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Fundamental of Operations Research

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Introduction

I'm delighted to share my experience of teaching the "Operations Research" module. This book is made up of a set of fundamental chapters that have been presented in the form of courses for several specialties in the field of economics. I have based my approach on simplicity and methodological presentation of the content, while staying as far away as possible from pure mathematical presentation, since this product is intended primarily for students of economics, business and management sciences.

Examples and exercises presented in all chapters are direct problems of economic application, this to better understand the use of models of operational research to the business enterprise itself.

This edition will be followed by other improvements including the introduction of new chapters concerning decision theory namely: the decision tree, ordering problem, project management etc.

Furthermore, the use of computer tools for solving problems of operational research is almost necessary to solving large-scale problems, which is why a new work will be prepared in this regard.

Let's address this work to our teaching colleagues and students. All that's left for me to do is ask readers to send me their comments or remarks on the content and methods used, so as to make it a certified reference.

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Summary

Chapter 1

1 Introduction to the operations research

1.1 Definition

There is no longer an exact definition of operations research, which is why researchers have defined it differently, depending on their research and application context. G.KIMBALL and P.MORSE define it as a scientific method providing executive management with a quantitative tool for decision-making. R.ACKOFF and G.CHORCHMAN have defined operations research as the use of scientific methods and models to solve problems related to operating systems, in order to provide decision-makers with optimal solutions, or practical decision theory, the use of scientific and mathematical methods that enable executives to solve problems, according to M.STARR and M.MILLER. Also, H.WANGER defined operational research as the use of scientific principles to solve executive managers' problems.

These definitions are common to a number of keywords, namely: scientific method, models and tools, which in fact constitute mathematical and statistical models, managers' problems, decisions, optimal solutions. These are essentially the result of the understanding and use of this discipline by researchers in different fields.

Overall, operational research can be defined as the set of rational methods and techniques for analyzing and synthesizing organizational phenomena, which can be used to develop better decisions.

1.2 History

As early as the 17th century, mathematicians such as Blaise Pascal tried to solve decision problems under uncertainty using mathematical expectation. Others, in the 18th and 19th centuries, solved combinatorial problems.

But it wasn't until the Second World War that the practice was first organized and acquired its name. In 1940, Patrick Blackett was called in by the British General Staff to lead the first operational research team, to solve problems such as the optimal siting of surveillance radars. The success of this research encouraged the Americans to use the various tools of operational research

The word "Operational" comes from the fact that the first application of a working group organized in this discipline was to military operations. The name has stuck ever since, even if the military is no longer the practice's main field of application.

After the war, techniques developed considerably, thanks in particular to the explosion in computer computing capacity. The range of applications also multiplied.

1.3 Applications fields

Operational research can help the decision-makers when faced with a problem belonging to one of the following types:

1.3.1 Combinatory problems

A problem is called a combinatory problem when it includes a large number of acceptable solutions among which one is looking for an optimal solution or close to the optimal one.

Typical example: determining where to locate 5 distribution centers out of 30 possible sites, so that transport costs between these centers and customers are kept to a minimum. This problem cannot be solved by simply listing the possible solutions in the human mind, because there are 30 * 29 * 28 * 27 * 26 = 17,100,720 (!). And even if a problem of this size could be solved by enumeration by a computer, decision-makers are regularly faced with infinitely more complex problems, where the number of acceptable solutions can be counted in billions of billions.

1.3.2 Random problems

A problem is called random if it consists of finding an optimal solution to a problem that is posed in uncertain terms.

Typical example: knowing the random distribution of the number of people entering a municipal administration in a minute and the random Distribution of a person's case processing time, determine the minimum number of checkpoints to open so that a person has less than 5% chance of having to wait more than 15 minutes.

1.3.3 Competitive problems

A problem is called a competitive problem if it is to find an optimal solution to a problem whose terms depend on the interrelationship between its own actions and those of other decision-makers.

Typical example: setting a sales price policy, knowing that the results of such a policy depend on the policy those competitors will adopt.

1.4 Relations with others disciplines

Operational research is at the crossroads of different sciences;

1.4.1 Economy

In addition to the fact that the main practical applications are in this area, economic analysis is often needed to define the objective to be achieved or to identify the constraints of a problem.

1.4.2 Computer science

Advances in computer science are closely linked to the growth in operational research applications. Significant computing power is required to solve large-scale problems. However, this power is far from being a panacea: it has been proven that certain problems, including some related to operational research, cannot be solved optimally by a computer in a reasonable time, even if we consider computers a billion times more powerful than those of today. For these problems, the operational researcher will most often call on techniques borrowed from artificial intelligence, enabling solutions close to the optimum to be found in an acceptable amount of time.

It can also be a field of application, in terms of locating servers, the number of servers to be set up to obtain a reasonable response time, etc....

1.4.3 Mathematics

Many of the methods used by operational research are derived from various mathematical theories. In this sense, operational research is a branch of applied mathematics.

1.5 Principle of operational research

Operational research is based on a scientific principle that begins with the creation of the appropriate model until its resolution and the application of the optimal decision. The use of operational research as a decision-making tool in the company must necessarily follow a certain path of operation:

- Determine all the components of the problem studied and the relationship of influence between them..
- Build the appropriate mathematical model.
- Examine the model introduced to take into account all the factors that influence the primary objective.
- Determine the optimal solution for the model and then check its feasibility.

Chapter 2

2 Linear Programming

There are mathematical methods that allow solving problems with defined objectives, such as maximizing profit or minimizing cost under a set of constraints. This technique is used to solve optimization problems.

Linear programming (PL) is one of the most powerful and most widely used tools in "industrial" application among decision-making technologies. It is used in several areas, namely: Production Planning; Resource Allocation; Production Choices; Investment Choices and Plans; Distribution Problems; Staff Affection and Management; Project Management, and many others. ...

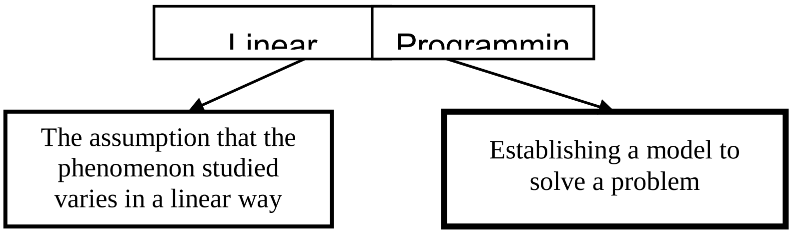


Fig 1 1 Linear programming

2.1 Definition

It is a mathematical technique used to deal systematically with complex problems where several activities compete for limited resources and an overall objective (profit maximization, cost minimization, etc.) is to be found.

2.2 Making Linear Programming model

Building the LP model is the key step in the overall process. In fact, it is the representation of real relationships by hypothetical mathematical functions, based on the examination and analysis of what already exists (the real problem). Such a model is created in just a few steps:

2.2.1 Define the decision variables

It presents the quantities that the decision-maker must know (unknown) and in addition these values have a direct influence on the value of the economic function. For example, the number of units produced, number of operating hours, ...

2.2.2 Economic function (or objective function)

It is a function that presents mathematically the main objective of the problem, it is characterized by a sense of preference that can be the maximization (benefit, production,...) or the minimization (cost, number of employees,...).

Exemple : $Max: Z = 3X_1 + 7X_2 + 8X_3$

with : X_1, X_2, X_3 *Decision variables*

3, 7, 8 *associated coefficients with the three variables that represent, for example, the sales price, the profit of one unit of each product.*

2.2.3 Functional Constraints

A set of constraints that mathematically translates into equations and/or inequations (inequalities), all the factors that have an impact on the components of the problem being studied. These are generally associated with the consumption of limited resources (raw materials, working hours, number of employees, etc.).

Exemple : $20X_1 - 10X_2 \leq 170$ a constraint in the form of an inequality "less than or equal to" a limited quantity of resources "170". Obviously, a constraint can be an equation or an inequality (≤,≥, <, >).

A linear program is generically expressed as:

$$
\begin{aligned}
Max: Z = c_1X_1 & + \cdots + c_jX_j + \cdots + c_nX_n \\
a_{11}X_1 & + \cdots + a_{1j}X_j + \cdots + a_{1n}X_n \leq b_1 \\
\vdots & \qquad \vdots \\
a_{i1}X_1 & + \cdots + a_{ij}X_j + \cdots + a_{in}X_n \leq b_i \\
\vdots & \qquad \vdots \\
a_{m1}X_1 & + \cdots + a_{mj}X_j + \cdots + a_{mn}X_n \leq b_m
\end{aligned}
$$

$\forall j \; ; \; X_j \geq 0$ *according to the case of non − negativity condition*

Or, in a more compact form:

$$
\begin{aligned}
Max: Z = \sum_{j=1}^{n} c_jX_j \\
\sum_{j=1}^{n} a_{ij}X_j \leq b_i \qquad i = 1 \ldots m
\end{aligned}
$$

$\forall j \; ; \; X_j \geq 0$ *according to the case of non − negativity condition*

Taking the following example to better understand the steps to follow in order to establish a linear program for a given problem:

Problem : A company produces several types of washing powder. This company received an order of 12000 kg of a type of powder that is constructed by the compilation of three products (A, B, C) and the characteristics of the powder requested are:

1. The quantity delivered contains at least 3000 kg of product B.
2. The quantity delivered does not contain more than 4000 kg of product A.
3. The quantity delivered contains a maximum of 2000 kg of product C.

If the cost of one kilogram of product B is 2 dinars, 3 dinars for the kg of product C and 4 dinars for the third product.

Write the linear program that minimizes total cost.

Solution :

The company received an order of 12000 kg on a single type of powder, so the overall problem is to minimize the purchase costs of the three raw materials (RM) while meeting the conditions required by this order and not forgetting the quantity requested since a quantity in addition to the RM entails additional charges.

Decision variables :

It is a matter of calculating the quantities of the three RMs, which directly influence the total cost value, so: we note by X_1, X_2, X_3, respectively the quantity of three raw materials A, B and C.

Economic Function :

The aim objective is to minimize raw material purchasing costs the preference sense is the minimization.

the cost of one kg of product A is 4 dinars

then the cost of the quantity X_1 kg of the product A, is $4X_1$ dinars

the cost of one kg of product B is 2 dinars

then the cost of the quantity X_2 kg of the product B, is $2X_2$ dinars

the cost of one kg of product C is 3 dinars

then the cost of the quantity X_3 kg of the product C, is $3X_3$ dinars

Which gives the total cost : $W = 4X_1 + 2X_2 + 3X_3$

The economic function is as follows: $MIN: W = 4X_1 + 2X_2 + 3X_3 \dots \dots \dots (1)$

The constraints :

1- constraint of product A

The condition of "the quantity delivered does not contain more than 4000 kg of product A" is mathematically translated as: $X_1 \leq 4000 \ldots\ldots\ldots(2)$

2- constraint of product B

The condition of "the quantity delivered contains at least 3000 kg of product B" is mathematically translated as: $X_2 \geq 3000 \ldots\ldots\ldots(3)$

3- constraint of product C

The condition of "the quantity delivered contains no more than 2000 kg of product C" is mathematically translated as: $X_3 \leq 2000 \ldots\ldots\ldots(4)$

4- constraint of requested quantity

The quantity requested is 12000 kg, which is mathematically translated as the sum of the three quantities: $X_1 + X_2 + X_3 = 12000$ But if we express this constraint in the form of equality, we do not have sufficient guarantees to ensure the production of 12000 kg, for this purpose this constraint will take the form of an inequality: $X_1 + X_2 + X_3 \geq 12000 \ldots\ldots\ldots(5)$

Adding the non-negative constraint (because of the positive quantities).

The model of the associated LP is the following:

$$
\begin{aligned}
&MIN: W = 4X_1 + 2X_2 + 3X_3 \\
&X_1 \leq 4000 \\
&X_2 \geq 3000 \\
&X_3 \leq 2000 \\
&X_1 + X_2 + X_3 \geq 12000 \\
&X_1, X_2, X_3 \geq 0
\end{aligned}
$$

Note: As the creation of the LP model represents a very important step, on which the action of decision-making on a given problem is articulated, it is necessary to present the resolution of a series of exercises in order to present as many cases as possible.

2.3 Exercises

Exercise 2.1 :

A national company produces two types of mattress (A and B) which each require three phases: the cutting phase, the folding phase in the form of rolls and the packaging phase.

The table below shows the time spent in minutes for the two types over the three production phases.

	Mattress types		Available time (min)
	A	B	
Phase 01	9	5	2100
Phase 02	5	7	1900
Phase 03	2	3	2500

It is given that the profit for each unit of type A is 1200 dinars and 800 dinars for that of type B. Write the linear program whose aim is to maximize the profit margin.

Exercise 2.2 A craftsman makes two items A and B, each requiring two operations: machining and heat treatment. Product A undergoes machining for 1 hour and heat treatment for 3 hours. Product B undergoes machining for 2 hours and heat treatment for 3 hours. In addition, 2kg of raw materials are used in the composition of A and 1kg in that of B. The manufacture of B ends with a finishing operation during 1 hour.

Every 3 weeks, the craftsman has the use of the machine shop for 80 hours and the furnace for 150 hours. In addition, during this period, he cannot devote more than 35 hours to finishing work or store more than 80kg of raw material.

What quantities of A and B must the craftsman make during this period if the profit margin is 30 euros for item A and 20 euros for item B?

Exercise 2.3 The VALAY quarry company supplies the Roads and Bridges Authority with gravel of various calibres; the contract, awarded for a global price, covers the following quantities: 13,500 tonnes of calibre 1 gravel, 11,200 tonnes of calibre 2 gravel and 5,000 tonnes of calibre 3 gravel. VALAY rents two quarries: P1 at 19.40 euros per tonne and P2 at 20 euros per tonne. After extraction, the stone is weighed and then crushed: each quarry supplies, per tonne of stone weighed, the quantities defined in the following table (the residual sand has no market value):

	calibre 1	calibre 2	calibre 3
stone of P1	0, 36t	0, 4t	0, 16t
stone of P2	0, 45t	0, 2t	0, 10t

The company wants to define its programme for extracting stones from P1 and P2 in such a way as to minimize the rental cost. Give a solution to this problem.

Exercise 2.4 In a cafeteria, 2 types of frozen dessert are served, based on exotic cocktails, ice cream and candied fruit: the Creole and the Tropical. The Creole requires 8cl of exotic cocktail, 2dl of ice cream and 15g of candied fruit.

The tropical requires 5cl of exotic cocktail, 2dl of ice cream and 25g of candied fruit. Each day, the patisserie workshop can prepare 1,600cl of exotic cocktail, 520dl of ice cream and 5kg of candied fruit. A Creole is sold for 1.2 euros and a tropical for 1 euro. Maximising profits.

Exercise 2.5 A business men specializes in producing leather shoes. He makes two types of shoe: hand-sewn A shoes and glued B shoes. His supply of leather does not allow him to make more than 120 shoes a week. What's more, he can't sell more than 100 shoes B and 70 shoes A during this period. It takes him 4 hours to make a pair of shoes A, and 1 hour to make a pair of shoes B. With his workers, he has 240 hours of work per week.

He can sell shoes A for 150 euros and shoes B for 50 euros. How many pairs of A and B shoes should he make per week to maximize his revenue?

Exercise 2.6 A company has 10,000 m² of cardboard in stock, it manufactures and markets 2 types of cardboard box. The manufacture of a type 1 or 2 cardboard box requires 1 and 2 m² of cardboard respectively, as well as 2 and 3 minutes of assembly time. Only 200 hours of work are available over the coming week. The boxes are stapled, and four times as many staples are needed for a type 2 of box as for a type 1 of box. The available stock of staples enables a maximum of 15,000 boxes of the first type to be assembled. The boxes are priced at 3 and 5 respectively.

a) Formulate the problem of finding a production plan which maximizes the company's turnover in the form of a linear program. Clearly specify the decision variables, the objective function and the constraints.

b) Determine an optimal production plan by solving the obtained linear program by graphical method (see the solutions section in Chapter 3).

Exercise solutions

Solution 2.1

(a) *Decision variables*

Let X_1 : be the number of mattresses type A;

X_2 : be the number of mattresses type B.

(b) *Economic function*

The aim is to maximize the profit margin.
The profit on one unit of mattress A is 1200 Dinars.
The profit on X_1 units of mattress A is 1200 X_1 Dinars.
The profit on one unit of mattress B is 800 Dinars
The profit on X_2 units of mattress B is 800 X_2 Dinars
the profit margin is : $Z = 1200X_1 + 800X_2$

So the economic function is : $Max: Z = 1200X_1 + 800X_2$

(b) Constraints
Phase 01 constraint
One unit of mattress A requires 9 minutes , so $9X_1$ minutes are needed for X_1 units.
One unit of mattress B takes 5 minutes, so $5X_2$ minutes are needed for X_2 units.
The duration of phase 01 is limited by 2100 minutes, which gives the following form of the constraint : $9X_1 + 5X_2 \leq 2100$
Phase 02 constraint
$5X_1$ minutes are needed for X_1 units type A, and $7X_1$ minutes are needed for X_1 units type B,
With a maximum duration of 1900 minutes, the constraint is written as :

$$5X_1 + 7X_2 \leq 1900$$

Phase 03 constraint
The time required to produce X_1, X_2 units of mattress A and B is: $2X_1 + 3X_2$ which must not exceed 2500 minutes, so : $2X_1 + 3X_2 \leq 2500$

This gives us the following linear program:

$$Max: Z = 1200X_1 + 800X_2$$

$$9X_1 + 5X_2 \leq 2100$$
$$5X_1 + 7X_2 \leq 1900$$
$$2X_1 + 3X_2 \leq 2500$$
$$X_1, X_2 \geq 0$$

Please refer to chapter 3 to see the resolution of this model,

Solution 2.2

(a) *Decision variables*

Let X_1 : be number of items A;

X_2 : be number of items B.

(b) *Economic function*

The aim is to maximize the profit margin.

The profit on one unit of items *type A is* 30 euros.

The profit on X_1 *units of mattress A is* $30X_1$€

The profit on one unit of items *B is* 20 euros

The profit on X_2 *units of* items *B is* $20X_2$€

the global profit margin is : $Z = 30X_1 + 20X_2$

So the economic function is : $Max: Z = 30X_1 + 20X_2$

(b) *Functional Contraints*

The machining constraint

A unit of product A is machined for one hour

Then X_1 *units of product A are machined for* X_1 *h*

A unit of product B is machined for 2 hours

Then X_2 *units of product B are machined for 2*X_2 *hours*

the craftsman has the machine shop at his disposal for 80 hours, so : $X_1 + 2X_2 \leq 80$

The heat treatment constraint

One unit of product A requires 3 hours, so 3 X_1 *hours are needed for* X_1 *units.*

One unit of product B requires 3 hours, so 3 X_2 *hours are needed for* X_2 *units.*

During this phase, the artisan does not have the furnace *available for more than 150 hours, which gives the following form of the constraint:* $3X_1+3X_2\leq150$

The finishing constraint

The manufacture of B ends with a finishing phase lasting 1 hour for each unit.

The duration of this phase is limited to 35 hours, so the constraint : $X_2\leq35$

The raw material constraint

2kg of raw material goes into the composition of A and 1kg into that of B.

So the quantity of the raw material consumed is: $2X_1+X_2$

The maximum storage threshold is 80kg: $2X_1+X_2\leq80$

We therefore have the following linear program:

$$Max: Z = 30X_1 + 20X_2$$
$$X_1 + 2X_2 \leq 80$$
$$3X_1 + 3X_2 \leq 150$$
$$X_2 \leq 35$$
$$2X_1 + X_2 \leq 80$$
$$X_1, X_2 \geq 0$$

Please refer to chapter 3 to see the resolution of this model,

Solution 2.3

The aim is to determine the quantities of stone in tonnes from the two quarries P1 and P2 required to meet VALAY's needs (which are 13,500 T, 11,200 T and 5,000 T of calibre 1, 2 and 3 respectively).

(a) Decision variables

Let X_1: quantity of stone extracted from the quarry P1 ;

X_2: quantity of stone extracted from the quarry P2.

(b) The economic function

The objective is to minimize the quarry rental costs,

The rental price for one tonne of P1 is €19.40

so the rental price of X_1 tonnes of P1 is $19.40X_1$ €.

The rental price for one tonne of P2 is €20

so the rental price for X_1 tonnes of P2 is $20X_1$ €.

The overall rental cost is: $W = 19.40X_1 + 20X_2$

The economic function is: $Min\text{: } W = 19.40X_1 + 20X_2$

(c) Constraints

The quantities required for the three calibers are defined, using information in the table:

- The caliber1 constraint

One tonne of P1 stone provides 0.36 t of caliber1 gravel

X_1 tonnes of P1 stone provides $0.36X_1$ tonnes of caliber1 gravel

One tonne of P2 stone provides 0.45 t of caliber1 gravel

X_2 tonnes of stone from P2 provides $0.45X_2$ tonnes of caliber1 gravel

Consequently, the quantity extracted from caliber1 is: $0.36X_1 + 0.45X_2$, which must be at least 13,500 tonnes (to meet requirements).

So the constraint is written as follows: $0.36X_1 + 0.45X_2 \geq 13500$

With the same explanation we retain :

- The caliber2 constraint:

$$0.41X_1 + 0.2X_2 \geq 11200$$

- The caliber3 constraint:

$$0.16X_1 + 0.11X_2 \geq 5000$$

therefore, we have the following linear program:

$$\begin{aligned} &Min\text{: } W = 19.40X_1 + 20X_2 \\ &\quad 0.36X_1 + 0.45X_2 \geq 13500 \\ &\quad 0.41X_1 + 0.2X_2 \geq 11200 \\ &\quad 0.16X_1 + 0.11X_2 \geq 5000 \\ &X_1, X_2 \geq 0 \end{aligned}$$

Please refer to chapter 3 to see the resolution of this model,

Solution 2.4

(a) Decision variables

Let X_1 be the number of Creole units;

X_2 : the number of tropical units .

(b) The economic function

The objective is to maximize the profit of this cafeteria.

The selling price of one unit of Creole is €1.2.

The selling price of X_1 units of Creole is €$1.2X_1$

The retail price of one unit of tropical is €1

The selling price of X_2 units of tropical is X_2 €.

The total profit is : $Z = 1.2X_1 + X_2$

So the economic function is : $Max: Z = 1.2X_1 + X_2$

(b) Constraints

- The exotic cocktail constraint

One unit of Creole requires 8cl, so $8X_1$cl are needed to prepare X_1 units.

One unit of tropical requires 5cl, so $5X_2$cl are needed to prepare X_2 units.

The patisserie can prepare 1600cl of exotic cocktail every day, which gives the constraint : $8X_1 + 5X_2 \leq 1600$

- *The ice cream constraint*

A unit of Creole requires 2 dl, so $2X_1$dl are needed to prepare X_1 units.

A tropical unit requires 2 dl, so $2X_2$dl are needed to prepare X_2units.

The pastry shop can prepare 520 dl of ice cream per day, which gives the constraint :

$$2X_1 + 2X_2 \leq 520$$

- The candied fruit constraint

One unit of Creole requires 15g, so $15X_1$g is needed to prepare X_1 units.

One unit of tropical requires 25 g, so $25X_2$g are needed to prepare X_1 units.

The pastry shop can prepare 5 kg (5000 g) of candied fruit per day, which gives the constraint : $15X_1 + 25X_2 \leq 5000$

We have the following linear program:

$$Max: Z = 1.2X_1 + X_2$$
$$8X_1 + 5X_2 \leq 1600$$
$$2X_1 + 2X_2 \leq 520$$
$$15X_1 + 25X_2 \leq 5000$$
$$X_1, X_2 \geq 0$$

Please refer to chapter 3 to see the resolution of this model,

Solution 2.5

Let be the following decision variables:
X_1: the quantity of shoes A produced;
X_2: the quantity of shoes B produced.
The objective function is to maximise the revenue obtained during a working week:
The selling price of a pair of shoes A is 150 euros and that of B is 50 euros, so the overall revenue is : $Z = 150X_1 + 50X_2$
So the objective function is given by : $Max: Z = 150X_1 + 50X_2$
The "leather" material constraint
The number of shoes produced per week must not exceed 120, so :

$$X_1 + X_2 \leq 120$$

- Sales quantity constraints

shoes Type A : $X_1 \leq 70$ i.e. we don't produce more than we can sell
shoes Type B : $X_2 \leq 100$

- The working hours constraint

240 hours of work is possible during the week and since a pair of shoes A and B require 4 and 1 hours of work respectively, this will give: $4X_1 + X_2 \leq 240$
therefore, we have the following linear program:

$$\begin{aligned} Max: Z = 150X_1 + 50X_2 \\ X_1 + X_2 \leq 120 \\ X_1 \leq 70 \\ X_2 \leq 100 \\ 4X_1 + X_2 \leq 240 \\ X_1, X_2 \geq 0 \end{aligned}$$

Please refer to chapter 3 to see the resolution of this model,

Solution 2.6

Let be the following decision variables:
X_1: number of type 1 boxes produced;
X_2: number of type 2 boxes produced.
We have the following constraints:
- on square meters of cardboard: $X_1 + 2X_2 \leq 10000$
- assembly time in minutes: $2X_1 + 3X_2 \leq 200 * 60$
- on the number of staples: $X_1 + 4X_2 \leq 15000$

The objective function to be maximized: corresponds to the turnover obtained from the sale of the boxes: $Max: Z = 3X_1 + 5X_2$

therefore, we have the following linear program:

$$
\begin{aligned}
&Max: Z = 3X_1 + 5X_2 \\
\text{S.c :}\quad & X_1 + 2X_2 \leq 10000 \\
& 2X_1 + 3X_2 \leq 12000 \\
& X_1 + 4X_2 \leq 15000 \\
& X_1, X_2 \geq 0
\end{aligned}
$$

Please refer to chapter 3 to see the resolution of this model,

Chapter 3

3 Graphical method and the resolution of LP model

3.1 Introduction

The graphical method is a simple method that is based on the graphical representation of the lines. However, its use is limited in some cases:

- If the number of variables exceeds two variables[1],
- If the number of constraints increases significantly.

