(t,x)$ *be locally Lipschitzian in* x *for each* $t \in \mathbb{T}$ *which is rd, and let*

$$D^+V^{\Delta}(t, x(t)) \leq g(t, V(t,x)),$$

where $g \in C_{rd}\left[\mathbb{T}^k \times \mathbb{R}_+, \mathbb{R}_+\right]$*,* $g(t,u)\mu(t) + u$ *is nondecreasing in* u *for each* $t \in \mathbb{T}$*. Let* $r(t) = r(t, t_0, u_0))$ *be the maximal solution of the scalar differential equation*

$$u^{\Delta} = g(t,u), \ u(t_0) = u_0 \geq 0$$

existing on $\mathbb{T}$*. Then,* $V(t_0, x_0) \leq u_0$ *implies that* $V(t, x(t)) \leq r(t, t_0, u_0),\ t \in \mathbb{T},\ \ t \geq t_0$.

Corollary 1.3. *The function* $g(t, u) \equiv 0$ *is admissible in Theorem 1.22 to yield*

$$V(t, x(t)) \leq V(t_0, x_0),\ t \in \mathbb{T}.$$

1.4.2 Stability Criteria

In this subsection, we shall consider some simple stability results

Theorem 1.23. *Assume that*

(i) $V \in C_{rd}\left[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}_+\right]$, $V(t, x)$ *is locally Lipschitzian in* x *for each right dense;*

(ii) $b(\| x \|) \leq V(t, x) \leq a(\| x \|),\ \ for\ \ (t, x) \in \mathbb{T} \times \mathbb{R}^n$,

where $a, b \in \mathcal{K} = \{\psi \in C\left[\mathbb{R}_+, \mathbb{R}_+\right] : \psi(0) = 0\ \ and\ \ \psi(u)\ \ is\ \ increasing\ \ in\ \ u\}$;

(iii) $f(t, 0) = 0$, $g \in C_{rd}\left[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}\right],\ g(t, u)\mu(t) + u$ *is nondecreasing in* u *for each* $t \in \mathbb{T}$*, and*

$$D^+ V^\Delta(t, x) \leq g(t, V(t, x)),\ \ (t, x) \in \mathbb{T} \times \mathbb{R}^n.$$

Then the stability properties of the trivial solution of

$$u^\Delta = g(t, u), \quad u(t_0) = u_0 \geq 0 \tag{1.10}$$

imply the corresponding stability properties of the trivial solution of (1.8).

Corollary 1.4. *:*

(i) The function $g(t, u) \equiv 0$ *is admissible in Theorem 1.23 to yield uniform stability of the zero solution of (1.8).*

(ii) The function $g(t, u) = -c(u),\ c \in \mathcal{K}$*, in Theorem 1.23 implies uniform asymptotic stability of the trivial solution of (1.8).*

Usually Lyapunov's theorem on uniform asymptotic stability should have the assumption $V^\Delta(t, x) \leq -c(|x|)$. However, it is easy to see that if V has an upper estimate as in (ii) of Theorem 1.23, one can obtain the assumption of Corollary 1.4,(ii).

Example 1.4.1. *Consider the system*

$$\begin{cases} x^{\Delta}(t) = -2x(t) - x(t)y(t), \\ y^{\Delta}(t) = -2y(t) + x^2(t), \end{cases} \tag{1.11}$$

where $t \in \mathbb{T}$.

Let $V(x,y) = x^2 + y^2$. *Then we have*

$$\begin{aligned} V^{\Delta}(x,y) &= (x^2 + y^2)^{\Delta} \\ &= x^{\Delta}(2x(t) + \mu(t)x^{\Delta}(t)) \\ &+ y^{\Delta}(t)\left(2y(t) + \mu(t)y^{\Delta}(t)\right) \\ &= (-2x(t) - x(t)y(t))(2x(t) + \mu(t)(-2x(t) - x(t)y(t))) \\ &+ (-2y(t) + x^2)(2y(t) + \mu(t)(-2y(t) + x^2(t))) \\ &= -4x^2(t) - 4y^2(t) + \mu(t)(4x^2(t) + x^2(t)y^2(t) \\ &+ 4x^2(t)y(t) + 4y^2(t) + x^4(t) - 4x^2(t)y(t)) \\ &= -4x^2(t) - 4y^2(t) + \mu(t)\left(4x^2(t) + x^2(t)y^2(t) + 4y^2(t) + x^4(t)\right). \end{aligned}$$

When $\mathbb{T} = \mathbb{R}$, *the system* (1.11) *becomes*

$$\begin{cases} x'(t) = -2x(t) - x(t)y(t), \\ y'(t) = -2y(t) + x^2(t), \end{cases} \tag{1.12}$$

for $t \geq t_0 = 0$, *say. For* $\mathbb{T} = \mathbb{R}$, $\mu(t) = 0, \forall t \in \mathbb{R}$ *and so* $V^{\Delta} = -4x^2 - 4y^2$ *is negative definite. By corollary 1.4 the trivial solution to (1.12) is asymptotically stable.*

If $\mathbb{T} = \mathbb{Z}$, $\mu(t) = 1$, *then*

$$V^{\Delta} = x^2y^2 + x^4 \geq 0,$$

and thus, by Corollary (1.4) the trivial solution to (1.11) is unstable.

Example 1.4.2. *Cnsider the following system*

$$\begin{cases} x^{\Delta}(t) = -x(t) - \dfrac{y}{1+x^2+y^2}, \\ y^{\Delta}(t) = -y(t) + \dfrac{x}{1+x^2+y^2}, \end{cases} \tag{1.13}$$

on time scales $\mathbb{T}$

Let $V(x,y) = x^2 + y^2$. *Then we have*

$$\begin{aligned} V^{\Delta}(x,y) &= (x^2+y^2)^{\Delta} \\ &= x^{\Delta}(2x(t) + \mu(t)x^{\Delta}(t)) \\ &+ y^{\Delta}(t)\left(2y(t) + \mu(t)y^{\Delta}(t)\right) \\ &= 2x\left(-x(t) - \frac{y}{1+x^2+y^2}\right) + 2y\left(-y(t) + \frac{x}{1+x^2+y^2}\right) \\ &+ \mu\left[\left(-x(t) - \frac{y}{1+x^2+y^2}\right)^2 + \left(-y(t) + \frac{x}{1+x^2+y^2}\right)^2\right] \\ &= -2x^2 - 2y^2 + \mu\left(x^2+y^2 + \frac{x^2+y^2}{(1+x^2+y^2)^2}\right) \\ &\le 2(\mu-1)V(x,y).. \end{aligned}$$

If $\mu(t) \le 1$ *then*

$$V^{\Delta}(x,y) \le 0$$

and thus by Corollary (1.4) , that system (1.13) is asymptotically stable.

If $\mu \ge 2$ *then*

$$V^{\Delta}(x,y) \ge \frac{x^2+y^2}{(1+x^2+y^2)^2} \ge 0,$$

From Corollary (1.4), the trivial solution to (1.13) is unstable.

Chapter 2

The non-population conserving SIR model on time scales

We study the non-population conserving SIR model on time scales based on the continuous-time SIR-NC model [12] , reformulating it in the general time scales and deriving its solution.

Theorem 2.1 (Variation of constants, see Theorems 2.74 and 2.77 of [7]). *Suppose $p \in \mathcal{R}$ and $f \in C_{rd}$. If $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}$ are given, then the unique solution of the IVP*

$$y^{\Delta} = p(t)y + f(t), \quad y(t_0) = y_0,$$

is given by

$$y(t) = e_p(t, t_0)y_0 + \int_{t_0}^{t} e_p(t, \sigma(s))f(s)\Delta s.$$

Similarly, the unique solution of the IVP

$$y^{\Delta} = -p(t)y^{\sigma} + f(t), \quad y(t_0) = y_0,$$

is given by

$$y(t) = e_{\ominus p}(t, t_0)y_0 + \int_{t_0}^{t} e_{\ominus p}(t, s)f(s)\Delta s.$$

Theorem 2.2 (See Theorem 2.39 of [7]). *If $p \in \mathcal{R}$ and $a, b, c \in \mathbb{T}$, then*

$$\int_a^b p(t)e_p(t,c)\Delta t = e_p(b,c) - e_p(a,c)$$

and

$$\int_a^b p(t)e_p(c,\sigma(t))\Delta t = e_p(c,a) - e_p(c,b).$$

2.1 The non-population conserving SIR model on time scales (SIR-NC)

On a given time scales $\mathbb{T}$, we propose the following SIR-NC model:

$$\begin{cases} S^\Delta(t) = -\lambda \dfrac{S(t)I^\sigma(t)}{S(t)+I(t)}, \\ I^\Delta(t) = \lambda \dfrac{S(t)I^\sigma(t)}{S(t)+I(t)} - \gamma I^\sigma(t), \\ R^\Delta(t) = \gamma I^\sigma(t), \end{cases} \tag{2.1}$$

where $S, I, R : \mathbb{T} \longrightarrow \mathbb{R}^+$ and $\lambda, \gamma > 0$, subject to given initial conditions

$$S(0) = S_0, \quad I(0) = I_0, \quad R(0) = R_0 \tag{2.2}$$

with $S_0 > 0$, $I_0 > 0$, and $R_0 \geq 0$. In the particular case $\mathbb{T} = \mathbb{R}$, problem (2.1)–(2.2) is studied in [12].

We begin by remarking that it is enough to solve the first two equations of system (2.1). Indeed, by knowing $I(t)$ we immediately get $R(t)$ from the 3rd equation of (2.1). For this reason, in the sequel we restrict ourselves to the two-dimensional IVP

$$\begin{cases} S^\Delta(t) = -\lambda \dfrac{S(t)I^\sigma(t)}{S(t)+I(t)}, \quad S(0) = S_0, \\ I^\Delta(t) = \lambda \dfrac{S(t)I^\sigma(t)}{S(t)+I(t)} - \gamma I^\sigma(t), \quad I(0) = I_0. \end{cases} \tag{2.3}$$

Let $N_0 := I_0 + S_0$ (we could also add R_0, but since $R(t)$ does not affect the evolution of (2.3), we consider here, without loss of generality, that $R_0 = 0$).

It is easy to see that in our SIR-NC model, the condition for an epidemic outbreak is given by

$$I^{\Delta}(0) > 0 \Rightarrow \frac{\lambda}{\gamma} > 1 + \frac{I_0}{S_0}.$$

Theorem 2.3. *Let* $C = \dfrac{S(0)}{I(0)}$. *If* $\gamma - \lambda, p(t) \in \mathcal{R}$, *then the unique solution to* (2.3) *is given by*

$$\begin{cases} S(t) = e_{(\ominus p(t)) \oplus (\gamma-\lambda)}(t,0) S(0), \\ I(t) = e_{\ominus p(t)}(t,0) I(0), \end{cases} \tag{2.4}$$

where $p(t) = \gamma - \dfrac{\lambda C}{e_{\ominus(\gamma-\lambda)}(t,0) + C}$.

