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Eigenvalues, numerical radius and norms of operators inequalities in Hilbert space

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Dedication

I would like to dedicate this modest work to:

My parent, who planted the seed of knowledge in my mind and nurtured it.

Mom and dad, I am nothing without both of you, no words can express how much I love you. My only wish is to repay you both for everything that you have done for me, and someday I hope to make you proud of me. I wish to thank my loving sisters (Rofaida, Chaimaa, Bouchra, Fatima) and brother (Mohammed) for their great support and encouragement that enable me to achieve my goal.

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Abstract

Our main target in this research is to refine some well-known numerical radius inequalities of operators on a Hilbert space or to obtain new bounds. In this thesis, we establish some bounds for the numerical radius of one operator and for the numerical radii of 2×2 operator matrices. Also, we provide new upper bounds for the numerical radii of $n \times n$ operator matrices. Applying some of our results, we have succeeded to give a new bound for the zeros of polynomials. Furthermore, we present an improvement of the triangle inequality for the operator norm. On the other hand, we establish several upper and lower bounds for the p-numerical radius of one operator and of 2×2 operator matrices as well as of $n \times n$ operator matrices. An application to 2-nilpotent operators is provided. A p-numerical radius inequality involving power is also given.

Key words: Numerical radius, normal operator, Schatten p-norm, p-numerical radius, inequality.

Notations

$\mathbb{N}$:	The set of natural numbers.
$\mathbb{R}^+$:	The set of positive real numbers.
$\mathbb{R}$:	The set of real numbers.
$\mathbb{C}$:	The set of complex numbers.
$\mathbb{M}_n(\mathbb{C})$:	$n \times n$ matrices in $\mathbb{C}$.
$\mathbb{K}$:	A field.
$\mathcal{H}$:	Complex Hilbert space.
$\mathbb{B}(\mathcal{H})$:	Bounded linear operators on a complex Hilbert space$\mathcal{H}$.
$\mathcal{K}(\mathcal{H})$:	The set of all compact linear operators on a complex Hilbert space $\mathcal{H}$.
$\mathbb{B}_p(\mathcal{H})$:	The Schatten p-classe.
$\langle \cdot, \cdot \rangle$:	Inner product.
$\| \cdot \|$:	The norm.
T, S, A, B, C, D :	Bounded linear operators.
T^* :	The adjoint of T.
$\|T\|$:	The absolute value of T.
$\tilde{T}$:	Aluthge transform of T.
I :	Identity operator.
$r(T)$:	The spectral radius of T.
$\sigma(T)$:	The spectrum of T.
$w(T)$:	The numerical radius of T.
$w_N(T)$:	The generalized numerical radius of T.
$w_p(T)$:	$p-$ numerical radius of T.
$W(T)$:	The numerical range of T.
$m(T)$:	The Crawford number of T.
$tr(T)$:	Trace of T.
$\Re(T)$:	The real part of T.
$\Im(T)$:	The imaginary part of T.
$Ker(T)$:	The kernal of T.
$ran(T)$:	The range of T.
$F^\perp$:	The orthogonal completement of F.
$\oplus$:	The sign of direct sum.

Introduction

Operator theory and matrix analysis play a central role in mathematics with many applications in different branches of pure and applied mathematics, such as approximation theory, numerical analysis, optimization theory, quantum computing, quantum information theory, and quantum control.

Many results in operator theory or matrix analysis appear in the form of inequalities. In fact, inequalities are used in almost all areas of pure and applied mathematics.

This thesis focuses on the well known concept the numerical radius inequalities for a bounded linear operator in Hilbert space as well as for the numerical radii of 2×2 operator matrices. The numerical radius is defined as

$$w(T) = \sup_{\|x\|=1} |\langle Tx, x\rangle| \quad \text{or} \quad w(T) = \max_{\lambda \in W(T)} |\lambda|,$$

where the $W(T) = \left\{\langle Tx, x\rangle, x \in \mathcal{H}, \|x\| = 1\right\}$ is the numerical range, which is a powerful tool in operator and matrix theory. The numerical range of a linear operator in a Hilbert space $\mathcal{H}$ localizes its spectrum. It was first studied by Toeplitz in 1918. One year later, Hausdorff showed that $W(T)$ is a convex set.

Many books discussed the numerical radius. We cite here some references of them [41], [46] and [47] .

Between 2003 and 2007, Kittaneh and his colleagues gave several inequalities for numerical radius of operators. In 2015, Kittaneh, Moslehian and Yamazaki gave a new identity of the numerical radius of operators concerning the Cartesian decomposition. Also, many estimations for the numerical radii of 2×2 operator matrices were established. New results concerning the numerical radii of $n \times n$ operator matrices were obtained, too. A related notions to the numerical radius have been introduced like the Hilbert-Schmidt numerical radius and p- numerical radius of a bounded linear operator.

This thesis is divided into four chapters.

In the first chapter, we present different mathematical objects, basic definitions and theorems needed for the following chapters.

In the second Chapter, we give new numerical radius inequalities which refine certain existing ones. Also, we provide some numerical radius inequalities for the sums and the products of two operators. Further, we establish some numerical radius inequalities which involve the Cartesian decomposition of an operator. We present an improvement of the triangle inequality for the operator norm.

In the third Chapter, we present some bounds for the numerical radii of 2×2 operator matrices. Also, we give upper bounds for the numerical radii of $n \times n$ operator matrices. Applying some of our results, we provide a new bound for the zeros of polynomials.

In the fourth Chapter, we establish several upper and lower bounds for the p-numerical radius of one operator as well as for $n \times n$ operator matrices. An application to 2-nilpotent operators is provided, and a p-numerical radius power inequality is also given. Also, we present some p-numerical radius inequalities for 2×2 operator matrices.

Contributions

The following articles are extracted from this thesis:

S. Soltani and A. Frakis, *Further refinements of some numerical radius inequalities for oprerators*, Operators and Matrices, 17 (2023), 245-257.

A. Frakis, F. Kittaneh, and S. Soltani, *Upper and lower bounds for the p-numerical radii of operators*, Results in Mathematics, DOI: 10.1007/s00025-023-02090-3

A. Frakis, F. Kittaneh, and S. Soltani, *Bounds for the numerical radii of 2×2 operator matrices*, Vietnam Journal of Mathematics. https://doi.org/10.1007/s10013-023-00638-y

A. Frakis, F. Kittaneh, and S. Soltani, *New numerical radius inequalities for operator-matrices and a bound for the zeros of polynomials*, Advances in Operator Theory, 8 (2023). https://doi.org/10.1007/s43036-022-00232-y

A. Frakis, F. Kittaneh, and S. Soltani, *On the p-numerical radii of 2×2 operator matrices*, Journal of Applied Mathematics and Computing. 70 (2024), 335-350.

Chapter 1

Preliminaries

In this Chapter, we give some basic concepts and results that will be used throughout this thesis. The most materials given in this chapter can be found in [14, 41, 46, 47].

1.1 Inner product and Hilbert space

Definition 1.1.1. *Let E be a vector space over a field $\mathbb{K} = (\mathbb{R}$ or $\mathbb{C})$. A norm on E is a function $\|\cdot\| : E \longrightarrow \mathbb{R}^{+}$ satisfies the following axioms, for all $x, y \in E$ and $\alpha \in \mathbb{C}$:*

1. *$\|x\| \geq 0$ for all $x \neq 0$ in E (positive).*
2. *$\|x\| = 0$ if and only if $x = 0$.*
3. *$\|\alpha x\| = |\alpha|\ \|x\|$ (homogeneous).*
4. *$\|x + y\| \leq \|x\| + \|y\|$ (the triangle inequality).*

A vector space equipped with a norm is called a normed vector space.

Definition 1.1.2. *Let $\mathcal{H}$ be a complex vector space. An **inner product** on $\mathcal{H}$ is a function $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{C}$ such that for all $x, y, z \in \mathcal{H}$ and $\alpha \in \mathbb{C}$:*

1. *$\langle x, x \rangle \geq 0$.*
2. *$\langle x, x \rangle = 0$ if and only if $x = 0$.*
3. *$\langle \alpha x + y, z \rangle = \alpha \langle x, z \rangle + \langle y, z \rangle$.*
4. *$\langle x, y \rangle = \overline{\langle y, x \rangle}$.*

*A complex vector space $\mathcal{H}$ equipped with an inner product $\langle \cdot, \cdot \rangle$ is called **Inner product space**.*

Theorem 1.1.1. *Any inner product space* $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ *is a normed space where* $\|x\| = \sqrt{\langle x, x\rangle}$, *which is called the associated norm with inner product.*

Definition 1.1.3. *A **Hilbert space** is an inner product space* $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ *such that the associated norm is complete.*

Proposition 1.1.2. *Let* $\mathcal{H}$ *be an inner product space, and let F be a subspace of* $\mathcal{H}$. *The orthogonal complement of F is the set*

$$F^{\perp} = \{x \in \mathcal{H} : \langle x, y\rangle = 0, \forall y \in F\}.$$

Theorem 1.1.3. *If F is a closed subspace of* $\mathcal{H}$, *then* $\mathcal{H}$ *is the direct orthogonal sum of F and* $F^{\perp}$, *i.e.,*

$$\mathcal{H} = F \oplus F^{\perp}.$$

1.2 Some inequalities

1.2.1 Cauchy-Schwarz inequality

Let $x, y \in \mathcal{H}$. Then

$$|\langle x, y\rangle| \leq \|x\| \, \|y\|.$$

If $x = (x_1, x_2, \cdots, x_n) \in \mathbb{C}^n$ and $y = (y_1, y_2, \cdots, y_n) \in \mathbb{C}^n$, then the Cauchy-Schwarz inequality is given by

$$\left|\sum_{i=1}^{n} x_i \overline{y_i}\right| \leq \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} |y_i|^2\right)^{\frac{1}{2}}.$$

1.2.2 Hôlder inequality

Let $x = (x_1, x_2, \cdots, x_n) \in \mathbb{C}^n$ and $y = (y_1, y_2, \cdots, y_n) \in \mathbb{C}^n$ and let $p, q \in [1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\left|\sum_{i=1}^{n} x_i y_i\right| \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |y_i|^q\right)^{\frac{1}{q}}.$$

1.2.3 Buzano's inequality

Let $x, y, e \in \mathcal{H}$ with $\|e\| = 1$. Then

$$|\langle x, e\rangle\langle e, y\rangle| \leq \frac{\|x\| \, \|y\| + |\langle x, y\rangle|}{2}.$$

1.2.4 The arithmetic-geometric mean inequality

Let a and b be positive real numbers. Then the arithmetic-geometric mean inequality is

$$\sqrt{ab} \leq \frac{a+b}{2}$$

or

$$ab \leq \frac{a^2+b^2}{2}.$$

1.2.5 Parallelogram law

Let $x, y \in \mathcal{H}$. Then

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right).$$

1.3 Bounded linear operators

In this section, we present several definitions and basic results of bounded linear operators acting on Hilbert space.

Definition 1.3.1. *Let $\mathcal{H}$ be a Hilbert space. An operator $T : \mathcal{H} \longrightarrow \mathcal{H}$ is a*

*1) **Linear operator** if*

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) \text{ for all } x, y \in \mathcal{H} \text{ and } \alpha, \beta \in \mathbb{C}.$$

*2) **Bounded linear operator** if there exists $c > 0$ such that*

$$\|Tx\| \leq c\|x\| \quad \text{for all} \quad x \in \mathcal{H}.$$

Notation: Let $\mathbb{B}(\mathcal{H})$ be the set of all bounded linear operators on a complex Hilbert space $\mathcal{H}$.

Definition 1.3.2. *Let $T \in \mathbb{B}(\mathcal{H})$. Then*

- *The range of T is the set*

$$ran(T) = \{Tx : x \in \mathcal{H}\}.$$

- *The kernel of T is the set*

$$Ker(T) = \{x \in \mathcal{H} : Tx = 0\}.$$

1.3.1 Generalized Polarization Identity

Proposition 1.3.1. *Let $T \in \mathbb{B}(\mathcal{H})$ and let $x, y \in \mathcal{H}$. Then*

$$\langle Tx, y\rangle = \frac{1}{4}\Big(\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle + i\langle T(x+iy), x+iy\rangle - i\langle T(x-iy), x-iy\rangle\Big).$$

1.3.2 Adjoint of operator

Theorem 1.3.2. *Let $T \in \mathbb{B}(\mathcal{H})$. Then there exists a unique $T^* \in \mathbb{B}(\mathcal{H})$ such that*

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \quad \text{for all} \quad x, y \in \mathcal{H}.$$

*The operator T^* is called **the adjoint** of T.*

Definition 1.3.3. *Let $T \in \mathbb{B}(\mathcal{H})$. Then T is said to be*

- ***self-adjoint*** *if and only if $T^* = T$.*
- ***normal*** *if and only if $T^*T = TT^*$.*
- ***unitary*** *if and only if $T^*T = TT^* = I$, where I is the identity operator*
- ***positive operator*** *if T is a self-adjoint operator and $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{H}$.*

Remark 1.3.1.

- *The sum of two positive operators is positive.*
- *The product of two positive operators is a positive if and only if the two operators commute.*
- *Each eigenvalue of a positive operator is positive real number.*
- *A self-adjoint operator T is positive if and only if all eigenvalues of T are positive real numbers.*

Definition 1.3.4. *Let $T \in \mathbb{B}(\mathcal{H})$, the absolute value of T, which is denoted by $|T|$, is the unique positive square root of the positive operator T^*T, i.e.,*

$$|T| = (T^*T)^{\frac{1}{2}}.$$

1.4 Unitarily invariant norms

In this section, we review the basic definitions of unitarily invariant norms.

Definition 1.4.1. *A function* $\|\cdot\| : \mathbb{B}(\mathcal{H}) \longrightarrow \mathbb{R}^+$ *is called a norm if for all* $T, S \in \mathbb{B}(\mathcal{H})$ *and* $\alpha \in \mathbb{C}$*, it satisfies the following axioms:*

1. $\|T\| \geq 0.$

2. $\|T\| = 0$ *if and only if* $T = 0.$

3. $\|\alpha T\| = |\alpha| \, \|T\|.$

4. $\|T + S\| \leq \|T\| + \|S\|.$

Definition 1.4.2. *Let* $T \in \mathbb{B}(\mathcal{H})$*, a norm* $\|\cdot\|$ *is a unitarily invariant norm if* $\|UTV\| = \|T\|$ *for any unitary operators* $U, V \in \mathbb{B}(\mathcal{H})$*.*

Remark 1.4.1. *If a norm is a unitarily invariant norm, it will be denoted by* $\|\cdot\|_u$*. Let* $T \in \mathbb{B}(\mathcal{H})$*. Then*

$$\|T\|_u = \|T^*\|_u = \|\,|T|\,\|_u.$$

1.4.1 Operator norm

Definition 1.4.3. *Let* $T \in \mathbb{B}(\mathcal{H})$*. Then*

$$\|T\| = \sup_{\|x\|=1} \|Tx\|.$$

The norm $\|\cdot\|$ *on* $\mathbb{B}(\mathcal{H})$ *is called* ***the usual operator norm****. An equivalent definition of the operator norm is*

$$\|T\| = \sup_{\|x\|=\|y\|=1} |\langle Tx, y\rangle|.$$

Proposition 1.4.1. *Let* $T, S \in \mathbb{B}(\mathcal{H})$ *and* $n \in \mathbb{N}$*. Then*

- $\|Tx\| \leq \|T\|\|x\| \quad$ *for all* $x \in \mathcal{H}$*.*
- $\|TS\| \leq \|T\|\|S\|.$
- $\|T^n\| \leq \|T\|^n.$

Lemma 1.4.2. *Let* $T, S \in \mathbb{B}(\mathcal{H})$ *be positive semidefinite operators. Then*

$$\|(T+S)^r\| \leq \|T^r + S^r\| \quad \text{for } 0 < r \leq 1.$$

Proposition 1.4.3. *Let* $T \in \mathbb{B}(\mathcal{H})$*. Then*

- $\|T^*\| = \|T\|.$

- $\|T^*T\| = \|TT^*\| = \|T\|^2.$

- $\| |T| \| = \| |T^*| \| = \|T\|$.

Next, we present lemmas which are needed throughout this thesis. The following lemma is a consequence of the convexity of the absolute value function and it can be found in [50].

Lemma 1.4.4. *Let $T \in \mathbb{B}(\mathcal{H})$ be self-adjoint, and let $x \in \mathcal{H}$ be any vector. Then*

$$|\langle Tx, x\rangle| \leq \langle |T|x, x\rangle.$$

The next lemma follows from the spectral theorem for positive operators and Jensen's inequality which can be found in [12, 50].

Lemma 1.4.5. *Let $T \in \mathbb{B}(\mathcal{H})$ be positive semidefinite, and let $x \in \mathcal{H}$ be any vector. Then*

1. $\langle Tx, x\rangle^r \leq \langle T^r x, x\rangle \quad$ *for* $r \geq 1$.
2. $\langle T^r x, x\rangle \leq \langle Tx, x\rangle^r \quad$ *for* $0 < r \leq 1$.

The following lemma is known as the mixed Schwarz inequality, it can be found in [41, pp. 75-76].

Lemma 1.4.6. *Let $T \in \mathbb{B}(\mathcal{H})$. Then*

$$|\langle Tx, y\rangle| \leq \langle |T|x, x\rangle^{\frac{1}{2}} \langle |T^*|y, y\rangle^{\frac{1}{2}} \quad \text{for all } x, y \in \mathcal{H}.$$

The following lemma can be found in [50].

Lemma 1.4.7. *Let $T \in \mathbb{B}(\mathcal{H})$ and $0 < \alpha < 1$. Then*

$$|\langle Tx, y\rangle|^2 \leq \langle |T|^{2\alpha} x, x\rangle \langle |T^*|^{2(1-\alpha)} y, y\rangle \qquad \text{for all } x, y \in \mathcal{H}.$$

Definition 1.4.4. *An operator $U \in \mathbb{B}(\mathcal{H})$ is called a partial isometry if*

$$\|Ux\| = \|x\| \quad \text{for all } x \in (Ker(U))^{\perp}.$$

1.5 The Polar decomposition

Theorem 1.5.1. *Let $T \in \mathbb{B}(\mathcal{H})$. Then there exists a partial isometry operator $U \in \mathbb{B}(\mathcal{H})$ such that*

$$T = U|T|, \quad \text{where} \quad |T| = (T^*T)^{\frac{1}{2}}.$$

1.6 The Cartesian decomposition

Theorem 1.6.1. *Let $T \in \mathbb{B}(\mathcal{H})$. Then there exists self-adjoint operators B and C such that*

$$T = B + iC,$$

where $B = \dfrac{T + T^*}{2}$ and $C = \dfrac{T - T^*}{2i}$. This decomposition is called the Cartesian decomposition of T. The operators B and C are called real part and imaginary part of T, respectively.

1.7 The spectral radius

In this section, we introduce the concept of spectral radius.

1.7.1 The eigenvalues

Let $\mathbb{M}_n(\mathbb{C})$ denote the set of all $n \times n$ complex matrices.

Definition 1.7.1. *Let $T \in \mathbb{M}_n(\mathbb{C})$. Then a complex number λ is called an eigenvalue of T, if there exists a nonzero vector $x \in \mathbb{C}^n$ such that $Tx = \lambda x$. The vector x is called an eigenvector of T corresponding to λ.*

Remark 1.7.1. *Let $T \in \mathbb{M}_n(\mathbb{C})$ with eigenvalues λ_i, where $i \in \{1, \dots, n\}$, then the determinant and trace functions are defined by $det(T) = \prod_{i=1}^{n} \lambda_i$ and $tr(T) = \sum_{i=1}^{n} \lambda_i$, respectively.*

Definition 1.7.2. *Let $T \in \mathbb{M}_n(\mathbb{C})$. The polynomial $p(\lambda) = det(T - \lambda I)$ is called the characteristic polynomial of T. The set of all eigenvalues of T is called the spectrum of T and it is denoted by $\sigma(T)$.*

1.7.2 The spectral radius

Definition 1.7.3. *The spectral radius $r(T)$ of an operator $T \in \mathbb{B}(\mathcal{H})$ is defined as*

$$r(T) = \sup \{|\lambda| : \lambda \in \sigma(T)\}.$$

Theorem 1.7.1. *Let $T \in \mathbb{B}(\mathcal{H})$. Then*

$$r(T) = \lim_{n \longrightarrow +\infty} \|T^n\|^{\frac{1}{n}}.$$

Properties 1.7.2. *Let $T, S \in \mathbb{B}(\mathcal{H})$, $\alpha \in \mathbb{C}$ and let n be a positive integer.*

- $r(\alpha T) = |\alpha| r(T)$.
- $r(T) \leq \|T\|$.
- $r(T^n) = r^n(T)$.
- $r(T^*) = r(T)$.
- $r(TS) = r(ST)$.

Theorem 1.7.3. *Let $T, S \in \mathbb{B}(\mathcal{H})$ such that $TS + ST$. Then*

$$r(T + S) \leq r(T) + r(S)$$

and

$$r(TS) \leq r(T)r(S).$$

1.8 The numerical range

Definition 1.8.1. *[39, p. 1]The numerical range of an operator $T \in \mathbb{B}(\mathcal{H})$ is the subset of the complex numbers $\mathbb{C}$, defined by*

$$W(T) = \{\langle Tx, x \rangle : \ x \in \mathcal{H}, \ \|x\| = 1\}.$$

Theorem 1.8.1. *The numerical range of an operator $T \in \mathbb{B}(\mathcal{H})$ is a bounded non-empty set.*

Proof.

- Let $x \in \mathcal{H}$ with $x \neq 0_{\mathcal{H}}$, by taking $y = \frac{x}{\|x\|}$, then $\|y\| = \left\| \frac{x}{\|x\|} \right\| = 1$. Since $\langle Ty, y \rangle \in W(T)$, then $W(T) \neq \emptyset$.
- Let $\lambda \in W(T)$, then there exists $x \in \mathcal{H}$ with $\|x\| = 1$ such as $\lambda = \langle Tx, x \rangle$

$$|\lambda| = |\langle Tx, x \rangle| \leq \|Tx\| \, \|x\| \leq \|T\|.$$

Then $W(T)$ is a bounded non-empty set. ∎

The following properties of $W(T)$ follow immediately from the definition.

Properties 1.8.2. *Let $T \in \mathbb{B}(\mathcal{H})$. Then*

- $W(\alpha I + \beta T) = \alpha + \beta W(T)$ *for $\alpha, \beta \in \mathbb{C}$.*

- $W(T+S) \subseteq W(T) + W(S)$.
- $W(T^*) = \{\bar{\lambda},\ \lambda \in W(T)\}$, *where* T^* *is the adjoint operator of* T.
- $W(U^*TU) = W(T)$ *for any unitarily operator* $U \in \mathbb{B}(\mathcal{H})$.

Example 1.8.1. *Let* $T \in \mathbb{M}_2(\mathbb{C})$ *be such that* $T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. *If* $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{C}^2$ *with* $\|x\| = 1$, *then*

$$\begin{aligned} |\langle Tx, x\rangle| &= \left|\left\langle \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right\rangle\right| \\ &= \left|\left\langle \begin{bmatrix} x_2 \\ 0 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right\rangle\right| \\ &= |x_2\overline{x_1}| \\ &\leq \frac{|x_2|^2 + |x_1|^2}{2} \\ &= \frac{1}{2}. \end{aligned}$$

So,

$$W(T) \subseteq \left\{\lambda \in \mathbb{C}, |\lambda| \leq \frac{1}{2}\right\}.$$

Letting $z = re^{i\theta}$ *and* $0 \leq r \leq \frac{1}{2}$, *if we choose* $x = \left(\cos\alpha, e^{i\theta}\sin\alpha\right)$, *where* $\sin 2\alpha = 2r \leq 1$ *and* $0 \leq \alpha \leq \frac{\pi}{4}$, *we see that*

$$\langle Tx, x\rangle = e^{i\theta}\sin\alpha\cos\alpha = re^{i\theta}.$$

Thus, $W(T) = \left\{\lambda \in \mathbb{C}, |\lambda| \leq \frac{1}{2}\right\}$, *the full half-disk.*

In the following theorem, we cite a very important property of the numerical range of an operator, it can be found in [47].

Theorem 1.8.3. *Let* $T \in \mathbb{B}(\mathcal{H})$. *Then*

$$\sigma(T) \subseteq \overline{W(T)}.$$

1.8.1 Crawford number

The Crawford number of the operator $T \in \mathbb{B}(\mathcal{H})$ is defined by

$$m(T) = \inf\{|\langle Tx, x\rangle| : x \in \mathcal{H}, \|x\| = 1\}.$$

1.8.2 The numerical radius

Definition 1.8.2. *The numerical radius of the operator $T \in \mathbb{B}(\mathcal{H})$, denoted by $w(\cdot)$, is defined by*

$$w(T) = \sup_{\|x\|=1} |\langle Tx, x\rangle|.$$

It is well-known that $w(\cdot)$ is a norm on $\mathbb{B}(\mathcal{H})$. That is for all $T, S \in \mathbb{B}(\mathcal{H})$ and $\alpha \in \mathbb{C}$, we have

- $w(T) \geq 0$ *and* $w(T) = 0$ *if and only if* $T = 0$.
- $w(\alpha T) = |\alpha| w(T)$.
- $w(T + S) \leq w(T) + w(S)$.

The numerical radius is a weakly unitarily invariant norm, i.e.,

$$w(T) = w(UTU^*), \qquad \text{for any unitary operator } U \in \mathbb{B}(\mathcal{H}). \tag{1.1}$$

Proposition 1.8.4. *Let $T \in \mathbb{B}(\mathcal{H})$ and $x \in \mathcal{H}$, we have*

$$|\langle Tx, x\rangle| \leq w(T)\|x\|^2.$$

Proof. Let $x, y \in \mathcal{H}$. Then

$$|\langle Ty, y\rangle| \leq \sup_{\|y\|=1} |\langle Ty, y\rangle| = w(T).$$

We put $y = \frac{x}{\|x\|}$ with $x \in \mathcal{H}$, it follows that

$$\left|\left\langle T\frac{x}{\|x\|}, \frac{x}{\|x\|}\right\rangle\right| \leq w(T).$$

Hence,

$$|\langle Tx, x\rangle| \leq \|x\|^2 w(T).$$

■

The following theorem show that the numerical radius $w(\cdot)$ is equivalent to the operator norm. It can be found in [39, p. 9]

Theorem 1.8.5. *Let $T \in \mathbb{B}(\mathcal{H})$. Then*

$$\frac{1}{2}\|T\| \leq w(T) \leq \|T\|. \tag{1.2}$$

Proof. Let $x, y \in \mathcal{H}$ with $\|x\| = \|y\| = 1$, by the Cauchy-Schwarz inequality, we get

$$|\langle Tx, x\rangle| \leq \|Tx\| \leq \|T\|.$$

By taking the supremum in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$, we obtain the second inequality in (1.2).

To prove the first inequality, we use the generalized polarization identity

$$4\langle Tx, y\rangle = \langle T(x+y), (x+y)\rangle - \langle T(x-y), (x-y)\rangle + i\langle T(x+iy), (x+iy)\rangle - i\langle T(x-iy), (x-iy)\rangle.$$

Thus,

$$\begin{aligned} 4|\langle Tx, y\rangle| &\leq w(T)\left(\|x+y\|^2 + \|x-y\|^2 + \|x+iy\|^2 + \|x-iy\|^2\right) \\ &= 4w(T)\left(\|x\|^2 + \|y\|^2\right). \end{aligned}$$

By taking the supremum on both sides in the above inequality over $x, y \in \mathcal{H}$ with $\|x\| = \|y\| = 1$, we obtain

$$\frac{1}{2}\|T\| \leq w(T),$$

as required. ■

Theorem 1.8.6. *Let $T \in \mathbb{B}(\mathcal{H})$. Then*

$$r(T) \leq w(T) \leq \|T\|.$$

Remark 1.8.1. *Let $T \in \mathbb{B}(\mathcal{H})$. If T is normal, then*

$$w(T) = \|T\| = r(T).$$

If $T^2 = 0$, then

$$w(T) = \frac{\|T\|}{2}. \tag{1.3}$$

In the following theorem, we give a result du to Yamazaki [65], in which he provide another formula of the numerical radius.