3.2 Treatments

When we want to represent the solutions of the inequation in a plan *OXY* $3x + 3y \leq 150$, There are two tricks:

(1) We transform inequality into one of the four possible forms:

y < ax + b ; y > ax + b ; y ≤ ax + b ; y ≥ ax + b.

For example

3y ≤ -3X + 150 where y ≤ -x + 50. drawing the line represented by y = -x + 50.

The set of points M(x; y) whose coordinates verify the inequation is the closed half-plane located below the equation line y = -x + 50, line included.

(2) drawing the line represented by $3x + 3y = 150$. The set of points M(x; y) whose coordinates verify the inequation is the closed half-plane located below the equation line 3x+3y = 150, line included.

For the general form ' ax+by < c ; ax+by > c ; ax+by ≤ c ; ax+by ≥ c', The following figure shows the possible cases for determining the solutions:

[1] The use of the graphical method is possible in the case of three variables (each constraint is represented graphically by a plan) and then the feasible solution, if it exists, is presented by a volume.

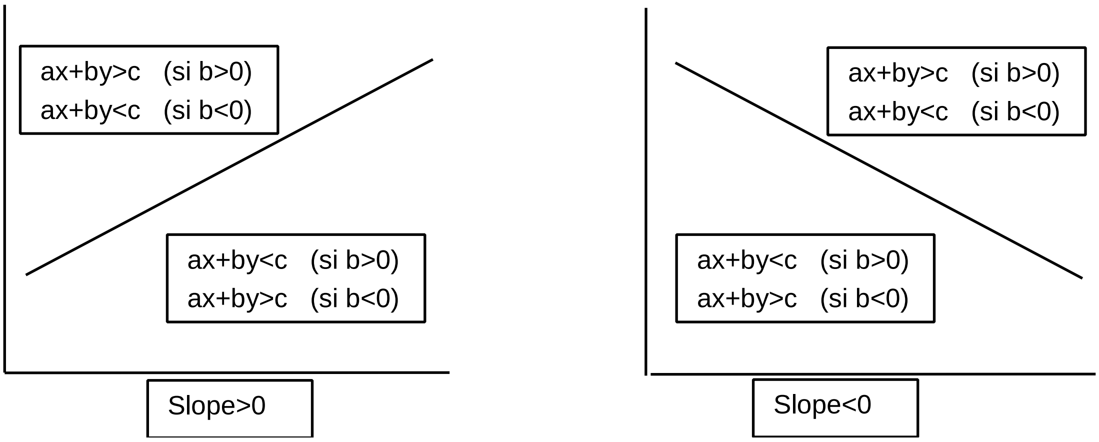


Fig 1 1 Feasible solutions of an inequality based on slope value

Consequently, to determine the solutions of a system of two unknown inequalities, it is sufficient to construct the corresponding straight and half-plans that designate the space verifying each inequality, the intersection of all half-planes is the representation of the feasible solutions.

Adding that for the resolution of a linear program, it remains the search for an optimal solution (which is described by the economic function) among the set of feasible solutions.

Let's take, by the two tips, example 2.5 from the previous lesson:

Whether the LP model is as follows: $Max: Z = 150X_1 + 50X_2$

$$X_1 + X_2 \leq 120$$
$$X_1 \leq 70$$
$$X_2 \leq 100$$
$$4X_1 + X_2 \leq 240$$
$$X_1, X_2 \geq 0$$

Tips 1 :

The variable X2 is isolated from the two inequations (1 and 4):

$$X_2 \leq -X_1 + 120$$
$$X_1 \leq 70$$
$$X_2 \leq 100$$
$$X_2 \leq -4X_1 + 240$$
$$X_1 \geq 0$$
$$X_2 \geq 0$$

First, we need to identify the pairs (x, y) verifying the system.

– The coordinate points (x, y) verifying inequality $X_2 \leq -X_1 + 120$ are the points belonging to one of the half-plane bounded by the line equation $X_2 = -X_1 + 120$. The suitable half-plane is the one below the line (since there is " ≤ ").

– The coordinate points (x, y) verifying inequality $X_2 \leq 100$ are the points belonging to one of the half-plane bounded by the line equation $X_2 = 100$ qui est parallèle à l'axe des abscisses.

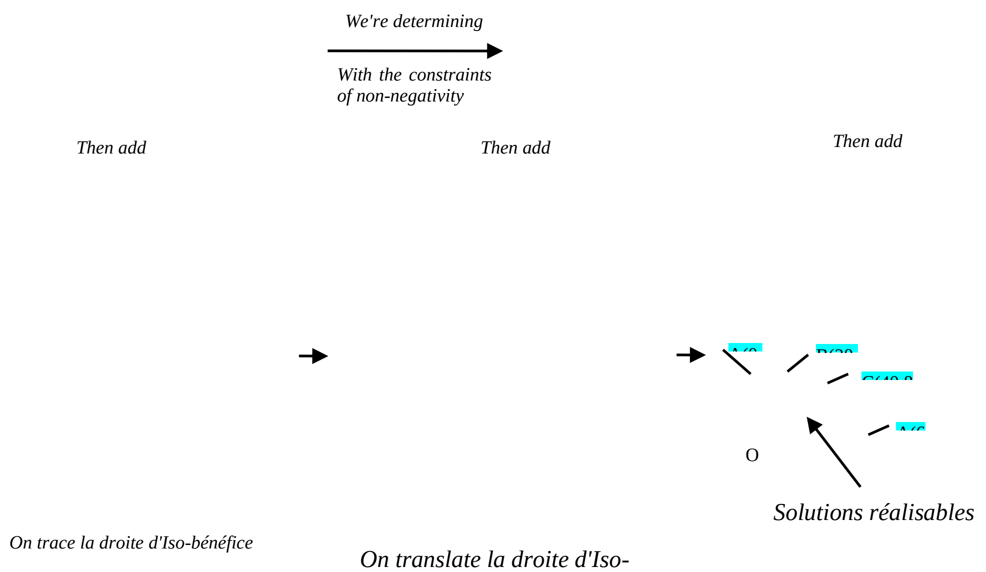
– The coordinate points (x, y) verifying inequality $X_1 \leq 70$ are the points belonging to one of the half-plane bounded by the line equation $X_1 = 70$ qui est parallèle à l'axe des ordonnées.

– The coordinate points (x, y) verifying inequality $X_2 \leq -4X_1 + 240$ are the points belonging to one of the half-plane bounded by the line equation $X_2 = -4X_1 + 240$

– The coordinate points (x, y) verifying inequality $X_1 \geq 0$ are the points belonging to the half-plan of the positive abscises,

– The coordinate points (x, y) verifying inequality $X_2 \geq 0$ are the points belonging to the half-plane of the positive ordinates,

We choose to color in gray the part that is not suitable.



On translate la droite d'Iso-bénéfice d'une manière parallèle, tous les points qui appartiennent à cette droite et coupe la zone des solutions réalisables représentent des points de même bénéfice.

La dernière intersection représente la solution optimale ici c'est le point ***C(40,80) qui donne un bénéfice de 10000 Dinars***

The figure above shows the steps involved in graphically solving the given LP modal. The search for the optimum has been based on the translation of the iso-benefit line (purely graphical) whatever is possible with calculations; so if the optimum solution of a PL exists, it is at the edges of the area of feasible solutions. To determine it, all we have to do is calculate the value of the economic function in all the corners of the feasible solution space, taking the highest value for maximization and the lowest for minimization.

Tips 2 : we plot the lines equation of the four constraints

$$X_1 + X_2 = 120$$
$$X_1 = 70$$
$$X_2 = 100$$
$$4X_1 + X_2 = 240$$

The information presented in figure is used to determine the space of feasible solutions for each constraint, and finally to determine the optimal solution.

3.3 Particular cases

1. If the constraints of the linear program are incompatible, the polygon of feasible solutions is empty: the linear program then has no solution

2. If the polygon is open upwards, a maximization problem has no solution, as the reference line can move upwards indefinitely.

3.The optimal solution, if any, is always at one of the corners of the polygon

4. If the coordinates of the point S found as a solution are not integer, then we must look for a point on the polygon with integer coordinates that is close to the point S.

5. If the reference line is parallel to one of the sides of the polygon, the problem has an infinite number of alternative solutions.

The straight lines defining the vertex correspond to resources that have been completely used up, known as scarce resources. Other resources not fully utilized are said to be in overabundance.

3.4 Resolution of the exercises of the previous chapter

Solution 2.1

Let's use the two tips for Example 2.1 in the previous course:

Let's consider the following LP model:

$$Max: Z = 1200X_1 + 800X_2$$
$$9X_1 + 5X_2 \leq 2100$$
$$5X_1 + 7X_2 \leq 1900$$
$$2X_1 + 3X_2 \leq 2500$$
$$X_1, X_2 \geq 0$$

Using tips 1 :

We isolate the variable X_2 from the three inequalities :

$X_2 \leq -\frac{9}{5}X_1 + 420$

$X_2 \leq -\frac{5}{7}X_1 + \frac{1900}{7}$

$X_2 \leq -\frac{2}{3}X_1 + \frac{2500}{3}$

$X_1 \geq 0 \qquad X_2 \geq 0$

First, we need to determine the (x, y) pairs that verify the system.

– The coordinate points (x, y) verifying inequality $X_2 \leq -\frac{9}{5}X_1 + 420$ are the points belonging to one of the half-plane bounded by the line equation $X_2 = -\frac{9}{5}X_1 + 420$. The suitable half-plane is the one below the line (since there is " $\leq$ ").

– The coordinate points (x, y) verifying inequality $X_2 \leq -\frac{5}{7}X_1 + \frac{1900}{7}$ are the points belonging to one of the half-plane bounded by the line equation $X_2 = -\frac{5}{7}X_1 + \frac{1900}{7}$

– The coordinate points (x, y) verifying inequality $X_2 \leq -\frac{2}{3}X_1 + \frac{2500}{3}$ are the points belonging to one of the half-plane bounded by the line equation $X_2 = -\frac{2}{3}X_1 + \frac{2500}{3}$

– The coordinate points (x, y) verifying inequality $X_1 \geq 0$ are the points belonging to one of the half-plane of the positive abscises,

– The coordinate points (x, y) verifying inequality $X_2 \geq 0$ are the points belonging to one of the half-plane of the positive ordinates,

We choose to color in grey the part that is not suitable.

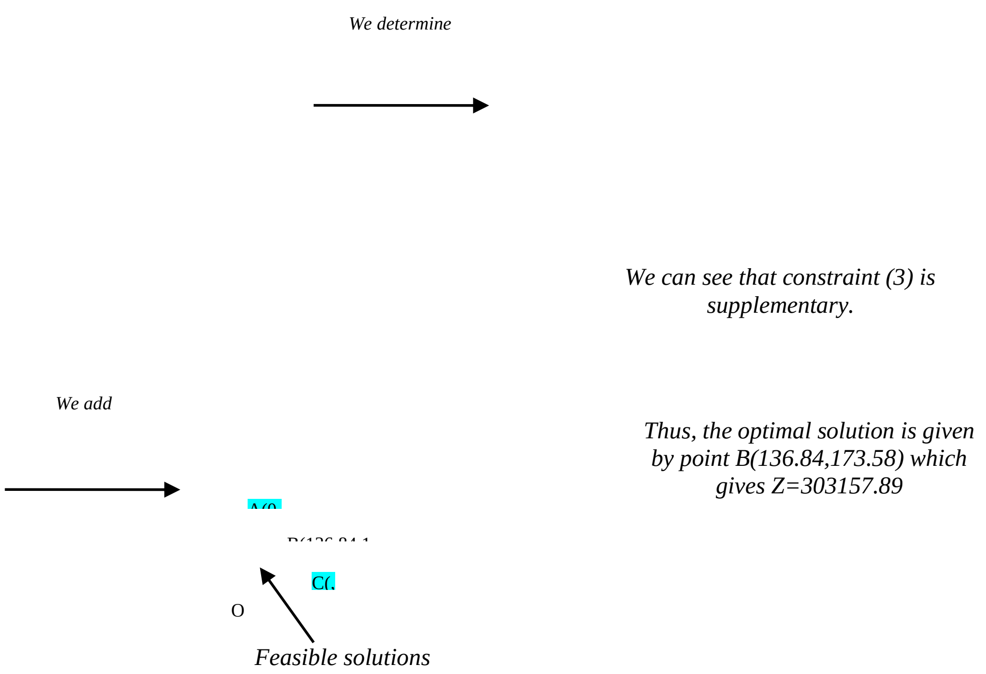


Fig 1 3 Steps of solution 2.1

Note that the values of the optimal solution are not integer values.

Solution 2.2

The aim is to maximize value of $Z = 30X_1 + 20X_2$ under the following constraints:
:

$$X_1 + 2X_2 \leq 80$$
$$3X_1 + 3X_2 \leq 150$$
$$X_2 \leq 35$$
$$2X_1 + X_2 \leq 80$$
$$X_1, X_2 \geq 0$$

We draw the straight lines corresponding to the four constraints, then determine the space of points that satisfy all the constraints. The solution steps are described in the figure below:

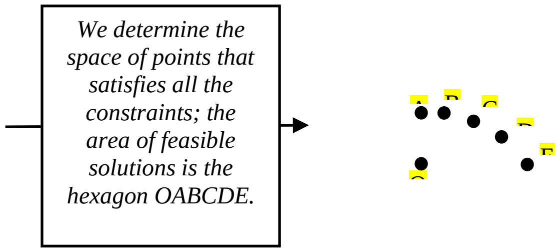


Fig 1 4 Graphical resolution (2.2)

The coordinates of the polygon points are determined: ABCDE.

	X1	X2	Z	
A	0	35	700	The maximum value of the economic function is 1300, which corresponds to the following solution : X_1=30, X_2=20.
B	10	35	1000	
C	20	30	1200	
D	**30**	**20**	**1300**	
E	40	0	1200	

Solution 2.3

$$Min:\ W = 19.40X_1 + 20X_2$$
$$0.36X_1 + 0.45X_2 \geq 13500$$
$$0.41X_1 + 0.2X_2 \geq 11200$$
$$0.16X_1 + 0.11X_2 \geq 5000$$
$$X_1, X_2 \geq 0$$

The resolution is shown in the following figure :

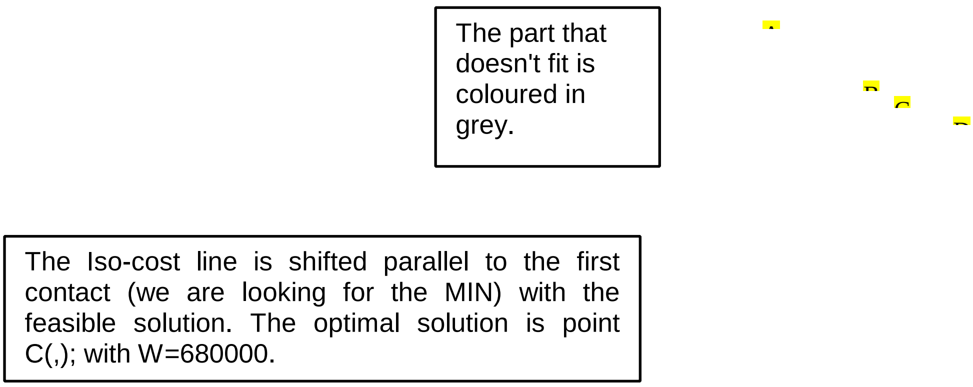


Fig 1 5 Graphical Solution of the problem (2.3)

Solution 2.4

$Max: Z = 1.2X_1 + X_2$

$$8X_1 + 5X_2 \leq 1600$$

$$2X_1 + 2X_2 \leq 520$$
$$15X_1 + 25X_2 \leq 5000$$

The part that doesn't fit is coloured in grey.

The Iso-cost line is shifted parallel to the last contact (MAX) with the feasible solution . The optimal solution is expressed by the point B(120,128); with Z=272.

$$X_1, X_2 \geq 0$$

Solution 2.6

Simply represent the admissible domain D of the linear program and find the point on the edge of D that maximizes the function $3X_1 + 5X_2$. This is done in the figure opposite, and we see that the optimal plan is to produce 600 type 1 boxes (x_1= 600) and 3600 type 2 boxes (x_2 = 3600), for sales worth 19,800 UM (z = 19,800).

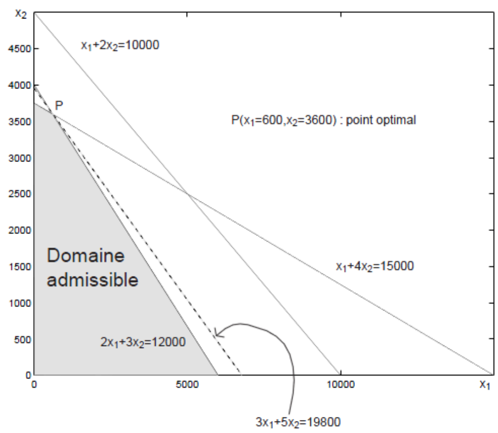


Fig. 1 7 Résolution graphique (2.6)

Chapter 4

4 Simplex method

4.1 Introduction

We now are ready to begin studying the simplex method, a general procedure for solving linear programming problems. Developed by George Dantzig in 1947, it has proved to be a remarkably efficient method that is used routinely to solve huge problems on today's computers. Except for its use on tiny problems, this method is always executed on a computer, and sophisticated software packages are widely available. Extensions and variations of the simplex method also are used to perform postoptimality analysis (including sensitivity analysis) on the model.

This chapter describes and illustrates the main features of the simplex method. The first section introduces its general nature, including its geometric interpretation. The following three sections then develop the procedure for solving any linear programming model that is in our standard form (maximization, all functional constraints in form, and nonnegativity constraints on all variables) and has only nonnegative right-hand sides bi in the functional constraints.

It will be necessary to bear in mind that the Simplex method only works with problem constraints whose constraints are of the "<=" type (less than or equal to) and whose independent coefficients are greater than or equal to 0. Thus, restrictions should be normalized to meet these requirements before starting the Simplex algorithm. In the event that constraints of type "≥" (greater than or equal to) or "=" (equal to) appear after this process, or cannot be modified, it will be necessary to use other solving methods, such as the Two-Phase method or the penalty method (known as Big-M).

4.2 The essence of the Simplex Method

The simplex method is an algebraic procedure, it is based on the determination of a feasible basic solution, so several iterations are triggered until the optimal solution is obtained, if it exists. The first step (initial basic solution) consists of choosing a number of variables (equal to the number of the functional constraints) known as basic variables (BV).

A basic variable must satisfy two conditions

- Its coefficient in the associated constraint is equal to "1",
- Its coefficient is zero in the other constraints.

At each iteration, a single BV is varied by choosing an incoming variable (from the non-basic variables) and an outgoing variable (from the basis variables). Obviously, the choice of these two types of variable is no longer random. An incoming variable is the one that most improves the EF value (increase in the case of maximization and decrease in the case of minimization). The outgoing variable is the one that has the least influence.

4.3 Adaptation model

Using the simplex method requires the linear model to be adapted into a mathematical equation model known as the standard model, with an objective function to be optimized.

The standard model must satisfy the following conditions:

1. The objective will be to maximize or minimize the value of the objective function (for example, increase profits or reduce losses, respectively).
2. All constraints must be equality equations (mathematical identities).
3. All variables (Xj) must be a positive or zero value (non-negativity condition).
4. The independent terms (bi) in each equation must be positive.

The general form of the standard model is given by :

Objective function : $Max: Z = c_1X_1 + \cdots + c_jX_j + \cdots + c_nX_n$

under constraints : $a_{11}X_1 + \cdots + a_{1j}X_j + \cdots + a_{1n}X_n + X_{n+1}^e = b_1$

$$\vdots \qquad \vdots$$

$$a_{i1}X_1 + \cdots + a_{ij}X_j + \cdots + a_{in}X_n + X_{n+i}^e = b_i$$

$$\vdots \qquad \vdots$$

$$a_{m1}X_1 + \cdots + a_{mj}X_j + \cdots + a_{mn}X_n + X_{n+m}^e = b_m$$

$$\forall\, i,j\ ;\ X_j, X_{n+i} \geq 0\ \ according\ to\ the\ non-negativity\ condition$$

With : X_{n+i}^e : slack variable for the respective functional constraint i.

we consider that the constraints of the linear model are inequalities of the type ≤, (less than or equal)

Thus, the simplex table is presented us follow:

Variables ☉		X_1	-	X_j	-	X_n	X_{n+1}^e	-	X_{n+m}^e		↓ Coefficients des VB
Coefficients of the EF	**Cj**	C_1		C_j		C_n	0		0	Bi	Cbi
Basic variables	X_{n+1}^e	a_{11}		a_{1j}		a_{1n}	1	-	0	b_1	Cj (X_{n+1}^e)
		---		---		---	---		---	-	---
	X_{n+i}^e	a_{i1}		a_{ij}		a_{in}	0	-	0	b_i	Cj (X_{n+i}^e)
		---		---		---	---		---	-	---
	X_{n+m}^e	a_{m1}		a_{mj}		a_{mn}	0	-	1	b_m	Cj (X_{n+m}^e)

Value of EF ✇	Z_j	Z_1		Z_j		Z_n	0		0	**Z**	
Marginal cost✇	ΔZ	ΔZ_1		ΔZ_j		ΔZ_n	0		0	↑ Available resources	

Tableau 1-4 Overview of the simplex table

Information can be extracted from the simplex table:

- The basic variables and their values: each BV is associated with a constraint, and they are shown in the first column of the table (Xn+1e, ..., Xn+me). Column Bi includes the values of the BVs (b1, ..., bm).
- Non-basic variables: represent the other variables that do not appear in the BV column. Their values are automatically reset to zero (0).
- The value of the objective function: is shown on the penultimate line in the box corresponding to the bi column (Z).
- Marginal costs: the marginal cost of a Non-basic variable (NBV) represents the effect on the optimal value of the EF resulting to a unit increase of this NBV. It is assumed that the marginal cost of a BV is null in all cases. These values are given in the last line of the table.
- Verification of the stopping criterion: this is the optimality condition which is based on the sign of the different values of the marginal costs according to the direction of the objective function:
  - For maximization case: marginal costs are negative or null.
  - For minimization case: marginal costs are positive or null.
- Marginal prices of constraints: represent the effect on the optimal value of the EF resulting to a unit increase in the available resource (bi) of this constraint.

4.4 Simplex algorithm

Take the following example:

Max $z = 2X_1 + 3X_2$ its standard form is given by : **Max** $z = 2X_1 + 3X_2$

s.c. $X_1 + 2X_2 \leq 20$ **s.c.** $X_1 + 2X_2 + S_1 = 20$

$X_1 + X_2 \leq 12$ $X_1 + X_2 + S_2 = 12$

$X_1 , X_2 \geq 0$ $X_1, X_2, S_1, S_2 \geq 0$

We have four variables and two functional constraints, according to the simplex algorithm we need to choose two variables as basic variables (which verify the two conditions in section 4.2). In this case, the slack variables S1 and S2 are the basic variables.

Note: At any iteration of the simplex method, step 2 uses the minimum ratio test to determine which basic variable drops to zero first as the entering basic variable is increased. Decreasing this basic variable to zero will convert it to a nonbasic variable for the next solution. Therefore, this variable is called the leaving basic variable for the current iteration (because it is leaving the basis).

The initial simplex table :

MAX		X_1	X_2	S_1	S_2		
Cj		2	3	0	0	Bi	Cbi
20/2=**10**	S_1	1	2	1	0	20	**0**
12/1=12	S_2	1	1	0	1	12	**0**
Zj		0	0	0	0	**0**	
ΔZ		2	**3**	0	0		

The marginal costs of the VHBs are positive and since we are dealing with a maximization problem, this means that we have the possibility to improve the current solution. So we apply the Danzig criteria to choose an entering basic variable (EBV) and a leaving basic variable (LBV).

The entering basic variable is the variable with the highest marginal cost (ΔZ) among the positive values, the X_2 variable (from now on, the X_2 column is known as the pivot column).

The leaving basic variable is the variable which minimizes the quotient bi/a_{i2} (we divide only on the positive values); with a_{i2} the coefficient of the entering basic variable in constraint i. which results in the leaving basic variable being S_1 (the row of variable S1 is called the pivot row).

- The transition to the next table is based on the following rules:
- The elements of the pivot line are divided by the value of the pivot.
- The elements of the other rows are obtained by :

  The new line = the old line - a_{i2}* the new pivot line (here a_{22}=1 for line 2)

i.e. : [1 1 0 1 12]
-(1)*[1/2 1 1/2 0 10]

[1/2 0 -1/2 1 2]

MAX		X_1	X_2	S_1	S_2		
Cj		2	3	0	0	Bi	Cbi
10/(0.5)=20	X_2	1/2	1	1/2	0	10	**3**
2/0.5=**4**	S_2	1/2	0	-1/2	1	2	**0**
Zj		3/2	3	3/2	0	**30**	
$\Delta Z = C_j - Z_j$		1/2	**0**	-3/2	0		

The marginal cost of the NBV X_1 is positive, which means that the current solution is no longer optimal, so we go through the same rules in the 2nd iteration :

MAX		X_1↓	X_2	S_1	S_2		
Cj		2	3	0	0	Bi	Cbi
10/0.5=20	**X_2**	1/2	1	1/2	0	10	**3**
2/0.5=**4**	←**S_2**	1/2	0	-1/2	1	2	**0**
Zj		3/2	3	3/2	0	**30**	
$\Delta Z = C_j - Z_j$		1/2	**0**	-3/2	0		

Entering basic variable (EBV) : X_1
Leaving basic variable (LBV) : S_1

MAX		X_1↓	X_2	S_1	S_2		
Cj		2	3	0	0	Bi	Cbi
	X_2	0	1	1	-1	8	**3**
	X_1	1	0	-1	2	4	**2**
Zj		2	3	1	1	**32**	
$\Delta Z = C_j - Z_j$		0	0	-1	-1		

The stopping criterion is verified in the table above (marginal costs are negative or null), adding that X_1 and X_2 have positive values (no contradiction with the nonnegative condition) then we say that the current solution is optimal with : X_1= 4, X_2=8, Z=32. (S_1=0, S_2=0 means that the available quantities of the two constraints have been totally consumed).

