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Presented by: ALEM Amina
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DEDICATION

To

My parents,Fatima and Ahmed.

My husband Hakim.

My son Ali Mohamed.

My brothers.

My sisters .

All my family.

My close friends .

For their love, endless supports and encouragements.

AMINA

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ABSTRACT

Polyharmonic maps of order k are a natural generalization of harmonic maps, for $k = 2$, this maps are called biharmonic maps. In this thesis we will study the biharmonicity of a vector field X on a pseudo-Riemannian manifold (M, g) viewed as a map $X : (M, g) \to (TM, g_S)$ where g_S is the Sasaki metric. More precisely, we establish the formula of the bitension field of X and we show characterization theorem for X to be biharmonic map, and we describe the relationship between vector fields X that are critical points of the bienergy functional E_2 restricted to variations through vector fields, equivalently X are biharmonic vector fields, and vector fields which are biharmonic maps. Moreover, several applications are included.
Key words: Tangent bundle, Sasaki metric, biharmonic map, vector fields.

RÉSUMÉ

Les applications polyharmoniques d'ordre k sont une généralisation naturelle des applications harmoniques, pour $k = 2$, ces applications sont appelées applications biharmoniques. Dans cette thèse nous étudierons la biharmonicité d'un champ de vecteurs X sur une variété pseudo-riemannienne (M, g) vue comme une application $X : (M, g) \to (TM, g_S)$ où g_S est la métrique de Sasaki. Plus précisément, nous établissons la formule du champ de bitension de X et nous montrons un théorème de caractérisation de X pour qu'il soit une application biharmonique, et nous décrivons la relation entre les champs de vecteurs X qui sont des points critiques de la fonctionnelle biénergie E_2 limité aux variations sur les champs de vecteurs, de manière équivalente X sont des champs de vecteurs biharmoniques, et les champs de vecteurs qui sont des aplications biharmoniques. De plus, plusieurs applications sont incluses.
Mots clés: Fibré tangent, metrique de Sasaki, application biharmonique, champ de vecteurs.

CONTENTS

INTRODUCTION

Let (M, g) and (N, h) be smooth pseudo-Riemannian manifolds of dimensions m and n respectively, and let $\varphi : (M, g) \to (N, h)$ be a smooth map between them. The energy functional or the Dirichlet energy of φ over a compact domain D of M is defined by

$$E(\varphi, D) = \frac{1}{2}\int_D \sum_{i=1}^{m} \varepsilon_i h(d\varphi(e_i), d\varphi(e_i))v_g, \tag{1}$$

where $\{e_i\}_{i=1}^m$ a local pseudo-orthonormal frame field of (M, g) with $\varepsilon_i = g(e_i, e_i) = \pm 1$ for all indices $i = 1, 2, \cdots, m$. If M is compact, we write $E(\varphi) = E(\varphi, M)$. The map φ is called harmonic if it is a critical point of the energy functional (1). The Euler-Lagrange equation of (1) is [3, 11]

$$\tau(\varphi) = \mathrm{Tr}_g(\nabla d\varphi) = \sum_{i=1}^{m} \varepsilon_i\{\nabla^{\varphi}_{e_i} d\varphi(e_i) - d\varphi(\nabla_{e_i} e_i)\} = 0.$$

Here $\tau(\varphi)$ is the tension field of φ and ∇^{φ} denotes the connection on the vector bundle $\varphi^{-1}TN \to M$ induced from the Levi-Civita connection ∇^N of (N, h) and ∇ the Levi-Civita connection of (M, g).

Now, denote by $\mathfrak{X}(M)$ the set of all smooth vector fields on M and by g_S the Sasaki metric on the tangent bundle TM. Any $X \in \mathfrak{X}(M)$ determines a smooth map from (M, g) to (TM, g_S). The energy of X is, by definition, the energy of the corresponding map. When M is compact and g is positive definite, it was proved in [17, 27] that $X : (M, g) \to (TM, g_S)$ is an harmonic map if and only if X is parallel, moreover this results remain true if X is a harmonic vector field i.e. X is a critical point of the energy functional E restricted to the set $\mathfrak{X}(M)$ see [13]. In contrast to the Riemannian case, it was shown in [5] the existence of non-parallel left-invariant vector fields which define harmonic maps on three dimensional unimodular and non-unimodular Lorentzian Lie groups.

One of the first generalizations of harmonic maps is the notion of polyharmonic maps of order $k(k \geq 2)$ between Riemannian manifolds introduced by Eells and Lemaire in [10]. Precisely, polyharmonic maps of order k are critical points of:

$$E_k(\varphi) = \frac{1}{2}\int_M |(d+\delta)^k \varphi|^2 v^M,$$

$$E_k(\varphi) = \begin{cases} \frac{1}{2}\int_M |W_\varphi^l|^2 v^M, \text{ if } k = 2l \\ \frac{1}{2}\int_M |\nabla^\varphi W_\varphi^l|^2 v^M, \text{if } k = 2l+1. \end{cases}$$

Here

$$W_\varphi^l = \underbrace{\Delta^\varphi \cdots \Delta^\varphi}_{\text{l-1}} \tau(\varphi) \in \varphi^{-1}TN,$$

where

$$\Delta^\varphi \tau(\varphi) = -\sum_{i=1}^m (\nabla^\varphi_{e_i}\nabla^\varphi_{e_i}\tau(\varphi) - \nabla^\varphi_{\nabla^M_{e_i}e_i}\tau(\varphi))$$

is the rough Laplacian on $\varphi^{-1}TN$. For $k = 2$, we obtain the bienergy of φ as the functional

$$E_2(\varphi) = \frac{1}{2}\int_D |\tau(\varphi)|^2 v_g,$$

and a smooth map φ is biharmonic if and only if it is a critical point of E_2. The associated Euler-Lagrange equation is established in [18]. By definition, it can be seen that every harmonic map is biharmonic. However, a biharmonic map can be non-harmonic in which case it is called proper biharmonic. We refer to [28, 31] for more information on results concerning the theory of biharmonic maps. The notion of biharmonic map between Riemannian manifolds has been extended to the case of pseudo-Riemannian manifolds. The corresponding critical point condition has been derived in [8] as follows

$$\tau_2(\varphi) = \sum_{i=1}^m \varepsilon_i \Big(\big(\nabla^\varphi_{e_i}\nabla^\varphi_{e_i} - \nabla^\varphi_{\nabla^M_{e_i}e_i}\big)\tau(\varphi) - R^N(d\varphi(e_i), \tau(\varphi))d\varphi(e_i) \Big) = 0,$$

where $\tau_2(\varphi)$ is the bitension field of φ and R^N is the curvature tensor of N.

On the other hand, when (M, g) is the pseudo-Riemannian manifold, Markellos and Urakawa [25] defined the bienergy of $X \in \mathfrak{X}(M)$ as the bienergy of the corresponding map (see [23] for the Riemannian case) and obtained the critical point of the bienergy functional E_2 restricted to the set $\mathfrak{X}(M)$ (equivalently, X is a biharmonic vector field, see [23] for the Riemannian case), further in [23] they proved that if g is positive definite and M is compact then X is biharmonic vector field (resp. biharmonic map) if and only if X is parallel. In this work, we will study the biharmonicity (polyharmonicity of order 2) of $X \in \mathfrak{X}(M)$ viewed as a map $X : (M, g) \to (TM, g_S)$ in both Riemannian case and pseudo-Riemannian case. More precisely, we address the problem of characterizing those vector fields which are biharmonic maps, and examine the relationship between vector fields X that are critical points of the functional E_2 restricted to variations through vector fields (equivalently, X are biharmonic vector fields) and vector fields which are biharmonic maps.

Let us now briefly describe the contents of the present work, organized into four chapters.

In the first chapter, after an introduction of basic material and definitions we discuss known facts about horizontal and vertical lifts of vector fields on a a differentiable manifold M. We calculate the Lie bracket, define a class of natural metrics $\bar{g}$ on TM and obtain formulae for its Levi-Civita connection $\bar{\nabla}$. Afterward we define the Sasaki metric as an example of a natural metric and calculate its Levi-Civita connection and its Riemann curvature tensor.

With the second chapter, we recall briefly the notions of harmonic and biharmonic mappings between pseudo-Riemannian manifolds, integrating them with some more details.

In the third chapter, we present our work on the biharmonicity of vector fields on Riemannian manifolds. We compute the expression of the bitension field of a vector field considered as a map from a Riemannian manifold (M, g) to its tangent bundle TM equipped with the Sasaki metric g_S. As a consequence, we show characterization theorem for a vector field to be biharmonic map. Moreover, we prove non-existence results for left-invariant vector fields which are biharmonic without being harmonic maps and non-harmonic biharmonic maps respectively on unimodular Lie groups of dimension three.

In the last chapter, we deal with the biharmonicity of a vector field X viewed as a map from a pseudo-Riemannian manifold (M, g) into its tangent bundle TM endowed with the Sasaki metric g_S. Precisely, we characterize those vector fields which are biharmonic maps, and find the relationship between them and biharmonic vector fields. Afterwards, we study the biharmonicity of left-invariant vector fields on the three dimensional Heisenberg group endowed with a left-invariant Lorentzian metric. Finally, we give examples of vector fields which are proper biharmonic maps on the Gödel universe.

CHAPTER 1

PRELIMINARIES

In this chapter, we give basic material and definitions needed later. The references used are: [3], [7],[21], [20], [22], [28], [33].

1.1 Differentiable manifold

1.1.1 Differentiable manifold

Definition 1.1.1. *Let M be a topological Hausdorff space with a countable basis. M is called a topological manifold, if there esists an $m \in \mathbb{N}$ and for every point $p \in M$ an open neighborhood U_p of p such that U_p is homeomorphic to some open subset $V_p \subset \mathbb{R}^m$. The integer m is called the dimension of M .*

Definition 1.1.2. *Let M^m be a topological manifold, U an open and connected subset of M and $\varphi : U \to \mathbb{R}^m$ a continuous map homeomorphic onto its image $\varphi(U)$. Then (U, φ) is called a local coordinate on M . A collection $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) \mid \alpha \in I\}$ of local coordinates on M is called a C^r-atlas if*

- *$M = \bigcup_\alpha U_\alpha$, and*
- *the corresponding transition maps*

$$\varphi_\beta \circ \varphi_\alpha^{-1} \mid_{\varphi_\alpha(U_\alpha \cap U_\beta)} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \mathbb{R}^m$$

are C^r for all $\alpha, \beta \in \mathbb{I}$.

If $\mathcal{A}$ is a C^r-atlas on M then a local coordinate (U, φ) on M is said to be compatible with $\mathcal{A}$ if $\mathcal{A} \cup (U, \varphi)$ is a C^r-atlas. A C^r-atlas $\hat{\mathcal{A}}$ is maximal if it contains all local coordinates compatible with it. It is also called a C^r-structure on M and the pair $(M, \hat{\mathcal{A}})$ is called a differentiable C^r-manifold. By smooth we mean C^∞ defined by $C^\infty = \cap_{k=1}^\infty C^k$. We write M^m to denote that M has dimension m.

1.1.2 Orientable manifold

Definition 1.1.3. *Let M^m be a smooth manifold. Two charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are orientation compatible if the transition map $\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}$ satisfies*

$$\det(d\varphi_{\alpha\beta})_x > 0,$$

for all $x \in \varphi_\alpha(U_\alpha \cap U_\beta)$. An orientation of M^m is an atlas $\mathcal{A} = \{(U_i, \varphi_i)\}_{i \in I}$ whose charts are pairwise orientation compatible. We say M is orientable if it has an orientation.

Theorem 1.1.1. *An n-dimensional manifold M is orientable if and only if M admits a nowhere vanishing n-form.*

1.1.3 Manifolds with boundary

Definition 1.1.4. *A smooth manifold with boundary is a Hausdorff space M with a countable basis of open sets and a differentiable structure $\mathcal{A} = \{U_\alpha, \varphi_\alpha\}$ where $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha) \subset \mathbb{H}^n$ is homeomorphism, such that :*

- $\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \ / \ x_1 \geq 0\}$. (*half-space*)
- *the union of U_α cover M*
- *If $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are two elements of $\mathcal{A}$ the $\varphi_\beta \circ \varphi_\alpha^{-1}$ and $\varphi_\alpha \circ \varphi_\beta^{-1}$ are diffeomorphisms of $\varphi_\alpha(U_\alpha \cap U_\beta)$ and $\varphi_\beta(U_\alpha \cap U_\beta)$, open subsets of $\mathbb{H}^n$*
- *$\mathcal{A}$ is maximal with respect first and third properties.*

1.2 Tangent Bundle

1.2.1 Tangent Space

Let M^m denote a C^∞ . Just as for $\mathbb{R}^n$, we define a germ of a C^∞ function at p in M to be an equivalence class of C^∞ functions defined in a neighborhood of p in M, two such functions being equivalent if they agree on some, possibly smaller, neighborhood of p. The set of germs of C^∞ real-valued functions at p in M is denoted by $C_p^\infty(M)$. The addition and multiplication of functions make $C_p^\infty(M)$ into a ring; with scalar multiplication by real numbers, $C_p^\infty(M)$ becomes an algebra over $\mathbb{R}$. choosing an arbitrary (U, φ) around p it is easily verified that $\varphi^* : C_{\varphi(p)}^\infty(M) \to C_p^\infty(M)$ given by $\varphi^*(f) = f \circ \varphi$ is an isomorphism of the algebra of germs of C^∞ function at $\varphi(p) \in \mathbb{R}^n$ onto the algebra $C_p^\infty(M)$.

Definition 1.2.1. *A tangent vector X_p at $p \in M$ is a map $X_p : C_p^\infty(M) \to \mathbb{R}$ such that*

(i) $X_p(\alpha \cdot f + \beta \cdot g) = \alpha \cdot X_p(f) + \beta \cdot X_p(g)$ *(linearity)*

(ii) $X_p(f \cdot g) = g(p) \cdot X_p(f) + f(p) \cdot X_p(f)$ *(Leibnitz rule)*

for all α, $\beta \in \mathbb{R}$ *and* $f, g \in C_p^\infty(M)$. *The set of all tangent vectors* X_p *and* $p \in M$ *is denoted by* T_pM *and is called the tangent space of* M *at* p.

The tangent space T_pM *is turned into a real vector space by defining the operations* $+$ *and* $\cdot$ *by*

(i) $(X_p + Y_p)(f) = X_p(f) + Y_p(f)$

(ii) $(\alpha \cdot X_p)(f) = \alpha \cdot X_p(f)$

for all $\alpha \in \mathbb{R}$ *and* $X_p, Y_p \in T_pM$.

Definition 1.2.2. *Let* $\varphi : M \to N$ *be a map between two manifolds. For a point* $p \in M$ *we define the map* $d\varphi_p : T_pM \to T_{\varphi(p)}N$ *by*

$$(d\varphi_p)(X_p)(f) = X_p(f \circ \varphi)$$

For all $X_p \in T_pM$ *and* $f \in C^\infty$ *The map* $d\varphi_p$ *is called the differential of* φ *at* $p \in M$.

Proposition 1.2.1. *Let* $\varphi : M \to \widetilde{M}$ *and* $\psi : M \to N$ *be two maps between smooth manifolds, then*

(i) *the map* $d\varphi_p : T_pM \to T_{\varphi(p)}\widetilde{M}$ *is linear,*

(ii) *if* id_M *is the identity map, then* $d(id_M)_p = id_{T_pM}$,

(iii) $d(\psi \circ \varphi)_p = d\psi_{\varphi(p)} \circ d\varphi_p$,

for all $p \in M$.

Proof. The first two points follow directly from the definition, so we only have to prove the **(iii)** . If $X_p \in T_pM$ and $f \in C^\infty$, then

$$\begin{aligned} (d\psi_\varphi(p) \circ d\varphi_p(X_p))(f) &= (d\varphi_p(X_p))(f \circ \psi) \\ &= X_p(f \circ \psi \circ \varphi) \\ &= d(\psi \circ \varphi)_p(X_p)(f) \end{aligned}$$

□

Corollary 1.2.1. *Let* $\varphi : M \to N$ *be a diffeomorphism with inverse* $\psi = \varphi^{-1} : N \to M$. *Then the differential* $d\varphi_p : T_pM \to T_\varphi(p)M$ *at* p *is bijective and* $(d\varphi_p)^{-1} = d\psi_{\varphi(p)}$.

Definition 1.2.3. *Let M^m be a manifold, (U, x) be a local coordinate on M and $\{e_k | k = 1, ..., m\}$ be the standard basis for $\mathbb{R}^m$. For $p \in M$, we define $(\frac{\partial}{\partial x_k})_p \in T_pM$ by*

$$(\frac{\partial}{\partial x_k})_p : f \mapsto (\frac{\partial f}{\partial x_k})(p) = \partial_{e_k}(f \circ x^{-1})(x(p))$$

Proposition 1.2.2. *The set $\{(\frac{\partial}{\partial x^k})_p | k = 1, ..., m\}$ is a basis for T_pM for all $p \in U$.*

Proof. Because M is smooth, it follows that the inverse of x is smooth and therefore the differential of the inverse satisfies

$$dx^{-1}_{x(p)}(\partial_{e_k})(f) = \partial_{e_k}(f \circ x^{-1})(x(p)) = ((\frac{\partial}{\partial x^k})_p)(f)$$

for all $f \in C^\infty$. □

The tangent space T_pM may be viewed in an alternative way. For this we use the set $C(p)$ of all equivalence classes of locally defined C^1-curves passing through the point $p \in M$. It is possible to identify T_pM with $C(p)$ being the set of all tangents to curves going through the point p. Then a vector $v \in T_pM$ can be described by

$$v(f) = \frac{d}{dt}(f \circ \gamma(t)) \mid_{t=0},$$

with $f : U \in M \to \mathbb{R}$ a function defined on U containing p and $\gamma : I \to U$ an arbitrary curve with $\gamma(0) = p$ and $\gamma(0) = v$

1.2.2 Tangent Bundle

Definition 1.2.4. *Let E and M be smooth manifolds and $\pi : E \to M$ be a continuous surjective map. If*

(i) *for each $p \in M$ the fiber $E_p = \pi^{-1}(p)$ is an n-dimensional vector space and,*

(ii) *for each $p \in M$ there exists a bundle chart $(\pi^{-1}(U), \psi)$ consisting of the pre image of π of an open neithborhood U of p and a homeomorphisme $\psi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ such that for all $q \in U$ the map $\psi_q = \psi \backslash_{E_q} : E_q \to q \times \mathbb{R}^n$ is a vector space isomorphism,*

then the triple $(E, M, \pi)$is called an n-dimensional topological vector bundle over M. It is said to be trivial if there exists a global bundle chart $\psi : E \to M \times \mathbb{R}^n$.

Definition 1.2.5. *Let (E, M, π) be a topological vector bundle. A continuous map $\sigma : M \to E$ is called a section of the bundle if $\pi \circ \sigma(p) = p$ for each $p \in M$.*

Definition 1.2.6. *A collection*

$$B = \{(\pi^{-1}(U_\alpha), \psi_\alpha) | \alpha \in I\}$$

of bundle charts is called a bundle atlas for (E, M, π) *if* $M = \bigcup_\alpha U_\alpha$. *For each pair* (α, β) *there exists a function* $A_{\alpha,\beta} : U_\alpha \cap U_\beta \to GL(\mathbb{R}^n)$, *into the general linear group* $GL(\mathbb{R}^n)$ *of* $\mathbb{R}^n$, *such that the corresponding continuous map*

$$\psi_\alpha \circ \psi_\beta^{-1}|_{(U_\alpha \cap U_\beta) \times \mathbb{R}^n} : (U_\alpha \cap U_\beta) \times \mathbb{R}^n \to (U_\alpha \cap U_\beta) \times \mathbb{R}^n$$

is given by

$$(p, v) \mapsto (p, (A_{\alpha,\beta})(v)).$$

The elements of $\{(A_{\alpha,\beta})(v) \backslash \alpha, \beta \in I\}$ *are called the transition maps of the bundle atlas* B.

Remark 1.2.1. *Since all the maps which we are using are smooth we call a topological vector bundle smooth, if* B *is maximal. A smooth section of* (E, M, π) *is called a vector field and we denote the set of all vector fields of* (E, M, π) *by* $\Gamma(E)$.

Definition 1.2.7. *By the following operations we make* $\Gamma(E)$ *into a* $C^\infty(M) = C^\infty(M, \mathbb{R})$ *module*

(i) $(v + w)_p = v_p + w_p$,

(ii) $(f \cdot v)_p = f(x) \cdot v_p$,

for all $v, w \in \Gamma(E)$ *and* $f \in C^\infty(M)$. *In particular,* $\Gamma(E)$ *is a vector space over* $\mathbb{R}$.

Definition 1.2.8. *Let* M *be a manifold and* (E, M, π) *be an* n*-dimensional vector bundle over* M. *A set* $F = \{v_1, ..., v_n\}$ *of vector fields*

$$v_1, ..., v_n : U \subset M \to E$$

is called a local frame for E *over* U *if for each* $p \in U$ *the set* $\{(v_1)_p, ..., (v_n)_p\}$ *is a basis for the vector space* E_p.

Definition 1.2.9. *Let* M^m *be a smooth manifold. The tangent bundle* TM *of* M *is given by*

$$TM = \{(p, u) | x \in M, u \in T_p M\}.$$

The bundle map $\pi : TM \to M$ *with* $\pi : (p, u) \mapsto p$ *is called the natural projection of* TM.

Theorem 1.2.1. *Let* M^m *be a smooth manifold with* C^∞*-atlas* A. *Then the tangent bundle* TM *is a smooth manifold of dimension* $2m$ *and* A *induces a* C^∞*-atlas* A^* *on* TM.

Proof. For every local coordinate $x : U \to \mathbb{R}^m$ in A we define $x^* : \pi^{-1}(U) \to \mathbb{R}^m \times \mathbb{R}^m$ by

$$x^* : (p, \sum_{k=1}^{m} u^k \frac{\partial}{\partial x^k}\Big|_t) \mapsto (x(p), (u^1, ..., u^m)).$$

Then the collection

$$(x^*)^{-1}(W) \subset TM|(U, x) \in \hat{A} \quad \text{and} \quad W \subset x(U) \times \mathbb{R}^m \quad \text{open}$$

is a basis for a topology T_{TM} on TM and $(\pi^{-1}(U), x^*)$ is a local coordinate on the $2m$-dimensional topological manifold (TM, T_{TM}). If (U, x) and (V, y) are two local coordinates in A such that $x \in U \cap V$, then the transition map

$$(y^*) \circ (x^*)^{-1} : x^*(\pi^{-1}(U \cap V)) \to \mathbb{R}^m \times \mathbb{R}^m$$

is given by

$$(p, u) \mapsto \Big(y \circ x^{-1}(p), \sum_{k=1}^{m} \frac{\partial y^1}{\partial x^k}(x^{-1}(p))u^k, ..., \sum_{k=1}^{m} \frac{\partial y^m}{\partial x^k}(x^{-1}(p))u^k\Big)$$

We are assuming that $y \circ x^{-1}$ is smooth, hence $(y^*) \circ (x^*)^{-1}$ is smooth and therefor $A^* = (\pi^{-1}(U), x^*\big|(U, x) \in \hat{A})$ is a C^∞-atlas on TM and $(TM, \hat{A}^*)$ is a smooth manifold. □

Remark 1.2.2. *For each point $p \in M$ the fiber $\pi^{-1}(p)$ of π is the tangent space T_pM of M at p and hence an m-dimensional vector space. For a local coordinate $x : U \to \mathbb{R}^m \in \hat{A}$ we define $\bar{x} : \pi^{-1}(U) \to U \times \mathbb{R}^m$ by*

$$\bar{x} : (p, \sum_{k=1}^{m} u^k \frac{\partial}{\partial x^k}\Big|_p) \mapsto (x, (u^1, ..., u^m)).$$

The restriction $\bar{x}_p = \hat{x}\big|_{T_xM} : T_xM \to x \times \mathbb{R}^m$ to T_xM is given by

$$\bar{x}_p : \sum_{k=1}^{m} u^k \frac{\partial}{\partial x^k}\Big|_p \mapsto (u^1, ..., u^m),$$

which obviously is a vector space isomorphism. Hence the $\bar{x} : \pi^{-1}(U) \to U \times \mathbb{R}^m$ is a bundle chart. This implies that

$$B = \{(\pi^{-1}(U), \bar{x})|(U, x) \in \hat{A}\}$$

is a bundle atlas transforming (TM, M, π) into an m-dimensional topological vector bundle. This implies that the vector bundle (TM, M, π) together with the maximal bundle atlas $\hat{B}$ induced by B is a smooth vector bundle. A smooth section of (TM, M, π) is called a vector field and we denote the set of all vector fields of (TM, M, π) by $\Gamma(TM)$ or $\mathfrak{X}(M)$.