Theorem 1.8.7. *[65] Let $T \in \mathbb{B}(\mathcal{H})$. Then*

$$w(T) = \sup_{\theta \in \mathbb{R}} \left\|Re(e^{i\theta}T)\right\| = \sup_{\theta \in \mathbb{R}} \left\|Im(e^{i\theta}T)\right\|. \tag{1.4}$$

Proof. Let $x \in \mathcal{H}$ be any unit vector. Then

$$\begin{aligned}\sup_{\theta\in\mathbb{R}} \left\|Re(e^{i\theta}T)\right\| &= \sup_{\theta\in\mathbb{R}} w(Re(e^{i\theta}T)) \\ &= \sup_{\theta\in\mathbb{R}} \sup_{\|x\|=1} |\langle Re(e^{i\theta}T)x, x\rangle| \\ &= \sup_{\|x\|=1} |\langle Tx, x\rangle| \\ &= w(T).\end{aligned}$$

By replacing T with (iT) in $w(T) = \sup_{\theta\in\mathbb{R}} \left\|Re(e^{i\theta}T)\right\|$, we get

$$w(T) = \sup_{\theta\in\mathbb{R}} \left\|Im(e^{i\theta}T)\right\|.$$

■

The following inequality is known as the power inequality of the numerical radius.

Theorem 1.8.8. *[39] Let $T \in \mathbb{B}(\mathcal{H})$ and $n \in \mathbb{N}$. Then*

$$w(T^n) \leq w^n(T).$$

Definition 1.8.3. *Let $T = U|T|$ be the polar decomposition of $T \in \mathbb{B}(\mathcal{H})$. The Aluthge transform of T, denoted by $\widetilde{T}$, is defined by*

$$\widetilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}.$$

Properties 1.8.9. *Let $T \in \mathbb{B}(\mathcal{H})$. Then*

- $\|\widetilde{T}\| \leq \|T\|$.
- $r(\widetilde{T}) = r(T)$.
- $w(\widetilde{T}) \leq w(T)$.

Next, the following lemma is very useful for computing the numerical radius of matrices. It can be found in [47, p. 44].

Lemma 1.8.10. *Let $T = [a_{ij}]$ be an $n \times n$ matrix such that $a_{ij} \geq 0$ for all $i, j = 1, \cdots, n$. Then*

$$w(T) = \frac{1}{2} r\left(\left[a_{ij} + a_{ji}\right]\right).$$

1.9 The Schatten $p-$norm

We start this section with the definition of a compact linear operator.

Definition 1.9.1. *Let T be a linear operator. T is a compact linear operator if the $(Tx_n)_{n\geq 1}$ has a convergent subsequence, for every bounded sequence $(x_n)_{n\geq 1}$ in $\mathcal{H}$.*

Notation: Let $\mathcal{K}(\mathcal{H})$ denote the set of all compact linear operators on a complex Hilbert space $\mathcal{H}$.

Let $T \in \mathbb{B}(\mathcal{H})$ and $1 \leq p \leq \infty$. The Schatten p-norm is defined as

$$\|T\|_p = (tr|T|^p)^{\frac{1}{p}}.$$

It should be mentioned here that for $p = \infty$ and $p = 2$, the Schatten p-norm is the usual operator norm $\|T\| = \sup_{\|x\|=1} \|Tx\|$ and the Hilbert-Schmidt norm $\|T\|_2^2 = tr|T|^2$, respectively.

Let $\mathbb{B}_p(\mathcal{H})$ be the set defined as follows.

$$\mathbb{B}_p(\mathcal{H}) = \left\{T \in \mathcal{K}(\mathcal{H}) \ : \ \|T\|_p < \infty\right\}.$$

In the following proposition, we present some important properties, it can be found in [9, 31].

Proposition 1.9.1. *Let $T \in \mathbb{B}_p(\mathcal{H})$, then*

1. *$\|\cdot\|_p$ is a norm [quasi-norm] on $\mathbb{B}_p(\mathcal{H})$ for $1 \leq p \leq \infty$ $[0 < p < 1]$.*
2. *$\|T\|_{rp}^r = \| \, |T|^r\|_p = \| \, |T^*|^r\|_p$ for $0 < p \leq \infty$.*
3. *$\|T\|_p = \|T^*\|_p$ for $0 < p \leq \infty$.*
4. *For any $S \in \mathbb{B}(\mathcal{H})$ and $0 < p \leq \infty$, we have*

$$\|ST\|_p \leq \|S\| \, \|T\|_p.$$

5. *For any $S \in \mathbb{B}_p(\mathcal{H})$ and $0 < p < \infty$, we have*

$$\left\|\begin{bmatrix} T & 0 \\ 0 & S \end{bmatrix}\right\|_p = \left\|\begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix}\right\|_p = \left(\|T\|_p^p + \|S\|_p^p\right)^{\frac{1}{p}}.$$

Lemma 1.9.2. *[52] Let $T, S \in \mathbb{B}_p(\mathcal{H})$. Then*

$$\|T + S\|_p^p \leq \|T\|_p^p + \|S\|_p^p \quad \text{for} \ \ 0 < p \leq 1.$$

1.10 Generalized numerical radius

In this section, we give a brief review for the basic results concerning the generalized numerical radius.

Definition 1.10.1. *A norm $N(\cdot)$ is called self-adjoint on $\mathbb{B}(\mathcal{H})$ if*

$$N(T) = N(T^*) \quad \text{for all } T \in \mathbb{B}(\mathcal{H}).$$

In [5], Abu-Omar and Kittaneh generalized the numerical radius as follows.

Definition 1.10.2. *[5] Let $N(\cdot)$ be a norm on $\mathbb{B}(\mathcal{H})$. The generalized numerical radius is a function defined as:*

$$\begin{aligned} w_N(\cdot): \quad \mathbb{B}(\mathcal{H}) &\longrightarrow \mathbb{R}^+ \\ T &\longmapsto w_N(T) = \sup_{\theta\in\mathbb{R}} N\left(Re(e^{i\theta}T)\right), \end{aligned}$$

where $w_N(\cdot)$ is a norm on $\mathbb{B}(\mathcal{H})$.

In particular, for the case $N(\cdot) = \|\cdot\|_p$, where $p \geq 1$, we have $w_p(T) = \sup_{\theta\in\mathbb{R}} \|Re(e^{i\theta}T)\|_p$, which is the $p-$numerical radius.

In [66], Zamani, Moslehian, Xu and Fu gave a new identity for the generalized numerical radius.

Theorem 1.10.1. *[66] Let $T \in \mathbb{B}(\mathcal{H})$ be with the Cartesian decomposition $T = B + iC$. Then*

$$w_N(T) = \sup_{\substack{\alpha,\beta\in\mathbb{R} \\ \alpha^2+\beta^2=1}} N\left(\alpha B + \beta C\right).$$

Theorem 1.10.2. *[5] Let $T \in \mathbb{B}(\mathcal{H})$. For a self-adjoint norm $N(\cdot)$, we have*

$$\frac{1}{2}N(T) \leq w_N(T) \leq N(T).$$

Theorem 1.10.3. *[66] Let $T \in \mathbb{B}(\mathcal{H})$. Then*

$$w_N(T) \leq \inf_{\theta\in\mathbb{R}} \sqrt{N^2(Re(e^{i\theta}T)) + N^2(Im(e^{i\theta}T))}.$$

Chapter 2

Numerical radius inequalities for a single operator

In this chapter, we present several upper and lower bounds for the numerical radius of one operator .

In the first section, we present basic numerical radius inequalities. In the second section, we give some results of the numerical radius inequalities that involve the Cartesian decomposition. In the last section, we derive new numerical radius inequalities of the product and the sum for two operators.

2.1 Numerical radius inequalities

Kittaneh improved the inequalities in (1.2) as follows.

Theorem 2.1.1. *[56] Let $T \in \mathbb{B}(\mathcal{H})$. Then*

$$\frac{1}{4}\|T^*T + TT^*\| \leq w^2(T) \leq \frac{1}{2}\|T^*T + TT^*\|. \tag{2.1}$$

Proof. Suppose that $T = B + iC$ is the Cartesian decomposition of the operator T. Let $x \in \mathcal{H}$ be any unit vector. It follows from the convexity of the function $f(t) = t^2$ that

$$\begin{aligned} |\langle Tx, x\rangle|^2 &= \langle Bx, x\rangle^2 + \langle Cx, x\rangle^2 \\ &\geq \frac{1}{2}\Big(|\langle Bx, x\rangle| + |\langle Cx, x\rangle|\Big)^2 \\ &\geq \frac{1}{2}\left|\langle (B \pm C)x, x\rangle\right|^2. \end{aligned}$$

By taking the supremum on both sides in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$, we obtain

$$w^2(T) \geq \frac{1}{2}\|B \pm C\|^2 = \frac{1}{2}\|(B \pm C)^2\|.$$

Hence,

$$\begin{aligned} 2w^2(T) &\geq \frac{1}{2}\left(\left\|(B+C)^2\right\| + \left\|(B-C)^2\right\|\right) \\ &\geq \frac{1}{2}\left\|(B+C)^2 + (B-C)^2\right\| \\ &= \left\|B^2 + C^2\right\| \\ &= \frac{1}{2}\|T^*T + TT^*\|. \end{aligned}$$

Therefore,

$$w^2(T) \geq \frac{1}{4}\|T^*T + TT^*\|,$$

which proves the first inequality in (2.1).

Next, let $x \in \mathcal{H}$ be any unit vector. It follows from the Cauchy-Schwarz inequality that

$$\begin{aligned} |\langle Tx, x\rangle|^2 &= \langle Bx, x\rangle^2 + \langle Cx, x\rangle^2 \\ &\leq \|Bx\|^2 + \|Cx\|^2 \\ &= \langle B^2x, x\rangle + \langle C^2x, x\rangle \\ &= \langle (B^2 + C^2)x, x\rangle. \end{aligned}$$

By taking the supremum on both sides in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$, we obtain

$$\begin{aligned} w^2(T) &\leq \|B^2 + C^2\| \\ &= \frac{1}{2}\|T^*T + TT^*\|, \end{aligned}$$

which proves the second inequality in (2.1). ∎

By using the generalized polarization identity and Aluthge transform, Yamazaki gave the following result in [65].

Theorem 2.1.2. *Let $T \in \mathbb{B}(\mathcal{H})$. Then*

$$w(T) \leq \frac{1}{2}\left(\|T\| + w(\widetilde{T})\right). \tag{2.2}$$

Proof. Let $\theta \in \mathbb{R}$ and let $T = U|T|$ be the polar decomposition of T and let $x \in \mathcal{H}$

with $\|x\| = 1$, then by using the generalized polarization identity, we have

$$\begin{aligned}
\left\langle e^{i\theta}Tx, x\right\rangle &= \left\langle e^{i\theta}|T|x, U^*x\right\rangle \\
&= \frac{1}{4}\left(\left\langle |T|\left(e^{i\theta}I + U^*\right)x, \left(e^{i\theta}I + U^*\right)x\right\rangle - \left\langle |T|\left(e^{i\theta}I - U^*\right)x, \left(e^{i\theta}I - U^*\right)x\right\rangle\right) \\
&+ \frac{i}{4}\left(\left\langle |T|\left(e^{i\theta}I + iU^*\right)x, \left(e^{i\theta}I + iU^*\right)x\right\rangle - \left\langle |T|\left(e^{i\theta}I - iU^*\right)x, \left(e^{i\theta}I - iU^*\right)x\right\rangle\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
Re\langle e^{i\theta}Tx, x\rangle &= \frac{1}{4}\left(\left\langle |T|\left(e^{i\theta}I + U^*\right)x, \left(e^{i\theta}I + U^*\right)x\right\rangle - \left\langle |T|\left(e^{i\theta}I - U^*\right)x, \left(e^{i\theta}I - U^*\right)x\right\rangle\right) \\
&\leq \frac{1}{4}\left\langle |T|\left(e^{i\theta}I + U^*\right)x, \left(e^{i\theta}I + U^*\right)x\right\rangle \\
&= \frac{1}{4}\left\langle \left(e^{-i\theta}I + U\right)|T|\left(e^{i\theta}I + U^*\right)x, x\right\rangle \\
&\leq \frac{1}{4}\left\|\left(e^{-i\theta}I + U\right)|T|\left(e^{i\theta}I + U^*\right)\right\| \\
&= \frac{1}{4}\left\|\left(e^{-i\theta}I + U\right)|T|^{\frac{1}{2}}\left(\left(e^{-i\theta}I + U\right)|T|^{\frac{1}{2}}\right)^*\right\| \\
&= \frac{1}{4}\left\|\left(\left(e^{-i\theta}I + U\right)|T|^{\frac{1}{2}}\right)^*\left(e^{-i\theta}I + U\right)|T|^{\frac{1}{2}}\right\| \\
&= \frac{1}{4}\left\||T|^{\frac{1}{2}}\left(e^{i\theta}I + U^*\right)\left(e^{-i\theta}I + U\right)|T|^{\frac{1}{2}}\right\| \\
&= \frac{1}{4}\left\|2|T| + e^{i\theta}|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} + e^{-i\theta}|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}}\right\| \\
&= \frac{1}{4}\left\|2|T| + e^{i\theta}\widetilde{T} + e^{-i\theta}(\widetilde{T})^*\right\| \\
&= \frac{1}{2}\left\||T| + Re(e^{i\theta}\widetilde{T})\right\| \\
&\leq \frac{1}{2}\left(\|T\| + \|Re(e^{i\theta}\widetilde{T})\|\right).
\end{aligned}$$

By taking the supremum on both sides in the above inequality over $\theta \in \mathbb{R}$, we obtain

$$\sup_{\theta\in\mathbb{R}} Re\langle e^{i\theta}Tx, x\rangle \leq \frac{1}{2}\left(\|T\| + w(\widetilde{T})\right).$$

By taking the supremum on both sides in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$, we get

$$w(T) \leq \frac{1}{2}\left(\|T\| + w(\widetilde{T})\right).$$

∎

Corollary 2.1.3. *Let $T \in \mathbb{B}(\mathcal{H})$. If $\widetilde{T} = 0$, then*

$$w(T) = \frac{\|T\|}{2}.$$

Now, we will improve the inequality (2.2). To do this, we need the following lemma, which can be found in [17, 33].

Lemma 2.1.4. *Let $T, S \in \mathbb{B}(\mathcal{H})$. Then*

$$w(TS + ST) \leq 2w(T)\|S\|.$$

Theorem 2.1.5. *Let $T \in \mathbb{B}(\mathcal{H})$. Then*

$$w^2(T) \leq \frac{1}{4}\left(\|T\|^2 + w^2(\widetilde{T}) + w(|T|\widetilde{T} + \widetilde{T}|T|)\right). \tag{2.3}$$

Proof. Let $T = U|T|$ be the polar decomposition of T and let $\theta \in \mathbb{R}$. For any unit vector $x \in \mathcal{H}$, we have

$$\begin{aligned} \langle e^{i\theta}Tx, x\rangle &= \langle e^{i\theta}|T|x, U^*x\rangle \\ &= \frac{1}{4}\left(\langle |T|(e^{i\theta} + U^*)x, (e^{i\theta} + U^*)x\rangle - \langle |T|(e^{i\theta} - U^*)x, (e^{i\theta} - U^*)x\rangle\right) \\ &+ \frac{i}{4}\left(\langle |T|(e^{i\theta} + iU^*)x, (e^{i\theta} + iU^*)x\rangle - \langle |T|(e^{i\theta} - iU^*)x, (e^{i\theta} - iU^*)x\rangle\right). \end{aligned}$$

Thus,

$$\begin{aligned} Re\langle e^{i\theta}Tx, x\rangle &= \frac{1}{4}\left(\langle |T|(e^{i\theta} + U^*)x, (e^{i\theta} + U^*)x\rangle - \langle |T|(e^{i\theta} - U^*)x, (e^{i\theta} - U^*)x\rangle\right) \\ &\leq \frac{1}{4}\|(e^{-i\theta} + U)|T|(e^{i\theta} + U^*)\| \\ &= \frac{1}{4}\left\|(e^{-i\theta} + U)|T|^{\frac{1}{2}}\left((e^{-i\theta} + U)|T|^{\frac{1}{2}}\right)^*\right\| \\ &= \frac{1}{4}\left\|\left((e^{-i\theta} + U)|T|^{\frac{1}{2}}\right)^*(e^{-i\theta} + U)|T|^{\frac{1}{2}}\right\| \\ &= \frac{1}{4}\||T|^{\frac{1}{2}}(e^{i\theta} + U^*)(e^{-i\theta} + U)|T|^{\frac{1}{2}}\| \\ &= \frac{1}{2}\left\||T| + Re(e^{i\theta}\widetilde{T})\right\|. \end{aligned}$$

Then,

$$\begin{aligned} Re\langle e^{i\theta}Tx, x\rangle &\leq \frac{1}{2}\left\|\left(|T| + Re(e^{i\theta}\widetilde{T})\right)^2\right\|^{\frac{1}{2}} \\ &= \frac{1}{2}\left\||T|^2 + Re^2(e^{i\theta}\widetilde{T}) + Re\left(e^{i\theta}(|T|\widetilde{T} + \widetilde{T}|T|)\right)\right\|^{\frac{1}{2}}. \end{aligned}$$

By taking the supremum on both sides in the above inequality over $\theta \in \mathbb{R}$, we obtain

$$w^2(T) \le \frac{1}{4}\left(\|T\|^2 + w^2(\widetilde{T}) + w(|T|\widetilde{T} + \widetilde{T}|T|)\right),$$

as required. ■

Remark 2.1.1. *By using Lemma 2.1.4, we obtain*

$$\begin{aligned} \frac{1}{4}\left(\|T\|^2 + w^2(\widetilde{T}) + w(|T|\widetilde{T} + \widetilde{T}|T|)\right) &\le \frac{1}{4}\left(\|T\|^2 + w^2(\widetilde{T}) + 2w(\widetilde{T})\|T\|\right) \\ &= \left(\frac{1}{2}\|T\| + \frac{1}{2}w(\widetilde{T})\right)^2. \end{aligned}$$

So, the inequality (2.3) is better than the inequality (2.2).

In [4], Abo-Omar and Kittaneh refined the second inequality in (1.2) and have obtained the following result.

Theorem 2.1.6. *Let $T \in \mathbb{B}(\mathcal{H})$. Then*

$$\frac{1}{2}\sqrt{\|T^*T + TT^*\| + 2m(T^2)} \le w(T) \le \frac{1}{2}\sqrt{\|T^*T + TT^*\| + 2w(T^2)}. \tag{2.4}$$

Proof. Let $x \in \mathcal{H}$ be any unit vector and let $\theta \in \mathbb{R}$ such that

$$e^{2i\theta}\langle T^2x, x\rangle = |\langle T^2x, x\rangle|.$$

We have

$$\begin{aligned} w(T) &\ge \|Re(e^{i\theta}T)\| \\ &= \frac{1}{2}\|e^{i\theta}T + e^{-i\theta}T^*\| \\ &= \frac{1}{2}\left\|\left(e^{i\theta}T + e^{-i\theta}T^*\right)^2\right\|^{\frac{1}{2}} \\ &= \frac{1}{2}\left\|T^*T + TT^* + 2Re(e^{2i\theta}T^2)\right\|^{\frac{1}{2}} \\ &\ge \frac{1}{2}\sqrt{|\langle(T^*T + TT^* + 2Re(e^{2i\theta}T^2))\,x, x\rangle|} \\ &= \frac{1}{2}\sqrt{|\langle(T^*T + TT^*)\,x, x\rangle + 2\,\langle Re(e^{2i\theta}T^2)x, x\rangle|} \\ &= \frac{1}{2}\sqrt{|\langle(T^*T + TT^*)\,x, x\rangle + 2Re\,(e^{2i\theta}\,\langle T^2x, x\rangle)|} \\ &= \frac{1}{2}\sqrt{\langle(T^*T + TT^*)\,x, x\rangle + 2\,|\langle T^2x, x\rangle|} \\ &\ge \frac{1}{2}\sqrt{\langle(T^*T + TT^*)\,x, x\rangle + 2m(T^2)}. \end{aligned}$$

By taking the supremum on both sides in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$, we get

$$w(T) \geq \frac{1}{2}\sqrt{\|T^*T + TT^*\| + 2m(T^2)},$$

which proves the first inequality in (2.4).

Now, to prove the second inequality in (2.4), we have

$$\begin{aligned} w(T) &= \sup_{\theta\in\mathbb{R}} \left\|Re(e^{i\theta}T)\right\| \\ &= \frac{1}{2}\sup_{\theta\in\mathbb{R}} \left\|e^{i\theta}T + e^{-i\theta}T^*\right\| \\ &= \frac{1}{2}\sup_{\theta\in\mathbb{R}} \left\|\left(e^{i\theta}T + e^{-i\theta}T^*\right)^2\right\|^{\frac{1}{2}} \\ &= \frac{1}{2}\sup_{\theta\in\mathbb{R}} \left\|T^*T + TT^* + 2Re(e^{2i\theta}T^2)\right\|^{\frac{1}{2}} \\ &\leq \frac{1}{2}\sqrt{\|T^*T + TT^*\| + 2\sup_{\theta\in\mathbb{R}}\|Re(e^{2i\theta}T^2)\|} \\ &= \frac{1}{2}\sqrt{\|T^*T + TT^*\| + 2w(T^2)}, \end{aligned}$$

which proves the second inequality in (2.4). ∎

To prove the next result, we need the following lemma, which can be found in [57].

Lemma 2.1.7. *Let $T_1, T_2, S_1, S_2 \in \mathbb{B}(\mathcal{H})$. Then*

$$\begin{aligned} r(T_1S_1 + T_2S_2) &\leq \frac{1}{2}(\|S_1T_1\| + \|S_2T_2\|) \\ &+ \frac{1}{2}\sqrt{(\|S_1T_1\| - \|S_2T_2\|)^2 + 4\|S_1T_2\|\|S_2T_1\|}. \end{aligned}$$

Theorem 2.1.8. *Let $T \in \mathbb{B}(\mathcal{H})$. Then*

$$w^2(T) \leq \frac{1}{8}\left(2\,w(T^2) + \|S\| + \sqrt{(2\,w(T^2) - \|S\|)^2 + 8\sup_{\theta\in\mathbb{R}}\|SRe(e^{2i\theta}T^2)\|}\right), \tag{2.5}$$

where $S = TT^* + T^*T$.

Proof. Let $x \in \mathcal{H}$ be any unit vector. It is well-known that

$$|\langle Tx, x\rangle|^2 = \frac{1}{4}\sup_{\theta\in\mathbb{R}}\left|e^{i\theta}\langle Tx, x\rangle + e^{-i\theta}\langle T^*x, x\rangle\right|^2.$$

By taking the supremum on both sides in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$, we obtain

$$\begin{aligned} w^2(2T) &\leq \sup_{\theta\in\mathbb{R}} \left\|(e^{i\theta}T + e^{-i\theta}T^*)^2\right\| \\ &= \sup_{\theta\in\mathbb{R}} \left\|2Re(e^{2i\theta}T^2) + TT^* + T^*T\right\| \\ &= \sup_{\theta\in\mathbb{R}} r\left(2Re(e^{2i\theta}T^2) + TT^* + T^*T\right). \end{aligned}$$

By choosing $T_1 = I, T_2 = 2Re(e^{2i\theta}T^2), S_1 = S$ and $S_2 = I$ in Lemma 2.1.7, we get

$$w^2(T) \leq \frac{1}{8}\sup_{\theta\in\mathbb{R}}\left(\|2Re(e^{2i\theta}T^2)\| + \|S\| + \sqrt{(\|2Re(e^{2i\theta}T^2)\| - \|S\|)^2 + 8\|SRe(e^{2i\theta}T^2)\|}\right).$$

Thus,

$$\begin{aligned} w^2(T) &\leq \frac{1}{4}\left\|\begin{bmatrix} \sup_{\theta\in\mathbb{R}}\|2Re(e^{2i\theta}T^2)\| & \sup_{\theta\in\mathbb{R}}\sqrt{\|2SRe(e^{2i\theta}T^2)\|} \\ \sup_{\theta\in\mathbb{R}}\sqrt{\|2SRe(e^{2i\theta}T^2)\|} & \sup_{\theta\in\mathbb{R}}\|S\| \end{bmatrix}\right\| \\ &= \frac{1}{4}\left\|\begin{bmatrix} 2w(T^2) & \sup_{\theta\in\mathbb{R}}\sqrt{\|2SRe(e^{2i\theta}T^2)\|} \\ \sup_{\theta\in\mathbb{R}}\sqrt{\|2SRe(e^{2i\theta}T^2)\|} & \|S\| \end{bmatrix}\right\| \end{aligned}$$

$$= \frac{1}{8}\left(2w(T^2) + \|S\|\right) + \frac{1}{8}\sqrt{(2w(T^2) - \|S\|)^2 + 8\sup_{\theta\in\mathbb{R}}\|SRe(e^{2i\theta}T^2)\|},$$

as required. ■

Remark 2.1.2. *Setting*

$$c_0 = \frac{1}{8}\left(2w(T^2) + \|S\|\right) + \frac{1}{8}\sqrt{(2w(T^2) - \|S\|)^2 + 8\sup_{\theta\in\mathbb{R}}\|SRe(e^{2i\theta}T^2)\|}.$$

Then

$$\begin{aligned} c_0 &\leq \frac{1}{8}\left(2\,w(T^2) + \|S\|\right) + \frac{1}{8}\sqrt{(2w(T^2) - \|S\|)^2 + 8w(T^2)\|S\|} \\ &= \frac{1}{2}w(T^2) + \frac{1}{4}\|TT^* + T^*T\|. \end{aligned}$$

This proves that the inequality (2.5) is an improvement of the second inequality in (2.4).

Theorem 2.1.9. *Let $T \in \mathbb{B}(\mathcal{H})$. Then for $\alpha, \beta > 0$*

$$\frac{1}{2} \sup_{\alpha^2+\beta^2=1} w\left(\alpha^2 T^2 + \beta^2 (T^*)^2\right) \leq w^2(T). \tag{2.6}$$

Proof. Let $x \in \mathcal{H}$ be any unit vector. Then

$$\begin{aligned} 2|\langle Tx, x\rangle|^2 &= |\langle Tx, x\rangle|^2 + |\langle T^*x, x\rangle|^2 \\ &= \sup_{\alpha^2+\beta^2=1} \left(\alpha|\langle Tx, x\rangle| + \beta|\langle T^*x, x\rangle|\right)^2 \\ &\geq \sup_{\alpha^2+\beta^2=1} \left|\langle(\alpha T \pm \beta T^*)x, x\rangle\right|^2. \end{aligned}$$

By taking the supremum on both sides in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$, we get

$$2w^2(T) \geq \sup_{\alpha^2+\beta^2=1} w^2\left(\alpha T \pm \beta T^*\right). \tag{2.7}$$

Thus,

$$\begin{aligned} 4w^2(T) &\geq \sup_{\alpha^2+\beta^2=1} \left(w\left(\alpha T + \beta T^*\right)^2 + w\left(\alpha T - \beta T^*\right)^2\right) \\ &\geq \sup_{\alpha^2+\beta^2=1} w\left((\alpha T + \beta T^*)^2 + (\alpha T - \beta T^*)^2\right). \end{aligned}$$

Hence,

$$2w^2(T) \geq \sup_{\alpha^2+\beta^2=1} w\left(\alpha^2 T^2 + \beta^2 (T^*)^2\right),$$

as required. ■

If we take $\alpha = \beta = \frac{1}{\sqrt{2}}$ in the inequality (2.6) and using the fact that the operator $(T^2 + (T^*)^2)$ is self-adjoint, then we obtain the following inequality, see [30],

$$\frac{1}{4}\left\|T^2 + (T^*)^2\right\| \leq w^2(T). \tag{2.8}$$

Therefore, we conclude that the inequality (2.6) is sharper than the inequality (2.8).