4.5 Particular cases

- After choosing an entering basic variable, if there is no leaving basic variable (the coefficients of all the BV in the pivot column are negative or null) then the solution is unbounded.
- Degenerate basic solution if one or more basic variables are null (no more bijection between admissible basic solutions and extreme points).
- The presence of an entering basic variable with null marginal cost gives rise to an infinite number of alternative solutions (this is justified by the fact that changing the basic variables has no change in the value of the objective function).

▪

4.6 Exercises

Exercise 4.1 :

Suppose the following linear program: **Max** $z = 2X_1 + 3X_2$

$$\begin{aligned} \textbf{subject to} \quad & 2X_1 + 3X_2 \geq 10 \\ & X_1 \leq 30 \\ & X_2 \leq 20 \\ & X_1 , X_2 \geq 0 \end{aligned}$$

- Define the LP model up to the second simplex step.
- Is the current solution feasible ? justify it ?
- Is the current solution optimal ? why ?
- Deduce the values of the decision variables.
- What is the value of the objective function ?

Exercice 4.2 :

Solve the following LP using the simplex method:

$$\textbf{Max } z = 2X_1 + X_2$$

$$\begin{aligned} \textbf{subject to} \quad & X_1 - 2X_2 \leq 2 \\ & -2X_1 + X_2 \leq 2 \\ & X_1 , X_2 \geq 0 \end{aligned}$$

Exercice 4.3 : Solve the following LP using the simplex method:

$$\textbf{Max } z = 2X_1 - X_2 + 3X_3$$

$$\begin{aligned} \textbf{subject to} \quad & X_1 - X_2 + 5X_3 \leq 10 \\ & -4X_1 + 2X_2 - 6X_3 \geq -80 \\ & X_1 , X_2 , X_3 \geq 0 \end{aligned}$$

Exercice 4.4 : Solve the following LP using the simplex method:

$$\textbf{Max } z = 3X_1 + 7X_2$$

$$\begin{aligned} \textbf{subject to} \quad & 2X_1 + 8X_2 \leq 16 \\ & 2X_1 + 4X_2 \leq 8 \\ & X_1 , X_2 \geq 0 \end{aligned}$$

Exercise solutions

Solution 4.1 :

The standard form of the linear program is us follow: **Max** $z = 2X_1 + 3X_2$

subject to $-2X_1 - 3X_2 + S_1 = -10$

$X_1 + S_2 = 30$

$X_2 + S_3 = 20$

$X_1, X_2, S_1, S_2, S_3 \geq 0$

MAX		X_1	X_2↓	S_1	S_2	S_3		
Cj		2	3	0	0	0	Bi	Cbi
/	S_1	-2	-3	1	0	0	-10	**0**
/	S_2	1	0	0	1	0	30	**0**
20/1=20	←S_3	0	**1**	0	0	1	20	**0**
Zj		0	0	0	0	**0**	**0**	
$\Delta Z = C_j - Z_j$		2	3	0	0	0		
		X_1	X_2↓	S_1	S_2	S_3		
Cj		2	3	0	0	0	Bi	Cbi
/	S_1	-2	0	1	0	3	50	**0**
/	S_2	1	0	0	1	0	30	**0**
	X_2	0	1	0	0	1	20	**3**
Zj		0	3	0	0	3	**60**	
$\Delta Z = C_j - Z_j$		2	0	0	0	-3		

EBV: X_2 (the highest marginal cost)

LBV: S_3 (negative or nil coefficients are not taken into account).

☹the pivot value = 1

Applying the transformation rules, we obtain the table opposite.

At this point, the stop criterion has not been met, and since only the first two tables are requested, we'll stop here.

- Is the current solution feasible? justify it?

Yes, it is feasible because there is no contradiction with the nonegative condition (X_1=0, X_2=20, S_1=50, S_2=30, S_3=0).

- Is the current solution optimal ? why ?

It is not optimal because there is still a positive marginal cost (that of X_1).

- Deduce the values of the decision variables.

Values of the basis variables are the corresponding values in the bi column:

X_2=20, S_1=50, S_2=30.

Values of the nonbasic variables are nils:

X_1=0, S_3=0.

- What is the value of the objective function ?

Z=60.

Solution 4.2 :

The standard form of the linear program is us follow:

Max $z = 2X_1 + X_2$

subject to $X_1 - 2X_2 + S_1 = 2$

$-2X_1 + X_2 + S_2 = 2$

$X_1, X_2, S_1, S_2 \geq 0$

MAX		**X_1↓**	**X_2**	**S_1**	**S_2**		
Cj		2	1	0	0	Bi	Cbi
2/1=2	**←S_1**	**1**	-2	1	0	2	**0**
/	**S_2**	-2	1	0	1	2	**0**
Zj		0	0	0	0	**0**	
ΔZ=**C_j-Z_j**		2	1	0	0		
		X_1	**X_2↓**	**S_1**	**S_2**		
Cj		2	1	0	0	Bi	Cbi
/	**X_1**	**1**	-2	1	0	2	**2**
/	**S_2**	0	-3	2	1	6	**0**
Zj		2	-4	2	0	**4**	
ΔZ=**C_j-Z_j**		0	5	-2	0		

EBV: X_1 (the highest marginal cost)

LBV: S_1 (negative or nil coefficients are not taken into account).

☹the pivot value = 1

Applying the transformation rules, we obtain the table opposite.

At this point, the stop criterion has not been met,

then we choose the EBV which is X2 BUT the choice of the LBV is impossible because of the negative coefficients (-2 and -3) ;

the solution is **unbounded**.

Solution 4.3 :

The standard form of the linear program is us follow:

Max $z = 2X_1 - X_2 + 3X_3$

subject to $X_1 - X_2 + 5X_3 + S_1 = 10$

$4X_1 - 2X_2 + 6X_3 + S_2 = 80$

$X_1, X_2, X_3, S_1, S_2 \geq 0$

MAX		**X_1**	**X_2**	**X_3↓**	**S_1**	**S_2**		
Cj		2	-1	3	0	0	Bi	Cbi
10/5=**2**	**←S_1**	1	-1	**5**	1	0	10	**0**
80/6=13.33	**S_2**	4	-2	6	0	1	80	**0**
Zj		0	0	0	0	0	**0**	
ΔZ=**C_j-Z_j**		2	-1	3	0	0		
		X_1↓	**X_2**	**X_3**	**S_1**	**S_2**		
Cj		2	-1	3	0	0	Bi	Cbi
2/1/5=**10**	**←X_3**	**1/5**	-1/5	1	1	0	2	**3**
68/14/5=24.2	**S_2**	14/5	-4/5	0	0	1	68	**0**
Zj		3/5	-3/5	3	3	0	**6**	
ΔZ=**C_j-Z_j**		7/5	-2/5	0	-3	0		
		X_1	**X_2↓**	**X_3**	**S_1**	**S_2**		
Cj		2	-1	3	0	0	Bi	Cbi

EBV: X_3 (the highest marginal cost)

LBV: S_1 (negative or nil coefficients are not taken into account).

the pivot value = 5

Applying the transformation rules, we obtain the table in opposite.

At this point, the stop criterion has not been met.

then we choose the EBV which is X_1 ;

the pivot value = 1/5.

Continuing with the choice of EBV (X_2) and the LBV (S_2), the

10/5=**2**	X_1	1	-1	5	5	0	10	**2**
80/6=13.33	←S_2	0	**2**	-14	-14	1	40	**0**
Zj		2	-2	10	10	0	**0**	
$\Delta Z = C_j - Z_j$		0	1	-7	-10	0	20	
		X_1↓	X_2	X_3	S_1	S_2		
Cj		2	-1	3	0	0	Bi	Cbi
2/1/5=**10**	X_1	**1**	0	-2	-2	1/2	30	**2**
68/14/5=24.2	X_2	0	1	-7	-7	1/2	20	**-1**
Zj		2	-1	3	3	1/2	**6**	
$\Delta Z = C_j - Z_j$		0	0	**0**	-3	-1/2		

pivot value = 2 ;

We note that there is a NBV(X_3) with nil marginal cost, this means that there are an infinite number of **alternative solutions**

Solution 4.4 : The standard form of the linear program is us follow:

Max $z = 3X_1 + 7X_2$

subject to $2X_1 + 8X_2 + S_1 = 16$

$2X_1 + 4X_2 + S_2 = 8$

$X_1, X_2, S_1, S_2 \geq 0$

MAX		X_1	X_2↓	S_1	S_2		
Cj		3	7	0	0	Bi	Cbi
16/8=**2**	←S_1	2	**8**	1	0	16	**0**
8/4=2	S_2	2	4	0	1	8	**0**
Zj		0	0	0	0	**0**	
$\Delta Z = C_j - Z_j$		3	7	0	0		

EBV: X_2 (the highest marginal cost)

LBV: In this case, we have two possibilities, since the quotients are the same. We chose S_1.

the pivot value = 8

		X_1↓	X_2	S_1	S_2		
Cj		3	7	0	0	Bi	Cbi
2/1/4=8	X_2	1/4	1	1/8	0	2	**7**
0/1=**0**	←S_2	**1**	0	-1/2	1	0	**0**
Zj		7/4	7	7/8	0	**14**	
$\Delta Z = C_j - Z_j$		5/4	0	-7/8	0		

Applying the transformation rules, we obtain the table in opposite.

At this point, the stopping criterion is not verified, so we choose the EBV, which is X_1, and the LBV, which is S_2; the value pivot = 1.

		X_1	X_2↓	S_1	S_2		
Cj		3	7	0	0	Bi	Cbi
	X_2	0	1	1/4	-1/4	2	**7**
	X_1	1	0	-1/2	1	0	**3**
Zj		3	7	1/4	5/4	**14**	
$\Delta Z = C_j - Z_j$		0	0	-1/4	-5/4		

The stopping criterion is then verified, but we note the values of the solution (X_1=0, X_2=2, Z=14) have not changed between the last two tables: the solution is said to be **degenerate**. This situation results from the fact that a constraint is in begging.

Chapter 5

5 The Big M and two-phases method

5.1 Limits of the simplex method

The independent terms (bi) of each constraint must be non-negative in order to use the Simplex method. To this end, if any of the constraints have a negative independent term, it must be multiplied by "-1" (bearing in mind that this operation also affects the type of constraint).

It can be found that in the constraints where we have to change the signs of the constants, the types of inequalities are "≤" (after the operation, they will stay of type "≥"), so it will be necessary to develop other methods. This disadvantage is not controllable, but otherwise it could occur in the opposite case and be beneficial if the independent terms are negative in all inequality constraints of type "≥". If there are constraints of type "=" they do not lead to advantages or disadvantages because it would still be necessary to apply one of the two methods: the Two Phases method or the penalty method (known as Big-M).

In summary, there are cases where the application of the simplex becomes impossible because of two essential points:

- If it is impossible to determine the basic variables (no variables verify the BV conditions),
- If the solution is not feasible (there is a contradiction with the non-negativity condition).

5.2 Functional constraints normalization

● **Constraint of type (≤):** To normalize an inequation constraint of type "≤", we add a new positive variable, called the slack variable X^e or S_i . This new variable does not appear in the objective function because it is a variable that expresses the difference between the two sides of the inequation (implicit variable).

$$a_{11}X_1 + \cdots + a_{1n}X_n \leq b_1 \Leftrightarrow a_{11}X_1 + \cdots + a_{1n}X_n + X^e = b_1$$

● **Constraint of type (≥):** In this case, we have to subtract a positive quantity which is represented by a new variable called the slack variable. This new variable does not appear in the objective function, Inequations containing the "≥" type of inequality would be:

$$a_{11}X_1 + \cdots + a_{1n}X_n \geq b_1 \Leftrightarrow a_{11}X_1 + \cdots + a_{1n}X_n - X^e = b_1$$

During the first iteration of the Simplex method, it will be impossible to choose the base variables (the non-negativity condition is no longer verified). We need to add another X^R variable, called the artificial variable, which will appear in the objective function depending on the method used and by adding it to the corresponding constraint. As follows:

$$\text{✇}\ a_{11}X_1 + \cdots + a_{1n}X_n - X^e + X^R = b_1$$

● **Constraint type (=) :** despite being identities between the two sides of the equation, we also need to add artificial variables X^R in order to allow the choice of BV. Logically, this operation is false (adding a more value to an equality to one side !), although it is mandatory to start the simplex algorithm, but it will be necessary to ensure that these artificial variables have a nil value at the end of the solution. The constraint is as follows:

$$a_{11}X_1 + \cdots + a_{1n}X_n = b_1 \quad \text{✇}\ a_{11}X_1 + \cdots + a_{1n}X_n + X^R = b_1$$

When the artificial variables appear in the standard or canonical form of the problem, it will be evident that one of the following methods is used:

5.3 Two phases method

In the first phase, all the artificial variables are driven to zero (because of the penalty of M per unit for being greater than zero) in order to reach an initial basic feasible (BF) solution for the real problem. In the second phase, all the artificial variables are kept at zero (because of this same penalty) while the simplex method generates a sequence of BF solutions for the real problem that leads to an optimal solution. The two-phase method described next is a streamlined procedure for performing these two phases directly, without even introducing M explicitly.

The first phase aims to solve the auxiliary problem W' which is based on the artificial variables in order to bring the sum of the artificial variables to zero (in order to avoid mathematical inconsistencies). Once the first problem has been solved, and as long as the result is as expected, the resulting table is reorganized for use in the second phase of the original problem. Otherwise, the problem is not feasible, i.e. there is no solution, and it is not necessary to continue with the second phase.

To better understand this method, here is the LP:

Min $W = 2X_1 + X_2$	so, its standard form is :	**[*a new EF*]**
s.t. $X_1 + 3X_2 \geq 30$		**s.t.** $X_1 + 3X_2 - \mathbf{S_1} + \mathbf{R_1} = 30$
$4X_1 + 2X_2 \geq 40$		$4X_1 + 2X_2 - \mathbf{S_2} + \mathbf{R_2} = 40$
$X_1, X_2 \geq 0$		$X_1, X_2, R_1, R_2 \geq 0$

The first phase consists of solving the LP by modifying its objective function with a new minimisation function which is represented by the addition of the artificial variables values; i.e.: Min[2] $W'=R_1+R_2$ **the calculation rules are the same as for the simplex method except that for this phase, the objective is to bring the value of** W' **to zero.**

[2] In all cases, the new EF takes the direction of minimisation even if the original EF has the direction of maximization.

Phase 1

	MIN	X_1 ↓	X_2	S_1	S_2	R_1	R_2		
	Cj	0	0	0	0	1	1	**Bi**	**Cbi**
30/1=30	R_1	1	3	-1	0	1	0	30	**1**
40/4=**10**	←R_2	**4**	2	0	-1	0	1	40	**1**
	W'j	5	5	-1	-1	1	1	**70**	
	$\Delta W'$=C_j-W'_j	-5	-5	1	1	0	0		

LBV[3]**:** X_1

X_1: the EF value will be reduced by (-5)*10=-50

X_2: the EF value will be reduced by (-5)*10=-50

where the value 10 represents the smallest quotient resulting from the choice of the LBV.

LBV: R_2.

The pivot value = 4

	MIN	X_1	X_2 ↓	S_1	S_2	R_1	R_2	Bi	Cbi
20/(5/2)=**8**	←R_1	0	**5/2**	-1	1/4	1	-1/4	20	**1**
10/(1/2)=20	X_1	1	1/2	0	-1/4	0	1/4	10	**0**
	W'j	0	5/2	-1	1/4	**1**	**-1/4**	**20**	
	$\Delta W'$=C_j-W'_j	0	-5/2	1	-1/4	0	1/4		

EBV : X_2 the smallest negative value of the marginal cost.

LBV : R_1 the smallest quotient value.

the pivot value = 5/2.

	MIN	X_1	X_2	S_1	S_2	R_1	R_2	Bi	Cbi
	X_2	0	1	-2/5	1/10	2/5	-1/10	8	**0**
	X_1	1	0	1/5	-5/10	-1/5	3/10	6	**0**
	W'j	0	0	0	0	**0**	**0**	**0**	
	$\Delta W'$=C_j-W'_j	0	0	0	0	-1	-1		

☹Then the value of the FE W' is zero, which confirms that the LP has a solution and that the table opposite represents the initial table of the 2nd phase.

Phase 2

MIN	X_1	X_2	S_1	S_2	R_1	R_2		
Cj	2	1	0	0			**Bi**	**Cbi**
X_2	0	1	-2/5	1/10			8	**1**
X_1	1	0	1/5	-			6	**2**

☹The values in the table opposite are the same as in the previous table, except that the columns associated with the artificial

[3] The two variables X_1, X_2 have the same marginal cost, so the EF can be determined in an optional way, or the variable that minimizes the EF the most can be selected.

				5/10					variables are eliminated and the coefficients of the original EF (W) are used**.**
Wj	2	1	0	-1/2			**20**		
ΔW = C_j-W_j	0	0	0	1/2					

The stopping criterion is verified (in the case of Min, all the marginal costs are positive or nil), but we notice that a NBV variable (S1) with a marginal cost equal to zero; this means that there is a set of alternative optimal solutions[4].

5.4 Méthode du pénalité (Big-M)

The calculation rules remain the same as with the simplex method. Except that the effect of artificial variables (AVs) on the search for the EF's optimum must be eliminated. To do this, the VAs are added to the original EF as follows:

(1) The EF is to be maximized: all artificial variables are given the same coefficient (-M) in the EF.

(2) The EF is to be minimized: all artificial variables given the same coefficient (+M) in the EF.

Where : M represents a very large value (from which come the two words *Big-M* and *penalty*).

Exemple : case of two artificial variables

Min $W = 2X_1 + X_2$ => **Min** $W = 2X_1 + X_2 + MR_1 + MR_2$

Max $Z = 2X_1 + X_2$ => **Max** $Z = 2X_1 + X_2 - MR_1 - MR_2$

Taking the previous LP :

Min $W = 2X_1 + X_2$ so, its standard form is : **Min** $W = 2X_1 + X_2 + MR_1 + MR_2$

s.t. $X_1 + 3X_2 \geq 30$ **s.t.** $X_1 + 3X_2 - \mathbf{S_1} + \mathbf{R_1} = 30$

$4X_1 + 2X_2 \geq 40$ $4X_1 + 2X_2 - \mathbf{S_2} + \mathbf{R_2} = 40$

$X_1, X_2 \geq 0$ $X_1, X_2, R_1, R_2 \geq 0$

1	MIN	X_1	X_2↓	S_1	S_2	R_1	R_2		
Cj		2	1	0	0	+M	+M	**Bi**	**Cbi**
30/3=**10**	←R_1	1	**3**	-1	0	1	0	30	**+M**
40/2=20	R_2	4	2	0	-1	0	1	40	**+M**
W'j		5M	5M	-M	-M	M	M	**70M**	

4 Graphically, this set of points lies on the straight line bounded by the two points : A(8.6) and B(0.20) and the EF value remains stable (W=20).

$\Delta W'$=C_j-W'_j		2-5M	1-5M	M	M	0	0		
2	MIN	**X_1↓**	**X_2**	**S_1**	**S_2**	**R_1**	**R_2**	**Bi**	**Cbi**
10/(1/3)=30	**X_2**	1/3	1	-1/3	0	1/3	0	10	**1**
20/(10/3)=**6**	**←R_2**	**10/3**	0	2/3	-1	-2/3	1	20	**+M**
W'j		1/3+10/3 M	1	-1/3+2/3M	-M	**1/3-2/3M**	**M**	**10+20 M**	
$\Delta W'$=C_j-W'_j		$\frac{1}{3}+\frac{10}{3}M$	0	$\frac{1}{3}-\frac{2}{3}M$	M	$-\frac{1}{3}+\frac{5}{3}M$	0		
3	MIN	**X_1**	**X_2**	**S_1**	**S_2**	**R_1**	**R_2**	Bi	Cbi
	X_2	0	1	-2/5	1/10	2/5	-1/10	8	**1**
	X_1	1	0	1/5	-3/10	-1/5	3/10	6	**2**
W'j		2	1	0	-1/2	**0**	**1/5**	**20**	
$\Delta W'$=C_j-W'_j		0	0	0	1/2	M	M-1/5		

All marginal costs are positive or nil (stopping criterion is verified) but the marginal cost of the nonbasic variable S_1 is nil, which is the case for alternative optimal solutions.

5.5 Exercises

Exercise 5.1 (reflection)

Given the following LP :

- Check whether the LP has a solution?
- If so, determine it without solving the LP.

$$\textbf{Min}\ W = 3X_1 + 2X_2$$
$$\textbf{s.c.}\ X_1 + X_2 \geq 2$$
$$2X_1 + 3X_2 \leq 4$$
$$X_1, X_2 \geq 0 \qquad [A]$$

Exercise 5.2

A maximization problem is presented in the form of the following LP:

- Solve this LP using the penalty method.
- Explain economically the solution obtained.

$$\textbf{Max}\ z = 4X_1 + 5X_2$$
$$\textbf{s.c.}\ -X_1 + 3X_2 \leq 2$$
$$X_1 + X_2 \geq 2$$
$$X_2 = 3$$
$$X_1, X_2 \geq 0 \qquad [B]$$

Exercise 5.3

We want to solve the LP shown opposite using the two-phase method:

- Determine the condition for choosing the

$$\textbf{Max}\ z = 2X_1 + 3X_2$$
$$\textbf{s.c.}\ 5/2X_1 + 2X_2 \leq 5$$
$$5X_1 + 4X_2 \geq 20$$
$$X_1, X_2 \geq 0 \qquad [C]$$

entering basic variable for the first phase. Justify your choice ?

- Solve the LP.
- How does the value of the EF vary (does it increase or decrease)? Why?

Exercise 5.4

$$\begin{aligned} \textbf{Max } & z = 2X_1 + X_2 \\ \textbf{s.c. } & 3X_1 + 2X_2 \le 6 \\ & 2X_1 + 3X_2 \ge 12 \\ & X_1, X_2 \ge 0 \qquad [D] \end{aligned}$$

- Deduce that the LP [D] has no solution?
- Check the answer graphically?
- Solve LPs [D] and [E] using the penalty or two-phase methods.

-

$$\begin{aligned} \textbf{Min } & z = 2X_1 + 2X_2 + 6X_3 \\ \textbf{s.c. } & 2X_1 - 2X_2 - X_3 \le 1 \\ & X_1 + 2X_2 \ge 1 \\ & X_1, X_2, X_3 \ge 0 \qquad [E] \end{aligned}$$

Exercise 5.5

Solve the following LPs using the two-phase method and then the penalty method:

$$\begin{aligned} \textbf{Max } & z = 4X_1 + 14X_2 \\ \textbf{s.c. } & 2X_1 + 7X_2 \le 21 \\ & 7X_1 + 2X_2 \ge 21 \\ & X_1, X_2 \ge 0 \end{aligned}$$

$$\begin{aligned} \textbf{Max } & z = X_1 - X_2 + X_3 \\ \textbf{s.c. } & 2X_1 - X_2 + 2X_3 \le 4 \\ & 2X_1 - 3X_2 + X_3 \le -5 \\ & -X_1 + X_2 - 2X_3 \le -1 \\ & X_1, X_2, X_3 \ge 0 \end{aligned}$$

Solutions

Solution 5.1 :

$$\begin{aligned} \textbf{Min } & W = 3X_1 + 2X_2 \\ \textbf{s.c. } & X_1 + X_2 \ge 2 \\ & 2X_1 + 3X_2 \le 4 \\ & X_1, X_2 \ge 0 \qquad [A] \end{aligned}$$

● If the solution to the LP shown here exists, it lies in the first quarter of the representation space. This is justified by the fact that we have two variables X_1, X_2 which have positive values (the non-negativity condition). So all we need to do is to check whether the constraints intersect in the first quarter:

$$X_1 + X_2 = 2$$
$$2X_1 + 3X_2 = 4 \quad \text{cela donne : } X_1 = 2,\ X_2 = 0$$

● Since you are asked to determine the optimal solution without solving the LP, we'll show you this trick: the slopes of the lines representing the two constraints are respectively P_1=-1, P_2=-2/3 and since (1) the two constraints are opposite, (2) the constraint with the greater slope value is expressed as an inequality of type (≤), (3) the intersection is located on the x-axis; we deduce that the optimal solution is indeed the point of intersection ($X_1 = 2$, $X_2 = 0$).

