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On certain operator modules and the normality of non-commutative varieties

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Dedication

I dedicate this dissertation

To my parents,and my husband whose endless love, support, and encouragement have been my guiding light. Their belief in my abilities has inspired me to strive for excellence.

A special dedicate to my children Abdelhadi, Wassim and my honey Amani

To my sisters, whose constant support, and companionship have made this journey more enjoyable and manageable .

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Abstract

Within the framework of the theory of operator spaces and noncommutative geometry, we study problems well known in literature namely, we give a positive answer to the hyperinvariant subspace problem for the operators intertwined with weighted bilateral shifts and other partials results in this context. In the category of Hilbert spaces we study the similarity problem of certain Hilbert modules by mean of the cohomology groups and we introduce a new class of operator (bi)modules wich generalizes the Hilbert C*-modules. Finaly, we study the noncommutative version of Serre-Swan theorem in the setting of Banach and Hilbert categories and we initiate the study of the classification of the Cuntz-Pimsner algebras by mean of thier associates C*-correspondences.

Key words: Operatos modules, invariant subspace problem, Hilbert modules, Hilbert C*-modules, Bundles of Banach spaces, Cuntz-PImsner algebras.

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Introduction

A major trend in modern mathematics, largely inspired by physics, is towards "non-commutative" or "quantized" phenomena. In the vast field of functional analysis, this trend has appeared notably under the name of operator spaces and operator modules. This young field covers the theory of Hilbert modules by Douglass and Paulsen [17], the theory of C*-Hilbert modules by Kaplansky [38,39,46], noncommutative spaces by E. Efros-Z. Ruan [23,24], D. Blecher [7] V. Paulsen [55,56] and its interactions with noncommutative geometry by A. Conne [12]. In this thesis we focus on problems which are well known in the literature, namely the problem of subspace (hyper), the problem of the similarity of operators in the category of Hilbert modules, the representation of some (bi)modules of operators. Also, we study the noncommutative version of Serre-Swan [69] theorem in the setting of Banach and Hilbert bundles and within the framework of noncommutative spaces (noncommutative varieties), we study certain C*-algebra defined from the theory of Hilbert C*-modules, sayed the Cuntz-Pimsner algebras, and we try to answer to the question: If two C*-correpondences X, Y over a C*-algebras A and B, respectively, are related in a particular way, what can be said about their Cuntz-Pimsner algebras O_X, O_Y?.

The invariant subspace problem is a major unsolved problem in operator theory and functional analysis. The problem asks whether every operator $T \in B(H)$ has a nontrivial invariant subspace. In a similar fashion, the Hyperinvariant Subspace Problem is whether every bounded linear operator such that $T \neq \alpha I$ has a non trivial hyperinvariant subspace. The invariant subspace problem was initially posed by David Hilbert in 1900 and has since captured the attention of mathematicians due to its profound implications. The study of (Hyper)invariant subspaces not only provides insights into the structure of linear operators but also has significant applications in various branches of mathematics and physics. Understanding the behavior of linear operators and the existence of invariant subspaces has profound consequences in areas such as spectral theory, operator algebras, noncommutative geometry and representation of groups and algebras. Over the years there have been many partial solutions, answering the question for certain types of Banach space and for certain classes of operators. Famously, the most general case was answered by Per Enflo in the 1980's when he showed that there exists an operator on a Banach space possessing no nontrivial closed invariant subspace. Today the variation of the problem which remains elusive is the case of operators on a separable Hilbert space. Intertwining relations of operators with respect to this problem are applied during long time. For example, it is known from 1970's that if two operators are quasisimilar, and one of them has nontrivial hyperinvariant subspace, then the other has nontrivial hyperinvariant subspace, too. On the other hand, if $UX = XT$, for some nonzero operator X and U is an absolutely continuous unitary operator, then the (Hyper)invariant Subspace Problem is still open even under the assumption that T is a contraction.

On the other hand, the topology and geometry of a space X can be studied using only algebraic information. For example, the Serre-Swan theorem [40] tells us that there is a bijective correspondence between finitely generated projective C*-modules and vector bundles over X. Another landmark result is the Gelfand-Naimark theorem [2, 12, 57], published in 1943, which states that locally compact Haussdorf spaces can be reconstructed, up to homeomorphism, from the commutative C*-algebra $C_0(X)$ of continuous functions vanishing at infinity and, vice versa, in the category lunguage we can say that there is a complete equivalence between the category of (locally) compact Hausdorff spaces and (proper and) continuous maps and the category of commutative (non necessarily) unital C^*-algebras and $*$-homomorphisms. If we now consider non-commutative C*-algebras instead, we can still use these algebraic formulations of topological features despite our algebra lacking an underlying topological space. We can refer to the "virtual" topological space underlying our non-commutative algebra as a non-commutative space, this is the main topic of the nonocommutative geometry [12]. A noncommutative C^*-algebra will be now thought of as the algebra of continuous functions on some 'virtual' noncommutative space. The most tautological way to resolve the problem of filling the gap in the noncommutative version of Gelfand-Naimark theorem is to define the category of non-commutative spaces $NCTop$ as equal to the category Alg^{op}, a non-commutative space is just a non-commutative algebra.

This thesis is organized in four chapters. In the first Chapter, we provide some basic background on wieghted spaces and weighted shift operators, Banach algebra, C*-algebras, modules, Hilber C*-modules, operator modules and category theory. Our aim in this chapter is to provide some basic notations, definitions and theorems.

Following the line of researche and the problems explained before, in the second chapter we give a positive answer to the hyperinvariant subspace problem for an operator T which intertwines a weighted bilateral shift and other partial results. Also, in the category of Hilbert modules, by mean of the cohomology groups we give some partial results concerning the similarity of Hilbert modules over the Disc algebra.

In the third chapter, we introduce a new class of normed modules over C*-algebras, sayed semi-inner product (bi)modules, which generalizes the Hilbert C*-modles and we show that it is representable and it is an operator space, in particular it ia an operator module.

In the last chapter, on the one hand, we study the noncommutative version of Serre-Swan theorem and we show that the semiinner product (bi)modules represents the noncommutative version of the continuous fields of Banach spaces in the sens of Fell. On the other hand, within the framework of the theory of noncommutative spaces we are gathering the material necessary to study and determine the category of noncomuutative spaces (noncommutaive varieties) $NCTop$ and we initiate the study of the classification of the Cuntz-Pimsner algebras by mean of thier associate C*-correspondences.

Chapter 1

Preliminairies

1.1 Hilbert spaces

Throughout this thesis, H denotes an infinite dimensional complex separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and $B(H, K)$ denotes the space of all bounded linear operators acting from H to K. The range of an operator T will be denoted by $RanT$ and the closed subspace spanned by a set A is denoted by $\vee\{a : a \in A\}$.

The commutant of T, denoted by $\{T\}^{\backslash}$, is the algebra of all operators $A \in B(H)$ such that $AT = TA$. A closed subspace $M \subseteq H$ is called a nontrivial hyperinvariant subspace for T if $0 \neq M \neq H$and $AM \subseteq M$ for every $A \in \{T\}^{\backslash}$. If $TM \subseteq M$, then the subspace M, $0 \neq M \neq H$, is called a nontrivial invariant subspace for T.

Definition 1.1.1 *1. Let A, B be bounded linear operators on the Hilbert spaces H and K respectively. We say A and B are similar if there exists an invertible operator X such that $XA = BX$.*

2. *An operator $T \in B(H)$ is said to be contraction if $\|T\| \leq 1$.*

3. *An operator $T \in B(H)$ is said to be polynomially bounded if there exists $C > 0$ such that $\|P(T)\| \leq C\|P\|_\infty$ for every polynomial P, where $\|P\|_\infty = sup_{|z|<1}|P(z)|$.*

It is well known, by von Neumann inequality, that every contraction operator is polynomially bounded. Recall that a unitary operator is singular (resp. absolutely continuous) if its spectral measure is singular (resp. absolutely continuous) with respect to the Lebesgue measure on the unit circle. And any contraction T can be decomposed uniquely as the direct sum $T = U_s \oplus U_a \oplus T_0$, where U_s, U_a are singular and absolutely continuous unitary operators respectively and T_0 is a completely nonunitary contraction. T is said to be absolutely continuous if in this decomposition U_s is absent. For this type of decomposition for polynomially bounded operators, see [10, 21, 36, 71].

We denote by $\mathbb{D}$ the open unit disc and by $\mathbb{T}$ the unit circle. Let m denote the normalized Lebesgue measure on the unit circle $\mathbb{T}$ (i.e., $m = \frac{d\theta}{2\pi}$) and let $L^2 = L^2(\mathbb{T})$ denote the space of all complex-valued Lebesgue measurable functions on $\mathbb{T}$ such that $\|f\|^2 = \int_{\mathbb{T}} |f(t)|^2 dm(t)$ is finite. As such, L^2 is a Hilbert space, a simple calculation using the fact that $m(\mathbb{T}) = 1$ shows that this space has a canonical orthonormal basis $\{z^n : n \in \mathbb{Z}\}$ given by $z^n(\xi) = \xi^n$, for all $n \in \mathbb{Z}$; $\mathbb{Z}$ being the set of integers and z denotes the identity function, i.e., $z(\xi) = \xi$; $\xi \in \mathbb{T}$ and in the sequel we set $\mathbf{1} \equiv z^0$.

The Hardy space $\mathbb{H}^2 = \mathbb{H}^2(\mathbb{T})$ is the closed linear span of $\{z^n : n = 0, 1, ...\}$. The operators of multiplication by the identity function z on the spaces $\mathbb{H}^2$ and $\mathbb{H}^2_- = L^2 \ominus \mathbb{H}^2$ are the unilateral forward shift S in $\mathbb{H}^2$ defined by $(Sf)(\xi) := \xi.f(\xi)$ and the unilateral forward shift S_- in $\mathbb{H}^2_-$

defined by $(S_-f)(\xi) := \overline{\xi}.f(\xi)$. It is clear that the bilateral forward shift U on L^2 has the following form with respect to the decomposition $L^2 = \mathbb{H}^2 \oplus \mathbb{H}^2_-$:

$$U = \begin{bmatrix} S & \mathbf{1} \otimes z^{-1} \\ 0 & S_- \end{bmatrix}. \tag{1.1}$$

For a Borel set $\alpha \subset \mathbb{T}$, we write $L^2(\alpha) = L^2(\alpha, m), L^\infty(\alpha) = L^\infty(\alpha, m)$ and the operator of multiplication by the identity function z on the space $L^2(\alpha)$ will be denoted by U_α.

Definition 1.1.2 *1. An inner function is a bounded analytic function f on $\mathbb{D}$ such that $|f(z)| = 1$ for almost every z in $\mathbb{T}$, where $f(z)$ is the radial limit of f (i.e, $f(z) = \lim_{r\to 1^-} f(rz)$).*

2. Let μ be a positive, finite singular (with respect to the Lebesgue measure m) Borel measure on $\mathbb{T}$. A singular inner function is an analytic function defined by

$$\phi_\mu(z) = exp(-\int \tfrac{\zeta+z}{\zeta-z} d_\mu\zeta),\ z \in \mathbb{D}.$$

If $\mu = \delta_1$ denotes the point mass at $\zeta = 1$ then

$$\phi_{\delta_1}(z) = exp(\tfrac{z+1}{z-1}),\ z \in \mathbb{D}.$$

This type of inner function is called a (singular) atomic inner function.

3 An outer function is an analytic function F on $\mathbb{D}$ of the form

$$F(z) = exp(i\gamma + \int \tfrac{\zeta+z}{\zeta-z}\phi(\zeta) d_m(\zeta))$$

where γ is a real constant and ϕ is a real-valued function in L^1.

We recall some well known notations: for every analytic function f in $\mathbb{D}$ the function $\tilde{f}$ defined on $\mathbb{D}$ by $\tilde{f}(z) = \overline{f(\overline{z})}$ is analytic in $\mathbb{D}$ and $\widehat{\tilde{f}}(n) = \overline{\widehat{f}}(n), n \geq 0$.

For more details, see [33, 43, 66].

1.2 Weighted spaces

Let $\omega : \mathbb{Z} \to (0, \infty)$ be a nonincreasing function. Set

$$l^2(\omega) = l^2_{\mathbb{Z}}(\omega) = \{f = (f_n)_{n\in\mathbb{Z}} : \|f\|^2_\omega = \sum_{-\infty}^{\infty} |f_n|^2\omega_n^2 < \infty\}$$

$$l^2(\omega^+) = \{f \in l^2(\omega) : f_n = 0, n \leq -1\};$$

$$l^2(\omega^-) = \{f \in l^2(\omega) : f_n = 0, n \geq 0\}.$$

The bilateral weighted shift S_ω on $l^2(\omega)$ is defined by

$$\forall f \in l^2(\omega) : (S_\omega f)_n = f_{n-1}, n \in \mathbb{Z}.$$

It is clear that $l^2(\omega^+)$ is an invariant subspace for S_ω and the restriction $S_\omega|_{l^2(\omega^+)}$ of S_ω on the subspace $l^2(\omega^+)$ is a unilateral shift. If $\omega_n = 1$, for all $n \geq 0$, we write $l^2(\omega^+) = l^2_+$. By identifying elements of the Hardy space $\mathbb{H}^2(\mathbb{D})$ to their radial limits defined a.e. on $\mathbb{T}$, we can identify $\mathbb{H}^2(\mathbb{D})$ with l^2_+. Then the restriction $S_\omega|_{l^2_+}$ is the simple unilateral forward shift S and the compression S_{ω^-} of S_ω acts on the subspace $l^2(\omega^-)$ by the following formula:

$$\forall f \in l^2(\omega^-) : (S_{\omega^-} f)_n = f_{n-1}, n \leq -1.$$

Since the function ω is nonincreasing then the operator S_ω is a contraction and if ω is a non-constant function then we can check that S_ω is a completely nonunitary. In particular, the $\mathbb{H}^\infty$-functional calculus is defined for S_ω.

For $t \in \mathbb{T}$ and $f \in l^2(\omega)$ set $(f)_t = (f_n t^n)_{n\in\mathbb{Z}}$.

For $n \in \mathbb{Z}$ by z^n the sequence $f \in l^2(\omega)$ is denoted such that $f_n = 1$ and $f_k = 0$ for $k \in \mathbb{Z}$, $k \neq n$.

Let $\phi \in \mathbb{H}^\infty$. Then

$$\phi(S_\omega)z^n = (\hat{\phi}(k-n))_{k\in\mathbb{Z}} = \sum_{k=0}^{\infty} \hat{\phi}(k) z^{k+n}.$$

Definition 1.2.1 *[28] A dissymetric weight is a nonincreasing, unbounded function $\omega : \mathbb{Z} \to [1, \infty[$ satisfying the following conditions:*

1. $\omega(n) = 1, n \geq 0.$

2. $\limsup \dfrac{\omega(n-1)}{\omega(n)} < \infty.$

3. $\omega(-n)^{\frac{1}{n}} \to 1$ *when* $n \to \infty$.

Remark 1.2.2 *We note here that if ω is a dissymetric weight then the operator S_ω is an invertible completely nonunitary contraction, the invertibily of S_ω follows from the condition 2 of the previous definition. If the dissymetric weight $\omega_n = 1$, for all $n \in \mathbb{Z}$ then we get the classical spaces $l^2 = l^2(\omega)$, $\mathbb{H}^2_- = l^2(\omega^-)$ and $S_\omega = U$, the usual bilateral shift on l^2.*

If T is absolutely continuous polynomially bounded operator and $f \in \mathbb{H}^\infty$ then $f(T^*)^* = \tilde{f}(T)$. For $\phi \in \mathbb{H}^\infty$ and for $t \in \mathbb{T}$ set $\phi_t(z) = \phi(tz)$, for every $z \in \mathbb{D}$. Then, $(\phi_t)^\sim = (\tilde{\phi})_{\overline{t}}$, where $(\tilde{\phi})_{\overline{t}}(z) = \sum_{n=0}^{\infty} \overline{\widehat{\phi}(n)}\overline{t}^n z^n, z \in \mathbb{D}$ and

$$\phi_{\overline{t}}(S_\omega^*)(f)_t = (\phi(S_\omega^*)f)_t$$

for $f \in l^2(\omega), t \in \mathbb{T}$. For more details, see [28, 29, 71].

1.3 Banach algebras and modules

Definition 1.3.1 *An algebra $\mathcal{A}$ over $\mathbb{K} = \mathbb{R}, \mathbb{C}$ is a linear space over $\mathbb{K}$ with product operation . such that:*

1. *. is associative,*

2. *$a.(b+c) = ab + ac$, $(a+b).c = ac + bc$ for $a, b, c \in \mathcal{A}$.*

A is said a unitary algebra if there is a unitary element 1: $a.1 = 1.a = a$, for all $a \in \mathcal{A}$. $\mathcal{A}$ is said a commutative algebra if the product . is commutative.

Definition 1.3.2 ***1.**Let h be a map on a $\mathbb{K}$-linear space $\mathcal{X}$, $\mathbb{K} = (\mathbb{R}$ or $\mathbb{C})$. h is said to be an involution on $\mathcal{X}$ if, for all $x, y \in \mathcal{X}$, $\alpha \in \mathbb{C}$the following conditions are satisfied:*

(i) Additive: $h(x+y) = h(x) + h(y)$;

(ii) Homogene if $\mathbb{K} = \mathbb{R}$: $h(\alpha x) = \alpha h(x)$; anti-homogene if $\mathbb{K} = \mathbb{C}$: $h(\alpha x) = \overline{\alpha} h(x)$;

(iii) Involutive: $h(h(x)) = h^2(x) = x$.

In the sequel h will be noted by $*$.

2. *A normed algebra $\mathcal{A}$ with norm $\|.\|$ is an algebra and a normed space satisfying the inequality $\|xy\| \leq \|x\| \, \|y\|$, for all x, $y \in \mathcal{A}$.*

3.If $\mathcal{A}$ is completed for this norm, it will be said a Banach algebra.

4. $\mathcal{A}$ is said to be an involutive algebra or $*$- algebra if it has an involution $*$ such that $(xy)^* = y^*x^*$, for all x, $y \in \mathcal{A}$.

***5.**$\mathcal{A}$ is said to be an involutive normed (Banach) algebra or $*$- normed (Banach) algebra if it is an involutive algebra and the involution $*$ is isometric with respect to the norm given in $\mathcal{A}$.*

Remark 1.3.3 *1. If $\mathcal{A}$ is a normed algebra with unit 1, we suppose in the sequel that $\|1\| = 1$ (we can find an equivalent norm $\|.\|^{`}$ to the original one such that $\|1\|^{`} = 1$).*

2. We note if $\mathcal{A}$ is $*$-algebra with unit 1, then we from the properties of $*$ that $1^* = 1$. Also, it's completion $\overline{\mathcal{A}}$ is an involutive normed algebra.

Definition 1.3.4

1. Let $\mathcal{A}$ be a non-unitary algebra on $\mathbb{K}$. On the linear space $\mathcal{A}_{un} = \mathcal{A} \times \mathbb{K}$ we define the following operation:

$$\forall a, b \in \mathcal{A}, \forall \alpha, \beta \in \mathbb{K} : (a, \alpha) \cdot (b, \beta) = (ab + \alpha b + \beta a, \alpha\beta)$$

2. Let $(e_\alpha)_\alpha \subseteq \mathcal{A}$. $(e_\alpha)_\alpha$ is said to be a left (resp. a right) approximate in $\mathcal{A}$ if for all $x \in \mathcal{A} : e_\alpha x \to x$ ($x e_\alpha \to x$). $(e_\alpha)_\alpha$ is said to be an approximate identity if it is a left and right approximate identity $\mathcal{A}$.

Theorem 1.3.5 *1. $\mathcal{A}_{un}$ is a unitary algebra on $\mathbb{K}$ with unit $\mathbf{1} = (0, 1)$. The algebra $\mathcal{A}$ can be identified as a subalgebra of $\mathcal{A}_{un}$ under the ambedding map $\imath : \mathcal{A} \to \mathcal{A}_{un}; a \mapsto (a, 0)$. $(\mathcal{A}_{un}, \imath)$ is* *said to be the unitisation of* $\mathcal{A}$.

2. If $\mathcal{A}$ *is an* $*$*-nonunitary algebra, then its unitisation* $\mathcal{A}_{un}$ *has one involution* $\dagger$ *derived from the* $*$ *of* $\mathcal{A}$ *as follows:*

$$\forall a \in \mathcal{A}, \forall \alpha \in \mathbb{K} : (a, \alpha)^{\dagger} = (a^{*}, \overline{\alpha})$$

3. If $\mathcal{A}$ *is a normed nonunitary algebra, then its unitisation* $\mathcal{A}_{un}$ *is a normed algebra with norm* $\|.\|_{un}$ *given by*

$$\forall a \in \mathcal{A}, \forall \alpha \in \mathbb{K} : \|(a, \alpha)\|_{un} = \|a\| + |\alpha|$$

Definition 1.3.6 *1. Let* $\mathcal{A}, \mathcal{B}$ *be algebras (resp.* $*-$*algebras) and* f *be a linear map from* $\mathcal{A}$ *to* $\mathcal{B}$. f *is said to be algebra (resp.* $*-$*algebras) if, for all* $a, b \in \mathcal{A}$,

$$f(ab) = f(a)f(b), \text{ (resp. } f(a^{*} = f(a)^{*})\text{)}.$$

2. Let $\mathcal{A}$ *be a Banach algebra. A representation of* $\mathcal{A}$ *is a continuous* $*$*-algebra morphism from* $\mathcal{A}$ *to* $B(H)$*, for some Hilbert space.*

3. Let π *be a representation of* $\mathcal{A}$.

. π *is said to be a faithfull if it is injective;*

. π *is said to be irreducible if it has trivial invariant closed subspaces of* H.

. *A vector* $\varphi \in H$ *is said cyclic if the subspace spanned by the set* $\{\pi(x)\varphi : x \in \mathcal{A}\}$ *is dense in* H.

Definition 1.3.7 *Let* $\mathcal{A}$ *be an algebra and* $M \subseteq \mathcal{A}$.

1. The commutant of M *is the set*

$$M' = \{x \in \mathcal{A} : xy = yx, \forall y \in M\}$$

2. The bicommutant de M is the commutant of M', denoted by M''.

Definition 1.3.8 *let $\mathcal{A}$ be a Banach $*$-algebra and $\mathcal{A}^{\dagger}$ its topologic dual.*

1. *A functional $\varphi \in \mathcal{A}^{\dagger}$ is said a positive functional, $\varphi \geq 0$, if $\varphi(a^*a) \geq 0$, for all $a \in \mathcal{A}$.*
2. *The space of states of $\mathcal{A}$ is the set $St(\mathcal{A}) = \left\{\varphi \in \mathcal{A}^{\dagger} : \varphi \geq 0 \text{ and } \|\varphi\| = 1\right\}$.*
3. *The weak*-topology on a normed space $\mathcal{X}$ is defined by:*

   $(\varphi_n)_n \subseteq \mathcal{X}^{\dagger}$, $\varphi \in \mathcal{X}^{\dagger}$:$\varphi_n \to \varphi$ *iff* $\forall x \in \mathcal{X} : \varphi_n(x) \to \varphi(x)$.
4. *the extremums points of $St(\mathcal{A})$ are said the pur states.*

Theorem 1.3.9 *1. The unit ball of $\mathcal{X}^{\dagger}$ is compact for the weak *-topology.*

*2. $St(\mathcal{A})$ is convex and compact for the weak *-topology (it is the convex envelop of the extremum points (Krein-Milman's theorem).*

Definition 1.3.10 *A C^*-algebra ,$\mathcal{A}$, is a Banach algebra over the field of complex numbers with an involution $* : a \to a^*$ satisfying the C^*-identity $\|a^*a\| = \|a\|^2$, for $a \in \mathcal{A}$. We say $\mathcal{A}$ is unital, if there exists a unit element $1 \in \mathcal{A}$.*

Example 1.3.11 *Let X be a locally Hausdorff space and $C_0(X)$ be a space of complex valued continuous function vanishing at infinity on X.*

We define an involution on $C_0(X)$ by $f^*(x) = \overline{f(x)}$ for $x \in X$. Then $C_0(X)$ is a commutative C^*-algebra. It is unital if and only if X is compact.