Corollary 2.1.10. *Let $T \in \mathbb{B}(\mathcal{H})$. Then for $\alpha, \beta > 0$*

$$\frac{1}{\sqrt{2}} \max\left\{\sup_{\alpha^2+\beta^2=1} w\left(\alpha T - \beta T^*\right), \sup_{\alpha^2+\beta^2=1} w\left(\alpha T + \beta T^*\right)\right\} \leq w(T). \tag{2.9}$$

Proof. The inequality (2.9) follows from the inequality (2.7). ■

If we choose in the inequality (2.9), $\alpha = \beta = \frac{1}{\sqrt{2}}$ and taking into account that $T + T^*$ and $T - T^*$ are normal, then we get the following inequalities, see [58],

$$\frac{1}{2}\|T - T^*\| \leq w(T) \tag{2.10}$$

and

$$\frac{1}{2}\|T + T^*\| \leq w(T). \tag{2.11}$$

Therefore, one can conclude that the inequality (2.9) is a refinement of both previous inequalities.

Also, El-Haddad and Kittaneh in [32] proved that.

Theorem 2.1.11. *Let $T \in \mathbb{B}(\mathcal{H})$, $0 \leq \alpha \leq 1$ and $r \geq 1$. Then*

$$w^r(T) \leq \frac{1}{2}\||T|^{2\alpha r} + |T^*|^{2(1-\alpha)r}\|. \tag{2.12}$$

El-Haddad and Kittaneh used the following lemma to prove Theorem 2.1.11 can be found in [43].

Lemma 2.1.12. *Let $a, b \geq 0$ and $0 \leq \alpha \leq 1$. Then*

$$a^\alpha b^{1-\alpha} \leq \alpha a + (1-\alpha)b \leq \left(\alpha a^r + (1-\alpha)b^r\right)^{\frac{1}{r}} \quad \text{for } r \geq 1.$$

Proof. Let $x \in \mathcal{H}$ be any unit vector. Then

$$\begin{aligned}
|\langle Tx, x\rangle| &\leq \langle |T|^{2\alpha}x, x\rangle^{\frac{1}{2}} \langle |T^*|^{2(1-\alpha)}x, x\rangle^{\frac{1}{2}} \text{ (by Lemma 1.4.7)} \\
&\leq \left(\frac{\langle |T|^{2\alpha}x, x\rangle^r + \langle |T^*|^{2(1-\alpha)}x, x\rangle^r}{2}\right)^{\frac{1}{r}} \text{ (by Lemma 2.1.12)} \\
&\leq \left(\frac{\langle |T|^{2\alpha r}x, x\rangle + \langle |T^*|^{2(1-\alpha)r}x, x\rangle}{2}\right)^{\frac{1}{r}} \text{ (by Lemma 1.4.5 (1))}.
\end{aligned}$$

Thus,

$$|\langle Tx, x\rangle|^r \leq \frac{1}{2}\langle (|T|^{2\alpha r} + |T^*|^{2(1-\alpha)r})x, x\rangle,$$

by taking the supremum on both sides in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$, we get the desired result. ∎

Theorem 2.1.13. *[32] Let $T \in \mathbb{B}(\mathcal{H})$, $0 \leq \alpha \leq 1$ and $r \geq 1$. Then*

$$w^{2r}(T) \leq \|\alpha|T|^{2r} + (1-\alpha)|T^*|^{2r}\|. \tag{2.13}$$

Proof. Let $x \in \mathcal{H}$ be any unit vector. Then

$$\begin{aligned}
|\langle Tx, x\rangle|^2 &\leq \langle |T|^{2\alpha} x, x\rangle \langle |T^*|^{2(1-\alpha)} x, x\rangle \text{ (by Lemma 1.4.7)} \\
&\leq \langle |T|^2 x, x\rangle^{\alpha} \langle |T^*|^2 x, x\rangle^{(1-\alpha)} \text{ (by Lemma 1.4.5 (2))} \\
&\leq \left(\alpha \langle |T|^2 x, x\rangle^r + (1-\alpha)\langle |T^*|^2 x, x\rangle^r\right)^{\frac{1}{r}} \text{ (by Lemma 2.1.12)} \\
&\leq \left(\alpha \langle |T|^{2r} x, x\rangle + (1-\alpha)\langle |T^*|^{2r} x, x\rangle\right)^{\frac{1}{r}} \text{ (by Lemma 1.4.5 (1))} \\
&= \langle \left(\alpha |T|^{2r} + (1-\alpha)|T^*|^{2r}\right) x, x\rangle^{\frac{1}{r}}.
\end{aligned}$$

By taking the supremum on both sides in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$, we get the desired result. ∎

2.2 Cartesian decomposition and numerical radius inequalities

In this section, we present some equalities and inequalities for the numerical radius by using the Cartesian decomposition of an operator.

In [58], Kittaneh, Moslehian and Yamazaki gave a new identity of the numerical radius of an operator.

Theorem 2.2.1. *[58] Let $T \in \mathbb{B}(\mathcal{H})$ be with the Cartesian decomposition $T = B + iC$. Then*

$$\sup_{\substack{\alpha,\beta \in \mathbb{R} \\ \alpha^2+\beta^2=1}} \|\alpha B + \beta C\| = w(T). \tag{2.14}$$

In particular,

$$\frac{1}{2}\|T + T^*\| \leq w(T) \quad \textit{and} \quad \frac{1}{2}\|T - T^*\| \leq w(T). \tag{2.15}$$

Proof. We have

$$\sup_{\theta \in \mathbb{R}} \|Re(e^{i\theta} T)\| = \sup_{\theta \in \mathbb{R}} w(Re(e^{i\theta} T)) = w(T).$$

By the Cartesian decomposition of T, we get

$$
\begin{aligned}
Re(e^{i\theta}T) &= \frac{e^{i\theta}T + e^{-i\theta}T^*}{2} \\
&= \frac{(\cos(\theta) + i\sin(\theta))T + (\cos(\theta) - i\sin(\theta)T^*)}{2} \\
&= \cos(\theta)\frac{T + T^*}{2} - \sin(\theta)\frac{T - T^*}{2i} \\
&= \alpha B + \beta C. \qquad (2.16)
\end{aligned}
$$

If we put $\alpha = \cos(\theta)$ and $\beta = \sin(\theta)$ in (2.16), the we get the equality (2.14). Especially, by choosing $(\alpha, \beta) = (1, 0)$ and $(\alpha, \beta) = (0, 1)$, we obtain the inequalities (2.15). ■

Lemma 2.2.2. *Let $a, b \geq 0$. Then*

$$(a^s + b^s)^{\frac{1}{s}} \leq (a^r + b^r)^{\frac{1}{r}} \quad \text{for } 0 < r \leq s.$$

In [32], El-Haddad and Kittaneh have obtained the following result.

Theorem 2.2.3. *[32] Let $T \in \mathbb{B}(\mathcal{H})$ be with the Cartesian decomposition $T = B + iC$. Then*

$$w^r(T) \leq \| \, |B|^r + |C|^r \| \quad \text{for } 0 < r \leq 2 \qquad (2.17)$$

and

$$w^r(T) \leq 2^{\frac{r}{2}-1} \| \, |B|^r + |C|^r \| \quad \text{for } r \geq 2. \qquad (2.18)$$

Proof. Let $x \in \mathcal{H}$ be any unit vector and let $1 \leq r \leq 2$. Then

$$
\begin{aligned}
|\langle Tx, x\rangle| &= \left(\langle Bx, x\rangle^2 + \langle Cx, x\rangle^2\right)^{\frac{1}{2}} \\
&\leq (|\langle Bx, x\rangle|^r + |\langle Cx, x\rangle|^r)^{\frac{1}{r}} \quad \text{(by Lemma (2.2.2))} \\
&\leq (\langle |B|x, x\rangle^r + \langle |C|x, x\rangle^r)^{\frac{1}{r}} \quad \text{(by Lemma (1.4.4))} \\
&\leq (\langle |B|^r x, x\rangle + \langle |C|^r x, x\rangle)^{\frac{1}{r}} \quad \text{(by Lemma (1.4.5))} \\
&= \langle (|B|^r + |C|^r)x, x\rangle^{\frac{1}{r}}.
\end{aligned}
$$

By taking the supremum on both sides in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$, we obtain

$$w^r(T) \leq \| \, |B|^r + |C|^r \| .$$

For the case $0 < r \leq 2$, we have

$$\begin{aligned} w^r(T) &\leq \|B^2 + C^2\|^{\frac{r}{2}} \\ &= \left\| \left(B^2 + C^2\right)^{\frac{r}{2}} \right\| \\ &\leq \|\, |B|^r + |C|^r \| \ \text{(by Lemma (1.4.2))}. \end{aligned}$$

For the case $r \geq 2$, we have

$$\begin{aligned} \frac{|\langle Tx, x\rangle|}{\sqrt{2}} &= \left(\frac{\langle Bx, x\rangle^2 + \langle Cx, x\rangle^2}{2}\right)^{\frac{1}{2}} \\ &\leq \left(\frac{|\langle Bx, x\rangle|^r + |\langle Cx, x\rangle|^r}{2}\right)^{\frac{1}{r}} \\ &\leq 2^{-\frac{1}{r}} \left(\langle |B|x, x\rangle^r + \langle |C|x, x\rangle^r\right)^{\frac{1}{r}} \\ &\leq 2^{-\frac{1}{r}} \left(\langle |B|^r x, x\rangle + \langle |C|^r x, x\rangle\right)^{\frac{1}{r}}. \end{aligned}$$

By taking the supremum on both sides in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$, we get

$$w^r(T) \leq 2^{\frac{r}{2}-1} \|\, |B|^r + |C|^r \|,$$

as required. ∎

Theorem 2.2.4. *[32] Let $T \in \mathbb{B}(\mathcal{H})$ be with the Cartesian decomposition $T = B + iC$ and let $r \geq 2$. Then*

$$2^{(-\frac{r}{2}-1)} \|\, |B + C|^r + |B - C|^r \| \leq w^r(T) \leq \frac{1}{2} \|\, |B + C|^r + |B - C|^r \|. \tag{2.19}$$

Proof. We have

$$w^2(T) \geq \frac{1}{2} \|(B \pm C)^2\|.$$

Thus,

$$w^r(T) \geq 2^{-\frac{r}{2}} \|(B \pm C)^2\|^{\frac{r}{2}} = 2^{-\frac{r}{2}} \||B \pm C|^r\|,$$

and so,

$$\begin{aligned} w^r(T) &\geq 2^{-\frac{r}{2}-1} \left(\|\, |B + C|^r\| + \|\, |B - C|^r\|\right) \\ &\geq 2^{-\frac{r}{2}-1} \|\, |B + C|^r + |B - C|^r \|, \end{aligned}$$

which proves the first inequality in (2.19).

Now, we prove the second inequality in (2.19), let $x \in \mathcal{H}$ be any unit vector. Then

$$\begin{aligned} |\langle Tx, x\rangle|^r &= \left(\langle Bx, x\rangle^2 + \langle Cx, x\rangle^2\right)^{\frac{r}{2}} \\ &= 2^{-\frac{r}{2}}\left(\langle (B+C)x, x\rangle^2 + \langle (B-C)x, x\rangle^2\right)^{\frac{r}{2}} \\ &\le 2^{-\frac{r}{2}}2^{\frac{r}{2}-1}\left(|\langle (B+C)x, x\rangle|^r + |\langle (B-C)x, x\rangle|^r\right) \\ &\quad \left(\text{by the convexity of the function } f(t) = t^{\frac{r}{2}}\right) \\ &\le \frac{1}{2}\left(\langle |B+C|x, x\rangle^r + \langle |B-C|x, x\rangle^r\right) \ (\text{by Lemma (1.4.4)}) \\ &\le \frac{1}{2}\left(\langle |B+C|^r x, x\rangle + \langle |B-C|^r x, x\rangle\right) \ (\text{by Lemma (1.4.5)}) \\ &= \frac{1}{2}\left\langle \left(|B+C|^r + |B-C|^r\right) x, x\right\rangle . \end{aligned}$$

By taking the supremum on both sides in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$, we get

$$w^r(T) \le \frac{1}{2}\| \, |B+C|^r + |B-C|^r \| ,$$

which completes the proof of the theorem. ■

Theorem 2.2.5. *Let $T \in \mathbb{B}(\mathcal{H})$ have the Cartesian decomposition $T = B + iC$. Then for $\alpha, \beta > 0$*

$$\sup_{\alpha^2+\beta^2=1} \left\|\alpha^2 B^2 + \beta^2 C^2\right\| \le w^2(T). \tag{2.20}$$

Proof. Let $x \in \mathcal{H}$ be any unit vector. Then

$$\begin{aligned} |\langle Tx, x\rangle|^2 &= \langle Bx, x\rangle^2 + \langle Cx, x\rangle^2 \\ &= \sup_{\alpha^2+\beta^2=1} \left(\alpha|\langle Bx, x\rangle| + \beta|\langle Cx, x\rangle|\right)^2 \\ &\ge \sup_{\alpha^2+\beta^2=1} \left|\langle (\alpha B \pm \beta C)x, x\rangle\right|^2 . \end{aligned}$$

By taking the supremum on both sides in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$, we get

$$w^2(T) \ge \sup_{\alpha^2+\beta^2=1} \|\alpha B \pm \beta C\|^2. \tag{2.21}$$

Thus,

$$\begin{aligned} 2w^2(T) &\geq \sup_{\alpha^2+\beta^2=1} \left(\left\| (\alpha B + \beta C)^2 \right\| + \left\| (\alpha B - \beta C)^2 \right\| \right) \\ &\geq \sup_{\alpha^2+\beta^2=1} \left\| (\alpha B + \beta C)^2 + (\alpha B - \beta C)^2 \right\|. \end{aligned}$$

Hence,

$$w^2(T) \geq \sup_{\alpha^2+\beta^2=1} \left\| \alpha^2 B^2 + \beta^2 C^2 \right\|,$$

as required. ■

Remark 2.2.1. *Taking $\alpha = \beta = \frac{1}{\sqrt{2}}$ in the inequality (2.20), gives*

$$\frac{1}{4}\|T^*T + TT^*\| \leq \sup_{\alpha^2+\beta^2=1} \|\alpha^2 B^2 + \beta^2 C^2\| \leq w^2(T).$$

This proves that the inequality (2.20) is an improvement of the first inequality in (2.1).

Corollary 2.2.6. *Let $T \in \mathbb{B}(\mathcal{H})$ have the Cartesian decomposition $T = B + iC$. Then for $\alpha, \beta > 0$,*

$$\max\left\{ \sup_{\alpha^2+\beta^2=1} \left\| \alpha B - \beta C \right\|, \sup_{\alpha^2+\beta^2=1} \left\| \alpha B + \beta C \right\| \right\} \leq w(T). \tag{2.22}$$

Proof. The inequality (2.22) follows from the inequality (2.21). ■

Now, taking $\alpha = \beta = \frac{1}{\sqrt{2}}$ in the inequality (2.22), gives the following inequality, see [30].

$$\frac{\sqrt{2}}{2} \max\left\{ \left\| \frac{(1-i)T + (1+i)T^*}{2} \right\|, \left\| \frac{(1+i)T + (1-i)T^*}{2} \right\| \right\} \leq w(T). \tag{2.23}$$

Therefore, one can conclude that the inequality (2.22) is a refinement of the inequality (2.23).

A generalization of Theorem 2.2.5 can be stated as follows.

Theorem 2.2.7. *Let $T \in \mathbb{B}(\mathcal{H})$ have the Cartesian decomposition $T = B + iC$, and let $r \geq 2$. Then for $\alpha, \beta > 0$*

$$\sup_{\alpha^2+\beta^2=1} \frac{1}{2} \left\| \, |\alpha B + \beta C|^r + |\alpha B - \beta C|^r \right\| \leq w^r(T). \tag{2.24}$$

Proof. From the inequality (2.21), we get

$$\begin{aligned} w^r(T) &\geq \sup_{\alpha^2+\beta^2=1} \left\|(\alpha B \pm \beta C)^2\right\|^{\frac{r}{2}} \\ &= \sup_{\alpha^2+\beta^2=1} \| \, |\alpha B \pm \beta C|^r \|. \end{aligned}$$

Hence

$$w^r(T) \geq \sup_{\alpha^2+\beta^2=1} \frac{1}{2} \left\| \, |\alpha B + \beta C|^r + |\alpha B - \beta C|^r \right\|.$$

■

Remark 2.2.2. *If we take $\alpha = \beta = \frac{1}{\sqrt{2}}$ in the inequality (2.24), then we obtain*

$$\sup_{\alpha^2+\beta^2=1} \frac{1}{2} \left\| \, |\alpha B + \beta C|^r + |\alpha B - \beta C|^r \right\| \geq 2^{\frac{-r}{2}-1} \| \, |B + C|^r + |B - C|^r \| .$$

This means that the inequality (2.24) is a refinement of the first inequality in (2.19).

Theorem 2.2.8. *Let $T \in \mathbb{B}(\mathcal{H})$ have the Cartesian decomposition $T = B + iC$. Then*

$$w^2(T) \leq \frac{1}{2} \left\{ \|B\|^2 + \|C\|^2 + \sqrt{(\|B\|^2 - \|C\|^2)^2 + \|BC + CB\|^2} \right\}. \tag{2.25}$$

Proof. We have

$$\begin{aligned} w^2(T) &= \sup_{\alpha^2+\beta^2=1} \|\alpha B + \beta C\|^2 \\ &= \sup_{\alpha^2+\beta^2=1} \|\alpha^2 B^2 + \beta^2 C^2 + \alpha\beta(BC + CB)\| \\ &\leq \sup_{\alpha^2+\beta^2=1} \left(\alpha^2 \|B\|^2 + \beta^2 \|C\|^2 + |\alpha\beta| \|BC + CB\|\right) \\ &= \frac{1}{2} \left\{ \|B\|^2 + \|C\|^2 + \sqrt{(\|B\|^2 - \|C\|^2)^2 + \|BC + CB\|^2} \right\}, \end{aligned}$$

as required. ■

2.3 Numerical radius inequalities for the product and the sum for two operators

In this sections, we present some numerical radius inequalities of the product and the sum for two operators.

Theorem 2.3.1. *[48] Let $T, S \in \mathbb{B}(\mathcal{H})$. Then*

$$w(TS) \leq 4w(T)w(S).$$

If T and S are commute, then

$$w(TS) \leq 2w(T)w(S).$$

Proof. Let $T, S \in \mathbb{B}(\mathcal{H})$. Then

1.

$$\begin{aligned} w(TS) &\leq \|TS\| \\ &\leq \|T\| \, \|S\| \\ &\leq 4w(T)w(S). \end{aligned}$$

2. We may assume that $w(T) = w(S) = 1$ and prove that $w(TS) \leq 2$. We have

$$\begin{aligned} w(TS) &= \frac{1}{4}w\left((T+S)^2 - (T-S)^2\right) \\ &\leq \frac{1}{4}\left[w\left((T+S)^2\right) + w\left((T-S)^2\right)\right] \\ &\leq \frac{1}{4}\left(w^2(T+S) + w^2(T-S)\right) \\ &\leq \frac{1}{2}\left(w(T) + w(S)\right)^2 \\ &= 2. \end{aligned}$$

∎

Sattari, Moslehian and Yamazaki have obtained another upper bound for $w(S^*T)$ as follows.

Theorem 2.3.2. *[62] Let $T, S \in \mathbb{B}(\mathcal{H})$. Then*

$$w^r(S^*T) \leq \frac{1}{4}\left\| |T^*|^{2r} + |S^*|^{2r} \right\| + \frac{1}{2}w^r(TS^*) \quad \text{for all } r \geq 1. \tag{2.26}$$

Proof. Let $x \in \mathcal{H}$ be any unit vector, we have

$$\begin{aligned}
Re\langle e^{i\theta}Tx, Sx\rangle &= \frac{1}{4}\left\|(e^{i\theta}T + S)x\right\|^2 - \frac{1}{4}\left\|(e^{i\theta}T - S)x\right\|^2 \\
&\leq \frac{1}{4}\left\|(e^{i\theta}T + S)x\right\|^2 \\
&\leq \frac{1}{4}\left\|e^{i\theta}T + S\right\|^2 \\
&= \frac{1}{4}\left\|(e^{i\theta}T + S)(e^{-i\theta}T^* + S^*)\right\| \\
&= \frac{1}{4}\left\|TT^* + SS^* + 2Re(e^{i\theta}TS^*)\right\| \\
&\leq \frac{1}{4}\|TT^* + SS^*\| + \frac{1}{2}\left\|Re(e^{i\theta}TS^*)\right\| \\
&\leq \frac{1}{4}\|TT^* + SS^*\| + \frac{1}{2}w(TS^*).
\end{aligned}$$

By taking the supremum on both sides in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$ and over $\theta \in \mathbb{R}$, we obtain

$$w(S^*T) \leq \frac{1}{4}\|TT^* + SS^*\| + \frac{1}{2}w(TS^*).$$

By the convexity of the function $f(t) = t^r$ on $[0, \infty)$, where $r \geq 1$, we have

$$\begin{aligned}
w^r(S^*T) &\leq \left(\frac{1}{2}\left\|\frac{TT^* + SS^*}{2}\right\| + \frac{1}{2}w(TS^*)\right)^r \\
&\leq \frac{1}{2}\left\|\frac{TT^* + SS^*}{2}\right\|^r + \frac{1}{2}w^r(TS^*) \\
&\leq \frac{1}{2}\left\|\frac{(TT^*)^r + (SS^*)^r}{2}\right\| + \frac{1}{2}w^r(TS^*),
\end{aligned}$$

as required. ∎

Theorem 2.3.3. *Let $T, S \in \mathbb{B}(\mathcal{H})$. Then*

$$w^2(S^*T) \leq \frac{1}{4}w^2(TS^*) + \frac{1}{8}w(PTS^* + TS^*P) + \frac{1}{16}\|P\|^2, \tag{2.27}$$

where $P = TT^* + SS^*$.

Proof. Let $x \in \mathcal{H}$ be any unit vector. For any $\theta \in \mathbb{R}$, we have

$$\begin{aligned}
Re\langle e^{i\theta} S^* Tx, x\rangle &= Re\langle e^{i\theta} Tx, Sx\rangle \\
&= \frac{1}{4}\left\|(e^{i\theta}T + S)x\right\|^2 - \frac{1}{4}\left\|(e^{i\theta}T - S)x\right\|^2 \\
&\leq \frac{1}{4}\left\|e^{i\theta}T + S\right\|^2 \\
&= \frac{1}{4}\left\|P + 2Re(e^{i\theta}TS^*)\right\| \\
&= \frac{1}{4}\left\|P^2 + 4Re^2(e^{i\theta}(TS^*)) + 2Re(e^{i\theta}(PTS^* + TS^*P))\right\|^{\frac{1}{2}} \\
&\leq \frac{1}{4}\left(\|P\|^2 + 4\|Re(e^{i\theta}(TS^*))\|^2 + 2\|Re(e^{i\theta}(PTS^* + TS^*P))\|\right)^{\frac{1}{2}}.
\end{aligned}$$

By taking the supremum on both sides in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$ and over $\theta \in \mathbb{R}$, we deduce the desired inequality. ■

Remark 2.3.1. *Using Lemma 2.1.4, it follows that*

$$\begin{aligned}
\frac{1}{4}w^2(TS^*) + \frac{1}{8}w(PTS^* + TS^*P) + \frac{1}{16}\|P\|^2 &\leq \frac{1}{4}w^2(TS^*) + \frac{1}{4}w(TS^*)\|P\| + \frac{1}{16}\|P\|^2 \\
&= \left(\frac{1}{4}\|P\| + \frac{1}{2}w(TS^*)\right)^2.
\end{aligned}$$

This proves that the inequality (2.27) is sharper than the inequality (2.26) for $r = 1$.

Theorem 2.3.4. *Let $T, S \in \mathbb{B}(\mathcal{H})$. Then*

$$w(ST) \leq \min\left\{\frac{1}{2}\| T^*|S| + |S^*| \|, \frac{1}{2}\| S|T^*|S^* + |T| \|\right\}.$$

Proof. Let $x \in \mathcal{H}$ be any unit vector. Using Lemma 1.4.6, we have

$$\begin{aligned}
|\langle STx, x\rangle| &\leq \langle |S|Tx, Tx\rangle^{\frac{1}{2}} \langle |S^*|x, x\rangle^{\frac{1}{2}} \\
&= \langle T^*|S|Tx, x\rangle^{\frac{1}{2}} \langle |S^*|x, x\rangle^{\frac{1}{2}} \\
&\leq \frac{1}{2}\left(\langle T^*|S|Tx, x\rangle + \langle |S^*|x, x\rangle\right).
\end{aligned}$$

By taking the supremum on both sides in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$, we get

$$w(ST) \leq \frac{1}{2}\|T^*|S|T + |S^*|\|.$$

Again, we have $w(ST) = w(T^*S^*) \leq \frac{1}{2}\|S|T^*|S^* + |T|\|$. This completes the proof. ■

Employing Buzano's inequality, Abu-Omar and Kittaneh [2] have proved the following inequality.

Theorem 2.3.5. *Let $T, S \in \mathbb{B}(\mathcal{H})$. Then*

$$w(T+S) \leq \sqrt{w(T)^2 + w(S)^2 + \|T\|\,\|S\| + w(S^*T)}. \tag{2.28}$$

Proof. Let $x \in \mathcal{H}$ be any unit vector, we have

$$\begin{aligned}
|\langle (T+S)x, x\rangle|^2 &\leq (|\langle Tx, x\rangle| + |\langle Sx, x\rangle|)^2 \\
&= |\langle Tx, x\rangle|^2 + |\langle Sx, x\rangle|^2 + 2|\langle Tx, x\rangle||\langle Sx, x\rangle| \\
&= |\langle Tx, x\rangle|^2 + |\langle Sx, x\rangle|^2 + 2|\langle Tx, x\rangle||\langle x, Sx\rangle| \\
&\leq |\langle Tx, x\rangle|^2 + |\langle Sx, x\rangle|^2 + \|Tx\|\,\|Sx\| + |\langle Tx, Sx\rangle| \\
&= |\langle Tx, x\rangle|^2 + |\langle Sx, x\rangle|^2 + \|Tx\|\,\|Sx\| + |\langle S^*Tx, x\rangle|.
\end{aligned}$$

By taking the supremum on both sides in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$, we get

$$w(T+S) \leq \sqrt{w(T)^2 + w(S)^2 + \|T\|\,\|S\| + w(S^*T)}.$$

■

Theorem 2.3.6. *Let $T, S \in \mathbb{B}(\mathcal{H})$. Then*

$$w^2(T+S) \leq 2w(TS) + \|TT^* + S^*S\|. \tag{2.29}$$

Proof. Let $x \in \mathcal{H}$ be any unit vector. Then

$$\begin{aligned}
|\langle (T+S)x, x\rangle| &\leq |\langle Tx, x\rangle| + |\langle Sx, x\rangle| \\
&= \sup_{\theta\in\mathbb{R}} \left| e^{i\theta}\langle Tx, x\rangle + e^{-i\theta}\langle S^*x, x\rangle \right| \\
&= \sup_{\theta\in\mathbb{R}} \left| \langle (e^{i\theta}T + e^{-i\theta}S^*)x, x\rangle \right|.
\end{aligned}$$

By taking the supremum on both sides in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$, we obtain

$$\begin{aligned}
w^2(T+S) &\leq \sup_{\theta\in\mathbb{R}} \left\| (e^{i\theta}T + e^{-i\theta}S^*)(e^{i\theta}T + e^{-i\theta}S^*)^* \right\| \\
&= \sup_{\theta\in\mathbb{R}} \left\| 2Re(e^{2i\theta}TS) + TT^* + S^*S \right\| \\
&\leq \sup_{\theta\in\mathbb{R}} \left\| 2Re(e^{2i\theta}TS) \right\| + \|TT^* + S^*S\| \\
&= 2w(TS) + \|TT^* + S^*S\|,
\end{aligned}$$

as required. ∎

If we take $S = T$ in the inequality (2.29), then we reobtain the second inequality in (2.4).