Solution 5.2 :

● The standard form of the LP is given by :

$$\begin{aligned} \textbf{Max } & z = 4X_1 + 5X_2 - MR_1 - MR_2 \\ \textbf{s.c. } & -X_1 + 3X_2 + S_1 = 2 \\ & X_1 + X_2 - S_2 + R_1 = 2 \\ & X_2 + R_2 = 3 \\ & X_1, X_2, S_1, S_2, R_1, R_2 \ge 0 \end{aligned}$$

1	MAX	**X_1**	**X_2↓**	**S_1**	**S_2**	**R_1**	**R_2**		
Cj		4	5	0	0	-M	-M	**Bi**	**Cbi**
2/3	←**S_1**	-1	**3**	1	0	0	0	2	**0**
2/1=2	**R_1**	1	1	0	-1	1	0	2	**-M**
3/1=3	**R_2**	0	1	0	0	0	1	3	**-M**
Zj		**-M**	**-2M**	**0**	**M**	**-M**	**-M**	**-5M**	
ΔZ=**C_j-Z_j**		4+M	5+2M	0	-M	0	0		
2	MAX	**X_1↓**	**X_2**	**S_1**	**S_2**	**R_1**	**R_2**	**Bi**	**Cbi**
/	**X_2**	-1/3	1	1/3	0	0	0	2/3	**5**
(4/3)/(4/3)=**1**	←**R_1**	**4/3**	0	-1/3	-1	1	0	4/3	**-M**
(7/3)/(1/3)=7	**R_2**	1/3	0	-1/3	0	0	1	7/3	**-M**
Zj		**-5/3+5/3M**	**5**	**5/3+2/3M**	**M**	**-M**	**-M**	$\frac{10}{3}-\frac{11}{3}M$	
ΔZ=**C_j-Z_j**		$\frac{17}{3}+\frac{5}{3}M$	0	$-\frac{5}{3}-\frac{2}{3}M$	-M	0	0		

3	MAX	**X_1**	**X_2**	**S_1**	**S_2↓**	**R_1**	**R_2**	Bi	Cbi
/	**X_2**	0	1	1/4	-1/4	2/5	0	1	**5**
/	**X_1**	1	0	-1/4	-3/4	-1/5	0	1	**4**
2/(1/4)=**8**	←**R_2**	0	0	-1/4	**1/4**	-1/4	1	2	**-M**
Zj		4	5	$\frac{1}{4}+\frac{M}{4}$	$\frac{-17}{4}-\frac{M}{4}$	$\frac{17}{4}+\frac{M}{4}$	**-M**	**9-2M**	
ΔZ=**C_j-Z_j**		0	0	$-\frac{1}{4}-\frac{M}{4}$	$\frac{17}{4}+\frac{M}{4}$	$-\frac{17}{4}-\frac{5M}{4}$	0		
4	MAX	**X_1**	**X_2**	**S_1↓**	**S_2**	**R_1**	**R_2**	Bi	Cbi
/	**X_2**	0	1	0	0	0	0	3	**5**
/	**X_1**	1	0	-1	0	0	0	7	**4**
/	**S_2**	0	0	-1	1	-1	4	8	**0**
Zj		4	5	-4	0	**0**	**0**	**43**	
ΔZ=**C_j-Z_j**		0	0	4	0	-M	-M		

In Table 4, variable S_1 is chosen as the entering basic variable but it is impossible to choose the leaving basic variable, so this is the case of an **unbounded solution**.

● The solution obtained is expressed in seven units of X_1 and three units of X_2, giving a profit of 43. But in reality this solution represents the smallest value of the EF, i.e. since the space of feasible solutions is not bounded on the high side, there may be other solutions which give a larger value of the EF. Consequently, the decision-maker alone has the capacity (power, experience, etc.) to determine the maximum level that can be reached depending on what already exists.

Solution 5.3 :

To use the two-phase method, we define an economic function represented by the sum of the artificial variables. To determine the condition for choosing the entranting basic variable for the first phase, it should be noted that the direction of preference of the new function will be minimized in all cases, even if the original EF is to be maximized (as in this case). Consequently, the entranting basic variable is the one with the smallest negative value among the negative values of the marginal costs.

This condition is justified by the fact that the objective of the first phase is to br the artificial variables values to zero, i.e. the value of the new EF will be zero, which is why we deduce that the LP receives a solution.

- La the standard form is :

We define a new EF expressed by the sum of the artificial variables.

Min $W' = R_1$

s.c. $5/2X_1 + 2X_2 + S_1 = 5$

$5X_1 + 4X_2 - S_2 + R_1 = 20$

$X_1 , X_2 , , S_1, S_2 , R_1 \geq 0$ [C]

1	MIN	X_1	$X_2\downarrow$	S_1	S_2	R_1		
Cj		0	0	0	0	1	**Bi**	**Cbi**
5/(5/2)=**2**	$\leftarrow S_1$	**5/2**	**2**	1	0	0	5	**0**
20/5=4	R_1	5	4	0	-1	1	20	**1**
W'j		**5**	**4**	**0**	**-1**	**1**	**20**	
$\Delta W'$=C_j-W'_j		-5	-4	0	1	0		
2	MIN	$X_1\downarrow$	X_2	S_1	S_2	R_1	**Bi**	**Cbi**
/	X_2	1	4/5	2/5	0	0	2	**0**
/	R_1	**0**	-1	-2/5	-1	1	10	**1**
Wj		**0**	**-1**	**-2/5**	**-1**	**1**	**10**	
$\Delta W'$=C_j-W'_j		0	1	2/5	1	0		

All the marginal costs are positive or nil, so there are no variables that minimize the EF; this is the end of the first phase. But we notice that there is an artificial variable with value zero (R_1=10), which implies that the LP has no solution.

For the first phase, the value of the EF is decreasing (20 then 10), because the algorithm of this method aims to make all the artificial variables zero in the first phase in order to correct the suppliant that was added at the start.

Chapter 6

6 Duality theory

6.1 Introduction

One of the most important discoveries in the early development of linear programming was the concept of duality and its many important ramifications. This discovery revealed that every linear programming problem has associated with it another linear programming problem called the dual. The relationships between the dual problem and the original problem (called the primal) prove to be extremely useful in a variety of ways.

6.2 Economic illustration [5]

(a) A family uses 6 food products as a source of vitamin A and C

	Products (unit/kg)						Demand (unit)
	1	2	3	4	5	6	
Vitamin A	1	0	2	2	1	2	9
Vitamin C	0	1	3	1	3	2	19
Price per Kg	35	30	60	50	27	22	

The aim is to satisfy vitamin requirements at minimum total cost.

(b) A product salesman wants to convince the family to buy his products.

What selling price W_A and W_C?

- to be competitive (with food products)
- and maximize profit.

The situation in cases 'a' and 'b' can be generalized as follows:

Primal problem (product demand): what quantity Xi of resource i should be purchased to satisfy demand at minimum cost?

$$(C_j * X_i) \qquad \text{S.T} : a_{ij}X_i \geq b_i\ C_j \quad \forall j$$

Dual problem (product seller): at what price should the products be offered to maximize profit while remaining competitive?

$$((b_i \quad * Y_j) \qquad \text{S.T} : a_{ji}Y_j \leq \ C_j \quad \forall i$$

[5] Example taken from "Cours de recherche opérationnelle" Nadia Brauner

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6.3 Transformation steps

1) The preference sens of the EF will be reversed (Max□Min and vice versa).

2) The coefficients of the FE of the primal represent the independent terms of the constraints (bi).

3) The values of the independent terms of the primal represent the coefficients of the FE of the dual.

4) The transpose of the matrix of coefficients associated with the primal constraints represents the coefficients of the dual variables.

5) The constraints of the dual will be represented by inequalities (≤, ≥) or equalities (=) depending on the non-negativity condition of the primal.

6) The condition of non-negativity of the dual is based on the type (≤, ≥, =) of the constraints of the primal.

Then the links between the primal program and its dual are as follows:

Primal program	Dual program
Maximization	Minimization
n basic variables	*n* functional constraints
m functional constraints	*m* basic variables
Aij	A^T =Aji
Cj	Bi
Bi	Cj
constraints≤	variables ≥
variables ≥	constraints≥ (as inequalities)
contraintes = (as equalities)	variables <>0
variables libres <>0	contraintes = (as equalities)

Example 1 : let the following LP:

Min $W = 4X_1 + X_2$
s.c. $30X_1 + 10X_2 \geq 100$
$125X_1 + 12X_2 \geq 200$
$120X_1 + 15X_2 \geq 150$
$X_1 , X_2 \geq 0$

The formulation of the dual model is as follows :

Min $W = 4X_1 + X_2$ **PRIMAL**
s.c. $30X_1 + 10X_2 \geq 100$ $\ldots Y_1$
$125X_1 + 12X_2 \geq 200$ $\ldots Y_2$
$120X_1 + 15X_2 \geq 150$ $\ldots Y_3$
$X_1 , X_2 \geq 0$

- The number of basic variables in the dual is three (Y_1, Y_2, Y_3) □ it represent the primal constraints number.
- X_1 and X_2 are positive □ all dual constraints are inequalities of type (≤ since all primal constraints are of type ≥)
- Dual EF is to maximize (since primal EF is to minimize.
- The primal EF coefficients are the values of dual independent terms (4, 1).

Max $Z = 100Y_1 + 200Y_2 + 150Y_3$ **DUAL**
s.c. $30X_1 + 125Y_2 + 120Y_3 \leq 4$ $\ldots X_1$
$10X_1 + 12Y_2 + 15Y_3 \leq 1$ $\ldots X_2$
$Y_1 , Y_2, Y_3 \geq 0$

- The primal independent terms represent dual EF coefficients (100, 200, 150).
- The constraint coefficients of the dual are obtained from the transpose matrix of constraint coefficients of the primal.
- All primal constraints are inequalities than the non-negativity condition is imposed for all dual variables.

Example 2 : Let the following LP:

$$\textbf{Max } Z = X_1 + X_2 - X_3 - X_4 \quad \textbf{PRIMAL}$$
$$\begin{aligned} \textbf{s.c.}\quad & 3X_1 - 2X_2 + X_3 + 5X_4 \leq 18 \\ & 5X_1 + 6X_3 = 20 \\ & X_1 - X_2 + 4X_3 + X_4 \geq 9 \\ & X_1 , X_2 , X_4 \geq 0 \\ & X_3 < > 0 \ \text{(signe libre)} \end{aligned}$$

Before we begin, we need to consider these points:

- First, we need to standardize the inequality of constraints in type ≤ since the EF is Max.
- The second constraint is an equation
- The variable X_3 may be positive or negative.

So, here are the steps of the transformation:

$$\textbf{Max } Z = X_1 + X_2 - X_3 - X_4 \quad \textbf{PRIMAL}$$
$$\begin{aligned} \textbf{s.c.}\quad & 3X_1 - 2X_2 + X_3 + 5X_4 \leq 18 && \ldots Y_1 \\ & 5X_1 + 6X_3 = 20 && \ldots Y_2 \\ & - X_1 + X_2 - 4X_3 - X_4 \leq -9 && \ldots Y_3 \\ & X_1 , X_2 , X_4 \geq 0 \\ & X_3 < > 0 \ \text{(en signe libre)} \end{aligned}$$

we have three dual basic variables (Y_1, Y_2, Y_3) it is the primal functional constraints number.

- we have four dual functional constraints it is the primal basic variables number.
- X_1, X_2 and X_4 are positive values all corresponding functional constraints in the dual are represented by inequalities of type (≥ since the primal functional constraints are of type ≤)
- Dual EF is to minimize (since primal EF is to maximize.
- The dual EF coefficients are the independants terms in the primal functional constraints (18, 20, -9).
- The dual independants terms are coefficients of the primal EF (1, 1, -1, -1).
- The dual constraint coefficients are obtained from the transpose matrix of primal constraint coefficients.
- The third constraint of the dual is an equation because the variable X_3 of the primal is in free sign

$$\textbf{Min } W = 18Y_1 + 20Y_2 - 9Y_3 \quad \textbf{DUAL}$$
$$\begin{aligned} \textbf{s.c.}\quad & 3Y_1 + 5Y_2 - Y_3 \geq 1 && \ldots X_1 \\ & -2Y_1 + Y_3 \geq 1 && \ldots X_2 \\ & Y_1 + 6Y_2 - 4Y_3 = -1 && \ldots X_3 \\ & 5Y_1 - Y_3 \geq -1 && \ldots X_4 \\ & Y_1 , Y_3 \geq 0 \end{aligned}$$

- The variable Y_2 is in free sign since the corresponding constraint in the primal is an equation.

- Les variables Y_1, Y_2 sont non-négatives puisque leurs contraintes correspondantes dans le primal sont des inéquations.
- The variables Y_1, Y2 are non-negative since their corresponding constraints in the primal are inequalities.

6.4 Duality conditions

Depending on their general form, LPs can be classified into three groups:

6.4.1 First group

6.4.1.1 Presentation

Le the LP :

Max Z = $\sum_{j=1}^{n} C_j X_j$
s.c. $\sum_{j=1}^{n} a_{ij} X_j = b_i$; $i = 1, \dots, m$
$X_j \geq 0$; $j = 1, \dots, n$

PRIMAL (I)

Min W = $\sum_{i=1}^{m} b_i Y_i$
s.c. $\sum_{i=1}^{m} a_{ji} Y_i \geq C_j$; $j = 1, \dots, n$
$Y_i \leq\geq 0$; $i = 1, \dots, m$

DUAL (II)

These two LPs are dual and as a result, on the one hand, if there is a solution to program I, a solution to program II is found by writing its m inequalities (which correspond to the VBs) in the form of equations, and on the other hand, if there is a solution to program II, a solution to program I is found by determining a basis grouping m variables corresponding to the m inequalities expressed in the form of equalities.

Converting inequalities into equations using the slack variables gives:

Max Z = $\sum_{j=1}^{n} C_j X_j$
s.c. $\sum_{j=1}^{n} a_{ij} X_j = b_i$; $i = 1, \dots, m$
$X_j \geq 0$; $j = 1, \dots, n$

PRI
MAL (I)

Min W = $\sum_{i=1}^{m} b_i Y_i$
s.c. $\sum_{i=1}^{m} a_{ji} Y_i - Y_{m+j} = C_j$; $j = 1, \dots, n$
$Y_i \leq\geq 0$, $Y_{m+j} \geq 0$; $i = 1, \dots, m$

DUAL (II) under his standard form

So the duality conditions are the following:

$\sum_{j=1}^{n} X_j Y_{m+j} = 0$ (1)
$\sum_{j=1}^{n} a_{ij} X_j = b_i$ (2)
$\sum_{i=1}^{m} a_{ji} Y_i - Y_{m+j} = C_j$ (3)
$X_j \geq 0$ (4)
$Y_{m+j} \geq 0$ (5)

The first condition results from the fact that if the slack variable value Y_{m+j} is zero (probably a non-basic variable), the variable X_j will not be zero (probably a base variable) and vice versa. The other conditions result from the two LPs (the primal and its dual).

The same conditions apply in the following case:

Min W = $\sum_{j=1}^{n} C_j X_j$	**Max** Z = $\sum_{i=1}^{m} b_i Y_i$
s.c. $\sum_{j=1}^{n} a_{ij} X_j = b_i$; $i = 1, \dots, m$	**s.c.** $\sum_{i=1}^{m} a_{ji} Y_i \le C_j$; $j = 1, \dots, n$
$X_j \ge 0$; $j = 1, \dots, n$	$Y_i <> 0$; $i = 1, \dots, m$
PRIMAL (I)	**DUAL (II)**

6.4.1.2 Exercises and solutions

Exercise 6.1 :

A primal program and its dual are given as follows:

Max Z = $5X_1 + 3X_2 - 4X_3$	**Min W** = $2Y_1 + 7Y_2$
s.c. $2X_1 + X_2 - X_3 = 2$ $\quad Y_1$	**s.c.** $2Y_1 + Y_2 \ge 5$ $\quad X_1$
$X_1 + 2X_2 + 3X_3 = 7$ $\quad Y_2$	$Y_1 + 2Y_2 \ge 3$ $\quad X_2$
$X_i \ge 0$: $\forall i$	$-Y_1 + 3Y_2 \ge -4$ $\quad X_3$
PRIMAL (I)	**DUAL (II)**

If the optimal primal solution is given by:

$Z^* = 5$; $X_1 = 0$ (NBV); $X_2 = 3$ (BV); $X_3 = 1$ (BV)

Deduce the optimal solution from its dual.

Solution 6.1 :

The dual standard for is:

$$\begin{aligned} &\text{Min W} = 2Y_1 + 7Y_2 \\ &\textbf{s.c.}\ \ 2Y_1 + Y_2 - \boldsymbol{Y_3} = 5 \qquad X_1 \\ &\quad Y_1 + 2Y_2 - \boldsymbol{Y_4} = 3 \qquad X_2 \\ &\quad -Y_1 + 3Y_2 - \boldsymbol{Y_5} = -4 \qquad X_3 \\ &\quad Y_i \ge 0 \quad ; \forall i \end{aligned}$$

According to the optimal solution of the primal model, the variables X_2 and X_3 are basic variables, so according to the first duality condition the slack variables of the dual model associated with the constraints corresponding to the variables X_2 and X_3 are nonbasic variables (NBV) so:

$Y_4 = Y_5 = 0$ (1). Then: $\begin{cases} Y_1 + 2Y_2 = 3 \\ -Y_1 + 3Y_2 = -4 \end{cases}$ so: $Y_1 = \frac{11}{3}$; $Y_2 = -\frac{1}{3}$

By replacing the two values in the first constraint:

$$2\left(\frac{11}{3}\right) + \left(\frac{-1}{3}\right) - Y_3 = 5 \implies Y_3 = 2 \quad ; \quad W^* = 2\left(\frac{11}{3}\right) + 7\left(\frac{-1}{3}\right) = 5.$$

The optimal solution of the dual is: $W^* = 5$; $Y_1 = \frac{11}{3}$; $Y_2 = -\frac{1}{3}$; $Y_3 = 2$; $Y_4 = Y_5 = 0$

The values of the slack variables mean that the available quantities in constraints 2 and 3 are fully exploited, but there are still two (02) unconsumed units in the first constraint.

Exercise 6.2:

A primal program and its dual are given as follows:

PRIMAL (I)		DUAL (II)	
Min W = $3X_1 + 8X_2 + 5X_3$		**Max Z** = $Y_1 + Y_2$	
s.c. $3X_1 + X_2 - 5X_3 = 1$	Y_1	**s.c.** $3Y_1 + Y_2 \leq 3$	X_1
$X_1 + 2X_2 + X_3 = 1$	Y_2	$Y_1 + 2Y_2 \leq 8$	X_2
$X_i > 0 \quad : \forall i$		$-5Y_1 + Y_2 \leq 5$	X_3

The optimal solution of the dual is given by :

$Y_1 = -\frac{1}{4}$ (BV); $Y_2 = \frac{15}{4}$ (VB); $Y_3 = Y_5 = 0$ (NBV)

- Give the values of $\boldsymbol{Z^*}$ **et** $\boldsymbol{Y_4}$.
- Deduce the optimal solution of the primal model.

Solution 6.2:

By numerical application we find: $Z^* = -\frac{1}{4} + \frac{15}{4} = \frac{7}{2}$

For the second constraint: $Y_1 + 2Y_2 + Y_4 = 8 \iff Y_4 = 8 - Y_1 - 2Y_2 \iff Y_4 = \frac{3}{4}$

To deduce the optimal solution, we have :

Since the variables Y_3 et Y_5 are nonbasic variables, then X_1 et X_3 are basic variables. Thus, since the number of basic variables is equal to the number of constraints; the case of the primal is two (02) variables; then the variable X_2 will be a nonbasic variable.

We replace $X_2 = 0$ in the two constraints of primal model:

$$\begin{cases} 3X_1 - 5X_3 = 1 \\ X_1 + X_3 = 1 \end{cases} \Rightarrow X_1 = \frac{3}{4} \;;\; X_3 = \frac{1}{4} \quad \text{et} \quad W^* = 3\left(\frac{3}{4}\right) + 8(0) + 5\left(\frac{1}{4}\right) = \frac{7}{2}$$

6.4.2 Seconde group

6.4.2.1 Presentation

Let the LP model:

PRIMAL (I)		DUAL (II)	
Max Z = $\sum_{j=1}^{n} C_j X_j$		**Min** W = $\sum_{i=1}^{m} b_i Y_i$	
s.c. $\sum_{j=1}^{n} a_{ij} X_j \leq b_i$	$; i = 1, \dots, m$	**s.c.** $\sum_{i=1}^{m} a_{ji} Y_i \geq C_j$	$; j = 1, \dots, n$
$X_j \geq 0$	$; j = 1, \dots, n$	$Y_i \geq 0$	$; i = 1, \dots, m$

Transforming the inequalities into equalities using the slack variables gives:

Max Z = $\sum_{j=1}^{n} C_j X_j$
s.c. $\sum_{j=1}^{n} a_{ij} X_j + X_{n+i} = b_i$; $i = 1, \dots, m$
$X_j \geq 0, X_{n+i} \geq 0$; $i = 1, \dots, n$

PRIMAL (I)

Min W = $\sum_{i=1}^{m} b_i Y_i$
s.c. $\sum_{i=1}^{m} a_{ji} Y_i - Y_{m+j} = C_j$; $j = 1, \dots, n$
$Y_i \geq 0, Y_{m+j} \geq 0$; $i = 1, \dots, m$

DUAL (II) standard form

The duality conditions are as follows:

$\sum_{j=1}^{n} X_j Y_{m+j} = 0$ (1)
$\sum_{i=1}^{m} Y_i X_{n+i} = 0$ (2)
$\sum_{j=1}^{n} a_{ij} X_j + X_{n+i} = b_i$ (3)
$\sum_{i=1}^{m} a_{ji} Y_i - Y_{m+j} = C_j$ (4)
$X_j \geq 0, Y_i \geq 0$

The first two conditions mean that if the slack variable Y_{m+j} is nonbasic variable, the variable X_j will be a basic variable and vice versa. We also deduce that if the slack variable X_{n+i} is nonbasic variable, the variable Y_i will be a basic variable and vice versa.

6.4.2.2 Exercises and solutions

Exercise 6.3:

Let the LP as follows:

PRIMAL (I)

Max Z = $2X_2 + X_3$
s.t. $X_1 + X_2 + X_3 \leq 48$ Y_1
$-7X_1 + 6X_2 - X_3 \leq 0$ Y_2
$X_i \geq 0$; $\forall i$

- Write its dual model.

The primal optimal solution is given by:

$Z^* = \frac{384}{7}$; $X_1 = 0$ (NBV); $X_2 = \frac{48}{7}$ (BV); $X_3 = \frac{288}{7}$ (BV)

- Deduce the optimal solution of its dual model.

Solution 6.3:

Applying the transformation rules from primal to dual, here is the dual model of the given program:

Min W = $48Y_1$
s.c. $Y_1 - 7Y_2 \geq 0$ X_1
$Y_1 + 6Y_2 \geq 2$ X_2
$Y_1 - Y_2 \geq 1$ X_3

Writing the dual in its standard form gives:

Min W = $48Y_1$
s.c. $Y_1 - 7Y_2 - \boldsymbol{Y_3} = 0$ X_1
$Y_1 + 6Y_2 - \boldsymbol{Y_4} = 2$ X_2
$Y_1 - Y_2 - \boldsymbol{Y_5} = 1$ X_3
$Y_i \geq 0$; $\forall i$
$Y_3 \geq 0, Y_4 \geq 0, Y_5 \geq 0$

According to the first two duality conditions:

Given that X_2, X_3 are basic variables, this means that the slack variables Y_4, Y_5 will be NBV in the optimal solution, i.e. we just need to solve the following system of equations:

$$\begin{cases} Y_1 - 7Y_2 - \boldsymbol{Y_3} = 0 \\ Y_1 + 6Y_2 = 2 \\ Y_1 - Y_2 = 1 \end{cases} \qquad \begin{cases} \boldsymbol{Y_3} = 1/7 \\ \Leftrightarrow \boldsymbol{Y_1} = 8/7 \\ \boldsymbol{Y_2} = 1/7 \end{cases} \quad \Rightarrow \quad W^* = 48\left(\frac{8}{7}\right) = \frac{384}{7}$$

so, all duality conditions are verified.

6.4.3 Third group

6.4.3.1 Presentation

Consider the LP shown opposite:

The constraints are in the form of equalities and inequalities.

PRIMAL (I)

$$\textbf{Max } Z = \sum_{j=1}^{k} C_j X_j + \sum_{j=k+1}^{n} C_j X_j$$

$$\textbf{s.c.} \quad \sum_{j=1}^{k} a_{ij} X_j + \sum_{j=k+1}^{n} a_{ij} X_j \leq b_i \;\; ; i = 1, \dots, L$$

$$\sum_{j=1}^{k} a_{ij} X_j + \sum_{j=k+1}^{n} a_{ij} X_j = b_i \;\; ; i = L+1, \dots, m$$

$$X_j \geq 0 \qquad ; j = 1, \dots, k$$

$$X_j \leq\geq 0 \qquad ; j = k+1, \dots, n$$

Its dual is:

The constraints are also expressed in the form of equalities and inequalities BUT on the basis of the nonnegative condition of the primal.

DUAL (II)

$$\textbf{Min } W = \sum_{i=1}^{L} b_i Y_i + \sum_{i=L+1}^{m} b_i Y_i$$

$$\textbf{s.c.} \quad \sum_{i=1}^{L} a_{ji} Y_i + \sum_{i=L+1}^{m} a_{ji} Y_i \geq C_j \;\; ; j = 1, \dots, k$$

$$\sum_{i=1}^{L} a_{ji} Y_i + \sum_{i=L+1}^{m} a_{ji} Y_i = C_j \;\; ; j = k+1, \dots, n$$

$$Y_i \geq 0 \qquad ; i = 1, \dots, L$$

$$Y_i \leq\geq 0 \qquad ; i = L+1, \dots, m$$

Programs I and II are dual and solving one of them leads to the solution of the other. Note that when writing the constraints of the dual, constraint j is written as an equality if the corresponding variable has a free sign in the primal (no nonnegative condition), otherwise constraint j is written as an inequality.

We can also see that a variable in the dual is no longer subject to the nonnegative constraint if it corresponds to a constraint in the form of equality in the primal. On the other hand, the variable is subject to the nonnegative constraint if it corresponds to a constraint in the form of an inequality.