Definition 1.2.10. *Let M be a manifold and $X, Y \in \Gamma(TM)$ be vector fields on M. Then the Lie bracket $[X, Y]_p$ of X and Y at $p \in M$ is defined by*

$$[X, Y]_p(f) = X_p(Y(f)) - Y_p(X(f))$$

where $f \in C^\infty(M)$.

1.2.3 Pullback tangent bundle

Definition 1.2.11. *Let $\varphi : M \to N$ a map of class C^∞ between two differentiable manifolds. The pullback tangent bundle is defined by:*

$$\varphi^{-1}TN = \{(x, v) | x \in M, v \in T_{\varphi(x)}N\}.$$

A section on $\varphi^{-1}TN$ is a map of class C^∞, $V : M \to TN$ such that $V(x) \in T_{\varphi(x)}N, \forall x \in M$. Denote by $\Gamma(\varphi^{-1}TN)$ the set of sections of $\varphi^{-1}TN$.

Example 1.2.1. *Let $\varphi : M \to N$ a map of class C^∞ between two differentiable manifolds.*

1. *For all $Y \in \Gamma(TN)$, $Y \circ \varphi : M \to TN$ is a section on $\varphi^{-1}TN$.*
2. *For all $X \in \Gamma(TM), d\varphi(X) \in \Gamma(\varphi^{-1}TN)$.*
3. *The vector fields along a curve γ in a differentiable manifold M are sections of $\varphi^{-1}TM$.*

1.3 Lie groups

Definition 1.3.1. *A Lie group is a group G with a structure of differential manifold, such that the map*

$$\begin{array}{rcl} \theta : G \times G & \longrightarrow & G \\ (x, y) & \longmapsto & xy^{-1} \end{array}$$

is smooth.

Example 1.3.1.

- $\mathrm{S}^1 = \{e^{i\theta}, \theta \in \mathbb{R}\}$, *considered as a group under multiplication.*
- *Linear Lie groups with matrix multiplications*

$$GL(n, \mathbb{R}) = \{M \in M(n, \mathbb{R}) / \det M \neq 0\},$$

$$SL(n, \mathbb{R}) = \{M \in GL(n, \mathbb{R}) / \det M = 1\},$$

$$O(n) = \{M \in GL(n, \mathbb{R}) / {}^t M M = 1\}.$$

1.3.1 Lie algebra of a Lie group

Definition 1.3.2. *A Lie algebra $\mathfrak{g}$ of dimension n on $\mathbb{K}$, is an n-dimensional vector space on $\mathbb{K}$ with a bilinear map, $[,] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ called Lie bracket which has the following properties:*

1. $[X, X] = 0$ *for each* $X \in \mathfrak{g}$.
2. $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ *(Jaccobi identity),*

for each $X, Y, Z \in \mathfrak{g}$.

Definition 1.3.3. *A Lie algebra morphism is a Linear map T between two Lie algebras which preserves the brackets i.e.* $T([,]) = [T(), T()]$.

Definition 1.3.4. *Let G be a Lie group, we define the two smooth maps*

$$\begin{array}{rcl} L_g : G & \longrightarrow & G \\ x & \longmapsto & L_g(x) = gx \end{array}$$

and,

$$\begin{array}{rcl} R_g : G & \longrightarrow & G \\ x & \longmapsto & R_g(x) = xg \end{array}$$

L_g*(resp.* R_g*) is called left translation (resp. right translation).*

Definition 1.3.5. *Let G be a Lie group, a vector field $X \in \mathcal{X}(G)$ is said to be left invariant if:*

$$(L_y)_* X = X \quad (\forall y \in G).$$

Definition 1.3.6. *The Lie algebra of the Lie group G is the space of all left invariant vector fields on G equipped with the Lie bracket of vector fields.*

1.4 Pseudo-Riemannian manifolds

Definition 1.4.1. *A pseudo-Riemannian metric tensor g on a manifold M is a symmetric non-degenerate $(0, 2)$ tensor on M of constant index, i.e., g assigns to each point $x \in M$ a scalar product g_x on T_xM and the index of g_x is the same for all $x \in M$.*

Definition 1.4.2. *A pseudo-Riemmannian manifold M^m is an m-dimensional manifold equipped with a pseudo-Riemannian metric tensor g. The common value s, $0 \leq s \leq m$, of index on M is called the index of M. If $s = 0$, M is called a Riemannian manifold. In this case, each g_x is a positive definite inner product on T_xM. A pseudo-Riemannian manifold (resp. metric) is also known as a semi-Riemannian* *(resp. metric). A pseudo-Riemanniann metric on an even-dimensional manifold M is called a neutral metric if its index is equal to $\frac{1}{2}\dim M$. If the index of M is one, M is called a Lorentz manifold and the corresponding metric is called Lorentzian. A manifold of dimension ≥ 2 admits a Lorentzian metric if and only if it admits a 1-dimensional distribution.*

Definition 1.4.3. *Let $\varphi : M \to N$ be a map of class C^∞ between two differentiable manifolds, and h be a pseudo-Riemannian metric on N. Then h induces a a pseudo-Riemannian metric on $\Gamma(\varphi^{-1}TN)$ given by $h(V,W)(x) = h_{\varphi(x)}(V_x, W_x)$, for all $x \in M$ and $V, W \in \Gamma(\varphi^{-1}TN)$.*

Definition 1.4.4. *Let G be a Lie group. A pseudo-Riemannian g on G is left invariant if*

$$g(X,Y)_x = g((L_a)_*X, (L_a)_*Y).$$

$\forall a, x \in G$, and $X, Y \in T_xG$, that is, L_a is an isometry.

1.5 Levi-Civita connexion

Definition 1.5.1. *A linear connection ∇ on a manifold M is a function:*

$$\begin{aligned} \nabla : \Gamma(TM) \times \Gamma(TM) &\to \Gamma(TM) \\ (X,Y) &\to \nabla_X Y \end{aligned}$$

such that for every $X, Y, Z \in \Gamma(TM)$ and $f \in C^\infty(M)$, we have:

1. $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$.
2. $\nabla_X(fY) = X(f)Y + f(\nabla_X Y)$.
3. $\nabla_{X+fY}(Z) = \nabla_X Z + f(\nabla_Y Z)$.

$\nabla_X Y$ is called the covariant derivative of Y with respect to X. The torsion tensor T of a linear connection ∇ is a tensor of type $(1,2)$ defined by $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$.

Remark 1.5.1. *With respect to a local coordinate system (x_i) on M, ∇ is entirely defined by the Christoffel symbols defined as follows:*

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \Gamma_{ij}^k \frac{\partial}{\partial x_k}.$$

Let $X = X^i \frac{\partial}{\partial x_i}$ et $Y = Y^j \frac{\partial}{\partial x_j}$, then :

$$\nabla_X Y = X^i \big(\frac{\partial Y^k}{\partial x_i} + \Gamma_{ij}^k Y^j \big) \frac{\partial}{\partial x_k}.$$

The following theorem shows that on a pseudo-Riemannian manifold there exists a unique connection sharing two further properties.

Theorem 1.5.1. *On a pseudo-Riemannian manifold (M, g), there exists a unique linear connection ∇ such that*

1. *∇ is torsion free, i.e., $[X, Y] = \nabla_X Y - \nabla_Y X$, and*

2. *$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$ for all $X, Y, Z \in \Gamma(TM)$.*

This unique linear connection ∇ is called the Levi-Civita connection of (M, g) and it is characterized by the Koszul formula:

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\ &\quad + g(Z, [X, Y]) + g(Y, [Z, X]) - g(X, [Y, Z]). \end{aligned} \tag{1.1}$$

1.5.1 Riemann curvature tensor

Remark 1.5.2. *For a pseudo-Riemannian manifold (M, g) with Levi-Civita connection ∇, the function $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ defined by*

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

is a $(1, 3)$ tensor field, called the Riemann curvature tensor.

Proposition 1.5.1. *The Riemann curvature tensor R satisfies the following properties:*

1. *$R(X, Y)Z = -R(Y, X)Z$.*

2. *$g(R(X, Y)Z, W) = -g(R(X, Y)W, Z)$.*

3. *$g(R(X, Y)Z, W) = g(R(Z, W)X, Y)$.*

4. *$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$.*

For all $X, Y, Z, W \in \mathfrak{X}(M)$.

1.6 The vertical and horizontal lifts

Let M^m be smooth manifold, TM the tangent bundle of M, and $\pi : TM \to M$ the canonical projection. For any $\xi \in TM$; $(d\pi)_\xi : T_\xi TM \to T_{\pi(\xi)}M$ is an epimorphism, or equivalently $\pi : TM \to M$ is a submersion. Explicitly , in local coordinates $(x^i)_{1\leq i\leq m}$ and $(x^i, y^i)_{1\leq i\leq m}$, for any $\xi \in TM$ and any $X \in T_\xi TM$, putting

$$X = X^i \frac{\partial}{\partial x^i}\Big|_\xi + X^{m+i} \frac{\partial}{\partial y^i}\Big|_\xi,$$

where $(X^i, X^{m+i})_{1\leq i\leq m} \in \mathbb{R}^{2m}$, we have

$$(d\pi)_\xi(X) = X^i \frac{\partial}{\partial x^i}\Big|_{\pi(\xi)}$$

Therefore, the vertical subspace $\mathcal{V}|_\xi = \ker(d\pi)_\xi$ of $T_\xi TM$ is given by

$$\mathcal{V}\Big|_\xi = Span\Big\{\big(\frac{\partial}{\partial y^i}\Big|_\xi\big)_{1\leq i\leq m}\Big\}$$

Let us now suppose that the manifold M is endowed with a linear connection ∇ whose components with respect to the local chart (U, φ) are Γ^i_{jk}; for any $\xi \in \pi^{-1}(U)$ is well defined the map

$$K_\xi : T_\xi TM \to T_{\pi(\xi)}M$$

such that for any $X \in T_\xi TM$, with $X = X^i \frac{\partial}{\partial x^i}|_\xi + X^{m+i}\frac{\partial}{\partial y^i}|_\xi$:

$$K_\xi(X) = \big(X^{m+i} + \Gamma^i_{jk}(\pi(\xi))X^j y^k(\xi)\big)\frac{\partial}{\partial x^i}\Big|_{\pi(\xi)}$$

which is called the **Dombrowski map**. The horizontal subspace $\mathcal{H}$ of $T_\xi TM$ is now defined by

$$\mathcal{H} = \ker(K_\xi),$$

and one can easily prove that, for any $\xi \in TM$

$$T_\xi TM = \mathcal{V}_\xi \oplus \mathcal{H}_\xi. \tag{1.2}$$

For example, if we consider a local chart (U, φ) at $x = \pi(\xi)$ of normal coordinates, the expression of K_ξ reduces to

$$K_\xi(X) = X^{m+i}\frac{\partial}{\partial x^i}\Big|_{\pi(x)};$$

from witch (1.2) follows immediately, we obtain , in this way, the vertical and the horizontal distributions on TM

$$\mathcal{V}TM = (\mathcal{V}_\xi)_{\xi\in TM} \qquad \text{and} \qquad \mathcal{H}TM = (\mathcal{H}_\xi)_{\xi\in TM}$$

and the natural projection operators, which we shall denote, respectively, with $P_\mathcal{V} : \Gamma(TTM) \to \Gamma(\mathcal{V}TM)$ and $P_\mathcal{H} : \Gamma(TTM) \to \Gamma(\mathcal{H}TM)$. It is of a certain importance to point out the possibility of defining some special types of vector fields on TM, starting with a vector field on M, Namely, if we take $X \in \Gamma(TM)$, it is defined the vertical lift of X, as the unique vector field $X^v \in \Gamma(TTM)$, such that, for any $\xi \in TM$

$$(d\pi)_\xi(X^v_\xi) = 0 \qquad \text{and} \qquad K_\xi(X^v_\xi) = X_{\pi(\xi)}$$

and the horizontal lift of X as the unique vector field X, such that, for any $X^h \in \Gamma(TTM)$

$$(d\pi)_\xi(X^h_\xi) = X_{\pi(\xi)} \qquad \text{and} \qquad K_\xi(X^v_\xi) = 0$$

in local coordinates $(x^i)_{1\leq i\leq m}$ and $(x^i, y^i)_{1\leq i\leq m}$, putting $X|_U = X^i\frac{\partial}{\partial x^i}$, with $X^i \in C^\infty(U)$, the expression of X^v and X^h are

$$X^v\big|_{\pi^{-1}(U)} = (X^i \circ \pi)\frac{\partial}{\partial y^i}$$

and

$$X^h\big|_{\pi^{-1}(U)} = (X^i \circ \pi)\frac{\partial}{\partial x^i} - (\Gamma^i_{jk} \circ \pi)(X^j \circ \pi)y^k\frac{\partial}{\partial y^i}$$

from which, in particular, we get

$$\Big(\frac{\partial}{\partial x^i}\Big)^v\big|_{\pi^{-1}(U)} = \frac{\partial}{\partial y^i}$$

and

$$\Big(\frac{\partial}{\partial x^i}\Big)^h\big|_{\pi^{-1}(U)} = \frac{\partial}{\partial x^i} - (\Gamma^r_{ik} \circ \pi)y^k\frac{\partial}{\partial y^r}$$

from the previous relations, for any $\xi \in TM$

$$\mathcal{V}_\xi = Span\Big\{\Big(\big(\tfrac{\partial}{\partial x^i}\big)^v_\xi\Big)_{1\leq i\leq m}\Big\} \qquad \text{and} \qquad \mathcal{H}_\xi = Span\Big\{\Big(\big(\tfrac{\partial}{\partial x^i}\big)^h_\xi\Big)_{1\leq i\leq m}\Big\}$$

therefore obtaining, recalling (1.2)

$$T_\xi TM = Span\Big\{\Big(\big(\frac{\partial}{\partial x^i}\big)^v_\xi, \big(\frac{\partial}{\partial x^i}\big)^h_\xi\Big)_{1\leq i\leq m}\Big\}$$

Remark 1.6.1. *Note that the maps $X \longmapsto X^h$ and $X \longmapsto X^v$ are isomorphisms between the vector space T_xM and the subspaces $\mathcal{H}_{(x,u)}$ and $\mathcal{V}_{(x,u)}$, respectively. Each tangent vector $Z \in T_{(x,u}TM$ can then be written as*

$$Z = X^h + Y^v,$$

where X and Y are uniquely determined by $X = d\pi(Z)$ and $Y = K(Z)$. It follows that if $f : M \longrightarrow \mathbb{R}$ is a smooth real valued function on M, then

$$X^h(f \circ \pi) = X(f) \circ \pi \text{ and } X^v(f \circ \pi) = 0,$$

for all $X \in \Gamma(TM)$.

1.7 The Lie Bracket

In this section we use the vertical and horizontal lifts to calculate the Lie bracket on the tangent bundle.

Theorem 1.7.1. *Let (M^m, g) be a pseudo-Riemannian manifold, ∇ be the Levi-Civita connection and R be the Riemann curvature tensor of ∇. Then the Lie bracket on the tangent bundle TM of M satisfies the following:*

(i) $[X^v, Y^v]_\zeta = 0$

(ii) $[X^h, Y^v]_\zeta = (\nabla_X Y)^v_\zeta$

(iii) $[X^h, Y^h] = -(R(X,Y)Z)^v_\zeta + [X,Y]^h_\zeta,$

for any $X, Y \in \Gamma(TM)$ and any $\zeta = (x,u) \in TM$, where $Z \in \Gamma(TM)$ such that $Z_{\pi(\zeta)} = \zeta$.

Proof. Using the inclusion map i, we see that there exist vector fields $\widetilde{X}, \widetilde{Y} \in C^\infty(T_u T_x M)$ which are i-related to X^v and Y^h, respectively i.e.

$$X^v_{(x,u)} = di(\widetilde{X}_u) \quad \text{and} \quad Y_{(x,u)} = di(\widetilde{Y}_u)$$

for all $u \in T_p M$. Hence we get

$$[X^v, Y^v]_Z = di([\widetilde{X}, \widetilde{Y}]_u)$$

By the definition of the **Dombrowski map** we know that $K(X^v_{(x,u)}) = X_x$ for all $u \in T_p M$. Therefore $\widetilde{X}$ and $\widetilde{Y}$ are right-invariant vector fields on $T_x M$ in its capacity as a Lie group. Hence the right-hand side of the formula vanishes, since $T_p M$ is an abelian Lie group. This proves (i). We Know that $d\pi(X^h_Z) = X_{\pi(Z)}$ and $d\pi(Y^v_Z) = 0$. Hence $d\pi([X^h, Y^v]) = [d\pi(X^h), d\pi(Y^v)] = 0$ and $d\pi((\nabla_X Y)^v) = 0$, we get

$$d\pi([X^h, Y^v]) = d\pi((\nabla_X Y)^v)$$

and

$$d\pi([X^h, Y^v]) = [X, Y].$$

So we only have to compute the function K of the right-hand sides in the last two parts of the theorem. To calculate them we will again use our previous abbreviation $X^i = \frac{\partial}{\partial x^i}$ where $(x^1, ..., x^m)$ are local coordinates for M . It is sufficient to calculate both terms just for $X, Y \in \{\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^m}\}$, because all our functions are linear in every argument. Including the abbreviations is this corollary, and using $[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}] = 0$ and $\frac{\partial}{\partial y^i}(y^j) = \delta_{ij}$ for all $i, i \in \{1, ..., 2m\}$, with δ_{ij} the Kronecker symbol, we obtain

$$\Big[\big(\frac{\partial}{\partial x^i}\big)^h, \big(\frac{\partial}{\partial y^j}\big)^v\Big] = \sum_{k=1}^m (\Gamma^k_{ij} \circ \pi) \frac{\partial}{\partial y^k}$$

By the definition of the **Dombrowski map** we obtain:

$$K\Big(\Big[(\frac{\partial}{\partial x^i})^h, (\frac{\partial}{\partial y^j})^v\Big]_{(x,u)}\Big) = \Big(\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial y^j}\Big)_x.$$

This provides us with **(ii)**. In the same way as above we will now, calculate.

$$\begin{aligned}\Big[(\frac{\partial}{\partial x^j})^h, (\frac{\partial}{\partial y^j})^h\Big] &= \sum_{k,l,n=1}^{m} \{\frac{\partial}{\partial x^j}(\Gamma_{il}^k \circ \pi) - \frac{\partial}{\partial x^i}(\Gamma_{jl}^k \circ \pi) + (\Gamma_{il}^n \circ \pi)(\Gamma_{jn}^k \circ \pi) \\ &\quad -(\Gamma_{jl}^n \circ \pi)(\Gamma_{in}^k \circ \pi) y^l \frac{\partial}{\partial y^k}\} \\ &= -\sum_{k,l=1}^{m} (R_{lij}^k \circ \pi) y^l \frac{\partial}{\partial y^k}\end{aligned}$$

Again, by using the **Dombrowski map**, we obtain for $Z = (x, u)$

$$K\Big(\Big[(\frac{\partial}{\partial x^i})^h, (\frac{\partial}{\partial x^j})^h\Big]_Z\Big) = -R\Big(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\Big)Z$$

This proves **(iii)** and completes the proof. □

1.8 Natural Metrics

Definition 1.8.1. *Let (M^m, g) be a pseudo-Riemannian manifold. A pseudo-Riemannian metric $\bar{g}$ on the tangent bundle TM of M is said to be natural with respect to g if :*

(i) $\bar{g}_{(x,u)}(X^h, Y^h) = g_x(X, Y)$,

(ii) $\bar{g}_{(x,u)}(X^h, Y^v) = 0$,

for all vector fields $X, Y \in \Gamma(TM)$.

We can now use the Koszul formula to compute the Levi-Civita connection $\bar{\nabla}$ for the tangent bundle $(TM, \bar{g})$ equipped with a natural metric $\bar{g}$ with respect to g on M.

Lemma 1.8.1. *Let (M^m, g) be a pseudo-Riemannian manifold and TM be the tangent bundle of M . Then for each $(x, u) \in TM$ and every natural metric $\bar{g}$ on TM the corresponding Levi-Civita connection $\bar{\nabla}$ satisfies.*

1. $\bar{g}(\bar{\nabla}_{X^h} Y^h, W^h) = g(\nabla_X Y, W)$,
2. $\bar{g}(\bar{\nabla}_{X^h} Y^h, W^v) = -\frac{1}{2}\bar{g}((R(X, Y)Z)^v, W^v)$,
3. $\bar{g}(\bar{\nabla}_{X^h} Y^v, W^h) = -\frac{1}{2}\bar{g}(Y^v, (R(W, X)Z)^v)$,
4. $\bar{g}(\bar{\nabla}_{X^h} Y^v, W^v) = \frac{1}{2}(X^h(\bar{g}(Y^v, W^v) + \bar{g}(W^v, (\nabla_X Y)^v)) - \bar{g}(Y^v, (\nabla_X W)^v))$,

5. $\bar{g}(\bar{\nabla}_{X^v}Y^h, W^h) = \frac{1}{2}\bar{g}(X^v, (R(Y,W)Z)^v)$,
6. $\bar{g}(\bar{\nabla}_{X^v}Y^h, W^v) = \frac{1}{2}(Y^h(\bar{g}(W^v, X^v) - \bar{g}(W^v, (\nabla_Y X)^v)) - \bar{g}(X^v, (\nabla_Y W)^v))$,
7. $\bar{g}(\bar{\nabla}_{X^v}Y^v, W^h) = \frac{1}{2}(-W^h(\bar{g}(X^v, Y^v) + \bar{g}(Y^v, (\nabla_W X)^v)) + \bar{g}(X^v, (\nabla_W Y)^v))$,
8. $\bar{g}(\bar{\nabla}_{X^h}Y^h, W^h) = \frac{1}{2}(X^v(\bar{g}(Y^v, W^v)) + Y^v(\bar{g}(W^v, Y^v)) - W^v(\bar{g}(X^v, Y^v)))$,

for all $X, Y, W \in \Gamma(TM)$, where $Z \in \Gamma(TM)$ such that $Z_{\pi(x,u)} = (x,u)$ and R is the curvature tensor field of ∇.