Proof 1. *Let*

$$\begin{cases} x(t) = \dfrac{S(t)}{S(t)+I(t)}, \\ y(t) = \dfrac{I(t)}{S(t)+I(t)}. \end{cases}$$

By the assumption that $I(0) > 0$ *we have* $x(t) + y(t) = 1$ *and* $x(0) < 1$. *We can rewrite* $x^{\Delta}(t)$ *and* $y^{\Delta}(t)$ *as follows:*

$$\begin{aligned} x^{\Delta}(t) &= \frac{S^{\Delta}(t)(S(t)+I(t)) - (S^{\Delta}(t)+I^{\Delta}(t))S(t)}{(S(t)+I(t))(S^{\sigma}(t)+I^{\sigma}(t))} \\ &= \frac{-\lambda S(t) I^{\sigma}(t) + \gamma S(t) I^{\sigma}(t)}{(S(t)+I(t))\,(S^{\sigma}(t)+I^{\sigma}(t))} \\ &= (\gamma-\lambda)\frac{S(t)}{(S(t)+I(t))}\frac{I^{\sigma}(t)}{(S^{\sigma}(t)+I^{\sigma}(t))} \\ &= (\gamma-\lambda) x(t) y^{\sigma}(t) \\ &= (\gamma-\lambda)(1 - x^{\sigma}(t)) x(t). \end{aligned} \tag{2.5}$$

Applying the substitution $z = \frac{1}{x}$, we obtain the linear first-order dynamic equation

$$\begin{aligned} z^\Delta &= \frac{-x^\Delta}{xx^\sigma} \\ &= \frac{-(\gamma - \lambda)(1 - x^\sigma)x}{xx^\sigma} \\ &= -(\gamma - \lambda)z^\sigma + (\gamma - \lambda). \end{aligned} \tag{2.6}$$

Its solution is

$$z(t) = e_{\ominus(\gamma-\lambda)}(t, 0)z(0) + \int_0^t e_{\ominus(\gamma-\lambda)}(t, s)(\gamma - \lambda)\Delta s. \tag{2.7}$$

Integrating yields

$$z(t) = e_{\ominus(\gamma-\lambda)}(t, 0)(z(0) - 1) + 1$$

and, re-substituting $z = \frac{1}{x}$, we obtain that

$$x(t) = \frac{x(0)}{e_{\ominus(\gamma-\lambda)}(t, 0)(1 - x(0)) + x(0)}.$$

We have

$$\begin{aligned} I^\Delta(t) &= I^\sigma(t)\left(\lambda \frac{S(t)}{S(t) + I(t)} - \gamma\right) \\ &= -(\gamma - \lambda x(t))I^\sigma(t). \end{aligned}$$

Letting $p(t) = \gamma - \lambda x(t) \in \mathcal{R}$ it follows that

$$I^\Delta(t) = -p(t)I^\sigma(t).$$

Clearly,

$$I(t) = e_{\ominus p(t)}(t, 0)I(0), \tag{2.8}$$

where

$$\begin{aligned} p(t) &= \gamma - \lambda x(t) \\ &= \gamma - \frac{\lambda x(0)}{e_{\ominus(\gamma-\lambda)}(t,0)(1-x(0)) + x(0)} \\ &= \gamma - \frac{\lambda C}{e_{\ominus(\gamma-\lambda)}(t,0) + C}, \end{aligned}$$

and

$$\begin{aligned} S(t) &= \frac{-I(t)x(t)}{x(t)-1} \\ &= \frac{I(t)\frac{C}{e_{\ominus(\gamma-\lambda)}(t,0)+C}}{\frac{C}{e_{\ominus(\gamma-\lambda)}(t,0)+C} - 1} \\ &= e_{(\ominus p(t))\oplus(\gamma-\lambda)}(t,0)S(0), \end{aligned} \tag{2.9}$$

where $C = \dfrac{S(0)}{I(0)}$.

The proof for the time dependent γ *and* λ *goes exactly the same way.*

Remark 2.1.1. *If* $\gamma = \lambda$, *then* $\gamma - \lambda \in \mathcal{R}$ *and, by Theorem 2.3, the solution of system (2.3) is*

$$\begin{cases} S(t) = e_{\ominus p(t)}(t,0)S(0), \\ I(t) = e_{\ominus p(t)}(t,0)I(0), \end{cases} \tag{2.10}$$

where $p(t) = \frac{\lambda}{1+C}$ *with* $C = \frac{S(0)}{I(0)}$.

As a corollary, we apply Theorem 2.3 to solve the discrete epidemic model

$$\begin{cases} S(t+1) = S(t) - \dfrac{\lambda S(t)I(t+1)}{S(t)+I(t)}, \\ I(t+1) = I(t) + \dfrac{\lambda S(t)I(t+1)}{S(t)+I(t)} - \gamma I(t+1), \end{cases} \tag{2.11}$$

$t \in \mathbb{Z}$, with initial conditions $S(0) = S_0 > 0$, $I(0) = I_0 > 0$, $R(0) = R_0 \geq 0$. Note that for any

$t \in \mathbb{Z}$ we have

$$\begin{aligned} e_{(\gamma-\lambda)}(t,0) &= exp\left(\int_0^t log(1+(\gamma-\lambda))\Delta s\right) \\ &= exp\left(\sum_{s=0}^{t-1} log(1+(\gamma-\lambda))\right) \\ &= exp\left(t.log(1+(\gamma-\lambda))\right) \\ &= (1+(\gamma-\lambda))^t = \delta(t) \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} p(t) &= \gamma - \frac{\lambda C}{e_{\ominus(\gamma-\lambda)}(t,0)+C} \\ &= \gamma - \frac{\lambda C}{\frac{1}{e_{(\gamma-\lambda)}(t,0)}+C} \\ &= \frac{\gamma+(\gamma-\lambda)C\delta(t)}{1+C\delta(t)}. \end{aligned}$$

and

$$\begin{aligned} e_{p(t)} &= exp\left(\int_0^t log(1+\frac{\gamma+(\gamma-\lambda)C\delta(\tau)}{1+C\delta(\tau)})\Delta\tau\right) \\ &= \prod_{\tau=0}^{t-1}\left(\frac{1+\gamma+(1+\gamma-\lambda)C\delta(\tau)}{1+C\delta(\tau)}\right) \end{aligned}$$

Corollary 2.1. *If $1+\gamma-\lambda, 1+\lambda \neq 0$ for all $t \in \mathbb{Z}$, then the unique solution to system (2.11) is given by*

$$\begin{cases} S(t) = S(0)\delta(t)\left[\prod_{i=0}^{t-1}\left(\frac{(1+\gamma)+(1+\gamma-\lambda)C\delta(i))}{1+C\delta(i)}\right)\right]^{-1} \\ I(t) = I(0)\left[\prod_{i=0}^{t-1}\left(\frac{(1+\gamma)+(1+\gamma-\lambda)C\delta(i))}{1+C\delta(i)}\right)\right]^{-1} \end{cases} \tag{2.13}$$

where $C = \frac{S(0)}{I(0)}$ and $\delta(t) = (1+(\gamma-\lambda))^t$.

For the more general case of time-dependent $\lambda(t), \gamma(t), t \geq 0$, we similarly have

$$\begin{aligned} e_{(\gamma(t)-\lambda(t))}(t,0) &= exp\left(\int_0^t log(1+(\gamma(t)-\lambda(t)))\Delta s\right) \\ &= exp\left(\sum_{s=0}^{t-1} log(1+(\gamma-\lambda)(s))\right) \\ &= \prod_{s=0}^{t-1}(1+(\gamma-\lambda)(s)), \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} p(t) &= \gamma(t) - \frac{\lambda(t)C}{e_{\ominus(\gamma-\lambda)(t)}(t,0)+C} \\ &= \gamma(t) - \frac{\lambda C}{\frac{1}{e_{(\gamma-\lambda)(t)}(t,0)}+C} \\ &= \gamma(t) - \frac{\lambda C}{\frac{1}{\prod_{s=0}^{t-1}(1+(\gamma-\lambda)(s)))}+C} \\ &= \gamma(t) - \frac{\lambda(t)C\prod_{s=0}^{t-1}(1+(\gamma-\lambda)(s))}{1+C\prod_{s=0}^{t-1}(1+(\gamma-\lambda)(s))} \\ &= \frac{\gamma(t)+C(\gamma-\lambda)(t)\prod_{s=0}^{t-1}(1+(\gamma-\lambda)(s))}{1+C\prod_{s=0}^{t-1}(1+(\gamma-\lambda)(s))} \\ &= \frac{\gamma(t)+C(\gamma-\lambda)(t)\delta(t)}{1+C\delta(t)}. \end{aligned}$$

then

$$\begin{aligned} e_{p(t)} &= exp\left(\int_0^t log(1+\frac{\gamma(i)+C(\gamma-\lambda)(i)\delta(i)}{1+C\delta(i)})\Delta i\right) \\ &= \prod_{i=0}^{t-1}\left(1+\frac{\gamma(i)+C(\gamma-\lambda)(i)\delta(i)}{1+C\delta(i)}\right) \\ &= \prod_{i=0}^{t-1}\frac{1+\gamma(i)+C(1+(\gamma-\lambda)(i))\delta(i)}{1+C\delta(i)} \end{aligned}$$

Where $\delta(t) = \prod_{s=0}^{t-1}(1 + (\gamma - \lambda)(s))$

Corollary 2.2. *If $1 + \gamma(t) - \lambda(t), 1 + \lambda(t) \neq 0$ for all $t \in \mathbb{Z}$, then the unique solution to system (2.11) is given by*

$$\begin{cases} S(t) = S(0)\delta(t)\left[\prod_{i=0}^{t-1} \dfrac{1 + \gamma(i) + C(1 + (\gamma - \lambda)(i))\delta(i))}{1 + C\delta(i)}\right]^{-1} \\ I(t) = I(0)\left[\prod_{i=0}^{t-1}\left(\dfrac{(1 + \gamma(i)) + C(1 + (\gamma - \lambda)(i))\delta(i))}{1 + C\delta(i)}\right)\right]^{-1} \end{cases}.$$

Example 2.1.1. *Let $\mathbb{T} = \mathbb{R}$ and $\gamma, \lambda \in \mathbb{R}$ with $\gamma \neq \lambda$. Then, by Theorem 2.3, the solution to system (2.1) is given with*

$$\begin{aligned} S(t) &= S(0)\frac{e^{\int_0^t (\gamma - \lambda)ds}}{e^{\int_0^t p(s)ds}} \\ &= S(0)e^{(\gamma-\lambda)t} e^{-\int_0^t \gamma ds} e^{\int_0^t \frac{\lambda C e^{(\gamma-\lambda)s}}{1 + Ce^{(\gamma-\lambda)s}}ds} \\ &= S(0)e^{-\lambda t} e^{\frac{\lambda}{\gamma - \lambda}\left[\ln(1 + Ce^{(\gamma-\lambda)s})\right]_0^t} \\ &= S(0)e^{-\lambda t}\left(\frac{1 + Ce^{(\gamma-\lambda)t}}{1 + C}\right)^{\frac{\lambda}{\gamma-\lambda}} \end{aligned}$$

and

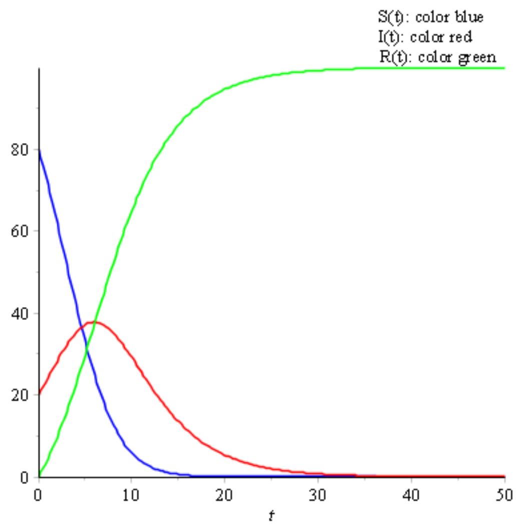
$$I(t) = I(0)e^{-\gamma t}\left(\frac{1 + Ce^{(\gamma-\lambda)t}}{1 + C}\right)^{\frac{\lambda}{\gamma-\lambda}}.$$

If $\gamma = \gamma(t), \lambda = \lambda(t), t \geq 0,\ t \in \mathbb{R}$, then the solution to system (2.1) is given with

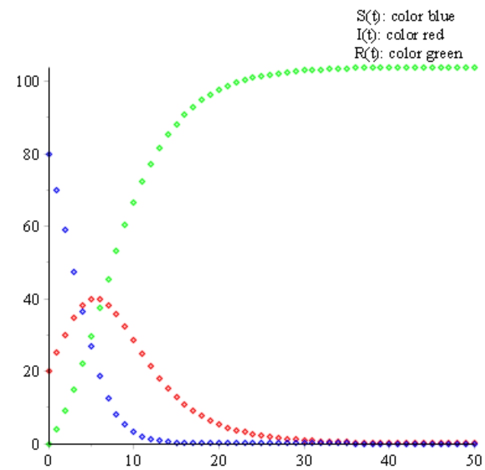
$$S(t) = S(0)e^{\int_0^t (\gamma - \lambda)(s)ds} e^{\int_0^t \left(-\gamma(s) + \frac{\lambda(s)Ce^{\int_0^s (\gamma-\lambda)(\tau)d\tau}}{1 + Ce^{\int_0^s (\gamma-\lambda)(\tau)d\tau}}\right)ds}$$

Example 2.1.2. *where $\lambda = 0.5, \gamma = 0.2, S(0) = 80, I(0) = 20$.*

If $\mathbb{T} = \mathbb{R}$ the solution of (2.1) is



(a) $\mathbb{T}=\mathbb{R}$



(b) $\mathbb{T}=\mathbb{Z}$

Figure 2.1: Numerical solutions of (2.1) with $\lambda = 0.5, \gamma = 0.2, S(0) = 80, I(0) = 20$. (a), we show the solution in the discrete-time case $\mathbb{T} = \mathbb{Z}$; and (b) we plot the solution to (2.1) for the continuous time scales.

$$\begin{cases} S(t) = 80.e^{-0.5t}\left(\dfrac{1+4.e^{-0.3t}}{5}\right)^{\frac{-5}{3}} \\ I(t) = 20.e^{-0.2t}\left(\dfrac{1+4.e^{-0.3t}}{5}\right)^{\frac{-5}{3}}, \\ R(t) = 0.2\int_0^t = 20.e^{-0.2t}\left(\dfrac{1+4.e^{-0.3t}}{5}\right)^{\frac{-5}{3}}, \end{cases}$$