6.4.3.2 Exercises and solutions

Exercice 6.4 :

Let the LP model as follows:

Write its dual model.

The dual optimal solution is given by:

$Y_2 = 3$ (BV); $Y_3 = 1$ (BV); $Y_5 = 6$ (BV)

$$\textbf{Min } W = 2X_1 - 4X_2 + 11X_3$$

$$\textbf{s.c.} \quad 2X_1 + X_2 + X_3 \geq 1 \qquad Y_1$$

$$X_1 - X_2 + X_3 \geq 7 \qquad Y_2$$

$$-X_1 - X_2 + 2X_3 = 1 \qquad Y_3$$

Deduce the optimal solution of this model.

Solution 6.4 :

The dual program is as follows:

$$\begin{aligned} &\textbf{Max } Z = Y_1 + 7Y_2 + Y_3 \\ &\textbf{s.c.} \quad 2Y_1 + Y_2 - Y_3 \le 2 \qquad X_1 \\ &\quad Y_1 - Y_2 - Y_3 = -4 \qquad X_2 \\ &\quad Y_1 + Y_2 + 2Y_3 \le 11 \qquad X_3 \\ &\quad Y_i \ge 0 \quad ; \forall i = 1,2 \end{aligned}$$

From the given solution, we can deduce that the variables $\boldsymbol{Y_1}, \boldsymbol{Y_4}$ (the slack variables) are nonbasic variables.

The standard form of the given primal program is :

$$\begin{aligned} &\textbf{Min } W = 2X_1 - 4X_2 + 11X_3 \\ &\textbf{s.c.} \quad 2X_1 + X_2 + X_3 - \boldsymbol{X_4} = 1 \qquad Y_1 \\ &\quad X_1 - X_2 + X_3 - \boldsymbol{X_5} = 7 \qquad Y_2 \\ &\quad -X_1 - X_2 + 2X_3 = 1 \qquad Y_3 \\ &\quad X_j \ge 0 \quad ; \forall j = 1,3 \end{aligned}$$

Y_2 is a basic variable $\Leftrightarrow$ $\mathrm{X}_5 = 0$ (Nonbasic variable)

Y_5 is a basic variable $\Leftrightarrow$ $\mathrm{X}_3 = 0$ (Nonbasic variable),

(N.A) :

$$\begin{cases} 2X_1 + X_2 - \boldsymbol{X_4} = 1 \\ X_1 - X_2 = 7 \\ -X_1 - X_2 = 1 \end{cases} \quad \text{Then} \quad \begin{cases} \boldsymbol{X_4} = 1 \\ X_2 = -4 \\ X_1 = 3 \end{cases}$$

so : $W^* = 2(3) - 4(-4) + 11(0) = 22$

Chapter 7

7 Dual Simplex method

7.1 Introduction

The final solution of a linear program can be determined if two conditions are verified:

- The solution must be feasible, i.e. it must satisfy the nonnegative constraint.
- The solution must be optimal, i.e. the stopping criterion is verified (no entering basic variable).

In some cases, the stopping condition is verified (optimality) but the solution is no longer feasible due to at least one negative value of one of the variables if the model does not accept negative values. In this particular case, the simplex algorithm is limited since one of the conditions for its application is that the independent coefficients associated with the constraints must be positive. The implication of the weakness of the simplex algorithm to other principles has given rise to a new method called Dual Simplex.

The dual simplex method treats the LP from two sides: the primal side (to achieve the stopping criterion based on marginal costs) and the dual side (to ensure that the base variables do not receive values that are in conflict with the nonnegative constraint). Consequently, the user of this method must initially choose the direction of treatment, i.e. : Primal then dual or dual then primal, depending on the case.

7.2 Essence of dual simplex

(a) *Primal then dual:* the program is processed on both sides until the stopping criterion based on marginal costs is satisfied first and then the criterion of the feasibility of the solution (which concerns the bi values).

(b) *Dual then primal:* this is the opposite of the previous case, i.e. the program is processed on both sides until the feasibility criterion is reached first and the stopping criterion based on marginal costs is reached second.

Using the following linear program as an example, to explain dual method steps:

Min W = $X_1 + 3X_2$
s.c. $X_1 - 2X_2 \leq 1$
$2X_1 + 3X_2 \geq 3$
$X_1, X_2 \geq 0$

The standard form is
The aim is to add the slack variables without worrying about negative values of Bi

Min W = $X_1 + 3X_2$
s.c. $X_1 - 2X_2 + \boldsymbol{X_3} = 1$
$-2X_1 - 3X_2 + \boldsymbol{X_4} = -3$
$X_1, X_2 \geq 0$

The first simplex table is given as follows:

MIN		**X₁↓**	**X₂**	**S₁**	**S₂**		
Cj		1	3	0	0	Bi	Cbi
	S_1	1	-2	1	0	1	**0**
	←**S_2**	**<u>-2</u>**	-3	0	1	-3	**0**
Wj		0	0	0	0	**0**	
$\Delta W = C_j - W_j$		1	3	0	0		

Remember that we treat a minimization problem and we can see that the stopping criterion has already been verified (the marginal costs $\Delta \boldsymbol{W}$ are all positive or zero (there is no question of treating the primal side of the program). We can also see that the value of the slack variable S2 is negative (it must necessarily be positive according to the nonnegative condition), so it's clear that we first need to solve the dual side of the problem (eliminate the negative values of Bi that contradict the nonnegative constraint).

In the case of dual-primal -particularly for the dual part- we need to determine the leaving basic variable (LBV) first and then the entering basic variable. To do this, we must follow a specific logic:

The leaving basic variable is determined by: $Max\,\{b_i;\,\forall b_i \leq 0\}$ the variable corresponding to the largest value of bi must be chosen from the negative values only. This criterion remains valid for the maximization and minimization cases.

The entering basic variable is determined by: $Min\left\{\left|\frac{\Delta w}{a_{kj}}\right|;\,\forall a_{kj} < 0\right\}$ we must choose the variable corresponding to the smallest quotient in absolute value of the marginal costs over the negative values of the line corresponding to the leaving basic variable. This criterion is valid for the maximization and minimization cases.

. In the case of this exercise, the LBV is X_1 (a single negative value in bi) and the EBV is S_2 ($Min\left\{\left|\frac{1}{-2}\right|,\left|\frac{3}{-3}\right|\right\} = \left\{\frac{1}{2}\right\}$), so the pivot value is (-2).

Applying the same rules for moving from one solution (table) to another presented in Chapter 4 (the simplex method), here is the following simplex table:

	MIN	X_1	$X_2\downarrow$	S_1	S_2		
Cj		1	3	0	0	Bi	Cbi
	$\leftarrow S_1$	0	**<u>-7/2</u>**	1	1/2	-1/2	**0**
	X_1	1	3/2	0	-1/2	3/2	**1**
Wj		1	3/2	0	-1/2	**3/2**	
ΔW=Cj-Wj		0	3/2	0	1/2		
	MIN	X_1	X_2	S_1	S_2		
Cj		1	3	0	0	Bi	Cbi
	X_2	0	1	-2/7	-1/7	1/7	**3**
	X_1	1	0	3/7	-2/7	9/7	**1**
Wj		1	3	-3/7	-5/7	**12/7**	
ΔW=Cj-Wj		0	0	3/7	5/7		

Note that there is also a negative value in bi, so we continue the dual:
LBV : S_1 (a single negative value)
EBV : X_2 (a single negative value at the pivot line) ➔ pivot=-7/2

Note that all the values of bi are positive, so the improvement of the dual is complete. Turning now to the primal, we see that all the marginal costs are positive or zero, so the stopping criterion for the primal is satisfied. ➔ the solution is optimal

Note:

In the case of dual treatment, we see that the variation of the economic function takes place in the opposite way; for minimization the value of the EF increases and it decreases for maximization. This is justified by the fact that the dual treatment is simply a correction (or improvement) applied to the linear program to make the solution feasible (so that it does not contradict the non-negativity constraint).

7.3 Exercises and solutions

Exercise 7.1

Solve the following LP using the dual simplex method (Primal-Dual and Dual-Primal):

$$\textbf{Max } Z = -6X_1 - 5X_2 + 2X_3$$
$$\textbf{s.c.} \quad -X_1 - X_2 + X_3 \leq -2$$
$$-X_1 - 2X_2 - 3X_3 \geq -3$$
$$X_1, X_2, X_3 \geq 0$$

Solution 7.1

The standard form is:

$$\textbf{Max } Z = -6X_1 - 5X_2 + 2X_3$$
$$\textbf{s.c.} \quad -X_1 - X_2 + X_3 \leq -2$$
$$-X_1 - 2X_2 - 3X_3 \geq -3$$
$$X_1, X_2, X_3 \geq 0$$

$$\textbf{Max } Z = -6X_1 - 5X_2 + 2X_3$$
$$\textbf{s.c.} \quad -X_1 - X_2 + X_3 + \boldsymbol{X_4} = -2$$
$$-X_1 - 2X_2 - 3X_3 + \boldsymbol{X_5} = -3$$
$$X_1, X_2, X_3, X_4, X_5 \geq 0$$

(1). Primal-Dual:

The first simplex table is given as follows:

	MAX	X_1	X_2	X_3↓	X_4	X_5		
Cj		-6	-5	2	0	0	Bi	Cbi
-2/1=**-2**	←X_4	-1	-1	**<u>1</u>**	1	0	-2	**0**
-	X_5	-1	-2	-3	0	1	-3	**0**
Zj		0	0	0	0	0	**0**	
ΔZ=Cj-Zj		-6	-5	2	0	0		

<u>Primal</u>

EBV: X_3 (the single positive value of ΔZ).

LBV: X_4 (the single positive value)

	MAX	X_1	X_2↓	X_3	X_4	X_5		
Cj		-6	-5	2	0	0	Bi	Cbi
-2/1=**-2**	←X_3	-1	**<u>-1</u>**	1	1	0	-2	**2**
-	X_5	-4	-5	0	3	1	-9	**0**
Zj		-2	-2	2	2	0	**-4**	
ΔZ=Cj-Zj		-4	-3	0	-2	0		

<u>Primal</u>

When the values of ΔZ are all negative or zero, the primal .stopping criterion is complete

Moving on to **<u>Dual</u>** :

LBV: X_3 (the greatest negative value of bi).

EBV: X_2 (Min $\left\{\left|\frac{-4}{-1}\right|, \left|\frac{-3}{-1}\right|\right\}$)

	MAX	X_1	X_2	X_3	X_4	X_5		
Cj		-6	-5	2	0	0	Bi	Cbi
	X_2	1	1	-1	-1	0	**2**	**-5**
	X_5	1	0	-5	-2	1	**1**	**0**
Zj		-5	-5	5	5	0	**-10**	
ΔZ=Cj-Zj		-1	0	-3	-5	0		

<u>Dual</u>

All values of bi are positive. In addition, all marginal costs are negative or zero, so this is the optimal solution.

The solution is: $X_1=X_3=0$, $X_2=2$, $Z^*=-10$. ($X_5=1$ - i.e. there is one unit that is not consumed from the second constraint -, $X_4=0$ – the value available for the first constraint has been completely used-).

(2). Dual-Primal :

	MAX	X_1	$X_2\downarrow$	X_3	X_4	X_5		
Cj		-6	-5	2	0	0	Bi	Cbi
-2/1=-**2**	$\leftarrow X_4$	-1	**-1**	1	1	0	-2	**0**
-	X_5	-1	-2	-3	0	1	-3	**0**
Zj		0	0	0	0	0	**0**	
$\Delta Z = C_j - Z_j$		-6	-5	2	0	0		
	MAX	X_1	$X_2\downarrow$	X_3	X_4	X_5		
Cj		-6	-5	2	0	0	Bi	Cbi
	$\leftarrow X_2$	1	1	-1	-1	0	**2**	**-5**
	X_5	1	0	-5	-2	1	**1**	**0**
Zj		-5	-5	5	5	0	**-10**	
$\Delta Z = C_j - Z_j$		-1	0	-3	-5	0		

Dual

LBV: X_3 (the greatest negative value of bi).

EBV: X_2 (Min $\left\{\left|\frac{-6}{-1}\right|, \left|\frac{-5}{-1}\right|\right\}$)

➔Pivot=-1

Dual

All values of bi are positive. In addition, all marginal costs are negative or zero, so this is the optimal solution.

Exercice 7.2

- Solve the LP below using the graphical method.
- Solve it using the dual simplex method (Primal-Dual or Dual-Primal).
- Compare the obtained results.

$$
\begin{aligned}
\textbf{Max } Z &= 2X_1 + X_2 \\
\textbf{s.c.} \quad X_1 - 2X_2 &\leq -1 \\
X_2 &\leq 2 \\
-X_1 + X_2 &\leq 1 \\
-2X_1 + 6X_2 &\leq 9 \\
X_1, X_2 &\geq 0
\end{aligned}
$$

- Write the dual program.
- Deduce its optimal solution based on the duality conditions

Solution 7.1

Graphical resolution: the lines corresponding to the constraints are represented graphically, then the set of feasible solutions to the problem is determined, and finally the optimal solution is defined if it exists

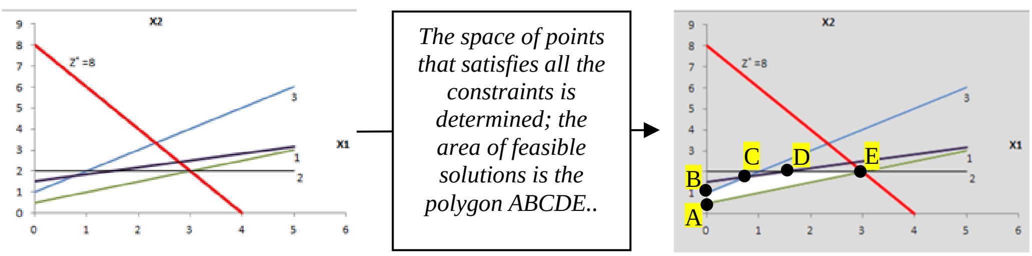


Figure 7.1- Steps of solution 7.1

According to the following table, the optimal solution is represented by point E ; **X1=3, X2=2, Z*=8.**

Points	X_1	X_2	EF Values
A	0	1/2	0.5
B	0	1	1
C	3/4	7/4	13/4
D	3/2	2	5
E	3	2	**8**

Solving with the dual simplex method: the standard form is as follows:

$$\textbf{Max } Z = 2X_1 + X_2$$
$$\textbf{s.c.} \quad X_1 - 2X_2 + \boldsymbol{X_3} = -1$$
$$X_2 + \boldsymbol{X_4} = 2$$
$$-X_1 + X_2 + \boldsymbol{X_5} = 1$$
$$-2X_1 + 6X_2 + \boldsymbol{X_6} = 9$$
$$X_1, X_2, X_3, X_4, X_5, X_6 \geq 0$$

The steps involved in solving the problem are shown in the tables below. Note here that we will start with the dual :

MAX		X_1	X_2↓	X_3	X_4	X_5	X_6		
Cj		2	1	0	0	0	0	Bi	Cbi
-	←X_3	1	**-2**	1	0	0	0	-1	**0**
-	X_4	0	1	0	1	0	0	2	**0**
-	X_5	-1	1	0	0	1	0	1	**0**
-	X_6	-2	6	0	0	0	1	9	**0**
Zj		0	0	0	0	0	**0**	**0**	
$\Delta Z = C_j - Z_j$		2	1	0	0	0			

Dual

LBV: X_3 (greatest negative value of bi).

VE: X_2 (Min $\left\{\left|\frac{1}{-2}\right|\right\}$)

➔Pivot=-2

MAX		X_1↓	X_2	X_3	X_4	X_5	X_6		
Cj		2	1	0	0	0	0	Bi	Cbi
-	X_2	-1/2	1	-1/2	0	0	**0**	**1/2**	**1**
(3/2)/(1/2)=3	←X_4	**1/2**	0	1/2	1	0	**0**	**3/2**	**0**
-	X_5	-1/2	0	1/2	0	1	**0**	**1/2**	**0**
6/1=6	X_6	1	0	3	0	0	**1**	**6**	**0**
Zj		-1/2	1	-1/2	0	0	**0**	**1/2**	
$\Delta Z = C_j - Z_j$		5/2	0	1/2	0	0	0		

Dual

All bi values are positive. Changing to PrimalEBV: X_1

LBV: X_4

➔Pivot=1/2

MAX		X_1↓	X_2	X_3	X_4	X_5	X_6		
Cj		2	1	0	0	0	0	Bi	Cbi
-	X_2	0	1	0	1	0	**0**	**2**	**1**
(3/2)/(1/2)=3	←X_1	1	0	1	2	0	**0**	**3**	**2**
-	X_5	0	0	1	1	1	**0**	**2**	**0**
6/1=6	X_6	0	0	2	-2	0	**1**	**3**	**0**

Primal

All marginal costs are negative or zero. So the solution is optimal

$X_1=3, X_2=2, Z^*=8$

zj	2	1	2	5	0	**0**	**8**	
ΔZ=Cj-Zj	0	0	-2	-5	0	0		

Comparison: note that each simplex table is represented by a point on the graph; table 1 represents the origin O(0,0) of the graph, then the simplex algorithm traverses the perimeter of the zone of feasible solutions because table 2 represents the point A(0,1/2). Moving from one table to another ensure that the right sense towards convergence to the optimal.

The dual program:

Min W = $-Y_1 + 2Y_2 + Y_3 + 9Y_4$

s.c. $Y_1 - Y_3 - 2Y_4 \geq 2 \qquad X_1$

$-2Y_1 + Y_2 + Y_3 + 6Y_4 \geq 1 \qquad X_2$

$Y_1, Y_2, Y_3, Y_4 \geq 0$

Min W = $-Y_1 + 2Y_2 + Y_3 + 9Y_4$

s.c. $Y_1 - Y_3 - 2Y_4 - Y_5 = 2 \qquad X_1$

$-2Y_1 + Y_2 + Y_3 + 6Y_4 - Y_6 = 1 \qquad X_2$

$Y_1, Y_2, Y_3, Y_4 \geq 0$

Standard form

The primal optimal solution is : X_1=3, X_2=2, X_5=2, X_6=3, Z^*=8, X_3=X_4=0 (VHB).

X_1, X_2 are BV ➔ Y_5,Y_6 are NBV So: Y_5=Y_6=0

X_5, X_6 are BV ➔ Y_3,Y_4 are NBV So: Y_3=Y_4=0 replacing values in constraints:

$Y_1 = 2$

$-2Y_1 + Y_2 = 1$ Donc: $Y_1 = 2$, $Y_2 = 5$ et W*=8

Chapter 8

8 Integer Programming

8.1 Introduction

You saw several examples of the numerous and diverse applications of linear programming. However, one key limitation that prevents many more applications is the assumption of divisibility, which requires that noninteger values be permissible for decision variables. In many practical problems, the decision variables actually make sense only if they have integer values. For example, it is often necessary to assign people, machines, and vehicles to activities in integer quantities. If requiring integer values is the only way in which a problem deviates from a linear programming formulation, then it is an integer programming (IP) problem. (The more complete name is integer linear programming, but the adjective linear normally is dropped except when this problem is contrasted with the more esoteric integer nonlinear programming problem)

The mathematical model for integer programming is the linear programming model with the one additional restriction that the variables must have integer values. If only some of the variables are required to have integer values (so the divisibility assumption holds for the rest), this model is referred to as mixed integer programming (MIP). When distinguishing the all-integer problem from this mixed case, we call the former pure integer programming.

Searching for the optimal solution of a model IP requires two essential steps:

(1) Solving the original LP with the appropriate method and if the optimal solution is not integer we move on to the second step.

(2) Separating the solution found in the previous step, with the aim of finding the closest integer solution to the original solution.

In this context, there are methods such as:

- Branch-and-Bound method.
- Gomory cutting method.

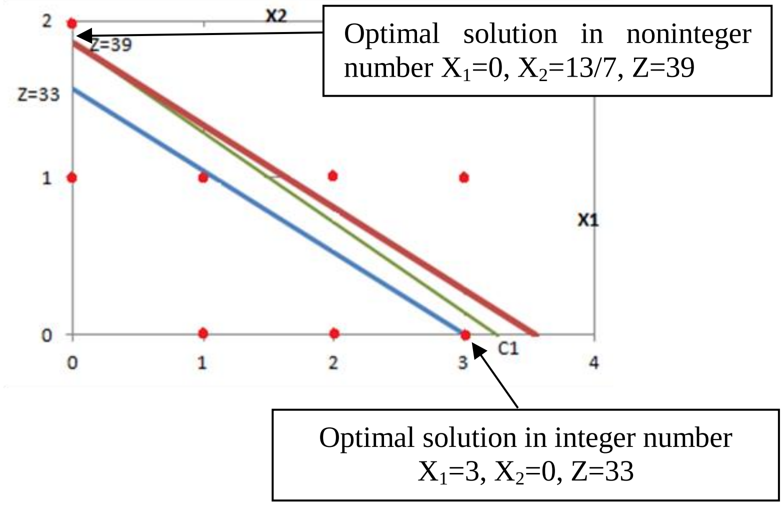
8.2 Comments on the IP

Note that in all cases of the IP, the integer solution is always poorer in terms of EF value than the original (non-integer) solution. In maximization, the integer solution represents the lower bound solution. For minimization, it represents the upper bound solution.

Let the following example: Max $Z=11X_1+22X_2$

S.C : $4X_1+7X_2 \leq 13$

$X_1, X_2 \geq 0$ are integer values



Without the integrity constraint, the optimum solution is at the point X_1= 0 and X_2 = 13/7. If we consider only points with integer coordinates, the optimum is reached at the point X_1 = 3 and X_2=0, which obviously cannot be obtained by rounding the real solution. In addition, the optimum value for the objective is only 33, whereas it is 39 in the case of real variables.

8.3 Branch-and-Bound method

The LP is solved without the integrity condition and if the optimal solution is not an integer, new models are constructed by adding new constraints according to the following logic:

Given (X_j) in real value represented by ***d_i***, the value of X_j lies in [d_{i1}, d_{i2}] with d_{i1}, d_{i2} representing two consecutive integer values ($d_{i1} \leq X_j \leq d_{i2}$).

For example, the value 4.63 lies between the two integer values 4 and 5.

So to exclude the value of X_j to be real value, we add two new constraints:

(a) $Xj \leq di1$ (b) $Xj \geq di2$, which results in two new LPs. we continue the separation procedure until we reach an optimal solution in integer values.

Exercice 8.1: Let the following LP:

Max Z=20 X_1+2 X_2

S.C : $4X_1+10X_2 \leq 22$

$X_1,X_2 \geq 0$ are integer.

Using the simplex method, the solution is: **X_1=11/2 ; X_2=0 ; Z*=110.**

	MAX	**X_1**	**X_2**	**S_1**		
Cj		20	2	0	Bi	Cbi
	X_1	1	5/2	1/4	11/2	**20**
Zj		20	50	5	**110**	
$\Delta Z_= C_j - Z_j$		0	-48	-5		

X_1=11/2 so : 5 ≤X_1≤6 and to exclude X_1 from this interval, two constraints are necessary:

X_1 ≤5 and X_1≥6 which results in two new linear programs to solve.

LP 1.2	LP 1.3
Max Z = $20X_1 + 2X_2$ **s.c.** $4X_1 + 10X_2 \le 22$ $X_1 \le 5$ $X_1, X_2 \ge 0$	**Max** Z = $20X_1 + 2X_2$ **s.c.** $4X_1 + 10X_2 \le 22$ $X_1 \ge 6$ $X_1, X_2 \ge 0$

We solve the two LPs:

LP 1.2 :

	MAX	**X₁**	**X₂**	**S₁**	**S₂**		
Cj		20	2	0	0	Bi	Cbi
	X_2	0	1	1/10	-4/10	1/5	**2**
	X_1	1	0	0	1	5	**20**
zj		20	2	1/5	**96/5**	**502/5**	
$\Delta Z = C_j - Z_j$		0	0	-1/5	-96/5		

The optimal solution is: X_1=5 ; X_2=1/5 ; Z*= 502/5.

LP 1.3: this program has no solution because the minimum value of X_1 is 6 according to constraint 2. If we replace this value in the first constraint, the constraint is no longer verified.

So we continue with the resolution of LP 1.2: the value of X2 is between 0 and 1, hence the addition of the constraints; X_2 ≤0 et X_2≥1 and two new LP are being introduced:

LP 1.2.1	LP 1.2.2
Max Z = $20X_1 + 2X_2$ **s.c.** $4X_1 + 10X_2 \le 22$ $X_1 \le 5$ $X_2 \le 0$ $X_1, X_2 \ge 0$	**Max** Z = $20X_1 + 2X_2$ **s.c.** $4X_1 + 10X_2 \le 22$ $X_1 \le 5$ $X_2 \ge 1$ $X_1, X_2 \ge 0$

The LP 1.2.1 presents a contradiction with the nonnegative constraint.

Solving LP 1.2.2 gives the following solution:

	MAX	**X₁**	**X₂**	**S₁**	**S₂**	**S₃**		
Cj		20	2	0	0	0	Bi	Cbi
	X_1	1	0	1/4	0	10/4	3	**20**
	S_2	0	0	-1/4	1	-10/4	2	**0**
	X_2	0	1	0	0	-1	1	**2**

Z_j	20	2	5	0	48	**62**	
$\Delta Z = C_j - Z_j$	0	0	-5	0	-48		

So **:** X_1=3 ; X_2=1 ; Z*= 62. Integer optimal solution.