Example 1.3.12 *The $*$-algebra of n-by-n matrices $\mathcal{M}_n(\mathbb{C})$, with the operator norm, is a non-commutative C^*-algebra.*

Example 1.3.13 *An important example of a Banach algebra which is not a C^*-algebra is the disk algebra $\mathbb{A}(\mathbb{D})$, consisting of all continuous functions $f \in C(\overline{\mathbb{D}})$ on the closed disk, which are analytic in the open disk $\mathbb{D}$.*

Definition 1.3.14 *A $*$-homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ between two C^*-algebras is an algebra homomorphism such that $\varphi(a^*) = \varphi(a)^*$ for all $a \in \mathcal{A}$.*

Theorem 1.3.15 *(Gelfand-Naimark-Seagal) Let $\mathcal{A}$ be a $C*$-algebra. Then there exist a Hilbert space H and an injective $*$-homomorphism $\pi : \mathcal{A} \to B(H)$.*

Every C^*-algebra can be embedded into the bounded operators on a Hilbert space. Idea Motivated from Gelfand duality, look at noncommutative C^*-algebras of operators as algebras of functions on some noncommutative space.

For more details on Banach algebra and C*-algebras theory, See [15, 16, 18, 72, 57].

Definition 1.3.16 *A normed space M will be said to be a left module for the (unital) algebra $\mathcal{A}$ if there exists a map (an action) $A \times M \to M$ which satisfies:*

1. $1h = h$, *for* $h \in M$.

2. $a(bh) = (ab)h$ *for* $a, b \in A$ *and* $h \in M$.

3. $a(\alpha_1 h_1 + \alpha_2 h_2) = \alpha_1 a h_1 + \alpha_2 a h_2$ *for* $a \in A$ *and* $\alpha_1, \alpha_2 \in \mathbb{C}$ *and* $h_1, h_2 \in M$, *and*

4. $(a + b)h = ah + bh$ *for* $a, b \in \mathcal{A}$ *and* $h \in M$

5 $\|ah\| \leq \|a\|\|h\|$ *for* $a \in \mathcal{A}$ *and* $h \in M$

If M is a Hilbert space then M is a left Hilbert $\mathcal{A}$-module. If M is a Banach space then M is a left Banach $\mathcal{A}$-module.

A right $\mathcal{A}$-module M is an $\mathcal{A}$-module where the action on M is defined on the right.

A normed $\mathcal{A},\mathcal{B}$-module M is a left normed $\mathcal{A}$-module and a right normed $\mathcal{B}$-module with the compatibilty of the actions: $(ax)b = a(xb)$ and $||axb|| \leq ||a||||x||||b||$, for all $a \in \mathcal{A}, b \in \mathcal{B}, x \in M$. In addition, we always assume that M is essential in the sense that $\overline{AM} = M = \overline{MB}$. See [17].

1.4 Operator Spaces and Operator Algebras

Definition 1.4.1 *A concrete operator space is a closed linear subspace of* $B(\mathcal{H})$*. To determine a "matrix norm" on the space* $\mathbb{M}_n(E)$ *(* $\mathbb{M}_n(E)$ *denotes the linear space of* $n \times n$ *matrices over* E *with no other assumed structure) we first look at the natural inclusions*

$$\mathbb{M}_n(E) \subset \mathbb{M}_n(B(\mathcal{H})) \simeq B(\mathcal{H}^n).$$

This determines a norm $\|.\|_n$ *on* $\mathbb{M}_n(E)$ *and we will denote this normed space by* $M_n(E)$*. An important observation is that we have no real analogue of () since many distinct operator spaces may have the same underlying normed space. Thus, we can not hope to relate the norm of a matrix with the norm of its individual entries. Ruan's axioms for operator spaces are motivated by the concrete observation.*

Theorem 1.4.2 *Suppose* $E \subset B(\mathcal{H})$ *is a concrete operator space. The following properties are satisfied:*

1. ($\mathcal{R}_1$): *if* $v \in M_m(E), w \in M_n(E)$ *then*

$$\|v \oplus w\|_{m+n} = \max\{\|v\|_m, \|w\|_n\}$$

2. ($\mathcal{R}_2$): *if* $\alpha \in M_{n,m}$, $\beta \in M_{m,n}$, *and* $v \in M_m(E)$, *we have*

$$\|\alpha v \beta\|_n \leq \|\alpha\|\|v\|_m\|\beta\|$$

Definition 1.4.3 *(Ruan's Axioms) Given a linear space* E*, a matrix norm* $\|.\| = \{\|.\|_n\}_n$ *is an assignment of a norm* $\|.\|_n$ *on each linear space* $\mathbb{M}_n(E)$ *for* $n \in \mathbb{N}$*. An abstract operator space in the pair* $(E, \{\|.\|_n\}_n)$ *where* E *is a linear space and* $\{\|.\|_n\}_n$ *is a matrix norm that satisfies the following properties:*

1. $\mathcal{R}_1$:$\|v \oplus w\|_{m+n} = \max\{\|v\|_m, \|w\|_n\}$,

2. $\mathcal{R}_2$:$\|\alpha v \beta\|_n \leq \|\alpha\|\|v\|_m\|\beta\|$, *for all* $v \in M_m(E), w \in M_n(E), \alpha \in M_{n,m}$, *and* $\beta \in M_{m,n}$.

In this case we call $\|.\| = \{\|.\|_n\}_n$ *an operator space matrix norm. We refer to* $\mathcal{R}_1, \mathcal{R}_2$ *as Ruan's axioms.*

We may view the linear space $\mathbb{M}_{m,n}(E)$ as a subspace of $M_p(E)$ where $p = \max\{m, n\}$, and denote the inherited normed linear space as $\mathbb{M}_{m,n}(E)$. It is important that we point some properties that do constrain the properties of our matrix norms. Given $v \in M_m(E)$ we see that

$$\|v_{ij}\| = \|E_i v E_j^*\| \leq \|v\|$$

$$\|v\| = \|\sum E_i^* v_{ij} E_j\| = \|\sum v_{ij}\|$$

Here we have let $E_j \in M_{1,n}$ denote the row matrix with 1 in the jth entry and 0's elsewhere. What precisely do these two equations tell us? We see that a sequence $v(k) \in M_n(E)$ converges if and only if $v_{ij}(k)$ converges for all i and j.Furtheremore, any two such norms on $\mathbb{M}_n(E)$ must be equivalent. Finally, we see that E will be complete if and only if $M_n(E)$ is complete for all $n \in \mathbb{N}$.

A final remark is that a map $F = [F_{ij}] : \mathcal{V} \to M_n$ is continuous if and only if each F_{ij} is continuous and thus we define the pairing

$$\langle ., . \rangle : M_n(E) \times M_n(E^*) \to \mathbb{C} : \langle v, f \rangle := \sum_{i,j} f_{ij}(v_{ij})$$

to identify the linear space $M_n(E^*)$ with the Banach dual $M_n(E)^*$.

See [23].

Definition 1.4.4 *An operator algebra is a norm-closed subalgebra of $B(H)$ for some Hilbert space H.*

Let $\mathcal{B}$ be an operator algebra and $T : \mathcal{B} \to B(H)$ be a bounded, linear map.

For each $n \geq 0$ consider the map

$$T \otimes I_n : \mathcal{B} \otimes M_n \to B(H) \otimes M_n,$$

where M_n denotes the $n \times n$ complex, matrices and I_n the identity matrix. After identifying $\mathcal{B} \otimes M_n \cong M_n(\mathcal{B})$ and $B(H) \otimes M_n \cong M_n(B(H)) = B(H^n), T \otimes I_n$ equals the map

$$T^{(n)}([b_{ij}]) = [T(b_{ij})].$$

Definition 1.4.5 *Let $T : \mathcal{B} \to B(H)$ be a bounded, linear map .*

1. *T is completely bounded if*

$$\|T \otimes I_n\|_{cb} := \sup_n \|T \otimes I_n\| < \infty.$$

2. *T is completely contractive if $\|T\|_{cb} \leq 1$.*

See [7]

1.5 Hilbert C*-modules

Hilbert C*-modules were first introduced in the work of Kaplansky. His idea was to generalize Hilbert spaces by allowing the inner product to take values in a commutative C*-algebra rather than the field of complex numbers $\mathbb{C}$. In fact, if A be a commutative C*-algebra; then using the Gelfand-Naimark theorem, A can be identified with $\mathcal{C}(X)$, for some (locally) compact Hausdorff space. We note that if X is a Riemannian manifold or any Euclidean space, one can analyze it by geometric techniques, among which is the study of vector bundles, see the example 3 below.

Let H be a right module over the C*-algebra A. We denote the action of $a \in A$ and $x \in H$ by $x.a$.

Definition 1.5.1 *A pre-Hilbert $\mathcal{A}$-module is a right $\mathcal{A}$-module H equipped with a $\mathcal{A}$-sesquilinear form $\langle , \rangle : HxH \to \mathcal{A}$ satisfying the following properties:*

1. $\langle x, x\rangle \geq 0$ *for every* $x \in H$*;*

2. $\langle x, x\rangle = 0$ *only in the case* $x = 0$*;*

3. $\langle x, y\rangle = \langle y, x\rangle^*$ *for every* $x, y \in H$*;*

4. $\langle x, y.a\rangle = \langle x, y\rangle a$ *for every* $x, y \in H$ *and* $a \in \mathcal{A}$*.*

The map $\langle , \rangle$ is called an $\mathcal{A}$-valued inner product.

The function

$$\|.\| : H \to \mathbb{R}_+, \|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}} \tag{1.2}$$

defines a norm on the $\mathcal{A}$-module H. Further, this function satisfies the following properties.

1. $\|x.a\| \leq \|x\|\|a\|$ for every $x \in H$ and $a \in \mathcal{A}$;

2. $\langle x, y\rangle\langle y, x\rangle \leq \|y\|^2\langle x, x\rangle$ for all $x, y \in H$;

3. $\|\langle x, y\rangle\| \leq \|x\|\|y\|$ for every $x, y \in H$.

Definition 1.5.2 *A Hilbert C^*-module H over a C^*-algebra $\mathcal{A}$ is a pre-Hilbert $\mathcal{A}$-module and a Banach space with respect to the norm defined by the relation (1.2).*
A Hilbert C^*-module over $\mathcal{A}$ is said to be full if $\overline{\langle H, H\rangle} = \mathcal{A}$.

We note here that the action of the algebra A on H and the $\mathcal{A}$-inner product on H extend to the completion $\tilde{H}$, which thus becomes a Hilbert module over $\mathcal{A}$. In the sequel we denote by the same symbol H for its completion.

Example 1.5.3 *1. Let $\mathcal{A}$ be a C^*-algebra and J be a right ideal of $\mathcal{A}$. It is clear that J is an $\mathcal{A}$-module and is a pre-Hilbert $\mathcal{A}$-module with respect to the $\mathcal{A}$-sesquilinear form : $\langle x, y\rangle = x^*y$ for all $x, y \in J$. Furthermore, J is a Hilbert $\mathcal{A}$-module with respect the induced norm of $\mathcal{A}$.*

In particular, every C^*-algebra $\mathcal{A}$ is a free Hilbert $\mathcal{A}$-module with one generator.

2. *Let $(H_i)_{i\in\mathbb{N}}$ be a countable set of Hilbert $\mathcal{A}$-Hilbert modules. Then the $\mathcal{A}$-module $\oplus H_i$ of all sequences (x_i) such that $\Sigma_i\langle x_i, x_i\rangle$ is norm convergent in $\mathcal{A}$, called the direct sum $\mathcal{A}$-module, is a Hilbert $\mathcal{A}$-module with respect to the $\mathcal{A}$-inner product*

$$\langle x, y\rangle = \Sigma_i\langle x_i, y_i\rangle \text{ for all } x, y \in \oplus H_i.$$

2. *The direct sum of a countable number of copies of a Hilbert $\mathcal{A}$-module H we shall denote by $l_2(H)$. If $H = \mathcal{A}$ then the Hilbert module $l_2(\mathcal{A})$ is called the standard Hilbert module over $\mathcal{A}$. If the C^*-algebra $\mathcal{A}$ is unital then the Hilbert module $l_2(\mathcal{A})$ possesses the standard basis $\{e_i : i \in \mathbb{N}$ where $e_i = (0, ..., 0, 1, 0, ...)$ with the unit at the i-th place.*

3. *Let X be a compact Hausdorff space and H be a Hilbert space. The space $E = \mathcal{C}(X, H)$ of continuous functions from X to H such that $s(t) \in H_t$, where H_t is a subspace of H, for every $t \in X$ and $s \in E$ is a vector bundle over X.*

   Furthermore, E is a $\mathcal{C}(X)$-module with respect to the action:$(s.f)(t) = s(t)f(t)$ for all $s \in E, f \in \mathcal{C}(X)$ and E is naturally endowed with a $\mathcal{C}(X)$-valued inner product:

$$\langle \xi, \zeta\rangle(t) = \langle \xi(t), \zeta(t)\rangle_H \text{ for all } \xi, \zeta \in E, t \in X.$$

   This means that E is Hilbert $\mathcal{C}(X)$-module. If $H = l_2(\mathbb{C})$ then E may be considered as the

Hilbert module $l_2(\mathcal{C}(X))$ which is the completion of the algebraic tensor product $\mathcal{C}(X) \otimes_{\mathbb{C}} l_2$ with respect to the norm $\|x\| = \|\langle x, x\rangle\|_{\infty}^{\frac{1}{2}}$, i.e. $l_2(\mathcal{C}(X))$ is the space of the sequences (f_n) such that $\Sigma_n f_n f_n^*$ is convergent.

Definition 1.5.4 *Let H be a right Hilbert C^*-module over a a C^*-algebra $\mathcal{A}$ and $x, y \in H$. We say that x and y are orthogonal with respect to $\mathcal{A}$-valued inner product $\langle ., . \rangle$ if $\langle x, y\rangle = 0$. We write $x \perp y$.*

It is clear that $x \perp y \Rightarrow x \perp_B y$ for all $x, y \in H$.

See [38,39,46].

1.6 Category theory

Definition 1.6.1 *A category $\mathcal{C}$ consists of*

- *A collection $Ob(\mathcal{C})$ of objects.* • *For every two objects X and Y in $\mathcal{C}$ a set of morphisms* $\hom_{\mathcal{C}}(X, Y)$.

- *For any triple of objects X, Y, Z a composition function*

$$\begin{aligned} \circ : \hom_C(Y, Z) \times \hom_C(X, Y) &\longrightarrow \hom_C(X, Z) \\ g, f &\longmapsto g \circ f \end{aligned}$$

subject to the following conditions:

$(C1)$ *The composition is associative : Given morphisms*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$

we have that $h \circ (g \circ f) = (h \circ g) \circ f$.

$(C2)$ *For every object* X *in* $\mathcal{C}$ *there is an identity morphism* $1_X : X \longrightarrow X$ *with the property that*

$$1_X \circ f = f, \qquad g \circ 1_X = g$$

for any morphisms $f : Y \longrightarrow X$ *and* $g : X \longrightarrow Z$.

$(C3)$ *The sets* $\hom_{\mathcal{C}(X,Y)}$ *and* $\hom_{\mathcal{C}(X',Y')}$ *are disjoint unless* $X = X'$ *and* $Y = Y'$.

Example 1.6.2 • $\mathcal{S}$ *is the category of sets , whose objects are sets and whose morphisms are functions between sets.*

• ${}_RMod$ *is the category of left modules over an associative unital ring* R*, whose objects are all left* R*-modules and whose morphisms are homomorphisms of left* R*-modules. Similarly,* Mod_R *denotes the category of right modules over* R.

• *Top is the category of topological spaces, whose objects are all topological spaces and whose morphisms are continuous maps.*

Definition 1.6.3 *A category* $\mathcal{C}$ *is called small if the collection of objects Ob* $\mathcal{C}$ *forms a set. The* *category of sets is not a small category , but the category of subsets of a given set would be an example of a small category.*

Definition 1.6.4 *A pre-additive category is a category $\mathcal{A}$ where the set of morphisms* $\hom_{\mathcal{A}}(X, Y)$ *between any two objects X and Y has the structure of an abelian group, and moreover the composition*

$$\hom_{\mathcal{A}}(Y, Z) \times \hom_{\mathcal{A}}(X, Y) \to \hom_{\mathcal{A}}(X, Z)$$

is bilinear, i.e. $(g + g') \circ f = g \circ f + g' \circ f$ and $g \circ (f + f') = g \circ f + g \circ f'$ for any morphisms $f, f' : X \to Y$ and $g, g' : Y \to Z$.

Definition 1.6.5 *An additive functor between pre-additive categories $\mathcal{A}$ and $\mathcal{B}$ is a functor $F : \mathcal{A} \to \mathcal{B}$ such that for every two objects X and Y in $\mathcal{A}$, the function*

$$\hom_{\mathcal{A}}(X, Y) \to \hom_{\mathcal{B}}(F(X), F(Y))$$

is a homomorphism of abelian groups, i.e. $F(f + g) = F(f) + F(g)$ for any morphisms $f, g : X \to Y$.

Definition 1.6.6 *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A (covariant) functor $F : \mathcal{C} \to \mathcal{D}$ consists of*

- *For every object X in $\mathcal{C}$ an object $F(X)$ in $\mathcal{D}$.*
- *For any object X in $\mathcal{C}$, we have that $F(1_X) = 1_{F(X)}$.*

Definition 1.6.7 *The opposite category, or dual category, of $\mathcal{C}$ is the category $\mathcal{C}^{op}$ whose objects are the same as those of $\mathcal{C}$ but where morphisms are reversed in the sense that*

$$\hom_{\mathcal{C}^{op}}(X, Y) = \hom_{\mathcal{C}}(Y, X)$$

for any objects X and Y. The composition

$$\circ_{op} : \hom_{\mathcal{C}^{op}}(Y, Z) \times \hom_{\mathcal{C}^{op}}(X, Y) \to \hom_{\mathcal{C}}(X, Z)$$

is defined using the composition in $\mathcal{C}$: Given composable morphisms

$$X \stackrel{f}{\longrightarrow} Y \stackrel{g}{\longrightarrow} Z$$

in $\mathcal{C}^{op}$, this is by definition the same thing as morphisms

$$X \stackrel{f}{\longleftarrow} Y \stackrel{g}{\longleftarrow} Z$$

in $\mathcal{C}$, and we define

$$g \circ_{op} f = f \circ g$$

This is a morphism from Z to X in $\mathcal{C}$, in other words a morphism from X to Z in $\mathcal{C}^{op}$.

Definition 1.6.8 *A contravariant functor $F : \mathcal{C} \to \mathcal{D}$ is a functor $F : \mathcal{C}^{op} \to \mathcal{D}$. A contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ can also be thought of as a functor from $\mathcal{C}$ to $\mathcal{D}^{op}$.*

Definition 1.6.9 *A morphism $f : X \longrightarrow Y$ is called an isomorphism if there is a morphism $g : Y \longrightarrow X$ such that $f \circ g = 1_Y$ and $g \circ f = 1_X$.*

A morphism f in a category $\mathcal{C}$ is called a monomorphism if $f \circ g = f \circ h$ implies $g = h$.It is called an epimorphism if $g \circ f = h \circ f$ implies $g = h$.

Any isomorphism is necessarily both a monomorphism and an epimorphism, but the converse *need not be true. A category is called balanced if any morphism which is both a monomorphism and an epimorphism is an isomorphism.*

See [75].

Chapter 2

Hilbert modules

2.1 Cohohmology groups and Derivations

Let X be a compact, separable, metric space and let $C(X)$ denote the Banach algebra of all continuous complex-valued functions on X with respect to the sup-norm.

Definition 2.1.1 *A function algebra on X is a closed subalgebra of $C(X)$, which contains the constants functions and separates points of X, i.e, for every $x \neq y$ there is f in the algebra such that $f(x) \neq f(y)$.*

Example 2.1.2 *The disc algebra $\mathbb{A} = A(\mathbb{D})$ is the closure in $C(\overline{\mathbb{D}})$ of the polynomials in the coordinate function z. We give here some properties of the algebra $\mathbb{A}$ as follows:*

-The functions in $\mathbb{A}$ can be characterized as the functions in $C(\overline{\mathbb{D}})$ which are holomorphic on $\mathbb{D}$.

-The closed subalgebras $\mathcal{A}_f$ generated by a nonconstant function $f \in \mathbb{A}$ are isometricaly isomorphic to $\mathbb{A}$

- The algebra $\mathbb{A}$ ia a maximal closed subalgebra of $C(\overline{\mathbb{D}})$.

See [34].

Definition 2.1.3 *A Hilbert module H over a function algebra $\mathcal{A}$ is a Hilbert space together with a bounded, unital homomorphism $\phi : \mathcal{A} \to B(H)$.*

Such a map is called a representation of the algebra $\mathcal{A}$ on H.

Given a representation ϕ, one defines the module action on H by a $a.h = \phi(a)h$.

It is easy to see that every Hilbert module action arises this way. In fact, if $\pi : A \times H \to H$ defines a bounded module action on H then $\phi(a) = \pi(a, h)$ defines a representation of $\mathcal{A}$ on H.

Definition 2.1.4 *Given two Hilbert $\mathcal{A}$-modules, (H, ϕ) and $(K\psi)$, an operator $T \in B(H, K)$ is called a Hilbert module map if $T\phi(a) = \psi(a)T$ for all $a \in \mathcal{A}$.*

For more details, see [17].

In [17], Douglas and Paulsen presented a first systematic study of Hilbert modules and reformulated several interesting operator theoretic concepts and problems in the language of module theory. Therefore, in studying Hilbert modules, as in studying any algebraic structure, the standard procedure is to look at submodules and associated quotient modules. This suggested the use of cohomological methods to study the extension problem: given two Hilbert modules H and K over a function algebra $\mathcal{A}$ what Hilbert module $\mathcal{J}$ may be constructed with submodule H and associated quotient module $\mathcal{J}$? We then have a short exact sequence

$$E : 0 \to H \to^{\tau} \mathcal{J} \to^{\sigma} K \to 0$$

of Hilbert $\mathcal{A}$-modules, where τ, σ are Hilbert module maps. Such a sequence is called an extension of K by H. The set of equivalence classes of extensions of Ķ by Ḩ, denoted by $Ext^1_{\mathcal{A}}(K, H)$ has a natural $\mathcal{A}$-module structure and is said the cohomology group []. The problem then is how to compute this group and what is its usefulness in the field of operator theory?. If $H = \mathbb{H}^2(\mathbb{D})$ is the Hardy module zith respect to the multiplication operator (shift operator) $S(f)z = zf(z)$ over the disc algebra $\mathbb{A} = \mathcal{A}(\mathbb{D})$, the characterization of $Ext^1_{\mathbb{A}}(K, \mathbb{H}^2(\mathbb{D}))$ was given by Clarck-Clarck [8] and by Ferguson [31] if K is a weighted Hardy module on the unit disk; as an application she gave a simple proof of a result due to Bourgain. Using cohomological techniques to study certain backward shift invariant operator ranges contained in vector-valued Hardy space; in particular the de Branges-Rovnyak spaces, Ferguson showed a connection between the extension problem for Hankel operator and the operator corona problem.