If $\mathbb{T}=\mathbb{Z}$ the solution of (2.1) is

$$\begin{cases} S(t) = 80.(0.7)^t\left[\prod_{s=0}^{t-1}\dfrac{1.2+4.(0.7)^{s+1}}{1+4.(0.7)^s}\right]^{-1} \\ I(t) = 20.(0.7)^t\left[\prod_{s=0}^{t-1}\dfrac{1.2+4.(0.7)^{s+1}}{1+4.(0.7)^s}\right]^{-1}, \\ R(t) = 100 - 80.(0.7)^t\left[\prod_{s=0}^{t-1}\dfrac{1.2+4.(0.7)^{s+1}}{1+4.(0.7)^s}\right]^{-1} - 20.(0.7)^t\left[\prod_{s=0}^{t-1}\dfrac{1.2+4.(0.7)^{s+1}}{1+4.(0.7)^s}\right]^{-1}, \end{cases}$$

2.2 The SIR-NC model with imported infections

On a general time scales $\mathbb{T}$, we propose the following SIR-NC model with imported infections:

$$\begin{cases} S^{\Delta}(t) = -\lambda \dfrac{S(t)I^{\sigma}(t)}{S(t)+I(t)} - \nu S^{\sigma}(t), \\ I^{\Delta}(t) = \lambda \dfrac{S(t)I^{\sigma}(t)}{S(t)+I(t)} + \nu S^{\sigma}(t) - \gamma I^{\sigma}(t), \\ R^{\Delta}(t) = \gamma I^{\sigma}(t), \end{cases} \tag{2.15}$$

where $S, I, R : \mathbb{T} \longrightarrow \mathbb{R}^{+}$ and $\lambda, \gamma, \nu > 0$. Similarly to Section 2.1, we restrict our attention, without loss of generality, to the first two equations of (2.15) and we set $N(t) := S(t) + I(t)$.

We observe that here the condition for an epidemic to break out is given by

$$\lambda \frac{I(0)}{N(0)} + \nu > \gamma \frac{I(0)}{S(0)}.$$

Theorem 2.4. *If $\nu, \gamma - \lambda \in \mathcal{R}$, then the solution to the SIR-NC system (2.15) with imported infections is given as follows:*

$$\begin{cases} S(t) = \dfrac{x(t)}{1-x(t)} e_{\ominus g(t)}(t,0) I(0), \\ I(t) = e_{\ominus g(t)}(t,0) I(0), \end{cases} \tag{2.16}$$

$t \in \mathbb{T}$, where $S, I : \mathbb{T} \longrightarrow \mathbb{R}^{+}$, $\lambda, \gamma, \nu > 0$, and

$$\begin{aligned} g(t) &= -\lambda x(t) + \gamma - \frac{\nu x^{\sigma}(t)}{1 - x^{\sigma}(t)}, \\ x(t) &= \frac{x(0)}{e_{\nu \ominus (\gamma - \lambda)}(t,0)\left(1 + \frac{x(0)(\gamma-\lambda)}{\nu - \gamma + \lambda}\right) - \frac{x(0)(\gamma - \lambda)}{\nu - \gamma + \lambda}}. \end{aligned}$$

Proof 2. *Defining* $x(t) = \dfrac{S(t)}{N(t)}$*, we have*

$$\begin{aligned} x^{\Delta}(t) &= \frac{S^{\Delta}(t)N(t) - N^{\Delta}(t)S(t)}{N(t)N^{\sigma}(t)} \\ &= \frac{-\lambda S(t)I^{\sigma}(t) - \nu S^{\sigma}(t)N(t) + \gamma I^{\sigma} S}{N(t)N^{\sigma}(t)} \\ &= -\lambda x(t)y^{\sigma}(t) - \nu x^{\sigma}(t) + \gamma x(t)y^{\sigma}(t) \\ &= (\gamma-\lambda)\left(\frac{\nu}{\lambda-\gamma} - x(t)\right)x^{\sigma}(t) + (\gamma-\lambda)x(t). \end{aligned} \tag{2.17}$$

Since $x(t) = x^{\sigma}(t) - \mu(t)x^{\Delta}(t)$*, then*

$$x^{\Delta}(t) = (\gamma-\lambda)\left(\frac{\nu}{\lambda-\gamma} - x(t)\right)x^{\sigma}(t) + (\gamma-\lambda)\left(x^{\sigma}(t) - \mu(t)x^{\Delta}(t)\right),$$

and from it

$$\left((\gamma-\lambda)\mu(t)+1\right)x^{\Delta}(t) = (\gamma-\lambda)\left(\frac{\nu}{\lambda-\gamma} - x(t)\right)x^{\sigma}(t) + (\gamma-\lambda)x^{\sigma}(t).$$

Then

$$x^{\Delta}(t) = \frac{\gamma-\lambda}{1+\mu(t)(\gamma-\lambda)}\left(\frac{\nu}{\lambda-\gamma} - x(t)\right)x^{\sigma}(t) + \frac{\gamma-\lambda}{1+\mu(t)(\gamma-\lambda)}x^{\sigma}(t).$$

Applying the substitution $z = \frac{1}{x}$ *it yields*

$$z^{\Delta}(t) = \frac{\nu-(\gamma-\lambda)}{1+\mu(t)(\gamma-\lambda)}z(t) + \frac{\gamma-\lambda}{1+\mu(t)(\gamma-\lambda)},$$

which has the solution

$$z(t) = e_{\alpha}(t,0)z(0) + \int_0^t e_{\alpha}(t,\sigma(s))\frac{\gamma-\lambda}{1+\mu(t)(\gamma-\lambda)}\Delta s$$

with $\alpha = \nu \ominus (\gamma-\lambda)$*. The solution is equivalent to*

$$\begin{aligned} z(t) &= e_{\alpha}(t,0)z(0) + \frac{\gamma-\lambda}{\nu-(\gamma-\lambda)}\int_0^t \alpha e_{\alpha}(t,\sigma(s))\Delta s \\ &= e_{\alpha}(t,0)z(0) + \frac{\gamma-\lambda}{\nu-\gamma+\lambda}(e_{\alpha}(t,0)-1). \end{aligned}$$

Re-substituting $x = \frac{1}{z}$ one has

$$x(t) = \frac{x(0)}{e_\alpha(t,0)\left(1 + \frac{x(0)(\gamma-\lambda)}{\nu-\gamma+\lambda}\right) - \frac{x(0)(\gamma-\lambda)}{\nu-\gamma+\lambda}},$$

and

$$\begin{aligned} I^\Delta(t) &= \lambda I^\sigma(t)x(t) + \nu S^\sigma(t) - \gamma I^\sigma(t) \\ &= (\lambda x(t) - \gamma)I^\sigma(t) + \nu S^\sigma(t). \end{aligned}$$

From $x(t) = \dfrac{S(t)}{S(t)+I(t)}$ it follows that $S(t) = \dfrac{I(t)x(t)}{1-x(t)}$ and

$$\begin{aligned} I^\Delta(t) &= (\lambda x(t) - \gamma)I^\sigma(t) + \frac{\nu x^\sigma(t)}{1-x^\sigma(t)} I^\sigma(t) \\ &= -\left(-\lambda x(t) + \gamma - \frac{\nu x^\sigma(t)}{1-x^\sigma(t)}\right) I^\sigma(t) \\ &= -g(t)I^\sigma(t), \end{aligned}$$

where $g(t) = -\lambda x(t) + \gamma - \dfrac{\nu x^\sigma(t)}{1-x^\sigma(t)}$. The solution is equivalently expressed as in (2.16).

As a corollary, we apply Theorem 2.4 to solve the discrete epidemic model

$$\begin{cases} S(t+1) = S(t) - \lambda \dfrac{S(t)I(t+1)}{S(t)+I(t)} - \nu S(t+1), \\ I(t+1) = I(t) + \lambda \dfrac{S(t)I(t+1)}{S(t)+I(t)} + \nu S(t+1) - \gamma I(t+1), \end{cases} \tag{2.18}$$

$t \in \mathbb{Z}$, with initial conditions $S(0) = S_0 > 0,\ I(0) = I_0 > 0$. Not that for any $t \in \mathbb{Z}$ we have

$$\begin{aligned} x(t) &= \frac{bC_1}{e_\alpha(t,0) + C_1} \\ &= \frac{bC_1}{e_{\left(\frac{\nu-\gamma+\lambda}{1+\gamma-\lambda}\right)}(t,0) + C_1} \\ &= \frac{bC_1}{\left(\dfrac{1+\nu}{1+\gamma-\lambda}\right)^t + C_1} \\ &= \frac{bC_1}{\delta(t) + C_1}, \end{aligned}$$

and

$$g(t) = \frac{-\lambda b C_1}{\delta(t) + C_1} + \gamma - \frac{\nu b C_1}{\delta(t+1) + (1-b)C_1},$$

then

$$\begin{aligned} e_{g(t)}(t,0) &= exp\left(\int_0^t log\left(1 - \frac{\lambda b C_1}{\delta(s) + C_1} + \gamma - \frac{\nu b C_1}{\delta(s+1) + (1-b)C_1}\right) \Delta s\right) \\ &= exp\left(\sum_{s=0}^{t-1} log\left(1 - \frac{\lambda b C_1}{\delta(s) + C_1} + \gamma - \frac{\nu b C_1}{\delta(s+1) + (1-b)C_1}\right)\right) \\ &= \prod_{s=0}^{t-1}\left(1 - \frac{\lambda b C_1}{\delta(s) + C_1} + \gamma - \frac{\nu b C_1}{\delta(s+1) + (1-b)C_1}\right), \end{aligned}$$

Corollary 2.3. *If $\nu - \gamma + \lambda \neq 0, 1 + \gamma - \lambda \neq 0$ for all $t \in \mathbb{Z}$, then the unique solution to system (2.18) is given by*

$$\begin{cases} S(t) = \dfrac{bC_1 I(0)}{\delta(t) + (1-b)C_1}\left[\displaystyle\prod_{s=0}^{t-1}\left(1 - \frac{\lambda b C_1}{\delta(s) + C_1} + \gamma - \frac{\nu b C_1}{\delta(s+1) + (1-b)C_1}\right)\right]^{-1}, \\ I(t) = \left[\displaystyle\prod_{s=0}^{t-1}\left(1 - \frac{\lambda b C_1}{\delta(s) + C_1} + \gamma - \frac{\nu b C_1}{\delta(s+1) + (1-b)C_1}\right)\right]^{-1} I(0), \end{cases} \tag{2.19}$$

where $C_1 = \dfrac{(\lambda - \gamma)x(0)}{(\lambda - \gamma + \nu) + (\gamma - \lambda)x(0)}$, $\delta(t) = \left(\dfrac{1+\nu}{1+\gamma-\lambda}\right)^t$, $b = \dfrac{\nu - \gamma + \lambda}{\lambda - \gamma}$.

As in Example 2.1.1, a simple application is obtained from our Theorem 2.4 when we restrict ourselves to the continuous case.