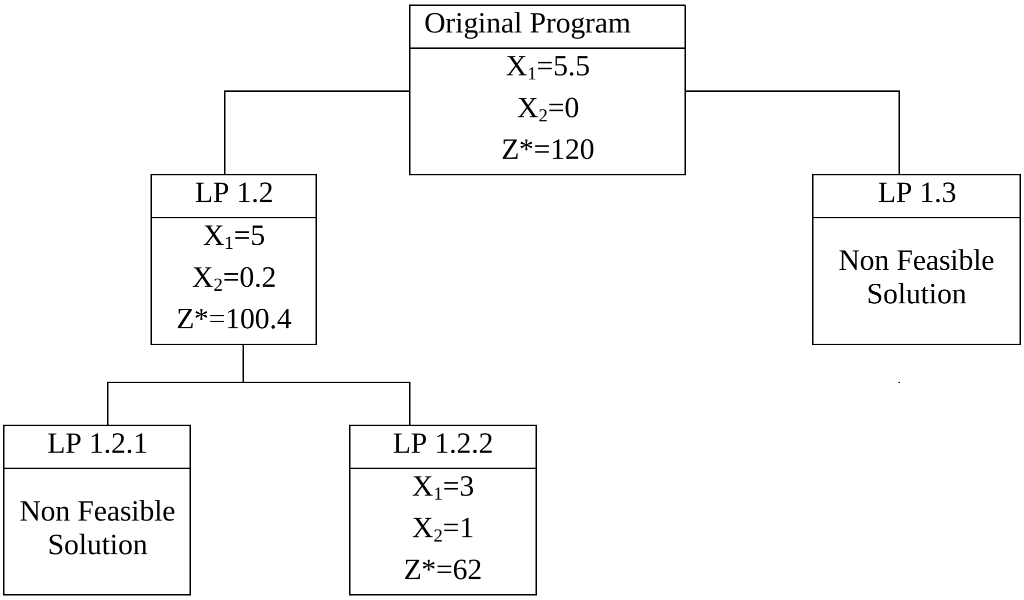


Figure 8-2 : Branch and bound procedure.

Chapter 9

9 Sensibility Analysis

9.1 Definition

One of the key uses of duality theory lies in the interpretation and implementation of *sensitivity analysis*. Sensitivity analysis is a very important part of almost every linear programming study. Because most of the parameter values used in the original model are just estimates of future conditions, the effect on the optimal solution if other conditions prevail instead needs to be investigated. Furthermore, certain parameter values (such as resource amounts) may represent managerial decisions, in which case the choice of the parameter values may be the main issue to be studied, which can be done through sensitivity analysis.

Sensitivity analysis is used to examine the stability of the optimal solution of the linear program following the variation of one of its parameters, i.e.:

(a) Variation of the objective function coefficients (cj).

(b) Variation in the resources available in the constraints (bi).

(c) Variation of the variable coefficients in the constraints (aij).

(d) Addition of a new activity (new variable).

(e) Addition of a new constraint.

The following example will be used to present the sensitivity analysis on the various parameters of the linear program:

$$\textbf{Max } Z = 3X_1 + 2X_2 + X_3$$
$$\textbf{s.c.} \quad 2X_1 + X_2 + 3X_3 \leq 6$$
$$X_1 + 4X_2 + 2X_3 \leq 4$$
$$X_1, X_2, X_3 \geq 0$$

The optimal solution is shown in the table below:

	MAX	**X_1**	**X_2**	**X_3**	**X_4**	**X_5**		
Cj		3	2	1	0	0	Bi	Cbi
working times	**X_1**	1	0	10/7	4/7	-1/7	20/7	**3**
Aluminium plates	**X_2**	0	1	1/7	-1/7	2/7	2/7	**2**
Zj		3	2	32/7	10/7	**1/7**	**64/7**	
$\Delta Z = C_j - Z_j$		0	0	-25/7	-10/7	-1/7	← marginal coast	

P'j is the vector associated with the variable Xj in the optimal solution table:

$$P'_1 = \begin{pmatrix}1\\0\end{pmatrix}; P'_2 = \begin{pmatrix}0\\1\end{pmatrix}; P'_3 = \begin{pmatrix}10/7\\1/7\end{pmatrix}; P'_4 = \begin{pmatrix}4/7\\-1/7\end{pmatrix}; P'_5 = \begin{pmatrix}-1/7\\2/7\end{pmatrix} \; et \; b'_i = \begin{pmatrix}20/7\\2/7\end{pmatrix}$$

And P_j is the vector associated with the variable X_j in the standard form of the LP :

$$P_1 = \begin{pmatrix}2\\1\end{pmatrix}; P_2 = \begin{pmatrix}1\\4\end{pmatrix}\ ; P_3 = \begin{pmatrix}3\\2\end{pmatrix}; P_4 = \begin{pmatrix}1\\0\end{pmatrix}; P_5 = \begin{pmatrix}0\\1\end{pmatrix}\ et\ \ b_i = \begin{pmatrix}6\\4\end{pmatrix}$$

9.2 Make Change on the EF coefficients

Depending on the solution of the given LP, there are two basic variables (X_1 and X_2) as well as the variable X3 which is nonbasic variable. The interval of variation of the values is then determined in two distinct ways for the case of a basic or nonbasic variable:

(a) Coefficients of basic variables

- Variable X_1 :

The coefficient of the variable X_1 represents the benefit of one unit of product 1, so an increase or decrease in this value means a proportional variation in the EF value (the overall benefit). But in reality, there are thresholds of variation for the solution to remain optimal, i.e. all marginal costs remain zero or negative (the case of maximization). To determine the variation thresholds, we have to do is express the marginal costs (C_j') of the NBVs as a function of C_1 (coefficient of X_1 in the EF).

The marginal cost is given by:

$$C_j' = C_j - Z_j \Longrightarrow \ \ C_j' = C_j - (C_{bi} * P_j')$$

1- marginal cost of X_3 :

$$C_3' = C_3 - (C_{bi} * P_3') \iff C_3' = C_3 - (C_1 \quad 2) * \begin{pmatrix}10/7\\1/7\end{pmatrix} \Rightarrow C_3' = \frac{-10}{7}C_1 + \frac{5}{7} \text{...............(1)}$$

2- marginal cost of X_4 :

$$C_4' = C_4 - (C_{bi} * P_4') \iff C_4' = C_4 - (C_1 \quad 2) * \begin{pmatrix}4/7\\-1/7\end{pmatrix} \Rightarrow C_4' = \frac{-4}{7}C_1 + \frac{2}{7} \text{...............(2)}$$

3- marginal cost of X_5 :

$$C_5' = C_5 - (C_{bi} * P_5') \iff C_5' = C_5 - (C_1 \quad 2) * \begin{pmatrix}-1/7\\2/7\end{pmatrix} \Rightarrow C_5' = \frac{1}{7}C_1 - \frac{4}{7} \text{...............(3)}$$

The solution remains optimal only if:

$$\forall j\ ;\ C_j' \leq 0 \Leftrightarrow \begin{cases} -\frac{10}{7}C_1 + \frac{5}{7} \ \leq 0 \\ -\frac{4}{7}C_1 + \frac{2}{7} \ \leq 0 \\ \frac{1}{7}C_1 - \frac{4}{7} \ \leq 0 \end{cases} \Leftrightarrow \begin{cases} C_1 \geq \dfrac{1}{2} \\ C_1 \geq \dfrac{1}{2} \\ C_1 \leq 4 \end{cases} \Rightarrow \frac{1}{2} \leq C_1 \leq 4$$

So we can vary the profit of product 1 in the interval $[\frac{1}{2}, 4]$ and the solution remains optimal even though the overall profit will have changed (the values of the variables will also have changed).

- Variable X_2 :

In the same way, we will determine the variation interval of product 2 profit:

1- marginal cost of X_3 :

$$C_3' = C_3 - (C_{bi} * P_3') \iff C_3' = C_3 - (3 \quad C_2) * \begin{pmatrix} 10/7 \\ 1/7 \end{pmatrix} \Rightarrow C_3' = \frac{-1}{7} C_2 - \frac{23}{7} \text{...............(1)}$$

2- marginal cost of X_4 :

$$C_4' = C_4 - (C_{bi} * P_4') \iff C_4' = C_4 - (3 \quad C_2) * \begin{pmatrix} 4/7 \\ -1/7 \end{pmatrix} \Rightarrow C_4' = \frac{1}{7} C_2 - \frac{12}{7} \text{...............(2)}$$

3- marginal cost of X_5 :

$$C_5' = C_5 - (C_{bi} * P_5') \iff C_5' = C_5 - (3 \quad C_2) * \begin{pmatrix} -1/7 \\ 2/7 \end{pmatrix} \Rightarrow C_5' = \frac{-2}{7} C_2 + \frac{3}{7} \text{...............(3)}$$

The solution remains optimal only if:

$$\forall \text{j} ; \ C_j' \leq 0 \Leftrightarrow \begin{cases} \frac{-1}{7} C_2 - \frac{23}{7} \leq 0 \\ \frac{1}{7} C_2 - \frac{12}{7} \leq 0 \\ \frac{-2}{7} C_2 + \frac{3}{7} \leq 0 \end{cases} \Leftrightarrow \begin{cases} C_2 \geq -23 \\ C_2 \leq 12 \\ C_2 \geq \frac{3}{2} \end{cases} \Rightarrow \frac{3}{2} \leq C_2 \leq 12$$

So we can vary the profit of product 2 in the interval $[\frac{3}{2}, 12]$ and the solution remains optimal even though the overall profit will have changed (the values of the variables will also have changed).

(b) Coefficients of nonbasic variables

This section only concerns the original variables of the LP model wich are nonbasic variable at the optimal solution, i.e. X_3, whereas the variance variables X_4 and X_5 are not affected (zero coefficients in the EF). If the coefficient of X_3 changes, this only causes the change of C_3' in the simplex table, so :

$$C_3' = C_3 - (C_{bi} * P_3') \iff C_3' = C_3 - (3 \quad 2) * \begin{pmatrix} 10/7 \\ 1/7 \end{pmatrix} \Rightarrow C_3' = C_3 - \frac{32}{7}$$

The solution remains optimal only if:: $C_3' \leq 0 \Leftrightarrow C_3 - \frac{32}{7} \leq 0$ donc $C_3 \leq \frac{32}{7}$

If the profit on product 3 is less than $\frac{32}{7}$, it is no longer economical for the company to produce it.

Suppose that the profit of product 3 equals 5, in which case its marginal cost is positive. This will lead to a change in the basic variables, since the solution is not optimal.

(c) Coefficients of basic and nonbasic variables at the same time

Suppose the EF is as follows: Max Z = $X_1 + 3X_2 + 2X_3$. Although the change is made in the three coefficients within the ranges determined in sections a and b, we can’t affirm that the solution remains optimal. Consequently, recalculation of all marginal costs is strongly recommended.

1- marginal cost of X_1 :

$$C_1' = C_1 - (C_{bi} * P_1') \iff C_1' = 1 - (1 \quad 3) * \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow C_1' = 0 \text{..........................(1)}$$

2- marginal cost of X_2 :

$C_2' = C_2 - (C_{bi} * P_2') \Leftrightarrow C_2' = 3 - (1 \quad 3) * \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow C_2' = 0$...........................(2)

3- marginal cost of X_3 :

$C_3' = C_3 - (C_{bi} * P_3') \Leftrightarrow C_3' = 2 - (1 \quad 3) * \begin{pmatrix} 10/7 \\ 1/7 \end{pmatrix} \Rightarrow C_3' = \frac{1}{7}$....................(3)

4- marginal cost of X_4 :

$C_4' = C_4 - (C_{bi} * P_4') \Leftrightarrow C_4' = 0 - (1 \quad 3) * \begin{pmatrix} 4/7 \\ -1/7 \end{pmatrix} \Rightarrow C_4' = -\frac{1}{7}$...............(4)

5- marginal cost of X_5 :

$C_5' = C_5 - (C_{bi} * P_5') \Leftrightarrow C_5' = 0 - (1 \quad 3) * \begin{pmatrix} -1/7 \\ 2/7 \end{pmatrix} \Rightarrow C_5' = -\frac{5}{7}$...............(5)

So the solution is not optimal because of the positive value of the marginal cost of X_3.

9.3 Make Change on the available resources of constraints (bi)

Changing one of the values of bi means changing the value of the base variables (b_i') and subsequently the value of FE. So, as long as the values of b_i' do not contradict the nonnegative constraint, the solution remains feasible.

Then we define a matrix relationship that links each column of the initial simplex table of an LP with it corresponding column in the table of the optimal solution as follows.

$$P_j' = B^{-1} * P_j \ ;$$

With : P_j, P_j' columns associated with the variable X_j respectively in the initial and the optimal solution tables.

B^{-1} : inverse matrix of the transition matrix.

For b_i' values, the matrix relationship is done by: $b_i' = B^{-1} * b_i$

How to determine the B and B^{-1} matrix values?

Values of B matrix are taken from the initial simplex table; BV of optimal table ↓

BV of → initial table

B=

	X_1	X_2
X_4	**2**	**1**
X_5	**1**	**4**

Values of B^{-1} matrix are taken from the optimal simplex table; BV of initial table ↓

BV of → optimal table

B^{-1}=

	X_4	X_5
X_1	$\frac{4}{7}$	$-\frac{1}{7}$
X_2	$-\frac{1}{7}$	$\frac{2}{7}$

9.3.1 Making change in working time values *(first constraint)*

We suppose that the available value for the first constraint is unknown (b_1) ;

$$b_i' = B^{-1} * b_i \iff b_i' = \begin{vmatrix} \frac{4}{7} & -\frac{1}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{vmatrix} x \begin{pmatrix} b_1 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{4}{7} b_1 - \frac{4}{7} \\ \frac{-1}{7} b_1 + \frac{8}{7} \end{pmatrix}$$

The solution is feasible if: $\forall i, b_i' \geq 0 \iff \begin{cases} \frac{4}{7} b_1 - \frac{4}{7} \geq 0 \\ \frac{-1}{7} b_1 + \frac{8}{7} \geq 0 \end{cases} \iff \begin{cases} b_1 \geq 1 \\ b_1 \leq 8 \end{cases} \Rightarrow \mathbf{1 \leq b_1 \leq 8}$

9.3.2 Making change on Aluminium plates values *(second constraint)*

We suppose that the available value for the second constraint is unknown (b_2) ;

$$b_i' = B^{-1} * b_i \iff b_i' = \begin{vmatrix} \frac{4}{7} & -\frac{1}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{vmatrix} x \begin{pmatrix} 6 \\ b_2 \end{pmatrix} = \begin{pmatrix} \frac{-1}{7} b_2 + \frac{24}{7} \\ \frac{2}{7} b_2 - \frac{6}{7} \end{pmatrix}$$

The solution is feasible if: $\forall i, b_i' \geq 0 \iff \begin{cases} \frac{-1}{7} b_2 + \frac{24}{7} \geq 0 \\ \frac{2}{7} b_2 - \frac{6}{7} \geq 0 \end{cases} \iff \begin{cases} b_2 \leq 24 \\ b_2 \geq 3 \end{cases} \Rightarrow \mathbf{3 \leq b_2 \leq 24}$

Case 1: we suppose that the number of plates is equal to 5, this value belongs to the interval [3, 24] i.e. the solution remains optimal but its value changes.

$$b_i' = B^{-1} * b_i \iff b_i' = \begin{vmatrix} \frac{4}{7} & -\frac{1}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{vmatrix} x \begin{pmatrix} 6 \\ \mathbf{5} \end{pmatrix} = \begin{pmatrix} \frac{19}{7} \\ \frac{4}{7} \end{pmatrix}$$

The optimal solution is : $X_1=\frac{19}{7}$, $X_2=\frac{4}{7}$, , $X_3=0$, $Z^*= \frac{65}{7}$.

Although the value of FE has increased by $\frac{1}{7}$, it is necessary to check whether this increase is beneficial or not?

The cost[6] of increasing one unit from bi of the second constraint is: $\frac{2}{7}$..

So the profit recorded is**:** $\frac{65}{7} - \frac{64}{7} = \frac{1}{7}$ this is less than the cost $\frac{2}{7}$ so this increase is not beneficial for the company.

Case 1 : we suppose that the number of plates equals 25, this value is at the back of the interval [3, 24] i.e. the solution will not be feasible (therefore not optimal). In this case we use the dual simplex method.

9.4 Make change in constraints matrix values

(a) change variable coefficients in constraints (a_{ij})

[6] This is shown in the box where the constraint line intersects the column of the slack variable used for this constraint.

The characteristics of a product can be changed so that the solution remains optimal. In particular, we're talking about the needs of a unit in terms of number of hours or number of aluminium panels. There are often two cases:

- Variation in the information of a nonbasic variable:

It is assumed that the aluminium panel's requirements of product 3 (X_3) are reduced to a single panel, so $P_3\binom{3}{1}$, the aim is to measure the impact of this change on the optimality of the program.

$$P'_3 = B^{-1} * P_3 \iff P'_3 = \begin{vmatrix} \frac{4}{7} & -\frac{1}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{vmatrix} x \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{11}{7} \\ \frac{-1}{7} \end{pmatrix}$$

So, the marginal cost is :

$$C'_3 = C_3 - (C_{bi} * P'_3) \iff C'_3 = 1 - (1 \quad 3) * \begin{pmatrix} 11/7 \\ -1/7 \end{pmatrix} \Rightarrow C'_3 = \frac{-24}{7}$$

The solution remains optimal, we conclude that the production of product 3 is not yet benefic.

- Variation in the information of a basic variable:

In this case, it is preferable to solve the LP again because the change in the coefficients of VB (the case of X_1 and X_2) in the constraints influences all the values of the optimal table

(b) Add a new activity (new variable)

We suppose that the company is aiming to produce a new product, so the decision-maker is looking to see whether the new product - depending on its characteristics - will be economical for the company.

For example, the following characteristics are proposed:

The new product (X_6) requires two (02) units of working hours and two units of aluminium panals, the unit profit is 2 monetary units. Therefore, $C_6 = 2$ et $P_6\binom{2}{2}$,

$$P'_6 = B^{-1} * P_6 \iff P'_6 = \begin{vmatrix} \frac{4}{7} & -\frac{1}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{vmatrix} x \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{6}{7} \\ \frac{2}{7} \end{pmatrix}$$

So : $C'_6 = C_6 - (C_{bi} * P'_6) \iff C'_6 = 2 - (1 \quad 3) * \begin{pmatrix} 6/7 \\ 2/7 \end{pmatrix} \Rightarrow C'_6 = \frac{-8}{7}$

The marginal cost of NBV X_6 is negative, i.e. producing the new product is uneconomical. We can deduce that its production will reduce overall profit by value[7] ($\frac{-8}{7}$).

(c) Add a new constraint

Before adding a constraint, it is first necessary to check whether this constraint is verified by the optimal solution of the LP. There are two possible cases:

[7] The impact of a NBV on the value of EF is the product of its marginal cost and the smallest ratio of the leaving variable, in our case: $\frac{-8}{7} x1 = \frac{-8}{7}$

(a) the constraint is verified; it is therefore considered to be supplicant to the LP and can therefore be ignored.

(b) the optimal solution does not satisfy the constraint, so it has to be added in the program and consequently the solution will be modified.

Example 1 : add the following constraint : $2X_1+X_2+2X_3 \leq 8$.

(N.A): $2(\frac{20}{7})+\frac{2}{7}+2(0) = \frac{42}{7}=6$ which is less than 8. Then the constraint is verified and can be ignored.

Example 2 : Let adding the constraint $2X_1+X_2+2X_3 \leq 5$. According to the previous example, this constraint is no longer verified by the optimal solution. Here are the necessary steps:

We transform the constraint into an equality: $2X_1+X_2+2X_3 = 5$ (with X_6 as the slack variable). The constraint is added to the optimal simplex table, but we notice that there are some anomalies to be sorted out; we need to check the characteristics of the basic variables, i.e. a basic variable must have a coefficient equal to one (1) in the associated constraint and zero for the rest of the constraints. This approach concerns the variables $\mathbf{X_1, X_2, X_6}$.

MAX		X_1	X_2	X_3	X_4	X_5	X_6		
Cj		3	2	1	0	0	0	Bi	Cbi
Heures de travail	X_1	1	0	10/7	4/7	-1/7	0	20/7	**3**
Plaques en aluminium	X_2	0	1	1/7	-1/7	2/7	0	2/7	**2**
Nouvelle contrainte	X_6	**2**	**1**	**2**	**0**	**0**	**1**	**5**	**0**
Zj		3	2	32/7	10/7	**1/7**		**64/7**	
$\Delta Z = C_j - Z_j$		0	0	-25/7	-10/7	-1/7		← Coûts marginaux	

So we multiply the first constraint by (-2) and the second by (-1) and we add up the three constraints according to the following table:

Constraint1x(-2)	-2	0	-20/7	-8/7	2/7	0	-40/7
Constraint2x (-1)	0	-1	-1/7	1/7	-2/7	0	-2/7
New Constraint	2	1	2	0	0	1	5
By addition:	**0**	**0**	**-1**	**-1**	**0**	**1**	**-1**

So the resulting simplex table will be as follows:

MAX		X_1	X_2	X_3	X_4↓	X_5	X_6			This solution is not feasible (negative value of X_6)
Cj		3	2	1	0	0	0	Bi	Cbi	
working hours	X_1	1	0	10/7	4/7	-1/7	0	20/7	**3**	
Aluminium panals	X_2	0	1	1/7	-1/7	2/7	0	2/7	**2**	**Dual Simplex**
New Constraint	←X_6	0	0	-1	**-1**	0	1	-1	**0**	LBV: X_6
Zj		3	2	32/7	10/7	**1/7**	0	**64/7**		EBV: X_4
$\Delta Z = C_j - Z_j$		0	0	-25/7	-10/7	-1/7	0	← marginal		Pivot = -1

	MAX	X_1	X_2	X_3	X_4	X_5	X_6	cost		
Cj		3	2	1	0	0	0	Bi	Cbi	Optimal and feasible solution
working hours	X_1	1	0	6/7	0	-1/7	4/7	16/7	**3**	
Aluminium panals	X_2	0	1	2/7	0	2/7	-1/7	3/7	**2**	
New Constraint	X_4	0	0	1	1	0	-1	1	**0**	
Zj		3	2	22/7	0	**1/7**	10/7	**54/7**		
$\Delta Z = C_j - Z_j$		0	0	-15/7	0	-1/7	-10/7	← marginal cost		

In this example, we have covered practically all the possible scenarios that can be presented when analyzing the sensitivity of the results of a given LP. Sensitivity analysis is an essential step for decision-makers, as it provides key information for analyzing the optimal solution and, of course, for future forecasts.

Exercise:

A company produces two types of cosmetic products using two machines. The information required is shown in the following table:

	Product1	Product2	Available hours
Machine 1	4	2	60
Machine 1	2	4	48
Profil	8	6	

- Define the quantity to be produced to maximize profit.
- Define the interval of variation of the profit associated with product1.
- Define the variation thresholds for the available hours for machine1.
- Study the possibility to add a new product which is characterized by: the unit profit is 7 u.m, it requires 3h, 2h respectively on machine1 and machine2.

Solution :

The LP for this problem is expressed as:

$$\textbf{Max } Z = 8X_1 + 6X_2$$
$$\textbf{s.c. } 4X_1 + 2X_2 \leq 60$$
$$2X_1 + 4X_2 \leq 48$$
$$X_1, X_2 \geq 0$$

The optimal solution is as follows:

MAX		X_1↓	X_2	X_3	X_4		
Cj		8	6	0	0	Bi	Cbi
Machine1	←X_3	**4**	2	1	0	60	**0**
Machine2	X_4	2	4	0	1	48	**0**
Zj		0	0	0	0	**0**	

$\Delta Z = C_j - Z_j$		8	6	0	0		
MAX		**X_1**	**X_2↓**	**X_3**	**X_4**		
Machine1	**X_1**	1	1/2	1/4	0	15	**8**
Machine2	**←X_4**	0	**3**	-1/2	1	18	**0**
Zj		8	4	2	0	**120**	
$\Delta Z = C_j - Z_j$		0	6	-2	0		
MAX		**X_1**	**X_2**	**X_3**	**X_4**		
Machine1	**X_1**	1	0	1/3	-1/6	12	**8**
Machine2	**X_2**	0	1	-1/6	1/3	6	**6**
Zj		8	6	5/3	2/3	**132**	
$\Delta Z = C_j - Z_j$		0	6	-2	-2/3		

The quantities are 12 units of product1 and 6 units of product2 which a profit of 132 u.m..

- *Define the interval of variation of the profit associated with product1*

To define the thresholds for this variation, we have to express the marginal costs (C_j') of the N as a function of C_1 (coefficient of X_1 in the EF).

The marginal cost is: $C_j' = C_j - Z_j \Longrightarrow C_j' = C_j - (C_{bi} * P_j')$

1- marginal cost of X_3 :

$$C_3' = C_3 - (C_{bi} * P_3') \iff C_3' = 0 - (C_1 \quad 6) * \begin{pmatrix} 1/3 \\ -1/6 \end{pmatrix} \Rightarrow C_3' = \frac{-1}{3} C_1 + 1 \text{................(1)}$$

2- marginal cost of X_4 :

$$C_4' = C_4 - (C_{bi} * P_4') \iff C_4' = 0 - (C_1 \quad 6) * \begin{pmatrix} -1/6 \\ 1/3 \end{pmatrix} \Rightarrow C_4' = \frac{1}{6} C_1 - 2 \text{................(2)}$$

The solution remains optimal only if:

$$\forall j \,;\; C_j' \leq 0 \iff \begin{cases} \dfrac{-1}{3} C_1 + 1 \leq 0 \\ \dfrac{1}{6} C_1 - 2 \leq 0 \end{cases} \iff \begin{cases} C_1 \geq 3 \\ C_1 \leq 12 \end{cases} \Longrightarrow 3 \leq C_1 \leq 12$$

So we can vary the benefit of product 1 in the interval [3,12] and the solution remains optimal.