Firstly we interpret the cohomolgy groupe $Ext^1_{\mathbb{A}}(K, H)$ in the language of operator theory. Let T be a bounded operator on a Hilbert space H and $\mathcal{A}$ be a function algebra. If $f(T)$ is defined for every $f \in \mathcal{A}$ then we get a representation of $\mathcal{A}$ on H: $\mathcal{A} \to B(H); f \mapsto f(T)$ and therefore a Hilbert $\mathcal{A}$-module action $f.h = f(T)h$; for every $h \in H$ and every $f \in \mathcal{A}$.

Recall that T is polynomially bounded if and only if the map $p \mapsto p(T)$ extends to a representation of the disk algebra, $\mathbb{A}$, on H. On the other hand, given a representation $\pi : \mathbb{A} \to B(H)$, the operator $T = \pi(z)$ is polynomially bounded, where z is the function $z \mapsto z$ and $\pi(f) = f(T)$, for every $f \in \mathcal{A}$.

Because of this correspondence, in the sequel, we will write H_T for the Hilbert module H with multiplication by z determined by the operator T.

We denote by $\mathcal{H}(\mathbb{A})$ for the category of Hilbert $\mathbb{A}$-modules over the disc algebra $\mathbb{A}$, and by

$\mathcal{CH}(\mathbb{A})$ for the category of cramped Hilbert modules, the object are Hilbert $\mathcal{A}$-module similar to the Hilbert $\mathcal{A}$-module H_T where T is a contraction operator. The category $\mathcal{H}(\mathbb{A})$ with objects are the Hilbert $\mathcal{A}$-modules H_T, T is a contraction, will be said the category of contractive Hilbert $\mathcal{A}$-modules.

Definition 2.1.5 *Let M be an $\mathcal{A}$-bimodule over an algebra $\mathcal{A}$ and let D be a map from $\mathcal{A}$ to M.*

1. *D is said to be a derivation if D is linear and satissfies the following formula:*

$$D(ab) = aD(b) + D(a)b, \text{ for all } a, b \in \mathcal{A}.$$

2. *D is said to be an inner derivarion if there is $m \in M$ such that:*

$$D(a) = am - ma, \text{ for all } a \in \mathcal{A}.$$

It is clear that the space of derivations has a structure of an $\mathcal{A}$-bimodule:

$$(a.D)(x) = aD(x) \text{ and } (D.a)(x) = D(x)a, \text{ for all } a, x \in \mathcal{A},$$

and the space of inner derivations is a subbimodule of the $\mathcal{A}$-bimodule of derivations.

We can interpret the cohomology group $Ext_{\mathcal{H}(\mathbb{A})}(H_B, K_A)$ as the group of derivations modulo inner derivations as follows: Each extension of Hilbert module H_A by a Hilbert module H_B in the category $\mathcal{H}(\mathbb{A})$ gives rise a bounded derivation D from $\mathbb{A}$ to the $\mathbb{A}$-bimodule $B(H \oplus K)$. It follows from the second propertie of the disc algebra $\mathbb{A}$ cited in the exemple 2.1.2 and the fact that the polynomial algebra is generated by the coordinate function $z \mapsto z$ that every derivation D can be obtained from the function $z \mapsto z$. Hence, if π and ρ are representations of $\mathbb{A}$ on H and K respectively such that $\pi(z) = A, \rho(z) = B$ then

$$R = \begin{bmatrix} \pi & D \\ 0 & \rho \end{bmatrix}$$

is a representation of $\mathbb{A}$ on the space $H \oplus K$ and

$$R(z) = \begin{bmatrix} A & D(z) \\ 0 & B \end{bmatrix} = R_X = \begin{bmatrix} A & X \\ 0 & B \end{bmatrix}$$

Furthermore we can check easly that there is a coresspondance bijective betwwen the representaion of $\mathbb{A}$ on $H \oplus K$ and the bounded derivations from $\mathbb{A}$ to $B(H \oplus K)$. Threfore, the cohomology group $Ext_{\mathcal{H}(\mathbb{A})}(H_B, K_A)$ is isomorphic to the group $\mathcal{D}/\mathcal{I}$, where $\mathcal{D}$ is the space of operators $X : K \longrightarrow H$ for which the operator R_X is polynomially bounded (i.e, $(H \oplus K)_{R_X}$ is a Hilbert $\mathbb{A}$-module), and $\mathcal{I}$ is the space of operators of the form $AL - LB$ for some bounded operator $L : K \longrightarrow H$.

Remark 2.1.6 *1. Cohomology groups are invariant under similarity: if H_B is Hilbert modules similar to M_C, then the groups $Ext_{\mathcal{H}(\mathbb{A})}(H_B, K_A)$ and $Ext_{\mathcal{H}(\mathbb{A})}(M_C, K_A)$ are isomorphic.*
2. The cohomology group $Ext_{\mathcal{H}(\mathbb{A})}(H_B, K_A)$ is in fact an $\mathbb{A}$-modules with the action:
(although not a Hilbert $\mathbb{A}$-module). However, the groupe $\mathcal{D}/\overline{\mathcal{I}}$ is a normed space. In particular, according to the theorem 3, if $A = S$ and $H = \mathbb{H}^2$ then the cohomology groupe $Ext_{\mathcal{H}(\mathbb{A})}(H_B, \mathbb{H}^2)$ is isomorphic to the groupe $\{x \in H : \Sigma_{n=0}^{\infty} |\langle T^n h, x\rangle|^2 < \infty; h \in H\}$ modulo $\{x \in H : 1 \otimes x \in \mathcal{I}\}$

It was shown in [31], in the category of cramped Hilbert $\mathbb{A}$-module, that $Ext^1_{\mathbb{A}}(K_T, \mathbb{H}^2) = \{0\}$ if and only the Hilbert module K_T is similar to the isometric Hilbert module. However, it is not known if the previous equivalence is still holds in the category of Hilbert $\mathbb{A}$-module $\mathcal{H}(\mathbb{A})$. In this section we introduce a subgroup of $Ext^1_{\mathbb{A}}(K,H)$, sayed the reduced cohomolgy groupe and noted by $Ext^r_{\mathbb{A}}(K,H)$ and then we establish some properties of similarties. As a consequence, we give positive answer to the previous question.

Definition 2.1.7 *Let E be a separable Hilbert space with dimension $m; 0 < m \leq \infty$. We define the following sets:*

1. *The set of cocycles:*

   - $1 < m \leq \infty$.

     $Z_{T,m} = \{X \in B(H,E) : \|\Gamma_X\|^2 = \Sigma_{n=0}^{\infty}\|XT^n h\|^2 < \infty\};$

   - $m = 1$.

     $Z_T = \{x \in H : \|\Gamma_x\| < \infty\}.$

2. *The set of coboundries:*

   - $1 < m \leq \infty$.

     $B_{T,m} = \{X \in B(H,E) : \exists L \in B(H, \mathbb{H}^2(E)); X = S_m L - LT\},$

   - $m = 1$.

     $B_T = \{k \in H : \exists L; 1 \otimes k = SL - LT\}.$

3. *The set of closed coboundries:*

- $1 < m \leq \infty$.

  $\overline{B}_{T,m} = \{X \in Z_{T,m} : \exists (L_n)_n; X = \lim_{n\to\infty}(S_m L_n - L_n T)\}$; *where* $(L_n)_n$ *is a bounded sequence in* $B(H, \mathbb{H}^2(E))$,

- $m = 1$.

  $\overline{B}_T = \{k \in Z_T : \exists (L_n)_n \text{ bounded}; 1 \otimes k = \lim_{n\to\infty}(SL_n - L_n T)\}$.

4. *The reduced cohomology group:*

   $1 \leq m \leq \infty$. $Ext^r_{\mathcal{H}(\mathbb{A})}(H_T, \mathbb{H}^2(E)) = Z_{T,m}/\overline{B}_{T,m}$,

It is clear, by the definition of the closed coboundries, that $Ext^r_{\mathcal{H}(\mathbb{A})}(H_T, \mathbb{H}^2(E))$ *is a subgroup of* $Ext^1_{\mathcal{H}(\mathbb{A})}(H_T, \mathbb{H}^2(E))$.

Definition 2.1.8 (45) *Let* $A \in B(K), B \in B(H)$ *and let* $(Q_n)_n$ *be a sequence of bounded operators in* $B(H, K)$.

1. $(Q_n)_n$ *is said to be invertibely bounded sequence if* $(Q_n^{-1})_n$ *is a bounded sequence and* $Sup\{Q_n, Q_n^{-1} : n = 1, 2, ...\} < \infty$;

2. *A is said to be approximately similar to B if there is an invertibly bounded sequence of operators* $(Q_n)_n$ *in* $B(H, K)$ *such that* $\|Q_n^{-1} A Q_n - B\| \to 0$

If $dimE = 1$ we set $R_{1\otimes k} = R_k$.

Lemma 2.1.9 *Let E be a separable complex Hilbert space of dimension m and*

$$R_X = \begin{bmatrix} S_m & X \\ 0 & T \end{bmatrix}$$

be an operator on $\mathbb{H}^2(E) \oplus H$ with S_m the forward unilateral shift of multiplicity m on $\mathbb{H}^2(E)$.

1. *$Ext_{\mathcal{H}(\mathbb{A})}(H_T, \mathbb{H}^2(E)) = 0$ if and only if $S \oplus T$ is similar to R_k.*

2. *if $Ext^r_{\mathcal{H}(\mathbb{A})}(H_T, \mathbb{H}^2(E)) = 0$ then $S \oplus T$ is assymptoticaly similar to R_k.*

Proof.

- It follows by the definition of the cohomology group $Ext_{\mathcal{H}(\mathbb{A})}(H_T, \mathbb{H}^2)$; see [8].

- If If $Ext^r_{\mathcal{H}(\mathbb{A})}(H_T, \mathbb{H}^2(E)) = 0$ then $Z_{T,m} = \overline{B_{T,m}}$ and therefore for every $X \in Z_{T,m}$ there is a bounded sequence $(L_n) \subset B(H, \mathbb{H}^2(E))$ such that $\|SL_n - L_nT - X\| \to 0$.

  Choose the sequence
  $$Q_n = \begin{bmatrix} I & L_n \\ 0 & I \end{bmatrix}$$
  it is obvious that $(Q_n)_n$ is invertibly bounded sequence with
  $$Q_n^{-1} = \begin{bmatrix} I & -L_n \\ 0 & I \end{bmatrix}.$$
  An easy computation shows that $\|R_X Q_n - Q_n R_0\| \to 0$, which means that $R_0 = S \oplus T$ is approximately similar to R_X.

□

We need the following characterization given by Nagy-Foias [70].

Lemma 2.1.10 *Let $T \in B(H)$. Then, T is similar to an isometry if and only if there is a positive number α such that*

$$\|x\|/\alpha \leq \|T^n x\| \leq \alpha\|x\|; \text{ for all } x \in H \text{ and all } n \geq 0.$$

Proposition 2.1.11 *Let E be a complex separable Hilbert space with $dimE = m$. If $Ext^r_{\mathcal{H}(\mathbb{A})}(H_T, \mathbb{H}^2) =$ 0 then $Ext^r_{\mathcal{H}(\mathbb{A})}(H_T, \mathbb{H}^2(E)) = 0$*

Proof. Let $\{e_k : k \geq 0\}$ be an orthonormal basis of E and X be a bounded operator from H to E. Then, $Xh = \Sigma_{k\geq 0}\langle h, x_k\rangle e_k$; where $x_k = X^* e_k$.

Let $\epsilon > 0$. Then, there is an integer N such that

$$\|Xh - \Sigma_{k\geq 0}^{N}\langle h, x_k\rangle e_k\|^2 = \Sigma_{k\geq N+1}|\langle h, x_k\rangle|^2 < \frac{\epsilon}{2}\|h\|^2. \tag{2.1}$$

By the hypothesis $H/\overline{B_T} = \{0\}$ it follows that for each $x_k \in H$ there is a sequence of bounded operators $(L_{nk}) \in B(H, \mathbb{H}^2)$ and an integer N_k such that

$$\forall n > N_k, \forall h \in H : \|SL_{nk}h - L_{nk}Th + \langle h, x_k\rangle\|^2 < \frac{\epsilon}{2(N+1)}\|h\|^2; \tag{2.2}$$

Since $(L_{nk}) \in B(H, \mathbb{H}^2)$ then the operators (L_{nk}) have the following form

$$L_{nk}h = \Sigma_{j\geq 0}\langle h, y_{nkj}\rangle z^j; y_{nkj} = L_{nk}^* z^j \tag{2.3}$$

Therefore, the equation (2.2) leads to

$$\Sigma_{j\geq 1}|\langle h, y_{nk(j-1)} - T^* y_{nkj}\rangle|^2 + |\langle h, x_k - T^* y_{nk0}\rangle|^2 < \frac{\epsilon}{2(N+1)}\|h\|^2. \tag{2.4}$$

We choose a sequence of bounded operators $(Y_n)_n \subset B(H, \mathbb{H}^2(E))$ as follows:

$$Y_n h = \Sigma_{j\geq 0} Y_{nj} h Z^j ; Y_{nj} h = \Sigma_{k\geq 0}^{N} \langle h, y_{nkj}\rangle e_k ; \forall h \in H. \tag{2.5}$$

Hence, by (2.3) we get

$$\|Y_n h\|^2 = \Sigma_{j\geq 0}\|Y_{nj}h\|^2 = \Sigma_{j\geq 0}\Sigma_{k\geq 0}^{N}|\langle h, y_{nkj}\rangle|^2 = \Sigma_{k\geq 0}^{N}\|L_{nk}h\|^2 < \infty, , \forall h \in H$$

Since $(L_{nk})_n$ is a bounded sequence of operators for each k so that, by the definition of $\overline{B_T}$ and the uniform boundedness theorem, it follows that $(Y_n)_n$ is a bounded sequence of operators in $B(H, \mathbb{H}^2(E))$.

On the other hand, an easy computation shows that

$$\|(X + S_m Y_n - Y_n T)h\|^2 = \Sigma_{j\geq 1}\|Y_{n(j-1)}h - Y_{nj}Th\|^2 + \|(X - Y_{n0})h\|^2$$

Using (2.5) we get

$\|(X + S_m Y_n - Y_n T)h\|^2 =$

$$= \Sigma_{j\geq 1}\Sigma_{k=0}^{N}|\langle h, y_{nk(j-1)} - T^* y_{nkj}\rangle|^2 + \Sigma_{k=0}^{N}|\langle h, x_k - T^* y_{nk0}\rangle|^2 + \Sigma_{k\geq N+1}|\langle h, x_k\rangle|^2$$

$$= \Sigma_{k=0}^{N}[\Sigma_{j\geq 1}|\langle h, y_{nk(j-1)} - T^* y_{nkj}\rangle|^2 + |\langle h, x_k - T^* y_{nk0}\rangle|^2] + \Sigma_{k\geq N+1}|\langle h, x_k\rangle|^2.$$

Finaly, it follows from the equations (2.1),(2.4) and the previous one that

$$\|(X + S_m Y_n - Y_n T)h\|^2 < (\Sigma_{k\geq 0}^{N} \frac{\epsilon}{2(N+1)} + \frac{\epsilon}{2})\|h\|^2 = \epsilon\|h\|^2.$$

Which means that $X \in \overline{B}_{T,m}$

□

Proposition 2.1.12 *Let K_R be an isometric Hilbert $\mathbb{A}$-module and $M = M_S$ be a submodule (i.e, an R-invariant subspace) of K with $S = R|_M$. Setting $H = K \ominus M$ and let T be the compression of R on H. Then,*

$Ext^r_{\mathcal{H}(\mathbb{A})}(H_T, \mathbb{H}^2) = 0$ if and only if the Hilbert $\mathbb{A}$-module H_T is similar to an isometric module.

Proof. If H_T is similar to an isometric module then; by [8,31], $Ext_{\mathcal{H}(\mathbb{A})}(H_T, \mathbb{H}^2) = 0$. By the definition 6, it is clear that $Ext^r_{\mathcal{H}(\mathbb{A})}(H_T, \mathbb{H}^2)$ is a subgroup of $Ext_{\mathcal{H}(\mathbb{A})}(H_T, \mathbb{H}^2)$. Thus, $Ext^r_{\mathcal{H}(\mathbb{A})}(H_T, \mathbb{H}^2) = \{0\}$.

The converse. If $Ext^r_{\mathcal{H}(\mathbb{A})}(H_T, \mathbb{H}^2) = \{0\}$ then by theorem 3 and lemma 9 $R_0 = S \oplus T$ is assymptoticaly similar to the isometry R. Hence, there is an invertibly bounded sequence of operators $(Q_k)_k$ in $B(K)$ such that $\|Q_k^{-1} R Q_k - R_0\| \to 0$. Let $\alpha = Sup\{Q_k.Q_k^{-1}\}$, so by an easy computation we get

$$\|x\|/\alpha \leq \|Q_k^{-1} R^n Q_k x\| \leq \alpha\|x\|; \text{ for all } x \in K \text{ and all } n, k \geq 0.$$

Therefore,

$$\|x\|/\alpha \leq \|R_0^n x\| \leq \alpha\|x\|; \text{ for all } x \in K \text{ and all } n \geq 0.$$

So by the previous lemma it follows that R_0 is similar to an isometry V.

Thus, there is an invertible (positive) operator D such that $R_0 = D^{-1}VD$. If $(.,.)$ is the original inner product in H, we set $(h,k)_* = (Dh, DK)$ for any $h, k \in H$. It is clear that $(.,.)_*$ defines a new inner product in the linear space K. We note by K_* for the Hilbert space K with respect to $(.,.)_*$ and by $\|.\|_*$ for its associate norm. Hence, the idendity map $\tau : K \to K_*$ is continuous and invertible. Setting $R_{0*} = \tau R_0 \tau^{-1}$, we get

$$\|R_{0*}h\|_* = \|\tau R_0 \tau^{-1} h\|_* = \|DR_0 h\| = \|VDh\|. \tag{2.6}$$

Since V is an isometry then $\|R_{0*}h\|_* = \|VDh\| = \|Dh\| = \|h\|_*$.

Hence, R_{0*} is an isometry. Therefore, $T = R_{0*}|_H$ is an isometry; which means that the Hilbert $\mathbb{A}$-module H_T is similar to an isometric Hilbert module.

□

2.2 Submodules for Hilbert modules

The Invariant Subspace Problem asks whether every operator $T \in B(H)$ has a nontrivial invariant subspace. In a similar fashion, the Hyperinvariant Subspace Problem is whether every bounded linear operator such that $T \neq \alpha I$ has a non trivial hyperinvariant subspace.

The (Hyper)invariant Subspace Problem is old open question in operator theory. Intertwining relations of operators with respect to this problem are applied during long time. For example, it is known from 1970s that if two operators are quasisimilar, and one of them has nontrivial hyperinvariant subspace, then the other has nontrivial hyperinvariant subspace, too. If S is a simple unilateral forward shift, $T \in B(H)$ and there exists a nonzero operator X such that $SX = XT$ then the set of eigenvalues of T^* contains the open unit disk. Consequently, T has nontrivial hyperinvariant subspaces. If, in addition, T is an absolutely continuous polynomially bounded operator, then the closure of $Ran\phi(T)$ is a nontrivial hyperinvariant subspace for every inner function ϕ. On the other hand, if $UX = XT$, for some nonzero operator X and U is an absolutely continuous unitary operator, then the (Hyper)invariant Subspace Problem is still open even under the assumption that T is a contraction.

We introduce the theory of representable operator bi-modules and Hochschild cohomology groups we convert the intertwining relations between operators on the existance of an isometric representation of an $\mathcal{A}, \mathcal{B}$-bimodules on certain Hilbert spaces. In fact, let H_T and K_R be Hilbert modules in the category $\mathcal{H}(\mathbb{A})$ then the algebras $\mathcal{A}$ has isometric representations on the Hilbert spaces H and K respectively, i.e,

$$\phi : \mathcal{A} \longrightarrow B(H), \phi(f) = f(T)$$

$$\psi : \mathcal{A} \longrightarrow B(K), \phi(f) = f(R)$$

So that if there exists an isometric representation of an $\mathcal{A}, \mathcal{B}$-bimodule M on $B(H, K)$:

$$\pi : M \longrightarrow B(H, K), \pi(fxg) = \phi(f)\pi(x)\psi(R)$$

for all $f, g \in \mathcal{A}, x \in M$ and the first Hochschild cohomology group $H^0(A, M)$, see [37], is non trivial then we get the $\mathcal{A}$-bimodule $Hom_{\mathcal{A}}(K, H)$ of morphisms from K to H is non trivial. The $\mathcal{A}$-bimodule $Hom_{\mathcal{A}}(K, H)$ will be called: the bimodule morphisms.

2.2.1 The existence of nontrivial Hilbert submodules intertwined with weighted modules.

We recall the following definitions and notations.

Definition 2.2.1 *Let H_T be Hilbert module over the algebra $\mathbb{H}^\infty$ and let $H_0 = \{x : \|T^n x\| \to 0; n \to \infty\}$ be a subspace of H_T. Then,*

1. *H_T is of class $C_{0.}$, that is strongly stable, if $H_0(T) = H$.*
2. *H_T is of class $C_{1.}$ if $H_0(T) = \{0\}$.*
3. *H_T is of class $C_{.j} : j = 0, 1$ if H_{T^*} is of class $C_{j.}; j = 0, 1$.*
4. *H_T is of class $C_{ij} : i, j = 0, 1$ if $H_T \in C_{i.} \cap C_{.j}$.*

For more details, see [45, 71].

Remark 2.2.2 1. *It is well known that H_0 is a hypersubmodule for H_T.*

2. *Let δ be a bounded operator on $B(H, K)$ defined by $\delta_{T,R}(X) = TX - XR$, where $T \in B(H), R \in B(K)$ and H, K are Hilbert spaces, it is called a generalized derivation. It is obvious that $Ker\delta_{T,R}$ is a closed subspace of $B(H, K)$, in fact it is a concrete operator space and if H_T, K_R are Hilbert $\mathcal{A}$-modules then $Ker\delta_{T,R}$ is a Banach $\mathcal{A}$-bimodule representable on $B(H, K)$.*

We note here that Nagy-Foias and Kerchy [71,42], have shwon that if H_T is of class C_{01} then $Hom_{\mathbb{H}^\infty}(H_T, L^2)$ is non trivial, that is $Ker\delta_{T,U}$ is an $\mathbb{H}^\infty$-bimodule representable on $B(H, L^2)$, where U is the unitary operator on $L^2(\mathbb{T})$. But the (hyper)invariant subspace problem is still open in this case.

In what follows we show that if $Ker\delta_{T,S_\omega^\infty} \neq \{0\}$, S_ω^∞ is a weighted bilateral operator on certain weighted space, then we give a positive answer to the hyperinvariant subspace problem. We note also that the algebra $\mathcal{A}$ considered in this section is the algebra $\mathbb{H}^\infty$.