Remark 2.3.2. *If T and S are normal, then the inequality (2.29) is a refinement for the triangle inequality of the numerical radius. Indeed,*

$$\begin{aligned} w^2(T+S) \leq 2w(TS) + \|TT^* + S^*S\| &\leq 2w(T)w(S) + w(TT^* + S^*S) \\ &\leq w^2(T) + w^2(S) + 2w(T)w(S) \\ &= (w(T)+w(S))^2 . \end{aligned}$$

Theorem 2.3.7. *Let $T, S \in \mathbb{B}(\mathcal{H})$ and let $r \geq 1$. Then*

$$w^{2r}(T+S) \leq 2^{2r-1}\left(w^r(TS) + \frac{1}{2}\|(TT^*)^r + (S^*S)^r\|\right).$$

Proof. Using the previous theorem, it follows that

$$\begin{aligned} w^{2r}(T+S) &\leq (2\,w(TS) + \|TT^* + S^*S\|)^r \\ &\leq 2^{r-1}\left(2^r w^r(TS) + 2^r \left\|\left(\frac{TT^* + S^*S}{2}\right)^r\right\|\right) \\ &\leq 2^{2r-1}\left(w^r(TS) + \frac{1}{2}\|(TT^*)^r + (S^*S)^r\|\right). \end{aligned}$$

∎

Corollary 2.3.8. *Let $T \in \mathbb{B}(\mathcal{H})$ and let $r \geq 1$. Then*

$$w^{2r}(T) \leq \frac{1}{2}\left(w^r(T^2) + \frac{1}{2}\|(TT^*)^r + (T^*T)^r\|\right). \tag{2.30}$$

Theorem 2.3.9. *Let $T, S \in \mathbb{B}(\mathcal{H})$. Then*

$$w^4(T+S) \leq 4w^2(ST) + 2w(STP + PST) + \|P\|^2,$$

where $P = T^*T + SS^*$.

Proof. We have

$$w(T+S) \leq \sup_{\theta\in\mathbb{R}} \|e^{i\theta}T + e^{-i\theta}S^*\|.$$

Let $\psi(\theta) = \|e^{i\theta}T + e^{-i\theta}S^*\|$. Then

$$\begin{aligned}
\psi(\theta) &= \|(e^{i\theta}T + e^{-i\theta}S^*)^*(e^{i\theta}T + e^{-i\theta}S^*)\|^{\frac{1}{2}} \\
&= \|P + 2Re(e^{2i\theta}(ST))\|^{\frac{1}{2}} \\
&= \|P^2 + 4Re^2(e^{2i\theta}(ST)) + 2Re(e^{2i\theta}(STP + PST))\|^{\frac{1}{4}} \\
&\leq \left(\|P\|^2 + 4\|Re(e^{2i\theta}(ST))\|^2 + 2\|Re(e^{2i\theta}(STP + PST))\|\right)^{\frac{1}{4}}.
\end{aligned}$$

By taking the supremum on both sides in $\psi(\theta)$ over $\theta \in \mathbb{R}$, we obtain the desired inequality. ■

Remark 2.3.3. *If we put $T = S$ in the previous theorem, then we get*

$$w^4(T) \leq \frac{1}{4}w^2(T^2) + \frac{1}{8}w(T^2R + RT^2) + \frac{1}{16}\|R\|^2,$$

where $R = T^*T + TT^*$.
This inequality has been given in [17].

Corollary 2.3.10. *Let $T, S \in \mathbb{B}(\mathcal{H})$. Then*

$$w^4(T + S) \leq 4w^2(TS) + 2w(TSL + LTS) + \|L\|^2,$$

where $L = S^*S + TT^*$.

Theorem 2.3.11. *Let $T, S \in \mathbb{B}(\mathcal{H})$. Then*

$$\|T + S\|^2 \leq \min\{\|TT^* + SS^*\| + 2w(TS^*), \|T^*T + S^*S\| + 2w(S^*T)\}. \quad (2.31)$$

Proof. Let $x, y \in \mathcal{H}$ be two vectors with $\|x\| = \|y\| = 1$. Then

$$\begin{aligned}
|\langle (T + S)x, y\rangle| &\leq |\langle Tx, y\rangle| + |\langle Sx, y\rangle| \\
&= \sup_{\theta\in\mathbb{R}} \left|e^{i\theta}\langle Tx, y\rangle + e^{-i\theta}\langle Sx, y\rangle\right| \\
&\leq \sup_{\theta\in\mathbb{R}} \left\|e^{i\theta}T + e^{-i\theta}S\right\| \|x\| \|y\|.
\end{aligned}$$

By taking the supremum on both sides in the above inequality over $x, y \in \mathcal{H}$ with $\|x\| = 1$ and $\|y\| = 1$, we obtain

$$\|T + S\| \leq \sup_{\theta\in\mathbb{R}} \left\|e^{i\theta}T + e^{-i\theta}S\right\|.$$

Using the fact that $\|XX^*\| = \|X^*X\| = \|X\|^2$ and $w(X) = \sup_{\theta\in\mathbb{R}} \left\|Re\left(e^{i\theta}X\right)\right\|$ for any operator X, the desired result is obtained. ■

The inequality (2.31) is a refinement of the triangle inequality. Indeed,

$$\|T+S\|^2 \leq \|T^*T+S^*S\| + 2w(S^*T) \leq \|T\|^2 + \|S\|^2 + 2\|T\|\|S\| = (\|T\| + \|S\|)^2.$$

Corollary 2.3.12. *Let $T, S \in \mathbb{B}(\mathcal{H})$. Then*

$$w^2(T+S) \leq \min\left\{\|TT^*+SS^*\| + 2w(TS^*), \|T^*T+S^*S\| + 2w(S^*T)\right\}. \tag{2.32}$$

Chapter 3

Numerical radius inequalities for operator matrices

In this chapter, we give some upper and lower bounds of the numerical radius for 2×2 operator matrices as well as for $n \times n$ operator matrices. Also, we derive a new bound for the zeros of polynomials.

3.1 Bounds for the numerical radii of 2×2 operator matrices

To begin this section, let's review a few widely recognized fact, about the numerical radius and operator norm of 2×2 operator matrices.

Let $A, B, C, D \in \mathbb{B}(\mathcal{H})$. Then the operator matrix $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ can be considered as an operator on $\mathcal{H} \oplus \mathcal{H}$ and it is defined as $Tx = \begin{bmatrix} Ax_1 + Bx_2 \\ Cx_1 + Dx_2 \end{bmatrix}$, where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathcal{H} \oplus \mathcal{H}$.

3.1.1 Numerical radius inequalities for 2×2 operator matrices

We start this section with the following lemma, which can be found in [33] .

Lemma 3.1.1. *Let $A, B, C, D \in \mathbb{B}(\mathcal{H})$. Then*

1. $w\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right) = \max\{w(A), w(D)\}$.

2. $w\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right) \leq w\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right)$ *and* $w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \leq w\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right)$.

3. $w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) = \frac{1}{2}\sup_{\theta\in\mathbb{R}} \|e^{i\theta}B + e^{-i\theta}C^*\| \quad$ *for all* $\theta \in \mathbb{R}$.

4. $w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) = w\left(\begin{bmatrix} 0 & C \\ B & 0 \end{bmatrix}\right)$.

5. $w\left(\begin{bmatrix} 0 & B \\ -B & 0 \end{bmatrix}\right) = w\left(\begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix}\right) = w(B)$.

6. $w\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right) = \max\{w(A+B), w(A-B)\}$.
   In particular,

$$w\left(\begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix}\right) = w(B).$$

Proof. 1. We have

$$\begin{aligned} w\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right) &= \sup_{\theta\in\mathbb{R}} \left\| Re\left(e^{i\theta}\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right)\right\| \\ &= \sup_{\theta\in\mathbb{R}} \left\|\begin{bmatrix} Re(e^{i\theta}A) & 0 \\ 0 & Re(e^{i\theta}D) \end{bmatrix}\right\| \\ &= \sup_{\theta\in\mathbb{R}} \max\left\{\|Re(e^{i\theta}A)\|, \|Re(e^{i\theta}D)\|\right\} \\ &= \max\left\{\sup_{\theta\in\mathbb{R}}\|Re(e^{i\theta}A)\|, \sup_{\theta\in\mathbb{R}}\|Re(e^{i\theta}D)\|\right\} \\ &= \max\{w(A), w(D)\}. \end{aligned}$$

2. Let $U = \left(\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}\right)$ be a unitary operator matrix and $T = \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right)$. Then

$$\begin{aligned} w\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right) &= \frac{1}{2}w(T + U^*TU) \\ &\leq \frac{w(T) + w(U^*TU)}{2} \\ &= w(T). \end{aligned}$$

For the second inequality, we have

$$\begin{aligned} w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) &= \frac{1}{2} w(T - U^* T U) \\ &\leq \frac{w(T) + w(U^* T U)}{2} \\ &= w(T). \end{aligned}$$

3. We have

$$\begin{aligned} w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) &= \sup_{\theta \in \mathbb{R}} \left\| Re\left(e^{i\theta} \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)\right\| \\ &= \frac{1}{2} \sup_{\theta \in \mathbb{R}} \left\| e^{i\theta} \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} + e^{-i\theta} \begin{bmatrix} 0 & C^* \\ B^* & 0 \end{bmatrix} \right\| \\ &= \frac{1}{2} \sup_{\theta \in \mathbb{R}} \left\| \begin{bmatrix} 0 & e^{i\theta} B + e^{-i\theta} C^* \\ e^{-i\theta} B^* + e^{i\theta} C & 0 \end{bmatrix} \right\| \\ &= \frac{1}{2} \sup_{\theta \in \mathbb{R}} \| e^{i\theta} B + e^{-i\theta} C^* \|. \end{aligned}$$

4. Let $U = \left(\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\right)$ be a unitary operator matrix. Then

$$w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) = w\left(U^* \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} U\right) = w\left(\begin{bmatrix} 0 & C \\ B & 0 \end{bmatrix}\right).$$

5. Let $U = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ be a unitary operator matrix. Then

$$\begin{aligned} w(B) = w\left(\begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix}\right) &= w\left(\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix} \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}\right) \\ &= w\left(\begin{bmatrix} 0 & B \\ -B & 0 \end{bmatrix}\right). \end{aligned}$$

6. Let $U = \frac{1}{\sqrt{2}}\begin{bmatrix} I & I \\ -I & I \end{bmatrix}$ be a unitary operator matrix. Then

$$\begin{aligned} w\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right) &= w\left(U\begin{bmatrix} A & B \\ B & A \end{bmatrix}U^*\right) \\ &= \frac{1}{2}w\left(\begin{bmatrix} I & I \\ -I & I \end{bmatrix}\begin{bmatrix} A & B \\ B & A \end{bmatrix}\begin{bmatrix} I & -I \\ I & I \end{bmatrix}\right) \\ &= w\left(\begin{bmatrix} A+B & 0 \\ 0 & A-B \end{bmatrix}\right) \\ &= \max\{w(A+B), w(A-B)\}. \end{aligned}$$

■

In the following theorem Hirzallah, Kitteneh and Shabrawi gave an upper bound for $w\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right)$.

Theorem 3.1.2. *[45] Let $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with $A, B, C, D \in \mathbb{B}(\mathcal{H})$. Then*

$$w(T) \leq \max\{w(A), w(D)\} + \frac{\|B\| + \|C\|}{2}. \tag{3.1}$$

Proof. Since $\begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, it follows by the identity (1.3) that

$$\begin{aligned} w\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) &\leq w\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right) + w\left(\begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}\right) + w\left(\begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix}\right) \\ &= \max\{w(A), w(D)\} + \frac{\left\|\begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}\right\| + \left\|\begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix}\right\|}{2} \\ &= \max\{w(A), w(D)\} + \frac{\|B\| + \|C\|}{2}, \end{aligned}$$

as required. ■

Recently, Hajmohamadi, Lashkaripour and Bakherad [40] have obtained the following inequality.

Theorem 3.1.3. *Let* $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *with* $A, B, C, D \in \mathbb{B}(\mathcal{H})$. *Then*

$$w(T) \leq \frac{1}{2} \max\{\||B| + |C^*|\|, \||B^*| + |C|\|\} + \frac{1}{2} \max\{\||A| + |A^*|\|, \||D| + |D^*|\|\}. \tag{3.2}$$

Proof. Let $x \in \mathcal{H}$ be any unit vector. Then

$$\begin{aligned}
|\langle Tx, x\rangle| &\leq \left|\left\langle \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} x, x \right\rangle\right| + \left|\left\langle \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} x, x \right\rangle\right| \\
&\leq \left\langle \left|\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right| x, x \right\rangle^{\frac{1}{2}} \left\langle \left|\begin{bmatrix} A^* & 0 \\ 0 & D^* \end{bmatrix}\right| x, x \right\rangle^{\frac{1}{2}} + \left\langle \left|\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right| x, x \right\rangle^{\frac{1}{2}} \left\langle \left|\begin{bmatrix} 0 & C^* \\ B^* & 0 \end{bmatrix}\right| x, x \right\rangle^{\frac{1}{2}} \\
&\leq \frac{1}{2}\left(\left\langle \left|\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right| x, x \right\rangle + \left\langle \left|\begin{bmatrix} A^* & 0 \\ 0 & D^* \end{bmatrix}\right| x, x \right\rangle + \left\langle \left|\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right| x, x \right\rangle + \left\langle \left|\begin{bmatrix} 0 & C^* \\ B^* & 0 \end{bmatrix}\right| x, x \right\rangle \right) \\
&= \frac{1}{2} \left\langle \begin{bmatrix} |A| + |A^*| & 0 \\ 0 & |D| + |D^*| \end{bmatrix} x, x \right\rangle + \frac{1}{2} \left\langle \begin{bmatrix} 0 & |B| + |C^*| \\ |C| + |B^*| & 0 \end{bmatrix} x, x \right\rangle.
\end{aligned}$$

Now taking the supremum over $x \in \mathcal{H}$ with $\|x\| = 1$ in the above inequality, we get the desired result. ■

The following lemma has been given in [53]. It is another improvement of the second inequality in (1.2).

Lemma 3.1.4. *Let* $T \in \mathbb{B}(\mathcal{H})$. *Then*

$$w(T) \leq \frac{1}{2} \|\, |T| + |T^*| \,\|. \tag{3.3}$$

Theorem 3.1.5. *Let* $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *with* $A, B, C, D \in \mathbb{B}(\mathcal{H})$. *Then*

$$w(T) \leq \max\left\{ w(A) + \frac{1}{2} \|\, |C| + |B^*| \,\|, w(D) + \frac{1}{2} \|\, |C^*| + |B| \,\| \right\}. \tag{3.4}$$

Proof. Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathcal{H} \oplus \mathcal{H}$ be any unit vector. Then

$$\begin{aligned}
|\langle Tx, x\rangle| &\leq |\langle Ax_1, x_1\rangle| + |\langle Bx_2, x_1\rangle| + |\langle Cx_1, x_2\rangle| + |\langle Dx_2, x_2\rangle| \\
&\leq w(A)\|x_1\|^2 + w(D)\|x_2\|^2 + \langle |B|x_2, x_2\rangle^{\frac{1}{2}}\langle |B^*|x_1, x_1\rangle^{\frac{1}{2}} \\
&+ \langle |C|x_1, x_1\rangle^{\frac{1}{2}}\langle |C^*|x_2, x_2\rangle^{\frac{1}{2}} \qquad \text{(by Lemma 1.4.6)} \\
&\leq w(A)\|x_1\|^2 + w(D)\|x_2\|^2 + \langle (|B| + |C^*|)\, x_2, x_2\rangle^{\frac{1}{2}}\langle (|B^*| + |C|)\, x_1, x_1\rangle^{\frac{1}{2}} \\
&\quad \text{(by the Cauchy-Schwarz inequality)} \\
&\leq w(A)\|x_1\|^2 + w(D)\|x_2\|^2 + \frac{1}{2}\left[\| \, |B| + |C^*| \, \| \|x_2\|^2 + \| \, |C| + |B^*| \, \| \|x_1\|^2\right] \\
&= \left(w(A) + \frac{1}{2}\| \, |C| + |B^*| \, \|\right)\|x_1\|^2 + \left(w(D) + \frac{1}{2}\| \, |B| + |C^*| \, \|\right)\|x_2\|^2 \\
&\leq \max\left\{w(A) + \frac{1}{2}\| \, |C| + |B^*| \, \|, w(D) + \frac{1}{2}\| \, |C^*| + |B| \, \|\right\}\left(\|x_1\|^2 + \|x_2\|^2\right) \\
&= \max\left\{w(A) + \frac{1}{2}\| \, |C| + |B^*| \, \|, w(D) + \frac{1}{2}\| \, |C^*| + |B| \, \|\right\}.
\end{aligned}$$

By taking the supremum in the above inequality over $x \in \mathcal{H} \oplus \mathcal{H}$ with $\|x\| = 1$, we obtain the desired result. ■

Remark 3.1.1. *Setting*

$$\max\left\{w(A) + \frac{1}{2}\| \, |C| + |B^*| \, \|, w(D) + \frac{1}{2}\| \, |C^*| + |B| \, \|\right\} = c,$$

it follows that

$$\begin{aligned}
c &\leq \max\{w(A), w(D)\} + \frac{1}{2}\max\{\| \, |C| + |B^*| \, \|, \| \, |C^*| + |B| \, \|\} \\
&\leq \frac{1}{2}\max\{\| \, |A| + |A^*| \, \|, \| \, |D| + |D^*| \, \|\} + \frac{1}{2}\max\{\| \, |B| + |C^*| \, \|, \| \, |B^*| + |C| \, \|\} \\
&\quad \text{(by the inequality (3.3))}.
\end{aligned}$$

This proves that the inequality (3.4) is a refinement of the inequality (3.2).

Example 3.1.1. *Let* $A = D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. *Then the upper bounds in the inequalities (3.4) and (3.2) are equal i.e.,* $w(T) \leq 2$.

Remark 3.1.2. *If* $A = D = 0$ *in Theorem 3.1.5, then we obtain*

$$w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \leq \frac{1}{2}\max\{\| \, |B| + |C^*| \, \|, \| \, |B^*| + |C| \, \|\}. \tag{3.5}$$

It should be mentioned here that the inequality (3.5) was given in [11].

Remark 3.1.3. *If $A = D = 0$ and $C = B$ in Theorem 3.1.5, then we reobtain the inequality (3.3).*

Corollary 3.1.6. *Let $A \in \mathbb{B}(\mathcal{H})$. Then*

$$w^2(A) \leq \frac{1}{6}\left\| |A|^2 + |A^*|^2 \right\| + \frac{1}{6}\left\| |A| + |A^*| \right\|^2. \tag{3.6}$$

Proof. Taking $A = B = D = C$ in Theorem 3.1.5 and using Lemma 3.1.1 (6), the fact that $\|X + Y\|^2 \leq 2\|X^2 + Y^2\|$ for self-adjoint operators X, Y and the inequality (3.3), we get

$$\begin{aligned} w^2(A) &\leq \frac{1}{6}\left\| |A|^2 + |A^*|^2 \right\| + \frac{1}{3} w(A) \left\| |A| + |A^*| \right\| \\ &\leq \frac{1}{6}\left\| |A|^2 + |A^*|^2 \right\| + \frac{1}{6}\left\| |A| + |A^*| \right\|^2. \end{aligned}$$

■

Remark 3.1.4. *It easy to see that the inequality (3.6) is an improvement of the second inequality in (2.1).*

An upper bound for $w(T)$ due to Paul and Bag [24] states that

Theorem 3.1.7. *Let $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with $A, B, C, D \in \mathbb{B}(\mathcal{H})$. Then*

$$w(T) \leq \frac{1}{2}\left(w(A) + w(D) + \sqrt{(w(A) - w(D))^2 + (\|B\| + \|C\|)^2} \right). \tag{3.7}$$

Proof. Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathcal{H} \oplus \mathcal{H}$ be any unit vector, we have

$$\begin{aligned} |\langle Tx, x\rangle| &= \left| \left\langle \begin{bmatrix} Ax_1 + Bx_2 \\ Cx_1 + Dx_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle \right| \\ &= |\langle Ax_1, x_1\rangle + \langle Bx_2, x_1\rangle + \langle Cx_1, x_2\rangle + \langle Dx_2, x_2\rangle| \\ &\leq |\langle Ax_1, x_1\rangle| + |\langle Bx_2, x_1\rangle| + |\langle Cx_1, x_2\rangle| + |\langle Dx_2, x_2\rangle| \\ &\leq w(A)\|x_1\|^2 + w(D)\|x_2\|^2 + (\|B\| + \|C\|)\|x\|\,\|y\|. \end{aligned}$$

By taking the supremum in the above inequality over $x \in \mathcal{H} \oplus \mathcal{H}$ with $\|x\|^2 = \|x_1\|^2 + \|x_2\|^2 = 1$, we obtain the desired result. ■

Theorem 3.1.8. *Let* $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *with* $A, B, C, D \in \mathbb{B}(\mathcal{H})$ *and* $0 < \alpha < 1$. *Then*

$$w(T) \leq \frac{1}{2}\left(w(A) + w(D) + \sqrt{(w(A) - w(D))^2 + \| \, |B^*|^{2(1-\alpha)} + |C|^{2\alpha} \| \, \| \, |B|^{2\alpha} + |C^*|^{2(1-\alpha)} \|} \right).$$

Proof. Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathcal{H} \oplus \mathcal{H}$ be any unit vector and $0 < \alpha < 1$, we have

$$\begin{aligned}
|\langle Tx, x\rangle| &\leq |\langle Ax_1, x_1\rangle| + |\langle Bx_2, x_1\rangle| + |\langle Cx_1, x_2\rangle| + |\langle Dx_2, x_2\rangle| \\
&\leq w(A)\|x_1\|^2 + w(D)\|x_2\|^2 + \langle |B|^{2\alpha}x_2, x_2\rangle^{\frac{1}{2}} \langle |B^*|^{2(1-\alpha)}x_1, x_1\rangle^{\frac{1}{2}} \\
&+ \langle |C|^{2\alpha}x_1, x_1\rangle^{\frac{1}{2}} \langle |C^*|^{2(1-\alpha)}x_2, x_2\rangle^{\frac{1}{2}} \quad \text{(by Lemma 1.4.7)} \\
&\leq w(A)\|x_1\|^2 + w(D)\|x_2\|^2 + \left\langle \left(|B|^{2\alpha} + |C^*|^{2(1-\alpha)} \right) x_2, x_2 \right\rangle^{\frac{1}{2}} \\
&\quad \left\langle \left(|B^*|^{2(1-\alpha)} + |C|^{2\alpha} \right) x_1, x_1 \right\rangle^{\frac{1}{2}} \quad \text{(by the Cauchy-Schwarz inequality)} \\
&\leq w(A)\|x_1\|^2 + w(D)\|x_2\|^2 + \| \, |B^*|^{2(1-\alpha)} + |C|^{2\alpha} \|^{\frac{1}{2}} \| \, |B|^{2\alpha} + |C^*|^{2(1-\alpha)} \|^{\frac{1}{2}} \|x_1\| \, \|x_2\| \\
&= \left\langle \widetilde{T_\alpha} \mathrm{x}, \mathrm{x} \right\rangle, \text{ where } \mathrm{x} = \begin{bmatrix} \|x_1\| \\ \|x_2\| \end{bmatrix}
\end{aligned}$$

and

$$\widetilde{T_\alpha} = \begin{bmatrix} w(A) & \frac{1}{2}\sqrt{\| \, |B^*|^{2(1-\alpha)} + |C|^{2\alpha} \| \, \| \, |B|^{2\alpha} + |C^*|^{2(1-\alpha)} \|} \\ \frac{1}{2}\sqrt{\| \, |B^*|^{2(1-\alpha)} + |C|^{2\alpha} \| \, \| \, |B|^{2\alpha} + |C^*|^{2(1-\alpha)} \|} & w(D) \end{bmatrix}.$$

Since x is a unit vector, then

$$|\langle Tx, x\rangle| \leq w\left(\widetilde{T_\alpha}\right).$$

Therefore,

$$w(T) = \sup_{\|x\|=1} |\langle Tx, x\rangle| \leq w\left(\widetilde{T_\alpha}\right),$$

as required. ∎

Remark 3.1.5. *If* $A = D = 0$ *and* $C = B$ *in Theorem 3.1.8, then we have*

$$w^r(B) \leq \left(\frac{1}{2} \left\| |B|^{2\alpha} + |B^*|^{2(1-\alpha)} \right\| \right)^r \qquad \textit{for} \;\; r \geq 1.$$

In view of the convexity of the function $f(t) = t^r$ *on* $[0, \infty)$ *for* $r \geq 1$, *this inequality is sharper than the inequality (2.12).*

Corollary 3.1.9. *Let* $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *with* $A, B, C, D \in \mathbb{B}(\mathcal{H})$. *Then*

$$w(T) \leq \frac{1}{2}\left(w(A) + w(D) + \sqrt{(w(A) - w(D))^2 + \| \, |B| + |C^*| \, \| \; \| \, |B^*| + |C| \, \|} \right). \tag{3.8}$$

Remark 3.1.6. *The inequality (3.8) is better than the inequality (3.7).*

Example 3.1.2. *Let* $A = D = 0$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. *Then the upper bounds for* $w(T)$ *in the inequalities (3.8) and (3.7) are as follows:*
The inequality (3.8) gives $w(T) \leq 0.5$.
The inequality (3.7) gives $w(T) \leq 1$.

Remark 3.1.7. *If we take* $A = D = 0$ *and* $C = B$ *in Corollary 3.1.9, then we reobtain the inequality (3.3).*

Remark 3.1.8. *If we take* $A = D = 0$ *in Corollary 3.1.9, then we obtain*

$$w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right) \leq \frac{1}{2} \| \, |B| + |C^*| \, \|^{\frac{1}{2}} \, \| \, |B^*| + |C| \, \|^{\frac{1}{2}} ,$$

which was already given in [3].

In [12], Bani-Domi and Kittaneh have obtained the following result.