Example 2.2.1. *If $\mathbb{T} = \mathbb{R}$, then system (2.15) reduces to*

$$\begin{cases} S'(t) = -\lambda \dfrac{S(t)I(t)}{S(t) + I(t)} - \nu S(t), \\ I'(t) = \lambda \dfrac{S(t)I(t)}{S(t) + I(t)} + \nu S(t) - \gamma I(t), \\ R'(t) = \gamma I(t), \end{cases} \tag{2.20}$$

and, by Theorem 2.4, the solution to system (2.20) is

$$\begin{cases} S(t) = \dfrac{x(t)I(t)}{1-x(t)}, \\ I(t) = e^{\int_0^t -g(s)ds} I(0), \end{cases} \tag{2.21}$$

We have

$$\begin{aligned} -g(t) &= \lambda x(t) + \frac{\nu x(t)}{1-x(t)} - \gamma \\ &= \frac{\nu x(0)}{e^{\alpha t}\left(1+\frac{(\gamma-\lambda)x(0)}{\alpha}\right) - x(0)(\frac{\gamma-\lambda}{\alpha}-1)} + \frac{\lambda x(0)}{e^{\alpha t}\left(1+\frac{(\gamma-\lambda)x(0)}{\alpha}\right) - x(0)\frac{\gamma-\lambda}{\alpha}} - \gamma \\ &= \frac{\nu}{e^{\alpha t}\left(\frac{C+1}{C}+\frac{\gamma-\lambda}{\alpha}\right) - \frac{\nu}{\alpha}} + \frac{\lambda}{e^{\alpha t}\left(\frac{C+1}{C}+\frac{\gamma-\lambda}{\alpha}\right) - \frac{\gamma-\lambda}{\alpha}} - \gamma \\ &= \frac{\nu\alpha C e^{-\alpha t}}{(\alpha+\nu C) - \nu C e^{-\alpha t}} + \frac{\lambda\alpha C e^{-\alpha t}}{(\alpha+\nu C) - (\gamma-\lambda)C e^{-\alpha t}} - \gamma, \end{aligned}$$

and

$$\int_0^t -g(s)ds = \ln\left(\frac{(\alpha+\nu C) - \nu C e^{-\alpha t}}{\alpha}\right) + \ln\left(\frac{(\alpha+\nu C)-(\gamma-\lambda)Ce^{-\alpha t}}{(\alpha+\nu C)-(\gamma-\lambda)C}\right)^{-\frac{\lambda}{\lambda-\gamma}} - \gamma t$$

and

$$e^{\int_0^t -g(s)ds} = e^{-\gamma t}\left(\frac{(\alpha+\nu C) - \nu C e^{-\alpha t}}{\alpha}\right)\left(\frac{(\alpha+\nu C)-(\gamma-\lambda)Ce^{-\alpha t}}{(\alpha+\nu C)-(\gamma-\lambda)C}\right)^{-\frac{\lambda}{\lambda-\gamma}}.$$

Then

$$\begin{aligned} I(t) &= I(0)\left(\frac{(\alpha+\nu C) - \nu C e^{-\alpha t}}{\alpha}\right) e^{-\gamma t}\left(\frac{1+C_1e^{-\alpha t}}{1+C_1}\right)^{-\frac{\lambda}{\lambda-\gamma}} \\ &= I(0)\left(\frac{1+C_1e^{-\alpha t}}{1+C_1}\right)^{-\frac{\lambda}{\lambda-\gamma}} e^{-\gamma t} + \frac{\nu S(0)}{\alpha}\left(1-e^{-\alpha t}\right)\left(\frac{1+C_1e^{-\alpha t}}{1+C_1}\right)^{-\frac{\lambda}{\lambda-\gamma}} e^{-\gamma t} \\ &= \left(\frac{1+C_1e^{-\alpha t}}{1+C_1}\right)^{-\frac{\lambda}{\lambda-\gamma}} e^{-\gamma t}\left[I(0) + \frac{\nu S(0)}{\alpha}\left(1-e^{-\alpha t}\right)\right] \end{aligned} \tag{2.22}$$

Where $C_1 = \frac{C(\lambda-\gamma)}{(\lambda-\gamma+\nu)+C\nu}$, $C = \frac{S(0)}{I(0)}$, $\alpha = \nu - (\gamma - \lambda)$

Letting

$$S(t) = \frac{x(t)I(t)}{1-x(t)},$$

where

$$\frac{x(t)}{1-x(t)} = \frac{\alpha C e^{-\alpha t}}{\nu C + \alpha - \nu C e^{-\alpha t}},$$

we have

$$\begin{aligned} S(t) &= I(0)e^{-\gamma t} C e^{-\alpha t} \left(\frac{1+C_1 e^{-\alpha t}}{1+C_1}\right)^{-\frac{\lambda}{\lambda-\gamma}} \\ &= S(0)e^{-(\lambda+\nu)t} \left(\frac{1+C_1 e^{-\alpha t}}{1+C_1}\right)^{-\frac{\lambda}{\lambda-\gamma}} \end{aligned}$$

Example 2.2.2. *: Let us consider the $SIR-NC$ model (2.15) with*

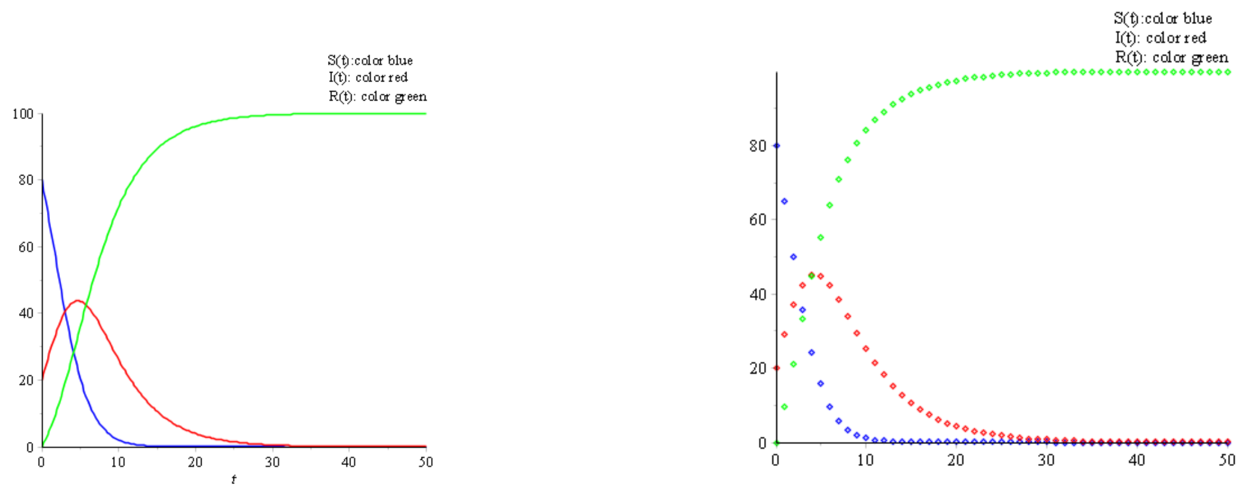
where $\lambda = 0.5, \gamma = 0.2, \nu = 0.05, S(0) = 80, I(0) = 20.$

If $\mathbb{T} = \mathbb{Z}$ the solution of (2.15) is

$$\begin{cases} S(t) = \dfrac{560}{11 \times 1.5^t - 4} \left[\displaystyle\prod_{s=0}^{t-1} \left(1.2 - \frac{14}{11 \times 1.5^s + 24} - \frac{14}{11 \times 1.5^{s+1} - 4}\right)\right]^{-1}, \\ I(t) = 20. \left[\displaystyle\prod_{s=0}^{t-1} \left(1.2 - \frac{14}{11 \times 1.5^s + 24} - \frac{14}{11 \times 1.5^{s+1} - 4}\right)\right]^{-1}, \\ R(t) = 100 - \dfrac{560}{11 \times 1.5^t - 4} \left[\displaystyle\prod_{s=0}^{t-1} \left(1.2 - \frac{14}{11 \times 1.5^s + 24} - \frac{14}{11 \times 1.5^{s+1} - 4}\right)\right]^{-1} \\ \qquad -20. \left[\displaystyle\prod_{s=0}^{t-1} \left(1.2 - \frac{14}{11 \times 1.5^s + 24} - \frac{14}{11 \times 1.5^{s+1} - 4}\right)\right]^{-1}, \end{cases}$$

If $\mathbb{T} = \mathbb{R}$ the solution of (2.15) is

$$\begin{cases} S(t) = 80.e^{-0.55.t} \left(\dfrac{1 + \dfrac{1.2}{0.55}.e^{-0.35t}}{1 + \dfrac{1.2}{0.55}}\right)^{\frac{-5}{3}} \\ I(t) = e^{-0.2t} \left(20 + \dfrac{4}{0.35}(1 - e^{-0.35t})\right) \left(\dfrac{1 + \dfrac{1.2}{0.55}.e^{-0.35t}}{1 + \dfrac{1.2}{0.55}}\right)^{\frac{-5}{3}}, \\ R(t) = 0.2 \int_0^t e^{-0.2t} \left(20 + \dfrac{4}{0.35}(1 - e^{-0.35t})\right) \left(\dfrac{1 + \dfrac{1.2}{0.55}.e^{-0.35t}}{1 + \dfrac{1.2}{0.55}}\right)^{\frac{-5}{3}}, \end{cases}$$



(a) $\mathbb{T} = \mathbb{R}$ (b) $\mathbb{T} = \mathbb{Z}$

Figure 2.2: Numerical solutions of (2.15) with $\lambda = 0.5, \gamma = 0.2, \nu = 0.05, S(0) = 80, I(0) = 20$. (a), we show the solution in the discrete-time case $\mathbb{T} = \mathbb{Z}$; and (b) we plot the solution to (2.15) for the continuous time scales

Chapter 3

Existence and uniqueness of solution for a fractional order SIR model

We consider an extension of the SIR model.

$$\begin{cases} \frac{dx(t)}{dt} = -\alpha y(t)x(t) \\ \frac{dy(t)}{dt} = \alpha y(t)x(t) - \beta y(t) \\ \frac{dz(t)}{dt} = \beta y(t), \end{cases} \tag{3.1}$$

where $N(t) = x(t) + y(t) + z(t)$ the total population at time t. Not that, in our natation, $(x(t), y(t), z(t))$ is interpreted as $(S(t), I(t), R(t))$. The initial condition is given by

$$x(0) = x_0,\ y(0) = y_0,\ z(0) = z_0.$$

The complete model that describes a system of caputo fractional differential equation is presented as follows:

$$\begin{cases} \frac{{}^c d^\alpha x(t)}{dt^\alpha} = -\alpha y(t)x(t) \\ \frac{{}^c d^\alpha y(t)}{dt^\alpha} = \alpha y(t)x(t) - \beta y(t) \\ \frac{{}^c d^\alpha z(t)}{dt^\alpha} = \beta y(t) \end{cases} \tag{3.2}$$

Where $0 < \alpha \leq 1$.

3.1 Main results

We will apply some basic theorems like Banach contraction and Schauder theorems to receives our required result as

$$\begin{cases} \frac{{}^{c}d^{\alpha}x(t)}{dt^{\alpha}} = \Phi_1(x(t), y(t), z(t), t) \\ \frac{{}^{c}d^{\alpha}y(t)}{dt^{\alpha}} = \Phi_2(x(t), y(t), z(t), t) \\ \frac{{}^{c}d^{\alpha}z(t)}{dt^{\alpha}} = \Phi_3(x(t), y(t), z(t), t) \end{cases} \tag{3.3}$$

Upon integration for $0 < \alpha \leq 1$ to equation (3.3), we get the given system as:

$$\begin{cases} x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Phi_1(x(t), y(t), z(t), t) ds \\ y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Phi_2(x(t), y(t), z(t), t) ds \\ z(t) = z_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Phi_3(x(t), y(t), z(t), t) ds \end{cases} \tag{3.4}$$

Now we will take $0 < t < T < \infty$ and define Banach space as $E = C(\mathbb{R}^3 \times [0, T], \mathbb{R}_+)$, then $E = E_1 \times E_2 \times E_3$ will also be the Banach space having the norm $\|(x, y, z)\| = \max_{t \in [0, T]} |x(t)| + \max_{t \in [0, T]} |y(t)| + \max_{t \in [0, T]} |z(t)|$. Expressing the system (3.4) as

$$W(t) = W_0(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha-1} \phi(W(\zeta), \zeta) d\zeta,$$

where

$$W(t) = \begin{cases} x(t) \\ y(t) \\ z(t) \end{cases}, W_0(t) = \begin{cases} x_0(t) \\ y_0(t) \\ z_0(t) \end{cases},$$

and

$$\phi(W(t), t) = \begin{cases} \Phi_1(x(t), y(t), z(t), t) \\ \Phi_2(x(t), y(t), z(t), t) \\ \Phi_3(x(t), y(t), z(t), t) \end{cases}.$$

We take the conditions of growth non-linear vector operator $\phi : \mathbb{R}^3 \times [0,\, T] \longrightarrow \mathbb{R}_+$ as:

$(A1)$ There is a constant $L_\phi > 0$; $\forall (W_1(t),\, W_2(t)) \in \mathbb{R} \times \mathbb{R}$;

$$|\phi(W_1(t), t) - \phi(W_2(t), t)| \leq L_\phi \, |W_1(t) - W_2(t)| \, .$$

$(A2)$ there is a constants $C_\phi > 0, M_\phi > 0$;

$$|\phi(W(t), t)| \leq C_\phi \, |W| + M_\phi.$$

Definition 3.1.1. *(See p. 118 of [18])*

Let X, Y be topological spaces. A map $f : X \to Y$ is called compact if $f(X)$ is contained in a compact subset of Y .

3.2 Existence and uniqueness of solution

Theorem 3.1. *Under the continuity of ϕ together with assumption $(A2)$, system (3.3) has at least one solution.*

Proof 3. *With the help of "Schauder fixed point theorem", we will prove that system (3.3) has solution. Let take a function $A : E \longrightarrow E$ is define by:*

$$A(W(t)) = W_0(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha-1} \phi(W(\zeta), \zeta) d\zeta.$$

The proof is given in several steps.

Step1: A is continuous. Let W_n be a sequence such that $W_n \longrightarrow W$ in E.