Proof. For any vector fields $X, Y, W \in \Gamma(TM)$ and $i, j, k \in \{h, v\}$

$$\begin{aligned} 2\bar{g}(\bar{\nabla}_{X^i}Y^j, W^k) &= X^i(\bar{g}(Y^j, W^k)) + Y^j(\bar{g}(W^k, X^i)) - W^k(\bar{g}(X^i, Y^i)) \\ &\quad +\bar{g}(W^k, [X^i, Y^j]) + \bar{g}(Y^j, [W^k, X^i]) - \bar{g}(X^i, [Y^j, W^k]), \end{aligned}$$

(1) This s a consequence of Theorem 1.7.1, Definition 1.8.1 and the following computations

$$\begin{aligned} 2\bar{g}(\bar{\nabla}_{X^h}Y^h, W^h) &= X^h(\bar{g}(Y^h, W^h)) + Y^h(\bar{g}(W^h, X^h)) \\ &\quad -W^h(\bar{g}(X^h, Y^h)) - \bar{g}(X^h, [Y^h, W^h]) \\ &\quad +\bar{g}(Y^h, [W^h, X^h]) + \bar{g}(W^h, [X^h, Y^h]) \\ &= X(g(Y,W)) + Y(g(W,Y))) - W(g(X,Y)) \\ &\quad -\bar{g}(X^h, [Y,W]^h) + \bar{g}(Y^h, [W,X]^h) + \bar{g}(W^h, [X,Y]^h) \\ &\quad +\bar{g}(X^h, (R(Y,W)Z)^v) - \bar{g}(Y^h, (R(W,X)Z)^v) \\ &\quad -\bar{g}(W^h, (R(X,Y)Z)^v) \\ &= 2g(\nabla_X Y, W). \end{aligned}$$

(2) The second assertion of the lemma is obtained as follows

$$\begin{aligned} \bar{g}(\bar{\nabla}_{X^h}Y^h, W^v) &= X^h(\bar{g}(Y^h, W^v)) + Y^h(\bar{g}(W^v, X^h)) \\ &\quad -W^v(\bar{g}(X^h, Y^h)) - \bar{g}(X^h, [Y^h, W^v]) \\ &\quad +\bar{g}(Y^h, [W^v, X^h]) + \bar{g}(W^v, [X^h, Y^h]) \\ &= -W^v g(X,Y) - \bar{g}(X^h, [Y^h, W^v]) \\ &\quad +\bar{g}(Y^h, [W^v, X^h]) + \bar{g}(W^v, [X^h, Y^h]). \end{aligned}$$

The first term vanishes, because differentiating a horizontal vector field in a vertical direction gives zero. The second and third terms also vanish, because the Lie bracket of a horizontal vector field is vertical, therefore

$$\begin{aligned} 2\bar{g}(\bar{\nabla}_{X^h}Y^h, W^h) &= \bar{g}(W^v, [X^h, Y^h]) \\ &= \bar{g}(W^v, [X,Y]^h - (R(X,Y)Z)^v) \end{aligned}$$

$$= -\bar{g}(W^v, (R(X,Y)Z)^v).$$

(3) This is analogous to the proof of part (2).
(3) The Koszul formula gives

$$\begin{aligned}\bar{g}(\bar{\nabla}_{X^h}Y^v, W^v) &= X^h(\bar{g}(Y^v, W^v)) + Y^v(\bar{g}(W^v, X^h)) \\ &\quad -W^v(\bar{g}(X^h, Y^v)) - \bar{g}(X^h, [Y^v, W^v]) \\ &\quad +\bar{g}(Y^v, [W^v, X^h]) + \bar{g}(W^v, [X^h, Y^v]) \\ &= X^h(\bar{g}(Y^v, W^v) - \bar{g}(X^h, [Y^v, W^v]) \\ &\quad +\bar{g}(Y^v, [W^v, X^h]) + \bar{g}(W^v, [X^h, Y^v]).\end{aligned}$$

But the Lie bracket of two vertical fields is equal to zero and hence the result is proven. (5) this is analogous to the proof of part (2), (6) and (7) are analogous to (4), (8) this is direct consequence of the fact that Lie bracket of two vertical vector fields vanishes. □

Corollary 1.8.1. *Let (M^m, g) be a pseudo-Riemannian manifold and $\bar{g}$ be a natural metric on the tangent bundle TM of M. Then the corresponding Levi-Civita connection satisfies.*

$$(\bar{\nabla}_{X^h}Y^h)_\zeta = (\nabla_X Y)^h_\zeta - \frac{1}{2}(R(X,Y)Z)^v$$

for all $X, Y \in \Gamma(TM)$ and any $\zeta = (x,u) \in TM$, where $Z \in C\infty(TM)$ such that $Z_{\pi(\zeta)} = \zeta$ and R is the curvature tensor field of ∇.

Definition 1.8.2. *Let (M^m, g) be a pseudo-Riemannian Manifold and let $\bar{\nabla}$ be the Levi-Civita connection on the tangent bundle $(TM, \bar{g})$, equipped with a natural metric $\bar{g}$. Let $F : TM \to TM$ be a differentiable map preserving the fibers and linear on each of them. Then we define the vertical and horizontal lifts F^v and F^h by*

$$F(\eta)^v = \textstyle\sum_{i=1}^m \eta_i F(\frac{\partial}{\partial x^i})^v \quad \textit{and} \quad F(\eta)^h = \sum_{i=1}^m \eta_i F(\frac{\partial}{\partial x^i})^h$$

where $\eta = \sum_{i=1}^m \eta_i \frac{\partial}{\partial x^i} \in \pi^{-1}(V)$ is a local representation of $\eta \in \Gamma(TM)$.

Lemma 1.8.2. *For any vector field $X \in \Gamma(TM)$, $\zeta = (x,u) \in TM$ and $\eta = \sum_{i=1}^m \eta_i \frac{\partial}{\partial x^i} \in \pi^{-1}(V)$ we have*

(i) $(\hat{\nabla}_{X^v}F^v)_\zeta = F(X_x)^v_\zeta + \sum_{i=1}^m \eta_i \bar{\nabla}_{X^v} F(\frac{\partial}{\partial x^i})^v$,

(ii) $(\hat{\nabla}_{X^v}F^h)_\zeta = F(X_x)^h_\zeta + \sum_{i=1}^m \eta_i \bar{\nabla}_{X^v} F(\frac{\partial}{\partial x^i})^h$,

(iii) $(\hat{\nabla}_{X^h}F^v)_\zeta = (\hat{\nabla}_{X^h}(F(u))^v)_\zeta$,

(iv) $(\hat{\nabla}_{X^h}F^h)_\zeta = (\hat{\nabla}_{X^h}(F(u))^h)_\zeta$

Proof. Let $(x^1, ..., x^m)$ be a local coordinate of M^m in a neighborhood V of x. Then, we have $X^v \cdot dx_i = dx^i(X)$ for $i \in \{1, ..., n\}$. Hence we get

$$\begin{aligned}
(\hat{\nabla}_{X^v} F(\eta)^v) &= \sum_{i=1}^{m} \nabla_{X^v}(\eta_i F(\frac{\partial}{\partial x^i})^v) \\
&= \sum_{i=1}^{m} X^v(\eta_i) F(\frac{\partial}{\partial x^i})^v + \eta_i \hat{\nabla}_{X^v} F(\frac{\partial}{\partial x^i})^v \\
&= \sum_{i=1}^{m} \eta_i(X) F(\frac{\partial}{\partial x^i})^v + \eta_i \hat{\nabla}_{X^v} F(\frac{\partial}{\partial x^i})^v \\
&= F(X_x)^v_\xi + \sum_{i=1}^{m} \eta_i \hat{\nabla}_{X^v} F(\frac{\partial}{\partial x^i})^v.
\end{aligned}$$

The second step follows by the product rule. Similarily we compute:

$$\begin{aligned}
(\hat{\nabla}_{X^v} F(\eta)^h) &= \sum_{i=1}^{m} \nabla_{X^v}(\eta_i F(\frac{\partial}{\partial x^i})^h) \\
&= \sum_{i=1}^{m} X^v(\eta_i) F(\frac{\partial}{\partial x^i})^h + \eta_i \hat{\nabla}_{X^v} F(\frac{\partial}{\partial x^i})^h \\
&= \sum_{i=1}^{m} \eta_i(X) F(\frac{\partial}{\partial x^i})^h + \eta_i \hat{\nabla}_{X^v} F(\frac{\partial}{\partial x^i})^h \\
&= F(X_x)^h_\xi + \sum_{i=1}^{m} \eta_i \hat{\nabla}_{X^v} F(\frac{\partial}{\partial x^i})^h.
\end{aligned}$$

For the two remaining equations of the Lemma we use a differentiable curve $\gamma : [0,1] \to M$ such that $\gamma(0) = x$ and $\gamma'(0) = X_x$ to get a differentiable curve $U \circ \gamma : [0,1] \to TM$ such that $(U \circ \gamma)(0) = \zeta$ and $(U \circ \gamma)'(0)) = X^h_\zeta$. By the definition of our functions F^v and F^h we obtain $F^v \mid_{U\circ\gamma} = (F \circ U)^v \mid_{U\circ\gamma}$ and $F^h \mid_{U\circ\gamma} = (F \circ U)^h \mid_{U\circ\gamma}$. This proves part (iii) and (iv) of the Lemma. □

1.9 The Sasaki Metric

Definition 1.9.1. *Let (M^m, g) be a pseudo-Riemannian manifold. Then the Sasaki metric g_S on the tangent bundle TM is natural metric given by*

$$\begin{cases}
g_S(X^h, Y^h) = g(X, Y) \circ \pi, \\
g_S(X^v, Y^h) = g_S(X^h, Y^v) = 0, \\
g_S(X^v, Y^v) = g(X, Y) \circ \pi,
\end{cases}$$

for all vector fields $X, Y \in \Gamma(TM)$. The Sasaki metric of g with respect to the local coordinates (x^i, y^i) of TM is [33]

$$g_S = g_{ij}dx^i dx^j + g_{ij}\partial^*_{y^i}\partial^*_{y^j}, \tag{1.3}$$

where $\partial^*_{y^i} = dy^i + y^h\Gamma^i_{hj}dx^j$.

It is easy to prove that, if (p, q) is the signature of the metric g, then $(2p, 2q)$ is the signature of the Sasaki metric g_S.

Example 1.9.1. *We consider the Lorentzian manifold $(\mathbb{R}^4, g)$, where*

$$g = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 - (dx^4)^2. \tag{1.4}$$

Then, by virtue of (1.3) and (1.4), the Sasaki metric of g with respect to the local coordinates (x^i, y^i) of the tangent bundle of $\mathbb{R}^4$ is given by:

$$g_S = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 - (dx^4)^2 + (dy^1)^2 + (dy^2)^2 + (dy^3)^2 - (dy^4)^2.$$

Example 1.9.2. *The Egorov spaces are Lorentzian manifolds $(\mathbb{R}^m, g_f)$, $m \geq 3$, where f is a positive smooth function of a real variable and*

$$g_f = f(x^m)\sum_{i=1}^{m-2}(dx^i)^2 + 2dx^{m-1}dx^m. \tag{1.5}$$

These manifolds are named after I. P. Egorov, who first introduced and studied them in [12]. If $(\mathbb{R}^m, g_f)$ is an Egorov space , $m \geq 3$, the only possible non-vanishing Christoffel symbols are the following ones [4]:

$$\Gamma^{m-1}_{ii} = \frac{-f'}{2}, \quad \Gamma^i_{im} = \frac{f'}{2f}, \quad i = 1, \cdots, m-2. \tag{1.6}$$

Then, by virtue of (1.3), (1.5) and (1.6), the Sasaki metric of g_f with respect to the local coordinates (x^i, y^i) of the tangent bundle of $\mathbb{R}^m$ is given by:

$$\begin{aligned}(g_f)_S = {} & f\sum_{i=1}^{m-2}(dx^i)^2 + 2dx^{m-1}dx^m + f\sum_{i=1}^{m-2}(dy^i)^2 + 2dy^{m-1}dy^m \\ & + f'\sum_{i=1}^{m-2}y^i dx^m dy^i - f'\sum_{i=1}^{m-2}y^i dy^m dx^i + \frac{(f')^2}{4f}\sum_{i=1}^{m-2}(y^i)^2(dx^m)^2.\end{aligned}$$

We can now calculate the Levi-Civita connection of the tangent bundle with respect to g_S.

Proposition 1.9.1. *Let $\widetilde{\nabla}$ be the Levi-civita connection of (TM, g_S) equipped with the Sasaki metric g_S. Then*

(i) $(\widetilde{\nabla}_{X^h} Y^h)_{(x,u)} = (\nabla_X Y)^h_{(x,u)} - \frac{1}{2}(R(X,Y)Z)^v_{(x,u)}$,

(ii) $(\widetilde{\nabla}_{X^h} Y^v)_{(x,u)} = (\nabla_X Y)^v_{(x,u)} + \frac{1}{2}(R(Z,Y)X)^h_{(x,u)}$,

(iii) $(\widetilde{\nabla}_{X^v} Y^h)_{(x,u)} = \frac{1}{2}(R(Z,X)Y)^h_{(x,u)}$,

(iv) $(\widetilde{\nabla}_{X^v} Y^v)_{(x,u)} = 0$,

for any $X, Y \in \Gamma(TM)$ and any $(x,u) \in TM$, where $Z \in \Gamma(TM)$ such that $Z_{\pi(x,u)} = (x,u)$ and R is the curvature tensor field of ∇.

Proof. (i) This is nothing but Corollary 1.8.1.

(ii) The part (iii) of Lemma 1.8.1 .

$$\begin{aligned} 2g_S(\widetilde{\nabla}_{X^h} Y^v, W^h) &= -g_S((R(W,X)Z)^v, Y^v) \\ &= -g(R(Z,Y)W, X) \\ &= -g(R(Z,Y)X, W). \end{aligned}$$

Part (iv) of Lemma 1.8.1 implies

$$\begin{aligned} 2g_S(\widetilde{\nabla}_{X^h} Y^v, W^v) &= X^h(g_S(Y^v, W^v)) + g_S(W^v, (\nabla_X Y)^v) \\ &\quad -g_S(Y^v, (\nabla_X W)^v) \\ &= X(g(Y,W)) + g(W, \nabla_X Y) - g(Y, \nabla_X W) \\ &= g(W, \nabla_X Y) + g(Y, \nabla_X W) + g(Z, \nabla_X Y) - g(Y, \nabla_X W) \\ &= 2g(\nabla_X Y, W). \end{aligned}$$

The last important step follows by the definition of a metric connection.

(iii) As above we use part (v) of Lemma 1.8.1 we get

$$\begin{aligned} 2g_S(\widetilde{\nabla}_{X^v} Y^h, W^h) &= g_S(X^v, (R(Y,W)Z)^v) \\ &= g(X, R(Y,W)Z) \\ &= g(R(Z,X)Y, W). \end{aligned}$$

Part (vi) of Lemma 1.8.1 gives us further

$$\begin{aligned} 2g_S(\widetilde{\nabla}_{X^v} Y^h, W^v) &= (Y^h(g_S(W^v, X^v)) \\ &\quad -g_S(W^v, (\nabla_Y X)^v) - g_S(X^v, (\nabla_Y W)^v) \\ &= Y(g(W,X)) - g(W, \nabla_Y X) - g(X, \nabla_Y W) - \\ &= g(W, \nabla_Y X) + g(X, \nabla_Y W) \\ &\quad -g(W, \nabla_Y X) - g(X, \nabla_Y W) \\ &= 0 \end{aligned}$$

(*iv*) As usual we use Lemma 1.8.1 to get

$$\begin{aligned} 2g_S(\widetilde{\nabla}_{X^v}Y^v, W^h) &= (-W^h(g_S(X^v, Y^v))) \\ &\quad + g_S(Y^v, (\nabla_W X)^v) + g_S(X^v, (\nabla_W Y)^v) \\ &= -W(g(X,Y)) + g(Y, \nabla_W X) + g(X, \nabla_W Y) - \\ &\quad - g(Y, \nabla_W X) - g(X, \nabla_W Y) + g(Y, \nabla_W X) \\ &\quad + g(X, \nabla_W Y) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} 2g_S(\widetilde{\nabla}_{X^v}Y^v, W^v) &= X^v(g_S(Y^v, W^v)) + Y^v(g_S(W^v, X^v)) \\ &\quad - W^v(g_S(X^v, Y^v)) \\ &= X^v(g(Y,W)) + Y^v(g(W,X)) - W^v(g(X,Y)) \\ &= 0 \end{aligned}$$

The last equation we have, because differentiating a horizontal vector field in the direction of a vertical one gives zero and by definition of the metric holds $g(X,Y) = g_S(X^h, Y^h)$. This completes the proof. □

1.10 The curvature tensor

For calculating the curvature tensor we need the following result, which is a direct consequence of Lemma 1.8.2.

Corollary 1.10.1. *Let (M^m, g) ba a pseudo-Riemannian manifold and let $\widetilde{\nabla}$ be the Levi-Civita connection on the tangent bundle TM, equipped with the Sasaki metric. Let $F : TM \to TM$ be a differentiable map preserving the fibers and linear on each of them. Then for any $x \in M$ and $\eta \in \Gamma(TTM)$ we have*

$$\widetilde{\nabla}_{X^v} F(\eta)^v = F(X)^v$$

$$\widetilde{\nabla}_{X^v} F(\eta)^h = F(X)^h + \frac{1}{2}(R(Z,X)F(\eta))^h.$$

Proposition 1.10.1. *Let (M^m, g) be a pseudo-Riemannian manifold and $\widetilde{R}$ be the curvature tensor of the tangent bundle (TM, g_S) equipped with the Sasaki metric. Then we have the following formulae*

(i) $\widetilde{R}_{(x,u)}(X^v, Y^v)W^v = 0$

(ii)

$$\widetilde{R}_{(x,u)}(X^v,Y^v)W^h = (R(X,Y)W + \frac{1}{4}R(Z,Y)(R(Z,Y)W) - \frac{1}{4}R(Z,Y)(R(Z,X)W))^h_{(x,u)},$$

(iii) $\widetilde{R}_{(x,u)}(X^h,Y^v)W^v = -(\frac{1}{2}R(Y,W)X + \frac{1}{4}R(Z,Y)(R(Z,W)X))^h_{(x,u)}$,

(iv)

$$\widetilde{R}_{(x,u)}(X^v,Y^v)W^h = (\frac{1}{4}R(R(Z,Y)W,X)Z + \frac{1}{2}R(X,W)Y)^v_{(x,u)} + \frac{1}{2}((\nabla_X R)(Z,Y)W)^h_{(x,u)}$$

(v)

$$\widetilde{R}_{(x,u)}(X^h,Y^h)W^v = = (R(X,Y)W + \frac{1}{4}R(R(Z,W)Y,X)Z - \frac{1}{4}R(R(Z,W)X,Y)Z)^v_{(x,u)} + \frac{1}{2}((\nabla_X R)(Z,W)Y - (\nabla_Y R)(Z,W)X)^h_{(x,u)},$$

(vi)

$$\widetilde{R}_{(x,u)}(X^h,Y^h)W^h = \frac{1}{2}((\nabla_W R)(X,Y)Z)^v_{(x,u)} + (R(X,Y)Z + \frac{1}{4}R(Z,R(W,Y)Z)X + \frac{1}{4}R(Z,R(X,W)Z)Y + \frac{1}{2}R(Z,R(X,Y)Z)W)^h_{(x,u)}$$

For any $X,Y,W \in \Gamma(TM)$ and any $(x,u) \in TM$, where $Z \in \Gamma(TM)$ such that $Z_{\pi(x,u)} = (x,u)$.

Proof. **(i)** The first part of the proposition follows directly by the last part of proposition 1.9.1 and the fact that the Lie bracket of two vertical vector fields vanishes.

(iii) The last part of Proposition 1.9.1 and the fact that

$$[X^h, Y^v] = (\nabla_X Y)^v,$$

by theorem 1.7.1, provide us with

$$\widetilde{R}(X^h,Y^v)W^v = \widetilde{\nabla}_{X^h}\widetilde{\nabla}_{Y^v}W^v - \widetilde{\nabla}_{Y^v}\widetilde{\nabla}_{X^h}W^v - \widetilde{\nabla}_{[X^h,Y^v]}W^v$$

$$
\begin{aligned}
&= -\widetilde{\nabla}_{Y^v}\widetilde{\nabla}_{X^h}W^v \\
&= -\widetilde{\nabla}_{Y^v}((\nabla_X W)^v + F(u)^h),
\end{aligned}
$$

where $F : TM \to TM$ is the linear fiber preserving map

$$F : u \mapsto \frac{1}{2}R(u, W_x)X_x,$$

for any $(x, u) \in TM$. By the last part of Proposition 1.9.1 we know that $\widetilde{\nabla}_{Y^v}(\nabla_X W)^v = 0$ and according to Corollary 1.10.1 we have

$$\widetilde{\nabla}_{Y^v}F(u)^h = F(Y)^h + \frac{1}{2}(R(u, Y)F(u))^h.$$

Therefore we obtain

$$
\begin{aligned}
\widetilde{R}(X^h, Y^v)W^v &= -\widetilde{\nabla}_{Y^v}\widetilde{\nabla}_{X^h}W^v \\
&= \widetilde{\nabla}_{Y^v}(\nabla_X W)^v + F(u)^h \\
&= -\widetilde{\nabla}_{Y^v}F(u)^h \\
&= -F(Y)^h - \frac{1}{2}(R(u, Y)F(u))^h \\
&= -(\frac{1}{2}R(Y, W)X + \frac{1}{4}R(u, Y)(R(u, W)X))^h.
\end{aligned}
$$

Hence the third part of the proposition is proven.

(i) Using the 1^{st} Bianchi identity

$$\widetilde{R}(X^v, Y^v)W^h = \widetilde{R}(W^h, Y^v)X^v - \widetilde{R}(W^h, Y^v)Y^v,$$

we get by using part (iii)

$$
\begin{aligned}
\widetilde{R}(X^v, Y^v)W^h &= (-\frac{1}{2}R(Y, X)W - \frac{1}{4}R(Z, Y)R(Z, X)W))^h \\
&\quad +(\frac{1}{2}R(X, Y)W + \frac{1}{4}R(Z, X)(R(Z, Y)W))^h.
\end{aligned}
$$

From which the statement follows.

(iv) As above we now introduce two mappings $F_1 : TM \to TM$ and $F_2 : TM \to TM$ by

$$F_1(u) \mapsto \frac{1}{2}R(u, Y_x)W_x$$

and

$$F_2(u) \mapsto -\frac{1}{2}R(X_x, W_x)u,$$

the third part of proposition 1.9.1 becomes

$$\widetilde{\nabla}_{Y^v}W^h = F_1(u)^h$$

By the definition of the curvature tensor we obtain

$$
\begin{aligned}
\widetilde{R}(X^h, Y^v)W^h &= \widetilde{\nabla}_{X^h}\widetilde{\nabla}_{Y^v}W^h - \widetilde{\nabla}_{Y^v}\widetilde{\nabla}_{X^h}W^h - \widetilde{\nabla}_{[X^h,Y^v]}W^h \\
&= \widetilde{\nabla}_{X^h}F_1(u)^h - \widetilde{\nabla}_{Y^v}((\nabla_X W)^h + F_2(u)^v) - \widetilde{\nabla}_{(\nabla_X Y)^v}W^h \\
&= (\nabla_X(F_1(u)))^h - \frac{1}{2}(R(X, F_1(u))u)^v \\
&\quad -\frac{1}{2}(R(u,Y)\nabla_X W)^h - F_2(Y)^v - \frac{1}{2}(R(u,\nabla_X Y)W)^h \\
&= (\frac{1}{4}R(R(u,Y)W,X)u + \frac{1}{2}R(X,W)Y)^v \\
&\quad +\frac{1}{2}((\nabla_X R)(u,Y)W)^h.
\end{aligned}
$$

The last step is only inserting the mappings F_1,F_2 and the definition of a covariant derivative. The middle step uses proposition 1.9.1 and Corollary 1.10.1.

(v) Using part (iv) and the 1^{st} Bianchi identity

$$\widetilde{R}(X^h, Y^h)W^v = \widetilde{R}(X^h, W^v)Y^h - \widetilde{R}(Y^h, W^v)X^h,$$

therefore

$$
\begin{aligned}
\widetilde{R}(X^h, Y^h)W^v &= (\frac{1}{4}R(R(u,W)Y,X)u)^v + \frac{1}{2}((\nabla_X R)(u,W)Y)^h \\
&\quad -(\frac{1}{4}R(R(u,W)X,Y)u)^v - \frac{1}{2}((\nabla_Y R)(u,W)X)^h \\
&\quad +\frac{1}{2}(R(X,Y)W - R(Y,X)W)^v.
\end{aligned}
$$

Which implies the result.