Lemma 2.2.3 *Let f be a nonzero function in $l^2(\omega)$. If there is no n_0 such that $f_n = 0$ for every $n \geq n_0$, then $\vee_{k\geq 0, t\in\mathbb{T}} S_\omega^{*k}(f)_t = l^2(\omega)$.*

Proof. Let $g \in l^2(\omega)$ be such that $\langle S_\omega^{*k}(f)_t, g\rangle = 0$ for all $k \geq 0, t \in \mathbb{T}$. Let k be fixed. Then

$$0 = \langle S_\omega^{*k}(f)_t, g\rangle = \langle (f)_t, S_\omega^k g\rangle = \Sigma_{n\in\mathbb{Z}} f_n t^n \overline{g}_{n-k}\omega^2(n)$$

for every $t \in \mathbb{T}$. Since $\Sigma_{n\in\mathbb{Z}}|f_n||g_{n-k}|\omega^2(n) < \infty$, we obtain that $f_n\overline{g}_{n-k} = 0$ for every $n \in \mathbb{Z}$. Since $0 \leq k$ was arbitrary, we conclude that $f_n\overline{g}_{n-k} = 0$ for every $n \in \mathbb{Z}$ and $k \geq 0$.

Let $s \in \mathbb{Z}$. By the assumption on f, there exists $n \in \mathbb{Z}$ such that $n \geq s$ and $f_n \neq 0$. Set $k = n - s$. Then $k \geq 0$, and $g_s = 0$. Thus, $g = 0$. □

Lemma 2.2.4 *Let ϕ be a nonzero function in $\mathbb{H}^\infty$ and let f be a nonzero function in $l^2(\omega)$. Suppose that $\phi(S_\omega^*)f = 0$. Then f satisfies the assumptions of the previous lemma.*

Proof. Suppose that there exists $n_0 \in \mathbb{Z}$ such that $f_n = 0$ for every $n > n_0$ and $f_{n_0} \neq 0$. Since $\phi \neq 0$ there exists $n_1 \geq 0$ such that $\hat{\phi}(n_1) \neq 0$ and $\hat{\phi}(n) = 0$ for every $0 \leq n < n_1$. Then

$$\begin{aligned} 0 = \langle \phi(S_\omega^*)f, z^{n_0-n_1}\rangle &= \langle f, \tilde{\phi}(S_\omega)z^{n_0-n_1}\rangle \\ &= \langle \sum_{k\leq n_0} f_k z^k, \sum_{k=n_1}^{\infty} \hat{\tilde{\phi}}(k) z^{k+n_0-n_1}\rangle \\ &= f_{n_0}\hat{\phi}(n_1)\omega^2(n_0), \end{aligned}$$

a contradiction. □

Recall that a contractive Hilbert A-module H_T is said to be of class C_0 if T is a completely nonunitary and there exists a nonzero function $\theta \in \mathbb{H}^\infty$ such that $\theta(T) = 0$. It is well known that a C_0-contraction has nontrivial hyperinvariant subspaces, see [71]. With the same reasoning as for C_0-Hilbert modules one can show the existence of $\psi \in \mathbb{H}^\infty$ such that the closure of $Ran\psi(T)$ is a nontrivial hypersubmodules for C_0-Hilbert A-modules. So that in what follows we suppose T is not a C_0-operator.

We recall some well known facts. The infinite countable orthogonal sum $H^{(\infty)}$ of copies of the Hilbert space H is defined by

$$H^{(\infty)} = \{(h_n)_{n\in\mathbb{N}} : h_n \in H, \|(h_n)_{n\in\mathbb{N}}\|^2 = \sum_{n\in\mathbb{N}} \|h_n\|^2 < \infty\},$$

and for every $T \in B(H)$ one can consider the operator $T^{(\infty)}$,

$$T^{(\infty)}(h_n)_{n\in\mathbb{N}} = (Th_n)_{n\in\mathbb{N}}, (h_n)_{n\in\mathbb{N}} \in H^{(\infty)}.$$

For $k \in H$ and $j \in \mathbb{N}$ let $(k)_{(j)}$ be the sequence from $H^{(\infty)}$ such that $h_j = k$ and $h_n = 0$ for $n \neq 0, n \in \mathbb{N}$.

It is clear that if T is an absolutely continuous polynomially bounded operator, then $T^{(\infty)}$ is an absolutely continuous polynomially bounded operator, too, and then $\phi(T^{(\infty)}) = (\phi(T))^{(\infty)}$ for every $\phi \in \mathbb{H}^\infty$.

It is obvious that if H_T is a Hilbert $\mathcal{A}$-modules then $H^\infty_{T^\infty}$ is a Hilbert $\mathcal{A}$-module. If $T = S_\omega$ then

H_ω^∞ will denote the Hilbert $\mathcal{A}$-module for the bilateral weight operator S^∞.

Theorem 2.2.5 *Suppose that H_T is an absolutely continuous Hilbert $\mathcal{A}$-module, $\omega : \mathbb{Z} \to (0, \infty)$ is a nonconstant nonincreasing function, there exists a nonzero function $\phi \in \mathbb{H}^\infty$ such that $ker\phi(S_\omega^*) \neq \{0\}$, and $Hom_\mathcal{A}(H_T, H_\omega^\infty) \neq \{0\}$. Then H_T has nontrivial hyperinvariant subspaces, that are the closures of $Ran\psi(T)$ for some $\psi \in \mathbb{H}^\infty$.*

Proof. As it is mentioned above, one may to assume that H_T is not a C_0-Hilbert $\mathcal{A}$-module. Let f be a nonzero function in $ker\phi(S_\omega^*)$.

By Lemmas 2.2.3 and 2.2.4, $l^2(\omega) = \vee_{k\geq 0, t\in\mathbb{T}} S_\omega^{*k}(f)_t$. Therefore,

$$\vee_{n\in\mathbb{N}, k\geq 0, t\in\mathbb{T}} (S_\omega^{*k}(f)_t)_{(n)} = (l^2(\omega))^{(\infty)}.$$

Let X be a nonzero operator in $Hom_\mathcal{A}(H_T, H_\omega^\infty)$.

There exist

$j \in \mathbb{N}, k \geq 0, t \in \mathbb{T}$ such that $X^*(S_\omega^{*k}(f)_t)_{(j)} \neq 0$.

Since, $\phi_{\bar{t}}(S_\omega^*)(f)_t = (\phi(S_\omega^*)f)_t$, we have

$$\begin{aligned}
\phi_{\bar{t}}(T^*)X^*(S_\omega^{*k}(f)_t)_{(j)} &= X^*\phi_{\bar{t}}((S_\omega^{(\infty)})^*)(S_\omega^{*k}(f)_t)_{(j)} \\
&= X^*(\phi_{\bar{t}}(S_\omega^*)(S_\omega^{*k}(f)_t)_{(j)} \\
&= X^*(S_\omega^{*k}\phi_{\bar{t}}(S_\omega^*)(f)_t)_{(j)} \\
&= X^*(S_\omega^{*k}(\phi(S_\omega^*)f)_t)_{(j)} \\
&= 0.
\end{aligned}$$

Thus, $ker\phi_{\bar{\iota}}(T^*) \neq \{0\}$. □

Remark 2.2.6 *According to the results of [32], if $\theta \in \mathbb{H}^\infty$ is an outer function and H_T is an absolutely continuous Hilbert module then $\theta(T)$ is injective and $Ran\theta(T)$ is dense. Also, if $\theta \in \mathbb{H}^\infty$ is a Blaschke product such that $Ran\theta(T)$ is not dense, then, according to [32], T^* has eigenvalue. Hence, in the previous theorem, we can consider only the singular inner functions in $\mathbb{H}^\infty$.*

Corollary 2.2.7 *Suppose that $\omega : \mathbb{Z} \to (0, \infty)$ is a nonconstant nonincreasing function and there exists a nonzero function $\phi \in \mathbb{H}^\infty$ such that $ker\phi(S_\omega^*) \neq \{0\}$. If $Hom_A(H_T, N_\omega^\infty) \neq \{0\}$, N_ω^∞ is a submodule for H_ω^∞). Then H_T has nontrivial hypersubmodules, that are the closures of $Ran\psi(T)$ for some $\psi \in \mathbb{H}^\infty$.*

If there exists a nonzero operator $X \in Hom_{\mathcal{A}}(H_T, H_\omega)$ then $X \oplus 0 \in Hom_A(H_T, H_\omega^\infty)$. Therefore, we obtain the following result.

Corollary 2.2.8 *Suppose that H_T is an absolutely continuous Hilbert module, $\omega : \mathbb{Z} \to (0, \infty)$ is a nonconstant nonincreasing function, there exists a nonzero function $\phi \in \mathbb{H}^\infty$ such that $ker\phi(S_\omega^*) \neq \{0\}$, and $I(T, S_\omega) \neq \{0\}$. Then H_T has nontrivial hypersubmodules, that are the closures of $Ran\psi(T)$ for some $\psi \in \mathbb{H}^\infty$.*

The following examples of weighted shifts operators S_ω satisfies the assumption $ker\phi(S_\omega^*) \neq \{0\}$ for some nonzero function $\phi \in \mathbb{H}^\infty$:

1. S_ω is a dissymetric weighted shift from [28].

2. The weighted shifts S_ω from [41] are not dissymetric.

3. The weighted shifts S_ω from [29], this class of operators is more general than that of [28].

In the remaining part of the paper, as an application of Corollary 2.2.8, we show that if there is an equivalence class $[X] \in Ext_{\mathcal{H}(\mathbb{A})}(H_T, \mathbb{H}^2)$ for some non zero $X^* \in Hom_{\mathcal{A}}(\mathbb{H}^2, H_{T^*})$ then H_T has a

nontrivial hypersubmodules.

We note here that the operator $\begin{bmatrix} S^* & \Gamma \\ 0 & S \end{bmatrix}$, Γ is a Hankel operator, was used to solve the problem of whether each polynomially bounded operator on Hilbert space is similar to a contraction. We refer the reader to [58, 56] for a survey of these classical results.

Proposition 2.2.9 *[8] Let H_T be a Hilbert $\mathcal{A}$-module. Then $x \in B_T$ if and only if there exists nonzero operators $X \in Hom_{\mathcal{A}}(H_T, L^2)$ such that $X^* z^{-1} = x$.*

Lemma 2.2.10 *Suppose that H_T is Hilbert $\mathcal{A}$-module and T^* has no eigenvalues. Then $Hom_{\mathcal{A}}(H_T, l^2_\omega) \neq \{0\}$, S_ω is a dissymetric weight shift, if and only if there exists a nonzero $x \in B_T$ and a nonzero bounded operator $X_\omega \in Hom_{\mathcal{A}}(H_T, l^2_{\omega^-})$ such that $X^*_\omega z^{-1} = x$.*

Proof. It is clear that S_ω has the following matrix form

$$S_\omega = \begin{bmatrix} S & \mathbf{1} \otimes z^{-1} \\ 0 & S_{\omega^-} \end{bmatrix};$$

with respect to the decomposition $l^2(\omega) = \mathbb{H}^2 \oplus l^2(\omega^-)$.

So that if $X \in Hom_{\mathcal{A}}(H_T, l^2_\omega)$ then

$$X = \begin{bmatrix} X_+ \\ X_\omega \end{bmatrix},$$

and by simple computation we get

$$\mathbf{1} \otimes X^*_\omega z^{-1} = X_+T - SX_+, \text{ and } X_\omega \in Hom_{\mathcal{A}}(H_T, l^2_{\omega^-}).$$

It is easy to check that $\widehat{X_\omega h}(-n) = \langle h, T^{*n-1}X^*_\omega z^{-1}\rangle$, for every $n \geq 1$, $h \in H$.

Suppose that $X^*_\omega z^{-1} = \{0\}$ then $X_\omega = 0$ and $X_+ \neq 0$, because of $X \neq 0$. Thus, $Hom_{\mathcal{A}}(H_T, \mathbb{H}^2) \neq \{0\}$. Therefore, T^* has non zero eigenvectors, a contradiction. Consequently, there exists a non zero $x = X^*_\omega z^{-1} \in B_T$ and $X_\omega \in Hom_{\mathcal{A}}(H_T, l^2_{\omega^-})$. □

Remark 2.2.11 *It is easy to check that the converse holds without using the assumption: T^* has no eigenvalues.*

Theorem 2.2.12 *Let H_T be an absolutely continuous Hilbert module and $[X] \in Ext_{\mathcal{H}(\mathbb{A})}(H_T, \mathbb{H}^2_{S^*})$, S^* is the backward unilateral shift on the Hardy space $\mathbb{H}^2$. where $X(.) = \sum_{n=0}^{\infty}\langle ., T^{*n}x\rangle z^n$ and $x \in B_T$.*

Then there exists a singular inner function ψ such that $ker\psi(T^*) \neq \{0\}$.

Proof. By Remark 2.1.6, and the definion of cocycles, we conclude that X is a bounded operator from H to $\mathbb{H}^2$. Since $S^*X = XT$ then

$$R^n = \begin{bmatrix} S^{*n} & nXT^{n-1} \\ 0 & T^n \end{bmatrix}, n \geq 1.$$

If there exists a non zero $x \in B_T$ such that $[X] \in Ext_{\mathcal{H}(\mathbb{A})}(H_T, \mathbb{H}^2_{S^*})$ then R is a polynomially bounded operator, then R^* is a polynomially bounded operator, too. In particular, $\sup_{n\geq 1} \|R^{*n}(1 \oplus 0)\| < \infty$. Therefore,

$$C = \sup_{n\geq 1} n\|T^{*(n-1)}X^*(1)\| = \sup_{n\geq 1} n\|T^{*(n-1)}x\| < \infty.$$

It is clear that the map $\omega : \mathbb{Z} \to [1, \infty)$ defined by

$$\omega(-n) = n^a, n \geq 1 \text{ and } \omega(n) = 1, n \geq 0,$$

for a fixed number $a : 0 < a < \frac{1}{2}$, is a dissymetric weight.

According to the section 1.2, we consider the weighted space $l^2(\omega^-)$ and the compression $S_{\omega-}$ of the bilateral weight shift S_ω.

Define an operator Y from H to $l^2(\omega^-)$ by the formulas

$$(Yh)(-n) = \langle h, T^{*(n-1)}x\rangle, n \geq 1, h \in H.$$

Then

$$\begin{aligned} \|Yh\|^2_\omega = \sum_{n=1}^{\infty} |\langle h, T^{*n-1}x\rangle|^2 n^{2a} &\leq \|h\|^2 \sum_{n=1}^{\infty} \|T^{*(n-1)}x\|^2 n^{2a} \\ &\leq C^2 \sum_{n=1}^{\infty} n^{2a-2}\|h\|^2. \end{aligned}$$

Since $\sum_{n=1}^{\infty} n^{2a-2} < \infty$, then Y is a bounded operator from H to $l^2(\omega^-)$ and $Y \in Hom_A(H_T, l^2_{\omega^-})$ follows from an easy calculation. The conclusion of the theorem follows from Lemma 2.2.10, and Corollary 2.2.8.

□

It is well known that the commutant $\{T\}^{`}$ of every quasianalytic contraction T of class C_{10} contains an operator similar to

$$R = \begin{bmatrix} S & 1 \otimes X^* z^{-1} \\ 0 & A \end{bmatrix}.$$

and the hyperinvariant subspace problem for T is equivalent to that of R, see [44]. The following result gives a partial result in this context.

Corollary 2.2.13 *Let H_T be an absolutely continuous Hilbert module, $X \in Hom_{\mathcal{A}}(H_T, \mathbb{H}^2_-)$, $\mathbb{H}^2_-$ is the module quotient of L^2 by the submodule $\mathbb{H}^2$. If $X^* z^{-1}$ is cyclic vector for H_T and if $[X] \in Ext_{\mathcal{H}(\mathbb{A})}(H_T, \mathbb{H}^2_{S^*})$ (i.e,$X^* z^{-1}$ defines a bounded derivation from $\mathbb{A}$ to $B(\mathbb{H}_2, H)$) then the following operator*

$$R = \begin{bmatrix} S & 1 \otimes X^* z^{-1} \\ 0 & T \end{bmatrix}.$$

defines a Hilbert A-module H_R in the category $\mathcal{H}(\mathbb{A})$ and H_R has nontrivial hypersubmodules of the form $\overline{Ran\psi(T^*)}$, for some singular inner function ψ.

Remark 2.2.14 *If $\|T\| < 1$ then $X^* z^{-1}$ always defines a bounded derivation from $\mathbb{A}$ to $B(\mathbb{H}_2, H)$ and therefore the previous corollary holds true without the assumption $[X] \in Ext_{\mathcal{H}(\mathbb{A})}(H_T, \mathbb{H}^2_{S^*})$.*

2.2.2 Some sufficent conditions for nonvanishing of the bimodules morphismes

In what follows we give some sufficient conditions such that $Hom_A(H_T, l^{2\infty}_{\omega}) \neq \{0\}$.

Lemma 2.2.15 *If H_T is of class $C_{.0}$ then there is a singular inner function ϕ such that*

$$\sum_{n=0}^{\infty} |\hat{\frac{1}{\phi}}(n)|^2 \|T^{*n}h\|^2 < \infty. \tag{2.7}$$

Proof. It follows from [32], that there exists a dissymetric weight ω such that

$$\omega(-n-1) \leq \frac{1}{\|T^{*n}h\|} \text{ for sufficiently large } n.$$

By J. Esterle theorem [28], there exists a singular inner function ϕ such that

$$\sum_{n=0}^{\infty} \frac{1}{\omega^2(-n-1)} |\widehat{\frac{1}{\phi}}(n)|^2 < \infty.$$

So that, for sufficiently large m, we get

$$\sum_{n=m}^{\infty} |\widehat{\frac{1}{\phi}}(n)|^2 \|T^{*n}h\|^2 \leq \sum_{n=m}^{\infty} \frac{1}{\omega^2(-n-1)} |\widehat{\frac{1}{\phi}}(n)|^2.$$

Thus,

$$\sum_{n=0}^{\infty} |\widehat{\frac{1}{\phi}}(n)|^2 \|T^{*n}h\|^2 < \infty, \text{ for every } h \in H.$$

□

Proposition 2.2.16 *If H_T is a $C_{.0}$-contractive Hilbert module then there is an increasing sequence of positive numbers $(\alpha_n)_{n\geq 0}$: $\alpha_0 = 0$ and $\alpha_n \to \infty$ such that*

$$\sum_{n=0}^{\infty} \alpha_{n+1} \|RT^{*n}x\|^2 < \infty.$$

for every $x \in H$ and $R \in \{D_T, D_{T^*}, [T^*, T]\}$.

Proof. Since $TD_T = D_{T^*}T$ and

$D_{T^*}^2 - D_T^2 = [T^*, T]$ then it suffices to consider only the case $R = D_{T^*}$. Then,

$$\begin{aligned} \|D_{T^*}T^{*n}x\|^2 &= \langle x, T^n D_{T^*}^2 T^{*n}x\rangle \\ &= \|T^{*n}x\|^2 - \|T^{*n+1}x\|^2, \forall x \in H. \end{aligned}$$

Let $(\alpha_n)_n$ be a sequence defined by:

$$\alpha_0 = 0;\ \alpha_{n+1} = \sum_0^n |\widehat{\frac{1}{\phi}}(n)|^2, n \geq 0.$$

It is clear that $(\alpha_n)_n$ is a positive increasing sequence. Since ϕ is an inner not an outer then $\frac{1}{\phi} \notin \mathbb{H}^2$. That is, $(\alpha_n)_n$ is an unbounded sequence ($\alpha_n \to \infty$). An easy computation shows, for every $n \geq 0$, that

$$\begin{aligned} & \sum_{k=0}^n \alpha_{k+1}\|D_{T^*}T^{*k}x\|^2 \\ \leq & \sum_{k=0}^n \alpha_{k+1}(\|T^{*k}x\|^2 - \|T^{*k+1}x\|^2) \\ \leq & \sum_{k=0}^n (\alpha_{k+1} - \alpha_k)\|T^{*k}x\|^2 - c\alpha_{n+1}\|T^{*n}x\|^2 \\ = & \sum_{k=0}^n |\widehat{\frac{1}{\phi}}(k)|^2\|T^{*k}x\|^2 - \alpha_{n+1}\|T^{*n}x\|^2 \end{aligned}$$

Since $\alpha_{n+1}\|T^{*n}x\|^2 > 0$ then

$$\sum_{k=0}^{n} \alpha_{k+1}\|D_{T^*}T^{*k}x\|^2 \leq \sum_{k=0}^{n} |\hat{\frac{1}{\phi}}(k)|^2\|T^{*k}x\|^2.$$

for every $n \geq 0$. Hence, by the previous Lemma, we get.

$$\sum_{k=0}^{\infty} \alpha_{k+1}\|D_{T^*}T^{*k}h\|^2 < \infty.$$

□

Set

$$\mathcal{C} = \{T : \exists A \in \{T\}', RanA \subseteq RanR\} \text{ and } R \in \{D_T, D_{T^*}, [T^*, T]\}.$$

We recall that an operator $T \in B(H)$ is said to be a quasinormal if

$$TT^*T = T^*TT.$$

Lemma 2.2.17 *The class $\mathcal{C}$ contains the quasinormal operators.*

Proof. It is clear that if T is a quasinormal operator then $TD_T^2 = D_T^2 T$. Thus, $D_T^2 \in \{T\}'$. Hence, $T \in \mathcal{C}$. □

Lemma 2.2.18 *[19]. Let $A, B \in B(H, K)$. Then the following statements are equivalent:*

1. *$RanA \subseteq RanB$,*

2. *$A = BC$ for some bounded linear operator*

   $C \in B(H)$,

3. $AA^* \leq \lambda BB^*$ *for some* $\lambda \geq 0$.

Theorem 2.2.19 *Let* H_T *be a* $C_{.0}$*-contractive Hilbert module. If one of the following conditions is satisfied:*

1. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^t}\|T^{*n}x\|^2 < \infty$ *for some* $x \in H$ *and* $0 \leq t < 1$,

2. $T \in \mathcal{C}$.

Then there is a bounded operator $X \in I(T, S_{\omega^-})$ *from* H *to* $l^2(\omega^-)$.

Proof. 1. Let $f(z) = \displaystyle\sum_{n=0}^{\infty} \frac{1}{n^t}\|T^{*n}x\|z^n$ be such that $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^t}\|T^{*n}x\|^2 < \infty$ for some $x \in H$ and $0 \leq t < 1$. Then, $f \in \mathbb{H}^2$.

Let $\phi = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s} z^n$, $0 < s \leq 1-t$. It is clear that $\phi \in \mathbb{H}^\infty$. Thus, $\phi.f \in \mathbb{H}^2$.
By the definition of the product of two function we get

$$(\phi.f)(z) = \sum_{n=1}^{\infty}(\sum_{k=1}^{n} \frac{1}{k^s}\frac{1}{(n-k)^t}\|T^{*n-k}x\|)z^n.$$

Since T is a contraction, then $\|T^{*n}x\| \leq \|T^{*n-k}x\|$ for every $0 \leq k \leq n$.