Theorem 3.1.10. *Let* $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *with* $A, B, C, D \in \mathbb{B}(\mathcal{H})$. *Then*

$$\begin{aligned} \|T\|^2 \;\; &\leq \;\; \max\left\{\|A\|^2, \|D\|^2\right\} + \max\left\{\|B\|^2, \|C\|^2\right\} \\ &+ \;\; \max\{\|A\|, \|D\|\} \max\{\|B\|, \|C\|\} + w\left(\begin{bmatrix} 0 & C^*D \\ B^*A & 0 \end{bmatrix} \right). \end{aligned} \tag{3.9}$$

Proof. Let $x, y \in \mathcal{H} \oplus \mathcal{H}$ be any two unit vectors. Then

$$\begin{aligned}
|\langle Tx, y\rangle|^2 &= \left|\left\langle\left(\begin{bmatrix} A & 0 \\ 0 & D\end{bmatrix} + \begin{bmatrix} 0 & B \\ C & 0\end{bmatrix}\right)x, y\right\rangle\right|^2 \\
&= \left|\left\langle\left(\begin{bmatrix} A & 0 \\ 0 & D\end{bmatrix}\right)x, y\right\rangle + \left\langle\left(\begin{bmatrix} 0 & B \\ C & 0\end{bmatrix}\right)x, y\right\rangle\right|^2 \\
&\leq \left(\left|\left\langle\begin{bmatrix} A & 0 \\ 0 & D\end{bmatrix}x, y\right\rangle\right| + \left|\left\langle\begin{bmatrix} 0 & B \\ C & 0\end{bmatrix}x, y\right\rangle\right|\right)^2 \\
&= \left|\left\langle\begin{bmatrix} A & 0 \\ 0 & D\end{bmatrix}x, y\right\rangle\right|^2 + \left|\left\langle\begin{bmatrix} 0 & B \\ C & 0\end{bmatrix}x, y\right\rangle\right|^2 + 2\left|\left\langle\left(\begin{bmatrix} A & 0 \\ 0 & D\end{bmatrix}\right)x, y\right\rangle\right|\left|\left\langle\left(\begin{bmatrix} 0 & B \\ C & 0\end{bmatrix}\right)x, y\right\rangle\right| \\
&= \left|\left\langle\begin{bmatrix} A & 0 \\ 0 & D\end{bmatrix}x, y\right\rangle\right|^2 + \left|\left\langle\begin{bmatrix} 0 & B \\ C & 0\end{bmatrix}x, y\right\rangle\right|^2 + 2\left|\left\langle\left(\begin{bmatrix} A & 0 \\ 0 & D\end{bmatrix}\right)x, y\right\rangle\right|\left|\left\langle y, \left(\begin{bmatrix} 0 & B \\ C & 0\end{bmatrix}\right)x\right\rangle\right| \\
&\leq \left|\left\langle\begin{bmatrix} A & 0 \\ 0 & D\end{bmatrix}x, y\right\rangle\right|^2 + \left|\left\langle\begin{bmatrix} 0 & B \\ C & 0\end{bmatrix}x, y\right\rangle\right|^2 + \left\|\begin{bmatrix} A & 0 \\ 0 & D\end{bmatrix}x\right\|\left\|\begin{bmatrix} 0 & B \\ C & 0\end{bmatrix}x\right\| \\
&+ \left|\left\langle\begin{bmatrix} A & 0 \\ 0 & D\end{bmatrix}x, \begin{bmatrix} 0 & B \\ C & 0\end{bmatrix}x\right\rangle\right| \text{ (by Buzano's ineqyality)} \\
&= \left|\left\langle\begin{bmatrix} A & 0 \\ 0 & D\end{bmatrix}x, y\right\rangle\right|^2 + \left|\left\langle\begin{bmatrix} 0 & B \\ C & 0\end{bmatrix}x, y\right\rangle\right|^2 + \left\|\begin{bmatrix} A & 0 \\ 0 & D\end{bmatrix}x\right\|\left\|\begin{bmatrix} 0 & B \\ C & 0\end{bmatrix}x\right\| \\
&+ \left|\left\langle\begin{bmatrix} 0 & C^*D \\ B^*A & 0\end{bmatrix}x, x\right\rangle\right|.
\end{aligned}$$

Now, taking the supremum in the above inequality over $x, y \in \mathcal{H} \oplus \mathcal{H}$ with $\|x\| = \|y\| = 1$, we obtain the desired result. ■

Theorem 3.1.11. *Let* $T = \begin{bmatrix} A & B \\ C & D\end{bmatrix}$ *with* $A, B, C, D \in \mathbb{B}(\mathcal{H})$. *Then*

$$\|T\|^2 \leq \min\{a, b\}, \tag{3.10}$$

where

$$a = \max\left\{\| \, |A^*|^2 + |B^*|^2\|, \| \, |D^*|^2 + |C^*|^2\|\right\} + 2w\left(\begin{bmatrix} 0 & AC^* \\ DB^* & 0\end{bmatrix}\right)$$

and

$$b = \max\left\{\| \, |A|^2 + |C|^2\|, \| \, |D|^2 + |B|^2\|\right\} + 2w\left(\begin{bmatrix} 0 & A^*B \\ D^*C & 0\end{bmatrix}\right).$$

Proof. Let x, y be any unit two vectors in $\mathcal{H} \oplus \mathcal{H}$. Then

$$\begin{aligned} \left|\langle Tx, y\rangle\right| &\le \left|\left\langle \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} x, y \right\rangle\right| + \left|\left\langle \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} x, y \right\rangle\right| \\ &= \sup_{\theta\in\mathbb{R}} \left| e^{i\theta} \left\langle \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} x, y \right\rangle + e^{-i\theta} \left\langle \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} x, y \right\rangle \right|. \end{aligned}$$

By taking the supremum in the above inequality over $x \in \mathcal{H}\oplus\mathcal{H}$ with $\|x\| = \|y\| = 1$, we get

$$\begin{aligned} \|T\| &\le \sup_{\theta\in\mathbb{R}} \left\| e^{i\theta} \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} + e^{-i\theta} \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right\| \\ &= \sup_{\theta\in\mathbb{R}} \left\| \left(e^{i\theta} \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} + e^{-i\theta} \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right) \left(e^{i\theta} \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} + e^{-i\theta} \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right)^* \right\|^{\frac{1}{2}} \\ &= \sup_{\theta\in\mathbb{R}} \left\| \begin{bmatrix} |A^*|^2 + |B^*|^2 & 0 \\ 0 & |D^*|^2 + |C^*|^2 \end{bmatrix} + 2Re\left(e^{2i\theta} \begin{bmatrix} 0 & AC^* \\ DB^* & 0 \end{bmatrix} \right) \right\|^{\frac{1}{2}}. \end{aligned}$$

Hence,

$$\begin{aligned} \|T\|^2 &\le \max\left\{ \| \, |A^*|^2 + |B^*|^2 \|, \| \, |D^*|^2 + |C^*|^2 \| \right\} + 2w\left(\begin{bmatrix} 0 & AC^* \\ DB^* & 0 \end{bmatrix} \right) \\ &= a. \end{aligned}$$

Applying the same method to $T^* = \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix}$, and observing that $\|T\| = \|T^*\|$, we obtain

$$\begin{aligned} \|T^*\|^2 &\le \max\left\{ \| \, |A|^2 + |C|^2 \|, \| \, |D|^2 + |B|^2 \| \right\} + 2w\left(\begin{bmatrix} 0 & A^*B \\ D^*C & 0 \end{bmatrix} \right) \\ &= b. \end{aligned}$$

Hence, the result follows immediately. ∎

Remark 3.1.9. *Let* $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathcal{H} \oplus \mathcal{H}$ *be any unit vector. Then*

$$\begin{aligned}
\min\{a, b\} &\le b \\
&\le \max\left\{\|A\|^2 + \|C\|^2, \|D\|^2 + \|B\|^2\right\} + w\left(\begin{bmatrix} 0 & C^*D \\ B^*A & 0 \end{bmatrix}\right) \\
&+ w\left(\begin{bmatrix} 0 & A^*B \\ D^*C & 0 \end{bmatrix}\right) \\
&\le \max\left\{\|A\|^2, \|D\|^2\right\} + \max\left\{\|C\|^2, \|B\|^2\right\} + w\left(\begin{bmatrix} 0 & C^*D \\ B^*A & 0 \end{bmatrix}\right) \\
&+ \sup_{\|x_1\|^2 + \|x_2\|^2 = 1} \left|\left\langle \begin{bmatrix} 0 & A^*B \\ D^*C & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle\right| \\
&\le \max\left\{\|A\|^2, \|D\|^2\right\} + \max\left\{\|C\|^2, \|B\|^2\right\} + w\left(\begin{bmatrix} 0 & C^*D \\ B^*A & 0 \end{bmatrix}\right) \\
&+ \frac{1}{2}\left(\|A\|\,\|B\| + \|D\|\,\|C\|\right) \text{(by the arithmetic-geometric mean inequality)} \\
&\le \max\left\{\|A\|^2, \|D\|^2\right\} + \max\left\{\|C\|^2, \|B\|^2\right\} + w\left(\begin{bmatrix} 0 & C^*D \\ B^*A & 0 \end{bmatrix}\right) \\
&+ \max\{\|A\|, \|D\|\} \max\{\|B\|, \|C\|\}.
\end{aligned}$$

This proves that the inequality (3.10) is a refinement of the inequality (3.9).

Example 3.1.3. *Let* $A = D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. *Then the upper bounds for* $\|T\|^2$ *in the inequalities (3.10) and (3.9) are as follows:*
The inequality (3.10) gives $\|T\|^2 \le 2$.
The inequality (3.9) gives $\|T\|^2 \le 3$.

Corollary 3.1.12. *Let* $A = D$ *and* $B = C$ *in the above theorem. Then*

$$\left\|\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right\|^2 = \max\left\{\|A + B\|^2, \|A - B\|^2\right\} \le \|\,|A|^2 + |B|^2\| + 2w(A^*B).$$

Now, it is easy to see that the last inequality is sharper than the triangle inequality for the usual operator norm. The previous corollary can be used to give a necessary condition for the equality case in the triangle inequality for the usual operator norm. See, also [12] and [26].

Corollary 3.1.13. *Let* $A, B \in \mathbb{B}(\mathcal{H})$. *If* $\|A + B\| = \|A\| + \|B\|$, *then* $w(A^*B) = \|A\|\,\|B\|$.

Theorem 3.1.14. *[11] Let $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with $A, B, C, D \in \mathbb{B}(\mathcal{H})$. Then*

$$w^2(T) \leq 2\max\left\{w^2(A), w^2(D)\right\} + \max\{w(BC), w(CB)\} + \frac{1}{2}\max\left\{\| |C|^2 + |B^*|^2\|, \| |C^*|^2 + |B|^2\|\right\}.$$

Proof. Let $x \in \mathcal{H} \oplus \mathcal{H}$ be any unit vector, and let $T_1 = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$, $T_2 = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$, we have

$$\begin{aligned}
|\langle Tx, x\rangle|^2 &= |\langle (T_1 + T_2)x, x\rangle|^2 \\
&\leq (|\langle T_1x, x\rangle| + |\langle T_2x, x\rangle|)^2 \\
&\leq 2\left(|\langle T_1x, x\rangle|^2 + |\langle T_2x, x\rangle|^2\right) \\
&= 2\left(|\langle T_1x, x\rangle|^2 + |\langle T_2x, x\rangle\langle x, T_2^*x\rangle|\right) \\
&= 2|\langle T_1x, x\rangle|^2 + \|T_1x\|\,\|T_2^*\| + |\langle T_2x, T_2^*x\rangle| (\text{by Buzano's inequality}) \\
&= 2|\langle T_1x, x\rangle|^2 + |\langle T_2^2x, x\rangle| + \langle |T_1|^2x, x\rangle^{\frac{1}{2}}\langle |T_2|^2x, x\rangle^{\frac{1}{2}} (\text{by Lemma 1.4.6}) \\
&\leq 2|\langle T_1x, x\rangle|^2 + |\langle T_2^2x, x\rangle| + \frac{\langle |T_1|^2x, x\rangle + \langle |T_2|^2x, x\rangle}{2} \\
&\quad (\text{by the arithmetic-geometric mean inequality}) \\
&= 2|\langle T_1x, x\rangle|^2 + |\langle T_2^2x, x\rangle| + \frac{1}{2}\langle (|T_1|^2 + |T_2|^2)x, x\rangle.
\end{aligned}$$

So,

$$|\langle Tx, x\rangle|^2 \leq 2\left|\left\langle \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} x, x\right\rangle\right|^2 + \left|\left\langle \begin{bmatrix} BC & 0 \\ 0 & CB \end{bmatrix} x, x\right\rangle\right| + \frac{1}{2}\left\langle \begin{bmatrix} |C|^2 + |B^*|^2 & 0 \\ 0 & |C^*|^2 + |B|^2 \end{bmatrix} x, x\right\rangle.$$

By taking the supremum over $x \in \mathcal{H} \oplus \mathcal{H}$ with $\|x\| = 1$ in the above inequality, we get the desired result. ■

In the the following theorem, we provide a lower bound of $w\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right)$.

Theorem 3.1.15. *Let $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with $A, B, C, D \in \mathbb{B}(\mathcal{H})$. Then*

$$\frac{1}{4}\sqrt{\max\{\| |A^*|^2 + |B^*|^2\|, \| |D^*|^2 + |C^*|^2\|\} + 2m\left(\begin{bmatrix} 0 & AC^* \\ DB^* & 0 \end{bmatrix}\right)} \leq w(T).$$

Proof. Let x be any unit vector in $\mathcal{H} \oplus \mathcal{H}$ and let $\theta \in \mathbb{R}$ such that

$$e^{2i\theta}\left\langle \begin{bmatrix} 0 & AC^* \\ DB^* & 0 \end{bmatrix} x, x \right\rangle = \left| \left\langle \begin{bmatrix} 0 & AC^* \\ DB^* & 0 \end{bmatrix} x, x \right\rangle \right|.$$

Using the identity $\sup_{\theta\in\mathbb{R}} \left| e^{i\theta} a + e^{-i\theta} \overline{b} \right| = |a| + |b|$, where $a, b \in \mathbb{C}$, it follows that

$$\begin{aligned}
2w(T) &\geq w\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right) + w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \\
&\geq \sup_{\|x\|=1} \left(\left| \left\langle \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} x, x \right\rangle \right| + \left| \left\langle \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} x, x \right\rangle \right| \right) \\
&= \sup_{\|x\|=1} \sup_{\theta\in\mathbb{R}} \left| e^{i\theta} \left\langle \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} x, x \right\rangle + e^{-i\theta} \left\langle \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} x, x \right\rangle \right| \\
&= \sup_{\theta\in\mathbb{R}} w\left(e^{i\theta} \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} + e^{-i\theta} \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right) \\
&\geq \frac{1}{2} \sup_{\theta\in\mathbb{R}} \left\| e^{i\theta} \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} + e^{-i\theta} \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right\| \\
&= \frac{1}{2} \sup_{\theta\in\mathbb{R}} \left\| \left(e^{i\theta} \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} + e^{-i\theta} \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right) \left(e^{i\theta} \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} + e^{-i\theta} \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right)^* \right\|^{\frac{1}{2}} \\
&= \frac{1}{2} \sup_{\theta\in\mathbb{R}} \left\| \begin{bmatrix} |A^*|^2 + |B^*|^2 & 0 \\ 0 & |D^*|^2 + |C^*|^2 \end{bmatrix} + 2Re\left(e^{2i\theta} \left(\begin{bmatrix} 0 & AC^* \\ DB^* & 0 \end{bmatrix} \right) \right) \right\|^{\frac{1}{2}} \\
&\geq \frac{1}{2} \sup_{\theta\in\mathbb{R}} \left| \left\langle \left(\begin{bmatrix} |A^*|^2 + |B^*|^2 & 0 \\ 0 & |D^*|^2 + |C^*|^2 \end{bmatrix} + 2Re\left(e^{2i\theta} \begin{bmatrix} 0 & AC^* \\ DB^* & 0 \end{bmatrix} \right) \right) x, x \right\rangle \right|^{\frac{1}{2}} \\
&= \frac{1}{2} \left(\left\langle \begin{bmatrix} |A^*|^2 + |B^*|^2 & 0 \\ 0 & |D^*|^2 + |C^*|^2 \end{bmatrix} x, x \right\rangle + 2 \left| \left\langle \begin{bmatrix} 0 & AC^* \\ DB^* & 0 \end{bmatrix} x, x \right\rangle \right| \right)^{\frac{1}{2}}.
\end{aligned}$$

Hence,

$$w(T) \geq \frac{1}{4} \sqrt{\max\left\{\| \, |A^*|^2 + |B^*|^2 \|, \| \, |D^*|^2 + |C^*|^2 \| \right\} + 2m\left(\begin{bmatrix} 0 & AC^* \\ DB^* & 0 \end{bmatrix} \right)}.$$

■

3.1.2 Numerical radius inequalities for 2×2 off-diagonal operator matrices

In this section, we present several numerical radius inequalities of 2×2 off-diagonal operator matrices.

First, we need the following lemma, which can be found in [44].

Lemma 3.1.16. *Let f be a non-negative, convex function on $[0, \infty)$ and let $B, C \in \mathbb{B}(\mathcal{H})$ be positive operators. Then*

$$\left\| f\left(\frac{B+C}{2}\right) \right\| \leq \left\| \frac{f(B)+f(C)}{2} \right\|.$$

In particular,

$$\left\| \left(\frac{B+C}{2}\right)^r \right\| \leq \left\| \frac{B^r+C^r}{2} \right\| \qquad \textit{for} \;\; r \geq 1.$$

Theorem 3.1.17. *Let $B, C \in \mathbb{B}(\mathcal{H})$. Define*

$\delta_\theta(B,C) = \frac{1}{2}\left| \left\| Re\left(e^{i\theta}\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)\right\|^2 - \left\| Im\left(e^{i\theta}\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)\right\|^2 \right|$*, where $\theta \in \mathbb{R}$. Then*

$$\frac{1}{4}\max\left\{\| \, |B|^2 + |C^*|^2\|, \| \, |B^*|^2 + |C|^2\|\right\} + \sup_\theta \delta_\theta(B,C) \leq w^2\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right). \tag{3.11}$$

Proof. We have

$$\begin{aligned} w^2\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) &\geq \max\left\{\left\| Re\left(e^{i\theta}\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)\right\|^2, \left\| Im\left(e^{i\theta}\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)\right\|^2\right\} \\ &= \frac{1}{2}\left\| Re\left(e^{i\theta}\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)\right\|^2 + \left\| Im\left(e^{i\theta}\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)\right\|^2 + \delta_\theta(B,C) \\ &\geq \frac{1}{2}\left\| Re^2\left(e^{2i\theta}\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) + Im^2\left(e^{2i\theta}\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)\right\| + \delta_\theta(B,C) \\ &\geq \frac{1}{4}\max\left\{\||B|^2 + |C^*|^2\|, \||B^*|^2 + |C|^2\|\right\} + \sup_\theta \delta_\theta(B,C), \end{aligned}$$

as required. ∎

A similar inequality was given recently in [28].

Remark 3.1.10. *If $C = B$ in Theorem 3.1.17, then the inequality (3.11) becomes*

$$\frac{1}{4}\left\| \, |B|^2 + |B^*|^2 \right\| + \frac{1}{2}\left| \|Re(B)\|^2 - \|Im(B)\|^2 \right| \leq w^2(B), \tag{3.12}$$

which refines the first inequality in (2.1).

Example 3.1.4. *Let $B = I + 2iI$. Then the lower bounds for $w^2(B)$ in the inequalities (3.12) and (2.1) are as follows:*
The inequality (3.12) gives $w^2(B) \geq 4$.
The inequality (2.1) gives $w^2(B) \geq 2.5$.

It should be mentioned here that the inequality (3.12) appeared recently in [23].

Theorem 3.1.18. *[19] Let $B, C \in \mathbb{B}(\mathcal{H})$. Then*

$$w^4\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \leq \frac{1}{16}\|S\|^2 + \frac{1}{4}w^2(CB) + \frac{1}{8}w(CBS + SCB), \tag{3.13}$$

where $S = |B|^2 + |C^*|^2$.

Proof. Let $\phi(\theta) = \frac{1}{2}\|e^{i\theta}B + e^{-i\theta}C^*\|$, we have

$$\begin{aligned} \phi(\theta) &= \frac{1}{2}\|(e^{i\theta}B + e^{-i\theta}C^*)^*(e^{i\theta}B + e^{-i\theta}C^*)\|^{\frac{1}{2}} \\ &= \frac{1}{2}\|(e^{-i\theta}B^* + e^{i\theta}C)(e^{i\theta}B + e^{-i\theta}C^*)\|^{\frac{1}{2}} \\ &= \frac{1}{2}\left\|S + 2Re(e^{2i\theta}CB)\right\|^{\frac{1}{2}} \\ &= \frac{1}{2}\left\|(S + 2Re(e^{2i\theta}CB))^2\right\|^{\frac{1}{4}} \\ &= \frac{1}{2}\left\|S^2 + 4(Re(e^{2i\theta}CB))^2 + 2Re(e^{2i\theta}(SCB + CBS))\right\|^{\frac{1}{4}}. \end{aligned}$$

Hence,

$$\phi^4(\theta) \leq \frac{1}{16}\|S\|^2 + \frac{1}{4}\left\|Re(e^{2i\theta}CB)\right\|^2 + \frac{1}{8}\left\|Re(e^{2i\theta}(SCB + CBS))\right\|.$$

Now, taking the supremum over $\theta \in \mathbb{R}$ in the above inequality, we obtain

$$w^4\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \leq \frac{1}{16}\|S\|^2 + \frac{1}{4}w^2(CB) + \frac{1}{8}w(CBS + SCB).$$

■

Utilizing Lemma 3.1.1 (4) and Theorem 3.1.18, we get the next result.

Corollary 3.1.19. *[19] Let $B, C \in \mathbb{B}(\mathcal{H})$. Then*

$$w^4\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \leq \frac{1}{16}\|P\|^2 + \frac{1}{4}w^2(CB) + \frac{1}{8}w(CBP + PCB), \tag{3.14}$$

where $P = |B^*|^2 + |C|^2$.

In particular, assuming $C = B$ in Corollary 3.1.19, we obtain the following inequality.

Corollary 3.1.20. *[17] Let* $B \in \mathbb{B}(\mathcal{H})$*. Then*

$$w^4(B) \le \frac{1}{16}\|P\|^2 + \frac{1}{4}w^2(B^2) + \frac{1}{8}w(B^2P + PB^2), \tag{3.15}$$

where $P = |B|^2 + |B^*|^2$.

In the following theorem Hirzallah, Kittaneh and Shebrawi gave another upper and lower estimate for the numerical radius of $w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)$.

Theorem 3.1.21. *[45] Let* $B, C \in \mathbb{B}$*. Then*

$$w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \le \frac{w(B+C) + w(B-C)}{2} \tag{3.16}$$

and

$$w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \ge \frac{\max\{w(B+C) + w(B-C)\}}{2}. \tag{3.17}$$

Proof. Let $U = \frac{1}{\sqrt{2}}\begin{bmatrix} I & -I \\ I & I \end{bmatrix}$. Then U is unitary, we have

$$\begin{aligned}
w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) &= w\left(U^*\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}U\right) \\
&= \frac{1}{2}w\left(\begin{bmatrix} B+C & B-C \\ -(B-C) & -(B+C) \end{bmatrix}\right) \\
&\le \frac{1}{2}\left(w\left(\begin{bmatrix} B+C & 0 \\ 0 & -(B+C) \end{bmatrix}\right) + w\left(\begin{bmatrix} 0 & B-C \\ -(B-C) & 0 \end{bmatrix}\right)\right) \\
&= \frac{w(B+C) + w(B-C)}{2},
\end{aligned}$$

which proves the first inequality.

To prove the second inequality, we have

$$\begin{aligned} w(B+C) &= w\left(\begin{bmatrix} 0 & B+C \\ B+C & 0 \end{bmatrix}\right) \\ &\leq w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) + w\left(\begin{bmatrix} 0 & C \\ B & 0 \end{bmatrix}\right) \\ &= 2w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right). \end{aligned}$$

So,

$$\frac{w(B+C)}{2} \leq w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right). \tag{3.18}$$

By replacing C by$(-C)$ in the inequality (3.18), we get

$$\frac{w(B-C)}{2} \leq w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right). \tag{3.19}$$

Now, by the inequalities (3.18) and (3.19), we get the second inequality (3.17). ■

Theorem 3.1.22. *Let $B, C \in \mathbb{B}(\mathcal{H})$ and let $0 < \alpha < 1$. Then*

$$w^r\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \leq \frac{1}{2}\max\left\{\left\| |B|^{2r\alpha} + |C^*|^{2r(1-\alpha)}\right\|, \left\| |C|^{2r\alpha} + |B^*|^{2r(1-\alpha)}\right\|\right\}$$

for $r \geq 1$.

Proof. Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathcal{H} \oplus \mathcal{H}$ be any unit vector and let $0 < \alpha < 1$, $r \geq 1$. Using Lemma 1.4.7, we have

$$\begin{aligned} \left|\left\langle \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} x, x \right\rangle\right| &\leq |\langle Bx_2, x_1\rangle| + |\langle Cx_1, x_2\rangle| \\ &\leq \left\langle |B|^{2\alpha}x_2, x_2\right\rangle^{\frac{1}{2}} \left\langle |B^*|^{2(1-\alpha)}x_1, x_1\right\rangle^{\frac{1}{2}} + \left\langle |C|^{2\alpha}x_1, x_1\right\rangle^{\frac{1}{2}} \left\langle |C^*|^{2(1-\alpha)}x_2, x_2\right\rangle^{\frac{1}{2}} \\ &\leq \frac{1}{2}\left(\left\langle\left(|B|^{2\alpha} + |C^*|^{2(1-\alpha)}\right)x_2, x_2\right\rangle + \left\langle\left(|C|^{2\alpha} + |B^*|^{2(1-\alpha)}\right)x_1, x_1\right\rangle\right) \\ &\leq \frac{1}{2}\left\| |C|^{2\alpha} + |B^*|^{2(1-\alpha)}\right\| \, \|x_1\|^2 + \left\| |B|^{2\alpha} + |C^*|^{2(1-\alpha)}\right\| \, \|x_2\|^2 \\ &\leq \frac{1}{2}\max\left\{\left\| |C|^{2\alpha} + |B^*|^{2(1-\alpha)}\right\|, \left\| |B|^{2\alpha} + |C^*|^{2(1-\alpha)}\right\|\right\}\left(\|x_1\|^2 + \|x_2\|^2\right) \\ &= \frac{1}{2}\max\left\{\left\| |C|^{2\alpha} + |B^*|^{2(1-\alpha)}\right\|, \left\| |B|^{2\alpha} + |C^*|^{2(1-\alpha)}\right\|\right\}. \end{aligned}$$

By taking the supremum in the above inequality over $x \in \mathcal{H} \oplus \mathcal{H}$ with $\|x\| = 1$, we get

$$w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \leq \frac{1}{2} \max\left\{\left\| |B|^{2\alpha} + |C^*|^{2(1-\alpha)} \right\|, \left\| |C|^{2\alpha} + |B^*|^{2(1-\alpha)} \right\|\right\}.$$

Using Lemma 3.1.16, it follows that

$$w^r\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \leq \frac{1}{2} \max\left\{\left\| |B|^{2r\alpha} + |C^*|^{2r(1-\alpha)} \right\|, \left\| |C|^{2r\alpha} + |B^*|^{2r(1-\alpha)} \right\|\right\},$$

as required. ■

Remark 3.1.11. *If $C = B$ in Theorem 3.1.22, then we reobtain the inequality (2.12).*

Corollary 3.1.23. *Let $B, C \in \mathbb{B}(\mathcal{H})$. Then*

$$w^{2r}\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \leq \frac{1}{2} \max\left\{\left\| |B|^{2r} + |C^*|^{2r} \right\|, \left\| |C|^{2r} + |B^*|^{2r} \right\|\right\} \qquad \text{for } r \geq \frac{1}{2}.$$

Remark 3.1.12. *If $C = B$ in Corollary 3.1.23, then we obtain*

$$w^{2r}(B) \leq \frac{1}{2}\left\| |B|^{2r} + |B^*|^{2r} \right\| \qquad \text{for } r \geq \frac{1}{2}.$$

This inequality is stronger than the inequality (2.13) for $\alpha = \frac{1}{2}$.

It should be mentioned here that a similar result of Corollary 3.1.23 can be found in [18] for $r = 1$.

Theorem 3.1.24. *[45] Let $B, C \in \mathbb{B}(\mathcal{H})$ have polar decompositions $B = U|B|$ and $C = V|C|$. Then*

$$w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \leq \frac{1}{2}\left(\max(\|B\|, \|C\|) + w\left(\begin{bmatrix} 0 & |C|^{\frac{1}{2}}U|B|^{\frac{1}{2}} \\ |B|^{\frac{1}{2}}V|C|^{\frac{1}{2}} & 0 \end{bmatrix}\right)\right). \tag{3.20}$$

Proof. Let $T = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$, we have $\|T\| = \max\{\|B\|, \|C\|\}$ and $\widetilde{T} = \begin{bmatrix} 0 & |C|^{\frac{1}{2}}U|B|^{\frac{1}{2}} \\ |B|^{\frac{1}{2}}V|C|^{\frac{1}{2}} & 0 \end{bmatrix}$. Now, the result follows by applying the inequality (2.2) to the operator T. ■

Lemma 3.1.25. *[45] Let $B \in \mathbb{B}(\mathcal{H})$. Then*

$$w\left(\begin{bmatrix} B & B \\ -B & -B \end{bmatrix}\right) = \frac{1}{2}\left\|\begin{bmatrix} B & B \\ -B & -B \end{bmatrix}\right\| = \|B\|. \tag{3.21}$$

In the following theorem Hirzallah, Kittaneh and Shebrawi gave new inequality connecting $w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)$, $w(B)$ and $w(C)$.