(vi) For the last part we have the following standard computations

$$
\begin{aligned}
\widetilde{R}(X^h, Y^h)W^h &= \widetilde{\nabla}_{X^h}\widetilde{\nabla}_{Y^h}W^h - \widetilde{\nabla}_{Y^h}\widetilde{\nabla}_{X^h}W^h - \widetilde{\nabla}_{[X^h,Y^h]}W^h \\
&= \widetilde{\nabla}_{X^h}((\nabla_Y W)^h - \frac{1}{2}(R(Y,W)u)^v) \\
&\quad -\widetilde{\nabla}_{Y^h}((\nabla_X W)^h - \frac{1}{2}(R(X,W)u)^v) \\
&\quad -\widetilde{\nabla}_{[X,Y]^h}W^h + \widetilde{\nabla}_{(R(X,Y)u)^v})W^h \\
&= (\nabla_X Y \nabla_Y W)^h - \frac{1}{2}(R(X,\nabla_Y W)u)^v \\
&\quad -\frac{1}{2}(\nabla_X R(Y,W)u)^v - \frac{1}{4}(R(Z,R(Y,W)u)X)^h \\
&\quad (\nabla_Y \nabla_X W)^h - \frac{1}{2}(R(Y,\nabla_X W)u)^v
\end{aligned}
$$

$$
\begin{aligned}
& \frac{1}{2}(\nabla_Y R(X,W)u)^v + \frac{1}{4}(R(u,R(X,W)u)Y)^h \\
& -(\nabla_{[X,Y]}W)^h + \frac{1}{2}(R([X,Y],W)u)^V \\
& +\frac{1}{2}(R(u,R(X,Y)u)W)^h \\
= \; & \frac{1}{2}((\nabla_W R)(X,Y)u)^v + (R(X,Y)W)^h \\
& +\frac{1}{4}(R(u,R(W,Y)u)X)^h + \frac{1}{4}(R(u,R(X,W)u)Y)^h \\
& +\frac{1}{2}(R(u,R(X,Y)u)W)^h.
\end{aligned}
$$

The last part of these computations follows by the 2^{nd} Bianchi identity, which tells us that

$$(\nabla_X R)(Y,W)u + (\nabla_Y R)(W,X)u + (\nabla_W R)(X,Y)u = 0.$$

$\square$

1.11 Induced Connection on the Tangent Bundle

Definition 1.11.1. *Let $\varphi : M \longrightarrow N$ be a smooth map between two differentiable manifolds M and N and let ∇^N be a linear connection on N, then the Pull-back connection on the tangent bundle $\varphi^{-1}TN$ is defined by:*

$$
\begin{aligned}
\nabla^{\varphi} : \Gamma(TM) \times \Gamma(\varphi^{-1}TN) & \longrightarrow \Gamma(\varphi^{-1}TN), \\
(X,V) & \longrightarrow \nabla^{\varphi}_X V = \nabla^N_{d\varphi(X)} \widetilde{V}
\end{aligned}
\tag{1.7}
$$

where $\widetilde{V} \in \Gamma(TN)$ such that $\widetilde{V} \circ \varphi = V$.

Locally:

$$
\begin{aligned}
\nabla^{\varphi}_X V &= \nabla^{\varphi}_{X^i \frac{\partial}{\partial x^i}} V^{\alpha}(\frac{\partial}{\partial y^{\alpha}} \circ \varphi) \\
&= X^i \left\{ \frac{\partial V^{\alpha}}{\partial x^i}(\frac{\partial}{\partial y^{\alpha}} \circ \varphi) + V^{\alpha} \nabla^{\varphi}_{\frac{\partial}{\partial x^i}} (\frac{\partial}{\partial y^{\alpha}} \circ \varphi) \right\}
\end{aligned}
$$

Note that :

$$
\begin{aligned}
\nabla^{\varphi}_{\frac{\partial}{\partial x^i}}(\frac{\partial}{\partial y^{\alpha}} \circ \varphi) &= \nabla^N_{d\varphi(\frac{\partial}{\partial x^i})} \frac{\partial}{\partial y \alpha} \\
&= \frac{\partial \varphi_{\beta}}{\partial x^i} \left(\nabla^N_{\frac{\partial}{\partial y^{\beta}}} \frac{\partial}{\partial y^{\alpha}} \right) \circ \varphi
\end{aligned}
$$

$$= \frac{\partial \varphi_\beta}{\partial x^i}\left(\Gamma^\gamma_{\alpha\beta}\frac{\partial}{\partial y^\gamma}\right)\circ\varphi$$

So that

$$\nabla^\varphi_X V = X^i\left\{\frac{\partial V^\gamma}{\partial x^i} + V^\alpha \frac{\partial \varphi_\beta}{\partial x^i}\left(\Gamma^\gamma_{\alpha\beta}\circ\varphi\right)\right\}\left(\frac{\partial}{\partial y^\gamma}\circ\varphi\right)$$

Then the relation (1.7) is independent of the choice of $\widetilde{V}$ i.e. this connection is well defined.

Definition 1.11.2. *If $\varphi : M \longrightarrow N$ is a map between differeniable manifolds, then two vector fields $X \in \Gamma(TM)$, $\widetilde{X} \in \Gamma(TN)$ are said to be φ-related if*

$$d\varphi_x(X) = \widetilde{X}_{\varphi(x)} \ \ \forall \ x \in M.$$

In that case we write $\widetilde{X} = d\varphi(X)$.

Proposition 1.11.1. *Let $\varphi : M \longrightarrow N$ be a smooth map and let ∇^N be a linear connection compatible with the Riemaniann metric h on N, then the linear connection ∇^φ is compatible with the induce Riemannian metric on $\varphi^{-1}TN$, that is*

$$X(h(V,W)) = h(\nabla^\varphi_X V, W) + h(V, \nabla^\varphi_X W),$$

for all $X \in \Gamma(TM)$ and V, $W \in \Gamma(\varphi^{-1}TN)$.

Proof. Let $X \in \Gamma(TM), V, W \in \Gamma(\varphi^{-1}TN)$ and $\widetilde{X}, \widetilde{V}, \widetilde{W} \in \Gamma(TN)$, such that

$$d\varphi(X) = \widetilde{X}\circ\varphi, \widetilde{V}\circ\varphi = V \text{ and } \widetilde{W}\circ\varphi = W$$

Then:

$$\begin{aligned}
X(h(V,W)) &= X(h(\widetilde{V}\circ\varphi, \widetilde{W}\circ\varphi)) \\
&= X(h(\widetilde{V}, \widetilde{W})\circ\varphi) \\
&= d(h(\widetilde{V}, \widetilde{W})\circ\varphi)(X) \\
&= dh(\widetilde{V}, \widetilde{W})(d\varphi(X)) \\
&= d\varphi(X)(h(\widetilde{V}, \widetilde{W})) \\
&= \widetilde{X}(h(\widetilde{V}, \widetilde{W}))\circ\varphi \\
&= h(\nabla^N_{\widetilde{X}}\widetilde{V}, \widetilde{W})\circ\varphi + h(\widetilde{V}, \nabla^N_{\widetilde{X}}\widetilde{W})\circ\varphi \\
&= h(\nabla^N_{\widetilde{X}\circ\varphi}\widetilde{V}, \widetilde{W}\circ\varphi) + h(\widetilde{V}\circ\varphi, \nabla^N_{\widetilde{X}\circ\varphi}\widetilde{W}) \\
&= h(\nabla^\varphi_X V, W) + h(V, \nabla^\varphi_X W).
\end{aligned}$$

$\square$

Proposition 1.11.2. *Let ∇^N be a torsion free connection on N, then*

$$\nabla^{\varphi}_X d\varphi(Y) = \nabla^{\varphi}_Y d\varphi(X) + d\varphi([X,Y]),$$

For all $X, Y \in \Gamma(TM)$.

Proof. Let $V, W \in \Gamma(TN)$ be a φ-related with X and Y respectively, then:

$$\begin{aligned} [V,W]\circ\varphi &= d\varphi\circ[X,Y] \\ \nabla^N_V W &= \nabla^N_W V + [V,W]. \end{aligned}$$

From where:

$$\begin{aligned} \nabla^{\varphi}_X d\varphi(Y) &= \nabla^{\varphi}_X W\circ\varphi \\ &= \nabla^N_{d\varphi(X)} W \\ &= \big(\nabla^N_V W\big)\circ\varphi \\ &= \big(\nabla^N_W V + [V,W]\big)\circ\varphi \\ &= \nabla^{\varphi}_Y d\varphi(X) + d\varphi([X,Y]). \end{aligned}$$

□

1.11.1 Divergence Theorem

Proposition 1.11.3. *Let D be a compact domain with boundary in a Riemannian manifold (M,g). Let ω differential a 1-forme and X a vector fields defined on a neighborhood included in D. Then :*

$$\int_D (\operatorname{div}^M \omega) v^D = \int_{\partial D} \omega(\mathfrak{n}) v^{\partial D} \qquad \textit{and} \qquad \int_D (\operatorname{div}^M \omega) v^D = \int_{\partial D} g(X,\mathfrak{n}) v^{\partial D},$$

where ∂D is the boundary of D, and $\mathfrak{n} = \mathfrak{n}(x)$ is the unit vector normal to ∂D.

Corollary 1.11.1. *For all ω a differential 1-form and X a compact supported vector field in a domain D,then :*

$$\int_D (\operatorname{div}\omega) v^D = 0 \qquad \textit{and} \qquad \int_D (\operatorname{div} X) v^D = 0.$$

1.11.2 Green Theorem

Theorem 1.11.1. *Let (M,g) a compact orientable and without boundary Riemannian manifold ($\partial M = \emptyset$). Then, $\forall X \in \Gamma(TM)$, $\forall w \in \Gamma(T^*M)$, we have :*

$$\int_M (\operatorname{div}^M X) v^g = 0, \quad \int_M (\operatorname{div}^M w) v^g = 0,$$

where $v^g = \sqrt{det(g_{ij})} dx^1 \wedge \ldots \wedge dx^m$.

CHAPTER 2

HARMONIC AND BIHARMONIC MAPPINGS

This chapter is devoted to recall briefly the notions of harmonic and biharmonic mappings between pseudo-Riemannian manifolds, integrating them with some more details. We shall follow [3, 8, 10, 11, 18, 22, 28, 31, 32], to which we refer the reader for more details.

2.1 Harmonic maps

Definition 2.1.1. *Let $\varphi : (M^m, g) \to (N^n, h)$ be a smooth map between two Riemannian manifolds, the energy functional of φ is defined by*

$$E(\varphi; D) = \frac{1}{2} \int_D |d\varphi|^2 \, v^g, \tag{2.1}$$

where D is a compact domain in M, $|d\varphi|$ the Hilbert-Schmidt norm of the differential $d\varphi$, and v^g the volume element on (M^m, g)

Remark 2.1.1. *The Hilbert-Schmidt norm of the differential of φ is given by*

$$|d\varphi|^2 = \sum_{i=1}^{m} h(d\varphi(e_i), d\varphi(e_i)),$$

with $\{e_1, \ldots, e_m\}$ be an orthonormal frame, the local expression of the Hilbert-Schmidt norm is :

$$\begin{aligned} |d\varphi|^2 &= \sum_{i=1}^{m} h(d\varphi(e_i), d\varphi(e_i)) \\ &= \sum_{i,a,b=1}^{m} h(d\varphi(e_i^a \frac{\partial}{\partial x^a}), d\varphi(e_i^b \frac{\partial}{\partial x^b})) \end{aligned}$$

$$= \sum_{a,b=1}^{m} g^{ab} \sum_{\alpha,\beta=1}^{n} h\left(\frac{\partial\varphi^\alpha}{\partial x^a}(\frac{\partial}{\partial y^\alpha}\circ\varphi), \frac{\partial\varphi^\beta}{\partial x^b}(\frac{\partial}{\partial y^\beta}\circ\varphi)\right)$$

$$= \sum_{a,b=1}^{m} g^{ab} \sum_{\alpha,\beta=1}^{n} \frac{\partial\varphi^\alpha}{\partial x^a}\frac{\partial\varphi^\beta}{\partial x_b} h\left(\frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial y^\beta}\right)\circ\varphi$$

$$= \sum_{i,j=1}^{m} \sum_{\alpha,\beta=1}^{n} g^{ij} \frac{\partial\varphi^\alpha}{\partial x^i}\frac{\partial\varphi^\beta}{\partial x^j}(h^{\alpha\beta}\circ\varphi).$$

Definition 2.1.2. *A variation of φ with support in a compact domain $D \subset M$, is a smooth family maps $(\varphi_t)_{t\in(-\epsilon,\epsilon)} : M \longrightarrow N$, such that $\varphi_0 = \varphi$ and $\varphi_t = \varphi$ on $M\backslash\mathrm{int}(D)$.*

Definition 2.1.3. *A map φ is called harmonic if it is a critical point of the energy functional over any compact subset D of M. i.e.*

$$\frac{d}{dt}E(\varphi_t; D)\Big|_{t=0} = 0.$$

2.1.1 First variation of energy

Definition 2.1.4. *Let $\varphi : (M, g) \longrightarrow (N, h)$ be a smooth map between two Riemannian manifolds. The trace of the second fundamental form of φ is called the tension field of φ, denoted by*

$$\tau(\varphi) = \mathrm{trace}_g \nabla d\varphi. \tag{2.2}$$

Local expression of tension field

Let a smooth map $\varphi : (M, g) \longrightarrow (N, h)$, we have

$$\tau(\varphi) = \sum_{i,j=1}^{m} g^{ij}(\nabla d\varphi)(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$$

$$= \sum_{i,j=1}^{m}\sum_{\gamma=1}^{n}\left(\frac{\partial^2\varphi_\gamma}{\partial x^i\partial x^j} + \sum_{\alpha,\beta=1}^{n} \frac{\partial\varphi^\alpha}{\partial x^i}\ \frac{\partial\varphi^\beta}{\partial x^j}\ {}^N\Gamma^\gamma_{\alpha\beta}\circ\varphi - \sum_{k=1}^{m} \frac{\partial\varphi^\gamma}{\partial x^k}\ {}^M\Gamma^k_{ij}\right)\frac{\partial}{\partial y^\gamma}\circ\varphi.$$

$(\frac{\partial}{\partial x^i})$ (resp. $(\frac{\partial}{\partial y^\alpha})$) is a local frame of vector fields on M (resp. on N)

Theorem 2.1.1. *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map and let $(\varphi_t)_{t\in(-\epsilon,\epsilon)}$ be a smooth variation of φ supported in D. Then*

$$\frac{d}{dt}E(\varphi_t; D)\Big|_{t=0} = -\int_D h(v, \tau(\varphi))\, v^g,$$

where $v = \frac{d\varphi_t}{dt}\Big|_{t=0}$ denotes the variation vector field of $\{\varphi_t\}$, and v^g the volume element on (M^m, g)

Proof. Let $\{e_1,\dots e_m\}$ an orthonormal frame on (M,g) and $\dfrac{d}{dt}$ a frame of vector field on $]-\varepsilon,\varepsilon[$. Thus, $\{(e_i,0),(0,\dfrac{d}{dt})\}_{i=1}^{m}$ becomes an orthonormal frame fields for the diagonal metric $g+dt^2$ on the product manifold $M\times]-\varepsilon,\varepsilon[$. We have

$$\left[(e_i,0),(0,\frac{d}{dt})\right]=0,\quad \forall i\in\{1,\dots,m\}.$$

Define

$$\begin{aligned}\phi: M\times]-\varepsilon,\varepsilon[&\longrightarrow N.\\ (x,t) &\longmapsto \phi(x,t)=\phi_t(x)\end{aligned}$$

By the Leibniz's formula, and :

$$\begin{aligned}\phi_x :]-\varepsilon,\varepsilon[&\longrightarrow N;\\ t &\longmapsto \phi_x(t)=\phi(x,t)=\varphi_t(x)\end{aligned}$$

$$\begin{aligned}\phi_t : M &\longrightarrow N.\\ x &\longmapsto \phi_t(x)=\phi(x,t)=\varphi_t(x)\end{aligned}$$

We get that :

$$\begin{aligned}d\phi(e_i,0)_{(x,0)} &= d_x\phi_0(e_i|_x)+d_0\phi_x(0)\\ &= d_x\phi_0(e_i|_x)\\ &= d_x\varphi(e_i|_x);\end{aligned}$$

$$\begin{aligned}d\phi(0,\frac{d}{dt})_{(x,0)} &= d_x\phi_0(0)+d_0\phi_x(\frac{d}{dt}|_{t=0})\\ &= d\phi_x(\frac{d}{dt}|_{t=0})\\ &= v(x).\end{aligned}$$

Thus:

$$d\phi(e_i,0)=d\varphi(e_i)\quad \text{et}\quad d\phi(0,\frac{d}{dt})=v\quad \text{en}\quad t=0.$$

Let ∇^{ϕ} be the Pull-back connection associated with the map ϕ we compute:

$$\begin{aligned}\frac{d}{dt}E(\varphi_t;D)\bigg|_{t=0} &= \frac{1}{2}\frac{d}{dt}\int_D |d\varphi_t|^2 v^g\bigg|_{t=0}\\ &= \frac{1}{2}\int_D \frac{\partial}{\partial t}|d\varphi_t|^2\bigg|_{t=0} v^g\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\int_D \frac{\partial}{\partial t}\sum_{i=1}^m h(d\varphi_t(e_i), d\varphi_t(e_i))\Big|_{t=0} v^g \\
&= \int_D \sum_{i=1}^m h(\nabla^\phi_{(0,\frac{d}{dt})} d\phi(e_i,0), d\phi(e_i,0))\Big|_{t=0} v^g \\
&= \int_D \sum_{i=1}^m h(\nabla^\phi_{(0,e_i)} d\phi(0,\frac{d}{dt}), d\phi(e_i,0))\Big|_{t=0} v^g \\
&= \int_D \sum_{i=1}^m h(\nabla^N_{d\varphi(e_i)} v, d\varphi(e_i)) v^g \\
&= \int_D \sum_{i=1}^m h(\nabla^\varphi_{e_i} v, d\varphi(e_i))\, v^g \\
&= \int_D \sum_{i=1}^m \big[e_i\, h(v, d\varphi(e_i)) - h(v, \nabla^\varphi_{e_i} d\varphi(e_i))\big]\, v^g. \qquad (2.3)
\end{aligned}
$$

Define a 1-form ω with support in D by

$$\omega(X) = h(v, d\varphi(X)), \quad \forall X \in \Gamma(TM).$$

we find:

$$
\begin{aligned}
\operatorname{div}^M \omega &= \sum_{i=1}^m (\nabla_{e_i}\omega)(e_i) \\
&= \sum_{i=1}^m \{e_i(\omega(e_i)) - \omega(\nabla^M_{e_i} e_i)\} \\
&= \sum_{i=1}^m \{e_i h(v, d\varphi(e_i)) - h(v, d\varphi(\nabla^M_{e_i} e_i))\}. \qquad (2.4)
\end{aligned}
$$

By using formulas (2.3) and (2.4) we have :

$$\frac{d}{dt}E(\varphi_t, D)\Big|_{t=0} = \int_D (\operatorname{div}\omega)\, v^g - \int_D h(v, \tau(\varphi)).$$

By the divergence theorem, we obtain

$$\frac{d}{dt}E(\varphi_t; D)\Big|_{t=0} = -\int_D h(v, \tau(\varphi)) v^g,$$

this completes the proof. □

Therefore

Theorem 2.1.2. *The map $\varphi \in C^\infty(M,N)$ between two Riemannian manifolds is harmonic if and only if $\tau(\varphi) = \mathrm{trace}\nabla d\varphi = 0$.*

Example 2.1.1. *Let $\varphi : (\mathbb{R}^m, <\cdot,\cdot>) \longrightarrow (\mathbb{R}^n, <\cdot,\cdot>)$ with*

$$\varphi(X) = \varphi\begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^m \end{pmatrix} = \begin{pmatrix} A_1^1 & A_2^1 & \cdots & A_m^1 \\ A_1^2 & A_2^2 & \cdots & A_m^2 \\ A_1^3 & A_2^3 & \cdots & A_m^3 \\ \vdots & \vdots & \ddots & \vdots \\ A_1^n & A_2^n & \cdots & A_m^n \end{pmatrix}\begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^m \end{pmatrix},$$

i.e., $\varphi(X) = (A^1X^t, A^2X^t, \cdots, A^nX^t)$ be a linear map, where A^i denotes the row vectors of the representation matrix. $(\mathbb{R}^m, <\cdot,\cdot>)$ (resp. $(\mathbb{R}^n, <\cdot,\cdot>)$) denotes the space $\mathbb{R}^m$ (resp. $(\mathbb{R}^n, <\cdot,\cdot>)$) equipped with the standard inner product. Then φ is harmonic, in fact: as φ is a linear map, we have:

$$\begin{aligned} d\varphi(e_i) &= d\varphi(\frac{\partial}{\partial x^i}) \\ &= A_i^j \frac{\partial}{\partial x^j}. \end{aligned}$$

Therefore

$$\begin{aligned} \tau(\varphi) &= \sum_{i=1}^m \{\nabla^\varphi_{e_i} d\varphi(e_i) - d\varphi(\nabla^{\mathbb{R}^m}_{e_i} e_i)\} \\ &= \sum_{i=1}^m \nabla^\varphi_{e_i} A_i^j \frac{\partial}{\partial x^j} \\ &= 0. \end{aligned}$$

2.1.2 Second variation of energy

Theorem 2.1.3. *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ a harmonic map and D a compact domain of M, if $\{\varphi_{t,s}\}$ is a variation of φ with two parameters with compact support in D, then*

$$\frac{\partial^2}{\partial t \partial s} E(\varphi_{t,s}; D)\Big|_{(t,s)=(0,0)} = \int_D h(\Delta^\varphi V - \mathrm{trace}\ R^N(V, d\varphi)d\varphi, W)\, v^g,$$

where $V = \frac{\partial \varphi_{t,s}}{\partial t}\Big|_{(t,s)=(0,0)}$ and $W = \frac{\partial \varphi_{t,s}}{\partial s}\Big|_{(t,s)=(0,0)}$ denotes the variation vector fields. Here $\Delta^\varphi V = -tr_g(\nabla^\varphi)^2 V = -\sum_{i=1}^m (\nabla^\varphi_{e_i}\nabla^\varphi_{e_i} V - \nabla^\varphi_{\nabla^M_{e_i} e_i} V)$ is the rough Laplacian on $\varphi^{-1}TN$, and R^N is the Riemann curvature tensor of N.