Thus,

$$\begin{aligned} & \sum_{n=1}^{\infty}|\sum_{k=1}^{n} \frac{1}{k^s}\frac{1}{(n-k)^t}\|T^{*n-k}x\||^2 \\ \geq & \sum_{n=1}^{\infty}|\sum_{k=1}^{n} \frac{1}{k^s}\frac{1}{(n-k)^t}|^2\|T^{*n}x\|^2. \end{aligned}$$

$$\begin{aligned}\sum_{k=1}^{2n}\frac{1}{k^s}\frac{1}{(n-k)^t} &= \sum_{k=1}^{n-1}\frac{1}{k^s}\frac{1}{(n-k)^t}+\sum_{k=n}^{2n}\frac{1}{k^s}\frac{1}{(n-k)^t}\\ &\geq \sum_{k=1}^{n-1}\frac{1}{k^s}\frac{1}{(n-k)^t}+\sum_{k=n}^{2n}\frac{1}{k^{s+t}}\end{aligned}$$

Hence,

$$\sum_{k=1}^{2n}\frac{1}{k^s}\frac{1}{(n-k)^t}-\sum_{k=1}^{n-1}\frac{1}{k^s}\frac{1}{(n-k)^t}\geq\sum_{k=n}^{2n}\frac{1}{k^{s+t}}\geq\sum_{k=n}^{2n}\frac{1}{k}$$

Since the rest of the harmonic series does not converges to zero then the sequence

$$(\alpha_n)_n : \alpha_n = (\sum_{k=1}^{n}\frac{1}{k^s}\frac{1}{(n-k)^t})^2, n\geq 1$$

diverges and therefore there exists a sequence

$\alpha_n \to \infty$ such that

$$\sum_{k=0}^{\infty}\alpha_{n+1}\|T^{*k}x\|^2<\infty.$$

Then, it follows from [32], that there exists a dissymetric weight ω such that $\omega_n \leq \alpha_n$ for sufficiently large n. Thus,

$$\sum_{n=1}^{\infty}\omega_n|\langle h, T^{*n-1}x\rangle|^2<\infty, \text{ for every } h\in H$$

Therefore, the operator

$$X : Xh = \sum_{n=1}^{\infty}\langle h, T^{*n-1}x\rangle z^{-n}$$

is a bounded operator form H to $l^2(\omega^-)$ and $X\in I(T, S_{\omega^-})$.

2. If $T \in \mathcal{C}$ then there is $A \in \{T\}'$ such that

$$RanA \subseteq RanR \text{ and } R \in \{D_T, D_{T^*}, [T^*, T]\}.$$

Since $TD_T = D_{T^*}T$ and $D_{T^*}^2 - D_T^2 = [T^*, T]$ then it suffices to consider only the case $RanA \subseteq RanD_{T^*}$.

By Lemma 2.2.18, there exists a bounded operator C such that

$$\begin{aligned} \|T^{*n}A^*x\|^2 &= \|A^*T^{*n}x\|^2 \\ &= \|C^*D_{T^*}T^{*n}x\|^2 \\ &\leq \|C^*\|^2\|D_{T^*}T^{*n}x\|^2. \end{aligned}$$

According to the proposition 2.2.16, there exists a sequence α_n such that

$$\sum_{n=0}^{\infty} \alpha_{n+1}\|T^{*n}A^*x\|^2 \leq \|C^*\|^2 \sum_{n=0}^{\infty} \alpha_{n+1}\|D_{T^*}T^{*n}x\|^2 < \infty.$$

Hence, there exists a dissymetric weight ω such that $\omega_n \leq \alpha_n$ for sufficiently large n and the operator $X(.) = \sum_{n=1}^{\infty}\langle ., T^{*n-1}A^*x\rangle z^{-n} \in I(T, S_{\omega^-})$. □

Lemma 2.2.20 *Suppose that T is a polynomially bounded operator. If there exists a nonzero $x \in B_T$ and a nonzero bounded operator $X_\omega \in I(T, S_{\omega^-})$ such that $X_\omega^* z^{-1} = x$ then $I(T, S_\omega) \neq \{0\}$.*

Conversely, if $I(T, S_\omega) \neq \{0\}$ and T^* has no eigenvalues then $B_T \neq \{0\}$ and $I(T, S_{\omega^-}) \neq \{0\}$.

Proof. It is clear that S_ω has the following matrix form

$$S_\omega = \begin{bmatrix} S & \mathbf{1} \otimes z^{-1} \\ 0 & S_{\omega^-} \end{bmatrix};$$

with respect to the decomposition

$l^2(\omega) = \mathbb{H}^2 \oplus l^2(\omega^-)$. By the definition of B_T, if $x \in B_T$, then there exists an operator X_+ such that

$$1 \otimes x = X_+ T - S X_+.$$

So that if there exists an operator $X_\omega \in I(T, S_{\omega^-})$ of the form:

$$X_\omega h = \sum_{n=1}^{\infty} \langle h, T^{*n-1} X_\omega^* z^{-1} \rangle z^{-n}$$

such that $X_\omega^* z^{-1} = x$. Then,

$$X = \begin{bmatrix} X_+ \\ X_\omega \end{bmatrix} \in I(T, S_\omega)$$

Conversely, if $X \in I(T, S_\omega)$ then

$$X = \begin{bmatrix} X_+ \\ X_\omega \end{bmatrix},$$

and by simple computation we get

$$\mathbf{1} \otimes X_\omega^* z^{-1} = X_+ T - S X_+, \text{ and } X_\omega \in I(T, S_{\omega^-}).$$

It is easy to check that $\widehat{X_\omega h}(-n) = \langle h, T^{*n-1} X_\omega^* z^{-1} \rangle$, for every $n \geq 1$, $h \in H$.

Suppose that $X_\omega^* z^{-1} = \{0\}$ then $X_\omega = 0$ and

$X_+ \neq 0$, because of $X \neq 0$. Thus, $I(T, S) \neq \{0\}$. Therefore, T^* has non zero eigenvectors, a contradiction. Consequently, there exists a non zero

$x = X_\omega^* z^{-1} \in B_T$ and $X_\omega \in I(T, S_{\omega^-})$. □

Theorem 2.2.21 *Let H_T be a absolutely continuous C_{10}-contractive Hilbert module. If one of the following conditions is satisfied:*

1. $\sum_{n=0}^{\infty} \frac{1}{n^t} \|T^{*n}x\|^2 < \infty$ *for some* $x \in B_T$ *and* $0 \leq t < 1$,

2. $T \in \mathcal{C}$.

then there is a bounded operator $X \in I(T, S_\omega)$ from H to $l^2(\omega)$ and either the point spectrum of T^* is not empty or T has nontrivial hyperinvariant subspaces of the form $\overline{Ran\phi(T^*)}$, for some singular inner function ϕ.

Proof. The first statement follows from Theorem 2.2.19, and Lemma 2.2.20. According to Theorem 11, it suffices to show that there exists a singular inner function ϕ such that

$$\sum_{n=0}^{\infty} |\widehat{\frac{1}{\phi}}(n)| \|T^{*n}h\| < \infty,\, h \in B_T.$$

By Theorem 2.2.19, there exists a positive unbounded sequence $(\gamma_n)_n$ such that $\sum_{n=0}^{\infty} \alpha_n \|T^{*n}x\|^2 < \infty$ and by [32], there exists a dissymetric weight ω such that

$$\omega(-n-1) \leq \sqrt{\gamma_n}$$

for sufficiently large n. and by Esterle's theorem [28], we get

$$\begin{aligned}
& \sum_{n=0}^{\infty}\left|\frac{\hat{1}}{\phi}(n)\right|\|T^{*n}x\| \\
= & \sum_{n=0}^{\infty}|\frac{\hat{1}}{\phi}(n)|\frac{1}{\sqrt{\alpha_n}}\sqrt{\alpha_n}\|T^{*n}x\| \\
\leq & \sum_{n=0}^{\infty}\left|\frac{\hat{1}}{\phi}(n)\right|^2\frac{1}{\alpha_n}\sum_{n=0}^{\infty}\alpha_n\|T^{*n}x\|^2 \\
\leq & \sum_{n=0}^{\infty}|\frac{\hat{1}}{\phi}(n)|^2\frac{1}{\omega^2(-n-1)}\sum_{n=0}^{\infty}\alpha_n\|T^{*n}x\|^2 \\
< & \infty
\end{aligned}$$

The result follows by Theorem 11. $\square$

Lemma 2.2.22 *[32, 50] Let $T \in B(H)$.*

If T is polynomially bounded operator then there is a contraction operator A such that $A \prec T$. Conversely, if T is a contraction operator then there is a polynomially bounded operator A such that $A \prec T$.

We deduce from the previous Lemma and the Theorem 2.2.21 the following result.

Corollary 2.2.23 *Let H_T be an absolutely continuous Hilbert module of class C_{10}. If one of the following conditions is satisfied:*

1. $\sum_{n=0}^{\infty}\frac{1}{n^t}\|T^{*n}x\|^2 < \infty$ *for some* $x \in B_T$ *and* $0 \leq t < 1$,

2. $T \in \mathcal{C}$.

then there is a bounded operator $X \in I(T, S_\omega)$ from H to $l^2(\omega)$ and either the point spectrum of T^* is not empty or T has nontrivial hyperinvariant subspaces of the form $\overline{Ran\phi(T^*)}$, for some singular inner function ϕ.

We point out here that our results are very powerful compared to those given in [32]. Indeed, if we set $t = 0$ in the first statement of the previous corollary we get the the main result obtained in [32].

2.2.3 The existence of nontrivial submodules and representable Banach bimodules

Unsing the language of Banach bimodules representables, we can give another version of the theorem obtained in the first section of this chapter as follows.

Theorem 2.2.24 *If there exists a Banach $\mathbb{H}^\infty$-bimodle representable M on $B(H, l^{2\infty}_\omega)$ and the first Hochschild cohomology $H^0(\mathbb{H}^\infty, M)$ is not vanaishing then the Hilbert module H_T has nontrivial hypersubmodules.*

Proof. If there exists a Banach $\mathbb{H}^\infty$-bimodule representable M on $B(H, l^{2\infty}_\omega)$ with representation π then

$$\pi(fxg) = f(T)\pi(x)g(S^\infty) \text{ for all } f, g \in \mathbb{H}^\infty \text{ and } x \in M.$$

Since,

$$H^0(\mathbb{H}^\infty, M) = \{x \in M : fx = xg, f, g \in \mathbb{H}^\infty\}$$

Then,

$$\pi(H^0(\mathbb{H}^\infty, M)) = \{\pi(x) \in B(H, l^{2\infty}_\omega) : f(T)\pi(x) = \pi(x)g(S^\infty), f, g \in \mathbb{H}^\infty\}$$

Which means that $\pi(H^0(\mathbb{H}^\infty, M))$ is a subbimodule of $Hom_{\mathbb{H}^\infty}(H_T, l^{2\infty}_\omega)$.

Since π is injective then if $H^0(\mathbb{H}^\infty, M) \neq \{0\}$ then, by theorem 2.2.5, H_T has nontrivial submodules. $\square$

Chapter 3

Generalized C^*-Hilbert modules

In [63] C.Pop and [14] A.Delarouche introduced a new class of $C^*(\mathcal{A}, \mathcal{B})$-bimodules, called representable bimodule, and they showed that it met a condition of $C^*(\mathcal{A}, \mathcal{B})$-convexity; in addition they showed whereas this condition characterizes the class of $C^*(\mathcal{A}, \mathcal{B})$-bimodules. In the following, on the one hand, we introduce a new class of C^*-modules, said C^*-semiinner product and we prove that it is a representable bimodule and, as a consequence, it is an operator space. On the other hand, by establishing a contravariant equivalence between the category of representative bimodules and that of Banach bundles in the sense of Fell we show that this class of bimodules is the noncommutative version of Banach bundles in the sense of Fell [30].

3.1 C^*-semiinner product bimodules

Proposition 3.1.1 *Let $\mathcal{X}$ be an esenntial normed $(\mathcal{A},\mathcal{B})$-bimodule and $\mathcal{A},\mathcal{B}$ be Banach algebras. If $\mathcal{A}$ and $\mathcal{B}$ have approximate unites (a_α) and (b_β) respectively, then*

$$\forall x \in \mathcal{X} : x = \lim_\alpha a_\alpha x = \lim_\beta x b_\beta$$

Proof. This follows from the definition of an essential bimodules. □

Thus, if $\mathcal{A}$ (or $\mathcal{B}$) isn't unital and $\mathcal{X}$ is a normed $\mathcal{A},\mathcal{B}$-bimodule we can have a normed $\mathcal{A}_{un},\mathcal{B}$-bimodule structure on $\mathcal{X}$, where $\mathcal{A}_{un}$ is the unitisation algebra of $\mathcal{A}$, as follows,

$$\forall a \in \mathcal{A}, \forall b \in \mathcal{B}, \forall \lambda \in \mathbb{C}, \forall x \in \mathcal{X} : (a,\lambda)\, xb = axb + \lambda xb$$

Also, if $\mathcal{X}$ is a Banach space and $\mathcal{A}$ is a C^*- algebra, we can extend the action from $\mathcal{A}$ on $\mathcal{X}$ to the multiplier algebra $M(\mathcal{A})$ on $\mathcal{A}$ as follows. If $m = (L,R) \in M(\mathcal{A})$, $a \in \mathcal{A}$ and $x \in \mathcal{X}$; we set $m.(ax) = L(a)x$. Indeed, if $y \in \mathcal{X}$, $b \in \mathcal{A}$ such that $ax = by$ then, using the approximate unit (a_α) of $\mathcal{A}$ we get

$$\begin{aligned} m.(ax) &= L(a)x = \lim L(a_\alpha a)x = \lim L(a_\alpha)ax = \lim L(a_\alpha) \\ &= \lim L(a_\alpha b)y = L(b)y = m\,(by)\,. \end{aligned}$$

By Cohen factorisation theorem [11], we have an action of $M(\mathcal{A})$ on $\mathcal{X}$.

Definition 3.1.2 *Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{X}$ be a left algebraic module over $\mathcal{A}$. If there is a* *map*

$$[.,.] : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$$

such that following conditions are satisfied:

(i) $\forall x \in \mathcal{X} : [x, x] > 0$; $[x, x] = 0 \Rightarrow x = 0$;

(ii) $\forall x, y, z \in \mathcal{X}$ *et* $\forall \alpha, \beta \in \mathbb{K} : [\alpha x + \beta y, z] = \alpha[x, z] + \beta[y, z]$;

(iii) $\forall x, y \in \mathcal{X}$ *et* $a \in \mathcal{A} : [ay, x] = a[y, x]$;

(iv) $\forall x, y \in \mathcal{X} : \|[y, x]\| \leq \|[x, x]\|^{\frac{1}{2}} \|[y, y]\|^{\frac{1}{2}}$.

Then $\mathcal{X}$ will be said a left C^*-semiproduct A-module, denoted by left s.p.A-module. In the same manner we define a right s.p. $\mathcal{A}$-module, we subtitue (iii) by

$$\forall x, y \in \mathcal{X} \text{ and } a \in \mathcal{A} : [ya, x] = [y, x]a.$$

Example 3.1.3 *1. Let $\mathcal{A}$ be a C^*-algebra and*

$$[.,.] : \mathcal{A} \times \mathcal{A} \to \mathcal{A} : [y, x] = \begin{cases} \frac{1}{\|x\|^2} y\,(x^* x x^*) : x \neq 0 \\ 0 : x = 0 \end{cases}$$

2. Let X be a normed linear space and H a Hilbert C^*-module. The left $\mathcal{A}$-module $M = H \otimes X$ is a left semiinner product $\mathcal{A}$-module with respect to the following $\mathcal{A}$-semiinner product:

$$[.,.] : M \times M \to \ A,\ [\Sigma_{i=1,n} h_i \otimes y_i, \Sigma_{j=1,m} k_j \otimes x_j] = \Sigma_{i,j} \langle \phi_j(y_i) h_i, k_j \rangle$$

where $\phi \in X^*$ and $\phi(x) = \|x\|^2, \|\phi\| = \|x\|$, ϕ is said the support functional of x.

We can check easily that $[.,.]$ is a $\mathcal{A}$-s.p. but isn't an $\mathcal{A}$-inner product on $\mathcal{A}$.

It is obvious that the semiproduct defined by G. Lumer [47] and Gilles [35], on normed spaces, is a particular case of our $\mathcal{A}$-sp. We note that if the map $[.,.]$ is linear congugate (i.e. $\forall x, y \in \mathcal{X} : [y, x]^* = [x, y]$) we get the definition of C^* preHilbert module on $\mathcal{A}$.

Proposition 3.1.4 *Let $\mathcal{X}$ be a left $\mathcal{A}$-semiinner product. Then $\mathcal{X}$ is a left $\mathcal{A}_{un}$-semiinner product. If in addition, $\mathcal{X}$ is a Banach space then $\mathcal{X}$ is a left $M(A)$-semiinner product module.*

Proof. 1. Let $\lambda \in \mathbb{C}, x, y \in \mathcal{X}$ et $a \in \mathcal{A}$, we have

$$\begin{aligned} \langle (a,\lambda) .y, x\rangle &= \left\langle \lim_{\alpha} (aa_\alpha + \lambda a_\alpha) y, x \right\rangle = \left\langle \lim_{\alpha} aa_\alpha y, x \right\rangle + \left\langle \lim_{\alpha} \lambda a_\alpha y, x \right\rangle \\ &= \langle ay, x\rangle + \langle \lambda y, x\rangle = a \langle y, x\rangle + \lambda \langle y, x\rangle = (a, \lambda) \langle y, x\rangle . \end{aligned}$$

2. Let $m = (L, R) \in M(\mathcal{A})$, $a \in \mathcal{A}$ and $x \in \mathcal{X}$

$$\langle m (ay) , x\rangle = \langle L (a) y, x\rangle = \left\langle \lim_{\alpha} L (a_\alpha a) y, x \right\rangle = \lim_{\alpha} L(a_\alpha) \langle ay, x\rangle = m \langle ay, x\rangle .$$

□

In what follows we suppose that the algebra $\mathcal{A}$ is unital.

Proposition 3.1.5 *If $\mathcal{X}$ is a left $\mathcal{A}$-preHilbert module then $\mathcal{X}$ is a left $\mathcal{A}$-sp.*

Proof. It suffices to check (iv) of the definition. We suppose that $\mathcal{X}$ is a left A-preHilbert module with $\mathcal{A}$-produit $(.,.)$. We have $(ax - y, ax - y) \geq 0$, for all $a \in \mathcal{A}$ and all $x, y \in \mathcal{X}$. Hence, $a (x, x) a^* - a (x, y) - (y, x) a^* + (y, y) \geq 0$. We know that if $b \in \mathcal{A}$ is positive then, for all $a \in A$, $a^*ba \leq \|b\| a^*a$, thus $0 \leq \|x\|^2 aa^* - a (x, y) - (y, x) a^* + (y, y)$. We set $a = (y, x)$

and we suppose that $\|x\| = 1$, we get $(y, x)(x, y) \leq (y, y)$ and for $\|x\| \neq 1$, we deduce that $(x, y)^*(x, y) \leq \|x\|^2 (y, y)$.

It follows from the last inequality that $\|(x, y)\| \leq \|x\| \|y\|$. □

Proposition 3.1.6 *If $\mathcal{X}$ is a left $\mathcal{A}$-sp then $\mathcal{X}$ is a left $\mathcal{A}$-normed module.*

Proof. It suffices to check that the function $\|.\|$ defined by $\|\langle x, x\rangle\|^{\frac{1}{2}} = \|x\|$, for all $x \in \mathcal{X}$, is a norm on $\mathcal{X}$. It is clear, by Definition 3.1.2; (i), $\|x\| = 0 \Leftrightarrow x = 0$.

It follows from (ii) et (iii), that

$$\begin{aligned}\|x+y\|^2 &= \|\langle x+y, x+y\rangle\| = \|\langle x, x+y\rangle + \langle y, x+y\rangle\| \leq \|\langle x, x+y\rangle\| + \|\langle y, x+y\rangle\| \\ &\leq \|\langle x, x\rangle\|^{\frac{1}{2}} \|\langle x+y, x+y\rangle\|^{\frac{1}{2}} + \|\langle y, y\rangle\|^{\frac{1}{2}} \|\langle x+y, x+y\rangle\|^{\frac{1}{2}}\end{aligned}$$

By the first assertion, if $x + y \neq 0$ we get

$\|\langle x+y, x+y\rangle\|^{\frac{1}{2}} \leq \|\langle x, x\rangle\|^{\frac{1}{2}} + \|\langle y, y\rangle\|^{\frac{1}{2}}$ or $\|x+y\| \leq \|x\| + \|y\|$.

If $x + y = 0$, then

$$\|ax\|^2 = \|\langle ax, ax\rangle\| = \|a\langle x, ax\rangle\| \leq \|a\| \|\langle x, ax\rangle\| \leq \|a\| \|x\| \|ax\|. \tag{3.1}$$

If $ax = 0$ then it is obvious that $0 = \|ax\| \leq \|a\| \|ax\|$.

If $ax \neq 0$ then, by (3.1), we get $\|ax\| \leq \|a\| \|ax\|$. □

Definition 3.1.7 *1. Let $\mathcal{X}$ be a semiinner product $\mathcal{A}$-bimodule.*

$\mathcal{X}$ is said to be full if the linear subspace spanned by $\{\langle y, x\rangle : x, y \in \mathcal{X}\}$ is dense in $\mathcal{A}$. This subspace is denoted by $\langle \mathcal{X}, \mathcal{X}\rangle$.

2. $\mathcal{X}$ is said to be $\mathcal{A}$-homogene if and only is for all $x, y \in \mathcal{X}$ and $a \in \mathcal{A}$: $\langle y, ax \rangle = \langle y, x \rangle a^*$.

Proposition 3.1.8 *Let $\mathcal{X}$ be a full semiinner product $\mathcal{A}$-bimodule and let $a \in \mathcal{A}$. Then, $\forall x \in \mathcal{X} : ax = 0 \Leftrightarrow a = 0$.*

Proof. It is clear that $a = 0 \Rightarrow \forall x \in \mathcal{X} : ax = 0$.

Conversly. Let $b \in \mathcal{A}$. Since $\mathcal{X}$ is full then

$$\exists \zeta_n \in \langle \mathcal{X}, \mathcal{X} \rangle : b = \lim_n \zeta_n \text{ et } \zeta_n = \sum_{i=1}^{m_n} \alpha_{i_n} \langle y_{i_n}, x_{i_n} \rangle ;\ y_{i_n}, x_{i_n} \in \mathcal{X},\ \alpha_{i_n} \in \mathbb{C}.$$

Thus,

$$ab = a \lim_n \sum_{i=1}^{m_n} \alpha_{i_n} \langle y_{i_n}, x_{i_n} \rangle = \lim_n \sum_{i=1}^{m_n} \alpha_{i_n} \langle a y_{i_n}, x_{i_n} \rangle = 0,$$

hence, $ab = 0$. We choose $b = a^*$ we get $\|a\|^2 = \|aa^*\| = 0$. Therefore, $a = 0$. □

Definition 3.1.9 *Let $\mathcal{X}$ be a semiinner product $\mathcal{A}$-bimodule with $\mathcal{A}$-sp. $[.,.]$*

For fixed $x \in \mathcal{X}$, the map $\widehat{x} : \mathcal{X} \to \mathcal{A}$; $\widehat{x}(y) = [x, y]$ is $\mathcal{A}$-lineair and bounded (follows from the definition of $\mathcal{A}$-s.p.). Hence, we can define a left duality map as follows, ${}_{\mathcal{A}}d : \mathcal{X} \to Hom_{\mathcal{A}}(\mathcal{X}, \mathcal{A})$;

$$\forall x \in \mathcal{X} :_{\mathcal{A}} d(x) = \widehat{x}; \tag{3.2}$$

where $Hom_{\mathcal{A}}(\mathcal{X}, \mathcal{A})$ is the set of $\mathcal{A}$-bounded linear maps from $\mathcal{X}$ to $\mathcal{A}$. It is in fact a left $\mathcal{A}$-module. It will be noted by $\mathcal{X}^{\dagger_l}$.