Theorem 3.1.26. *[45] Let $B, C \in \mathbb{B}(\mathcal{H})$. Then*

$$w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \leq \min\{w(B), w(C)\} + \frac{\min\{\|B+C\|, \|B-C\|\}}{2}.$$

Proof. Let $U = \frac{1}{\sqrt{2}}\begin{bmatrix} I & -I \\ I & I \end{bmatrix}$. Then U is unitary, we have

$$\begin{aligned}
w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) &= w\left(U^* \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} U\right) \\
&= \frac{1}{2} w\left(\begin{bmatrix} B+C & B-C \\ -(B-C) & -(B+C) \end{bmatrix}\right) \\
&= \frac{1}{2} w\left(\begin{bmatrix} B+C & B+C \\ -(B+C) & -(B+C) \end{bmatrix} + \begin{bmatrix} 0 & -2C \\ 2C & 0 \end{bmatrix}\right) \\
&\leq \frac{1}{2}\left(w\left(\begin{bmatrix} B+C & B+C \\ -(B+C) & -(B+C) \end{bmatrix}\right) + w\left(\begin{bmatrix} 0 & -2C \\ 2C & 0 \end{bmatrix}\right)\right) \\
&= \frac{\|B+C\|}{2} + w(C) \quad \text{(by identity 3.21 and Lemma 3.1.1 (6))} \qquad (3.22)
\end{aligned}$$

Replacing C by $-C$ in the inequality (3.22), and using Lemma 3.1.1 (5), we get

$$w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \leq \frac{\|B-C\|}{2} + w(C). \qquad (3.23)$$

From the inequalities (3.22) and (3.23), it follows that

$$w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \leq \frac{\min\{\|B+C\|, \|B-C\|\}}{2} + w(C). \qquad (3.24)$$

Now, interchanging B and C in the inequality (3.24) and using Lemma 3.1.1 (4), we get

$$w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \leq \frac{\min\{\|B+C\|, \|B-C\|\}}{2} + w(B). \qquad (3.25)$$

From the inequalities (3.24) and (3.25), we get the desired result. ■

In [3], Abu-Omar and Kittaneh extended Theorem 2.1.6 as follows.

Theorem 3.1.27. *Let $B, C \in \mathbb{B}(\mathcal{H})$. Then*

$$\frac{1}{2}\sqrt{\| \, |B|^2 + |C^*|^2\| + 2m(CB)} \leq w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \leq \frac{1}{2}\sqrt{\| \, |B|^2 + |C^*|^2\| + 2w(CB)}. \tag{3.26}$$

Proof. Let $x \in \mathcal{H}$ be any unit vector and let $\theta \in \mathbb{R}$ such that

$$e^{2i\theta}\langle CBx, x\rangle = |\langle CBx, x\rangle|.$$

We have

$$\begin{aligned} w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) &\geq \frac{1}{2}\|e^{i\theta}B + e^{-i\theta}C^*\| \\ &= \frac{1}{2}\left\|(e^{i\theta}B + e^{-i\theta}C^*)^*(e^{i\theta}B + e^{-i\theta}C^*)\right\|^{\frac{1}{2}} \\ &= \frac{1}{2}\||B|^2 + |C^*|^2 + 2Re(e^{2i\theta}CB)\|^{\frac{1}{2}} \\ &\geq \frac{1}{2}\sqrt{|\langle(|B|^2 + |C^*|^2 + 2Re(e^{2i\theta}CB))x, x\rangle|} \\ &= \frac{1}{2}\sqrt{|\langle(|B|^2 + |C^*|^2)x, x\rangle + 2\langle Re(e^{2i\theta}CB))x, x\rangle|} \\ &= \frac{1}{2}\sqrt{|\langle(|B|^2 + |C^*|^2)x, x\rangle + 2|\langle CBx, x\rangle||} \\ &\geq \frac{1}{2}\sqrt{\langle(|B|^2 + |C^*|^2)x, x\rangle + 2m(CB)}. \end{aligned}$$

By taking the supremum in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$, we get the first inequality in (3.26).
To prove the second inequality in (3.26), we have

$$\begin{aligned} w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) &= \frac{1}{2}\sup_{\theta\in\mathbb{R}}\|e^{i\theta}B + e^{-i\theta}C^*\| \\ &= \frac{1}{2}\sup_{\theta\in\mathbb{R}}\left\|(e^{i\theta}B + e^{-i\theta}C^*)^*(e^{i\theta}B + e^{-i\theta}C^*)\right\|^{\frac{1}{2}} \\ &= \frac{1}{2}\sup_{\theta\in\mathbb{R}}\| \, |B|^2 + |C^*|^2 + 2Re(e^{2i\theta}CB)\|^{\frac{1}{2}} \\ &\leq \frac{1}{2}\sqrt{\| \, |B|^2 + |C^*|^2\| + 2\sup_{\theta\in\mathbb{R}}\|Re(e^{2i\theta}CB)\|} \\ &\leq \frac{1}{2}\sqrt{\| \, |B|^2 + |C^*|^2\| + 2w(CB)}, \end{aligned}$$

which completes the proof of the theorem. ∎

Abu-Omar and Kittaneh have obtained the following theorem.

Theorem 3.1.28. *[3] Let $B, C \in \mathbb{B}(\mathcal{H})$. Then*

$$w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \le \frac{1}{2}\sqrt{\|\,|B^*| + |C|\,\|\,\|\,|B| + |C^*|\,\|}. \tag{3.27}$$

Proof. Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathcal{H}$ be any unit vector, we have

$$\begin{aligned}
\left|\left\langle \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} x, x \right\rangle\right| &= |\langle Bx_2, x_1\rangle + \langle Cx_1, x_2\rangle| \\
&\le |\langle Bx_2, x_1\rangle| + |\langle Cx_1, x_2\rangle| \\
&\le \langle |B|x_2, x_2\rangle^{\frac{1}{2}}\langle |B^*|x_1, x_1\rangle^{\frac{1}{2}} + \langle |C|x_1, x_1\rangle^{\frac{1}{2}}\langle |C^*|x_2, x_2\rangle^{\frac{1}{2}} \\
&\le (\langle |B^*|x_1, x_1\rangle + \langle |C|x_1, x_1\rangle)^{\frac{1}{2}}(\langle |B|x_2, x_2\rangle + \langle |C^*|x_2, x_2\rangle)^{\frac{1}{2}} \\
&\quad \text{(by Cauchy-Schwarz inequality)} \\
&= (\langle (|B^*| + |C|)x_1, x_1\rangle)^{\frac{1}{2}}(\langle (|B| + |C^*|)x_2, x_2\rangle)^{\frac{1}{2}} \\
&\le \|\,|B^*| + |C|\,\|^{\frac{1}{2}}\|\,|B| + |C^*|\,\|^{\frac{1}{2}}\|x_1\|\,\|x_2\| \\
&\le \|\,|B^*| + |C|\,\|^{\frac{1}{2}}\|\,|B| + |C^*|\,\|^{\frac{1}{2}}\frac{\|x_1\|^2 + \|x_2\|^2}{2} \\
&\quad \text{(by the arithmetic-geometric mean inequality)} \\
&= \frac{1}{2}\|\,|B^*| + |C|\,\|^{\frac{1}{2}}\|\,|B| + |C^*|\,\|^{\frac{1}{2}},
\end{aligned}$$

which completes the proof of the theorem.

∎

3.2 New numerical radius inequalities for $n \times n$ operator matrices and a bound for the zeros of polynomials

In this section, we give some bounds for the numerical radii of $n \times n$ operator matrices. Also, we derive a new bound for the zeros of polynomials.

3.2.1 Numerical radius inequalities for $n \times n$ operator matrices

Let $\mathcal{H}_1, \mathcal{H}_2, \cdots, \mathcal{H}_n$ be complex Hilbert spaces, and consider $\mathcal{H}^{(n)} = \oplus_{k=1}^n \mathcal{H}_k$. With respect to this decomposition, every operator $T \in \mathbb{B}\left(\mathcal{H}^{(n)}\right)$ has an $n \times n$ operator matrix representation $T = [T_{kj}]$ with $T_{kj} \in \mathbb{B}(\mathcal{H}_j, \mathcal{H}_k)$, the space of all bounded linear operators from $\mathcal{H}_j$ to $\mathcal{H}_k$.

A very useful numerical radius inequality for operator matrices due to Hou and Du [49] asserts that if $T \in \mathbb{B}(\mathcal{H})$, then

$$w(T) \leq w([\|T_{kj}\|]). \tag{3.28}$$

We give some numerical radius inequalities for $n \times n$ operator matrices. Also, we refine the inequality (3.28). Other refinements of the inequality (3.28) can be found in [3] and references therein. Let $T = [T_{kj}]$ be an $n \times n$ operator matrix with $T_{kj} \in \mathbb{B}(\mathcal{H}_j, \mathcal{H}_k)$ and let $L_k = \sum_{j=1}^{n} \|T_{kj}\| - \|T_{kk}\|$, $C_j = \sum_{k=1}^{n} \|T_{kj}\| - \|T_{jj}\|$.

Our first numerical radius inequality for $n \times n$ operator matrices can be stated as follows.

Theorem 3.2.1. *Let $T = [T_{kj}]$, and let L_k, C_j be as described above. Assume that $L = \max_k (L_k)$, $C = \max_j (C_j)$, then*

$$w(T) \leq \max_k w(T_{kk}) + (LC)^{1/2}.$$

Proof. Let $x = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T \in \mathcal{H}^{(n)}$ be any unit vector. Then

$$\begin{aligned}
|\langle Tx, x\rangle| &= \left| \sum_{k,j=1}^{n} \langle T_{kj}x_j, x_k\rangle \right| \\
&\leq \sum_{k,j=1}^{n} |\langle T_{kj}x_j, x_k\rangle| \\
&= \sum_{k=1}^{n} |\langle T_{kk}x_k, x_k\rangle| + \sum_{\substack{j,k=1 \\ k \neq j}}^{n} |\langle T_{kj}x_j, x_k\rangle| \\
&\leq \sum_{k=1}^{n} w(T_{kk})\|x_k\|^2 + \sum_{\substack{j,k=1 \\ k \neq j}}^{n} \|T_{kj}\| \, \|x_j\| \, \|x_k\|.
\end{aligned}$$

By the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned}
|\langle Tx, x\rangle| &\leq \max_k w(T_{kk}) + \left(\sum_{\substack{j,k=1\\k\neq j}}^{n} \|T_{kj}\|\,\|x_k\|^2\right)^{1/2}\left(\sum_{\substack{j,k=1\\k\neq j}}^{n} \|T_{kj}\|\,\|x_j\|^2\right)^{1/2}\\
&= \max_k w(T_{kk}) + \left(\sum_{k=1}^{n} L_k\|x_k\|^2\right)^{1/2}\left(\sum_{j=1}^{n} C_j\|x_j\|^2\right)^{1/2}\\
&\leq \max_k w(T_{kk}) + \left(L\sum_{k=1}^{n} \|x_k\|^2\right)^{1/2}\left(C\sum_{j=1}^{n} \|x_j\|^2\right)^{1/2}\\
&= \max_k w(T_{kk}) + (LC)^{1/2}.
\end{aligned}$$

Since $w(T) = \sup_{\|x\|=1} |\langle Tx, x\rangle|$, then the desired inequality follows immediately. ∎

Other related numerical radius inequalities are given in the following results.

Theorem 3.2.2. *Let $T = [T_{kj}]$, L_k and C_k be as described above and let $S_k = \dfrac{L_k + C_k}{2}$, $S = \max_k S_k$. Then*

$$w(T) \leq \max_k w(T_{kk}) + S.$$

Proof. Let $x = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^t \in \mathcal{H}^{(n)}$ be any unit vector. Then

$$\begin{aligned}
|\langle Tx, x\rangle| &\leq \max_k w(T_{kk}) + \sum_{\substack{j,k=1\\k\neq j}}^{n} \|T_{kj}\|\,\|x_k\|\,\|x_j\|\\
&\leq \max_k w(T_{kk}) + \sum_{\substack{j,k=1\\k\neq j}}^{n} \|T_{kj}\|\left(\frac{\|x_k\|^2 + \|x_j\|^2}{2}\right)\\
&= \max_k w(T_{kk}) + \frac{1}{2}\sum_{k=1}^{n} L_k\|x_k\|^2 + \frac{1}{2}\sum_{j=1}^{n} C_j\|x_j\|^2\\
&= \max_k w(T_{kk}) + \sum_{k=1}^{n} S_k\|x_k\|^2\\
&\leq \max_k w(T_{kk}) + S.
\end{aligned}$$

The result follows directly by taking the supremum in the above inequality over $x \in \mathcal{H}^{(n)}$ with $\|x\| = 1$. ∎

Theorem 3.2.3. *Let $T = [T_{kj}]$ be an $n \times n$ operator matrix with $T_{kj} \in \mathbb{B}(\mathcal{H}_j, \mathcal{H}_k)$. Then*

$$w(T) \leq \max_k w(T_{kk}) + \left(\sum_{\substack{j,k=1 \\ k \neq j}}^{n} \|T_{kj}\|^2 \right)^{1/2}.$$

Proof. Let $x = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^t \in \mathcal{H}^{(n)}$ be any unit vector. Then

$$|\langle Tx, x \rangle| \leq \sum_{k=1}^{n} w(T_{kk}) \|x_k\|^2 + \sum_{\substack{j,k=1 \\ k \neq j}}^{n} \|T_{kj}\| \, \|x_k\| \, \|x_j\|.$$

Applying the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} |\langle Tx, x \rangle| &\leq \max_k w(T_{kk}) + \left(\sum_{\substack{j,k=1 \\ k \neq j}}^{n} \|T_{kj}\|^2 \right)^{1/2} \left(\sum_{\substack{j,k=1 \\ k \neq j}}^{n} \|x_k\|^2 \|x_j\|^2 \right)^{1/2} \\ &\leq \max_k w(T_{kk}) + \left(\sum_{\substack{j,k=1 \\ k \neq j}}^{n} \|T_{kj}\|^2 \right)^{1/2} \left(\sum_{k=1}^{n} \|x_k\|^2 \sum_{j=1}^{n} \|x_j\|^2 \right)^{1/2} \\ &\leq \max_k w(T_{kk}) + \left(\sum_{\substack{j,k=1 \\ k \neq j}}^{n} \|T_{kj}\|^2 \right)^{1/2}. \end{aligned}$$

The result follows directly by taking the supremum in the above inequality over $x \in \mathcal{H}^{(n)}$ with $\|x\| = 1$. ∎

Theorem 3.2.4. *Let $T = [T_{kj}]$ be an $n \times n$ operator matrix with $T_{kj} \in \mathbb{B}(\mathcal{H}_j, \mathcal{H}_k)$. Then*

$$w(T) \leq \max_k w(T_{kk}) + (n-1) \max_{k \neq j} \|T_{kj}\|.$$

Proof. Let $x = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^t \in \mathcal{H}^{(n)}$ be any unit vector. Then

$$\begin{aligned}
|\langle Tx, x\rangle| &= \left| \sum_{k,j=1}^{n} \langle T_{kj}x_j, x_k\rangle \right| \\
&\leq \sum_{k=1}^{n} |\langle T_{kk}x_k, x_k\rangle| + \sum_{\substack{j,k=1\\k\neq j}}^{n} \left|\langle T_{kj}x_j, x_k\rangle\right| \\
&\leq \sum_{k=1}^{n} w(T_{kk})\|x_k\|^2 + \sum_{\substack{j,k=1\\k\neq j}}^{n} \|T_{kj}\|\, \|x_k\|\, \|x_j\| \\
&\leq \max_k w(T_{kk}) + \max_{k\neq j} \|T_{kj}\| \left(\sum_{\substack{j,k=1\\k\neq j}}^{n} \|x_k\|\, \|x_j\| \right) \\
&= \max_k w(T_{kk}) + \max_{k\neq j} \|T_{kj}\| \left(\sum_{k<j} 2\|x_k\|\, \|x_j\| \right) \\
&\leq \max_k w(T_{kk}) + \max_{k\neq j} \|T_{kj}\| \left((n-1) \sum_{k=1}^{n} \|x_k\|^2 \right).
\end{aligned}$$

The desired result follows by taking the supremum in the above inequality over $x \in \mathcal{H}^{(n)}$ with $\|x\| = 1$. ∎

For the rest of this work, we need the following scalar inequality. For any two real numbers a and b, we have

$$|a+b| \leq \sqrt{2}|a+ib|. \tag{3.29}$$

By replacing T by $B + iC$ in the second inequality in (2.1), we get the following proposition.

Proposition 3.2.5. *Let $B, C \in \mathbb{B}(\mathcal{H})$ be self-adjoint. Then*

$$w^2(B+iC) \leq \|B^2 + C^2\|.$$

Theorem 3.2.6. *Let $T = [T_{kj}]$ be an $n \times n$ operator matrix with $T_{kj} \in \mathbb{B}(\mathcal{H}_j, \mathcal{H}_k)$. Then*

$$w(T) \leq w([\tilde{a}_{kj}]), \tag{3.30}$$

where

$$\tilde{a}_{kj} = \begin{cases} \frac{1}{\sqrt{2}} w(|T_{kk}| + i|T_{kk}^*|) & \text{if } k = j \\ \|T_{kj}\| & \text{if } k \neq j. \end{cases}$$

Proof. Let $x = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^t \in \mathcal{H}^{(n)}$ be any unit vector. Then

$$\begin{aligned}
|\langle Tx, x\rangle| &= \left| \sum_{k,j=1}^{n} \langle T_{kj} x_j, x_k \rangle \right| \\
&\leq \sum_{k,j=1}^{n} |\langle T_{kj} x_j, x_k \rangle| \\
&= \sum_{k=1}^{n} |\langle T_{kk} x_k, x_k \rangle| + \sum_{\substack{j,k=1 \\ k\neq j}}^{n} |\langle T_{kj} x_j, x_k \rangle| \\
&\leq \sum_{k=1}^{n} \langle |T_{kk}| x_k, x_k \rangle^{1/2} \langle |T_{kk}^*| x_k, x_k \rangle^{1/2} + \sum_{\substack{j,k=1 \\ k\neq j}}^{n} |\langle T_{kj} x_j, x_k \rangle| \\
&\quad \text{(by Lemma 1.4.6)} \\
&\leq \frac{1}{2} \sum_{k=1}^{n} \left\langle (|T_{kk}| + |T_{kk}^*|) x_k, x_k \right\rangle + \sum_{\substack{j,k=1 \\ k\neq j}}^{n} |\langle T_{kj} x_j, x_k \rangle| \\
&\quad \text{(by the arithmetic - geometric mean inequality)} \\
&\leq \frac{1}{\sqrt{2}} \sum_{k=1}^{n} \left| \left\langle (|T_{kk}| + i|T_{kk}^*|) x_k, x_k \right\rangle \right| + \sum_{\substack{j,k=1 \\ k\neq j}}^{n} \|T_{kj}\| \, \|x_k\| \, \|x_j\| \\
&\quad \text{(by the inequality (3.29))} \\
&\leq \frac{1}{\sqrt{2}} \sum_{k=1}^{n} w(|T_{kk}| + i|T_{kk}^*|) \|x_k\|^2 + \sum_{\substack{j,k=1 \\ k\neq j}}^{n} \|T_{kj}\| \, \|x_k\| \, \|x_j\| \\
&= \left\langle [\tilde{a}_{kj}] \tilde{x}, \tilde{x} \right\rangle, \text{ where } \tilde{x} = \begin{bmatrix} \|x_1\| & \|x_2\| & \cdots & \|x_n\| \end{bmatrix}^T.
\end{aligned}$$

Now, since $\tilde{x}$ is a unit vector, we have

$$|\langle Tx, x\rangle| \leq w([\tilde{a}_{kj}]).$$

Thus,

$$w(T) \leq w([\tilde{a}_{kj}]),$$

as required. ∎

Remark 3.2.1. *By Proposition 3.2.5, we get*

$$w^2\left(|T_{kk}| + i|T_{kk}^*|\right) \le \||T_{kk}|^2 + |T_{kk}^*|^2\|.$$

So, the inequality (3.30) is sharper than the inequality (3.28).

Using Lemma 1.8.10, we can reformulate the inequality (3.30) as follows.

Theorem 3.2.7. *Let $T = [T_{kj}]$ be an $n \times n$ operator matrix with $T_{kj} \in \mathbb{B}(\mathcal{H}_j, \mathcal{H}_k)$. Then*

$$w(T) \le r([s_{kj}]),$$

where

$$s_{kj} = \begin{cases} \frac{1}{\sqrt{2}} w(|T_{kk}| + i|T_{kk}^*|) & \text{if } k = j \\ \frac{1}{2}(\|T_{kj}\| + \|T_{jk}\|) & \text{if } k \ne j. \end{cases}$$

When $n = 2$, we have the following corollary.

Corollary 3.2.8. *Let $T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$ be an operator matrix in $\mathbb{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$. Then*

$$w(T) \le \frac{1}{2}\left(\alpha + \gamma + \sqrt{(\alpha - \gamma)^2 + 4\beta^2}\right),$$

where

$$\alpha = \frac{1}{\sqrt{2}} w(|T_{11}| + i|T_{11}^*|),\ \beta = \frac{1}{2}(\|T_{12}\| + \|T_{21}\|) \text{ and } \gamma = \frac{1}{\sqrt{2}} w(|T_{22}| + i|T_{22}^*|).$$

It should be mentioned here that the numerical radius inequalities related to Theorems 3.2.6 and 3.2.7 can be found in [3] and [25].

Lemma 3.2.9. *[46, p. 346] Let $T = [a_{ij}]$ be an $n \times n$ matrix. Then*

$$r(T) \le \min\left\{\max_i \sum_{j=1}^n |a_{ij}|, \max_j \sum_{i=1}^n |a_{ij}|\right\}.$$

Using Theorem 3.2.7, Lemma 3.2.9 and Remark 3.2.1, we obtain the following theorem.

Theorem 3.2.10. *Let $T = [T_{kj}]$ be an $n \times n$ operator matrix with $T_{kj} \in \mathbb{B}(\mathcal{H}_j, \mathcal{H}_k)$. Then*

$$w(T) \le \max_k \left(\frac{1}{\sqrt{2}} w(|T_{kk}| + i|T_{kk}^*|) + \frac{1}{2} \sum_{\substack{j=1 \\ k \ne j}}^n (\|T_{kj}\| + \|T_{jk}\|) \right). \tag{3.31}$$

The inequality (3.31) generalizes a related inequality for the numerical radii of matrices [47, p. 33].

We give some numerical examples in order to illustrate that the bounds of the numerical radius in Theorems 3.2.1, 3.2.2, 3.2.3, 3.2.4 and 3.2.10 are not comparable in general.

Example 3.2.1. *(1) Let* $T_1 = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$. *Then the upper bounds for* $w(T_1)$ *in the above theorems are as shown in the following table.*

Theorem 3.2.3	5.23
Theorems 3.2.1 and 3.2.4	5
Theorems 3.2.2 and 3.2.10	4.5

(2) Let $T_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. *Then the upper bounds for* $w(T_2)$ *in the above theorems are as shown in the following table.*

Theorem 3.2.4	3
Theorem 3.2.3	2.41
Theorems 3.2.1 and 3.2.2	2
Theorem 3.2.10	1.5

(3) Let $T_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. *Then the upper bounds for* $w(T_3)$ *in the above theorems are as shown in the following table.*

Theorem 3.2.4	4
Theorem 3.2.3	3
Theorems 3.2.2 and 3.2.10	2.5
Theorem 3.2.1	2.41

3.2.2 A bound for the zeros of polynomials

We provide a new bound for the zeros of polynomials.

Let $p(z) = z^n + a_n z^{n-1} + \cdots + a_2 z + a_1$ be a monic polynomial of degree $n \geq 2$ with complex coefficients $a_1, a_2, \ldots, a_n$. Then the Frobenius companion matrix of $p(z)$ is

$$C(p) = \begin{bmatrix} -a_n & -a_{n-1} & \ldots & -a_2 & -a_1 \\ 1 & 0 & \ldots & \ldots & 0 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & 1 & 0 \end{bmatrix}.$$

It is well known that the eigenvalue of $C(p)$ are exactly the zeros of the polynomial $p(z)$. If λ is any zero of the polynomial $p(z)$, then it follows that

$$|\lambda| \leq w(C(p)) \quad as \quad \sigma(C(p)) \subseteq W(C(p)). \tag{3.32}$$

The inequality (3.32) has attracted attention of many mathematicians, see, e.g., [2], [8] and [27]. So, they have provided several estimations of upper bounds for the zeros of the polynomial $p(z)$. We cite here some of them :

(1) Cauchy [46] provided the following estimate

$$|\lambda| \leq 1 + \max\left\{|a_k| : k = 1, 2, \cdots, n\right\}.$$

(2) Carmichael and Mason [46] gave the following estimate

$$|\lambda| \leq \sqrt{1 + \sum_{i=1}^{n} |a_i|^2}.$$

(3) Montel [46] presented the following estimate

$$|\lambda| \leq \max\left\{1, \sum_{i=1}^{n} |a_i|\right\}.$$

(4) Abu-Omar and Kittaneh [2] established the following estimate

$$|\lambda| \leq \frac{1}{2}\left(\frac{1}{2}(|a_n| + \alpha) + \cos\left(\frac{\pi}{n+1}\right) + \sqrt{\left(\frac{|a_n| + \alpha}{2} - \cos\left(\frac{\pi}{n+1}\right)\right)^2 + 4\beta}\right),$$

where $\alpha = \sqrt{\sum_{j=1}^{n} |a_j|^2}$ and $\beta = \sqrt{\sum_{j=1}^{n-1} |a_j|^2}$.

(5) Al-Dolat, Jaradat and Al-Husban [8] have obtained the following estimate

$$|\lambda| \leq \frac{1}{2}\left(|a_n| + 2\cos\left(\frac{\pi}{n}\right) + \sqrt{t^2|a_n|^2 + \sum_{i=1}^{n-1}|a_i|^2 + \sqrt{1+(1-t)^2|a_n|^2}}\right)$$

for all $t \in [0,1]$.

Let λ be a zero of the polynomial $p(z)$. Using the inequality (3.32), we can derive a new upper bound for $|\lambda|$.

Theorem 3.2.11. *Let λ be any zero of $p(z)$. Then*

$$|\lambda| \leq \frac{1}{4}\left(2 + \sqrt{2}w(|T_1| + i|T_1^*|) + \sqrt{\left(\sqrt{2}w(|T_1| + i|T_1^*|) - 2\right)^2 + 4\left(1 + \sqrt{\sum_{j=1}^{n-2}|a_j|^2}\right)^2}\right), \tag{3.33}$$

where $T_1 = \begin{bmatrix} -a_n & -a_{n-1} \\ 1 & 0 \end{bmatrix}$.

Proof. Partition $C(p)$ as

$$C(p) = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix},$$

where

$$T_1 = \begin{bmatrix} -a_n & -a_{n-1} \\ 1 & 0 \end{bmatrix},\ T_2 = \begin{bmatrix} -a_{n-2} & \cdots & -a_2 & -a_1 \\ 0 & \cdots & 0 & 0 \end{bmatrix},\ T_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}$$

and

$$T_4 = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Using Corollary 3.2.8, we have

$$\begin{aligned} w(C(p)) &\leq r\left(\begin{bmatrix} \frac{1}{\sqrt{2}}w(|T_1|+i|T_1^*|) & \frac{1}{2}(\|T_2\|+\|T_3\|) \\ \frac{1}{2}(\|T_2\|+\|T_3\|) & \frac{1}{\sqrt{2}}w(|T_4|+i|T_4^*|) \end{bmatrix}\right) \\ &= \frac{1}{2}r\left(\begin{bmatrix} \sqrt{2}w(|T_1|+i|T_1^*|) & 1+\sqrt{\sum_{j=1}^{n-2}|a_j|^2} \\ 1+\sqrt{\sum_{j=1}^{n-2}|a_j|^2} & 2 \end{bmatrix}\right) \\ &= \frac{1}{4}\sqrt{\left(\sqrt{2}w(|T_1|+i|T_1^*|)-2\right)^2+4\left(1+\sqrt{\sum_{j=1}^{n-2}|a_j|^2}\right)^2} \\ &\quad +\frac{1}{4}\left(2+\sqrt{2}w(|T_1|+i|T_1^*|)\right), \end{aligned}$$

as required. ■

Remark 3.2.2. *If we partition $C(p)$ as*

$$C(p)=\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix},$$

where

$$T_1=\begin{bmatrix}-a_n\end{bmatrix},\ T_2=\begin{bmatrix}-a_{n-1} & \cdots & -a_2 & -a_1\end{bmatrix},\ T_3=\begin{bmatrix}1\\0\\ \vdots \\0\end{bmatrix}$$

and

$$T_4=\begin{bmatrix} 0 & \cdots & 0 & 0 \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix},$$

then it follows directly by using Corollary 3.2.8 that

$$|\lambda| \leq \frac{1}{2}\left(1+|a_n|+\sqrt{(|a_n|-1)^2+\left(1+\sqrt{\sum_{j=1}^{n-1}|a_j|^2}\right)^2}\right). \tag{3.34}$$

The bound (3.34) has been obtained earlier in [54].