Proof. Let $\{e_1, \dots, e_m\}$ an orthonormal frame on (M^m, g). We put :

$$\begin{array}{rcccc} \phi & : & M \times (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) & \longrightarrow & N, \\ & & (x, t, s) & \longmapsto & \varphi_{t,s}(x) \end{array}$$

$$E_i = (e_i, 0, 0),\ \frac{\partial}{\partial t} = (0, \frac{d}{dt}, 0)\ et\ \frac{\partial}{\partial s} = (0, 0, \frac{d}{ds}).$$

Then :

$$\frac{\partial^2}{\partial t \partial s} E(\varphi_{t,s}; D)\Big|_{(t,s)=(0,0)} = \frac{1}{2} \int_D \sum_{i=1}^m \frac{\partial^2}{\partial t \partial s} h(d\phi(E_i), d\phi(E_i))\, v^g \Big|_{(t,s)=(0,0)}, \quad (2.5)$$

$$\begin{aligned} \frac{1}{2} \frac{\partial^2}{\partial t \partial s} h(d\phi(E_i), d\phi(E_i)) &= \frac{\partial}{\partial t} h(\nabla^\phi_{\frac{\partial}{\partial s}} d\phi(E_i), d\phi(E_i)) \\ &= h(\nabla^\phi_{\frac{\partial}{\partial t}} \nabla^\phi_{\frac{\partial}{\partial s}} d\phi(E_i), d\phi(E_i)) \\ &\quad + h(\nabla^\phi_{\frac{\partial}{\partial s}} d\phi(E_i), \nabla^\phi_{\frac{\partial}{\partial t}} d\phi(E_i)), \end{aligned} \quad (2.6)$$

and :

$$\begin{aligned} h(\nabla^\phi_{\frac{\partial}{\partial t}} \nabla^\phi_{\frac{\partial}{\partial s}} d\phi(E_i), d\phi(E_i)) &= h(\nabla^\phi_{\frac{\partial}{\partial t}} \nabla^\phi_{E_i} d\phi(\frac{\partial}{\partial s}), d\phi(E_i)) \\ &= h(R^N(d\phi(\frac{\partial}{\partial t}), d\phi(E_i)) d\phi(\frac{\partial}{\partial s}), d\phi(E_i)) \\ &\quad h(\nabla^\phi_{E_i} \nabla^\phi_{\frac{\partial}{\partial t}} d\phi(\frac{\partial}{\partial s}), d\phi(E_i)) \\ &\quad + h(\nabla^\phi_{[\frac{\partial}{\partial t}, E_i]} d\phi(\frac{\partial}{\partial s}), d\phi(E_i)). \end{aligned} \quad (2.7)$$

Define an $1-$form ω with support in D, by :

$$\omega(X) = h(\nabla^\phi_{\frac{\partial}{\partial t}} d\phi(\frac{\partial}{\partial s})\Big|_{(t,s)=(0,0)}, d\varphi(X)), \quad X \in \Gamma(TM).$$

As φ is a harmonic map, we get:

$$\begin{aligned} \mathrm{div}^M \omega &= \sum_{i=1}^m \{e_i(\omega(e_i)) - \omega(\nabla^M_{e_i} e_i)\} \\ &= \sum_{i=1}^m \{e_i(h(\nabla^\phi_{\frac{\partial}{\partial t}} d\phi(\frac{\partial}{\partial s})\Big|_{(t,s)=(0,0)}, d\varphi(e_i))) - h(\nabla^\phi_{\frac{\partial}{\partial t}} d\phi(\frac{\partial}{\partial s})\Big|_{(t,s)=(0,0)}, d\varphi(\nabla^M_{e_i} e_i))\} \\ &= \sum_{i=1}^m \{h(\nabla^\phi_{E_i} \nabla^\phi_{\frac{\partial}{\partial t}} d\phi(\frac{\partial}{\partial s})\Big|_{(t,s)=(0,0)}, d\varphi(e_i)) \end{aligned}$$

$$
\begin{aligned}
& +h(\nabla^{\phi}_{\frac{\partial}{\partial t}} d\phi(\frac{\partial}{\partial s})\Big|_{(t,s)=(0,0)}, \nabla^{\varphi}_{e_i} d\varphi(e_i)) - h(\nabla^{\phi}_{\frac{\partial}{\partial t}} d\phi(\frac{\partial}{\partial s})\Big|_{(t,s)=(0,0)}, d\varphi(\nabla^{M}_{e_i} e_i))\} \\
= & \sum_{i=1}^{m}\{h(\nabla^{\phi}_{E_i}\nabla^{\phi}_{\frac{\partial}{\partial t}} d\phi(\frac{\partial}{\partial s})\Big|_{(t,s)=(0,0)}, d\varphi(e_i)) + h(\nabla^{\phi}_{\frac{\partial}{\partial t}} d\phi(\frac{\partial}{\partial s})\Big|_{(t,s)=(0,0)}, \tau(\varphi)) \\
= & \sum_{i=1}^{m}\{h(\nabla^{\phi}_{E_i}\nabla^{\phi}_{\frac{\partial}{\partial t}} d\phi(\frac{\partial}{\partial s})\Big|_{(t,s)=(0,0)}, d\varphi(e_i)) \qquad (2.8)
\end{aligned}
$$

By (2.7) and (2.8), and since $\left[\frac{\partial}{\partial t}, e_i\right] = 0$, we get :

$$
\begin{aligned}
h(\nabla^{\phi}_{\frac{\partial}{\partial t}}\nabla^{\phi}_{\frac{\partial}{\partial s}} d\phi(E_i), d\phi(E_i))\Big|_{(t,s)=(0,0)} &= \sum_{i=1}^{m} h(R^N(V, d\varphi(e_i))W, d\varphi(e_i)) \\
&\quad +\mathrm{div}^M\omega. \qquad (2.9)
\end{aligned}
$$

The second term of the right hand side of the equality (2.6) is given by

$$
\begin{aligned}
h(\nabla^{\phi}_{\frac{\partial}{\partial s}} d\phi(E_i), \nabla^{\phi}_{\frac{\partial}{\partial t}} d\phi(E_i)) &= h(\nabla^{\phi}_{E_i} d\phi(\frac{\partial}{\partial s}), \nabla^{\phi}_{E_i} d\phi(\frac{\partial}{\partial t})) \\
&= e_i\left(h(d\phi(\frac{\partial}{\partial s}), d\phi(\frac{\partial}{\partial t}))\right) \\
&\quad -h(d\phi(\frac{\partial}{\partial s}), \nabla^{\phi}_{E_i}\nabla^{\phi}_{E_i} d\phi(\frac{\partial}{\partial t})). \qquad (2.10)
\end{aligned}
$$

if η is an 1-form with support in D, defined by

$$
\eta(X) = h(W, \nabla^{\varphi}_X V), \quad X \in \Gamma(TM).
$$

$$
\begin{aligned}
\Longrightarrow \mathrm{div}^M\eta &= \sum_{i=1}^{m}\{e_i(\eta(e_i)) - \eta(\nabla^M_{e_i} e_i)\} \\
&= \sum_{i=1}^{m}\{e_i(h(W, \nabla^{\varphi}_{e_i} V)) - h(W, \nabla^{\varphi}_{\nabla^M_{e_i} e_i} V)\}. \qquad (2.11)
\end{aligned}
$$

By using (2.10) and (2.11), we obtain

$$
\begin{aligned}
\sum_{i=1}^{m} h(\nabla^{\phi}_{\frac{\partial}{\partial t}} d\phi(E_i), \nabla^{\phi}_{\frac{\partial}{\partial s}} d\phi(E_i))\Big|_{(t,s)=(0,0)} &= \mathrm{div}^M\eta + \sum_{i=1}^{m} h(W, \nabla^{\varphi}_{\nabla^M_{e_i} e_i} V) \\
&\quad -\sum_{i=1}^{m} h(W, \nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i} V). \qquad (2.12)
\end{aligned}
$$

From the equations (2.5), (2.6), (2.9), (2.17), and the divergence theorem we find,

$$
\frac{\partial^2}{\partial t\partial s} E(\varphi_{t,s}; D)\Big|_{(t,s)=(0,0)} = \int_D \sum_{i=1}^{m}\left\{-h(R^N(V, d\varphi(e_i))d\varphi(e_i), W)\right.
$$

$$+ h(W, \nabla^{\varphi}_{\nabla^{M}_{e_i} e_i} V) - h(W, \nabla^{\varphi}_{e_i} \nabla^{\varphi}_{e_i} V) \Big\} \, v^g.$$

Therefore

$$\frac{\partial^2}{\partial t \partial s} E(\varphi_{t,s}; D) \Big|_{(t,s)=(0,0)} = \int_D h(\Delta^{\varphi} V - \text{trace } R^N(V, d\varphi) d\varphi, W) \, v^g,$$

□

2.2 Biharmonic maps

Definition 2.2.1. *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map between two Riemannian manifolds, and D a compact domain in M. The bienergy functional of φ on D is defined by*

$$\begin{array}{rccc} E_2 & : & C^{\infty}(M, N) & \longrightarrow & \mathbb{R}_+, \\ & & \varphi & \longmapsto & E_2(\varphi; D) = \dfrac{1}{2} \displaystyle\int_D |\tau(\varphi)|^2 v^g \end{array}$$

où $|\tau(\varphi)|^2 = h(\tau(\varphi), \tau(\varphi))$, and $\tau(\varphi)$ is the tension field of the map φ.

Definition 2.2.2. *The smooth map $\varphi : (M^m, g) \longrightarrow (N^n, h)$ between two Riemannian manifold is called biharmonic map if it is a critical point of the bienergy functional over any compact subset D of M.*

$$\frac{d}{dt} E_2(\varphi_t; D) \Big|_{t=0} = 0, \tag{2.13}$$

here $\{\varphi_t\}$ is a variation of φ with compact support in D.

2.2.1 The first variation of the bienergy

Theorem 2.2.1. *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map between two Riemannian manifolds, and $\{\varphi_t\}$ a smooth variation of φ with support in D. Then*

$$\frac{d}{dt} E_2(\varphi_t; D) \Big|_{t=0} = - \int_D h(v, \tau_2(\varphi)) v^g,$$

where $v = \frac{d\varphi_t}{dt}\big|_{t=0}$ is the field of variation associated with $\{\varphi_t\}$, and $\tau_2(\varphi) \in \Gamma(\varphi^{-1}TN)$ is called the bitension field defined by

$$\begin{aligned} \tau_2(\varphi) &= -\sum_{i=1}^{m} \{\nabla^{\varphi}_{e_i} \nabla^{\varphi}_{e_i} \tau(\varphi) - \nabla^{\varphi}_{\nabla^{M}_{e_i} e_i} \tau(\varphi)\} - \sum_{i=1}^{m} R^N(\tau(\varphi), d\varphi(e_i)) d\varphi(e_i) \\ &= \Delta^{\varphi} \tau(\varphi) - \sum_{i=1}^{m} R^N(\tau(\varphi), d\varphi(e_i)) d\varphi(e_i). \end{aligned}$$

Proof. Let $\phi : M \times (-\epsilon, \epsilon) \longrightarrow N$ a map defined by $\phi(x,t) = \varphi_t(x)$. Then

$$\frac{d}{dt}E_2(\varphi_t; D)\Big|_{t=0} = \int_D \sum_{i,j=1}^m h(\nabla^\phi_{(0,\frac{d}{dt})}\nabla d\phi((e_i,0),(e_i,0)), \nabla d\phi((e_j,0),(e_j,0)))v^g\Big|_{t=0}. \tag{2.14}$$

As

$$[(0,\frac{d}{dt}),(e_i,0)] = 0.$$

We have

$$\nabla^\phi_{(0,\frac{d}{dt})} d\phi(e_i,0) = \nabla^\phi_{(e_i,0)} d\phi(0,\frac{d}{dt}), \tag{2.15}$$

also

$$\nabla^\phi_{(0,\frac{d}{dt})} d\phi(\nabla^M_{e_i}e_i,0) = \nabla^\phi_{(\nabla^M_{e_i}e_i,0)} d\phi(0,\frac{d}{dt}). \tag{2.16}$$

We compute

$$\begin{aligned}
\nabla^\phi_{(0,\frac{d}{dt})}\nabla d\phi((e_i,0),(e_i,0)) &= \nabla^\phi_{(0,\frac{d}{dt})}\{\nabla^\phi_{(e_i,0)} d\phi(e_i,0) - d\phi(\nabla^{M\times(-\epsilon,\epsilon)}_{(e_i,0)}(e_i,0))\} \\
&= \nabla^\phi_{(0,\frac{d}{dt})}\nabla^\phi_{(e_i,0)} d\phi(e_i,0) - \nabla^\phi_{(0,\frac{d}{dt})} d\phi(\nabla^{M\times(-\epsilon,\epsilon)}_{(e_i,0)}(e_i,0)) \\
&= R^N(d\phi(0,\frac{d}{dt}), d\phi(e_i,0))d\phi(e_i,0) + \nabla^\phi_{(e_i,0)}\nabla^\phi_{(0,\frac{d}{dt})} d\phi(e_i,0) \\
&\quad + \nabla^\phi_{[(0,\frac{d}{dt}),(e_i,0)]} d\phi(e_i,0) - \nabla^\phi_{(0,\frac{d}{dt})} d\phi(\nabla^M_{e_i}e_i,0) \\
&= R^N(d\phi(0,\frac{d}{dt}), d\phi(e_i,0))d\phi(e_i,0) + \nabla^\phi_{(e_i,0)}\nabla^\phi_{(e_i,0)} d\phi(0,\frac{d}{dt}) \\
&\quad - \nabla^\phi_{(\nabla^M_{e_i}e_i,0)} d\phi(0,\frac{d}{dt}).
\end{aligned}$$

Thus

$$\begin{aligned}
&\sum_{i,j=1}^m h(\nabla^\phi_{(0,\frac{d}{dt})}\nabla d\phi((e_i,0),(e_i,0)), \nabla d\phi((e_j,0),(e_j,0)))\Big|_{t=0} \\
&= \sum_{i=1}^m h(R^N(v, d\varphi(e_i))d\varphi(e_i) + \nabla^\varphi_{e_i}\nabla^\varphi_{e_i}v - \nabla^\varphi_{\nabla^M_{e_i}e_i}v, \tau(\varphi)).
\end{aligned} \tag{2.17}$$

Let $w \in \Gamma(T^*M)$ be a 1-form with support in D defined by

$$w(X) = h(\nabla^\varphi_X v, \tau(\varphi)), \forall X \in \Gamma(TM).$$

$$\begin{aligned}
\Longrightarrow \operatorname{div}^M w &= \sum_{i=1}^m \{e_i(w(e_i)) - w(\nabla^M_{e_i}e_i)\} \\
&= \sum_{i=1}^m \{e_i(h(\nabla^\varphi_{e_i}v, \tau(\varphi))) - h(\nabla^\varphi_{\nabla^M_{e_i}e_i}v, \tau(\varphi))\}
\end{aligned} \tag{2.18}$$

$$= \sum_{i=1}^{m}\{h(\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}v,\tau(\varphi)) + h(\nabla^{\varphi}_{e_i}v,\nabla^{\varphi}_{e_i}\tau(\varphi)) - h(\nabla^{\varphi}_{\nabla^M_{e_i}e_i}v,\tau(\varphi))\}.$$

From (2.17) and (2.18), we obtain

$$\sum_{i,j=1}^{m} h(\nabla^{\phi}_{(0,\frac{d}{dt})}\nabla d\phi((e_i,0),(e_i,0)),\nabla d\phi((e_j,0),(e_j,0)))v^g\Big|_{t=0}$$
$$= \sum_{i=1}^{m} h(R^N(v,d\varphi(e_i))d\varphi(e_i),\tau(\varphi)) + \mathrm{div}^M w - \sum_{i=1}^{m} h(\nabla^{\varphi}_{e_i}v,\nabla^{\varphi}_{e_i}\tau(\varphi)). \tag{2.19}$$

Also let $\eta \in \Gamma(T^*M)$ be an 1-form to support in D defined by

$$\eta(X) = h(v,\nabla^{\varphi}_{X}\tau(\varphi)),\ \forall X \in \Gamma(TM).$$

$$\begin{aligned}\Longrightarrow \mathrm{div}^M\eta &= \sum_{i=1}^{m}\{e_i(\eta(e_i)) - \eta(\nabla^M_{e_i}e_i)\}\\ &= \sum_{i=1}^{m}\{e_i(h(v,\nabla^{\varphi}_{e_i}\tau(\varphi)) - h(v,\nabla^{\varphi}_{\nabla^M_{e_i}e_i}\tau(\varphi)))\} \qquad (2.20)\\ &= \sum_{i=1}^{m}\{h(\nabla^{\varphi}_{e_i}v,\nabla^{\varphi}_{e_i}\tau(\varphi)) + h(v,\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}\tau(\varphi)) - h(v,\nabla^{\varphi}_{\nabla^M_{e_i}e_i}\tau(\varphi))\}.\end{aligned}$$

Replacing (2.20) in (2.19), we get

$$\sum_{i,j=1}^{m} h(\nabla^{\phi}_{(0,\frac{d}{dt})}\nabla d\phi((e_i,0),(e_i,0)),\nabla d\phi((e_j,0),(e_j,0)))v^g\Big|_{t=0}$$
$$= \sum_{i=1}^{m} h(R^N\tau(\varphi),d\varphi(e_i))d\varphi(e_i),v) + \mathrm{div}^M w - \mathrm{div}^M\eta + h(v,\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}\tau(\varphi))$$
$$-h(v,\nabla^{\varphi}_{\nabla^M_{e_i}e_i}\tau(\varphi)). \tag{2.21}$$

From (2.14), (2.21), and the divergence theorem, we obtain

$$\frac{d}{dt}E_2(\varphi_t;D)\Big|_{t=0} = -\int_D \sum_{i=1}^{m} h(-R^N(\tau(\varphi),d\varphi(e_i))d\varphi(e_i) - \nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}\tau(\varphi)$$
$$+\nabla^{\varphi}_{\nabla^M_{e_i}e_i}\tau(\varphi),v)v^g.$$

Then

$$\frac{d}{dt}E_2(\varphi_t;D)\Big|_{t=0} = -\int_D h(v,\tau_2(\varphi))v^g,$$

where

$$\tau_2(\varphi) = \Delta^{\varphi}\tau(\varphi) - \sum_{i=1}^{m} R^N(\tau(\varphi), d\varphi(e_i))d\varphi(e_i)$$

□

Thus, we deduce

Theorem 2.2.2. *The map $\varphi \in C^{\infty}(M, N)$ between two Riemannian manifolds is biharmonic if and only if*

$$\tau_2(\varphi) = \Delta^{\varphi}\tau(\varphi) - \sum_{i=1}^{m} R^N(\tau(\varphi), d\varphi(e_i))d\varphi(e_i) = 0. \tag{2.22}$$

Remark 2.2.1. *The equation* (2.22) *is called the Euler-Lagrange equation associated with the bienergy functional.*

2.3 Harmonic and biharmonic maps between pseudo-Riemannian manifolds

The generalization of the concepts of harmonic and biharmonic maps between Riemannian manifolds to the case of pseudo-Riemannian manifolds is straightforward.

Let (M^m, g) and (N^n, h) be smooth pseudo-Riemannian manifolds, and let $\varphi : (M, g) \to (N, h)$ be a smooth map between them. The energy functional or the Dirichlet energy of φ over a compact domain D of M is defined by

$$E(\varphi, D) = \frac{1}{2}\int_D \sum_{i=1}^{m} \varepsilon_i h(d\varphi(e_i), d\varphi(e_i))v_g, \tag{2.23}$$

where $\{e_i\}_{i=1}^m$ is a local pseudo-orthonormal frame field of (M^m, g) with $\varepsilon_i = g(e_i, e_i) = \pm 1$ for all indices $i = 1, 2, \cdots, m$. If M is compact, we write $E(\varphi) = E(\varphi, M)$. The map φ is called harmonic if it is a critical point of the energy functional (2.23). The Euler-Lagrange equation of (2.23) is [3, 11]

$$\tau(\varphi) = \mathrm{Tr}_g(\nabla d\varphi) = \sum_{i=1}^{m} \varepsilon_i \{\nabla^{\varphi}_{e_i} d\varphi(e_i) - d\varphi(\nabla_{e_i} e_i)\} = 0.$$

The notion of biharmonic map between Riemannian manifolds has been extended to the case of pseudo-Riemannian manifolds as follows [8]:

Definition 2.3.1. *A map $\varphi : (M^m, g) \to (N^n, h)$ between pseudo-Riemannian manifolds is a biharmonic map if its bitension field vanishes identically, i.e.,*

$$\tau_2(\varphi) = \sum_{i=1}^{m} \varepsilon_i \Big(\big(\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i} - \nabla^{\varphi}_{\nabla^M_{e_i} e_i}\big)\tau(\varphi) - R^N(d\varphi(e_i), \tau(\varphi))d\varphi(e_i) \Big) = 0,$$

where $\{e_i\}_{i=1}^m$ *is a local pseudo-orthonormal frame field of* (M^m, g) *with* $\varepsilon_i = g(e_i, e_i) = \pm 1$ *for all indices* $i = 1, 2, \cdots, m$ *and* R^N *is the curvature tensor of* N*.*

Remark 2.3.1. *Note that the only difference of the tension (and the bitension) fields between the Riemannian and the pseudo- Riemannian cases lies in the definition of the trace of a bilinear form in these two different cases.*

Example 2.3.1. *Any harmonic mapping is trivially biharmonic. However, a biharmonic map can be non-harmonic in which case it is called proper biharmonic. We refer to [3, 8, 28, 31] for more examples on harmonic maps and biharmonic maps.*

CHAPTER 3

BIHARMONICITY OF VECTOR FIELDS ON RIEMANNIAN MANIFOLDS

This chapter presents our work on the biharmonicity of vector fields on Riemannian manifolds. We compute the expression of the bitension field of a vector field considered as a map from a Riemannian manifold (M, g) to its tangent bundle TM equipped with the Sasaki metric g_S. As a consequence, we show characterization theorem for a vector field to be biharmonic map. Moreover, we prove non-existence results for left-invariant vector fields which are biharmonic without being harmonic maps and non-harmonic biharmonic maps respectively on unimodular Lie groups of dimension three. The references used are: [1], [9],[10], [11], [18], [23], [25], [13], [14].

3.1 Harmonicity of vector fields on Riemannian manifolds

A vector field X on (M, g) can be viewed as the immersion $X : (M, g) \to (TM, g_S) : x \mapsto (x, X_x) \in TM$ into its tangent bundle TM equipped with the Sasaki metric g_S. If $Y \in \Gamma(TM)$ then, we have (see [9, pp. 50])

$$dX(Y) = \{Y^h + (\nabla_Y X)^v\} \circ X. \tag{3.1}$$

Theorem 3.1.1. *[13] Let (M, g) be a Riemannian manifold of dimension m and (TM, g_S) its tangent bundle equipped with the Sasaki metric, if $X : (M, g) \to (TM, g_S)$ is a smooth vector field then the tension field $\tau(X)$ is given by*

$$\tau(X) = (-S(X))^h + (-\bar{\Delta} X)^v, \tag{3.2}$$

where

$$S(X) = \sum_{i=1}^{m} R(\nabla_{e_i} X, X)e_i$$

and $\bar{\Delta}X$ is the rough Laplacian given by

$$\bar{\Delta}X = -tr_g(\nabla^2 X) = \sum_{i=1}^{m}(\nabla_{\nabla_{e_i}e_i}X - \nabla_{e_i}\nabla_{e_i}X).$$

Proof. Let $(x,u) \in TM$ and $\{e_i\}_{i=1}^m$ be a local orthonormal frame on M such that $\nabla_{e_i}e_j = 0$ at $x \in M$ and $u = X_x$, from the Proposition 1.9.1 and (3.1), we get

$$\begin{aligned}
\tau(X)\big|_{(x,X_x)} =& \sum_{i=1}^{m} \big[\nabla^X_{e_i} dX(e_i)\big]\big|_{(x,X_x)} \\
=& \sum_{i=1}^{m} \big[\widetilde{\nabla}_{e_i^h+(\nabla_{e_i}X)^v}(e_i^h + (\nabla_{e_i}X)^v)\big]\big|_{(x,X_x)} \\
=& \sum_{i=1}^{m} \big[\widetilde{\nabla}_{e_i^h} e_i^h + \widetilde{\nabla}_{e_i^h}(\nabla_{e_i}X)^v + \widetilde{\nabla}_{(\nabla_{e_i}X)^v} e_i^h + +\widetilde{\nabla}_{(\nabla_{e_i}X)^v}(\nabla_{e_i}X)^v\big]\big|_{(x,X_x)} \\
=& \sum_{i=1}^{m} \big[(-R(\nabla_{e_i}X, X)e_i)^h + (\nabla_{e_i}\nabla_{e_i}X)^v\big]\big|_{(x,X_x)} \\
=& \big[(-S(X))^h + (-\bar{\Delta}X)^v\big]\big|_{(x,X_x)}.
\end{aligned}$$

□

Theorem 3.1.2. *[17] Let (M,g) be a Riemannian manifold of dimension m and (TM, g_S) its tangent bundle equipped with the Sasaki metric, if $X : (M,g) \to (TM, g_S)$ is a smooth vector field then X is a harmonic map if and only if $\bar{\Delta}X = 0$ and $S(X) = 0$.*

Note that, for any smooth function f and vector field X of M, we have

$$S(fX) = f^2 S(X). \tag{3.3}$$

Definition 3.1.1 ([13])**.** *A vector field X is called harmonic vector field if it is a critical point of the energy functional (2.1), restricted to variations through vector fields.*

Theorem 3.1.3. *Let (M,g) be a compact oriented m-dimensional Riemannian manifold, $\{e_i\}_{i=1}^m$ a local orthonormal frame field of (M,g), X a tangent vector field on M and $E : \mathfrak{X}(M) \longrightarrow [0,+\infty)$ the energy functional restricted to the space of all vector fields. Then*

$$\frac{d}{dt}E(X_t)\bigg|_{t=0} = \int_M g(\bar{\Delta}X, V)v_g,$$

for any smooth 1-parameter variation $U : M \times (-\epsilon,\epsilon) \to TM$ of X through vector fields i.e., $X_t(z) = U(z,t) \in T_zM$ for any $|t| < \epsilon$ and $z \in M$, or equivalently $X_t \in \mathfrak{X}(M)$ for any $|t| < \epsilon$. Also, V is the tangent vector field on M given by

$$V(z) = \frac{d}{dt}X_z(0), \quad z \in M,$$

where $X_z(t) = U(z,t)$, $(z,t) \in M \times (-\epsilon,\epsilon)$.

Proof. Let $U : M \times (-\epsilon, \epsilon) \to TM$ be a smooth variation of X(i.e., $U(z,0) = X(z)$ for any $z \in M$) such that $X_t(z) = U(z,t) \in T_zM$ for any $z \in M$ and any $|t| < \epsilon$. We have

$$E(X_t) = \frac{1}{2}\int_M |dX_t|^2 v_g.$$

Then, from [10], we get

$$\frac{d}{dt}E(X_t)\Big|_{t=0} = -\int_M g_S(\mathcal{V}, \tau(X))v_g,$$

where $\mathcal{V}(z) = \frac{d}{dt}X_t(z)\big|_{t=0}$, $z \in M$, and from [9, pp. 58], we have

$$\mathcal{V} = V^v \circ X. \tag{3.4}$$

Taking into account (4.4) and the expression of $\tau(X)$ given by (3.2), we find

$$\begin{aligned}\frac{d}{dt}E(X_t)\Big|_{t=0} &= -\int_M g_S(V^v, \tau(X))v_g,\\ &= \int_M g(V, \bar{\Delta}X)v_g,\end{aligned}$$

as required. □

Then, we deduce the following [13].