In the same manner we define the right duality map,

$d_{\mathcal{A}} : \mathcal{X} \to Hom(\mathcal{X}, \mathcal{A})_{\mathcal{A}}$;

$$\forall x \in \mathcal{X} : d_{\mathcal{A}}(x) = \widehat{x}. \tag{3.3}$$

Also, $Hom(\mathcal{X},\mathcal{A})_{\mathcal{A}}$ is a right $\mathcal{A}$-module, noted by $\mathcal{X}^{\dagger_r}$.

Definition 3.1.10 *Let $\mathcal{X}$ be semiinner product $\mathcal{A}$-bimodule with $[.,.]$ sp. and $x \in \mathcal{X}$.*

1. *If $R \in \mathcal{X}^{\ddagger_L}$ then R is said to be $\mathcal{A}$-support at the point x if $R(x) = [x,x]^{\frac{1}{2}}$ and $\|R\| = 1$.*
2. *If there is one support at x, we say x is $\mathcal{A}$-smooth;*
3. *$\mathcal{X}$ is said to be a smooth $\mathcal{A}$-module if every $x \in \mathcal{X}$ is $\mathcal{A}$-smooth.*

We observe that the notion of smoothness is related to both the $\mathcal{A}$-norme $|.|^2 = [.,.]$ and $\mathbb{K}$-norme $\|.\|$, which means that A-smoothness generalizes the usuel $\mathbb{K}$-smoothness in the normed spaces [60].

Definition 3.1.11 [3, 6] *Let $\mathcal{X}$ be a left algebraic $\mathcal{A}$-module. $\mathcal{X}$ is said to be a pre-Finsler $\mathcal{A}$-module if there is a map $\digamma : \mathcal{X} \to \mathcal{A}^+$ such that*

(i) the function $\|.\|$ defined by: $\forall x \in \mathcal{X} : \|x\| = \|\digamma(x)\|^{\frac{1}{2}}$ is a norm in $\mathcal{X}$;

(ii) $\forall x \in \mathcal{X}$, $\forall a \in \mathcal{A} : \digamma(ax) = a\digamma(x)a^*$.

where $\mathcal{A}^+$ denotes the positive cone of $\mathcal{A}$.

If $\mathcal{X}$ is completed for this norm, $\mathcal{X}$ is said to be a Finsler $\mathcal{A}$-module.

We note here that the original definition of Finsler module is that X is assuming a Banach space for the standard given in (i), [9], Although, C. Akemann [3] showed that the completude of $\mathcal{A}$-pré-Finsler module with respect to the given norm is in fact a Finsler $\mathcal{A}$-module.

Proposition 3.1.12 *Let $\mathcal{X}$ be an algebraic $\mathcal{A}$-module.*

1. *If $\mathcal{X}$ is a pre-Finsler $\mathcal{A}$-module then it is a normed $\mathcal{A}$-module.*
2. *If $\mathcal{X}$ is an $\mathcal{A}$-homogene semiinner product $\mathcal{A}$-module then $\mathcal{X}$ is a pre-Finsler $\mathcal{A}$-module.*

Proof. 1. $\|ax\|^2 = \|F(ax)\| = \|aF(x)a^*\| \leq \|a\|^2 \|F(x)\| = \|a\|^2 \|x\|^2$.

Thus, $\|ax\| \leq \|a\| \|x\|$.

2. We suppose that $\mathcal{X}$ is a semiinnerproduct $\mathcal{A}$-bimodule with $[.,.]$ sp. We set $F(x) = [x,x]$, for all $x \in \mathcal{X}$. Then, by the definion of $[.,.]$, $\|F(x)\|$ is a norm in $\mathcal{X}$.

On the other hand, $F(ax) = \langle ax, ax\rangle = a\langle x,x\rangle a^* = aF(x)a^*$, thus $\mathcal{X}$ is a pre-Finsler $\mathcal{A}$-module. □

We note that duality in bimodules is more delicate than the one in the modules. Indeed, let's consider the algebraic case without using topology, if $\mathcal{X}$ is an algebraic $\mathcal{A}$-bimodule, the abelian group $Hom_{\mathcal{A}}(\mathcal{X},\mathcal{A})$ (the algebraic dual of $\mathcal{X}$) does not have a structure of an algebraic $\mathcal{A}$-bimodule if $\mathcal{A}$ is not commutative. To solve this problem, we consider the dual algebraic $Hom_{\mathcal{A}}(\mathcal{X},\mathcal{A}\otimes\mathcal{A})$, where $\otimes$ is the algebraic tensoriel product of $\mathcal{A}$,and consequently $Hom_{\mathcal{A}}(\mathcal{X},\mathcal{A}\otimes\mathcal{A})$ becomes an algebraic $\mathcal{A}-$bimodule relative to the follwing actions $a.(e\otimes g).b = aeb\otimes g$. Or we represent the given algebra on some Hilbert spaces, which is part of our work.

Remark 3.1.13 *Let $\mathcal{A}$ be a C^*- algebra, we denote by $Rep\mathcal{A}$ for the set of all nondegenrates representations $(\pi, H_{\mathcal{A}})$ of $\mathcal{A}$ on Hilbert space H_A.*

Depending on the representation of $\mathcal{A}$ on a Hilbert space, we reintroduce a semi-product in $\mathcal{X}$ related to the given representation while keeping the same conditions of the definition except the third is replaced by the following

(iii') $\forall x,y \in \mathcal{X}$ *et* $a,b \in \mathcal{A} : [ayb, x] = \pi(a)[y,x]\pi(b)$.

3.2 Representability of Semiinner product bimodules

Now we are ready to prove the main result of this subsection namely; we show that the class of semiinner C^*-bimodule coincides with the representable bimodule class (or $\mathcal{A}$-bimodules $\mathcal{A}$-convex) introduced by C.Pop [63], C.A.Delarouche [14] and B.J.Magajna [48].

First we fix some notations. Let $\mathcal{X}$ be a normed $\mathcal{A}$-bimodule and $(\pi, H_A) \in RepA$. We denote by $\mathcal{X}^{\ddagger} = Hom_{\mathcal{A}}(\mathcal{X}, B(H_{\mathcal{A}}))$ for the set of all continuous $\mathcal{A}$-morphismes from $\mathcal{X}$ to $B(H_A)$, $\left(\mathcal{X}^{\ddagger}\right)_1$ for the closed unit ball of $\mathcal{X}^{\ddagger}$, i.e, the subset of contractive $\mathcal{A}$-morphismes. And by $\ell^{\infty}\left(\left(\mathcal{X}^{\ddagger}\right)_1, B(H_{\mathcal{A}})\right)$ for the normed space of the suites $R = (R_T)_{T\in\left(\mathcal{X}^{\ddagger}\right)_1}$ such that $\|R\| = \sup_T \|R_T\| < \infty$, it is a normed $\mathcal{A}$-bimodule.

Lemma 3.2.1 *Let $\mathcal{X}$ be a normed $\mathcal{A}$-bimodule $\mathcal{A}$-convex. Then, for all $x \in \mathcal{X}$, there is a contractive morphism of $\mathcal{A}$-bimodules $T : \mathcal{X} \to B(H_{\mathcal{A}''})$ such that $T(x)$ is positive and $\|T(x)\| = \|x\|$.*

Proof. Let $0 \neq x_0 \in \mathcal{X}$. Then, there is $f \in \mathcal{X}^{\dagger}$ (the dual space of $\mathcal{X}$) such that $f(x_0) = \|x_0\|$. It is well known, [see 16, 57] that we can find a state φ on $\mathcal{A}$ such that

$$\begin{aligned} \forall a, b &\in \mathcal{A}, x \in \mathcal{X} : \\ |f(axb| &\leq \varphi(aa^*)^{\frac{1}{2}} \|x\| \varphi(b^*b)^{\frac{1}{2}}, \; f(axa^*) \geq 0 \end{aligned} \tag{3.4}$$

The extension φ^{**} of φ from A to $\mathcal{A}''$ is a normal state, denoted by the same symbol φ, and by [16] it is a vector state, that is there is a unit vector $\eta \in H_{\mathcal{A}''}$ with norm 1 such that $\varphi = \omega_{\eta}$. Let $x \in \mathcal{X}$, then the sesquilinear form on $\mathcal{A}\eta\times \mathcal{A}\eta$ given by $(a\eta, b\eta) \mapsto f(b^*xa)$ is definite and positive. Indeed, by the previous inequality, we get

$$|f(b^*xa| \leq \varphi(a^*a)^{\frac{1}{2}} \|x\| \varphi(b^*b)^{\frac{1}{2}} \leq \|y\| \|a\eta\| \|b\eta\| \text{ et } f(a^*xa) \geq 0$$

Thus, for each $x \in \mathcal{X}$, there exists a one operator $T(x) \in B(\overline{\mathcal{A}\eta})$ such that $\|T(x)\| \leq \|x\|$ and

$$\forall a, b \in \mathcal{A} : f(b^*xa) = (T(x)b\eta, a\eta)$$

Since $\overline{\mathcal{A}\eta} \subseteq H_{\mathcal{A}''}$ we can consider $T(x)$ as a morphism in $H_{\mathcal{A}''}$, denoted again with the same notation $T(x)$. We check that T is a morphism of $\mathcal{A}$-bimodules

$$(T(cxd)b\eta, a\eta) = f(b^*cxda) = (T(x)db\eta, c^*a\eta) = (cT(x)db\eta, a\eta)$$

Thus, $\forall c, d \in \mathcal{A} : T(cxd) = cT(x)d$. For $x = x_0$,

$$\forall a \in \mathcal{A} : f(a^*x_0a) = (T(x_0)a\eta, a\eta) \geq 0.$$

Thus, $T(x_0)$ is positive and it follows from $f(x_0) = \|x_0\|$ that $\|T(x_0)\| = \|x_0\|$. □

Theorem 3.2.2 *If $\mathcal{X}$ is a normed $\mathcal{A}$-bimodule $\mathcal{A}$-convex with norm $\|.\|$ such that all states of $\mathcal{A}$ are vectors with respect to $(\rho, H_{\mathcal{A}}) \in Rep\mathcal{A}$. Then there is $B(\mathcal{H}_{\mathcal{A}})$-s.p. $[.,.]$ on $\mathcal{X}$ generates the same norm $\|.\|$.*

Proof. If $\mathcal{X}$ is a normed $\mathcal{A}$-bimodule with norme $\|.\|$ and $\mathcal{A}$-convex then, by [63], the topological dual of $\mathcal{X}$ is $\mathcal{X}_{\rho}^{\ddagger} = Hom_{\mathcal{A}}(\mathcal{X}, B(\mathcal{H}_{\mathcal{A}})$.

The dual $\mathcal{X}_\rho^\ddagger$ has an obvious structure of $\rho\left(\mathcal{A}\right)'$-bimodule with respect to the following actions

$$\forall a, b \in \rho\left(\mathcal{A}\right)', \forall x \in \mathcal{X}, \forall T \in \mathcal{X}_\rho^\ddagger : \left(a.T.b\right)\left(x\right) = aT(x)b$$

By the previous lemma the normalized duality mapping, D_ρ: $\mathcal{X} \to \mathcal{X}_\rho^\ddagger$ has the following form, for all $x \in \mathcal{X}$,

$$D_\rho\left(x\right) = \left\{T \in \mathcal{X}_\rho^\ddagger : T\left(x\right) \in B^+\left(\mathcal{H}_\mathcal{A}\right), \;\; \left\|T\left(x\right)\right\| = \left\|x\right\|^2 \text{ et } \left\|T\right\| = \left\|x\right\|\right\},$$

where $B^+\left(\mathcal{H}_\mathcal{A}\right)$ is the positive cone of $B\left(\mathcal{H}_\mathcal{A}\right)$.

For each $x \in \mathcal{X}$ we take one selection T from $D_\rho\left(x\right)$ we can define a function $[.,.] : \mathcal{X} \times \mathcal{X} \to B\left(\mathcal{H}_\mathcal{A}\right)$,

$$\forall x, y \in \mathcal{X} : [y, x] = T(y).$$

It is clear that $[.,.]$ is a map well defined and, for all $x \in \mathcal{X}$, $[x, x] = T(x)$ is a positive operator. Also, $x = 0$ is equivalent to $[x, x] = 0$. Since each selection T, for each $x \in \mathcal{X}$, is $\mathcal{A}$-morphism then we get the others properties of $\mathcal{A}$-semiinner product. □

Remark 3.2.3 *If $\mathcal{A}$ is identified with its image in its standard form $\mathcal{A}''$ by the canonical inclusion, the dual (A-dual) of $\mathcal{X}$ is simply $\mathcal{X}^\ddagger = Hom_\mathcal{A}(\mathcal{X}, B\left(\mathcal{H}_{\mathcal{A}''}\right)$. It will be called the standard dual of $\mathcal{X}$.*

Since every state of the enveloping algebra $\mathcal{A}''$ are vectors with respect to the standard representation. Hence, by the previous theorem, we say that every normed A-bimodule A-convex is a normed semiinner product $\mathcal{A}$-bimodule and in this case the $\mathcal{A}$-bimodule has a standard $B\left(\mathcal{H}_\mathcal{A}\right)$-sp. $[.,.]$.

Theorem 3.2.4 *Let $\mathcal{X}$ be semiinner product A-bimodule. Then, there exists a Hilbert space H, tow faithfull nondegenerates representationsπ of $\mathcal{A}$ and J such that $J : \mathcal{X} \to B(H)$ is isometric and,*

$$\forall a, b \in \mathcal{A}, x \in \mathcal{X} : J(axb) = \pi(a)J(x)\pi(b)$$

Proof. Let $[.,.]$ be $\mathcal{A}$-sp. in $\mathcal{X}$ and $(\pi, H_{\mathcal{A}}) \in Rep\mathcal{A}$. Let $(0 \neq) x \in \mathcal{X}$, by the definition of $[.,.]$, the map $T : \mathcal{X} \to B(\mathcal{H}_{\mathcal{A}})$ given by $T(y) \in \frac{1}{\|x\|}[y, x]$ is $\mathcal{A}$-morphism and $\|T(x)\| = \|x\|$. Thus, for all $x_0 \in \mathcal{X}$ there exists a continuous contractive morphism ($\|T\| \leq 1$) T such that $\|T(x_0)\| = \|x_0\|$. Hence, $\|x_0\| = \sup\limits_{\|T\|\leq 1} \|T(x_0)\|$.
Let $U : \mathcal{X} \to \ell^{\infty}\left((\mathcal{X}^{\vee})_1, B(H_{\mathcal{A}})\right)$ be a map defined by $U(x)(T) = Tx$. It follow from the previous argument that U is an isometric morphism. We have the obvious inclusion $\ell^{\infty}\left((\mathcal{X}^{\vee})_1, B(H_{\mathcal{A}})\right) \subset B(H)$ where $H = \ell^2\left((\mathcal{X}^{\vee})_1\right) \otimes H_{\mathcal{A}}$.
We take $\rho = 1 \otimes \pi$ and we define a map $J : \mathcal{X} \to B(H)$ by

$$\forall x \in \mathcal{X} : J(x) = U(x)$$

we deduce that the representation ρ is faithfull and nondegenerate, and since U is an isometric morphism then J is again an isometric morphism. □

Remark 3.2.5 *1. We deduce from the previous arguments that every semiinner product $\mathcal{X}$ is an operator space, in particular is an operator module. In particular, the Hilbert C^*-modules are operator spaces.*
On the other hand, $J(\mathcal{X})$ is a semiinner product $\pi(\mathcal{A})$-bimodule. Indeed, we set $[y, x]_{\mathcal{X}} = [J(y), J(x)]_J$, for all $x, y \in \mathcal{X}$ where $[.,.]_{\mathcal{X}}$ is an $\mathcal{A}$-s.p. in $\mathcal{X}$. We can check easly that $[.,.]_J$ is an

$\mathcal{A}$-s.p. in $J(\mathcal{X})$ and the norm of $J(\mathcal{X})$ is the induced norm of $B(H_{\mathcal{A}})$.

2. Considering an involutive Banach algebra with approximate identity and non degenerate representation π we get, with the same argument as in the previous theorem, a nondegenerate representation $\rho = 1 \otimes \pi$ and an isometric morphism $J : \mathcal{X} \to B(H)$.

C. Pop [63] and B. Magajna [49] have showed separately the equivalence between the $\mathcal{A}$-$\mathcal{B}$-bimodules $\mathcal{A}$-$\mathcal{B}$-convexes and the $\mathcal{A}$-$\mathcal{B}$-bimodules representables. Thus, we deduce the following result.

Corollary 3.2.6 *Let $\mathcal{X}$ be a normed $\mathcal{A}$-bimodule such that every state of $\mathcal{A}$ is a vector with respect to $(\rho, H_{\mathcal{A}}) \in Rep\mathcal{A}$. Then, if $\mathcal{X}$ is an $\mathcal{A}$-bimodule $\mathcal{A}$-convex then $\mathcal{X}$ is a $B(H_{\mathcal{A}})$-semiinner product $\mathcal{A}$-bimodule. Conversly, every $\mathcal{A}$-semiinnerproduct $\mathcal{A}$-bimodule is an $\mathcal{A}$-convex $\mathcal{A}$-bimodule.*

Definition 3.2.7 *Let $\mathcal{A}$ be a C^*-algebra.*

(i) If $(\pi, H_{\mathcal{A}}) \in Rep\mathcal{A}$, then $(\pi, H_{\mathcal{A}})$ is said locally cyclic if for all $h_1, h_2, .., h_3 \in H_{\mathcal{A}}$ there exists $h \in H_{\mathcal{A}}$ with $h_i \in \overline{\pi(\mathcal{A})\, h}$, $i = 1, ., n$; [68].

(ii) $\mathcal{A}$ is said possed the property (LC) if $\forall (\pi, H_{\mathcal{A}}) \in Rep\mathcal{A}$, $(\pi, H_{\mathcal{A}})$ is locally cyclic; [63].

Theorem 3.2.8 *Let $\mathcal{X}$ be semiinner product $\mathcal{A}$-bimodule and $(\pi, H_{\mathcal{A}}) \in Rep\mathcal{A}$ such that $(\pi, H_{\mathcal{A}})$ is locally cyclic and every state of $\mathcal{A}$ is a vector with respect to $(\rho, H_{\mathcal{A}})$ and let $F \subseteq \mathcal{X}$ be a $\mathcal{A}$-subbimodule. If $T : F \to B(H_{\mathcal{A}})$ is a contractive morphism of $\mathcal{A}-$bimodules, then there exists a contractive morphism of $\mathcal{A}-$bimodules $\widetilde{T} : \mathcal{X} \to B(H_{\mathcal{A}})$ which extends T.*

Proof. Since $\mathcal{X}$ is a semiinner $\mathcal{A}$-bimodule with respect to given nondegenerate faithfull representation for A, then by theorem 3.1.7, $\mathcal{X}$ is A-convexe bimodule and, by extension theorem of C.Pop [63], there exists a contractive morphism of $\mathcal{A}-$bimodules $\widetilde{T}:\mathcal{X}\rightarrow B(H_{\mathcal{A}})$ witch extends T. □

Remark 3.2.9 *Since the standard representations of the enveloping Von Neumann algebra of A are locally cyclic then every semiinner product $\mathcal{A}$-bimodule $\mathcal{X}$ in its standard form satisfy the previous extension theorem.*

As exemple, every stable C^*-algebra has the property (LC). In [68], it was shown that if a Von Neumann algebra M acts on a Hilbert H, the inclusion of M in $B(H)$ is locally cyclic if every normal state of $M^{'}$ is a vector state. In particular, the standard representation of a von Neumann algebra M is locally cyclic. For a C^*- algebra $\mathcal{A}$, the standard representation of its enveloping Von Neumann algebra (on some Hilibert space) $\mathcal{A}^{''}$ is again locally cyclic. It is well known that the only representation of $\mathbb{C}$ that is locally cyclic is the trivial representation.

Proposition 3.2.10 *Let H be a Hilbert space and $M_n(\mathbb{C})$ be the C^*-algebra of n-square matrices on $\mathbb{C}$. Then, a representation $\pi:M_n(\mathbb{C})\rightarrow B(H)$ is locally cyclic if and only if π is the sum of not more than n copies of the identical representation.*

Proof. Let I be the identic representation. Then $\oplus_{i=1}^{m}I_i$ is a subrepresentation of the standard representation if $m\leq n$ thus, from the previous examples, $\oplus_{i=1}^{m}I_i$ is locally cyclic if $m\leq n$. We suppose that $m>n$ and $\pi=\oplus_{i=1}^{m}I_i$ if π is locally cyclic then, if $h_1,h_2,..,h_{nm}\in\mathbb{C}^n\otimes\mathbb{C}^m$ where the h_i are linearly independent and $h\in\mathbb{C}^n\otimes\mathbb{C}^m$ such that $h_i\in\pi(M_n(\mathbb{C}))h,$. Since $\dim\pi(M_n(\mathbb{C}))h\leq n^2$, we get a contradiction. □

Proposition 3.2.11 *Let $\mathcal{X}$ be a semiinner product with s.p. ${}_{\mathcal{A}}[.,.]$ and let $(\pi,H_{\mathcal{A}})\in Rep\mathcal{A}$ be a faithfull isometric representation of $\mathcal{A}$. Then, the map*

$$_{\pi}[.,.] : \mathcal{X} \times \mathcal{X} \to B(H); _{\pi}[y,x] = \pi\left(_{\mathcal{A}}[y,x]\right)$$

for all $x, y \in \mathcal{X}$, is a $\mathcal{A} - \mathbb{C}$-semiinner product in $\mathcal{X}$.

Proof. (i). Since $_{\mathcal{A}}[x,x] \geq 0$, for all x, then $\forall x \in \mathcal{X} :_{\pi} [x,x] = \pi\left(_{\mathcal{A}}[x,x]\right) \geq 0$. Since π is faithfull then $\pi\left(_{\mathcal{A}}[x,x]\right) = 0 \Rightarrow_{\mathcal{A}} [x,x] = 0$, thus $x = 0$.

(ii)The left linearity is obvious.

(iii) For all $a \in \mathcal{A}$ and all $x, y \in \mathcal{X}$, we get $_{\pi}[ay,x] = \pi\left(_{\mathcal{A}}[ay,x]\right) = \pi\left(a_{\mathcal{A}}[y,x]\right) = \pi(a)\,\pi\left(_{\mathcal{A}}[y,x]\right) = \pi(a)_{\pi}[y,x]$

(iv) $\|_{\pi}[x,x]\|_{B(H)} = \|\pi\left(_{\mathcal{A}}[x,x]\right)\|_{B(H)} = \|_{\mathcal{A}}[x,x]\|_{\mathcal{A}} = \|x\|^2$. Hence,

$\|_{\pi}[y,x]\|_{B(H)} = \|_{\mathcal{A}}[y,x]\|_{\mathcal{A}} \leq \|y\|\,\|x\|$. □

By analogy in the case of Hilbert C^*-modules, the existence of a semiinner product $[.,]$ in the $\mathcal{A}$-representable bimodules $\mathcal{X}$ allows us to introduce a notion of orthogonality in $\mathcal{X}$ with respect to $[.,.]$. So that we set the following definition.