Then for each $0 < t < T$, suppose that $B(t) = A(W_n(t)) - A(W(t))$ hence

$$|B(t)| \;\; \leq \;\; \frac{1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha-1} \, |\phi(W_n(\zeta), \zeta) - \phi(W(\zeta), \zeta)| \, d\zeta,$$

$$
\begin{aligned}
|B(t)| &\leq \frac{\|\phi(W_n(.),.)-\phi(W(.),.)\|}{\Gamma(\alpha)}\int_0^t (t-\zeta)^{\alpha-1}d\zeta, \\
&\leq \frac{T^\alpha\|\phi(W_n(.),.)-\phi(W(.),.)\|}{\Gamma(\alpha+1)}.
\end{aligned}
$$

Since ϕ is a continuous function, we have

$|A(W_n(t))-A(W(t))| \leq \frac{T^\alpha}{\Gamma(\alpha+1)}\|\phi(W_n(\cdot),\cdot)-\phi(W(\cdot),\cdot)\| \to 0$ *as* $n\to\infty$.

For the second of the proof we have to show that the set $A(E)$ is relatively compact. Let $V = A(W) \in A(E)$. Therefore, $\|A(W)\| \leq R$. By hypothesis, at any $W \in E$, follows

$$
|A(W(t))| \leq |W_0| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\zeta)^{\alpha-1}|\phi(W(\zeta),\zeta)|\,d\zeta,
$$

$$
\begin{aligned}
|A(W(t))| &\leq |W_0| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\zeta)^{\alpha-1}[C_\phi|W| + M_\phi]d\zeta, \\
&\leq |W_0| + \frac{T^\alpha}{\Gamma(\alpha+1)}[C_\phi\|W\| + M_\phi].
\end{aligned}
$$

wich implies that

$$
\|A(W)\| \leq |W_0| + \frac{T^\alpha}{\Gamma(\alpha+1)}[C_\phi\|W\| + M_\phi] \leq R,
$$

Thus $A(E) \subset E$. From this we say that operator A is closed and bounded. Next we go ahead to prove the result for completely continuous operator as:

Let $t_2 > t_1$ lies in $[0,T]$, suppose that $B = A(W)(t_2) - A(W)(t_1)$ and $K = C_\phi R + M_\phi$ Therefore

$$
|B| \leq \frac{1}{\Gamma(\alpha)}\left[\int_0^{t_2}(t_2-\zeta)^{\alpha-1}d\zeta - \int_0^{t_1}(t_1-\zeta)^{\alpha-1}d\zeta\right]K, \tag{3.5}
$$

$$
|A(W)(t_2) - A(W)(t_1)| \leq \frac{(C_\phi R + M_\phi)}{\Gamma(\alpha+1)}[t_2^\alpha - t_1^\alpha]. \tag{3.6}
$$

Now from (3.6), on can observe that as t_1 approaches to t_2, then right side also vanishes. So one concludes that $|A(W)(t_2) - A(W)(t_1)|$ tend to 0, as t_1 tends to t_2.

So $A(E)$ is equicontinuous. By using "Arzelà-Ascoli theorem", the operator A is completely con-

tinuous operator and also uniformly bounded proved already. By "Schauder's fixed point theorem" system has one or more than one solution.

Further we proceeds for uniqueness as:

Theorem 3.2. *Using* $(A1)$*, system has unique or one solution if* $\frac{T^\alpha}{\Gamma(\alpha+1)} L_\phi < 1$.

Proof 4. *Take* $A: E \longrightarrow E$*, consider* W *and* $\overline{W}$ *in* E *suppose that* $\overline{B} = A(W) - A(\overline{W})$ *as*

$$\begin{aligned} \left\|\overline{B}\right\| &= \max_{t\in[0,T]} \left|A(W)(t) - A(\overline{W}(t))\right| \\ &\leq \frac{T^\alpha}{\Gamma(\alpha+1)} L_\phi \left\|W - \overline{W}\right\|. \end{aligned} \tag{3.7}$$

From (3.7), follows

$$\left\|A(W) - A(\overline{W})\right\| \leq \frac{T^\alpha}{\Gamma(\alpha+1)} L_\phi \left\|W - \overline{W}\right\|.$$

Hence A is contarction. By "Banach contraction theorem" system has one solution. □

Chapter 4

Uniform Stability of Dynamic SICA HIV Transmission Models

In this chapter, we study the model SICA HIV Transmission Models on time scales, providing a Lyapunov function definition. The study aims to demonstrate the system's permanence, existence of solutions, and establish sufficient conditions guaranteeing a unique almost periodic solution that is uniformly asymptotically stable.

On a given time scales $\mathbb{T}$, we propose the following SICA model :

$$\begin{cases} x_1^{\Delta}(t) = \Lambda - \beta\lambda(t)x_1^{\sigma}(t) - \nu x_1^{\sigma}(t), \\ x_2^{\Delta}(t) = \beta\lambda(t)x_1(t) - (\rho + \phi + \nu)x_2^{\sigma}(t) + \gamma x_4(t) + \omega x_3(t), \\ x_3^{\Delta}(t) = \phi x_2(t) - (\omega + \nu)x_3^{\sigma}(t), \\ x_4^{\Delta}(t) = \rho x_2(t) - (\gamma + \nu + d)x_4^{\sigma}(t), \end{cases} \tag{4.1}$$

where $t \in \mathbb{T}^+$, with $\mathbb{T}^+$ a nonempty closed subset of $\mathbb{R}^+ =]0, +\infty[$.

4.1 Definitions

Let f be a function defined on $\mathbb{T}^+$. We set

$$f^L = \inf\left\{f(t) : t \in \mathbb{T}^+\right\} \quad \text{and} \quad f^U = \sup\left\{f(t) :\ t \in \mathbb{T}^+\right\}.$$

Lemma 4.1 (See [23]). *Assume that $a > 0$, $b > 0$, and that $-a \in \mathcal{R}^+$. Then,*

$$y^{\Delta}(t) \geq (\leq)\ b - ay^{\sigma}(t),\ y(t) > 0,\ \ t \in [t_0, \infty)_{\mathbb{T}}$$

implies that

$$y(t) \geq (\leq)\ \frac{b}{a}\left[1 + \left(\frac{ay(t_0)}{b} - 1\right) e_{(\ominus a)}(t, t_0)\right],\ t \in [t_0, \infty)_{\mathbb{T}}.$$

Definition 4.1.1 (See [32]). *A time scale $\mathbb{T}$ is called an almost periodic time scales if*

$$\Pi = \{\tau \in \mathbb{R} :\ t + \tau \in \mathbb{T}\ \textit{for all}\ \ t \in \mathbb{T}\} \neq \{0\}\,.$$

Definition 4.1.2 (See [32]). *Let $\mathbb{T}$ be an almost periodic time scales. A function $x \in C(\mathbb{T}, \mathbb{R}^n)$ is called an almost periodic function if the ε-translation set of x,*

$$E\{\varepsilon, x\} = \{\tau \in \Pi :\mid x(t+\tau) - x(t) \mid < \varepsilon\ \textit{for all}\ t \in \mathbb{T}\}\,,$$

is a relatively dense set in $\mathbb{T}$, that is, for all $\varepsilon > 0$ there exists a constant $l(\varepsilon) > 0$ such that each interval of length $l(\varepsilon)$ contains a $\eta(\varepsilon) \in E\{\varepsilon, x\}$ such that $|x(t+\tau) - x(t)| < \varepsilon$ for all $t \in \mathbb{T}$. Moreover, τ is called the ε-translation number of $x(t)$ and $l(\varepsilon)$ is called the inclusion length of $E\{\varepsilon, x\}$.

Definition 4.1.3 (See [32]). *Let $\mathbb{D}$ be an open set in $\mathbb{R}^n$ and let $\mathbb{T}$ be a positive almost periodic time scales. A function $f \in C(\mathbb{T} \times \mathbb{D}, \mathbb{R}^n)$ is called an almost periodic function in $t \in \mathbb{T}$, uniformly for $x \in \mathbb{D}$, if the ε-translation set of f,*

$$E\{\varepsilon, f, \mathbb{S}\} = \{\tau \in \Pi :\mid f(t+\tau, x) - f(t, x) \mid < \varepsilon\ \ \textit{for all}\ \ (t, x) \in \mathbb{T} \times \mathbb{S}\}\,,$$

is a relatively dense set in $\mathbb{T}$ for all $\varepsilon > 0$ and for each compact subset $\mathbb{S}$ of $\mathbb{D}$ there exists a constant $l(\varepsilon, \mathbb{S}) > 0$ such that each interval of length $l(\varepsilon, \mathbb{S})$ contains $\tau(\varepsilon, \mathbb{S}) \in E\{\varepsilon, f, \mathbb{S}\}$ such that

$$\mid f(t+\tau, x) - f(t, x) \mid < \varepsilon\ \ \textit{for all}\ \ (t, x) \in \mathbb{T} \times \mathbb{S}.$$

Consider the system

$$x^{\Delta}(t) = h(t, x) \tag{4.2}$$

where $h : \mathbb{T}^{+} \times \mathbb{S}_B \longrightarrow \mathbb{R}^n$, $\mathbb{S}_B = \{x \in \mathbb{R}^n : \|x\| < B\}$ and $h(t, x)$ is almost periodic in t uniformly for $x \in \mathbb{S}_B$ and continuous in x.

Lemma 4.2 (See [51]). *Suppose that there exists a Lyapunov function $V(t, x, z)$ defined on $\mathbb{T}^{+} \times \mathbb{S}_B \times \mathbb{S}_B$, that is, there exists a function $V(t, x, z)$ satisfying the following conditions:*

1. $a(\|x - z\|) \leq V(t, x, z) \leq b(\|x - z\|)$, *where* $a, b \in \mathbb{K}$ *with*

$$\mathbb{K} = \Big\{\alpha \in C(\mathbb{R}^{+}, \mathbb{R}^{+}) : \alpha(0) = 0, \text{ and } \alpha \text{ increasing } \Big\};$$

2. $|V(t, x, z) - V(t, x_1, z_1)| \leq L(\|x - x_1\| + \|z - z_1\|)$, *where $L > 0$ is a constant;*

3. $D^{+}V^{\Delta}(t, x, z) \leq -cV(t, x, z)$, *where $c > 0$ and $-c \in \mathcal{R}^{+}$.*

Furthermore, if there exists a solution $x(t) \in \mathbb{S}$ of system (4.2) for $t \in \mathbb{T}^{+}$, where $\mathbb{S} \cup \mathbb{S}_B$ is a compact set, then there exist a unique almost periodic solution $f(t) \in \mathbb{S}$ of system (4.2), which is uniformly asymptotically stable.

Definition 4.1.4 (See [37]). *System (4.1) is said to be permanent if there exist positive constants m and M such that*

$$m \leq \liminf_{t \longrightarrow \infty} x_i(t) \leq \limsup_{t \longrightarrow \infty} x_i(t) \leq M, \quad i = 1, 2, 3, 4,$$

for any solution $(x_1(t), x_2(t), x_3(t), x_4(t))$ of (4.1).

4.2 Permanence of positives solutions

The principal objective of this section is to establish sufficient conditions for system (4.1) to be permanent. Let $t_0 \in \mathbb{T}$ be a fixed positive initial time. We introduce the following assumption for (4.1):

(H_1) $\lambda(t)$ is a bounded and almost periodic function satisfying

$$0 < \lambda^L \leq \lambda(t) \leq \lambda^U.$$

Lemma 4.3. *Suppose hypothesis (H_1) holds. Then, for any positive solution $(x_1(t), x_2(t), x_3(t), x_4(t))$ of system (4.1), there exists positive constants M and T such that $x_i(t) \leq M$, $i = 1, 2, 3, 4$, for $t \geq T$.*

Proof 5. *Let $Z(t) = (x_1(t), x_2(t), x_3(t), x_4(t))$ be any positive solution of system (4.1). From the ith equation of system (4.1), we have*

$$\begin{cases} x_1^{\Delta}(t) \leq \Lambda - (\beta\lambda^L + \nu)x_1^{\sigma}(t), \\ x_2^{\Delta}(t) \leq \beta\lambda^U M_1 + (\gamma + \omega)\dfrac{\Lambda}{\nu} - (\rho + \phi + \nu)x_2^{\sigma}(t), \\ x_3^{\Delta}(t) \leq \phi M_2 - (\omega + \nu)x_3^{\sigma}(t), \\ x_4^{\Delta}(t) \leq \rho M_2 - (\gamma + \nu + d)x_4^{\sigma}(t). \end{cases} \tag{4.3}$$

Hence, by Lemma 4.1, there exist positive constants M_i and T_i such that for any positive solution $(x_1(t), x_2(t), x_3(t), x_4(t))$ of system (4.1), we have

$$x_1(t) \leq \frac{\Lambda}{\beta\lambda^L + \nu}\left[1 + \left(\frac{(\beta\lambda^L + \nu)x_1(t_0)}{\Lambda} - 1\right)e_{\ominus(\beta\lambda^L+\nu)}(t, t_0)\right].$$

If $-(\beta\lambda^L + \nu) < 0$, then $e_{\ominus(\beta\lambda^L+\nu)}(t, t_0) \longrightarrow 0$ as $t \longrightarrow \infty$ and

$$\begin{cases} x_1(t) \leq M_1 := \Lambda/(\beta\lambda^L + \nu), & \text{for } t \geq T_1, \\ x_2(t) \leq M_2 := \beta(\lambda^U M_1 + (\gamma + \omega)\dfrac{\Lambda}{\nu})/(\rho + \phi + \nu), & \text{for } t \geq T_2, \\ x_3(t) \leq M_3 := \phi M_2/(\omega + \nu), & \text{for } t \geq T_3, \\ x_4(t) \leq M_4 := \rho M_2/(\gamma + \nu + d), & \text{for } t \geq T_4. \end{cases} \tag{4.4}$$

Let $M = \max_{1\leq i\leq 4}\{M_i\}$ and $T = \max_{1\leq i\leq 4}\{T_i\}$. Then, $x_i(t) \leq M$, $i = 1, 2, 3, 4$, for all $t \geq T$.