- Define the variation thresholds for the available hours for machine1.

We suppose that the available value for the first constraint is unknown (b_1);

$$b_i' = B^{-1} * b_i \iff b_i' = \begin{vmatrix} \frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{1}{3} \end{vmatrix} x \begin{pmatrix} b_1 \\ 48 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} b_1 - 8 \\ \frac{-1}{6} b_1 + 16 \end{pmatrix}$$

The solution is feasible if: $\forall i \,, b_i' \geq 0 \iff \begin{cases} \frac{1}{3} b_1 - 8 \geq 0 \\ \frac{-1}{6} b_1 + 16 \geq 0 \end{cases} \iff \begin{cases} b_1 \geq 24 \\ b_1 \leq 96 \end{cases} \Longrightarrow \mathbf{24 \leq b_1 \leq 96}$

▪ Study the possibility to add a new product which is characterized by: the unit profit is 7 u.m, it requires 3h, 2h respectively on machine1 and machine2.
We have:

$$P_3' = B^{-1} * P_3 \iff P_3' = \begin{vmatrix} \frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{1}{3} \end{vmatrix} x \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{6} \end{pmatrix}$$

$C_5' = C_5 - (C_{bi} * P_5') \iff C_5' = 7 - (8 \quad 6) * \begin{pmatrix} \frac{2}{3} \\ \frac{1}{6} \end{pmatrix} \Rightarrow C_5' = \frac{2}{3}$. Production of the new product is beneficial (positive marginal cost).

Exercise without solution :

Let the following LP :

$$\begin{aligned} \textbf{Max } Z &= 5X_1 + 3X_2 \\ \textbf{s.c.} \quad X_1 - X_2 &\leq 2 \\ 2X_1 + X_2 &\leq 4 \\ -3X_1 + 2X_2 &\leq 6 \\ X_1, X_2 &\geq 0 \end{aligned}$$

The optimal solution is as presented :

MAX		**X1**	**X2**	**X3**	**X4**	**X5**		
Cj		5	3	0	0	0	Bi	Cbi
	X3	0	0	1	1/7	3/7	36/7	**0**
Working hours	**X1**	1	0	0	2/7	-1/7	2/7	**5**
Panals	**X2**	0	1	0	3/7	2/7	24/7	**3**
Zj		5	3	0	19/7	**1/7**	**82/7**	
$\Delta Z = C_j - Z_j$		0	0	0	-19/7	-1/7	← Marginal costs	

▪ Define the interval of variation in all EF coefficients.
▪ Define the variation thresholds for the available resources in all constraints.

Chapter 10

10 Transportation Problème –cass of minimization-

10.1 Définition

Previous Chapters emphasized the wide applicability of linear programming. We continue to broaden our horizons in this chapter by discussing a particularly important type of linear programming problems. It is called the transportation problem, received this name because many of its applications involve determining how to optimally transport goods. However, some of its important applications (e.g., production scheduling) actually have nothing to do with transportation.

A transport problem can be presented as follows:

- A product must be transported from sources (factories) to destinations (depots, customers).
- *Objective:* define the quantity sent from each source to each destination by minimizing transport costs. Costs are proportional to the quantities transported.
- *Contraintes* d'offre limitée aux sources et de demande à satisfaire aux destinations.

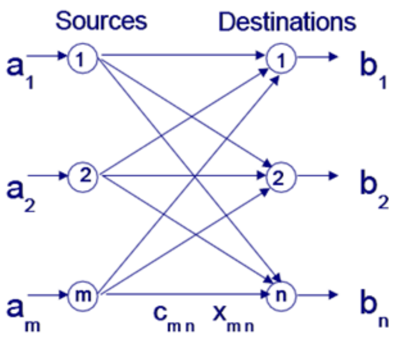


Fig 10.1: Network presentation

10.2 Presentation

A set of cities supplied with drinking water by pumping stations. Each station is characterised by the quantity of water pumped in m^3/day:

- Station1 delivers quantity S1.
- Station2 delivers the quantity S2. The stations represent the sources.
- Station3 delivers quantity S3.

The daily drinking water requirements of the towns represent the daily quantities demanded:

- City1 consumes the quantity D1.
- City2 consumes the quantity D2.
- City3 consumes the quantity D3. The cities represent the destinations.
- City4 consumes quantity D4.

Pumping costs are recorded and presented in the following form:

		Destinations			
		City 1	City 2	City 3	City 4
Sources	Station1	C_{11}	C_{12}	C_{13}	C_{14}
	Station2	C_{21}	C_{22}	C_{23}	C_{24}
	Station3	C_{31}	C_{32}	C_{33}	C_{34}

where: C_{ij} present the cost of 1 m^3 pumped from station 'i' to city 'j'.

The aim is to satisfy the daily drinking water needs of the cities while respecting the capacity of each station, and all this with a minimum overall cost.

We need to define the quantity of water in m^3 distributed from each station to the four cities. Xij is the quantity of water in m^3 distributed from station 'i' to city 'j', as shown in the table below:

	Ville1	Ville2	Ville3	Ville4
Station1	X_{11}	X_{12}	X_{13}	X_{14}
Station2	X_{21}	X_{22}	X_{23}	X_{24}
Station3	X_{31}	X_{32}	X_{33}	X_{34}

10.3 Transportation problem model

All the information can then be grouped together in a single table:

	Ville1	Ville2	Ville3	Ville4	
Station1	C_{11} X_{11}	C_{12} X_{12}	C_{13} X_{13}	C_{14} X_{14}	S_1
Station3	C_{21} X_{21}	C_{22} X_{22}	C_{23} X_{23}	C_{24} X_{24}	S_2
Station2	C_{31} X_{m1}	C_{32} X_{m2}	C_{33} X_{33}	C_{34} X_{34}	S_3
	D_1	D_2	D_3	D_4	

the transportation table is expressed as follows:

with:

s_i : sources

S_i : quantity provided by the source s_i.

d_j : destinations

D_j: quantity requested by the destination d_j.

C_{ij}: transportation cost of one unit from s_i to d_j

X_{ij}: quantity transported from s_i to d_j.

	d_1	d_2	……	d_n	
s_1	C_{11} X_{11}	C_{12} X_{12}	……	C_{1n} X_{1n}	S_1
s_2	C_{21} X_{21}	C_{22} X_{22}	……	C_{2n} X_{2n}	S_2
……	………	…………	……	………	……
s_m	C_{m1} X_{m1}	C_{m2} X_{m2}	……	C_{mn} X_{mn}	S_m
	D_1	D_2	……	D_n	

A transportation problem is described mathematically by

(1) a global cost W :

W= $\mathbf{C_{11}X_{11}+ C_{12}X_{12}+ C_{13}X_{13}+ C_{14}X_{14}+ C_{21}X_{21}+ C_{22}X_{22}+ C_{23}X_{23}+ C_{24}X_{24}+ C_{31}X_{31}+ C_{32}X_{32}+ C_{33}X_{33}+ C_{34}X_{34}}$

So, $w = \sum_{i=1}^{3}\sum_{j=1}^{4} C_{ij}X_{ij}$

(2) quantities transported from sources :

Source1 : $\mathbf{X_{11}+ X_{12}+ X_{13}+ X_{14}=S_1}$

Source2 : $\mathbf{X_{21}+ X_{22}+ X_{23}+ X_{24}=S_2}$

Source3 : $\mathbf{X_{31}+ X_{32}+ X_{33}+ X_{34}=S_3}$

So, $\sum_{j=1}^{4} X_{ij} = S_i\ ;\ i = 1,2,3$

(3) quantities transported to destinations :

City1 : $\mathbf{X_{11}+ X_{21}+ X_{31} =D_1}$

City2 : $\mathbf{X_{12}+ X_{22}+ X_{32} =D_2}$

City3 : $\mathbf{X_{13}+ X_{23}+ X_{33} =D_3}$

City4 : $\mathbf{X_{14}+ X_{24}+ X_{34} =D_3}$

So, $\sum_{i=1}^{3} X_{ij} = D_j\ ; j = 1,2,3,4$

The transportation model is expressed mathematically in the following general form: $Min\ w = \sum_{i=1}^{3}\sum_{j=1}^{4} C_{ij}X_{ij}$

$$\text{S.T :}\quad \begin{cases} \sum_{j=1}^{4} X_{ij} = S_i\ ; pour\ i = 1,2,3 \\ \sum_{i=1}^{3} X_{ij} = D_j\ ; pour\ j = 1,2,3,4 \\ \sum_{i=1}^{3} S_i = \sum_{j=1}^{4} D_j \\ X_{ij} \geq 0 \end{cases}$$

10.4 Solving a transportation problem

Before starting to solve a transport problem, it is almost necessary to check that the supply and demand quantities are equal. If this is not the case, more details and tricks are presented in section 10.5.

10.4.1 Search a initial basic feasible solution

10.4.1.1 North-West Corner Method

A simple and efficient technique to search an initial basic feasible solution for a transportation problem in operations research.

Its principle is to choose at each stage the variable located at the intersection of the first row and the first column of the reduced table. So, starting from the top left-hand corner of the table, here are the steps:

1. allocate as much as possible to the current cell and adjust supply and demand;
2. move one cell to the right (zero demand) or down (zero supply);
3. repeat until all the supply has been allocated.

10.4.1.2 Minimum cost method

At each stage, choose the variable C_{pq} corresponding to the lowest cost in the reduced table

$C_{pq} = \min_{ij}\left(C_{ij}\right)$

Here are the detailed steps:

Select the minimum cost cell.

1. allocate as much as possible to the current cell and adjust supply and demand;
2. Select the minimum cost cell with non-zero demand and supply;
3. Repeat until all the supply has been allocated.

10.4.1.3 Vogel's Approximation Method (VAM)

It is based on the calculation of penalties, generally the solution is very approximate to the optimal solution:

1. for each row (column) with non-zero supply (demand), calculate a penalty equal to the difference between the two lowest costs in the row (column);

2. select the row or column with the maximum penalty and select the cell with the minimum cost in the row or column;

3. allocate as much as possible to the current cell;

4. when only one row or column remains: select the cell with the lowest cost.

10.4.2 Research to the optimal solution

Having defined a basic solution that can be implemented using one of the three previous methods, we then try to find an optimal solution using one of the two methods:

10.4.2.1 Modified Distribution method (MODI)

The dual program of a transportation model is given by :

$$Max\!: Z = \sum_{i=1}^{m} a_i\, u_i + \sum_{j=1}^{n} b_j\, v_j$$

$$avec: \; u_i + v_j > C_{ij}\;; i = 1, \dots, m, j = 1, \dots, n$$

Note that u_i, v_j are free values calculated for all basic variables according to the formula:

$$C_{ij} = u_i + v_j;\; \forall X_{ij} > 0$$

Note: in the transportation table, cells with positive values represent basic variables cells, while empty cells represent nonbasic variables.

This gives (m + n -1) equalities with (m + n) unknowns quantities. We fix u1 = 0 to solve recursively the following system:

$C_{ij} = u_i + v_j;\; for\, X_{ij} > 0$ i.e. for filled cells.

The values of u_i, v_j are recorded in the transportation table at the beginning of the corresponding rows and columns.

Once you have calculated the value of the dual variables, it is easy to calculate the value of the components of the relative cost vector for each nonbasic variable:

$$C_{ij} - (u_i + v_j) \geq 0;\ for\ all\ NBV.$$

Thus, the basic solution is optimal only if:

$$C_{ij} - (u_i + v_j) \geq 0;\ for\ X_{ij} > 0$$

If one of these components is negative, then we are not at the optimum and must define the LBC and EBV to preserve eligibility:

Two objectives need to be checked:

1. supply and demand must continue to be satisfied;
2. all transported quantities must remain positive.

To achieve this we must:

- Define the Entering variable characterized by the smallest value of negative relative costs.
- Construct a cycle running through the basic variables, starting from and returning to the Entering variable (the cycle must contain only horizontal and vertical lines and we do not count the BV that are not at the corner of the cycle);
- Alternately mark with + and - the variables which make up the selected cycle, starting with the entering variable.
- Move along rows and columns, alternately adding and removing the smallest of the quantities preceded by a minus sign (-), corresponding to the leaving variable
- Recalculate the new values of the dual variables, then the values of the relative costs. If the latter are not negative, then the solution is optimal, otherwise the previous steps are repeated until the optimum is reached.

Example 1: Let the closet cycle: $X_{12} \rightarrow X_{13} \rightarrow X_{23} \rightarrow X_{22} \rightarrow X_{12}$, here, the entering variable is X_{12}, et X_{13}, X_{23}, X_{22} are basic variables with the following values respectively $50, 20, 65$: Applying the rules described above we have :

Cycle	$X_{12} \rightarrow$	$X_{13} \rightarrow$	$X_{23} \rightarrow$	$X_{22} \rightarrow$	X_{12}
Sign	**+**	**-**	**+**	**-**	
BV values		**50**	20	65	
The smallest value preceded by the minus sign (-) is **50 ➔ LV:** X_{13}					
Treatment	**+50**	**-50**	**+50**	**-50**	
BV new values	50		70	15	

10.4.2.2 Stepping stone method

The algorithm for this method is most similar to the previous method, although we first define the cycles for all the nonbasic variables and then calculate the relative cost of each cycle as follows:

- Taking the cycle in example 1 and assuming that the unit costs are respectively $9, 10.5, 12$ and 8 for the Entering variable, then the relative cost of the cycle is:

$$X_{12} \rightarrow X_{13} \rightarrow X_{23} \rightarrow X_{22} \rightarrow X_{12}$$
$$8 - 9 + 10.5 - 12 = -2.5$$

- The Entering variable corresponds to the cycle with the smallest negative cost.
- Alternately mark with + and - the variables which make up the selected cycle, starting with the *Entering variable*.
- Move along the rows and columns, alternately adding and removing the smallest of the quantities preceded by a minus sign (-), corresponding to the outgoing variable.
- Adjust the transport table with the new values.
- Reconstruct all the cycles associated with the NB variables, then their relative cost values. If the values are not negative, then the solution is optimal, otherwise the previous steps are repeated until the optimum is reached.

Exercise 10.1:

Three factories (A1, A2, A3) supply the same type of raw material to four micro-enterprises (S1, S2, S3, S4). The information on supply and demand and the unit cost of transport is shown in the table opposite:

Define the lowest-cost transportation strategy.

	B1	B2	B3	B4	Offer
A1	3	2	5	2	**30**
A2	6	5	8	7	**40**
A3	2	8	4	6	**150**
Demand	**30**	**90**	**80**	**20**	

Solution 10.1:

We will solve this example using all studied methods:

1- *Verifying the condition of equality between supply and demand* :

$\forall i,j; \sum_{i=1}^{3} S_i = \sum_{j=1}^{4} D_j$, Requested quantity =30+90+80+20=220

Offered quantity =30+40+150=220 so *the condition is satisfied*

The transportation table is shown below:

	B1	B2	B3	B4	
A1	3	2	5	2	30
A2	6	5	8	7	40
A3	2	8	4	6	150
	30	90	80	20	220 / 220

2- defining a basic feasible solution:

(a) North-West Corner Method:

- The cell in the North-West corner is C_{11}, the supply from source A1 is equal to the demand from customer B1, so cell C_{11} receives the quantity 30, i.e. X_{11}=30. We adjust the values in the table.

- The cell in the North-West corner is C_{22}, Min {90,40}={40} so X_{22}=40.

- The North-West corner cell is C_{32}, Min {50,150}={50} so X_{32}=50

- The North-West corner cell is C_{33}, Min {80,100}={80} so X_{33}=80.

- There remains only the quantity requested from customer B4 which, according to the table, corresponds to the quantity of supply remaining from source A3, so X_{34}=20.

	B1	B2	B3	B4	
A1	3 30	2 ε	5	2	~~30~~ 0
A2	6	5 40	8	7	~~40~~ 0
A3	2	8 50	4 80	6 20	~~150~~ ~~100~~ ~~20~~ 0
	~~30~~ 0	~~90~~ ~~50~~ 0	~~80~~ 0	~~20~~ 0	220 220

Important: the number of occupied cells (corresponding to the number of BV) is **5**, whereas it should be (m+n-1=6; with 'm' the number of sources and 'n' the number of customers). In this case, we add the quantity to one of the empty cells (in our case, cell C_{12}).

The global cost is : $W= C_{11}X_{11}+ C_{12}X_{12}+ C_{23}X_{23}+C_{32}X_{32}+ C_{33}X_{33}+ C_{34}X_{34}$

$$W= 3*30 +2*\varepsilon+5*40+8*50+4*80+6*20=\mathbf{1130+2\varepsilon}$$

(b) Minimum cost method:

For each iteration, the cell containing the smallest cost value is selected, with supply and demand being adjusted each time.

- The smallest cost value (=2) corresponds to the three cells [C_{12}, C_{14}, C_{31}], starting with X_{12} which will receive the quantity 30.
- The cell C_{31}, Min {30,150}={30} so X_{31}=30.
- We therefore look for the cell with the lowest cost without taking into account the first row (supply delivered) and the first column (demand satisfied). This is cell C_{33}; Min {80,120}={80} so X_{33}=80
- Cell C_{22} contains the minimum cost of the remaining cells, so Min{60,40}={40}, so X_{22}=40
- There are still two cells that need to be filled to satisfy the requests of customers B2 and B4. Therefore, X_{34}=20 and X_{32}=20.

	B1	B2	B3	B4	
A1	3	2 30	5	2	~~30~~ 0
A2	6	5 40	8	7	~~40~~ 0
A3	2 30	8 20	4 80	6 20	~~150~~ ~~120~~ ~~40~~ 0
	~~30~~ 0	~~90~~ ~~60~~ ~~20~~ 0	~~80~~ 0	~~20~~ 0	220 220

Important: the number of occupied cells (corresponding to the number of BV) is **6**, whereas it should be (m+n-1=6; with 'm' the number of sources and 'n' the number of customers). In this case, the condition is verified.

The global cost is : $W = C_{12}X_{12} + C_{22}X_{22} + C_{31}X_{31} + C_{32}X_{32} + C_{33}X_{33} + C_{34}X_{34}$

$$W = 2*30 + 5*40 + 2*30 + 8*20 + 4*80 + 6*20 = 920$$

(c) Méthode approximative de Vogel (VAM) :

In each case, we calculate the penalties for the rows and columns which have a non-zero quantity of supply or demand. For example, for the first row the lowest cost is 2, the next value is 3, so the penalty is P1=3-2=1. The result is that column B4 has received the largest penalty (=4), in this column the smallest cost corresponds to cell C_{14}, so $X_{14}=20$.

After adjusting supply and demand, we recalculate the penalties for the rows and columns with non-zero supply or demand quantities.

The largest penalty corresponds to the 2nd column for which the small cost equals 2. Then $X_{12}=10$.

Recalculating the penalties, we notice that the first and third columns have the same penalty (=4), we chose the first column because it contains the smallest cost (=2), so $X_{31}=30$.

And $X_{33}=80$, $X_{22}=40$, $X_{32}=40$.

	B1	B2	B3	B4	supplay	Penalties
A1	3	2 10	5	2 20	~~30~~ ~~10~~ 0	1 1 - -
A2	6	5 40	8	7	~~40~~ 0	1 1 1 3
A3	2 30	8 40	4 80	6	~~150~~ ~~120~~ ~~40~~ 0	2 2 2 4
Demand	~~30~~ 0	~~90~~ ~~80~~ ~~40~~ 0	~~80~~ 0	~~20~~ 0		
Penalties	1	3	1	4		
	1	3	1	-		
	4	3	4	-		
	-	3	4	-		

The number of BV equals (6), which is the same number given by the formula (m+n-1), so the condition is satisfied.

The overall cost is: $W = C_{12}X_{12} + C_{14}X_{14} + C_{22}X_{22} + C_{31}X_{31} + C_{32}X_{32} + C_{33}X_{33}$

$$W = 2*10 + 2*20 + 5*40 + 2*30 + 8*40 + 4*80 = 960$$

Note: three different basic feasible solutions have been selected, and the least-cost solution is the closest to the optimum solution.

2- determining the optimal solution as follows: For the initial solution, logically the solution closest to the optimal solution should be used (the lowest overall cost), but since the aim of this course is to explain the algorithms of the methods in detail, the initial solution is the one obtained by the north-west corner method.

(a) Modified Distribution method (MODI)

We calculate u_i, v_j accorfing to the formula: $C_{ij} = u_i + v_j;\ \forall X_{ij} > 0$

So, it result (m + n -1) equalities with (m + n) unknown values. We have set u1 = 0 and recursively solve the following system:

BV		$u_1 = 0$ than,
X_{11}	$C_{11} = u_1 + v_1$	$v_1 = 3$
X_{12}	$C_{12} = u_1 + v_2$	$v_2 = 2$
X_{22}	$C_{22} = u_2 + v_2$	$u_2 = 3$
X_{32}	$C_{32} = u_3 + v_2$	$u_3 = 6$
X_{33}	$C_{33} = u_3 + v_3$	v_3=-2
X_{34}	$C_{34} = u_3 + v_4$	$v_4 = 0$

The values of u_i, v_j are listed in the transportation table at the beginning of the corresponding rows and columns. Then, we calculate the value of the components of the relative cost vector for each non-basic variable: $C_{ij} - (u_i + v_j) \geq 0;\ for\ all\ NBV.$

We can see that the cost relating to NBV X_{31} is negative, so it is the entering variable (EV). So we construct a cycle starting from this cell and closing on it.

		B1	**B2**	**B3**	**B4**
	V_j / U_i	**3**	**2**	**-2**	**0**
A1	**0**	**3** **30**	**2** ε	**5** 7	**2** 2
A2	**3**	**6** 0	**5** **40**	**8** 7	**7** 4
A3	**6**	**2** -7	**8** **50**	**4** **80**	**6** **20**

The closed cycle will be: **X_{31}➔**X_{32} ➔X_{12}➔ X_{11}➔ **X_{31}**

Applying the principle of this method to define the leaving variable

Cycle	$X_{31} \rightarrow X_{32} \rightarrow X_{12} \rightarrow X_{11} \rightarrow X_{31}$			
Sign	**+**	**-**	**+**	**-**
BV values		**<u>50</u>**	ε	30
The smallest value preceded by the sign (-) is **30 ➔ LV:** X_{11}				
Treatment	**+30**	**-30**	**+30**	**-30**
New values of BV	30	20	30	

The second iteration is shown in the table below:

		B1	**B2**	**B3**	**B4**
	V_j / U_i	**-4**	**2**	**-2**	**0**
A1	**0**	**3** 7	**2** **30**	**5** 7	**2** 2
A2	**3**	**6** 7	**5** **40**	**8** 7	**7** 4
A3	**6**	**2** **30**	**8** **20**	**4** **80**	**6** **20**

Note that all relative costs are positive, i.e. the solution is optimal.

The overall cost is: W= $C_{12}X_{12}$+ $C_{22}X_{22}$+ $C_{31}X_{31}$+$C_{32}X_{32}$+ $C_{33}X_{33}$+ $C_{34}X_{34}$

W= 2*30 +5*40+2*30+8*20+4*80+6*20=920

(b) Stepping Stone method

We take the initial solution obtained by Vogel's approximate method as our starting point.

	B1	**B2**	**B3**	**B4**
A1	**3**	**2** **10**	**5**	**2** **20**
A2	**6**	**5** **40**	**8**	**7**
A3	**2** **30**	**8** **40**	**4** **80**	**6**

We define the closed cycles corresponding to the NBV and then calculate the relative costs for each cycle:

NBV	**Cycle**	**Cost**
X_{11}	$X_{11} \rightarrow X_{12} \rightarrow X_{32} \rightarrow X_{31} \rightarrow X_{11}$	3-2+8-2=7
X_{13}	$X_{13} \rightarrow X_{33} \rightarrow X_{32} \rightarrow X_{12} \rightarrow X_{13}$	5-4+8-2=7
X_{21}	$X_{21} \rightarrow X_{22} \rightarrow X_{32} \rightarrow X_{31} \rightarrow X_{21}$	6-5+8-2=7
X_{23}	$X_{32} \rightarrow X_{33} \rightarrow X_{32} \rightarrow X_{22} \rightarrow X_{32}$	8-4+8-5=7
X_{24}	$X_{24} \rightarrow X_{14} \rightarrow X_{12} \rightarrow X_{22} \rightarrow X_{24}$	7-2+2-5=2
X_{34}	$X_{34} \rightarrow X_{14} \rightarrow X_{12} \rightarrow X_{32} \rightarrow X_{34}$	6-2+2-8=-2

The entering variable is X_{34} :

Cycle	$X_{34} \rightarrow$	$X_{14} \rightarrow$	$X_{12} \rightarrow$	$X_{32} \rightarrow$	X_{34}
Sign	**+**	**-**	**+**	**-**	
BV values		**20**	10	40	
The smallest value preceded by the sign (-) is **20 ➔ LV:** X_{14}					
Treatment	**+20**	**-20**	**+20**	**-20**	
New values of BV	20		30	20	

The result are as follows :

	B1	**B2**	**B3**	**B4**
A1	**3**	**2** **30**	**5**	**2**
A2	**6**	**5** **40**	**8**	**7**
A3	**2** **30**	**8** **20**	**4** **80**	**6** **20**

Once again, we define the closed cycles corresponding to the NBV and then calculate the relative costs for each cycle:

VHB	Cycle	Coût relatif
X_{11}	$X_{11} \rightarrow X_{12} \rightarrow X_{32} \rightarrow X_{31} \rightarrow X_{11}$	3-2+8-2=7
X_{13}	$X_{13} \rightarrow X_{33} \rightarrow X_{32} \rightarrow X_{12} \rightarrow X_{13}$	5-4+8-2=7
X_{14}	$X_{14} \rightarrow X_{34} \rightarrow X_{32} \rightarrow X_{12} \rightarrow X_{14}$	2-6+8-2=2
X_{21}	$X_{21} \rightarrow X_{22} \rightarrow X_{32} \rightarrow X_{31} \rightarrow X_{21}$	6-5+8-2=7
X_{23}	$X_{32} \rightarrow X_{33} \rightarrow X_{32} \rightarrow X_{22} \rightarrow X_{32}$	8-4+8-5=7
X_{24}	$X_{24} \rightarrow X_{14} \rightarrow X_{12} \rightarrow X_{22} \rightarrow X_{24}$	7-2+2-5=2

All the relative costs are positive, which means that there is no NB variable that reduces the overall cost, so this solution is optimal.