Definition 3.2.12 *Let $\mathcal{X}$ be an $\mathcal{A}$-bimodule representable over a C^*-algebra $\mathcal{A}$ with $\mathcal{A}$-semiinner product $[.,.]$ and $x, y \in H$. We say that x and y are orthogonal with respect to the $\mathcal{A}$-valued semiinner product $[.,.]$ if $[x,y] = 0$. We write $x \perp y$.*

Theorem 3.2.13 *Let $\mathcal{X}$ be an $\mathcal{A}$-bimodule representable over a C^*-algebra $\mathcal{A}$ with $\mathcal{A}$-semiinner product $[.,.]$. Then, there exits a Hilbert C^*-module $\mathcal{X}_e$ over $\mathcal{A}$, as a bimodule quotient of $\mathcal{X}$ with respect to a subbimodules of $\mathcal{X}$, with $\mathcal{A}$-inner product $(x,y) = [x,e][y,e]^*$, for all $x, y \in \mathcal{X}_e$.*

Proof. Let e be a fixed element in $\mathcal{X}$ and let ϕ be a map from $\mathcal{X}$ to $\mathcal{A}$ defined by $\phi(x) = [x,e]$. By the definition of $[.,.]$, ϕ is a bounded $\mathcal{A}$-bimodule morphism with $Ker\phi = \{x \in \mathcal{X} : x \perp e\}$. Thus, $ker\phi$ is a subbimodule of $\mathcal{X}$.

Let $(.,.)$ be a map from $\mathcal{X} x \mathcal{X}$ to $\mathcal{A}$ given by $(x,y) = \phi(x)\phi(y)^*$. It is clear that $(.,.)$ is a positive semidefinite innerproduct that induces a seminorm. A Hilbert C^*-module, denoted $\mathcal{X}_e$, is obtained by dividing out by the closed subbimodule $Ker\phi$ and completing. □

Remark 3.2.14 *In the proof of Theorem 3.1.15, we constructed an $\mathcal{A}$-semiinner product $[.,.]$ from one selection of the normalized duality mapping. Thus, in the sequel, for every representable $\mathcal{A}$-bimodule $\mathcal{X}$ we choose a suitable vector $e \in \mathcal{X}$, a convenable selection T from the normalized duality $D_\rho(e)$ and then we consider an $\mathcal{A}$-inner product $(x,y)_T = T(x)T(y)^*$, for every $x, y \in \mathcal{X}_T$, where $\mathcal{X}_T$ is the Hilbert C^*-module obtained by passing to the quotient and completing.*

Chapter 4

Noncommutative bundles and noncommutative varieties

4.1 Banach bundles over commutative C^*-algebra

We recall the definitions of fields of Banach spaces.

Definition 4.1.1 *Let Ω be a Hausdorff. A fields on Ω is a triple $\Gamma = (\mathcal{X}, \Omega, P)$ where $\mathcal{X}$ is a Hausdorff topological space and $P : \mathcal{X} \to \Omega$ is an open continuous map.*

$\mathcal{X}$ is called the field of Γ, Ω is the base space and P is the field projection. For all $t \in \Omega$, $P^{-1}(t) = \mathcal{X}_t$, is the fibre over the point t.

A section of Γ is a function $f : \Omega \to \mathcal{X}$ such that $f(t) \in \mathcal{X}_t$, for all $t \in \Omega$.

We said Γ is full if for all $x \in \mathcal{X}$, there exists a continuous section f such that $x \in f(\Omega)$.

Definition 4.1.2 (30) *A continuous field of Banach spaces on Ω, in the sense of J.M.G. Fell,* *denoted (F)-continuous field of Banach spaces, is a field* $\Gamma = (\mathcal{X}, \Omega, P)$ *such that the stalks* $\mathcal{X}_t$ *are Banach spaces, for every* $t \in \omega$*, with the following conditions:*

1. *The function* $\mathcal{X} \ni x \mapsto \|x\|$ *is continuous.*

2. *The sum* $+$ *is a continuous function from* $\{(x, y) \in \mathcal{X}^2 : P(x) = P(y)\}$ *to* $\mathcal{X}$*.*

3. $\forall \lambda \in \mathbb{K}:$ *the map* $x \mapsto \lambda.x$ *is continuous.*

4. *If* $t \in \Omega$ *and* $(x_\alpha)_\alpha$ *is a net in* $\mathcal{X}$ *such that* $\|x_\alpha\| \to 0$ *and* $P(x_\alpha) \to t$ *in* Ω*, then* $x_\alpha \to 0_t$ *in* $\mathcal{X}$*.* 0_t *is the origin in each stalk* $\mathcal{X}_t$*.*

Definition 4.1.3 *A continuous field of Banach spaces on* Ω *in the sense of K.H. Hoffman, (H)-field of Banach spaces, has the same definition except the first assertion is subtitue by: the function* $\mathcal{X} \ni x \mapsto \|x_t\|$ *is upper semi continuous; i.e.* $\{t \in \Omega : \|x\| < \delta\}$ *is an open of* Ω *pour tout* $\delta > 0$*. For more details, see [22].*

Let Ω be Hausdorff compact space and let $C = C(\Omega)$ be a C^*-algebra of continuous functions on Ω. Let $\mathcal{X}$ be a essential Banach $C-$module. For each$t \in \Omega$, we set

$$\mathcal{N}_t = \{fx : x \in \mathcal{X}, f(t) = 0\}, \ \mathcal{X}_t = \mathcal{X} \diagup \mathcal{N}_t,$$

if $x \in \mathcal{X}$, we note x_t for the image of x by the projection $\mathcal{X} \to \mathcal{X}_t$. It is obvious that, $\mathcal{X}_t$ is a C-module. The action of C is defined by

$$fx_t = (fx)_t = f(t)x_t. \tag{4.1}$$

Thus, we have a family $(\mathcal{X}_t)_{t\in\Omega}$ of Banach spaces and a set $\widetilde{\mathcal{X}} = \{\widetilde{x} : x \in \mathcal{X}\} \subseteq \Pi_{t\in\Omega} X_t$ of vectors fields (sections) $\widetilde{x}\colon t \mapsto x_t$. If $X = H$ is a Hilbert C-module, we get a familly of Hilbert spaces $(H_t)_{t\in\Omega}$ and $\widetilde{H}$, the set of vectors fields (sections)

$$t \mapsto x_t$$

. $\tilde{X}$ is a submodule of $\Pi_{t\in\Omega} X_t$ (resp. $\tilde{H}$ is a sub module of $\Pi_{t\in\Omega} H_t$ with its $C-$product

$$(\widetilde{x}, \widetilde{y})(t) = (x_t, y_t)_{H_t}, \text{for all } t \in \Omega \text{ and } \widetilde{x}, \widetilde{y} \in \widetilde{H}.$$

and $\Phi : X \to X, x \mapsto \widetilde{x}$ is a surjective homomorphisme of C-modules and $\|\Phi(x)\|_\infty = \sup_{t\in\Omega} \|x_t\| \leq \|x\|$.

We ask the natural question: For what class of C-modules X the previous map becomes an isometry in such a way the module X be represented by (H) or (F)-fields of Banach spaces.

Definition 4.1.4 *Let $\mathcal{X}$ be a Banach C-module.* ***1.***[24]. *We said $\mathcal{X}$ is an abelian C-module if there is a commutative algebra C^*- B with an isometry $\pi : \mathcal{X} \to B$ and *-isomorphism $\varphi : C \to B$ such that*

$$\pi(fx) = \varphi(f)\, x : \forall x \in \mathcal{X}, \forall f \in C$$

2. *We said that $\mathcal{X}$ is a C-convex C-module if, $\mathcal{X}$ is $C - C$ bimodule by using*

$$\forall f \in C, \forall x \in \mathcal{X} : fx = xf$$

$\mathcal{X}$ is C convex.

Theorem 4.1.5 *[59] Let $\mathcal{X}$ be a Banach space.* ***1.*** *If $\mathcal{A} = C(\Omega)$ and $\mathcal{X}$ is a Banach $\mathcal{A}$-module. Then the following assertions are equivalent:*

i. $\mathcal{X}$ is an abelian C-module;

ii. $\mathcal{X}$ is a C-convex C-module;

iii. There exists (H)-fields of Banach spaces on Ω such that $\mathcal{X}$ is isomorph to a C-module of continuous fields.

2. *If $\mathcal{A} = L^{\infty}(\Omega)$ is the commutative Von Neumann algebra and $\mathcal{X}$ is a Banach $\mathcal{A}$-module. Then the following assertions are equivalent:*

i. $\mathcal{X}$ is an abelian $\mathcal{A}$-module;

ii. $\mathcal{X}$ is a Finsler $\mathcal{A}$-module with respect to the norm in $\mathcal{X}$.

It follows from the previous theorem that if $\mathcal{X}$ is a Banach C-module. Then, the condition $\mathcal{X}$ is semiinnerproduct $\mathcal{A}$-bimodule in the standard form is equivalents to the existance of (H)-fields of Banach spaces on Ω such that $\mathcal{X}$ is isomorphs to C-module of continuous fields.

If Ω is a Haussdorf locally compact space then the space $C_0 = C_0(\Omega)$ is a C^*-algebra of continuous functions on Ω vanishing to ∞. We note by Ω_∞ for the compactification of Ω at ∞ and $\mathcal{N}_\infty = \{fx : x \in \mathcal{X}, f(\infty) = 0\} = \mathcal{X}$ and then the fibre at ∞ is trivial. $\widetilde{\mathcal{X}}_{|\Omega}$ denotes the restriction of fields $\widetilde{\mathcal{X}}$ at Ω. We note also that we can see that every Banach $C_0(\Omega)$-module as a Banach $C(\Omega_\infty)$-module.

Remark 4.1.6 *Since every Banach $\mathcal{A}$-module is a Banach $\mathcal{A}_{un}$-module (if $\mathcal{A}$ is not unitary) in the same way if $\mathcal{X}$ is a $\mathcal{A}-$convex Banach $\mathcal{A}$-module then it is again a $\mathcal{A}_{un}-$ convex Banach $\mathcal{A}_{un}$-module. Thus, the previous theorem can be extended to the case of Banach C_0-module.*

We recall that, in [73], Takesaki has shown that every HilbertC_0-module is isomorph to (F)-fields of Hilbert spaces.

We observe that every Hilbert module represents the noncommutativity version of fields of Hilbert spaces represents, in general, the noncommutativity version of fields of Banach spaces. On the othe hand, the tangent space at point x of a riemanniann manifold is a linear space with an inner product. Thus, tangent fibre of a riemanniann manifold is a fields of Hilbert spaces. In other words, the class of Hilbert C^*-modules represents the noncommutativity of the riemanniann and hermitian geometry.

Theorem 4.1.7 *Let $\mathcal{X}$ be a Banach C_0-module. then the following assertions are equivalent:*

1. There is a C_0-s.p. $[.,.]$ which generates the same norm given in $\mathcal{X}$ and $\mathcal{X}$ is semiinner product $\mathcal{A}$-bimodule.

2.There exists (F)-fields of Banach spaces on Ω such that $\mathcal{X}$ is isomorphs to C_0-module of continuous fields vanishing at ∞.

Proof. (1)⇒(2): If there exists C_0-s.p.g $[.,.]$ such that $\mathcal{X}$ is semiinner product A-bimodule, then $\mathcal{X}$ is a representable $\mathcal{A}$-module. If Ω is a compact space then the second assertion holds. If Ω is not compact but locally compact, we note by Ω_∞ for the compactification of Ω at the point ∞. Hence, $\mathcal{X}$ is a $C(\Omega_\infty)$-bimodule and it is a Finsler $C(\Omega_\infty)$-module. Thus, $\mathcal{X}$ is isomorphs to $C(\Omega_\infty)$-module of continuous fields for certain (F)-fields of Banach spaces. Since $[x,x] \in C_0(\Omega)$ for all $x \in \mathcal{X}$, then the fibre at the point ∞ is trivial and then $\mathcal{X}$ is isomorphic to the restriction of $C_0(\Omega)$-module of continuous fields vanishing at ∞.

(2)⇒(1): It is obvious that (F)-fields of Banach spaces on Ω leads to the existence of (H)-fields of Banach spaces on Ω. Thus, by the previous theorem and Theorem 3.1.17, $\mathcal{X}$ is a semiinner product $\mathcal{A}$-bimodule with its $C_0(\Omega)$- s.p. $[.,.]$. We set, for all $t \in \Omega$, $[y,x](t) = [y_t, x_t]_{E_t}$ où $y, x : \Omega \to E$, continuous sections. Then, it follows that $[.,.]_{E_t}$ are semiinner products which generate the same norm in E_t for all $t \in \Omega$. By the definition of the semiinner product we deduce that $\mathcal{X}$ is a semiinner product bimodule. □

Definition 4.1.8 *Let Ω be a compact space. Ω is said hyper-Stonien if $C(\Omega)$ is isometric to the dual of another Banach space.*

We recall that if Ω is hyper-Stonnien then $C(\Omega)$ is a Von Neumann algebra. The character space Φ_μ of the commutative C^*-algebra $L^\infty(\Omega, \mu)$, μ is a positive measure on a compact space Ω, is a hyper-Stonien space.

We deduce from the previous results the following corollary.

Corollary 4.1.9 *If Ω is a hyper-Stonien space and $\mathcal{X}$ is a Banach $C(\Omega)$-module then every semiinner product $\mathcal{A}$-bimodule $\mathcal{X}$ is isomorphic to C_0-module of continuous fields of certain (F)-fields of Banach spaces on Ω.*

We can say, from the previous results, that the class of semiinner product $\mathcal{A}$-bimodules $\mathcal{A}$represents the noncommutativity version of the continuous fields of Banach spaces.

4.2 Toeplitz and Cuntz-Pimsner Algebras

Algebraically, a vector bundle $M \to X$ over a compact Haussdorff space(finite-dimensional manifold X is completely characterized by its continuous (smooth) sections $\Gamma(M, X)$. In this context, the space of sections is a (right) module over the algebra $C(X), (C^\infty(X))$ of continuous (smooth) functions over X. Indeed, by the Serre-Swan theorem finite-rank complex vector bundles over a compact Hausdorff space X correspond canonically to finite projective modules over the algebra $C(X)$, $(C^\infty(X))$. Indeed, by this theorem a $C^\infty(X)$-module E is isomorphic to a module $\Gamma(M, X)$ of smooth sections, if and only if it is finite projective. For a hermitian bundle there is extra structure: the hermitian inner product $\langle , \rangle_x$ on each fiber $Mx, x \in X$, gives a $C^\infty(X)$-valued hermitian map on the module $\Gamma(M, X)$. Let $End(M) \to X$ be the endomorphism bundle with corresponding sections $\Gamma(End(M), X)$. The latter is an algebra under composition and there is an identification $\Gamma(End(M), X) \simeq End_{C^\infty(X)}(\Gamma(M, X))$, with the algebra of $C^\infty(X)$-endomorphisms of the module $\Gamma(M, X)$. By its definition $End_{C^\infty(X)}(\Gamma(M, X))$ acts on the left on the module $\Gamma(M, X)$. Moreover, in parallel with $C^\infty(X)$-valued Hermitian map on $\Gamma(M, X)$ there is a $End_{C^\infty(X)}(\Gamma(M, X))$-valued Hermitian product on $\Gamma(M, X)$. The fact that $\Gamma(M, X)$ is a $(End_{C^\infty(X)}(\Gamma(M, X)), C^\infty(X))$-bimodule and is endowed with two Hermitian products which are compatible, put it in the context of Morita equivalence that we shall defined. On the other hand, one sees that the vector bundle $M \to X$ is a line bundle if and only if $End_{C^\infty(X)}(\Gamma(M, X)) \simeq C^\infty(X)$. This motivates calling noncommutative line bundle over the noncommutative algebra A (having the role of $C^\infty(X)$), a self-Morita equivalence bimodule for A, that is a A-bimodule E (having the role of $(\Gamma(M, X))$ with extra structures (roughly, two compatible A-valued Hermitian products on E).

In this section we illustrate how to naturally associate a Fock module over the (noncommutative) algebra A to any such a correspondance module (noncommutative line bundle) over the algebra A of the base space. The algebra of corresponding creation and annihilation operators acting on a Hilbert module (or rigged Hilbert space) can then be realised as the total space algebra of a noncommutative principal $U(1)$-bundle over the algebra $\mathcal{A}$.

4.2.1 Fock modules and Left-Right Creation Operators

Let $\mathcal{H}$ be an $n-$dimensional Hilbert space with orthonormal basis $e_1, e_2, ..., e_n$, $1 \leq n \leq \infty$. The full Fock space of $\mathcal{H}$ is defined as

$$\mathcal{F}^2(\mathcal{H}) := \bigoplus_{k\geq 0} \mathcal{H}^{\otimes k} = \mathbb{C} \oplus \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H}) \oplus (\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}) \oplus ... \tag{4.2}$$

with the following inner product, defined on elementary tensors:

$$\langle \zeta_0 \otimes \zeta_1 \otimes ... \otimes \zeta_j, \xi_0 \otimes \xi_1 \otimes ... \otimes \xi_k \rangle = \begin{cases} 0 & \text{if } j \neq k \\ \prod_{i=0}^{n} & \text{otherwise} \end{cases} \tag{4.3}$$

If the underlying Hilbert space is clear, we shall denote $\mathcal{F}^2(\mathcal{H})$ by $\mathcal{F}^2$ and $I_{\mathcal{H}}$ by I.

One representation of the full Fock space is given using the free semigroup on n generators. Let $\mathbb{F}_n^+$ be the unital free semigroup on n generators, $g_1, g_2, ..., g_n$, together with the identity g_0. Define the length of an element $\alpha \in \mathbb{F}_n^+$ as $|\alpha| := 0$ if $\alpha = g_0$ and $|\alpha| := k$ if $\alpha = g_{i_1} g_{i_2} ... g_{i_k}$, where $1 \leqslant i_1, i_2, ..., i_k \leqslant n$. For each $\alpha \in \mathbb{F}_n^+$, define

$$e_\alpha := \begin{cases} e_{g_{i_1}} \otimes e_{g_{i_2}} \otimes ... \otimes e_{g_{i_k}} & \text{if } \alpha = g_{i_1} g_{i_2} ... g_{i_k} \\ 1 & \text{if } \alpha = g_0 \end{cases} \tag{4.4}$$

It is clear that the set $\{e_\alpha : \alpha \in \mathbb{F}_n^+\}$ is an orthonormal basis for the full Fock space $\mathcal{F}^2$. Thus, we can identify the full Fock space $\mathcal{F}^2$ with the Hilbert space $\ell^2(\mathbb{F}_n^+)$. Finally, if $(T_1, T_2, ..., T_n)$

is an $n-$tuple of operators $T_i \in B(\mathcal{H})$ and $\alpha = g_{i_1}g_{i_2}...g_{i_k} \in \mathbb{F}_n^+$, define T_α to be the product $T_{i_1}T_{i_2}...T_{i_k}$.

The concept of a row contraction is fundamental in both the study of the full Fock spaces, modules and the domain algebras introduced by Popescu in [].

Definition 4.2.1 *Let $\mathcal{H}$ be an n dimensional Hilbert space. A row contraction on $\mathcal{H}$ is an $n-$tuple of operators $(T_1, T_2, ..., T_n)$, with $T_i \in B(\mathcal{H})$, $1 \leq i \leq n$, such that*

$$\sum_{i=1}^{n} T_i T_i^* \leq I_H \tag{4.5}$$

It is usually beneficial to think of the $n-$tuple $(T_1, T_2, ..., T_n)$ as the operator $T = [T_1 T_2 ... T_n]$ acting on $\mathcal{H}^n$. In this sense, $(T_1, T_2, ..., T_n)$ is a row contraction if and only if T is a contraction, since $\sum_{i=1}^{n} T_i T_i^* \leq I_{\mathcal{H}}$ holds exactly when $TT^* \leq I_{\mathcal{H}^n}$.

With this definition in mind, we can begin our discussion on the left and right creation operators.

Definition 4.2.2 *For each $i = 1, 2, ...n$, the left creation operators are defined by*

$$L_i : \ell^2(\mathbb{F}_n^+) \to \ell^2(\mathbb{F}_n^+), L_i e_\alpha = e_{g_i \alpha} \tag{4.6}$$

for every $\alpha \in \mathbb{F}_n^+$. Similarly, for each $i = 1, 2, ...n$, the right creation operators are defined by

$$R_i : \ell^2(\mathbb{F}_n^+) \to \ell^2(\mathbb{F}_n^+), R_i e_\alpha = e_{g_\alpha g_i} \tag{4.7}$$

for every $\alpha \in \mathbb{F}_n^+$.

The left and right creation operators have some important properties. We can check easly the following proposition.

Proposition 4.2.3 *The left (respectively, right) creation operators are isometries with orthogonal ranges. Furthermore, they are row contractions:*

$$\sum_{i=1}^{n} L_i L_i^* \leq I, \sum_{i=1}^{n} R_i R_i^* \leq I. \tag{4.8}$$

For more details, see [64].

Let X be a Hilbert C^*-module over $\mathcal{A}$ and Y a Hilbert C^*-module over $\mathcal{B}$. Further, let $\phi : A \to \mathcal{L}(Y)$ be a $*$-homomorphism, where $\mathcal{L}(Y)$ is the space of all adjointable operators on the Hilbert C^*-module Y with respect to the $\mathcal{B}$-product $\langle , \rangle_{\mathcal{B}}$.
The homomorphism ϕ makes Y a left A-module

$$a \cdot y = \phi(a)y \quad a \in A, y \in Y.$$

On the balanced algebraic tensor product $X \otimes_A Y$, there is a positive semi-definite inner product satisfying

$$\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = \langle y_1, \phi(\langle x_1, x_2 \rangle) y_2 \rangle \quad x_1, x_2 \in X, y_1, y_2 \in Y$$

that induces a seminorm.

Definition 4.2.4 *(Internal tensor product) A Hilbert C^*-module $X \otimes_\phi Y$ (called the interior* *tensor product and sometimes also denoted $X \otimes_A Y$) is obtained by dividing out by the null space*

$$N = \{z \in X \otimes_A Y : \langle z, z \rangle = 0\}$$

and completing.

Its right B-module structure is the obvious one:

$$(x \otimes y) \cdot b = x \otimes (yb).$$

Observe that Y is simultaneously a left-$\mathcal{A}$-module and a right $\mathcal{B}$-module. In fact, it is an $(\mathcal{A}-\mathcal{B})$-bimodule, meaning the left and right actions commute.

Definition 4.2.5 *A C^*-correspondence X from $\mathcal{A}$ to $\mathcal{B}$ is a right Hilbert C^*-module over a C^*-algebra $\mathcal{B}$ that is also a left $\mathcal{A}$-module whose action is given by a C^*-homomorphism $\phi : A \to \mathcal{L}(Y)$. In case $\mathcal{A} = \mathcal{B}$, X is called a C^*-correspondence over $\mathcal{A}$.*

The correspondence X from $\mathcal{A}$ to $\mathcal{B}$ is called nondegenerate when $\overline{\mathcal{A}.X} = X$, where $\mathcal{A}.X = \{\phi(a)x : a \in \mathcal{A}, x \in X\}$. By the Cohen-Hewitt factorization theorem, a C*-correspondence X is nondegenerate if and only if $X = \mathcal{A}X$.