We illustrate with an example that our bound can give a better estimation than the previous bounds mentioned above.

Example 3.2.2. *Consider the polynomial* $p(z) = z^5 + z + 4$. *Then the upper bounds for the zeros of the polynomial* $p(z)$ *estimated by the mathematicians above are as shown in the following table.*

Cauchy [46]	*5*
Carmichael and Mason [46]	*4.24*
Montel [46]	*5*
Abu-Omar and Kittaneh [2]	*3.62*
Al-Dolat, Jaradat and Al-Husban [8]	*3.56*

If λ *is a zero of the polynomial* $p(z)$, *then Theorem 3.2.11 gives* $|\lambda| \leq 3.41$, *which is better than all of the above estimations.*

Chapter 4

Upper and lower bounds for the p-numerical radii of operators

4.1 Inequalities involving p-numerical radius

In this section, we give several new upper and lower bounds for the p-numerical radii of operators as well as for $n \times n$ operator matrices. An application to 2-nilpotent operators is provided, and a p-numerical radius power inequality is also given.

In [31], Bottazzi and Conde have obtained the following results.

Theorem 4.1.1. *Let $T \in \mathbb{B}_p(\mathcal{H})$. Then*

$$2^{-\frac{1}{p}}\|T\|_p \leq w_p(T) \leq \|T\|_p \quad \textit{for } 1 \leq p \leq 2 \tag{4.1}$$

and

$$2^{\frac{1}{p}-1}\|T\|_p \leq w_p(T) \leq \|T\|_p \quad \textit{for } 2 \leq p \leq \infty. \tag{4.2}$$

In the next theorem, we give a lower and upper bound for the p-numerical radius.

Theorem 4.1.2. *[31] Let $T \in \mathbb{B}_p(\mathcal{H})$ and $p \geq 2$. Then*

$$\frac{1}{4}\|T^*T+TT^*\|_{p/2}+\frac{1}{2}\sup_{\theta\in\mathbb{R}}\left|\|Re(e^{i\theta}T)\|_p^2-\|Im(e^{i\theta}T)\|_p^2\right| \leq w_p^2(T) \leq \frac{1}{4}\|T^*T+TT^*\|_{p/2}+w_{p/2}(T^2). \tag{4.3}$$

Proof. To prove the first inequality, we have

$$\begin{aligned}
w_p^2(T) &\geq \max\left\{\|Re(e^{i\theta}T)\|_p^2, \|Im(e^{i\theta}T)\|_p^2\right\} \\
&= \frac{\|Re(e^{i\theta}T)\|_p^2 + \|Im(e^{i\theta}T)\|_p^2}{2} + \frac{1}{2}\sup_{\theta\in\mathbb{R}}\left|\|Re(e^{i\theta}T)\|_p^2 - \|Im(e^{i\theta}T)\|_p^2\right| \\
&= \frac{\|Re^2(e^{i\theta}T)\|_{p/2} + \|Im^2(e^{i\theta}T)\|_{p/2}}{2} + \frac{1}{2}\sup_{\theta\in\mathbb{R}}\left|\|Re(e^{i\theta}T)\|_p^2 - \|Im(e^{i\theta}T)\|_p^2\right| \\
&\geq \frac{\|Re^2(e^{i\theta}T) + Im^2(e^{i\theta}T)\|_{p/2}}{2} + \frac{1}{2}\sup_{\theta\in\mathbb{R}}\left|\|Re(e^{i\theta}T)\|_p^2 - \|Im(e^{i\theta}T)\|_p^2\right| \\
&= \frac{1}{4}\|T^*T + TT^*\|_{p/2} + \frac{1}{2}\sup_{\theta\in\mathbb{R}}\left|\|Re(e^{i\theta}T)\|_p^2 - \|Im(e^{i\theta}T)\|_p^2\right|,
\end{aligned}$$

which proves the first inequality in (4.3).

To prove the second inequality, we have

$$\begin{aligned}
w_p^2(T) &= \sup_{\theta\in\mathbb{R}}\|Re(e^{i\theta}T)\|_p^2 \\
&= \frac{1}{4}\sup_{\theta\in\mathbb{R}}\|e^{i\theta}T + e^{-i\theta}T^*\|_p^2 \\
&= \frac{1}{4}\sup_{\theta\in\mathbb{R}}\|(e^{i\theta}T + e^{-i\theta}T^*)^2\|_{p/2} \\
&= \frac{1}{4}\sup_{\theta\in\mathbb{R}}\|T^*T + TT^* + 2Re(e^{2i\theta}T^2)\|_{p/2} \\
&\leq \frac{1}{4}\|T^*T + TT^*\|_{p/2} + \frac{1}{2}\sup_{\theta\in\mathbb{R}}\|Re(e^{2i\theta}T^2)\|_{p/2} \\
&= \frac{1}{4}\|T^*T + TT^*\|_{p/2} + \frac{1}{2}w_{p/2}(T^2),
\end{aligned}$$

as required. ■

Benmakhlouf, Hirzallah and Kittaneh [13] have shown that.

Theorem 4.1.3. *Let $T \in \mathbb{B}_p(\mathcal{H})$. Then*

$$w_p^p(T) \leq \|Re(T)\|_p^p + \|Im(T)\|_p^p \qquad \text{for } 1 \leq p \leq 2 \tag{4.4}$$

and

$$w_p^p(T) \leq 2^{\frac{p}{2}-1}\left(\|Re(T)\|_p^p + \|Im(T)\|_p^p\right) \qquad \text{for } 2 \leq p < \infty. \tag{4.5}$$

Proof. By Theorem 1.10.1, we have

$$w_p(T) = \sup_{\substack{\alpha,\beta\in\mathbb{R}\\ \alpha^2+\beta^2=1}} \|\alpha Re(T) + \beta Im(T)\|_p \quad \text{for } 0 < p \le \infty.$$

In particular,

$$w_p(T) = \sup_{0\le\theta\le\pi} \|\cos(\theta)Re(T) + \sin(\theta)Im(T)\|_p.$$

So,

$$\begin{aligned} w_p(T) &\le \sup_{0\le\theta\le\pi} \left(|\cos(\theta)|\|Re(T)\|_p + |\sin(\theta)|\|Im(T)\|_p\right) \\ &\le \sup_{0\le\theta\le\pi} (|\cos(\theta)|^q + |\sin(\theta)|^q)^{\frac{1}{q}} \left(\|Re(T)\|_p^p + \|Im(T)\|_p^p\right)^{\frac{1}{p}} \text{ (by Hôlder's inequality)} \\ &= \sup_{0\le\theta\le\frac{\pi}{2}} (\cos^q(\theta) + \sin^q(\theta))^{\frac{1}{q}} \left(\|Re(T)\|_p^p + \|Im(T)\|_p^p\right)^{\frac{1}{p}} \end{aligned}$$

If $1 \le p \le 2$, then $2 \le q \le \infty$, which implies that $\sup_{0\le\theta\le\frac{\pi}{2}} (\cos^q(\theta) + \sin^q(\theta))^{\frac{1}{q}} = 1$, so we get

$$w_p^p(T) \le \|Re(T)\|_p^p + \|Im(T)\|_p^p.$$

If $2 \le p < \infty$, then $1 < q \le 2$, which implies that $\sup_{0\le\theta\le\frac{\pi}{2}} (\cos^q(\theta) + \sin^q(\theta))^{\frac{1}{q}} + 2^{\frac{1}{q}-\frac{1}{2}} = 2^{\frac{1}{2}-\frac{1}{p}}$, so we obtain

$$w_p^p(T) \le 2^{\frac{p}{2}-1}\left(\|Re(T)\|_p^p + \|Im(T)\|_p^p\right).$$

■

Theorem 4.1.4. *Let $T \in \mathbb{B}_p(\mathcal{H})$ and $4 \le p \le \infty$. Then*

$$w_p^4(T) \le \frac{1}{16}\|S\|_{p/2}^2 + \frac{1}{4}w_{p/2}^2(T^2) + \frac{1}{8}w_{p/4}(T^2S + ST^2),$$

where $S = T^*T + TT^*$.

Proof. For every $\theta \in \mathbb{R}$, we have

$$
\begin{aligned}
\|Re(e^{i\theta}T)\|_p^4 &= \frac{1}{16}\left\||e^{i\theta}T + e^{-i\theta}T^*|^2\right\|_{p/2}^2 \\
&= \frac{1}{16}\left\|e^{2i\theta}T^2 + e^{-2i\theta}(T^*)^2 + S\right\|_{p/2}^2 \\
&= \frac{1}{16}\left\||e^{2i\theta}T^2 + e^{-2i\theta}(T^*)^2 + S|^2\right\|_{p/4} \\
&= \frac{1}{16}\|4\left(Re(e^{2i\theta}T^2)\right)^2 + 2Re(e^{2i\theta}T^2)S + 2SRe(e^{2i\theta}T^2) + S^2\|_{p/4} \\
&= \frac{1}{16}\left\|4\left(Re(e^{2i\theta}T^2)\right)^2 + 2Re\left(e^{2i\theta}(T^2S + ST^2)\right) + S^2\right\|_{p/4} \\
&\leq \frac{1}{16}\left\|S^2\right\|_{p/4} + \frac{1}{4}\left\|Re(e^{2i\theta}T^2)\right\|_{p/2}^2 + \frac{1}{8}\left\|Re\left(e^{2i\theta}(T^2S + ST^2)\right)\right\|_{p/4}.
\end{aligned}
$$

By taking the supremum on both sides in the above inequality over $\theta \in \mathbb{R}$, we get the desired result. ■

Theorem 4.1.5. *Let $T \in \mathbb{B}_p(\mathcal{H})$. Then*

$$
w_p^p(T) \leq \inf_{\phi \in \mathbb{R}} \left(\|\varphi(\phi)\|_p^p + \|\varphi(\phi + \frac{\pi}{2})\|_p^p\right) \quad \textit{for } 1 \leq p \leq 2 \tag{4.6}
$$

and

$$
w_p^p(T) \leq 2^{\frac{p}{2}-1} \inf_{\phi \in \mathbb{R}} \left(\|\varphi(\phi)\|_p^p + \|\varphi(\phi + \frac{\pi}{2})\|_p^p\right) \quad \textit{for } 2 \leq p < \infty, \tag{4.7}
$$

where $\varphi(\phi) = Re(e^{i\phi}T)$.

Proof. Let $\phi \in [0, 2\pi]$. Then $\varphi(\theta) = \cos\theta Re(T) - \sin\theta Im(T)$. It follows that

$$
\begin{aligned}
\|\varphi(\theta + \phi)\|_p &= \|\cos(\theta + \phi)Re(T) - \sin(\theta + \phi)Im(T)\|_p \\
&= \|\cos\theta(\cos\phi Re(T) - \sin\phi Im(T)) - \sin\theta(\sin\phi Re(T) + \cos\phi Im(T))\|_p \\
&= \|\cos\theta(\cos\phi Re(T) - \sin\phi Im(T)) + \sin\theta(\cos(\phi + \frac{\pi}{2})Re(T) - \sin(\phi + \frac{\pi}{2})Im(T))\|_p \\
&= \left\|\cos\theta Re(e^{i\phi}T) + \sin\theta Re\left(e^{i(\phi + \frac{\pi}{2})}T\right)\right\|_p \\
&\leq \|\cos\theta\varphi(\phi)\|_p + \|\sin\theta\varphi(\phi + \frac{\pi}{2})\|_p \\
&\leq (|\cos\theta|^q + |\sin\theta|^q)^{\frac{1}{q}}\left(\|\varphi(\phi)\|_p^p + \|\varphi(\phi + \frac{\pi}{2})\|_p^p\right)^{\frac{1}{p}}
\end{aligned}
$$

(by the Holder's inequality).

By taking the supremum in the above inequality over $\theta \in \mathbb{R}$, we obtain

$$w_p(T) \leq \sup_{\theta\in\mathbb{R}} (|\cos\theta|^q + |\sin\theta|^q)^{\frac{1}{q}} \left(\|\varphi(\phi)\|_p^p + \|\varphi(\phi + \frac{\pi}{2})\|_p^p\right)^{\frac{1}{p}}.$$

If $1 \leq p \leq 2$, then $2 \leq q \leq \infty$, and so $\sup_{\theta\in\mathbb{R}} (|\cos\theta|^q + |\sin\theta|^q)^{\frac{1}{q}} = 1$. Hence,

$$w_p^p(T) \leq \inf_{\phi\in\mathbb{R}} \left(\|\varphi(\phi)\|_p^p + \|\varphi(\phi + \frac{\pi}{2})\|_p^p\right).$$

If $2 \leq p < \infty$, then $1 < q \leq 2$, and so $\sup_{\theta\in\mathbb{R}} (|\cos\theta|^q + |\sin\theta|^q)^{\frac{1}{q}} = 2^{\frac{1}{q}-\frac{1}{2}} = 2^{\frac{1}{2}-\frac{1}{p}}$. Hence,

$$w_p^p(T) \leq 2^{\frac{p}{2}-1} \inf_{\phi\in\mathbb{R}} \left(\|\varphi(\phi)\|_p^p + \|\varphi(\phi + \frac{\pi}{2})\|_p^p\right).$$

■

Remark 4.1.1. *It is easy to see that the inequalities (4.6) and (4.7) are better than the inequalities (4.4) and (4.5), respectively.*

Theorem 4.1.6. *Let $T \in \mathbb{B}_p(\mathcal{H})$ and $2 \leq p \leq \infty$. Then*

$$w_p^2(T) \leq \frac{1}{2} w_{p/2}(T^2) + \frac{1}{2}\|Re(T^2)\|_{p/2} + \min\left\{\|Re(T)\|_p^2, \|Im(T)\|_p^2\right\}.$$

Proof. For every $\theta \in \mathbb{R}$, we have

$$\begin{aligned}
\|Re(e^{i\theta}T)\|_p^2 &= \frac{1}{4}\left\||e^{i\theta}T + e^{-i\theta}T^*|^2\right\|_{p/2} \\
&= \frac{1}{4}\left\|e^{2i\theta}T^2 + e^{-2i\theta}T^{*2} + TT^* + T^*T\right\|_{p/2} \\
&= \frac{1}{4}\left\|(e^{2i\theta} - 1)T^2 + (e^{-2i\theta} - 1)T^{*2} + T^2 + T^{*2} + TT^* + T^*T\right\|_{p/2} \\
&= \frac{1}{2}\left\|Re\Big((e^{2i\theta} - 1)T^2\Big) + 2\Big(Re(T)\Big)^2\right\|_{p/2} \\
&\leq \frac{1}{2}\left\|Re\Big((e^{2i\theta} - 1)T^2\Big)\right\|_{p/2} + \left\|\Big(Re(T)\Big)^2\right\|_{p/2} \\
&\leq \frac{1}{2}\left\|Re(e^{2i\theta}T^2)\right\|_{p/2} + \frac{1}{2}\|Re(T^2)\|_{p/2} + \|Re(T)\|_p^2.
\end{aligned}$$

By taking the supremum on both sides in the above inequality over $\theta \in \mathbb{R}$, we obtain

$$w_p^2(T) \leq \frac{1}{2} w_{p/2}(T^2) + \frac{1}{2}\|Re(T^2)\|_{p/2} + \|Re(T)\|_p^2.$$

Applying a similar argument, we can prove that

$$w_p^2(T) \leq \frac{1}{2} w_{p/2}(T^2) + \frac{1}{2} \|Re(T^2)\|_{p/2} + \|Im(T)\|_p^2.$$

Therefore, we get the desired result. ■

Corollary 4.1.7. *Let $T \in \mathbb{B}_p(\mathcal{H})$ be a 2-nilpotent operator. Then*

$$w_p(T) = \|Re(T)\|_p = \|Im(T)\|_p \quad \text{for } 2 \leq p \leq \infty.$$

Remark 4.1.2. *A different proof of Corollary 4.1.7 which is valid for $0 < p \leq \infty$ goes as follows: Since $T^2 = 0$, it follows that we can write $T = \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}$ with respect to the decomposition $\mathcal{H} = \overline{ranT} \oplus (ranT)^{\perp}$, where $ranT$ denotes the range of T and A is an operator from $(ranT)^{\perp}$ into $\overline{ranT}$. Now,*

$$\begin{aligned} w_p(T) &= \sup_{\theta \in \mathbb{R}} \|Re(e^{i\theta} T)\|_p \\ &= \frac{1}{2} \sup_{\theta \in \mathbb{R}} \left\| \begin{bmatrix} 0 & e^{i\theta} A \\ e^{-i\theta} A^* & 0 \end{bmatrix} \right\|_p \\ &= 2^{\frac{1}{p}-1} \|A\|_p = \|Re(T)\|_p = \|Im(T)\|_p \quad \text{for } 0 < p \leq \infty. \end{aligned}$$

Remark 4.1.3. *We remark here that the converse of Corollary 4.1.7 is not true. To see this, consider the two-dimensional example $T = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$. Then $w_p(T) = \|Re(T)\|_p = \|Im(T)\|_p = 1 \quad$ for $2 \leq p \leq \infty$, see e.g.,[13], but $T^2 \neq 0$.*

Theorem 4.1.8. *Let $T \in \mathbb{B}_p(\mathcal{H})$ and $2 \leq p \leq \infty$. Then*

$$w_p^2(T) \leq \frac{1}{2}\Big(\|Re(T)\|_p^2 + \|Im(T)\|_p^2 + \sqrt{\Big(\|Re(T)\|_p^2 - \|Im(T)\|_p^2\Big)^2 + 4\|Re(T)Im(T)\|_{p/2}^2}\Big). \tag{4.8}$$

Proof. By theorem 1.10.1, we have

$$
\begin{aligned}
w_p^2(T) &= \sup_{\substack{\alpha,\beta\in\mathbb{R}\\ \alpha^2+\beta^2=1}} \|\alpha Re(T) + \beta Im(T)\|_p^2 \\
&= \sup_{\substack{\alpha,\beta\in\mathbb{R}\\ \alpha^2+\beta^2=1}} \left\|\alpha^2(Re(T))^2 + \beta^2(Im(T))^2 + \alpha\beta\Big(Re(T)Im(T) + Im(T)Re(T)\Big)\right\|_{p/2} \\
&\leq \sup_{\substack{\alpha,\beta\in\mathbb{R}\\ \alpha^2+\beta^2=1}} \left(\alpha^2\|Re(T)\|_p^2 + \beta^2\|Im(T)\|_p^2 + 2\alpha\beta\|Re(T)Im(T)\|_{p/2}\right) \\
&\leq \sup_{\substack{\alpha,\beta\in\mathbb{R}\\ \alpha^2+\beta^2=1}} \left\|\begin{bmatrix} \|Re(T)\|_p^2 & \|Re(T)Im(T)\|_{p/2} \\ \|Re(T)Im(T)\|_{p/2} & \|Im(T)\|_p^2 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix}\right\| \\
&\leq \left\|\begin{bmatrix} \|Re(T)\|_p^2 & \|Re(T)Im(T)\|_{p/2} \\ \|Re(T)Im(T)\|_{p/2} & \|Im(T)\|_p^2 \end{bmatrix}\right\| \\
&= \frac{1}{2}\left(\|Re(T)\|_p^2 + \|Im(T)\|_p^2 + \sqrt{\left(\|Re(T)\|_p^2 - \|Im(T)\|_p^2\right)^2 + 4\|Re(T)Im(T)\|_{p/2}^2}\right),
\end{aligned}
$$

as required. ■

It follows from Theorem 7 in [31] that if $T \in \mathbb{B}_p(\mathcal{H})$, then

$$\frac{1}{4}\|T^*T + TT^*\|_{p/2} + \frac{1}{8}\left|\|T + T^*\|_p^2 - \|T - T^*\|_p^2\right| \leq w_p^2(T) \quad \text{for } 2 \leq p \leq \infty.$$

For $0 < p \leq 2$, we have the following related inequality, we need the following lemma which is an application of Jensen's inequality and it can be found in [43].

Lemma 4.1.9. *Let $a, b \geq 0$ and let $0 \leq \alpha \leq 1$. Then*

$$(\alpha a^r + (1-\alpha)b^r)^{\frac{1}{r}} \leq (\alpha a^s + (1-\alpha)b^s)^{\frac{1}{s}} \quad \text{for } \ 0 < r \leq s.$$

Theorem 4.1.10. *Let $T \in \mathbb{B}_p(\mathcal{H})$. Then*

$$2^{-\frac{2}{p}-1}\|T^*T + TT^*\|_{p/2} + \frac{1}{8}\left|\|T + T^*\|_p^2 - \|T - T^*\|_p^2\right| \leq w_p^2(T) \quad \text{for } 0 < p \leq 2.$$

Proof. For $0 < p \le 2$, we have

$$\begin{aligned}
4w_p^2(T) &\ge \max\left\{\|T+T^*\|_p^2, \|T-T^*\|_p^2\right\} \\
&= \frac{1}{2}\left(\|T+T^*\|_p^2 + \|T-T^*\|_p^2 + \left|\|T+T^*\|_p^2 - \|T-T^*\|_p^2\right|\right) \\
&\ge \left(\frac{\|T+T^*\|_p^p + \|T-T^*\|_p^p}{2}\right)^{\frac{2}{p}} + \frac{1}{2}\left|\|T+T^*\|_p^2 - \|T-T^*\|_p^2\right| \\
&\quad \text{(by Lemma 4.1.9)} \\
&= \left(\frac{\||T+T^*|^2\|_{p/2}^{\frac{p}{2}} + \||T-T^*|^2\|_{p/2}^{\frac{p}{2}}}{2}\right)^{\frac{2}{p}} + \frac{1}{2}\left|\|T+T^*\|_p^2 - \|T-T^*\|_p^2\right| \\
&\ge 2^{1-\frac{2}{p}}\|T^*T + TT^*\|_{p/2} + \frac{1}{2}\left|\|T+T^*\|_p^2 - \|T-T^*\|_p^2\right| \ \text{(by Lemma 1.9.2)}.
\end{aligned}$$

Hence,

$$w_p^2(T) \ge 2^{-\frac{2}{p}-1}\|T^*T + TT^*\|_{p/2} + \frac{1}{2}\left|\|T+T^*\|_p^2 - \|T-T^*\|_p^2\right|.$$

∎

The following two lemmas can be found in [42] and [67], respectively.

Lemma 4.1.11. *Let* $T = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$, *where* $B, C \in \mathbb{B}_p(\mathcal{H})$ *and* $0 < p \le \infty$. *Then*

$$w_p(T) = 2^{\frac{1}{p}-1}\sup_{\theta\in\mathbb{R}}\left\|e^{i\theta}B + e^{-i\theta}C^*\right\|_p.$$

In particular,

$$w_p\left(\begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix}\right) = 2^{\frac{1}{p}} w_p(B).$$

Lemma 4.1.12. *Let* $T, S \in \mathbb{B}_p(\mathcal{H})$ *be positive. Then*

$$\|T - S\|_p \le \left(\|T\|_p^p + \|S\|_p^p\right)^{\frac{1}{p}} \quad \textit{for } 0 < p \le \infty.$$

Remark 4.1.4. *It is known that if* $T, S \in \mathbb{B}(\mathcal{H})$ *are positive, then*

$$\|T - S\| \le \max\{\|T\|, \|S\|\},$$

see e.g., [55].

Theorem 4.1.13. *Let $T, S \in \mathbb{B}_p(\mathcal{H})$. Then*

$$\max\left\{w_{p/2}(TS), w_{p/2}(ST)\right\} \leq w_p^2\left(\begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix}\right),$$

for $0 < p \leq \infty$.

Proof. From the identity $4Re(T^*S) = |T+S|^2 - |T-S|^2$, it follows that

$$\begin{aligned} \|4Re(T^*S)\|_{p/2} &= \left\||T+S|^2 - |T-S|^2\right\|_{p/2} \\ &\leq \left(\left\||T+S|^2\right\|_{p/2}^{p/2} + \left\||T-S|^2\right\|_{p/2}^{p/2}\right)^{\frac{2}{p}} \quad \text{(by Lemma 4.1.12).} \end{aligned}$$

Thus,

$$\|4Re(T^*S)\|_{p/2} \leq \left(\left\|T+S\right\|_p^p + \left\|T-S\right\|_p^p\right)^{\frac{2}{p}}. \tag{4.9}$$

Now, replacing T by $e^{i\theta}T$, and taking the supremum on both sides in the inequality (4.9) over $\theta \in \mathbb{R}$, we obtain

$$\begin{aligned} w_{p/2}(T^*S) &\leq 2^{-2}\Big(\sup_{\theta\in\mathbb{R}} \left\|e^{i\theta}T + S\right\|_p^p + \sup_{\theta\in\mathbb{R}} \left\|e^{i\theta}T - S\right\|_p^p\Big)^{\frac{2}{p}} \\ &= w_p^2\left(\begin{bmatrix} 0 & T \\ S^* & 0 \end{bmatrix}\right) \quad \text{(by Lemma 4.1.11).} \end{aligned}$$

Hence,

$$\max\left\{w_{p/2}(TS), w_{p/2}(ST)\right\} \leq w_p^2\left(\begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix}\right).$$

■

Remark 4.1.5. *If we choose $S = T$ in Theorem 4.1.13, then we get the following p-numerical radius power inequality*

$$w_{p/2}(T^2) \leq 2^{\frac{2}{p}} w_p^2(T),$$

which generalizes the usual numerical radius power inequality $w(T^2) \leq w^2(T)$, see e.g., [41, p 118].

In the following theorem, we present a lower bound for the p-numerical radius of a product of two operators.

Theorem 4.1.14. *Let $T, S \in \mathbb{B}_p(\mathcal{H})$. Then*

$$2^{1-\frac{1}{p}} w_{2p}^2\left(\begin{bmatrix} 0 & T \\ S^* & 0 \end{bmatrix}\right) - \frac{1}{2}\left\|T^*T + S^*S\right\|_p \leq w_p(T^*S)$$

for $1 \leq p \leq \infty$.

Proof. We have

$$\begin{aligned} \left\|T + S\right\|_{2p}^2 &= \left\||T + S|^2\right\|_p \\ &= \left\|T^*T + S^*S + 2Re(T^*S)\right\|_p \\ &\leq \left\|T^*T + S^*S\right\|_p + 2\left\|Re(T^*S)\right\|_p. \end{aligned}$$

Hence, for every $\theta \in \mathbb{R}$, we have

$$\left\|e^{i\theta}T + S\right\|_{2p}^2 \leq \left\|T^*T + S^*S\right\|_p + 2\left\|Re(e^{-i\theta}T^*S)\right\|_p.$$

By taking the supremum on both sides in the above inequality over $\theta \in \mathbb{R}$, and using Lemma 4.1.11, we get the desired result. ■

If we take $S^* = T$ in Theorem 4.1.14, and we use the convexity of the function $f(t) = t^r$ on $[0, \infty)$ for $r \geq 1$, then we obtain the following corollary.

Corollary 4.1.15. *Let $T \in \mathbb{B}_p(\mathcal{H})$. Then*

$$w_{2p}^{2r}(T) \leq \frac{1}{2}\left(\|T\|_{2p}^{2r} + w_p^r(T^2)\right) \tag{4.10}$$

for $r \geq 1$ and $1 \leq p \leq \infty$.

It should be mentioned here that the inequality (4.10) has been obtained in [62] for the case $p = \infty$, i.e., for the usual numerical radius.

Theorem 4.1.16. *Let $T, S \in \mathbb{B}_p(\mathcal{H})$. If $T^*S = TS^* = 0$, then*

$$2^{\frac{p}{2}-1} w_{p/2}^{p/2}\left(\begin{bmatrix} 0 & TS \\ ST & 0 \end{bmatrix}\right) \leq w_p^p(T + S) + w_p^p(T - S)$$

for $0 < p < \infty$.