The overall cost is : $W= C_{12}X_{12}+ C_{22}X_{22}+ C_{31}X_{31}+C_{32}X_{32}+ C_{33}X_{33}+ C_{34}X_{34}$

$$W= 2*30 +5*40+2*30+8*20+4*80+6*20=920$$

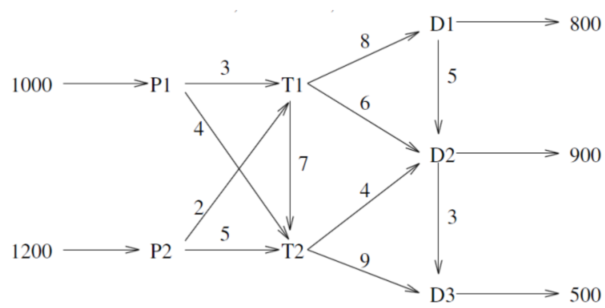
10.5 Unbalanced problem

The model is said to be unbalanced if supply does not equal demand. To this end, we introduce an artificial source (if demand exceeds supply) or an artificial destination (if supply exceeds demand). The artificial source and destination are characterised by zero costs.

10.6 Transshipment problem

A transportation problem allows only shipments that go directly from supply points to demand points. In many situations, shipments are allowed between supply points or between demand points. Sometimes there may also be points (called transshipment points) through which goods can be transshipped on their journey from a supply point to a demand point. The optimal solution to a transshipment problem can be found by solving a transportation problem.

For example, two plants P1 and P2 serve 3 vendors D1, D2 and D3, via two transit centers T1 and T2 (see diagram below).



Transformation en problème de transport :

Define a supply point to be a point that can send goods to another point but cannot receive goods from any other point.

Similarly, a demand point is a point that can receive goods from other points but cannot send goods to any other point.

A transshipment point is a point that can both receive goods from other points and send goods to other points.

– Transshipment points are both sources and destinations for the transport problem

– Buffer: quantity needed to transport all the demand through the transshipment point. In our example : B = 1000+1200=2200.

	T1	T2	D1	D2	D3	Supply
P1	3	4	M	M	M	1000
P2	2	5	M	M	M	1200
T1	0	7	8	6	M	2200
T2	M	0	M	4	9	2200
D1	M	M	0	5	M	2200
D2	M	M	M	0	3	2200
Demand	2200	2200	3000	3100	500	

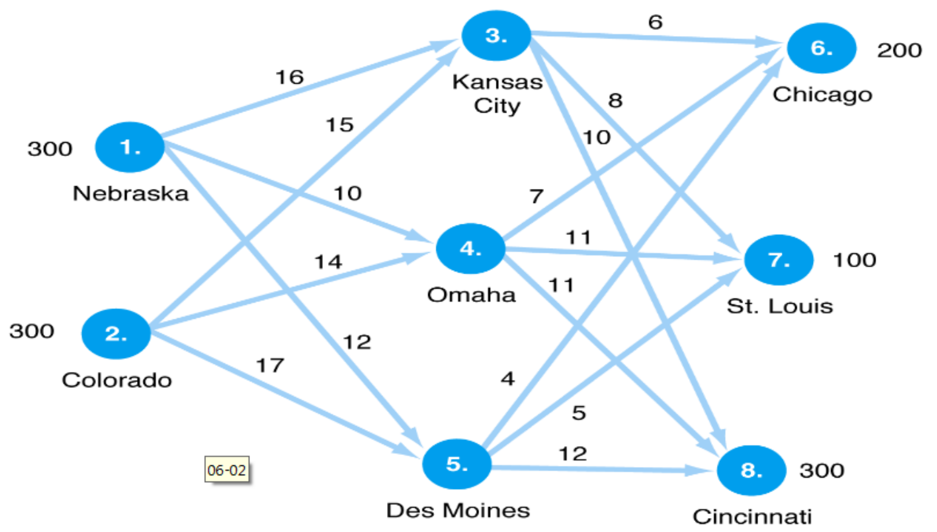
For T1 and T2, the quantity requested is the same since the two transshipment points receive the products from the two sources P1 and P2 (the sum of the quantities).

Customer D1 requests a quantity of 800, although he has the option of delivering the product to customer D2, so the buffer must be added, i.e. 800+2200=3000. The same applies to customer D2; 900+2200=3100 is the quantity requested. Customers D1 and D2 are transshipment points. Customer D3 represents a demand point, i.e. the quantity requested is 500.

The value M represents a very high transport cost, which really means that there is no relationship between the source and the destination. Using the methods described, we can easily reach the optimal solution.

Exercise:10.1

Let the following transshipment problem:



x_{ij} = amount transported from i to j , i = 1, 2, 3

Minimize Z = $16x_{13} + 10x_{14} + 12x_{15} + 15x_{23} + 14x_{24} + 17x_{25}$

$+ 6x_{36} + 8x_{37} + 10x_{38} + 7x_{46} + 11x_{47} + 11x_{48}$

$+ 4x_{56} + 5x_{57} + 12x_{58}$

subject to:

$x_{13} + x_{14} + x_{15} = 300$ (supply point constraints)

$x_{23} + x_{24} + x_{25} = 300$

$x_{36} + x_{46} + x_{56} = 200$ (demand point constraints)

$x_{37} + x_{47} + x_{57} = 100$

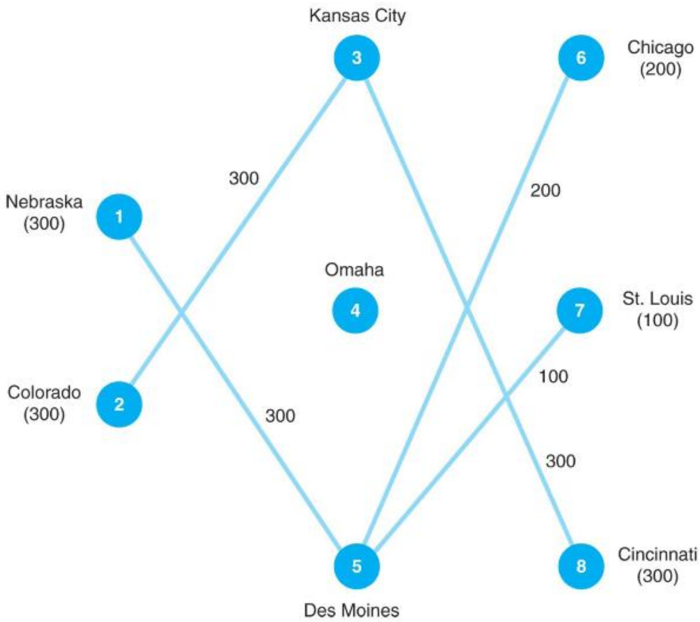
$x_{38} + x_{48} + x_{58} = 300$

$x_{13} + x_{23} - x_{36} - x_{37} - x_{38} = 0$ (transshipment point constraints)

$x_{14} + x_{24} - x_{46} - x_{47} - x_{48} = 0$

$x_{15} + x_{25} - x_{56} - x_{57} - x_{58} = 0$

$x_{ij} \geq 0$



10.7 Other uses of the transportation problem

The transportation model is not limited to the transportation of products between geographical sources and destinations. In fact, it can be adapted to other economic minimization problems such as financing projects, carrying out projects at minimum cost, investment choices, planning purchase or sale operations, supply problems and, of course, there are a multitude of cases that can easily be adapted to the transport model. By example, here are two cases:

Example 1 (Production planification)

A company manufactures rucksacks, for which demand from March to June is 100, 200, 180 and 300 units respectively. Production for these months is 50, 180, 280 and 270, respectively.

Demand can be satisfied:

1. by the current month's production ($40 / bag);
2. by the production of a previous month (+ $0.5 / bag / month for storage);
3. by the following month's production (+ $2 / bag / month late delivery penalty).

Correspondence with the transportation model:

Transport	Production – stocks	
Source i	Production period i	March, April, May, June
Destination j	Demand period j	March, April, May, June
Supply at the source i	Production capacity at period i	50, 180, 280, 270.
Demand at the destination j	Demand for period j	100, 200, 180, 300.
Transportation cost from i to j	(Unit cost (production + inventory + penalty for production in period i for period j	See the following table

Cost table:

	March	April	May	June	supply
March	40	42.5	45	47.5	50
April	42.5	40	42.5	45	180
May	45	42.5	40	42.5	280
June	47.5	45	42.5	40	270
Demand	100	200	180	300	

Exercice 10.2 (résolu) :

The sports activities league of the University of Mascara has organized a football competition. Three football pitches were selected (Africa unit stadium, d'El Bordj stadium and Tighennif stadium) to program one match per day. The Sociales office of the University of Mascara has been commissioned to transport participants and spectators from three starting points (the 2000-bed university campus, the Mamounia center and the Sid Said center) by mini-bus with a capacity of 30 seats. The following table describes the different transport costs.

	Stadium			Students number
	Africane unit	**El Bordj**	**Tighennif**	
City of university	5	7.5	6.5	**90**
Pole of Sidi Said	5	7.5	6.5	**90**
Pole of Mamounia	5.5	8	9	**95**
	120	**70**	**85**	

- Define a feasible basic solution to the problem.
- Define the optimal solution and calculate the overall transport cost.

Assume that the chosen solution involves a single journey (one way or return), and that the competition has been programmed over three days:

- What will the overall cost be?
- Define the number of buses needed for the transport according to the solution obtained.

Solution 10.2

Note that the problem is balanced, i.e. the number of students equals the number of places available: 90+90+95 = 120+70+85 = 275.

- *Defining a feasible basic solution*: using the Minimum cost method

From the obtained basic solution, we can see that the number of VBs is equal to (m+n-1), i.e. 5 basic variables.

	St. Mascara	St. El Bordj	St. Tighennif	Nb students
2000 lits	5 90	7.5	6.5	~~90~~ 0
Sidi Said	5 30	7.5	6.5 60	~~90~~ ~~60~~ 0
Mamounia	5.5	8 70	9 25	~~95~~ ~~25~~ 0
Capacité	~~120~~ ~~30~~ 0	~~70~~ 0	~~85~~ ~~25~~ 0	**275** / **275**

The overall cost is : W= 5*90+5*30+6.5*60+8*70+9*25=1775 um.

- *We then check the optimality of the solution* using the modified distribution method.

Calculate the values of the dual variables that satisfy the relationship: Cij=ui+vj for all the base variables. Assuming u1=0, we obtain recursively: u2=0, u3=2.5, v1=5, v2=5.5, v3=6.5.

If we now calculate the costs relating to the NBV, we can see that there are negative values, i.e. the solution is not optimal.

ui \ vj	**5**	**5.5**	**6.5**
0	5 90	7.5 **2.5**	6.5 **0**
0	5 30	7.5 **2.5**	6.5 60
2.5	5.5 **-1.5**	8 70	9 25

Constructing the closed cycle corresponds to NBV X_{31} :

X_{31} →	X_{33} →	X_{23} →	X_{21} → X_{31}
+	-	+	-
<u>25</u>	60	30	
+25	**-25**	**+25**	**-25**
25		**85**	**5**

The table is as follows:

<table>
<tr><td rowspan="4">The values of the dual variables that satisfy the relationship: Cij=ui+vj for all the base variables are calculated again. Assuming u1=0, we obtain recursively: u2=0, u3=0.5, v1=5, v2=7.5, v3=6.5.
The costs relating to the NBV are all positive, so the solution is optimal.</td><td>ui \ vj</td><td>5</td><td>7.5</td><td>6.5</td></tr>
<tr><td>0</td><td>5
90</td><td>7.5
0</td><td>6.5
0</td></tr>
<tr><td>0</td><td>5
5</td><td>7.5
0</td><td>6.5
85</td></tr>
<tr><td>0.5</td><td>5.5
25</td><td>8
70</td><td>9
2</td></tr>
<tr><td colspan="5">The overall cost is : W= 5*90+5*5+6.5*85+5.5*25+8*70=1725 um.</td></tr>
</table>

Remark:

There are NBV with a zero relative cost, which means that the current solution can be changed while keeping the same value of the economic function (1725 um), because the relative cost of a NBV represents its influence on the value of the EF if this variable is retained as an input variable in the base.

If we consider that the variable X_{12} is a EV, the solution will be as follows:

X_{11}=20, X_{12}=70, X_{21}=5, X_{23}=85, X_{31}=95. Ce qui donne W=1725 um.

It will be the same case if we choose X_{13} or X_{22} as the base input variable. We say that there are alternative solutions.

Supposing that the solution chosen corresponds to a single journey (outward or return), this gives six (6) journeys for duration of three days:

- Overall cost = 1725*6=10350 um
- The number of buses required for transport according to the solution obtained will be:
  1. To transport 90 students from the university campus to the African Unity Sports Centre, three (3) buses are needed (90/30=3).
  2. To transport 5 students from the Mamounia pole and 25 students from the Sidi Said pole all to the African Unity Omnisports, one (1) bus is required.
  3. To transport 70 students from the pole of Mamounia towards the stadium of El Bordj, it requires three (3) buses (70/30=2.33).
  4. To transport 85 students from the Sidi Said pole to the Tighennif stadium, three (3) buses are needed (85/30=2.83).

  So ten (10) buses are therefore needed to transport the students over three days.

Exercise 10.3:

A factory of plastic products has three warehouses located in Mascara, Oran and Algiers. Following a stock shortage of a type of raw material essential to the manufacture of a product which has been in high demand by its four major customers, the decision-makers are looking for a better delivery strategy in terms of cost and time. Information on supply and demand and the unit cost of transport is presented in the table below:

- Define the optimal transport strategy.
- How can you justify that this strategy is optimal in terms of time?

	Customer1	Customer 2	Customer 3	Customer 4	supply
Mascara	2	3	7	11	200
Oran	5	8	5	12	125
Algiers	14	13	3	4	75
Demand	100	20	80	200	

Chapter 11

11 Transportation problem – case of maximization-

11.1 Introduction

The use of transportation problems is not limited to minimization problems, as they can be applied to profit maximization problems, production maximization problems, etc. Overall, they are applicable to any maximization problem that can be adapted to a structure consistent with transportation models. Unlike the minimization problems, the economic function takes the direction of maximization and unit costs are replaced by unit profits as appropriate.

11.2 Mathematical form

The transportation model is expressed mathematically by the general form :

$$Max\ z = \sum_{i=1}^{m} \sum_{j=1}^{n} C_{ij} X_{ij}$$

$$\text{S.C :} \begin{cases} \sum_{j=1}^{n} X_{ij} = S_i \ ; pour\ i = 1,2, \dots, m \\ \sum_{i=1}^{m} X_{ij} = D_j \ ; pour\ j = 1,2, \dots, n \\ \sum_{i=1}^{m} S_i = \sum_{j=1}^{n} D_j \\ X_{ij} \geq 0, C_{ij} \geq 0 \end{cases}$$

With: "m" number of sources, "n" number of destinations, "Dj" quantities requested, "Si" supply quantities.

11.3 Transportation problem resolution

There is not much difference between solving maximization and minimization problems, except for a few adjustments to the methods presented in the previous chapter. Firstly, the constraint of equality of supply and demand quantities must always be checked. If this is not the case, artificial sources (or destinations) are added.

11.3.1 Searching of initial basic solution

a- North-West Corner Method: at each stage, choose the variable located at the intersection of the first row and the first column of the reduced table. So, starting from the top left-hand corner of the table, here are the steps:

1. allocate as much as possible to the current cell and adjust supply and demand ;
2. move one cell to the right (zero demand) or the bottom (zero supply);
3. Repeat until all the supply has been allocated.

b- Maximum profit method: at each stage, choose the variable C_{pq} corresponding to the greatest profit in the reduced table. . $C_{pq} = \max_{ij}(C_{ij})$

Here are the detailed steps:

Select the maximum profit cell.

1. Allocate as much as possible to the current cell and adjust supply and demand;
2. Select the maximum profit cell with non-zero demand and supply;

3. Repeat until all the supply has been allocated.

c- Vogel's Approximation Method (VAM): it is based on calculating the values corresponding to the difference between the two largest successive profits for each row (column) with non-zero supply (demand), then selecting the row or column with the maximum value and selecting the cell with the maximum profit in the row or column; and so on until all the supply is allocated.

11.3.2 Searching for optimal solution

The two methods described in the previous chapter are used:

a- Modified Distribution method (MODI)

The dual program of a transportation problem is given by:

$$Min: W = \sum_{i=1}^{m} a_i\, u_i + \sum_{j=1}^{n} b_j\, v_j$$

$$with:\; u_i + v_j > C_{ij}\;; i = 1, \dots, m, j = 1, \dots, n$$

Once the value of the dual variables has been calculated, it is easy to calculate the components value of the relative profit vector for each nonbasic variable:

$$\mathrm{C_{ij}} - (\mathrm{u_i} + \mathrm{v_j}) \leq 0;\; for\ all\ NBV.$$

Thus, the basic solution considered is optimal if and only if:

$$C_{ij} - (u_i + v_j) \leq 0;\; pour\ tout\ X_{ij} > 0$$

If one of these components is negative, then we are not at the optimum and must define the leaving and entering variables to preserve eligibility:

To do this we must:

- Define the entering variable which is characterized by the largest value of positive relative profits.
- Construct a cycle running through the basic variables, starting from and returning to the entering variable;
- Alternately mark with + and - the variables which make up the selected cycle, starting with the entering variable.
- Move along rows and columns, alternately adding and removing the smallest quantity among the quantities preceded by a minus sign (-), which corresponds to the leaving variable.
- Recalculate the new values of the dual variables, then the values of the relative profits. If the latter are not positive, then the solution is optimal, otherwise the previous steps are repeated until the optimum is reached.

b- Stepping stone method

We define the closed cycles for all the nonbasic variables and then calculate the relative benefit of each cycle as if we were in the minimization case.

- The Entering variable corresponds to the cycle with the greatest positive benefit.
- The rest of the steps are identical to the modified distribution method.

Exercice résolu :

A transportation company has received an order from a principal to transport potatoes from three ports to three stocks. The company's profit changes according to the distance covered. The information on the quantities in tonnes of supply and demand and the profit (x1000) in dinars are shown in the following table:

	Stock 1	**Stock 2**	**Stock 3**	**supply**

				(tonnes)
Port 1	9	3	1	200
Port 2	6	3	0.5	150
Port 3	4	0.5	8	250
Demand (tonnes)	280	220	100	

- Define the best transport strategy to ensure maximum profit.

Solution :

Note that the sum of the quantities demanded is equal to the sum of the quantities offered, so we can define a basic solution that can be achieved using the maximum profit method:

1. The highest profit is entered in cell (1,1), for which 200 T are offered and 280 T are requested, so variable X11 receives 200.
2. The next largest profit is in cell (3,3), so X33 receives 100, so the demand for stock 3 is fully met.
3. At this stage, the largest profit is that of cell (2,1) and since the remaining demand from stock 1 is 80, the supply from port 2 is 150, so variable X21 receives 80.
4. Next, cell (3,1) has the highest profit, but since the quantity requested has been completely paid for, we move on to cell (1,2) with a profit of 3,000 dinars, but we notice that the supply quantity has been completely consumed. We move on to cell (2.1) with a profit of 3,000 dinars, where the offer quantity is 70 and the requested quantity is 220. Variable X21 therefore receives 70, i.e. the offer for port 2 is consumed.
5. All that remains is cell (3,2) with available supply and demand quantities. So X32 receives 150. The basic solution is shown in the following table:

	Stock 1	**Stock 2**	**Stock 3**	**supply (tons)**
Port 1	9 200	3	1	~~200~~ 0
Port 2	6 80	3 70	0.5	~~150~~ ~~70~~ 0
Port 3	4	0.5 150	8 100	~~250~~ ~~150~~ 0
Demand	~~280~~ ~~80~~ 0	~~220~~ ~~150~~ 0	~~100~~ 0	**600** / **600**

The overall profit is: Z= $(9*200+6*80+3*70+0.5*150+8*100)*10^3 = 3365*10^3$ um.

We examine the optimality of this solution using the stepping stone method. We begin by calculating the relative benefits of empty cells (NBV):

NBV	Cycle	relative profits
X_{12}	$X_{12} \rightarrow X_{22} \rightarrow X_{21} \rightarrow X_{11} \rightarrow X_{12}$	3-3+6-9=-3
X_{13}	$X_{13} \rightarrow X_{33} \rightarrow X_{32} \rightarrow X_{22} \rightarrow X_{21} \rightarrow X_{11} \rightarrow X_{13}$	1-8+0.5-3+6-9=-12.5
X_{23}	$X_{23} \rightarrow X_{33} \rightarrow X_{32} \rightarrow X_{22} \rightarrow X_{23}$	0.5-8+0.5-3=-10
X_{31}	$X_{31} \rightarrow X_{21} \rightarrow X_{22} \rightarrow X_{32} \rightarrow X_{31}$	4-6+3-0.5=0.5

The solution is not optimal because there are NBV with a positive relative benefit, so the EV is X_{31} :

$X_{31} \rightarrow X_{21} \rightarrow X_{22} \rightarrow X_{32} \rightarrow X_{31}$				
+	-	+	-	
	80	70	150	
+80	-80	+80	-80	
80		150	70	

The solution is as follows:

	Stock 1	**Stock 2**	**Stock 3**	**Supply (tons)**
Port 1	9 200	3	1	200
Port 2	6	3 150	0.5	150
Port 3	4 80	0.5 70	8 100	250
Demand	280	220	100	**600** / **600**

Calculating the relative benefits of empty cells (NBV) :

NBV	Cycle	relative profits
X_{12}	$X_{12} \rightarrow X_{32} \rightarrow X_{31} \rightarrow X_{11} \rightarrow X_{12}$	3-0.5+4-9=-2.5
X_{13}	$X_{13} \rightarrow X_{33} \rightarrow X_{31} \rightarrow X_{11} \rightarrow X_{13}$	1-8+4-9=-12
X_{21}	$X_{21} \rightarrow X_{22} \rightarrow X_{32} \rightarrow X_{31} \rightarrow X_{21}$	6-3+0.5-4=-0.5
X_{23}	$X_{23} \rightarrow X_{33} \rightarrow X_{32} \rightarrow X_{22} \rightarrow X_{23}$	0.5-8+0.5-3=-10

All relative profits are negative, so there is no nonbasic variable that can increase overall profit, so the solution is optimal.

The overall profit is :Z= (9*200+3*150+4*80+0.5*70+8*100)*103 =3405*103 um.

This solution is expressed as follows:

The company has to transport a quantity of 200 tonnes from port1 to stock1, 150 tonnes from port2 to stock2 and 250 tonnes from port3, distributed among the three stocks in quantities of 80, 70 and 100 respectively.

The company will earn the sum of : 3 405 000 um.

Exercise:

A fish delivery company aims to satisfy the needs of its customers in the West while respecting the quantity requested and the delivery time. To this end, he has set up four depots, well distributed geographically in relation to his customers. He usually brings in fish from three fisheries (Oran, Mostaganem and Ain Témouchent). The earnings per load of fish as well as the supply quantities and capacities (per unit) of the depots are shown in the following table:

		Dépôt1	**Dépôt2**	**Dépôt3**	**Dépôt4**	**Offre**
Pêcheries	Oran	10	30	20	11	200
	Mostaganem	12	7	9	20	350
	Ain Témouchent	30	14	16	18	180
	Capacité d'accueil	260	300	120	50	

- Using all studied methods, to help this delivery driver in his mission.

- Explain the retuned solution.
- What is the total gain?
- Comment on the delivery time problem.

References

Hamdy A. Taha, Operations Research, an introduction, 8 edition. Prentice-Hall, 2007.

Hillier, Frederick S. Introduction to operations research. 2001. /Frederick S. Hillier, Gerald J. Lieberman; cases developed by Karl Schmedders and Molly Stephens; tutorial software developed by Mark Hillier and Michael O'Sullivan.—7th ed. ISBN 0-07-232169-5

Markov D. P., Predrag S., and Nebojsa V. A modification of revised simplex method. 2002 [163-169].

Ratoul M. Recherche Opérationnelle. 2006, 2ième édition.

Robert F., Bernard L., et Christophe P. Précis de recherche opérationnelle, Méthodes et exercices d'application. Collection : Sciences Sup, Dunod. 2014 - 7ème édition.

Sakarovitch, M. Linear programming. Springer-Verlag, New York-Berlin. 1983.

Werra, D., Liebling, T.-M., et Héche, J.-F. Recherche Opérationnelle pour Ingénieurs, Tome 1. Presses Polytechniques et Universitaires Romandes, 2003.

Wolsey, L. A. Integer Programming. Wiley-Interscience, 1998.

Support de cours

Brauner, N. Cours de recherche opérationnelle I, , Grenoble 2015

Bernard F. Recherche opérationnelle et applications. 2012

Poly de cours http://www.g-scop.grenoble-inp.fr/~braunern

Sites internet

http://www.phpsimplex.com/fr/theorie_methode_simplexe.htm#deux_phases

http://www2.informs.org/Resources/

http://www.ensta.fr/~diam/ro/

Logiciels pour la RO

http://www.coin-or.org/resources.html

http://www.wior.uni-karlsruhe.de/bibliothek/