Corollary 3.1.1. *A vector field X of an m-dimensional Riemannian manifold (M, g) is harmonic if and only if*

$$\bar{\Delta}X = 0, \tag{3.5}$$

where $\{e_i\}_{i=1}^m$ is a local orthonormal frame field of (M, g).

Remark 3.1.1. *Theorem 3.1.3 holds if (M, g) is a non-compact Riemannian manifold see [23].*

Combining Theorem 3.1.1 and Corollary 3.1.1, we get

Corollary 3.1.2. *A vector field X of an m-dimensional Riemannian manifold (M, g) is harmonic map if and only if X is harmonic vector field and $S(X) = 0$.*

Theorem 3.1.4. *Let (M, g) be a compact m-dimensional Riemannian manifold and $X \in \mathfrak{X}(M)$ a vector field. Then $X : (M, g) \longrightarrow (TM, g_S)$ is a harmonic vector field if and only if X is parallel.*

Proof. We assume that $X : (M, g) \longrightarrow (TM, g_S)$ is parallel, then from Corollary 3.1.1, X is a harmonic vector field. Conversely, we assume that $X : (M, g) \longrightarrow (TM, g_S)$ is a harmonic vector field, then $X \in \mathfrak{X}(M)$ be a critical point of $E \mid_{\mathfrak{X}(M)}$. Then $\frac{d}{dt}E(X_t)|_{t=0} = 0$ for any smooth 1-parameter variation $\{X_t\}_{|t|<\epsilon}$ of X. In particular we may consider the variation

$$X_t(x) = (1+t)X, \quad x \in M, \quad |t| < \epsilon,$$

hence

$$\begin{aligned} 0 = \frac{d}{dt}E(X_t)\bigg|_{t=0} &= \frac{d}{dt}\Big\{\frac{m}{2}Vol(M) + \frac{1}{2}\int_M |\nabla X_t|^2 v_g\Big\}_{t=0} \\ &= \frac{d}{dt}\Big\{\frac{(1+t)^2}{2}\int_M |\nabla X_t|^2 v_g\Big\}_{t=0} \\ &= \int_M |\nabla X|^2 v_g. \end{aligned}$$

Thus $\nabla X = 0$, i.e X is parallel. □

Theorem 3.1.5. *Let (M, g) be a compact m-dimensional Riemannian manifold and $X \in \mathfrak{X}(M)$ a vector field. Then $X : (M, g) \longrightarrow (TM, g_S)$ is a harmonic map if and only if X is parallel.*

Proof. We assume that $X : (M, g) \longrightarrow (TM, g_S)$ is a harmonic map, then from Corollary 3.1.2, X is a harmonic vector field and, hence X is parallel. Conversely, we assume that the vector field X is parallel, by virtue of Theorem 3.1.2, X is a harmonic map. □

Lemma 3.1.1 ([16])**.** *Let (M, g) be a Riemannian manifold and X a vector field of M. Then the following equation is satisfied:*

$$\bar{\Delta}(fX) = (\Delta f)X + f\bar{\Delta}X - 2\nabla_{\operatorname{grad} f}X, \tag{3.6}$$

where f being a smooth function of M and grad f *the gradient of f.*

3.2 Biharmonicity of vector fields on Riemannian manifolds

In what follows, we give the formula of the bitension field $\tau_2(X)$ of X. We prove the following Theorem:

Theorem 3.2.1. *Let (M, g) be a Riemannian manifold of dimension m and (TM, g_S) its tangent bundle equipped with the Sasaki metric, if $X : (M, g) \to (TM, g_S)$ is a smooth vector field then the bitension field of X is given by*

$$\tau_2(X) = \Big\{ -\bar{\Delta}\bar{\Delta}X - \sum_{i=1}^{m}[(\nabla_{e_i}R)(e_i, S(X))X + R(e_i, \nabla_{e_i}S(X))X$$

$$+2R(e_i,S(X))\nabla_{e_i}X\Big]\Big\}^v+\Big\{-\bar{\Delta}S(X)-R(X,\bar{\Delta}X)S(X)+\sum_{i=1}^m[R(X,\nabla_{e_i}\bar{\Delta}X)e_i$$
$$-R(\nabla_{e_i}X,\bar{\Delta}X)e_i-R(e_i,S(X))e_i-(\nabla_{S(X)}R)(\nabla_{e_i}X,X)e_i$$
$$+R(X,\nabla_{e_i}X)\nabla_{e_i}S(X)-R(X,R(e_i,S(X))X)e_i]\Big\}^h. \tag{3.7}$$

Proof. Let $(x,u)\in TM$ and $\{e_i\}_{i=1}^m$ be a local orthonormal frame on M such that $\nabla_{e_i}e_j=0$ at $x\in M$ and $u=X_x$, using the Proposition 1.9.1 and (3.1) and (3.2), we get

$$\begin{aligned}\sum_{i=1}^m\nabla^X_{e_i}\tau(X)\big|_{(x,X_x)}&=-\sum_{i=1}^m\big[\widetilde{\nabla}_{e_i^h+(\nabla_{e_i}X)^v}(S(X)^h+\bar{\Delta}X^v)\big]\big|_{(x,X_x)}\\&=-\Big\{\sum_{i=1}^m\big[\nabla_{e_i}S(X)+\frac{1}{2}R(X,\nabla_{e_i}X)S(X)+\frac{1}{2}R(X,\bar{\Delta}X)e_i\big]\Big\}^h_{(x,X_x)}\\&\quad-\Big\{\sum_{i=1}^m\big[\nabla_{e_i}\bar{\Delta}X-\frac{1}{2}R(e_i,S(X))X\big]\Big\}^v_{(x,X_x)}\end{aligned}$$

and

$$\begin{aligned}\Delta^X\tau(X)\big|_{(x,X_x)}&=-\sum_{i=1}^m\nabla^X_{e_i}\nabla^X_{e_i}\tau(X)\big|_{(x,X_x)}=\Big\{\sum_{i=1}^m\big[\nabla_{e_i}\nabla_{e_i}S(X)\\&+\frac{1}{2}\nabla_{e_i}R(X,\nabla_{e_i}X)S(X)+\frac{1}{2}\nabla_{e_i}R(X,\bar{\Delta}X)e_i+\frac{1}{2}R(X,\nabla_{e_i}X)\nabla_{e_i}S(X)\\&+\frac{1}{4}R(X,\nabla_{e_i}X)R(X,\nabla_{e_i}X)S(X)+\frac{1}{4}R(X,\nabla_{e_i}X)R(X,\bar{\Delta}X)e_i\\&+\frac{1}{2}R(X,\nabla_{e_i}\bar{\Delta}X)e_i-\frac{1}{4}R(X,R(e_i,S(X))X)e_i\big]\Big\}^h_{(x,X_x)}+\Big\{\sum_{i=1}^m\big[\nabla_{e_i}\nabla_{e_i}\bar{\Delta}X\\&-\frac{1}{2}\nabla_{e_i}R(e_i,S(X))X-\frac{1}{2}R(e_i,\nabla_{e_i}S(X))X-\frac{1}{4}R(e_i,R(X,\nabla_{e_i}X)S(X))X\\&-\frac{1}{4}R(e_i,R(X,\bar{\Delta}X)e_i)X\big]\Big\}^v_{(x,X_x)}.\end{aligned}\tag{3.8}$$

Let $\widetilde{R}$ the curvature tensor field of $\widetilde{\nabla}$. By Proposition 1.10.1, we find

$$\begin{aligned}-\sum_{i=1}^m\widetilde{R}(\tau(X),dX(e_i))dX(e_i)\big|_{(x,X_x)}&=\Big\{\sum_{i=1}^m\big[R(S(X),e_i)e_i\\&+\frac{3}{4}R(X,R(S(X),e_i)X)e_i+(\nabla_{S(X)}R)(X,\nabla_{e_i}X)e_i-\frac{1}{2}(\nabla_{e_i}R)(X,\nabla_{e_i}X)S(X)\\&-\frac{1}{4}R(X,\nabla_{e_i}X)R(X,\nabla_{e_i}X)S(X)-\frac{1}{2}(\nabla_{e_i}R)(X,\bar{\Delta}X)e_i+\frac{3}{2}R(\bar{\Delta}X,\nabla_{e_i}X)e_i\\&-\frac{1}{2}R(X,\bar{\Delta}X)S(X)-\frac{1}{4}R(X,\nabla_{e_i}X)R(X,\bar{\Delta}X)e_i\big]\Big\}^h_{(x,X_x)}\\&+\Big\{\sum_{i=1}^m\big[\frac{1}{2}(\nabla_{e_i}R)(S(X),e_i)X+\frac{3}{2}R(S(X),e_i)\nabla_{e_i}X\\&-\frac{1}{4}R(R(X,\nabla_{e_i}X)S(X),e_i)X-\frac{1}{4}R(R(X,\bar{\Delta}X)e_i,e_i)X\big]\Big\}^v_{(x,X_x)}.\end{aligned}\tag{3.9}$$

Summing (3.8) and (3.9) and using the following formulas we get the desired formula

$$\sum_{i=1}^{m} \nabla_{e_i} R(X, \nabla_{e_i} X)S(X) = \sum_{i=1}^{m} \left[(\nabla_{e_i} R)(X, \nabla_{e_i} X)S(X) + R(X, \nabla_{e_i} X)\nabla_{e_i} S(X) \right] - R(X, \bar{\Delta} X)S(X),$$

$$\sum_{i=1}^{m} \nabla_{e_i} R(X, \bar{\Delta} X)e_i = \sum_{i=1}^{m} \big[(\nabla_{e_i} R)(X, \bar{\Delta} X)e_i + R(\nabla_{e_i} X, \bar{\Delta} X)e_i + R(X, \nabla_{e_i} \bar{\Delta} X)e_i \big],$$

$$\sum_{i=1}^{m} \nabla_{e_i} R(e_i, S(X))X = \sum_{i=1}^{m} \big[(\nabla_{e_i} R)(e_i, S(X))X + R(e_i, \nabla_{e_i} S(X))X + R(e_i, S(X))\nabla_{e_i} X \big]. \square \tag{3.10}$$

Theorem 3.2.2. *Let (M, g) be a m-dimensional Riemannian manifold and $X \in \mathfrak{X}(M)$, then $X : (M, g) \to (TM, g_S)$ is a biharmonic map if and only if*

$$\bar{\Delta}\bar{\Delta} X + \sum_{i=1}^{m} [(\nabla_{e_i} R)(e_i, S(X))X + R(e_i, \nabla_{e_i} S(X))X + 2R(e_i, S(X))\nabla_{e_i} X] = 0,$$

and

$$\bar{\Delta} S(X) + R(X, \bar{\Delta} X)S(X) - \sum_{i=1}^{m} [R(X, \nabla_{e_i} \bar{\Delta} X)e_i - R(\nabla_{e_i} X, \bar{\Delta} X)e_i - R(e_i, S(X))e_i - (\nabla_{S(X)} R)(\nabla_{e_i} X, X)e_i + R(X, \nabla_{e_i} X)\nabla_{e_i} S(X) - R(X, R(e_i, S(X))X)e_i] = 0,$$

where $\{e_i\}_{i=1}^m$ is a local orthonormal frame field of (M, g).

Definition 3.2.1 ([23])**.** *Let (M, g) be a Riemannian manifold. A vector field $X \in \mathfrak{X}(M)$ is called biharmonic if the corresponding map $X : (M, g) \longrightarrow (TM, g_S)$ is a critical point for the bienergy functional E_2, only considering variations among maps defined by vector fields.*

Now, by virtue of the formula (3.7), we give another proof of the following Theorem given in [23].

Theorem 3.2.3. *Let (M, g) be a compact oriented m-dimensional Riemannian manifold, $\{e_i\}_{i=1}^m$ a local orthonormal frame field of (M, g), X a tangent vector field on M and $E_2 : \mathfrak{X}(M) \longrightarrow [0, +\infty)$ the bienergy functional restricted to the space of all vector fields. Then*

$$\frac{d}{dt} E_2(X_t)\bigg|_{t=0} = \int_M \Big\{ g(\bar{\Delta}\bar{\Delta} X + \sum_{i=1}^{m} \big[(\nabla_{e_i} R)(e_i, S(X))X + R(e_i, \nabla_{e_i} S(X))X$$

$$+ 2R(e_i, S(X))\nabla_{e_i}X\big], V)\Big\}v_g,$$

for any smooth 1*-parameter variation* $U : M \times (-\epsilon, \epsilon) \to TM$ *of* X *through vector fields i.e.,* $X_t(z) = U(z,t) \in T_zM$ *for any* $|t| < \epsilon$ *and* $z \in M$*, or equivalently* $X_t \in \mathfrak{X}(M)$ *for any* $|t| < \epsilon$*. Also,* V *is the tangent vector field on* M *given by*

$$V(z) = \frac{d}{dt}X_z(0), \quad z \in M,$$

where $X_z(t) = U(z,t)$*,* $(z,t) \in M \times (-\epsilon, \epsilon)$*.*

Proof. Let $U : M \times (-\epsilon, \epsilon) \to TM$ be a smooth variation of X(i.e., $U(z,0) = X(z)$ for any $z \in M$) such that $X_t(z) = U(z,t) \in T_zM$ for any $z \in M$ and any $|t| < \epsilon$. We have

$$E_2(X_t) = \frac{1}{2}\int_M |\tau(X_t)|^2 v_g.$$

Then, from [18], we get

$$\frac{d}{dt}E_2(X_t)\Big|_{t=0} = -\int_M g_S(\mathcal{V}, \tau_2(X))v_g,$$

where $\mathcal{V}(z) = \frac{d}{dt}X_t(z)\big|_{t=0}$, $z \in M$, and from [9, pp. 58], we have

$$\mathcal{V} = V^v \circ X. \tag{3.11}$$

Taking into account (3.11) and the expression of $\tau_2(X)$ given by (3.7), we find

$$\begin{aligned} \frac{d}{dt}E_2(X_t)\Big|_{t=0} &= -\int_M g_S(V^v, \tau_2(X))v_g, \\ &= \int_M \Big\{g(V, \bar{\Delta}\bar{\Delta}X + \sum_{i=1}^m \big[(\nabla_{e_i}R)(e_i, S(X))X \\ &\quad + R(e_i, \nabla_{e_i}S(X))X + 2R(e_i, S(X))\nabla_{e_i}X\big])\Big\}v_g, \end{aligned}$$

which completes the proof. □

Then, we deduce the following [23].

Corollary 3.2.1. *A vector field* X *of an* m*-dimensional Riemannian manifold* (M,g) *is biharmonic if and only if*

$$\begin{aligned} \bar{\Delta}\bar{\Delta}X + \sum_{i=1}^m &[(\nabla_{e_i}R)(e_i, S(X))X \\ &+ R(e_i, \nabla_{e_i}S(X))X + 2R(e_i, S(X))\nabla_{e_i}X] = 0, \end{aligned} \tag{3.12}$$

where $\{e_i\}_{i=1}^m$ *is a local orthonormal frame field of* (M,g)*.*

Remark 3.2.1. *Theorem 3.2.3 holds if (M, g) is a non-compact Riemannian manifold see [23].*

Remark 3.2.2. *Combining Theorem 3.2.2 and Corollary 3.2.1, we get that a vector field $X \in \mathfrak{X}(M)$ is biharmonic map if and only if X is biharmonic vector field and*

$$\begin{aligned}\bar{\Delta}S(X) + R(X, \bar{\Delta}X)S(X) - \sum_{i=1}^{m}[R(X, \nabla_{e_i}\bar{\Delta}X)e_i \\ - R(\nabla_{e_i}X, \bar{\Delta}X)e_i - R(e_i, S(X))e_i - (\nabla_{S(X)}R)(\nabla_{e_i}X, X)e_i \\ + R(X, \nabla_{e_i}X)\nabla_{e_i}S(X) - R(X, R(e_i, S(X))X)e_i] = 0.\end{aligned}$$

Theorem 3.2.4. *Let (M, g) be a compact m-dimensional Riemannian manifold and $X \in \mathfrak{X}(M)$ a vector field. Then $X : (M, g) \longrightarrow (TM, g_S)$ is a biharmonic map if and only if X is parallel.*

Proof. We assume that $X : (M, g) \longrightarrow (TM, g_S)$ is a biharmonic map, then from Remark 3.2.2, X is a biharmonic vector field and, hence X is parallel [23]. Conversely, we assume that the vector field X is parallel, by virtue of Theorem 3.2.2, X is a biharmonic map. □

Example 3.2.1. *Consider the solvable Lie group Sol_3 as the Cartesian 3-space $\mathbb{R}^3(x, y, z)$ equipped with the left-invariant metric g given by*

$$g = e^{2z}(dx)^2 + e^{-2z}(dy)^2 + (dz)^2.$$

The left-invariant vector fields

$$e_1 = e^{-z}\frac{\partial}{\partial x}, \quad e_2 = e^{z}\frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z},$$

constitute an orthonormal basis of the Lie algebra $\mathfrak{g}$ of Sol_3. The corresponding components of the Levi-Civita connection are determined by [29]

$$\begin{array}{lll}\nabla_{e_1}e_1 = -e_3, & \nabla_{e_1}e_2 = 0, & \nabla_{e_1}e_3 = e_1, \\ \nabla_{e_2}e_1 = 0, & \nabla_{e_2}e_2 = e_3, & \nabla_{e_2}e_3 = -e_2, \\ \nabla_{e_3}e_1 = 0, & \nabla_{e_3}e_2 = 0, & \nabla_{e_3}e_3 = 0.\end{array} \tag{3.13}$$

Also the curvature components are given by

$$\begin{array}{lll}R(e_1, e_2)e_1 = -e_2, & R(e_1, e_2)e_2 = e_1, & R(e_1, e_2)e_3 = 0, \\ R(e_2, e_3)e_1 = 0, & R(e_2, e_3)e_2 = e_3, & R(e_2, e_3)e_3 = -e_2, \\ R(e_3, e_1)e_1 = -e_3, & R(e_3, e_1)e_2 = 0, & R(e_3, e_1)e_3 = e_1.\end{array} \tag{3.14}$$

We consider the vector field $X = f(z)e_3$, where $f(z)$ is a smooth real function depending of the variable z. By using (3.13), *we get*

$$\bar{\Delta}e_1 = -\nabla_{e_1}\nabla_{e_1}e_1 + \nabla_{\nabla_{e_1}e_1}e_1 - \nabla_{e_2}\nabla_{e_2}e_1 + \nabla_{\nabla_{e_2}e_2}e_1 - \nabla_{e_3}\nabla_{e_3}e_1 = e_1, \tag{3.15}$$

and

$$\bar{\Delta}e_2 = e_2, \quad \bar{\Delta}e_3 = 2e_3. \tag{3.16}$$

Combining relations (3.3), (3.6), *and* (3.13)-(3.16), *we find*

$$\begin{aligned} \bar{\Delta}X &= \Delta f e_3 + f\bar{\Delta}e_3 - 2f'\nabla_{e_3}e_3 = \big(2f - f''\big)e_3, \\ \bar{\Delta}\bar{\Delta}X &= \Delta\big(2f - f''\big)e_3 + \big(2f - f''\big)\bar{\Delta}e_3 = (f'''' - 4f'' + 4f)e_3, \\ S(X) &= f^2 S(e_3) = f^2\big(R(e_1, e_3)e_1 - R(e_2, e_3)e_2\big) = 0, \end{aligned} \tag{3.17}$$

where $f' = \frac{df}{dz}$, $f'' = \frac{d^2 f}{dz^2}$ etc. On the other hand, using relations (3.13), (3.14) *and* (3.17), *we obtain*

$$\sum_{i=1}^{3} R(X, \nabla_{e_i}\bar{\Delta}X)e_i = 0, \quad \textit{and} \quad \sum_{i=1}^{3} R(\nabla_{e_i}X, \bar{\Delta}X)e_i = 0.$$

Then, from Theorem 3.2.2, we get that X is biharmonic map if and only if the function f satisfies the following homogeneous fourth order differential equation.

$$f'''' - 4f'' + 4f = 0. \tag{3.18}$$

The general solution of (3.18) *is*

$$f(z) = c_1 e^{\sqrt{2}z} + c_2 z e^{\sqrt{2}z} + c_3 e^{-\sqrt{2}z} + c_4 z e^{-\sqrt{2}z}, \tag{3.19}$$

where c_1, c_2, c_3 and c_4 are real constants. Note that $X = f(z)e_3$ is also biharmonic vector field, where $f(z)$ is given by (3.19).

3.3 Biharmonicity of vector fields of three-dimensional unimodular Lie groups

In this section, we investigate biharmonicity of left-invariant vector fields on three-dimensional unimodular Lie groups equipped with a left-invariant Riemannian metric. Let G be a three-dimensional unimodular Lie group and $\mathfrak{g}$ its Lie algebra, this is, $tr\, ad_X = 0$ for all $X \in \mathfrak{g}$, equip G with a left-invariant Riemannian metric $<,>$. Then, there exists an orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathfrak{g}$ such that

$$[e_2, e_3] = \lambda_1 e_1, \quad [e_3, e_1] = \lambda_2 e_2, \quad [e_1, e_2] = \lambda_3 e_3,$$

Signs of $\lambda_1, \lambda_2, \lambda_3$	Associated Lie groups
$+, +, +$	$SU(2)$ or $SO(3)$
$+, +, -$	$SL(2, \mathbb{R})$ or $O(1, 2)$
$+, +, 0$	$E(2)$
$+, 0, -$	$E(1, 1)$
$+, 0, 0$	$\mathbb{H}^3$
$0, 0, 0$	$\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$

Table 3.1: Three-dimensional unimodular Lie groups

where $\lambda_1, \lambda_2, \lambda_3$ are constants. According to the signs of $\lambda_1, \lambda_2, \lambda_3$, Milnor [26] classified three-dimensional unimodular Lie groups as described in Table 1: let $\theta^i, i = 1, 2, 3$, be the dual one forms of $\{e_i\}, i = 1, 2, 3$. Let $V = x_1e_1 + x_2e_2 + x_3e_3$ an arbitrary left-invariant vector field on G. The Levi-Civita connection ∇ of G is given by [15]

$$\begin{aligned}
\nabla e_1 &= \mu_3 e_2 \otimes \theta^3 - \mu_2 e_3 \otimes \theta^2, \\
\nabla e_2 &= -\mu_3 e_1 \otimes \theta^3 + \mu_1 e_3 \otimes \theta^1, \\
\nabla e_3 &= \mu_2 e_1 \otimes \theta^2 - \mu_1 e_2 \otimes \theta^1,
\end{aligned} \tag{3.20}$$

where

$$\mu_i = \frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3) - \lambda_i, \quad i = 1, 2, 3. \tag{3.21}$$

Using (3.20) we get [25]

$$\begin{aligned}
\nabla_{e_1} V &= \mu_1(x_2e_3 - x_3e_2), \\
\nabla_{e_2} V &= \mu_2(x_3e_1 - x_1e_3), \\
\nabla_{e_3} V &= \mu_3(x_1e_2 - x_2e_1).
\end{aligned}$$

While the Riemann curvature tensor is given by [25]

$$\begin{aligned}
R(e_1, e_2)e_2 &= (\lambda_3\mu_3 - \mu_1\mu_2)e_1, & R(e_1, e_3)e_3 &= (\lambda_2\mu_2 - \mu_1\mu_3)e_1, \\
R(e_2, e_1)e_1 &= (\lambda_3\mu_3 - \mu_1\mu_2)e_2, & R(e_2, e_3)e_3 &= (\lambda_1\mu_1 - \mu_2\mu_3)e_2, \\
R(e_3, e_1)e_1 &= (\lambda_2\mu_2 - \mu_1\mu_3)e_3, & R(e_3, e_2)e_2 &= (\lambda_1\mu_1 - \mu_2\mu_3)e_3.
\end{aligned} \tag{3.22}$$

Again from [25] we have

$$\bar{\Delta} V = (\mu_2^2 + \mu_3^2)x_1e_1 + (\mu_1^2 + \mu_3^2)x_2e_2 + (\mu_1^2 + \mu_2^2)x_3e_3 \tag{3.23}$$

and

$$S(V) = A_1x_2x_3e_1 + A_2x_1x_3e_2 + A_3x_1x_2e_3, \tag{3.24}$$

where $A_1 = \mu_2^2(\mu_3 - \mu_1) + \mu_3^2(\mu_1 - \mu_2)$, $A_2 = \mu_1^2(\mu_2 - \mu_3) + \mu_3^2(\mu_1 - \mu_2)$ and $A_3 = \mu_1^2(\mu_2 - \mu_3) + \mu_2^2(\mu_3 - \mu_1)$.