A C*-correspondence X from $\mathcal{A}$ to $\mathcal{B}$ is called injective if the left action $\phi : \mathcal{A} \to \mathcal{L}(Y)$ is injective. It is called proper if $\phi(\mathcal{A})$ is contained in the C^*-algebra $K(X)$, the space of compact adjointable operators in $\mathcal{L}(X)$.

A C^*-correspondence X is called regular if it is both injective and proper. For more detail, see [27,52].

Example 4.2.6 *1. Given an automorphism $\alpha : \mathcal{A} \to \mathcal{A}$ (assumed to be unital and adjoint - preserving). We set in the previous definition $X = \mathcal{A}$, the algebra $\mathcal{A}$ becomes a C^*-correspondence over $\mathcal{A}$ by defining the left action by $a \cdot b = \alpha(a)b$ for $a, b \in \mathcal{A}$. This correspondence is denoted by ${}_\alpha\mathcal{A}$.*

2. [52] Let $\sigma : \mathcal{A} \to B(H)$ be a C^*-representation and X a right Hilbert C^*-module over A. Observe that σ makes H C^*-correspondence from A to $\mathbb{C}$. Let $X \otimes_\sigma H$ be the internal tensor product, σ can be induced to a representation $\sigma^X : \mathcal{L}(X) \to B(X \otimes_\sigma H)$ defined by $\sigma^X(F)(x \otimes h) = F(x) \otimes h$. In other words, $\sigma^X(F) = F \otimes I_H$.

If J is an ideal in a C*-algebra $\mathcal{A}$, write $J^\perp$ for the orthogonal complement $\{a \in A : aJ = Ja =\}$.

Definition 4.2.7 *Let X be a C*-correspondence from $\mathcal{A}$ to $\mathcal{B}$ with structure map $\phi : \mathcal{A} \to \mathcal{L}(X)$. We define the ideal*

$$J = \phi^{-1}(K(X)) \cap (Ker\phi)^\perp$$

and call it Katsura's ideal.

Definition 4.2.8 *Let $\mathcal{A}$ be a C*-algebra. Let X be a right Hilbert $\mathcal{A}$-module with inner product $\langle , \rangle_X$. If X is also a left Hilbert $\mathcal{A}$-module with inner product ${}_X\langle , \rangle$ satisfying*

$$a(xb) = (ax)b \text{ and } {}_X\langle x, y\rangle z = x\langle y, z\rangle_X$$

for all $x, y, z \in X$ and $a, b \in \mathcal{A}$ then we call X a Hilbert $\mathcal{A}$-bimodule.

A Hilbert bimodule X is left-full if the closed span of ${}_\mathcal{A}\langle X, X\rangle$ is all of $\mathcal{A}$.

An imprimitivity bimodule X is a Hilbert bimodule that is full on both the left and the right.

For more details, see [27]. Every Hilbert $\mathcal{A}$-bimodule is a C^*-correspondence over $\mathcal{A}$. Indeed, it suffice to define the structure map $\phi : \mathcal{A} \to \mathcal{L}(X)$ to act by left multiplication, $\phi(a)x = ax$. It is shown in [lawrance] that ϕ indeed maps into the adjointable operators on X. If we restrict ϕ to KatsuraâĂŹs ideal J then it becomes a *-isomorphism. We can also go the opposite direction: Any C*-correspondence such that the structure map restricts to an isomorphism from KatsuraâĂŹs ideal to the compacts can be turned into a Hilbert $\mathcal{A}$-bimodule. Left multiplication is given by $ax = \phi(a)x$, and the inner product by ${}_X\langle x, y\rangle = \phi^{-1}(\theta_{x,y})$; where $\theta_{x,y}(z) = x\langle y, z\rangle_X$.

Example 4.2.9 *Let Ω be a compact Hausdorff space, V a vector bundle over Ω, and $\eta : \Omega \to \Omega$ a homeomorphism. We can turn the right Hilbert $C(\Omega)$-module $\Gamma(V)$ into a C*-correspondence by defining the left action*

$$\phi(f)x = x(fo\eta), \text{ for all } x \in \Gamma(V), f \in C(\Omega).$$

This C-correspondence $\Gamma(V, \eta)$ is studied in [1]. It is shown there that $\Gamma(V, \eta)$ is a Hilbert bimodule if and only if V is a line bundle*

Let X, Y be C^*-correspondence from $\mathcal{A}$ to $\mathcal{B}$ and from $\mathcal{C}$ to $\mathcal{D}$ respectively.

Definition 4.2.10 *A C^*-correspondence homomorphism is a triple $(\Phi, \phi_l, phi_r) : X \to Y$ consisting of a linear map $\Phi : X \to Y$ and *-homomorphisms $\phi_l : \mathcal{A} \to \mathcal{B}$ and $\phi_r : \mathcal{C} \to \mathcal{AD}$ satisfying:*

1. $\Phi(ax) = \phi_l(a)\Phi(x)$,
2. $\phi_r(\langle x, y\rangle)_{\mathcal{C}} = \langle \Phi(x), \Phi(y)\rangle_{\mathcal{D}}$

for all $x, y \in X, a \in \mathcal{A}$.

The triple (Φ, ϕ_l, ϕ_r) is called a C*-correspondence isomorphism if, in addition, Φ is bijective and ϕ_l, ϕ_r are isomorphisms. In the sequel, we primarily deal with the situation where $\mathcal{A} = \mathcal{B}$ and $\mathcal{C} = \mathcal{D}$. In that situation, we take ϕ_l and ϕ_r to be the identity maps on $\mathcal{A}$ and $\mathcal{C}$, respectively, and we simply denote the isomorphism $X \to Y$ by Φ instead of the triple (Φ, ϕ_l, ϕ_r).

Definition 4.2.11 *A morphism (π, Φ) of C*-correspondences from X to Y is called covariant if*

$$\pi(a) \in J_Y \text{ and } \phi_Y(\pi(a)) = \psi_\Phi(\phi_X(a)), \text{ for all } a \in J_X$$

*where X, Y are C^*-correspondence for A and B, respectively, and ψ_Φ is a *-homomorphism from $K(X)$ to $K(Y)$ such that*

$$\psi_\Phi(\theta_{x,y}) = \theta_{\Phi(x),\Phi(y)}, \text{ for all } x, y \in X.$$

Definition 4.2.12 *A covariant representation of a C^*-correspondence X on a C^*-algebra $\mathcal{B}$ is a morphism of C*-correspondences from X to $\mathcal{B}$.*

Explicitly, a representation of a C^*-correspondence X on a C^*-algebra $\mathcal{B}$ is given by a pair (π, T). The map π is a *-homomorphism from $\mathcal{A}$ to $\mathcal{B}$, and T is a linear map from X to $\mathcal{B}$ such that

$$\pi(\langle x, y\rangle_X) = T(x)^* T(y) \text{ and } \pi(a)T(x) = T(\phi(a)x), \text{ for all } x, y \in X, a \in A.$$

Fore more details, see [27, 62].

Definition 4.2.13 *(Morita equivalence) [65]: an $(\mathcal{A}, \mathcal{B})$-equivalence bimodule is a full $(\mathcal{A}, \mathcal{B})$-correspondence E where the left action $\phi : \mathcal{A} \to \mathcal{L}(E)$ is an isomorphism onto $K(E)$.*
One says that two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent if such an $(\mathcal{A}, \mathcal{B})$-equivalence bimodule exists.

In [51], Muhly and Solel introduced the notion of Morita equivalence for C^*-correspondences as follows: if X, Y are C^*-correspondences for $\mathcal{A}$ and $\mathcal{B}$, respectively, then X, Y are called Morita equivalent if there exists an imprimitivity $(\mathcal{A}, \mathcal{B})$-bimodule M such that $X \otimes_{\mathcal{A}} M \cong M \otimes_{\mathcal{B}} Y$.

Every full Hilbert $\mathcal{A}$-module E is a $(K(E); \mathcal{A})$-equivalence bimodule. Morita equivalence is a weaker equivalence relation than isomorphism. Indeed, given an isomorphism: $\mathcal{A} \to \mathcal{B}$, the C^*-correspondence is an $(\mathcal{A}, \mathcal{B})$-equivalence bimodule.

We note here that the Morita equivalence is an equivalence relation. Morita equivalence is a purely noncommutative notion. Indeed, Morita equivalent algebras have isomorphic centers [65], and therefore two commutative C^*-algebras are Morita equivalent if and only if they are isomorphic.

In noncommutative topology Morita equivalence is the most natural equivalence relation to consider: Morita equivalent C^*-algebras have, among other things, the same representation theory and the same K-theory and K-homology groups.

4.2.2 Pimsner Algebras

Pimsner associates in [62] a very natural and universal C^*-algebra. This important work has attracted a lot of attention and has been meanwhile generalized in several directions.

We will now define the Cuntz-Pimsner algebra $O(X)$ associated to a C^*-correspondence X.

Let X be a C^*-correspondence over a C^*-algebra $\mathcal{A}$. If (π, U) is a covariant representation of X on a C^*-algebra $\mathcal{B}$, then $C^*(\pi, U)$ denotes the C^*-subalgebra of $\mathcal{B}$ generated by the images of π and U. The covariant representation (π, U) is called universal if we have another covariant representation (ρ, L) there exists a surjective *-homomorphism Φ from $C^*(\pi, U)$ onto $C^*(\rho, L)$ such that $\rho = \Phi o \pi$ and $L = \Phi o U$. We denote the universal covariant representation of X by (π_X, U_X).

Definition 4.2.14 (62) *The C^*-algebra (π_X, U_X) is called the Cuntz-Pimsner algebra and is denoted by $O(X)$.*

By definition the universal covariant representation is unique if it exists. To show that it does exist one has to construct it explicitly, which also shows that the Cuntz-Pimsner algebra of X exists and is unique.

In his breakthrough paper [62], starting from a full C*-correspondence X such that the left action: $A \to \mathcal{L}(X)$ is an isometric -homomorphism, Pimsner constructed two C*-algebras: these are now referred to as the Toeplitz algebra T_X and the Cuntz-Pimsner algebra O_X of the C*-correspondence X. The former is actually an extension of the second, and can be thought of as a generalization of the Toeplitz algebra, while the latter encompasses a large class of examples, like Cuntz-Krieger algebras, graph C*-algebras and crossed products by the integers. Both algebras are characterized by universal properties and depend only on the isomorphism class of the C*-correspondence.

Let X be a C^*-correspondence over a unital C^*-algebra $\mathcal{A}$ with left action $\phi : \mathcal{A} \to \mathcal{L}(X)$. We let $X^{\otimes n} = X \otimes_{\mathcal{A}} X \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} X$ be the n-fold internal tensor product of X. The left A-action is given by the $*$-homomorphism $\phi_n : A \to \mathcal{L}(X^{\otimes n})$ satisfying

$$\phi_n(a)(x_1 \otimes x_2 \otimes \cdots x_n) = (\phi(a)x_1) \otimes x_2 \otimes \cdots \otimes x_n.$$

By convention we declare $X^{\otimes 0} = \mathcal{A}$ as a C^*-correspondence over itself, with the automorphism being the identity map $A \to A$.

Definition 4.2.15 *The full Fock space $\mathcal{F}(X)$ over X is the C^*-correspondence over $\mathcal{A}$*

$\oplus^{\infty_{n=0}} X^{\otimes n} = A \oplus X \oplus (X \otimes_{\mathcal{A}} X) \oplus \cdots$. *The left $\mathcal{A}$-module structure is $\oplus_n \phi_n$ which we denote by ϕ_∞. It can be represented by the diagonal matrix*

$$\phi_\infty(a) = \begin{bmatrix} a & & & & & \\ & \phi(a) & & & & \\ & & \phi_2(a) & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \cdot \end{bmatrix}, a \in \mathcal{A}$$

where $\phi_n(a)(\xi_1 \otimes \cdots \otimes \xi_n = (\phi(a)\xi_1) \otimes \cdots \xi_n)$. Looking at the $(1,1)$-entry, it is clear that ϕ_∞ is injective. Thus, we will often identity $\mathcal{A}$ with its image $\phi_\infty(a)$.

For each $x \in X$, we define the creation operator $T_x \in \mathcal{L}(\mathcal{F}(X))$ by

$$T_x = \begin{bmatrix} 0 & & & & & \\ T_x^{(1)} & 0 & & & & \\ & T_x^{(2)} & 0 & & & \\ & & T_x^{(3)} & 0 & & \\ & & & \ddots & \ddots & \\ & & & & \ddots & \ddots \end{bmatrix}$$

where $T_x^{(k)} : X^{\otimes k} \to X^{\otimes (k+1)}$ is given by the formula

$$T_x^{(k)}(x_1 \otimes \cdots \otimes x_k) = x \otimes x_1 \otimes \cdots \otimes x_k$$

Definition 4.2.16 *Let X be a C^*-correspondence over $\mathcal{A}$.*

1. The tensor algebra of (X, ϕ), denoted $\mathcal{J}_+(X)$ is the norm closed subalgebra of $\mathcal{L}(\mathcal{F}(X))$ generated by $\phi_\infty(a)$ and $T_x : x \in X$.

2. The Toeplitz algebra is the C^*-algebra generated by $\mathcal{J}_+(X)$ in $\mathcal{L}(\mathcal{F}(X))$.

If the image of ϕ is contained in $K(X)$ then the Cuntz-Pismner algebra O_X of a full C*-correspondence (X, ϕ) is a quotient of the Toeplitz algebra T_X appearing in the exact sequence

$$0 \to K(X) \to T_X \to O_X \to 0.$$

It is easy to check that since X is full, then $\mathcal{F}(X)$ is a full Hilbert module as well; hence $K(X)$ is by definition Morita equivalent to the algebra A.

Example 4.2.17 *Let $\mathcal{A} = \mathbb{C}$ and $X = \mathbb{C}^n$ and ϕ the left action by multiplication. If one chooses a basis for $\mathbb{C}$, then the tensor algebra $\mathcal{J}_+(X) = \mathcal{A}_n$ is Popescu's noncommutative disc algebra and the Toeplitz algebra of (X, ϕ) is generated by n isometries $V_1, ..., V_n$ satisfying $\Sigma_i V_i V_i^* \leq 1$. This is the Toeplitz extension for the Cuntz algebras O_n: $0 \to K(H) \to C^*(V_1, .., V_n) \to O_n \to 0$. In particular, for n = 1 one gets $\mathcal{F}(\mathbb{C}) \cong H^2$, the classical Hardy space, $\mathcal{F}_+(X) = A(\mathbb{D})$ is the classical disc algebra and the classical Toeplitz extension $0 \to K(H) \to T_{\mathbb{C}} \to C(\mathbb{S}^1) \to 0$.*

4.3 Main result

The subject of Cuntz-Pimsner algebras has grown into a rapidly expanding area of research, and determining how the structure of a Cuntz-Pimsner algebra is determined by its C*-correspondence has become a popular line of investigation. There is a naturel question which is: If two C*-correspondences X, Y over $\mathcal{A}$ and B, respectively, are related in a particular way, what can be said about their Cuntz-Pimsner algebras O_X, O_Y?. In [67], the author sparked the idea of investigating this problem in a categorical framework. Motivated by this idea, in [26], the authors construct a categorical framework that provides direct results, allowing one to immediately deduce relationships between Cuntz-Pimsner algebras from relationships between the defining C*-correspondences and they constructed a functor form suitable subcategory of Hilbert C*-module to their category C*-algebras. As a consequence, it was proved in [27] the following result.

Theorem 4.3.1 *If two C*-correspondences X and Y are Morita equivalent, then their Cuntz-Pimsner algebras O_X and O_Y are Morita equivalent (in the sense of Rieffel).*

On the other hand, it was shown in [25,40] that Hilbert module over a C*-algebra is equivalent to the notion of Hilbert bundle over the space of pure states of the C*-algebra provided that the purely topological structure of the Hilbert bundle is augmented with suitable holomorphic and uniform structures. (In the commutative case the additional structure is redundant.) They established the following result which is the noncommutative version of Serre-Swan theorem.

Theorem 4.3.2 (25,40) *The category of right Hilbert $\mathcal{A}$-modules is equivalent to the category of uniform holomorphic Hilbert bundles over $P_0(\mathcal{A})$ (the set of pure states of $\mathcal{A}$ together with the functional 0) of dual Hopf type.*

Since the Cuntz-Pimsner algebras represent a model for a noncommutative spaces and noncommutative varieties then, with the same manner in algebraic topology, the problem of their classifications is not so easy to do, it is a dificult task, it is still open and it is an active reserche area. Motivated by the results; theorems 4.2.15 and 4.2.16., we show that the Cuntz-Pimsner algebras can be represented as a bundles of C*-algebras having Cuntz's algebras fibers (stalks). Our results generalize the one obtained in [74, 13], where the C*-algebra $\mathcal{A}$ considered is commutative or a $C(\Omega)$-algebras. We expect that our results can be applied to the genral model theory developped by Gelu Popescu [64], In other words, we expect that we can get a fields of Popescu'models varing continuously with respect to the space of pure states. Also, we can used our result for studing and classifing the groups of K-homology of Cuntz-Pimsner algebras by using the topology of fiber bundles and characteristic classes developped in algebraic topology [].

Proposition 4.3.3 *Let X be a C^*-correspondence for a C^*-algebra $\mathcal{A}$ with the structure $\phi_X : A \to \mathcal{L}(X)$. Then, X is isomorphic to the C^*-correspondence $\Gamma(X)$ for a C^*-algebra $\mathcal{A}_\mu(P_0)$.*

Proof. By theorem 4.3.2, the Hilbert C*-module X is isomorphic to the Hilbert C*-modules of sections $\Gamma(X)$ of the Bundle $\Pi : E = (X_f)_{f\in P_0} \to P_0$ and from the proof of [40], we have $\mathcal{A}$ is equivalent to $\mathcal{A}_\mu(_{P_0})$. Then we can identify $\mathcal{A}$ with $\mathcal{A}_\mu(_{P_0})$.

For $x \in X$, we define the section s_x of $\Gamma(X)$ by $s_x(p) = [x]_p$ for $p \in P_0$. Then $\|s_x\| = \|x\|$ for every $x \in X$. Then we can define the linear map as follows

$$V(x) = s_x \text{ for all } x \in X.$$

We can deduce from the proof of [40], that V is an isometric isomorphism from X to $\Gamma(X)$. Hence, by the definition of C*-correspondence homomorphism 4.2.10,

$$\phi_{\Gamma(X)}(\xi_a) * V(x) = V(\phi_X(a)x), \text{ for all } a \in \mathcal{A}, x \in X, \xi_a \in \mathcal{A}_\mu(P_0).$$

Witch means that the pair (ξ, V) is an isomorphism of C*-correspondences from X to $\Gamma(X)$. □

Let X be a C*-correspondence for a C*-algebra $\mathcal{A}$ with the structure $\phi_X : \mathcal{A} \to \mathcal{L}(X)$.

Theorem 4.3.4 *The Toepliz algebra T_X and the Cuntz-Pimsner algebras O_X are represented by bundle of C*-algebras as a fields of bounded holomorphic operators.*

Proof. For each $f \in P_0(\mathcal{A})$ denote by X_f the Hilbert space arising from the positive sesquilinear form on X obtained by composing the $\mathcal{A}$-valued inner product with f. The sections of the bundle $(X_f)_{f\in P_0(\mathcal{A})}$ corresponding to the elements of X constitute the bounded uniformly continuous holomorphic sections of a unique uniform holomorphic Hilbert bundle, $\Gamma(X)$, over $P_0(\mathcal{A})$, necessarily of dual Hopftype. This space admits a unique structure of right Hilbert $\mathcal{A}_\mu(P_0)$-module: $s * \xi = s.l + i\triangle_{y_\xi} s$, $i = \sqrt{-1}$, $\triangle$ is a flat connection on $(X_f)_{f\in P_0(\mathcal{A})}$ and y_ξ is a vector field in $\mathcal{X}(P(H))$, $P(H)$ is the projective space for Hilbert space H. and then the space of adjointable operators on $\Gamma(X)$ is represented as a fields of bounded operators on the Hilbert spaces X_f, $f \in P_0(\mathcal{A})$. That is, for every adjointable operator $T \in B(\Gamma(X))$ has the following representations

$$T = (T_f)_{f\in P_0} \text{ and } (Ts)_f = T_f s_f, \text{ for all } f \in P_0.$$

We suppose that the sections of $\Gamma(X)$ satisfying the following condition

$$D_f = \{s(f) : s \in Gamma(X)\} \text{ is dense in } X_f, \text{ for every } f \in P_0 \ (**)$$

Since the fibre bundles $(X_f)_{f\in P_0(\mathcal{A})}$ has a uniform and holomorphic structure we deduce from $(**)$ that every $T \in B(\Gamma X)$ is represented by a uniform holomorphic fields of bounded linear operators on Hilbert spaces.

If we take the fock module $F(X)$ defined previously over a C*-correspondence X the, by the previous propostion, $F(X)$ is a C*-correspondence over $\mathcal{A}$ and it is isomorphic to the Fock modules of sections $F(\Gamma(X))$ (which is also isomorphic to the modules of sections of the fock module $\gamma(F(X))$), we can check easly that the representation of the left and right creation operators on $B(F(\Gamma(X)))$ has the form $L_s = (L_{s_f})_{f\in P_0}$, for all $s \in \Gamma(X)$, where L_f is a left creation operator on the Fock space $F(X_f)$ which, by Proposition 4.2.3, has nice properties.

Therefore, by the previous discution and the construction of Toeplitz and Cuntz-Pimsner algebras T_X and O_X from any C*-correspondence X, we can check easly that T_X and O_X have representation as algebras defined by uniform holomorphic fields of bounded (isometric)operators. This in fact are fields of C*-algebras. □ It follows from our results the following corollary, which generalizes the results that given in [74, 13].

Corollary 4.3.5 *If $\mathcal{A} = C(\Omega)$, where Ω is a locally compact space, or A is a $C(\Omega)$-algebra, then the Cuntz-Pimsner algebra of every C^*-correspondence over $\mathcal{A}$ is represented by a field of C^*-algebras having Cuntz algebras as fibers.*

Definition 4.3.6 *In the category C^*-alg$_{cor}$, the objects are C^*-algebras, and the morphisms from $\mathcal{A}$ to $\mathcal{B}$ are the isomorphism classes of $\mathcal{A} - \mathcal{B}$ correspondences. The composition of $[X] : \mathcal{A} \to \mathcal{B}$ with $[Y] : \mathcal{B} \to \mathcal{C}$ is the isomorphism class of the internal tensor product $X \otimes_{\mathcal{B}} Y$; the identity morphism on $\mathcal{A}$ is the isomorphism class of the identity correspondence $\mathcal{A}$, and the zero morphism $\mathcal{A} \to \mathcal{B}$ is $[O]$.*

We note here that a morphism $[X]$ is an isomorphism in C^*-alg$_{cor}$ if and only if X is an imprimitivity bimodule, for more details, see [].

Our next aim is to use the previous result in order to study the classifications of the Cuntz-Pimsner algebras in previous category by means of their C*-correspondences. This task is not yet finished.

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