Lemma 4.4. *Suppose that (H_1) holds. Then, system (4.1) is permanent.*

Proof 6. *Let $Z(t) = (x_1(t), x_2(t), x_3(t), x_4(t))$ be any positive solution of system (4.1). From the* *ith equation of system (4.1), we have*

$$\begin{cases} x_1^{\Delta}(t) \geq \Lambda - (\beta\lambda^U + \nu)x_1^{\sigma}(t), \\ x_2^{\Delta}(t) \geq \beta\lambda^L m_1 - (\rho + \phi + \nu)x_2^{\sigma}(t), \\ x_3^{\Delta}(t) \geq \phi m_2 - (\omega + \nu)x_3^{\sigma}(t), \\ x_4^{\Delta}(t) \geq \rho m_2 - (\gamma + \nu + d)x_4^{\sigma}(t). \end{cases} \tag{4.5}$$

From hypothesis (H_1) *and Lemma 4.1, there exists positive constants* $m_i > 0$ *such that for any positive solution* $(x_1(t), x_2(t), x_3(t), x_4(t))$ *of system (4.1) there exists* $\hat{T}_i$ *such that*

$$\begin{cases} x_1(t) \geq m_1 := \Lambda/(\beta\lambda^U + \nu), & \text{for } t \geq \hat{T}_1, \\ x_2(t) \geq m_2 := (\beta\lambda^L m_1)/(\rho + \phi + \nu), & \text{for } t \geq \hat{T}_2, \\ x_3(t) \geq m_3 := \phi m_2/(\omega + \nu), & \text{for } t \geq \hat{T}_3, \\ x_4(t) \geq m_4 := \rho m_2/(\gamma + \nu + d), & \text{for } t \geq \hat{T}_4. \end{cases} \tag{4.6}$$

Let $m = \min_{1\leq i\leq 4}\{m_i\}$ *and* $\hat{T} = \max_{1\leq i\leq 4}\left\{\hat{T}_i\right\}$. *We conclude that* $x_i(t) \geq m$, $i = 1, 2, 3, 4$, *for all* $t \geq \hat{T}$.

4.3 Uniform asymptotic stability

In this section, we prove sufficient conditions for the existence and uniform asymptotic stability of the unique positive almost periodic solution to system (4.1). Let us define

$$\Omega := \Big\{ Z(t) = (x_1(t), x_2(t), x_3(t), x_4(t)) \in (\mathbb{R}^+)^4 : (x_1(t), x_2(t), x_3(t), x_4(t)) \text{ is a solution of (4.1) with } 0 < m \leq x_i \leq M, i = 1, \ldots 4, \text{ and } N(t) \leq \frac{\Lambda}{\nu} \Big\}.$$

It is clear that Ω is an invariant set of system (4.1) and, by Lemma 4.4, we have $\Omega \neq \emptyset$. We introduce some more notation. Let

$$\begin{aligned}
a_1 &:= \beta\lambda^L + \nu, \\
a_2 &:= \rho + \phi + \nu, \\
a_3 &:= \omega + \nu, \\
a_4 &:= \gamma + \nu + d, \\
b_1 &:= \left(\beta\lambda^U + \frac{\beta^2 M}{2m}\right), \\
b_2 &:= (\rho + \phi), \\
b_3 &:= \left(\omega + \frac{\beta^2 M}{2m}\eta_C\right), \\
b_4 &:= \left(\gamma + \frac{\beta^2 M}{2m}\eta_A\right).
\end{aligned}$$

Moreover, let $\Gamma_1 := \min_{1\leq i\leq 4} a_i$ and $\Gamma_2 := \max_{1\leq i\leq 4} b_i$. In our next result (Theorem 4.1) we assume the following additional hypothesis:

(H_2) $\Gamma_2 < \Gamma_1$ with $\Gamma_1, \Gamma_2 \in \mathcal{R}^+$.

Theorem 4.1. *Suppose that (H_1) and (H_2) hold. Then the dynamic system (4.1) has a unique almost periodic solution $Z(t) = (x_1(t), x_2(t), x_3(t), x_4(t)) \in \Omega$ that is uniformly asymptotically stable.*

Proof 7. *According to Lemma 4.1, every solution*

$$Z(t) = (x_1(t), x_2(t), x_3(t), x_4(t))$$

of system (4.1) satisfies $x_i^L \leq x_i(t) \leq x_i^U, i = 1, \dots, 4,$ and $|x_i| \leq K_i, i = 1, \dots, 4.$ Denote

$$\begin{aligned}
\parallel Z \parallel &= \parallel (x_1(t), x_2(t), x_3(t), x_4(t)) \parallel \\
&= \sup_{t\in\mathbb{T}^+} \left(| x_1(t) | + | x_2(t) | + | x_3(t) | + | x_4(t) |\right).
\end{aligned}$$

Let $Z(t) = (x_1(t), x_2(t), x_3(t), x_4(t))$ and $\hat{Z}(t) = (\hat{x}_1(t), \hat{x}_2(t), \hat{x}_3(t), \hat{x}_4(t))$ be two positive solutions *of (4.1). Then* $\| Z \| \leq K$ *and* $\| \hat{Z} \| \leq K$*, where*

$$K = \sum_{i=1}^{i=4} K_i.$$

In view of (4.1), we have

$$\begin{cases} x_1^{\Delta}(t) = \Lambda - \beta\lambda(t)x_1^{\sigma}(t) - \nu x_1^{\sigma}(t), \\ x_2^{\Delta}(t) = \beta\lambda(t)x_1(t) - (\rho + \phi + \nu)x_2^{\sigma}(t) + \gamma x_4(t) + \omega x_3(t), \\ x_3^{\Delta}(t) = \phi x_2(t) - (\omega + \nu)x_3^{\sigma}(t), \\ x_4^{\Delta}(t) = \rho x_2(t) - (\gamma + \nu + d)x_4^{\sigma}(t), \end{cases}$$

and

$$\begin{cases} \hat{x}_1^{\Delta}(t) = \Lambda - \beta\hat{\lambda}(t)\hat{x}_1^{\sigma}(t) - \nu \hat{x}_1^{\sigma}(t), \\ \hat{x}_2^{\Delta}(t) = \beta\hat{\lambda}(t)\hat{x}_1(t) - (\rho + \phi + \nu)\hat{x}_2^{\sigma}(t) + \gamma \hat{x}_4(t) + \omega \hat{x}_3(t), \\ \hat{x}_3^{\Delta}(t) = \phi \hat{x}_2(t) - (\omega + \nu)\hat{x}_3^{\sigma}(t), \\ \hat{x}_4^{\Delta}(t) = \rho \hat{x}_2(t) - (\gamma + \nu + d)\hat{x}_4^{\sigma}(t). \end{cases}$$

Define the Lyapunov function $V(t, Z, \hat{Z})$ *on* $\mathbb{T}^+ \times \Omega \times \Omega$ *as*

$$V(t, Z, \hat{Z}) = \sum_{i=1}^{i=4} | x_i(t) - \hat{x}_i(t) | = \sum_{i=1}^{i=4} V_i(t),$$

where $V_i(t) = | x_i(t) - \hat{x}_i(t) |$*. The two norms*

$$\| Z(t) - \hat{Z}(t) \| = \sup_{t \in \mathbb{T}^+} \sum_{i=1}^{i=4} | x_i(t) - \hat{x}_i(t) |$$

and

$$\| Z(t) - \hat{Z}(t) \|_* = \sup_{t \in \mathbb{T}^+} \left(\sum_{i=1}^{i=4} (x_i(t) - \hat{x}_i(t))^2 \right)^{\frac{1}{2}}$$

are equivalent, that is, there exist two constants η_1 *and* $\eta_2 > 0$ *such that*

$$\eta_1 \| Z(t) - \hat{Z}(t) \|_* \leq \| Z(t) - \hat{Z}(t) \| \leq \eta_2 \| Z(t) - \hat{Z}(t) \|_* .$$

Hence,

$$\eta_1 \parallel Z(t) - \hat{Z}(t) \parallel_* \leq V(t, Z, \hat{Z}) \leq \eta_2 \parallel Z(t) - \hat{Z}(t) \parallel_* .$$

Let $a, b \in C(\mathbb{R}^+, \mathbb{R}^+)$, $a(x) = \eta_1 x$ *and* $b(x) = \eta_2 x$. *Then the assumption (i) of Lemma 4.2 is satisfied. Moreover,*

$$\begin{aligned}
\mid V(t, Z, \hat{Z}) - V(t, Z^*, \hat{Z}^*) \mid &= \left| \sum_{i=1}^{i=4} \mid x_i(t) - \hat{x}_i(t) \mid - \sum_{i=1}^{i=4} \mid x_i^*(t) - \hat{x_i^*}(t) \mid \right| \\
&\leq \sum_{i=1}^{i=4} \mid (x_i(t) - \hat{x}_i(t)) - \left(x_i^*(t) - \hat{x_i^*}(t) \right) \mid \\
&\leq \sum_{i=1}^{i=4} \mid (x_i(t) - x_i^*(t)) + \left(\hat{x^*}_i(t) - \hat{x}_i(t) \right) \mid \\
&\leq \sum_{i=1}^{i=4} \mid (x_i(t) - x_i^*(t)) \mid + \sum_{i=1}^{i=4} \mid \left(\hat{x^*}_i(t) - \hat{x}_i(t) \right) \mid \\
&\leq L \left(\parallel Z(t) - Z^*(t) \parallel + \parallel \hat{Z}(t) - \hat{Z}^*(t) \parallel \right),
\end{aligned}$$

where $L = 1$, *so that condition (ii) of Lemma 4.2 is also satisfied. Now, let* $v_i(t) = x_i(t) - \hat{x}_i(t)$, $i = 1, \ldots, 4$. *We compute and estimate the Dini derivative* D^+V^Δ *of* V *along the associated product system (4.1). Using Lemma 4.1 of [33], it follows that* $D^+V_1^\Delta(t) \leq sign(v_1^\sigma(t))(v_1(t))^\Delta$. *For more details on* $D^+V_1^\Delta(t)$ *see [22, 26]. Now, let us begin computing*

$$\begin{aligned}
D^+V_1^\Delta(t) &\leq sign(v_1^\sigma(t))(v_1(t))^\Delta \\
&= sign(v_1^\sigma(t))[-\beta\lambda(t)x_1^\sigma(t) - \nu x_1^\sigma(t) + \beta\hat{\lambda}(t)\hat{x}_1^\sigma(t) + \nu\hat{x}_1^\sigma(t)] \\
&= sign(v_1^\sigma(t))[-\beta\lambda(t)(x_1^\sigma(t) - \hat{x}_1^\sigma(t)) - \nu(x_1^\sigma(t) - \hat{x}_1^\sigma(t))] \\
&\quad -\beta\hat{x}_1^\sigma(t)(\lambda(t) - \hat{\lambda}(t))] \\
&= sign(v_1^\sigma(t))[-(\beta\lambda(t) + \nu)(x_1^\sigma(t) - \hat{x}_1^\sigma(t)) \\
&\quad -\beta\hat{x}_1^\sigma(t)(\lambda(t) - \hat{\lambda}(t))] \\
&\leq -(\beta\lambda(t) + \nu) \mid x_1^\sigma(t) - \hat{x}_1^\sigma(t)) \mid + \beta\hat{x}_1^\sigma(t) \mid \lambda(t) - \hat{\lambda}(t) \mid \\
&\leq -(\beta\lambda^L(t) + \nu) \mid x_1^\sigma(t) - \hat{x}_1^\sigma(t)) \mid + \beta\hat{x}_1^\sigma(t) \mid \lambda(t) - \hat{\lambda}(t) \mid \\
&\leq -(\beta\lambda^L(t) + \nu) \mid x_1^\sigma(t) - \hat{x}_1^\sigma(t)) \mid + \beta M \mid \lambda(t) - \hat{\lambda}(t) \mid,
\end{aligned}$$

that is,

$$D^+V_1^\Delta(t) \;\; \leq \;\; -(\beta\lambda^L(t)+\nu) \mid \hat{x}_1^\sigma(t) - \hat{x}_1^\sigma(t) \mid +(\beta M) \mid \lambda(t) - \hat{\lambda}(t) \mid .$$