The following theorem follows from (3.21) and (3.23).

Theorem 3.3.1. *A left-invariant vector field V on G is harmonic if and only if one of the following cases occurs:*

1. $\lambda_1 = \lambda_2 = \lambda_3 = 0$. *Then, any vector field $V \in \mathfrak{g}$ is harmonic.*
2. $\lambda_1 = \lambda_2 - \lambda_3 = 0$. *Then, $V = x_1 e_1$.*
3. $\lambda_2 = \lambda_1 - \lambda_3 = 0$. *Then, $V = x_2 e_2$.*
4. $\lambda_3 = \lambda_1 - \lambda_2 = 0$. *Then, $V = x_3 e_3$.*

Moreover, by (3.20)*, vectors listed above are parallel. Hence, they also define harmonic maps.*

Using (3.20) and (3.23) we obtain

$$\bar{\Delta}\bar{\Delta}V = (\mu_2^2 + \mu_3^2)^2 x_1 e_1 + (\mu_1^2 + \mu_3^2)^2 x_2 e_2 + (\mu_1^2 + \mu_2^2)^2 x_3 e_3. \tag{3.25}$$

Combining relations (4.5), (3.20), (3.22)-(3.25), a long but straightforward calculation gives that the vector field $V = x_1 e_1 + x_2 e_2 + x_3 e_3$ is biharmonic if and only if

$$\begin{aligned} x_1\{(\mu_2^2 + \mu_3^2)^2 + A_2^2 x_3^2 + A_3^2 x_2^2\} &= 0 \\ x_2\{(\mu_1^2 + \mu_3^2)^2 + A_1^2 x_3^2 + A_3^2 x_1^2\} &= 0 \\ x_3\{(\mu_1^2 + \mu_2^2)^2 + A_1^2 x_2^2 + A_2^2 x_1^2\} &= 0. \end{aligned} \tag{3.26}$$

The subcases $x_1 = x_2 = 0$, $x_2 = x_3 = 0$ and $x_1 = x_3 = 0$ give vector fields which define harmonic maps. We proceed as in [**?**], we deal with the six types of Lie groups described in Table 1.

Case 1 : $\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$. In this case, $\lambda_1 = \lambda_2 = \lambda_3 = 0$, therefore the system (3.26) implies that every left-invariant vector field is biharmonic and defines a biharmonic maps.

Case 2 : $\mathbb{H}^3$. We yield $A_1 = 0$, $A_2 = 2\mu_1^3$, $A_3 = -2\mu_1^3$, $\mu_1^2 = \mu_2^2 = \mu_3^2$ and the system (3.26) is transformed to

$$\begin{aligned} x_1\{1 + \mu_1^2(x_2^2 + x_3^2)\} &= 0 \\ x_2\{1 + \mu_1^2 x_1^2\} &= 0 \\ x_3\{1 + \mu_1^2 x_1^2\} &= 0. \end{aligned}$$

This system admits only the trivial solution $x_1 = x_2 = x_3 = 0$.

Case 3 : $E(1,1)$. In this case, we have $\mu_3 = -\mu_1 > 0$, $A_1 = -2\mu_1\mu_2^2 + \mu_1^2(\mu_1 - \mu_2)$, $A_2 = 2\mu_1^3 < 0$, $A_3 = -2\mu_1\mu_2^2 + \mu_1^2(\mu_1 + \mu_2)$ and the system (3.26) is reduces to

$$\begin{aligned} x_1\{(\mu_1^2 + \mu_2^2)^2 + A_3^2 x_2^2 + A_2^2 x_3^2\} &= 0 \\ x_2\{4\mu_1^2 + A_1^2 x_3^2 + A_3^2 x_1^2\} &= 0 \\ x_3\{(\mu_1^2 + \mu_2^2)^2 + A_1^2 x_2^2 + A_2^2 x_1^2\} &= 0. \end{aligned} \tag{3.27}$$

If $x_1, x_2, x_3 \neq 0$. We will prove that $A_1, A_3 \neq 0$. We suppose that $A_1 = 0$, the system (3.27) gives $\mu_1 = 0$ which is a contradiction. The case $A_3 = 0$ is treated similarly. Then $A_1, A_3 \neq 0$ and the system (3.27) has no solution.

Case 4 : $E(2)$. In this case, we have $\mu_2 = -\mu_1$, $A_1 = 2\mu_1\mu_3^2 + \mu_1^2(\mu_3 - \mu_1)$, $A_2 = 2\mu_1\mu_3^2 - \mu_1^2(\mu_1 + \mu_3)$, $A_3 = -2\mu_1^3$ and the system (3.26) is reduces to

$$\begin{aligned} x_1\{(\mu_1^2 + \mu_3^2)^2 + A_3^2x_2^2 + A_2^2x_3^2\} &= 0 \\ x_2\{(\mu_1^2 + \mu_3^2)^2 + A_1^2x_3^2 + A_3^2x_1^2\} &= 0 \\ x_3\{4\mu_1^2 + A_1^2x_2^2 + A_2^2x_1^2\} &= 0. \end{aligned}$$

If $\mu_1 = 0$, we obtain $A_1 = A_2 = A_3 = 0$ and we get $x_1 = x_2 = 0$ (the harmonic solution). If $x_1, x_2, x_3 \neq 0$, we do not have solution following the same procedure appeared in the case of $E(1,1)$.

Case 5 : $SL(2,\mathbb{R})$ or $O(1,2)$. We distinguish two cases:

• $\lambda_1 = \lambda_2 > \lambda_3$. We have $\mu_1 = \mu_2 = \frac{\lambda_3}{2}$, $\mu_3 = \frac{2\lambda_1 - \lambda_3}{2} > 0$, $A_1 = \mu_1^2(\mu_3 - \mu_1)$, $A_2 = -A_1$, $A_3 = 0$. So, the system (3.26) is reduced to

$$\begin{aligned} x_1\{(\mu_1^2 + \mu_3^2)^2 + A_1^2x_3^2\} &= 0 \\ x_2\{(\mu_1^2 + \mu_3^2)^2 + A_1^2x_3^2\} &= 0 \\ x_3\{4\mu_1^2 + A_1^2(x_1^2 + x_2^2)\} &= 0. \end{aligned} \tag{3.28}$$

The system (3.28) admits only the zero solution.

• $\lambda_1 > \lambda_2 > \lambda_3$ or $\lambda_2 > \lambda_1 > \lambda_3$. We have $\mu_3 > 0$. If $x_1, x_2, x_3 \neq 0$, the system (4.20) is reduced to

$$\begin{aligned} (\mu_2^2 + \mu_3^2)^2 + A_2^2x_3^2 + A_3^2x_2^2 &= 0 \\ (\mu_1^2 + \mu_3^2)^2 + A_1^2x_3^2 + A_3^2x_1^2 &= 0 \\ (\mu_1^2 + \mu_2^2)^2 + A_1^2x_2^2 + A_2^2x_1^2 &= 0. \end{aligned} \tag{3.29}$$

We will prove that $A_1, A_2, A_3 \neq 0$. We suppose that $A_1 = 0$, the system (3.29) gives $\mu_3 = 0$ which is a contradiction. Similarly for the cases $A_2 = 0$ and $A_3 = 0$. Then $A_1, A_2, A_3 \neq 0$ and the system (3.29) has no solution.

Case 6 : $SU(2)$ or $SO(3)$. We distinguish two cases:

• $\lambda_1 = \lambda_2 = \lambda_3$, $\lambda_1 = \lambda_2 \neq \lambda_3$, $\lambda_1 = \lambda_3 \neq \lambda_2$, $\lambda_2 = \lambda_3 \neq \lambda_1$. We get $x_1 = x_2 = x_3 = 0$ (the zero solution).

• $\lambda_1 \neq \lambda_2 \neq \lambda_3$. If $x_1, x_2, x_3 \neq 0$, the system (3.26) is (3.29). We will prove that $A_1, A_2, A_3 \neq 0$. We suppose that $A_1 = 0$, the system (3.29) gives $\mu_1 = \mu_2 = \mu_3 = 0$, equivalently, $\lambda_1 = \lambda_2 = \lambda_3 = 0$ which is a contradiction. Similarly for the cases $A_2 = 0$ and $A_3 = 0$. Therefore $A_1, A_2, A_3 \neq 0$ and the system (3.29) has no solution.

Summarizing, we yield

Theorem 3.3.2. *Let G be a three-dimensional unimodular Lie group. Then*

1. *The set of left-invariant biharmonic vector fields which do not define harmonic maps into TG is empty.*
2. *The set of left-invariant vector fields which are biharmonic maps and do not define harmonic maps into TG is empty.*

CHAPTER 4

BIHARMONICITY OF VECTOR FIELDS ON PSEUDO-RIEMANNIAN MANIFOLDS

This chapter presents our work on the biharmonicity of vector fields on pseudo-Riemannian manifolds. We deal with the biharmonicity of a vector field X viewed as a map from a pseudo-Riemannian manifold (M, g) into its tangent bundle TM endowed with the Sasaki metric g_S. Precisely, we characterize those vector fields which are biharmonic maps, and find the relationship between them and biharmonic vector fields. Afterwards, we study the biharmonicity of left-invariant vector fields on the three dimensional Heisenberg group endowed with a left-invariant Lorentzian metric. Finally, we give examples of vector fields which are proper biharmonic maps on the Gödel universe. The references used are: [2], [9],[10], [11], [8], [13], [14], [23], [25].

4.1 Harmonicity of vector fields on pseudo-Riemannian manifolds

Let (M, g) be a pseudo-Riemannian manifold of dimension m. We know that any vector field X on (M, g) can be viewed as the immersion $X : (M, g) \to (TM, g_S)$; $x \mapsto (x, X_x) \in TM$ into its tangent bundle TM equipped with the Sasaki metric g_S. The energy of X is, by definition, the energy of the corresponding map $X : (M, g) \to (TM, g_S)$, that is [14]

$$E(X) = \frac{1}{2}\int_M |dX|^2 v_g = \frac{m}{2} Vol(M) + \frac{1}{2}\int_M |\nabla X|^2 v_g \tag{4.1}$$

(assuming M compact; in the non-compact case, one works over compact domain).

Theorem 4.1.1. *[13] Let (M, g) be a pseudo-Riemannian manifold of dimension m and (TM, g_S) its tangent bundle equipped with the Sasaki metric, if $X : (M, g) \to$*

(TM, g_S) is a smooth vector field then the tension field $\tau(X)$ is given by

$$\tau(X) = \Big(-\sum_{i=1}^{m} \varepsilon_i R(\nabla_{e_i} X, X) e_i \Big)^h + \Big(\sum_{i=1}^{m} \varepsilon_i (\nabla_{e_i} \nabla_{e_i} X - \nabla_{\nabla_{e_i} e_i} X) \Big)^v,$$

where $\{e_i\}_{i=1}^m$ a local pseudo-orthonormal frame field of (M, g) with $\varepsilon_i = g(e_i, e_i) = \pm 1$ for all indices $i = 1, 2, \cdots, m$.

Proof. Let $(x, u) \in TM$ and $\{e_i\}_{i=1}^m$ be a local pseudo-orthonormal frame on M such that $\nabla_{e_i} e_j = 0$ at $x \in M$ and $u = X_x$, using 1.9.1, (3.1) and (4.2), we get

$$\begin{aligned}
\tau(X)\big|_{(x,X_x)} &= \sum_{i=1}^{m} \varepsilon_i \big[\nabla^X_{e_i} dX(e_i) \big]\big|_{(x,X_x)} \\
&= \sum_{i=1}^{m} \varepsilon_i \big[\widetilde{\nabla}_{e_i^h + (\nabla_{e_i} X)^v} (e_i^h + (\nabla_{e_i} X)^v) \big]\big|_{(x,X_x)} \\
&= \sum_{i=1}^{m} \varepsilon_i \big[\widetilde{\nabla}_{e_i^h} e_i^h + \widetilde{\nabla}_{e_i^h} (\nabla_{e_i} X)^v + \widetilde{\nabla}_{(\nabla_{e_i} X)^v} e_i^h + + \widetilde{\nabla}_{(\nabla_{e_i} X)^v} (\nabla_{e_i} X)^v \big]\big|_{(x,X_x)} \\
&= \sum_{i=1}^{m} \varepsilon_i \big[(-R(\nabla_{e_i} X, X) e_i)^h + (\nabla_{e_i} \nabla_{e_i} X)^v \big]\big|_{(x,X_x)} \\
&= \big[\big(-\sum_{i=1}^{m} \varepsilon_i R(\nabla_{e_i} X, X) e_i \big)^h + \big(\sum_{i=1}^{m} \varepsilon_i (\nabla_{e_i} \nabla_{e_i} X - \nabla_{\nabla_{e_i} e_i} X) \big)^v \big]\big|_{(x,X_x)}.
\end{aligned}$$

□

We can rewrite $\tau(X)$ as follows [25]:

$$\tau(X) = (-S(X))^h + (\nabla^* \nabla X)^v, \tag{4.2}$$

where

$$S(X) = \sum_{i=1}^{m} \varepsilon_i R(\nabla_{e_i} X, X) e_i,$$

and $\nabla^* \nabla X$ is the rough Laplacian given by

$$\nabla^* \nabla X = \sum_{i=1}^{m} \varepsilon_i (\nabla_{e_i} \nabla_{e_i} X - \nabla_{\nabla_{e_i} e_i} X).$$

Note that, for any smooth function f and vector field X of M, we have

$$S(fX) = f^2 S(X). \tag{4.3}$$

Theorem 4.1.2. *[13] Let (M, g) be a pseudo-Riemannian manifold of dimension m and (TM, g_S) its tangent bundle equipped with the Sasaki metric, if $X : (M, g) \to (TM, g_S)$ is a smooth vector field then X is a harmonic map if and only if $\nabla^*\nabla X = 0$ and $S(X) = 0$.*

Definition 4.1.1 ([23])**.** *A vector field X is called harmonic vector field if it is a critical point of the energy functional* (1)*, restricted to variations through vector fields.*

Theorem 4.1.3. *Let (M, g) be an m-dimensional pseudo-Riemannian manifold, $\{e_i\}_{i=1}^m$ a local pseudo-orthonormal frame field of (M, g), X a tangent vector field on M and $E : \mathfrak{X}(M) \longrightarrow [0, +\infty)$ the energy functional restricted to the space of all vector fields. Then*

$$\frac{d}{dt}E(X_t)\bigg|_{t=0} = \int_M g(\nabla^*\nabla X, V)v_g,$$

for any smooth 1*-parameter variation $U : M \times (-\epsilon, \epsilon) \to TM$ of X through vector fields i.e., $X_t(z) = U(z, t) \in T_zM$ for any $|t| < \epsilon$ and $z \in M$, or equivalently $X_t \in \mathfrak{X}(M)$ for any $|t| < \epsilon$. Also, V is the tangent vector field on M given by*

$$V(z) = \frac{d}{dt}X_z(0), \quad z \in M,$$

where $X_z(t) = U(z, t)$, $(z, t) \in M \times (-\epsilon, \epsilon)$.

Proof. Let $U : M \times (-\epsilon, \epsilon) \to TM$ be a smooth variation of X(i.e., $U(z, 0) = X(z)$ for any $z \in M$) such that $X_t(z) = U(z, t) \in T_zM$ for any $z \in M$ and any $|t| < \epsilon$. We have

$$E(X_t) = \frac{1}{2}\int_M |dX_t|^2 v_g.$$

Then, from [10], we get

$$\frac{d}{dt}E(X_t)\bigg|_{t=0} = -\int_M g_S(\mathcal{V}, \tau(X))v_g,$$

where $\mathcal{V}(z) = \frac{d}{dt}X_t(z)\big|_{t=0}$, $z \in M$, and from [9, pp. 58], we have

$$\mathcal{V} = V^v \circ X. \tag{4.4}$$

Taking into account (4.4) and the expression of $\tau(X)$ given by (3.1), we find

$$\begin{aligned}\frac{d}{dt}E(X_t)\bigg|_{t=0} &= -\int_M g_S(V^v, \tau(X))v_g,\\ &= \int_M g(V, \nabla^*\nabla X)v_g,\end{aligned}$$

as required. □

Then, we deduce the following [23].

Corollary 4.1.1. *A vector field X of an m-dimensional pseudo-Riemannian manifold (M, g) is harmonic if and only if*

$$\nabla^*\nabla X = 0, \tag{4.5}$$

where $\{e_i\}_{i=1}^m$ is a local pseudo-orthonormal frame field of (M, g).

Combining Theorem 4.1.1 and Corollary 4.1.1, we get

Corollary 4.1.2. *A vector field X of an m-dimensional pseudo-Riemannian manifold (M, g) is harmonic map if and only if X is harmonic vector field and $S(X) = 0$.*

4.2 Biharmonicity of vector fields on pseudo-Riemannian manifolds

In the next Theorem, we compute the bitension field $\tau_2(X)$ of X.

Theorem 4.2.1. *Let (M, g) be an m-dimensional pseudo-Riemannian manifold and (TM, g_S) its tangent bundle equipped with the Sasaki metric, if $X : (M, g) \to (TM, g_S)$ is a smooth vector field then the bitension field of X is given by*

$$\begin{aligned}
\tau_2(X) = &\Big\{(\nabla^*\nabla)^2 X + \sum_{i=1}^m \varepsilon_i[(\nabla_{e_i}R)(e_i, S(X))X + R(e_i, \nabla_{e_i}S(X))X \\
&+ 2R(e_i, S(X))\nabla_{e_i}X]\Big\}^v + \Big\{ - \nabla^*\nabla S(X) - R(X, \nabla^*\nabla X)S(X) + \sum_{i=1}^m \varepsilon_i[R(X, \nabla_{e_i}\nabla^*\nabla X)e_i \\
&- R(\nabla_{e_i}X, \nabla^*\nabla X)e_i + R(e_i, S(X))e_i - (\nabla_{S(X)}R)(X, \nabla_{e_i}X)e_i \\
&- R(X, \nabla_{e_i}X)\nabla_{e_i}S(X) + R(X, R(e_i, S(X))X)e_i]\Big\}^h.
\end{aligned} \tag{4.6}$$

Proof. Let $(x, u) \in TM$ and $\{e_i\}_{i=1}^m$ be a local pseudo-orthonormal frame on M such that $\nabla_{e_i}e_i = 0$ at $x \in M$ and $u = X_x$. If $Y \in \Gamma(TM)$ then, we have (see [9, pp. 50]) using 1.9.1, (3.1) and (4.2) one has

$$\begin{aligned}
\sum_{i=1}^m \nabla^X_{e_i}\tau(X)\big|_{(x,X_x)} =& \sum_{i=1}^m \big[\widetilde{\nabla}_{e_i^h+(\nabla_{e_i}X)^v}(-S(X)^h + \nabla^*\nabla X^v)\big]\big|_{(x,X_x)} \\
=& \Big\{\sum_{i=1}^m \big[- \nabla_{e_i}S(X) - \frac{1}{2}R(X, \nabla_{e_i}X)S(X) + \frac{1}{2}R(X, \nabla^*\nabla X)e_i\big]\Big\}^h_{(x,X_x)} \\
&+ \Big\{\sum_{i=1}^m \big[\nabla_{e_i}\nabla^*\nabla X + \frac{1}{2}R(e_i, S(X))X\big]\Big\}^v_{(x,X_x)}
\end{aligned}$$

and

$$\sum_{i=1}^m \varepsilon_i\nabla^X_{e_i}\nabla^X_{e_i}\tau(X)\big|_{(x,X_x)} = \Big\{\sum_{i=1}^m \varepsilon_i\big[- \nabla_{e_i}\nabla_{e_i}S(X)$$

$$
\begin{aligned}
&-\frac{1}{2}\nabla_{e_i}R(X,\nabla_{e_i}X)S(X)+\frac{1}{2}\nabla_{e_i}R(X,\nabla^*\nabla X)e_i-\frac{1}{2}R(X,\nabla_{e_i}X)\nabla_{e_i}S(X)\\
&-\frac{1}{4}R(X,\nabla_{e_i}X)R(X,\nabla_{e_i}X)S(X)+\frac{1}{4}R(X,\nabla_{e_i}X)R(X,\nabla^*\nabla X)e_i+\frac{1}{2}R(X,\nabla_{e_i}\nabla^*\nabla X)e_i\\
&+\frac{1}{4}R(X,R(e_i,S(X))X)e_i\big]\Big\}^h_{(x,X_x)}+\Big\{\sum_{i=1}^m\varepsilon_i\big[\nabla_{e_i}\nabla_{e_i}\nabla^*\nabla X+\frac{1}{2}\nabla_{e_i}R(e_i,S(X))X\\
&+\frac{1}{2}R(e_i,\nabla_{e_i}S(X))X+\frac{1}{4}R(e_i,R(X,\nabla_{e_i}X)S(X))X-\frac{1}{4}R(e_i,R(X,\nabla^*\nabla X)e_i)X\big]\Big\}^v_{(x,X_x)}. \qquad (4.7)
\end{aligned}
$$

Let $\widetilde{R}$ the curvature tensor field of $\widetilde{\nabla}$. On making use of Theorem 1 in [21], we find

$$
\begin{aligned}
&-\sum_{i=1}^m\varepsilon_i\widetilde{R}(dX(e_i),\tau(X))dX(e_i)\big|_{(x,X_x)}=\Big\{-\frac{1}{2}R(X,\nabla^*\nabla X)S(X)+\sum_{i=1}^m\varepsilon_i\big[R(e_i,S(X))e_i\\
&+\frac{3}{4}R(X,R(e_i,S(X))X)e_i-(\nabla_{S(X)}R)(X,\nabla_{e_i}X)e_i+\frac{1}{2}(\nabla_{e_i}R)(X,\nabla_{e_i}X)S(X)\\
&+\frac{1}{4}R(X,\nabla_{e_i}X)R(X,\nabla_{e_i}X)S(X)-\frac{1}{2}(\nabla_{e_i}R)(X,\nabla^*\nabla X)e_i+\frac{3}{2}R(\nabla^*\nabla X,\nabla_{e_i}X)e_i\\
&-\frac{1}{4}R(X,\nabla_{e_i}X)R(X,\nabla^*\nabla X)e_i\big]\Big\}^h_{(x,X_x)}+\Big\{\sum_{i=1}^m\varepsilon_i\big[\frac{1}{2}(\nabla_{e_i}R)(e_i,S(X))X \qquad (4.8)\\
&-\frac{3}{2}R(S(X),e_i)\nabla_{e_i}X+\frac{1}{4}R(R(X,\nabla_{e_i}X)S(X),e_i)X-\frac{1}{4}R(R(X,\nabla^*\nabla X)e_i,e_i)X\big]\Big\}^v_{(x,X_x)}.
\end{aligned}
$$

On the other hand, we have the following formulae

$$
\begin{aligned}
\sum_{i=1}^m\varepsilon_i\nabla_{e_i}R(X,\nabla_{e_i}X)S(X)=&\sum_{i=1}^m\varepsilon_i\big[(\nabla_{e_i}R)(X,\nabla_{e_i}X)S(X)+R(X,\nabla_{e_i}X)\nabla_{e_i}S(X)\big]\\
&+R(X,\nabla^*\nabla X)S(X), \qquad (4.9)
\end{aligned}
$$

$$
\begin{aligned}
\sum_{i=1}^m\varepsilon_i\nabla_{e_i}R(X,\nabla^*\nabla X)e_i=&\sum_{i=1}^m\varepsilon_i\big[(\nabla_{e_i}R)(X,\nabla^*\nabla X)e_i+R(\nabla_{e_i}X,\nabla^*\nabla X)e_i\\
&+R(X,\nabla_{e_i}\nabla^*\nabla X)e_i\big], \qquad (4.10)
\end{aligned}
$$

$$
\begin{aligned}
\sum_{i=1}^m\varepsilon_i\nabla_{e_i}R(e_i,S(X))X=&\sum_{i=1}^m\varepsilon_i\big[(\nabla_{e_i}R)(e_i,S(X))X+R(e_i,\nabla_{e_i}S(X))X\\
&+R(e_i,S(X))\nabla_{e_i}X\big]. \qquad (4.11)
\end{aligned}
$$

One can calculate $\tau_2(X)$ by summing up (4.7) and (4.8) and using the formulae (4.9)-(4.11). □

Then, we give the following characterization theorem.