Let $v_2(t) = x_2(t) - \hat{x_2}(t)$. *Similarly, we have*

$$\begin{aligned}
D^+V_2^\Delta(t) &\leq sign(v_2^\sigma(t))(v_2(t))^\Delta \\
&= sign(v_2^\sigma(t))[\beta\lambda(t)x_1(t) - (\rho+\phi+\nu)x_2^\sigma(t) + \gamma x_4(t) \\
&+ \omega x_3(t) - \beta\hat{\lambda}(t)\hat{x}_1(t) + (\rho+\phi+\nu)\hat{x}_2^\sigma(t) - \gamma\hat{x}_4(t) - \omega\hat{x}_3(t)] \\
&= sign(v_2^\sigma(t))[(\beta\lambda(t)x_1(t) - \beta\hat{\lambda}(t)\hat{x}_1(t)) - (\rho+\phi+\nu)(x_2^\sigma(t) - \hat{x}_2^\sigma(t)) \\
&+ \gamma(x_4(t) - \hat{x}_4(t)) + \omega(x_3(t) - \hat{x}_3(t))] \\
&= sign(v_2^\sigma(t))[\beta(\lambda(t)(x_1(t) - \hat{x}_1(t)) + \hat{x}_1(t)(\lambda(t) - \hat{\lambda}(t))) \\
&- (\rho+\phi+\nu)(x_2^\sigma(t) - \hat{x}_2^\sigma(t)) + \gamma(x_4(t) - \hat{x}_4(t)) + \omega(x_3(t) - \hat{x}_3(t))] \\
&= sign(v_2^\sigma(t))[\beta\lambda(t)(x_1(t) - \hat{x}_1(t)) + \beta\hat{x}_1(t)(\lambda(t) - \hat{\lambda}(t)) \\
&- (\rho+\phi+\nu)(x_2^\sigma(t) - \hat{x}_2^\sigma(t)) + \gamma(x_4(t) - \hat{x}_4(t)) + \omega(x_3(t) - \hat{x}_3(t))] \\
&\leq \beta\lambda^U \mid x_1(t) - \hat{x}_1(t) \mid + \beta M \mid \lambda(t) - \hat{\lambda}(t) \mid \\
&- (\rho+\phi+\nu) \mid x_2^\sigma(t) - \hat{x}_2^\sigma(t) \mid + \gamma \mid x_4(t) - \hat{x}_4(t) \mid \\
&+ \omega \mid x_3(t) - \hat{x}_3(t) \mid;
\end{aligned}$$

for $v_3(t) = x_3(t) - \hat{x_3}(t)$ *one has*

$$\begin{aligned}
D^+V_3^\Delta(t) &\leq sign(v_3^\sigma(t))(v_3(t))^\Delta \\
&= sign(v_3^\sigma(t))[\phi x_2(t) - (\omega+\nu)x_3^\sigma(t) - \phi\hat{x}_2(t) + (\omega+\nu)\hat{x}_3^\sigma(t)] \\
&\leq \phi \mid x_2(t) - \hat{x}_2(t) \mid -(\omega+\nu) \mid x_3^\sigma(t) - \hat{x}_3^\sigma(t) \mid;
\end{aligned}$$

and for $v_4(t) = x_4(t) - \hat{x_4}(t)$ *we have*

$$D^+V_4^\Delta(t) \;\; \leq \;\; \rho \mid x_2(t) - \hat{x}_2(t) \mid -(\gamma+\nu+d) \mid x_4^\sigma(t) - \hat{x}_4^\sigma(t) \mid .$$

Since

$$\lambda(t) = \frac{\beta}{N(t)}(x_1(t) + \eta_C x_3(t) + \eta_A)x_4(t),$$

it follows that

$$\begin{aligned} D^+V^\Delta(t) \leq & -(\beta\lambda^L(t) + \nu) \mid x_1^\sigma(t) - \hat{x}_1^\sigma(t)) \mid +(\beta M) \mid \lambda(t) - \hat{\lambda}(t) \mid \\ & + \beta\lambda^U \mid x_1(t) - \hat{x}_1(t) \mid +\beta M \mid \lambda(t) - \hat{\lambda}(t) \mid \\ & - (\rho + \phi + \nu) \mid x_2^\sigma(t) - \hat{x}_2^\sigma(t) \mid +\gamma \mid x_4(t) - \hat{x}_4(t) \mid \\ & + \omega \mid x_3(t) - \hat{x}_3(t) \mid +\phi \mid x_2(t) - \hat{x}_2(t) \mid \\ & - (\omega + \nu) \mid x_3^\sigma(t) - \hat{x}_3^\sigma(t) \mid \\ & + \rho \mid x_2(t) - \hat{x}_2(t) \mid -(\gamma + \nu + d) \mid x_4^\sigma(t) - \hat{x}_4^\sigma(t) \mid . \end{aligned} \tag{4.7}$$

Therefore,

$$\begin{aligned} |\lambda(t) - \hat{\lambda}(t)| \leq & \frac{\beta}{4m} [\mid x_1(t) - \hat{x}_1(t) \mid +\eta_C \mid x_3(t) - \hat{x}_3(t) \mid \\ & +\eta_A \mid x_4(t) - \hat{x}_4(t) \mid] , \end{aligned}$$

where $0 \leq \eta_C \leq 1$, $\eta_A \geq 1$, *and* $\beta > 0$. *The inequality (4.7) becomes*

$$\begin{aligned} D^+V^\Delta(t) \leq & -(\beta\lambda^L + \nu) \mid x_1^\sigma(t) - \hat{x}_1^\sigma(t) \mid -(\rho + \phi + \nu) \mid x_2^\sigma(t) - \hat{x}_2^\sigma(t) \mid \\ & - (\omega + \nu) \mid x_3^\sigma(t) - \hat{x}_3^\sigma(t) \mid -(\gamma + \nu + d) \mid x_4^\sigma(t) - \hat{x}_4^\sigma(t) \mid \\ & + 2\beta M \mid \lambda(t) - \hat{\lambda}(t) \mid +(\beta\lambda^U) \mid x_1(t) - \hat{x}_1(t) \mid +\gamma \mid x_4(t) - \hat{x}_4(t) \mid \\ & + \omega \mid x_3(t) - \hat{x}_3(t) \mid +(\rho + \phi) \mid x_2(t) - \hat{x}_2(t) \mid \\ = & - \left(\beta\lambda^L + \nu\right) \mid x_1^\sigma(t) - \hat{x}_1^\sigma(t) \mid -(\rho + \phi + \nu) \mid x_2^\sigma(t) - \hat{x}_2^\sigma(t) \mid \\ & - (\omega + \nu) \mid x_3^\sigma(t) - \hat{x}_3^\sigma(t) \mid -(\gamma + \nu + d) \mid x_4^\sigma(t) - \hat{x}_4^\sigma(t) \mid \\ & + \beta \left(\lambda^U + \frac{\beta M}{2m}\right) \mid x_1(t) - \hat{x}_1(t) \mid +(\rho + \phi) \mid x_2(t) - \hat{x}_2(t) \mid \\ & + \left(\omega + \frac{\beta^2 M}{2m}\eta_C\right) \mid x_3(t) - \hat{x}_3(t) \mid \\ & + \left(\gamma + \frac{\beta^2 M}{2m}\eta_A\right) \mid x_4(t) - \hat{x}_4(t) \mid . \end{aligned}$$

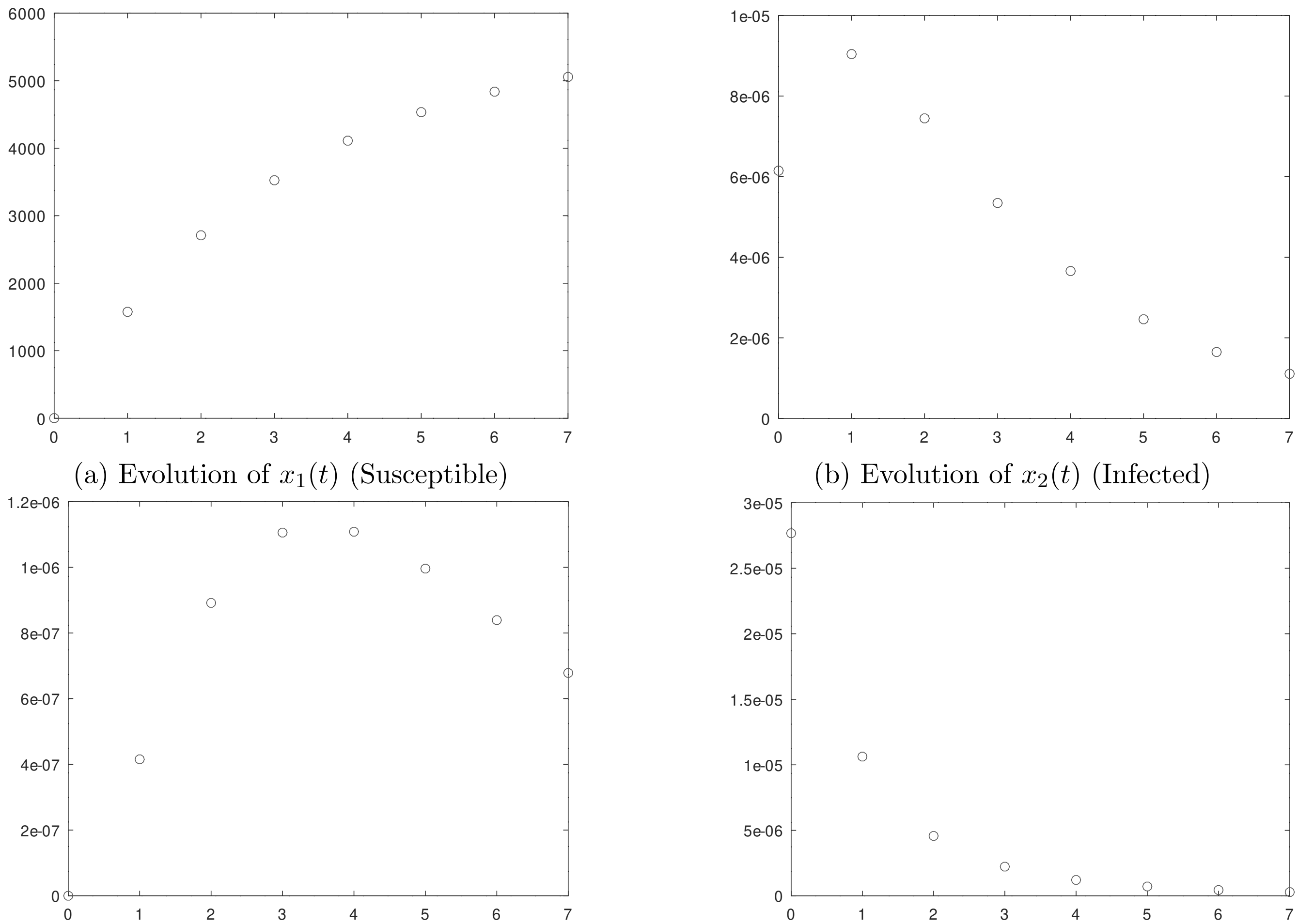
From the previous inequality we can write

$$\begin{aligned}
D^+V^\Delta(t) &\leq -(\beta\lambda^L+\nu) \mid x_1^\sigma(t)-\hat{x}_1^\sigma(t) \mid -(\rho+\phi+\nu) \mid x_2^\sigma(t)-\hat{x}_2^\sigma(t) \mid \\
&\quad -(\omega+\nu) \mid x_3^\sigma(t)-\hat{x}_3^\sigma(t) \mid -(\gamma+\nu+d) \mid x_4^\sigma(t)-\hat{x}_4^\sigma(t) \mid \\
&\quad +\left(\beta\lambda^U+\frac{\beta^2 M}{2m}\right) \mid x_1(t)-\hat{x}_1(t) \mid \\
&\quad +(\phi+\rho) \mid x_2(t)-\hat{x}_2(t) \mid \\
&\quad +\left(\omega+\frac{\beta^2 M}{2m}\eta_C\right) \mid x_3(t)-\hat{x}_3(t) \mid \\
&\quad +\left(\gamma+\frac{\beta^2 M}{2m}\eta_A\right) \mid x_4(t)-\hat{x}_4(t) \mid \\
&= -a_1 \mid x_1^\sigma(t)-\hat{x}_1^\sigma(t) \mid -a_2 \mid x_2^\sigma(t)-\hat{x}_2^\sigma(t) \mid \\
&\quad -a_3 \mid x_3^\sigma(t)-\hat{x}_3^\sigma(t) \mid -a_4 \mid x_4^\sigma(t)-\hat{x}_4^\sigma(t) \mid \\
&\quad +b_1 \mid x_1(t)-\hat{x}_1(t) \mid +b_2 \mid x_2(t)-\hat{x}_2(t) \mid \\
&\quad +b_3 \mid x_3(t)-\hat{x}_3(t)| + b_4 \mid x_4(t)-\hat{x}_4(t) \mid \\
&= -\Gamma_1 V(\sigma(t)) + \Gamma_2 V(t) \\
&= (\Gamma_2-\Gamma_1)\, V(t) - \Gamma_1\mu(t) D^+V^\Delta(t)
\end{aligned}$$

and it follows that $D^+V^\Delta(t) \leq \frac{\Gamma_2-\Gamma_1}{(1+\Gamma_1\mu)}V(t) = -\Psi V(t)$*. By hypothesis* (H_2)*, one has* $-\Psi \in \mathcal{R}^+$ *and* $\Psi = \frac{\Gamma_1-\Gamma_2}{(1+\Gamma_1\mu)} > 0$*. Thus, the assumption (iii) of Lemma 4.2 is satisfied. Furthermore, the conditions (i) and (ii) of Lemma 4.2 also hold. For condition (i) we consider two functions* $a, b \in C(\mathbb{R}^+)$ *with* $a(x) = \eta_1 x$ *and* $b(x) = \eta_2 x$*. For condition (ii) we put* $L = 1$*. So there exists a unique uniformly asymptotically stable almost periodic solution* $Z(t) = (x_1(t), x_2(t), x_3(t), x_4(t))$ *of the dynamic system (4.1) with* $Z(t) \in \Omega$*.* $\square$

Example 4.3.1. *Motivated by [34] and the case of Morocco, we consider system (4.1) for* $\mathbb{T} = \mathbb{Z}^+$*:*

$$\begin{cases}
\Delta x_1(t) = \Lambda - \beta\lambda(t)x_1(t+1) - \nu x_1(t+1), \\
\Delta x_2(t) = \beta\lambda(t)x_1(t) - (\rho+\phi+\nu)x_2(t+1) + \gamma x_4(t) + \omega x_3(t), \\
\Delta x_3(t) = \phi x_2(t) - (\omega+\nu)x_3(t+1), \\
\Delta x_4(t) = \rho x_2(t) - (\gamma+\nu+d)x_4(t+1),
\end{cases} \tag{4.8}$$



(a) Evolution of $x_1(t)$ (Susceptible)

(b) Evolution of $x_2(t)$ (Infected)

(c) Evolution of $x_3(t)$ (Chronic)

(d) Evolution of $x_4(t)$ (AIDS)

Figure 4.1: Example 4.3.1: solution of 4.8 during 7 years.

where

$$x_1(0) = 1 - \frac{11}{N_0}, \quad x_2(0) = \frac{2}{N_0}, \quad x_3(0) = 0, \quad x_4(0) = \frac{9}{N_0},$$

with $N_0 = 325235$, $\Lambda = 2190$, $\beta = 2.710^{-7}$, $\nu = 0.39$, $\rho = 0.2$, $\phi = 0.1$, $\gamma = 0.33$, $\omega = 0.09$, *and* $d = 1$. *System* 4.8 *is permanent. Furthermore,* $m_1 = 5615.381462$, $m_2 = 5.47870412010^{-9}$, $m_3 = 1.14139669210^{-9}$, $m_4 = 6.37058618610^{-10}$, $M_1 = 5615.384615$, $M_2 = 0.002773104793$, $M_3 = 0.0005777301652$, *and* $M_4 = 0.0003224540457$. *The conditions of Theorem* 4.1 *are verified with* $0.37 = \Gamma_2 < \Gamma_1 = 0.39$ *and* $\Psi = 0.01391941151$. *Therefore, system* 4.8 *has a unique positive almost periodic solution, which is uniformly asymptotic stable. In Figure* 4.1 *we plot the solution for the first 7 years with* $\eta_C = 0.5$ *and* $\eta_A = 1.5$.

4.4 Conclusion

We have investigated the uniform stability of the singular positive solution in an HIV/AIDS epidemic model, specifically the SICA model on an arbitrary time scale. The purpose of incorporating time scales is to integrate both continuous and discrete time models. We established the permanence of each solution and, using a suitable Lyapunov function, we derived a sufficient condition for uniform asymptotic stability of the solution. Additionally, we presented an illustrative example to substantiate our analytical results.

Chapter 5

Existence and uniqueness of solution for a fractional order SICA HIV Transmission models

On considere the model SICA suivant:

$$\begin{cases} \dfrac{dw(t)}{dt} = \Lambda - \beta\lambda(t)w(t) - \mu w(t) \\ \dfrac{dx(t)}{dt} = \beta\lambda(t)w(t) - (\rho + \phi + \mu)x(t) + \gamma z(t) + \omega y(t), \\ \dfrac{dy(t)}{dt} = \phi x(t) - (\omega + \mu)y(t), \\ \dfrac{dz(t)}{dt} = \rho x(t) - (\gamma + \mu + d)z(t). \end{cases}$$

$N(t) = w(t) + x(t) + y(t) + z(t)$ the total population at time t.

Let us consider the Caputo fractional order SICA model:

$$\begin{cases} {}_{t_0}^{C}D_t^{\alpha} w(t) = \Lambda - \beta(x(t) + \eta_y y(t) + \eta_z z(t))w(t) - \mu w(t), \\ {}_{t_0}^{C}D_t^{\alpha} x(t) = \beta(x(t) + \eta_y y(t) + \eta_z z(t))w(t) - \xi_3 x(t) + \omega y(t) + \gamma z(t), \\ {}_{t_0}^{C}D_t^{\alpha} y(t) = \phi x(t) - \xi_2 y(t), \\ {}_{t_0}^{C}D_t^{\alpha} z(t) = \rho x(t) - \xi_1 z(t). \end{cases}$$

5.1 Main results

We will apply some basic theorems like Banach contraction and Schauder theorems to receives our required result as

$$\begin{cases} \frac{{}^{c}d^{\alpha}w(t)}{dt^{\alpha}} = \omega_1(w(t), x(t), y(t), z(t), t) \\ \frac{{}^{c}d^{\alpha}x(t)}{dt^{\alpha}} = \omega_2(w(t), x(t), y(t), z(t), t) \\ \frac{{}^{c}d^{\alpha}y(t)}{dt^{\alpha}} = \omega_3(w(t), x(t), y(t), z(t), t) \\ \frac{{}^{c}d^{\alpha}z(t)}{dt^{\alpha}} = \omega_4(w(t), x(t), y(t), z(t), t). \end{cases} \tag{5.1}$$

Now we will take $0 < t < T < \infty$ and define Banach space as $E = C([0,T] \times \mathbb{R}^4, \mathbb{R}_+)$, then $E = E_1 \times E_2 \times E_3 \times E_4$ will also be the Banach space having the norm $\|(w,x,y,z)\| = \max_{t\in[0,T]} |w(t)| + \max_{t\in[0,T]} |x(t)| + \max_{t\in[0,T]} |y(t)| + \max_{t\in[0,T]} |z(t)|$.

$$U(t) = U_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha-1} \psi(U(\zeta), \zeta) d\zeta,$$

where

$$U(t) = \begin{cases} w(t) \\ x(t) \\ y(t) \\ z(t) \end{cases}, U_0(t) = \begin{cases} w_0 \\ x_0(t) \\ y_0(t) \\ z_0(t) \end{cases}, \psi(U(t), t) = \begin{cases} \omega_1(w(t), x(t), y(t), z(t), t) \\ \omega_2(w(t), x(t), y(t), z(t), t) \\ \omega_3(w(t), x(t), y(t), z(t), t) \\ \omega_4(w(t), x(t), y(t), z(t), t). \end{cases}$$

We take the conditions of growth non-linear vector operator $\psi : [0,T] \times \mathbb{R}^4 \longrightarrow \mathbb{R}_+$ as:

(A1) $\exists$ a constants $L_\psi > 0$; $\forall (U_1(t), U_2(t)) \in \mathbb{R} \times \mathbb{R}$;

$$|\psi(U_1(t), t) - \psi(U_2(t), t)| \leq L_\psi |U_1(t) - U_2(t)|$$

.

(A2) $\exists$ a constants $C_\psi > 0, M_\psi > 0$;

$$|\ \psi(U(t), t)\ | \leq C_\psi \ |\ U\ | + M_\psi$$

.

5.2 Existence and uniqueness of solution

Theorem 5.1. *Under the continuity of ψ together with assumption (A2), system(5.1) has at least one solution.*

Proof 8. *With the help of "Schauder fixed point theorem", we will derive our result. Let $D : E \longrightarrow E$ is define by:*

$$D(U(t)) = U_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha-1} \psi(U(\zeta), \zeta) d\zeta.$$

at any $U \in E$, follows

$$\begin{aligned} |D(U(t))| &\leq |U_0| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha-1} |\psi(U(\zeta), \zeta)| d\zeta \\ &\leq |U_0| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha-1} [C_\psi |U| + M_\psi] d\zeta \\ &\leq |U_0| + \frac{T^\alpha}{\Gamma(\alpha+1)} [C_\psi |U| + M_\psi], \end{aligned}$$

wich implies that

$$\begin{aligned} \|D(U)\| &\leq |U_0| + \frac{T^\alpha}{\Gamma(\alpha+1)} [C_\psi |U| + M_\psi] \\ &\leq \rho. \end{aligned}$$

From this we say that operator D is closed and bounded. Next we go ahead to prove the result for completely continuous operator as:
Let $t_2 > t_1$ lies in $[0, T]$, and take

$$\begin{aligned} |D(U)(t_2) - D(U)(t_1)| &= \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2-\zeta)^{\alpha-1} \psi(U(\zeta), \zeta) d\zeta - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-\zeta)^{\alpha-1} \psi(U(\zeta), \zeta) d\zeta \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \left[\int_0^{t_2} (t_2-\zeta)^{\alpha-1} d\zeta - \int_0^{t_1} (t_1-\zeta)^{\alpha-1} d\zeta \right] (C_\psi R + M_\psi), \end{aligned}$$

then

$$|D(U)(t_2) - D(U)(t_1)| \leq \frac{(C_\psi R + M_\psi)}{\Gamma(\alpha+1)}[t_2^\alpha - t_1^\alpha] \tag{5.2}$$

Now from (5.2), on can observe that as t_1 approaches to t_2, then right side also vanishes.So one concludes that $|D(U)(t_2) - D(U)(t_1)|$ tend to 0, as t_1 tends to t_2. So $D(E)$ is equicontinuous. By using "Arzelà-Ascoli theorem", the operator D is completely continuous operator and also uniformly bounded proved already. By "Schauder's fixed point theorem" system has one or more than one solution. □

Further we procceeds for uniqueness as

Theorem 5.2. *Using (A1), system (5.1) has unique or one solution if $\frac{T^\alpha}{\Gamma(\alpha+1)}L_\psi < 1$.*

Proof 9. *Take $D : E \longrightarrow E$, consider U and $\bar{U}$ in E as*

$$\begin{aligned} \left\|D(U) - D(\bar{U})\right\| &= \max_{t\in[0,T]}\left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-\zeta)^{\alpha-1}\psi(U(\zeta),\zeta)d\zeta - \frac{1}{\Gamma(\alpha)}\int_0^t (t-\zeta)^{\alpha-1}\psi(\bar{U}(\zeta),\zeta)d\zeta\right| \\ &\leq \frac{T^\alpha}{\Gamma(\alpha+1)}L_\psi\|U - \bar{U}\|. \end{aligned} \tag{5.3}$$

From (5.3), follows

$$\left\|D(U) - D(\bar{U})\right\| \leq \frac{T^\alpha}{\Gamma(\alpha+1)}L_\psi\|U - \bar{U}\|.$$

Hence D is contraction. By "Banach contraction theorem" system (5.1) has one solution. □

Perspective

Here, we provide a list of the main research perspectives that emerge at the end of this thesis:

- Study the stability of epidemic models on time scales.
- Comparison theorems of the stability of epidemic models on time scales.

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