Theorem 4.2.2. *Let (M,g) be an m-dimensional pseudo-Riemannian manifold and $X\in\mathfrak{X}(M)$, then $X:(M,g)\to(TM,g_S)$ is a biharmonic map if and only if*

$$
(\nabla^*\nabla)^2X+\sum_{i=1}^m\varepsilon_i[(\nabla_{e_i}R)(e_i,S(X))X+R(e_i,\nabla_{e_i}S(X))X+2R(e_i,S(X))\nabla_{e_i}X]=0,
$$

and

$$\begin{aligned}&-\nabla^*\nabla S(X) - R(X, \nabla^*\nabla X)S(X) + \sum_{i=1}^{m} \varepsilon_i[R(X, \nabla_{e_i}\nabla^*\nabla X)e_i \\ &- R(\nabla_{e_i}X, \nabla^*\nabla X)e_i + R(e_i, S(X))e_i - (\nabla_{S(X)}R)(X, \nabla_{e_i}X)e_i \\ &- R(X, \nabla_{e_i}X)\nabla_{e_i}S(X) + R(X, R(e_i, S(X))X)e_i] = 0,\end{aligned}$$

where $\{e_i\}_{i=1}^m$ *is a local pseudo-orthonormal frame field of* (M, g).

Definition 4.2.1 ([25]). *Let* (M, g) *be a pseudo-Riemannian manifold. A vector field* $X \in \mathfrak{X}(M)$ *is called biharmonic if the corresponding map* $X : (M, g) \longrightarrow (TM, g_S)$ *is a critical point for the bienergy functional* E_2*, only considering variations among maps defined by vector fields.*

By virtue of the formula (4.6), one obtain another proof of the next Theorem given in [23].

Theorem 4.2.3. *Let* (M, g) *be a compact oriented* m*-dimensional pseudo-Riemannian manifold,* $\{e_i\}_{i=1}^m$ *a local pseudo-orthonormal frame field of* (M, g)*,* X *a tangent vector field on* M *and* $E_2 : \mathfrak{X}(M) \longrightarrow [0, +\infty)$ *the bienergy functional restricted to the space of all vector fields. Then*

$$\begin{aligned}\frac{d}{dt}E_2(X_t)\Big|_{t=0} = \int_M \Big\{ g((\nabla^*\nabla)^2 X &+ \sum_{i=1}^{m} \varepsilon_i\big[(\nabla_{e_i}R)(e_i, S(X))X + R(e_i, \nabla_{e_i}S(X))X \\ &+ 2R(e_i, S(X))\nabla_{e_i}X\big], V)\Big\} v_g\end{aligned}$$

for any smooth 1*-parameter variation* $U : M \times (-\epsilon, \epsilon) \to TM$ *of* X *through vector fields i.e.,* $X_t(z) = U(z, t) \in T_zM$ *for any* $|t| < \epsilon$ *and* $z \in M$*, or equivalently* $X_t \in \mathfrak{X}(M)$ *for any* $|t| < \epsilon$*. Also,* V *is the tangent vector field on* M *given by*

$$V(z) = \frac{d}{dt}X_z(0), \quad z \in M,$$

where $X_z(t) = U(z, t)$*,* $(z, t) \in M \times (-\epsilon, \epsilon)$*.*

Proof. Let $U : M \times (-\epsilon, \epsilon) \to TM$ be a smooth variation of X(i.e., $U(z, 0) = X(z)$ for any $z \in M$) such that $X_t(z) = U(z, t) \in T_zM$ for any $z \in M$ and any $|t| < \epsilon$. We have

$$E_2(X_t) = \frac{1}{2}\int_M |\tau(X_t)|^2 v_g.$$

As in the Riemannian case [18], we can write

$$\frac{d}{dt}E_2(X_t)\Big|_{t=0} = \int_M g_S(\mathcal{V}, \tau_2(X))v_g,$$

where $\mathcal{V}(z) = \frac{d}{dt}X_t(z)\big|_{t=0}$, $z \in M$, however from [9, pp. 58], we have

$$\mathcal{V} = V^v \circ X. \tag{4.12}$$

On making use of the expression of $\tau_2(X)$ given by (4.6) and (4.12), we find

$$\begin{aligned} \frac{d}{dt}E_2(X_t)\bigg|_{t=0} &= \int_M g_S(V^v, \tau_2(X))v_g \\ &= \int_M \Big\{g(V, (\nabla^*\nabla)^2 X + \sum_{i=1}^m \varepsilon_i\big[(\nabla_{e_i}R)(e_i, S(X))X \\ &\quad + R(e_i, \nabla_{e_i}S(X))X + 2R(e_i, S(X))\nabla_{e_i}X\big])\Big\}v_g, \end{aligned} \tag{4.13}$$

which completes the proof. □

Then, we deduce the following [25].

Corollary 4.2.1. *A vector field X of an m-dimensional pseudo-Riemannian manifold (M, g) is biharmonic if and only if*

$$(\nabla^*\nabla)^2X + \sum_{i=1}^m \varepsilon_i[(\nabla_{e_i}R)(e_i, S(X))X + R(e_i, \nabla_{e_i}S(X))X + 2R(e_i, S(X))\nabla_{e_i}X] = 0,$$

where $\{e_i\}_{i=1}^m$ is a local pseudo-orthonormal frame field of (M, g).

Remark 4.2.1. *Theorem 4.2.3 holds if (M, g) is a non-compact pseudo-Riemannian manifold see [25].*

A reformulation of Theorem 4.2.2 is then

Corollary 4.2.2. *Let (M, g) be an m-dimensional pseudo-Riemannian manifold and $X \in \mathfrak{X}(M)$. Then X is a biharmonic map if and only if X is biharmonic vector field and*

$$\begin{aligned} &-\nabla^*\nabla S(X) - R(X, \nabla^*\nabla X)S(X) + \sum_{i=1}^m \varepsilon_i[R(X, \nabla_{e_i}\nabla^*\nabla X)e_i \\ &- R(\nabla_{e_i}X, \nabla^*\nabla X)e_i + R(e_i, S(X))e_i - (\nabla_{S(X)}R)(X, \nabla_{e_i}X)e_i \\ &- R(X, \nabla_{e_i}X)\nabla_{e_i}S(X) + R(X, R(e_i, S(X))X)e_i] = 0. \end{aligned}$$

4.3 Biharmonicity of left-invariant vector fields of Heisenberg group

The Heisenberg group H_3 can be seen as the Cartesian 3-space $\mathbb{R}^3(x, y, z)$ endowed with multiplication

$$(x, y, z)(\bar{x}, \bar{y}, \bar{z}) = (x + \bar{x}, y + \bar{y}, z + \bar{z} - x\bar{y}).$$

H_3 is three-dimensional Lie group. In [30], the authors proved that any left-invariant Lorentzian metric on H_3, is isometric to one of the subsequent metrics

$$\begin{aligned} g_1 &= -dx^2 + dy^2 + (xdy + dz)^2, \\ g_2 &= dx^2 + dy^2 - (xdy + dz)^2, \\ g_3 &= dx^2 + (xdy + dz)^2 - ((1 - x)dy - dz)^2. \end{aligned}$$

In this section we investigate biharmonicity of left-invariant vector fields on H_3 endowed with g_1, g_2 and g_3 respectively.

4.3.1 Biharmonicity of left-invariant vector fields on (H_3, g_1)

The aim of this subsection is to completely determine the set of left-invariant vector fields on (H_3, g_1) which are biharmonic and biharmonic maps respectively. The left-invariant vector fields

$$e_1 = \frac{\partial}{\partial z}, \;\; e_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}, \;\; e_3 = \frac{\partial}{\partial x},$$

constitute an orthonormal basis of the Lie algebra of H_3 with

$$g_1(e_1, e_1) = g_1(e_2, e_2) = 1, \;\; g_1(e_3, e_3) = -1,$$

for which, we have the Lie brackets:

$$[e_2, e_3] = e_1, \;\; [e_1, e_2] = 0, \;\; [e_1, e_3] = 0,$$

The components of the Levi-Civita connection of (H_3, g_1) are determined by [30]

$$\begin{aligned} &\nabla_{e_1} e_1 = 0, && \nabla_{e_1} e_2 = \frac{1}{2} e_3, && \nabla_{e_1} e_3 = \frac{1}{2} e_2, \\ &\nabla_{e_2} e_1 = \frac{1}{2} e_3, && \nabla_{e_2} e_2 = 0, && \nabla_{e_2} e_3 = \frac{1}{2} e_1, \\ &\nabla_{e_3} e_1 = \frac{1}{2} e_2, && \nabla_{e_3} e_2 = -\frac{1}{2} e_1, && \nabla_{e_3} e_3 = 0. \end{aligned} \tag{4.14}$$

Also the curvature components are given by

$$\begin{aligned} &R(e_1, e_2)e_1 = \frac{1}{4} e_2, && R(e_1, e_2)e_2 = -\frac{1}{4} e_1, && R(e_1, e_2)e_3 = 0, \\ &R(e_2, e_3)e_1 = 0, && R(e_2, e_3)e_2 = -\frac{3}{4} e_3, && R(e_2, e_3)e_3 = -\frac{3}{4} e_2, \\ &R(e_3, e_1)e_1 = -\frac{1}{4} e_3, && R(e_3, e_1)e_2 = 0, && R(e_3, e_1)e_3 = -\frac{1}{4} e_1. \end{aligned} \tag{4.15}$$

Let $X = \alpha e_1 + \beta e_2 + \gamma e_3$ an arbitrary left-invariant vector field on (H_3, g_1). By using (4.14) and (4.15), one has

$$\begin{aligned}
\nabla^*\nabla X &= \frac{\alpha}{2}e_1 + \frac{\beta}{2}e_2 + \frac{\gamma}{2}e_3,\\
(\nabla^*\nabla)^2 X &= \frac{\alpha}{4}e_1 + \frac{\beta}{4}e_2 + \frac{\gamma}{4}e_3,\\
S(X) &= \frac{\alpha\gamma}{4}e_2 + \frac{\alpha\beta}{4}e_3.
\end{aligned} \tag{4.16}$$

By virtue of (4.14)-(4.16), a long but straightforward calculation we get

Proposition 4.3.1. *Let $X = \alpha e_1 + \beta e_2 + \gamma e_3$ be a left-invariant vector field on the Lorentzian Lie group (H_3, g_1). Then,*

$$\begin{aligned}
&(\nabla^*\nabla)^2 X + \sum_{i=1}^{3} \varepsilon_i[(\nabla_{e_i} R)(e_i, S(X))X + R(e_i, \nabla_{e_i} S(X))X\\
&+ 2R(e_i, S(X))\nabla_{e_i} X] = \frac{\alpha(4-(\beta^2-\gamma^2))}{16}e_1 + \frac{\beta(4-\alpha^2)}{16}e_2 + \frac{\gamma(4-\alpha^2)}{16}e_3,
\end{aligned}$$

and

$$\begin{aligned}
&- \nabla^*\nabla S(X) - R(X, \nabla^*\nabla X)S(X) + \sum_{i=1}^{3} \varepsilon_i[R(X, \nabla_{e_i}\nabla^*\nabla X)e_i\\
&- R(\nabla_{e_i} X, \nabla^*\nabla X)e_i + R(e_i, S(X))e_i - (\nabla_{S(X)} R)(X, \nabla_{e_i} X)e_i\\
&- R(X, \nabla_{e_i} X)\nabla_{e_i} S(X) + R(X, R(e_i, S(X))X)e_i]\\
&= \frac{\alpha\gamma(-8-2(\gamma^2-\beta^2)-\alpha^2)}{16}e_2 + \frac{\alpha\beta(-8-2(\gamma^2-\beta^2)-\alpha^2)}{16}e_3.
\end{aligned}$$

From Proposition 4.3.1, we easily conclude that the vector field $X = \alpha e_1 + \beta e_2 + \gamma e_3$ is biharmonic map if and only if

$$\begin{cases}
\alpha(4-(\beta^2-\gamma^2)) = 0,\\
\beta(4-\alpha^2) = 0,\\
\gamma(4-\alpha^2) = 0,
\end{cases} \tag{4.17}$$

and

$$\begin{cases}
\alpha\gamma(-8-2(\gamma^2-\beta^2)-\alpha^2) = 0,\\
\alpha\beta(-8-2(\gamma^2-\beta^2)-\alpha^2) = 0.
\end{cases} \tag{4.18}$$

In particular, X is biharmonic vector field if and only if (4.17) holds. From the system (4.17), we obtain that the coordinates of X satisfy the equations of hyperbolas: $C_1 = \{\alpha = 2,\ \beta^2 - \gamma^2 = 4\}$ and $C_2 = \{\alpha = -2,\ \beta^2 - \gamma^2 = 4\}$. Summarizing, we yield

Theorem 4.3.1. *Let $X = \alpha e_1 + \beta e_2 + \gamma e_3$ be a left-invariant vector field on the Lorentzian Lie group (H_3, g_1). Then,*

1. *$X = \alpha e_1+\beta e_2+\gamma e_3$ is a biharmonic vector field which does not define biharmonic map if and only if the coordinates of X satisfy the equations of the equilateral hyperbolas C_1 and C_2.*

2. *The set of left-invariant vector fields which are proper biharmonic maps into TH_3 is empty.*

4.3.2 Biharmonicity of left-invariant vector fields on (H_3, g_2)

This subsection is devoted to the determination of the set of left-invariant vector fields on (H_3, g_2) which are biharmonic and biharmonic maps respectively. The left-invariant vector fields

$$e_1 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial x}, \quad e_3 = \frac{\partial}{\partial z},$$

constitute an orthonormal basis of the Lie algebra of H_3 with

$$g_2(e_1, e_1) = g_2(e_2, e_2) = 1, \quad g_2(e_3, e_3) = -1,$$

for which, we have the Lie brackets:

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = 0, \quad [e_2, e_3] = 0.$$

The components of the Levi-Civita connection of (H_3, g_2) are determined by [30]

$$\begin{aligned}
&\nabla_{e_1} e_1 = 0, && \nabla_{e_1} e_2 = \frac{1}{2}e_3, && \nabla_{e_1} e_3 = \frac{1}{2}e_2,\\
&\nabla_{e_2} e_1 = -\frac{1}{2}e_3, && \nabla_{e_2} e_2 = 0, && \nabla_{e_2} e_3 = -\frac{1}{2}e_1,\\
&\nabla_{e_3} e_1 = \frac{1}{2}e_2, && \nabla_{e_3} e_2 = -\frac{1}{2}e_1, && \nabla_{e_3} e_3 = 0.
\end{aligned} \tag{4.19}$$

Also the curvature components are given by

$$\begin{aligned}
&R(e_1, e_2)e_1 = -\frac{3}{4}e_2, && R(e_1, e_2)e_2 = \frac{3}{4}e_1, && R(e_1, e_2)e_3 = 0,\\
&R(e_2, e_3)e_1 = 0, && R(e_2, e_3)e_2 = \frac{1}{4}e_3, && R(e_2, e_3)e_3 = \frac{1}{4}e_2,\\
&R(e_3, e_1)e_1 = -\frac{1}{4}e_3, && R(e_3, e_1)e_2 = 0, && R(e_3, e_1)e_3 = -\frac{1}{4}e_1.
\end{aligned} \tag{4.20}$$

Let $X = \alpha e_1 + \beta e_2 + \gamma e_3$ an arbitrary left-invariant vector field on (H_3, g_2). By using (4.19) and (4.20), then one obtains

$$\begin{aligned}
\nabla^*\nabla X &= \frac{\alpha}{2}e_1 + \frac{\beta}{2}e_2 + \frac{\gamma}{2}e_3,\\
(\nabla^*\nabla)^2 X &= \frac{\alpha}{4}e_1 + \frac{\beta}{4}e_2 + \frac{\gamma}{4}e_3,
\end{aligned}$$

$$S(X) = \frac{-\beta\gamma}{4}e_1 + \frac{\alpha\gamma}{4}e_2. \tag{4.21}$$

By virtue of (4.19)-(4.21), a long but direct and easy calculations we get

Proposition 4.3.2. *Let $X = \alpha e_1 + \beta e_2 + \gamma e_3$ be a left-invariant vector field on the Lorentzian Lie group (H_3, g_2). Then,*

$$\begin{aligned}&(\nabla^*\nabla)^2 X + \sum_{i=1}^{3} \varepsilon_i[(\nabla_{e_i}R)(e_i, S(X))X + R(e_i, \nabla_{e_i}S(X))X\\&+ 2R(e_i, S(X))\nabla_{e_i}X] = \frac{\alpha(4+\gamma^2)}{16}e_1 + \frac{\beta(4+\gamma^2)}{16}e_2 + \frac{\gamma(4-(\alpha^2+\beta^2))}{16}e_3,\end{aligned}$$

and

$$\begin{aligned}&-\nabla^*\nabla S(X) - R(X, \nabla^*\nabla X)S(X) + \sum_{i=1}^{3}\varepsilon_i[R(X, \nabla_{e_i}\nabla^*\nabla X)e_i\\&- R(\nabla_{e_i}X, \nabla^*\nabla X)e_i + R(e_i, S(X))e_i - (\nabla_{S(X)}R)(X, \nabla_{e_i}X)e_i\\&- R(X, \nabla_{e_i}X)\nabla_{e_i}S(X) + R(X, R(e_i, S(X))X)e_i]\\&= \frac{\beta\gamma(16+5(\alpha^2+\beta^2)-3\gamma^2)}{32}e_1 + \frac{-\alpha\gamma(16+5(\alpha^2+\beta^2)-3\gamma^2)}{32}e_2.\end{aligned}$$

From Proposition 4.3.2, one conclude that the vector field $X = \alpha e_1 + \beta e_2 + \gamma e_3$ is biharmonic map if and only if

$$\begin{cases}\alpha(4+\gamma^2) = 0,\\ \beta(4+\gamma^2) = 0,\\ \gamma(4-(\alpha^2+\beta^2)) = 0,\end{cases} \tag{4.22}$$

and

$$\begin{cases}\beta\gamma(16+5(\alpha^2+\beta^2)-3\gamma^2) = 0,\\ \alpha\gamma(16+5(\alpha^2+\beta^2)-3\gamma^2) = 0.\end{cases} \tag{4.23}$$

In particular, X is biharmonic vector field if and only if (4.22) holds. From (4.22) and (4.23), one has

Theorem 4.3.2. *On the Lorentzian Lie group (H_3, g_2). We have*

1. *The set of left-invariant biharmonic vector fields which do not define harmonic maps into TH_3 is empty.*

2. *The set of left-invariant vector fields which are proper biharmonic maps into TH_3 is empty.*

4.3.3 Biharmonicity of left-invariant vector fields on (H_3, g_3)

In this subsection we aim to completely determine the set of left-invariant vector fields on (H_3, g_3) which are biharmonic and biharmonic maps respectively. The left-invariant vector fields

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y} + (1-x)\frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z},$$

constitute an orthonormal basis of the Lie algebra of H_3 with

$$g_3(e_1, e_1) = g_3(e_2, e_2) = 1, \quad g_3(e_3, e_3) = -1,$$

for which, we have the Lie brackets:

$$[e_2, e_3] = 0, \quad [e_3, e_1] = e_2 - e_3, \quad [e_2, e_1] = e_2 - e_3.$$

The components of the Levi-Civita connection of (H_3, g_3) are determined by [30]

$$\begin{array}{lll} \nabla_{e_1} e_1 = 0, & \nabla_{e_1} e_2 = 0, & \nabla_{e_1} e_3 = 0, \\ \nabla_{e_2} e_1 = e_2 - e_3, & \nabla_{e_2} e_2 = -e_1, & \nabla_{e_2} e_3 = -e_1, \\ \nabla_{e_3} e_1 = e_2 - e_3, & \nabla_{e_3} e_2 = -e_1, & \nabla_{e_3} e_3 = -e_1. \end{array} \tag{4.24}$$

Let $X = \alpha e_1 + \beta e_2 + \gamma e_3$ an arbitrary left-invariant vector field on (H_3, g_3). By using (4.24), we get that $\nabla^* \nabla X = 0$ and since g_3 is flat we deduce tat $S(X) = 0$. Then, we yield

Theorem 4.3.3. *On the Lorentzian Lie group (H_3, g_3), every left-invariant vector field is biharmonic maps.*

4.4 Gödel universe

An interesting space-time in general relativity is the classical Gödel universe [14]. This model is $\mathbb{R}^4$ endowed with the metric

$$\langle \cdot, \cdot \rangle_L = dx_1^2 + dx_2^2 - \frac{1}{2} e^{2\alpha x_1} dy^2 - 2e^{\alpha x_1} dy dt - dt^2,$$

where α is a positive constant. We denote by $\partial_{\bar{y}} = \sqrt{2}(e^{-\alpha x_1}\partial_y - \partial_t)$. The Levi-Civita connection in the pseudo-orthonormal frame field $\{e_1, e_2, e_3, e_4\}$ where $e_1 = \partial_{x_1}$, $e_2 = \partial_{x_2}$, $e_3 = \partial_{\bar{y}}$ and $e_4 = \partial_t$, is given by [14]

$$\begin{array}{lll} \nabla_{e_1} e_4 = -\frac{\alpha}{\sqrt{2}} e_3, & \nabla_{e_2} e_4 = 0, & \nabla_{e_3} e_4 = \frac{\alpha}{\sqrt{2}} e_1, \\ \nabla_{e_4} e_4 = 0, & \nabla_{e_1} e_1 = 0, & \nabla_{e_2} e_1 = 0, \\ \nabla_{e_3} e_1 = \frac{\alpha}{\sqrt{2}} e_4 + \alpha e_3, & \nabla_{e_2} e_2 = 0, & \nabla_{e_3} e_2 = 0, \end{array} \tag{4.25}$$

$$\nabla_{e_1} e_3 = -\frac{\alpha}{\sqrt{2}} e_4, \qquad \nabla_{e_3} e_3 = -\alpha e_1.$$

Taking the vector field $X = f(x_2)e_4$, where $f(x_2)$ is a smooth real function depending of the variable x_2. From [25] we have

$$R(e_1, e_4)e_3 = R(e_3, e_4)e_1 = 0, \tag{4.26}$$

$$\nabla^*\nabla X = (f'' + \alpha^2 f)e_4, \tag{4.27}$$

$$(\nabla^*\nabla)^2 X = (f'''' + 2\alpha^2 f'' + \alpha^4 f)e_4,$$

and

$$S(X) = 0,$$

where $f' = \frac{df}{dz}$, $f'' = \frac{d^2 f}{dz^2}$ etc. By virtue of relations (4.25), (4.26) and (4.27), we get

$$\sum_{i=1}^{3} \varepsilon_i R(X, \nabla_{e_i} \nabla^*\nabla X)e_i = 0, \text{ and } \sum_{i=1}^{3} \varepsilon_i R(\nabla_{e_i} X, \nabla^*\nabla X)e_i = 0.$$

Then, from Theorem **??**, it follows that X is biharmonic map if and only if the function f satisfies the subsequent differential equation.

$$f'''' + 2\alpha^2 f'' + \alpha^4 f = 0. \tag{4.28}$$

Note that (4.28) is homogeneous fourth order differential equation with general solution see [25]

$$f(x_2) = c_1 \cos(\alpha x_2) + c_2 \sin(\alpha x_2) + c_3 x_2 \cos(\alpha x_2) + c_4 x_2 \sin(\alpha x_2), \tag{4.29}$$

where c_1, c_2, c_3 and c_4 are real constants. Particulary, in [23] Markellos and Urakawa proved that $X = f(x_2)e_4$ is biharmonic vector field, where $f(x_2)$ is given by (4.29).

Proposition 4.4.1. *The vector fields* $X = x_2(c_3 \cos(\alpha x_2) + c_4 \sin(\alpha x_2))e_4$ *are proper biharmonic maps of* $(\mathbb{R}^4, \langle \cdot, \cdot \rangle_L)$.

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