Proof. Using the inequality (4.9) again, it follows that

$$\|4Re(e^{i\theta}T)Re(e^{i\theta}S)\|_{p/2} \leq \left(\|Re(e^{i\theta}(T + S))\|_p^p + \|Re(e^{i\theta}(T - S))\|_p^p\right)^{\frac{2}{p}}.$$

Thus,

$$\|e^{2i\theta}TS + e^{-2i\theta}T^*S^*\|_{p/2} \leq \left(\|Re(e^{i\theta}(T+S))\|_p^p + \|Re(e^{i\theta}(T-S))\|_p^p\right)^{\frac{2}{p}}.$$

By taking the supremum on both sides in the above inequality over $\theta \in \mathbb{R}$, and using Lemma 4.1.11, we get the required result. ∎

In order to give the rest of our results, we need the following lemmas, which can be found in [15] and [51], respectively. A generalization of the second lemma to unitarily invariant norms is well known.

Lemma 4.1.17. *Let $T = [T_{ij}]$ with $T_{ij} \in \mathbb{B}_p(\mathcal{H})$, $i, j = 1, \ldots, n$. Then*

(a) $\|T\|_p^p \leq \sum_{i,j=1}^{n} \|T_{ij}\|_p^p \leq n^{2-p}\|T\|_p^p$ *for* $1 \leq p \leq 2$.

(b) $n^{2-p}\|T\|_p^p \leq \sum_{i,j=1}^{n} \|T_{ij}\|_p^p \leq \|T\|_p^p$ *for* $2 \leq p < \infty$.

Lemma 4.1.18. *Let $T, S \in \mathbb{B}_p(\mathcal{H})$ be such that TS is self-adjoint. Then*

$$\|TS\|_p \leq \|Re(ST)\|_p \quad \text{for } 1 \leq p \leq \infty.$$

Theorem 4.1.19. *Let $T, S \in \mathbb{B}_p(\mathcal{H})$. Then*

$$w_p^p(S^*T) \leq \frac{1}{2^p}\left(2\|TS^*\|_p^p + \|T\|_{2p}^{2p} + \|S\|_{2p}^{2p}\right) \quad \text{for } 1 \leq p \leq 2,$$

and

$$w_p^p(S^*T) \leq \frac{1}{4}\left(2\|TS^*\|_p^p + \|T\|_{2p}^{2p} + \|S\|_{2p}^{2p}\right) \quad \text{for } 2 \leq p < \infty.$$

Proof. For every $\theta \in \mathbb{R}$, we have

$$\begin{aligned}
\left\|Re(e^{i\theta}S^*T)\right\|_p^p &= \frac{1}{2^p}\left\|e^{i\theta}S^*T + e^{-i\theta}T^*S\right\|_p^p \\
&= \frac{1}{2^p}\left\|\begin{bmatrix} e^{i\theta}S^* & e^{-i\theta}T^* \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T & 0 \\ S & 0 \end{bmatrix}\right\|_p^p \\
&\leq \frac{1}{2^p}\left\|Re\left(\begin{bmatrix} T & 0 \\ S & 0 \end{bmatrix}\begin{bmatrix} e^{i\theta}S^* & e^{-i\theta}T^* \\ 0 & 0 \end{bmatrix}\right)\right\|_p^p \quad \text{(by Lemma 4.1.18)} \\
&\leq \frac{1}{2^p}\left\|\begin{bmatrix} T & 0 \\ S & 0 \end{bmatrix}\begin{bmatrix} e^{i\theta}S^* & e^{-i\theta}T^* \\ 0 & 0 \end{bmatrix}\right\|_p^p \\
&= \frac{1}{2^p}\left\|\begin{bmatrix} e^{i\theta}TS^* & e^{-i\theta}TT^* \\ e^{i\theta}SS^* & e^{-i\theta}ST^* \end{bmatrix}\right\|_p^p.
\end{aligned}$$

Now, using Lemma 4.1.17, we get

$$\left\|Re(e^{i\theta}S^*T)\right\|_p^p \leq \frac{1}{2^p}\left(2\|TS^*\|_p^p + \|T\|_{2p}^{2p} + \|S\|_{2p}^{2p}\right) \quad \text{for } 1 \leq p \leq 2$$

and

$$\left\|Re(e^{i\theta}S^*T)\right\|_p^p \leq \frac{1}{4}\left(2\|TS^*\|_p^p + \|T\|_{2p}^{2p} + \|S\|_{2p}^{2p}\right) \quad \text{for } 2 \leq p < \infty.$$

By taking the supremum in the above inequalities over $\theta \in \mathbb{R}$, we get the desired results. ■

Proposition 4.1.20. *Let* $T = \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$ *with* $T_{1j} \in \mathbb{B}_p(\mathcal{H})$, $j = 1, \ldots, n$. *Then*

(a) $w_p(T) \leq \left(\|Re(T_{11})\|_p^p + \|Im(T_{11})\|_p^p + 2^{2-p}\sum_{j=2}^{n}\|T_{1j}\|_p^p\right)^{\frac{1}{p}}$ *for* $1 \leq p \leq 2$.

(b) $w_p(T) \leq (\sqrt{2}n)^{1-\frac{2}{p}}\left(\|Re(T_{11})\|_p^p + \|Im(T_{11})\|_p^p + 2^{2-p}\sum_{j=2}^{n}\|T_{1j}\|_p^p\right)^{\frac{1}{p}}$ *for* $2 \leq p < \infty$.

Proof. For $1 \leq p \leq 2$, using Lemma 3.5 (a), we obtain

$$\|Re(T)\|_p^p \leq \|Re(T_{11})\|_p^p + 2^{1-p}\sum_{j=2}^{n}\|T_{1j}\|_p^p$$

and

$$\|Im(T)\|_p^p \leq \|Im(T_{11})\|_p^p + 2^{1-p}\sum_{j=2}^{n}\|T_{1j}\|_p^p.$$

Hence, in view of the inequality (4.4), we have

$$w_p^p(T) \leq \|Re(T_{11})\|_p^p + \|Im(T_{11})\|_p^p + 2^{2-p}\sum_{j=2}^{n}\|T_{1j}\|_p^p.$$

Similarly, for $2 \leq p < \infty$, using Lemma 3.5 (b), we obtain

$$\|Re(T)\|_p^p \leq n^{p-2}\Big(\|Re(T_{11})\|_p^p + 2^{1-p}\sum_{j=2}^{n}\|T_{1j}\|_p^p\Big)$$

and

$$\|Im(T)\|_p^p \leq n^{p-2}\Big(\|Im(T_{11})\|_p^p + 2^{1-p}\sum_{j=2}^{n}\|T_{1j}\|_p^p\Big).$$

Hence, in view of the inequality (4.5), we have

$$w_p^p(T) \leq (\sqrt{2}n)^{p-2}\Big(\|Re(T_{11})\|_p^p + \|Im(T_{11})\|_p^p + 2^{2-p}\sum_{j=2}^{n}\|T_{1j}\|_p^p\Big).$$

■

Theorem 4.1.21. *Let $T = [T_{ij}]$ be an $n \times n$ operator matrix with $T_{ij} \in \mathbb{B}_p(\mathcal{H})$, $i, j = 1, \ldots, n$. Then*

(a) $w_p(T) \leq \sum_{k=1}^{n}\Big(\|Re(T_{kk})\|_p^p + \|Im(T_{kk})\|_p^p + 2^{2-p}\sum_{\substack{j=1\\ k\neq j}}^{n}\|T_{kj}\|_p^p\Big)^{\frac{1}{p}}$ *for* $1 \leq p \leq 2$.

(b) $w_p(T) \leq (\sqrt{2}n)^{1-\frac{2}{p}}\sum_{k=1}^{n}\Big(\|Re(T_{kk})\|_p^p + \|Im(T_{kk})\|_p^p + 2^{2-p}\sum_{\substack{j=1\\ k\neq j}}^{n}\|T_{kj}\|_p^p\Big)^{\frac{1}{p}}$ *for* $2 \leq p < \infty$.

Proof. Applying the triangle inequality for the p-numerical radius, we have

$$w_p(T) \leq w_p(T_1) + w_p(T_2) + \cdots + w_p(T_n),$$

where $T_1 = \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, T_2 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \ldots,$

$$T_n = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{bmatrix}.$$

For each $i = 2, 3, \ldots, n$, let U_i be the $n \times n$ permutation operator matrix, (U_i is a unitary operator), obtained by interchanging the first and ith rows of the identity operator matrix. Now, using the weakly unitarily invariant property of the p-numerical radius, we get

$$w_p(T) \le w_p(T_1) + w_p(U_2^* T_2 U_2) + \cdots + w_p(U_n^* T_n U_n).$$

This yields

$$w_p(T) \le w_p\left(\begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right) + w_p\left(\begin{bmatrix} T_{22} & T_{21} & \cdots & T_{2n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right) + \cdots + w_p\left(\begin{bmatrix} T_{nn} & T_{n2} & \cdots & T_{n1} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right).$$

Therefore, applying Proposition 4.1.20, we get the desired result immediately. ■

4.2 p-numerical radius inequalities for 2×2 operator matrices

In this section, we give some new p-numerical radii inequalities for 2×2 operator matrices.

In the following theorems, we give some new upper bounds for the numerical radii of the off-diagonal parts of 2×2 operator matrices. In the first theorem, we give an inequality relating $w_p(\cdot)$, $w_{p/2}(\cdot)$ and $w_{p/4}(\cdot)$ for $4 \le p < \infty$. Also, we extract some results from this theorem for particular cases.

Theorem 4.2.1. *Let $T, S \in \mathbb{B}_p(\mathcal{H})$ and $4 \le p \le \infty$. Then*

$$w_p^4\left(\begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix}\right) \le 2^{\frac{4}{p}-4}\|P\|_{p/2}^2 + 2^{\frac{4}{p}-2} w_{p/2}^2(ST) + 2^{\frac{4}{p}-3} w_{p/4}(STP + PST), \tag{4.11}$$

where $P = T^*T + SS^*$.

Proof. By Lemma 4.1.11, we have

$$\begin{aligned} w_p\left(\begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix}\right) &= 2^{\frac{1}{p}-1} \sup_{\theta\in\mathbb{R}} \|e^{i\theta}T + e^{-i\theta}S^*\|_p \\ &= 2^{\frac{1}{p}-1} \sup_{\theta\in\mathbb{R}} \|(e^{i\theta}T + e^{-i\theta}S^*)^*(e^{i\theta}T + e^{-i\theta}S^*)\|_{p/2}^{1/2} \\ &= 2^{\frac{1}{p}-1} \sup_{\theta\in\mathbb{R}} \|P + 2Re(e^{2i\theta}ST)\|_{p/2}^{1/2} \\ &= 2^{\frac{1}{p}-1} \sup_{\theta\in\mathbb{R}} \|(P + 2Re(e^{2i\theta}ST))^2\|_{p/4}^{1/4} \\ &= 2^{\frac{1}{p}-1} \sup_{\theta\in\mathbb{R}} \|P^2 + 4(Re(e^{2i\theta}ST))^2 + 2Re(e^{2i\theta}(STP + PST))\|_{p/4}^{1/4}. \end{aligned}$$

Thus,

$$w_p^4\left(\begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix}\right) \leq 2^{\frac{4}{p}-4}\|P\|_{p/2}^2 + 2^{\frac{4}{p}-2} \sup_{\theta\in\mathbb{R}} \|Re(e^{2i\theta}ST)\|_{p/2}^2 + 2^{\frac{4}{p}-3} \sup_{\theta\in\mathbb{R}} \|Re(e^{2i\theta}(STP+PST))\|_{p/4}.$$

Hence, we obtain the desired result. ∎

Putting $p = \infty$ in the inequality (4.11), gives the inequality (3.13)

$$w^4\left(\begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix}\right) \leq \frac{1}{16}\|P\|^2 + \frac{1}{4}w^2(ST) + \frac{1}{8}w(STP + PST).$$

Remark 4.2.1. *If we put $T = S$ in Theorem 4.2.1, then we obtain*

$$w_p^4(T) \leq \frac{1}{16}\|P\|_{p/2}^2 + \frac{1}{4}w_{p/2}^2(P^2) + \frac{1}{8}w_{p/4}^2(T^2P + PT^2), \tag{4.12}$$

where $P = T^*T + TT^*$. Taking $p = \infty$ in the inequality (4.12), gives

$$w^4(T) \leq \frac{1}{16}\|P\|^2 + \frac{1}{4}w^2(T^2) + \frac{1}{8}w^2(T^2P + PT^2),$$

which has also been given in [17] and [63]. If we choose $T^2 = 0$ in the inequality (4.12), then we get

$$w_p^2(T) \leq \frac{1}{4}\|T^*T + TT^*\|_{p/2}. \tag{4.13}$$

The reverse of the inequality (4.13) has been obtained by Benmakhlouf, Hirzallah and Kittaneh [13]. Hence, we get

$$w_p^2(T) = \frac{1}{4}\|T^*T + TT^*\|_{p/2}.$$

It should be mentioned here that this result has already appeared in [13] by a different approach.

Using an argument similar to that used in the proof of Theorem 4.2.1 and applying the proposition 1.9.1 (2), we get the following theorem.

Theorem 4.2.2. *Let $T, S \in \mathbb{B}_p(\mathcal{H})$ and $4 \le p \le \infty$. Then*

$$w_p^4\left(\begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix}\right) \le 2^{\frac{4}{p}-4}\|R\|_{p/2}^2 + 2^{\frac{4}{p}-2}w_{p/2}^2(TS) + 2^{\frac{4}{p}-3}w_{p/4}(TSR + RTS),$$

where $R = S^*S + TT^*$.

To give the rest of our results, we need the following lemma, which can be found in [42].

Lemma 4.2.3. *Let $T, S \in \mathbb{B}_p(\mathcal{H})$ and $0 < p \le \infty$. Then*

1. $w_p\left(\begin{bmatrix} 0 & T \\ e^{i\theta}S & 0 \end{bmatrix}\right) = w_p\left(\begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix}\right)$ *for all $\theta \in \mathbb{R}$.*
2. $w_p\left(\begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix}\right) = w_p\left(\begin{bmatrix} 0 & S \\ T & 0 \end{bmatrix}\right)$.

In the following lemma, we provide a formula for $w_p\left(\begin{bmatrix} T & T \\ -T & -T \end{bmatrix}\right)$.

Lemma 4.2.4. *Let $T \in \mathbb{B}_p(\mathcal{H})$ and $0 < p \le \infty$. Then*

$$w_p\left(\begin{bmatrix} T & T \\ -T & -T \end{bmatrix}\right) = 2^{\frac{1}{p}}\|T\|_p.$$

Proof. Let $U = \frac{1}{\sqrt{2}}\begin{bmatrix} I & I \\ -I & I \end{bmatrix}$. Then U is unitary, and so

$$\begin{aligned} w_p\left(\begin{bmatrix} T & T \\ -T & -T \end{bmatrix}\right) &= \frac{1}{2}w_p\left(\begin{bmatrix} I & I \\ -I & I \end{bmatrix}\begin{bmatrix} T & T \\ -T & -T \end{bmatrix}\begin{bmatrix} I & -I \\ I & I \end{bmatrix}\right) \\ &= w_p\left(\begin{bmatrix} 0 & 0 \\ -2T & 0 \end{bmatrix}\right) \quad \text{(by Lemma 4.1.11)} \\ &= 2^{\frac{1}{p}}\|T\|_p. \end{aligned}$$

■

Remark 4.2.2. *The cases $p = \infty$ and $p = 2$ yield*

$$w\left(\begin{bmatrix} T & T \\ -T & -T \end{bmatrix}\right) = \|T\| \ \textit{and} \ w_2\left(\begin{bmatrix} T & T \\ -T & -T \end{bmatrix}\right) = \sqrt{2}\|T\|_2.$$

Theorem 4.2.5. *Let $S, T \in \mathbb{B}_p(\mathcal{H})$ and $1 \le p \le \infty$. Then*

$$w_p\left(\begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix}\right) \le 2^{\frac{1}{p}-1} \min\left\{\|T+S\|_p, \|T-S\|_p\right\} + 2^{\frac{1}{p}} \min\left\{w_p(T), w_p(S)\right\}.$$

Proof. Let $U = \frac{1}{\sqrt{2}}\begin{bmatrix} I & I \\ -I & I \end{bmatrix}$. Then U is unitary, and so

$$\begin{aligned} w_p\left(\begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix}\right) &= w_p\left(U\begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix}U^*\right) \\ &= \frac{1}{2}w_p\left(\begin{bmatrix} T+S & T-S \\ -(T-S) & -(T+S) \end{bmatrix}\right) \\ &\le \frac{1}{2}\left(w_p\left(\begin{bmatrix} T+S & T+S \\ -(T+S) & -(T+S) \end{bmatrix}\right) + w_p\left(\begin{bmatrix} 0 & -2S \\ 2S & 0 \end{bmatrix}\right)\right) \\ &= 2^{\frac{1}{p}-1}\|T+S\|_p + 2^{\frac{1}{p}}w_p(S) \end{aligned}$$

(by Lemma 4.2.4, Lemma 4.2.3 and Lemma 4.1.11).

Thus,

$$w_p\left(\begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix}\right) \le 2^{\frac{1}{p}-1}\|T+S\|_p + 2^{\frac{1}{p}}w_p(S). \tag{4.14}$$

Replacing S by $(-S)$ in the inequality (4.14) and using Lemma 4.2.3, we get

$$w_p\left(\begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix}\right) \le 2^{\frac{1}{p}-1}\|T-S\|_p + 2^{\frac{1}{p}}w_p(S). \tag{4.15}$$

From the inequalities (4.14) and (4.15), we obtain

$$w_p\left(\begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix}\right) \le 2^{\frac{1}{p}-1} \min\left\{\|T+S\|_p, \|T-S\|_p\right\} + 2^{\frac{1}{p}}w_p(S). \tag{4.16}$$

By interchanging T and S in the inequality (4.16) and using Lemma 4.2.3, we get

$$w_p\left(\begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix}\right) \le 2^{\frac{1}{p}-1} \min\left\{\|T+S\|_p, \|T-S\|_p\right\} + 2^{\frac{1}{p}}w_p(T). \tag{4.17}$$

From the inequalities (4.16) and (4.17), we conclude that

$$w_p\left(\begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix}\right) \leq 2^{\frac{1}{p}-1} \min\left\{\|T+S\|_p, \|T-S\|_p\right\} + 2^{\frac{1}{p}} \min\left\{w_p(T), w_p(S)\right\},$$

as required. ∎

Remark 4.2.3. *In [42], Hamza and Issa have obtained the following inequality for* $2 \leq p \leq \infty$

$$w_p\left(\begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix}\right) \leq \min\left\{w_p(T+S)_p, w_p(T-S)\right\} + 2^{\frac{1}{p}} \min\left\{w_p(T), w_p(S)\right\}.$$

Clearly, Theorem 4.2.5 is better than the above inequality.

Remark 4.2.4. *Letting* $p = \infty$ *in Theorem 4.2.5, gives the inequality*

$$w\left(\begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix}\right) \leq \frac{1}{2} \min\left\{\|T+S\|, \|T-S\|\right\} + \min\left\{w(A), w(B)\right\},$$

which has been given in [61]. Now, letting $p = 2$ *in Theorem 4.2.5, gives the inequality*

$$w_2\left(\begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix}\right) \leq \frac{1}{\sqrt{2}} \min\left\{\|T+S\|_2, \|T-S\|_2\right\} + \sqrt{2} \min\left\{w_2(T), w_2(S)\right\}.$$

The following lemma can be found in [42].

Lemma 4.2.6. *Let* $T, S \in \mathbb{B}_p(\mathcal{H})$ *and* $0 < p \leq \infty$. *Then*

$$w_p\left(\begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix}\right) \leq 2^{\frac{1}{p}-1}\left(w_p(T+S) + w_p(T-S)\right).$$

In the following theorem, we give another upper bound for $w_p\left(\begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix}\right)$.

Theorem 4.2.7. *Let* $T, S \in \mathbb{B}_p(\mathcal{H})$ *and* $0 < p \leq \infty$. *Then*

$$w_p\left(\begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix}\right) \leq 2^{\frac{1}{p}}\left(w_p(T) + w_p(S) - \frac{1}{2}\left|w_p(T+S) - w_p(T-S)\right|\right).$$

Proof. By Lemma 4.2.6, we get

$$\begin{aligned} w_p\left(\begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix}\right) &\leq 2^{\frac{1}{p}-1}\big(w_p(T+S) + w_p(T-S)\big) \\ &= 2^{\frac{1}{p}}\Big(\max\big\{w_p(T+S), w_p(T-S)\big\} - \frac{1}{2}\big|w_p(T+S) - w_p(T-S)\big|\Big) \\ &\leq 2^{\frac{1}{p}}\Big(w_p(T) + w_p(S) - \frac{1}{2}\big|w_p(T+S) - w_p(T-S)\big|\Big). \end{aligned}$$

This completes the proof. ■

Remark 4.2.5. *In particular, if we choose $S = T^*$ in Theorem 4.2.7 and use Lemma 4.1.11, then we can deduce that*

$$\frac{1}{2}\Big(\|T\|_p + \big|\|Re(T)\|_p - \|Im(T)\|_p\big|\Big) \leq w_p(T). \tag{4.18}$$

Also, if we take $p = \infty$ in the inequality (4.18), then we obtain the inequality

$$\frac{1}{2}\Big(\|T\| + \big|\|Re(T)\| - \|Im(T)\|\big|\Big) \leq w(T),$$

which has been proved earlier in [61]. See also [22].

To provide the rest of our results, we need he following lemma, which can be found in [42].

Lemma 4.2.8. *Let $T, S \in \mathbb{B}_p(\mathcal{H})$ and $0 < p \leq \infty$. Then*

$$w_p\left(\begin{bmatrix} T & 0 \\ 0 & S \end{bmatrix}\right) \leq \big(w_p^p(T) + w_p^p(S)\big)^{\frac{1}{p}}.$$

In particular, if T and S are self-adjoint, then

$$w_p\left(\begin{bmatrix} T & 0 \\ 0 & S \end{bmatrix}\right) = \big(w_p^p(T) + w_p^p(S)\big)^{\frac{1}{p}}.$$

In the following two theorems, we give upper bounds for some specific 2×2 operator matrices.

Theorem 4.2.9. *Let $T, S \in \mathbb{B}_p(\mathcal{H})$ and $0 < p \leq \infty$. Then*

$$w_p\left(\begin{bmatrix} T & S \\ iS & T \end{bmatrix}\right) \leq \left(w_p^p\left(T + \frac{1+i}{\sqrt{2}}S\right) + w_p^p\left(T - \frac{1+i}{\sqrt{2}}S\right)\right)^{\frac{1}{p}}.$$

Proof. Let $U = \frac{1}{\sqrt{2}}\begin{bmatrix} I & I \\ \frac{1+i}{\sqrt{2}}I & -\frac{1+i}{\sqrt{2}}I \end{bmatrix}$. Then U is unitary, and so

$$\begin{aligned} w_p\left(\begin{bmatrix} T & S \\ iS & T \end{bmatrix}\right) &= w_p\left(U^*\begin{bmatrix} T & S \\ iS & T \end{bmatrix}U\right) \\ &= w_p\left(\begin{bmatrix} T + \frac{1+i}{\sqrt{2}}S & 0 \\ 0 & T - \frac{1+i}{\sqrt{2}}S \end{bmatrix}\right) \\ &\leq \left(w_p^p\left(T + \frac{1+i}{\sqrt{2}}S\right) + w_p^p\left(T - \frac{1+i}{\sqrt{2}}S\right)\right)^{\frac{1}{p}} \\ &\quad \text{(by Lemma 4.2.8)}, \end{aligned}$$

which gives the desired result. ■

Theorem 4.2.10. *Let $T, S \in \mathbb{B}_p(\mathcal{H})$ and $0 < p \leq \infty$. Then*

$$w_p\left(\begin{bmatrix} T & -S \\ S & T \end{bmatrix}\right) \leq \left(w_p^p(S - iT) + w_p^p(S + iT)\right)^{\frac{1}{p}}.$$

Proof. Let $U = \frac{1}{\sqrt{2}}\begin{bmatrix} I & iI \\ iI & I \end{bmatrix}$. Then U is unitary, and so

$$\begin{aligned} w_p\left(\begin{bmatrix} iT & -S \\ S & iT \end{bmatrix}\right) &= w_p\left(U^*\begin{bmatrix} iT & -S \\ S & iT \end{bmatrix}U\right) \\ &= w_p\left(\begin{bmatrix} -i(S-T) & 0 \\ 0 & i(S+T) \end{bmatrix}\right) \\ &\leq \left(w_p^p(S - T) + w_p^p(S + T)\right)^{\frac{1}{p}} \quad \text{(by Lemma 4.2.8)}. \end{aligned}$$

Replacing T by iT in the above inequality, we get the desired inequality. ■

Corollary 4.2.11. *Let $T \in \mathbb{B}_p(\mathcal{H})$ be with the Cartesian decomposition $T = B + iC$, and let $0 < p \leq \infty$. Then*

$$w_p\left(\begin{bmatrix} C & -B \\ B & C \end{bmatrix}\right) \leq 2^{\frac{1}{p}} w_p(T).$$

Theorem 4.2.12. *Let $T, S \in \mathbb{B}_p(\mathcal{H})$ and $2 \leq p \leq \infty$. Then*

$$w_p^2\left(\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}\right) \leq w_p^2(T) + 2^{\frac{2}{p}-1} \sup_{\theta \in \mathbb{R}} \|Re(e^{i\theta}T)S\|_{p/2} + 2^{\frac{2}{p}-2}\|S\|_p^2.$$

Proof. We have

$$\begin{aligned}\left\|Re\left(e^{i\theta}\begin{bmatrix}T & S\\ 0 & 0\end{bmatrix}\right)\right\|_p^2 &= \left\|\begin{bmatrix}\left(Re(e^{i\theta}T)\right)^2+\frac{1}{4}|S^*|^2 & \frac{1}{2}e^{i\theta}Re(e^{i\theta}T)S\\ \frac{1}{2}e^{-i\theta}S^*Re(e^{i\theta}T) & \frac{1}{4}|S|^2\end{bmatrix}\right\|_{p/2}\\ &\leq \left\|\begin{bmatrix}\left(Re(e^{i\theta}T)\right)^2 & 0\\ 0 & 0\end{bmatrix}\right\|_{p/2}+\left\|\begin{bmatrix}\frac{1}{4}|S^*|^2 & 0\\ 0 & \frac{1}{4}|S|^2\end{bmatrix}\right\|_{p/2}\\ &+\left\|\begin{bmatrix}0 & \frac{1}{2}e^{i\theta}Re(e^{i\theta}T)S\\ \frac{1}{2}e^{-i\theta}S^*Re(e^{i\theta}T) & 0\end{bmatrix}\right\|_{p/2}\\ &\leq \|Re(e^{i\theta}T)\|_p^2+2^{\frac{2}{p}-1}\|Re(e^{i\theta}T)S\|_{p/2}+2^{\frac{2}{p}-2}\|S\|_p^2.\end{aligned}$$

By taking the supremum on both sides in the above inequality over $\theta\in\mathbb{R}$, we get the desired result. ■

In the following theorems, we derive some upper bounds for 2×2 operator matrices.

Theorem 4.2.13. *Let $A,B,C,D\in\mathbb{B}_p(\mathcal{H})$ and $1\leq p\leq\infty$. Then*

$$\begin{aligned}w_p\left(\begin{bmatrix}A & B\\ C & D\end{bmatrix}\right) &\leq \frac{1}{2}\Big(w_p^p\big((C-B)-i(A+D)\big)+w_p^p\big((C-B)+i(A+D)\big)\Big)^{\frac{1}{p}}\\ &+2^{\frac{1}{p}-1}\big(w_p(B+C)+w_p(D-A)\big).\end{aligned}$$

Proof. Let $U = \frac{1}{\sqrt{2}}\begin{bmatrix} I & -I \\ I & I \end{bmatrix}$. Then U is unitary, and so

$$\begin{aligned}
w_p\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) &= w_p\left(U^*\begin{bmatrix} A & B \\ C & D \end{bmatrix}U\right) \\
&= \frac{1}{2}w_p\left(\begin{bmatrix} A+B+C+D & -A+B-C+D \\ -A-B+C+D & A-B-C+D \end{bmatrix}\right) \\
&\leq \frac{1}{2}w_p\left(\begin{bmatrix} A+D & -(C-B) \\ C-B & A+D \end{bmatrix}\right) + \frac{1}{2}w_p\left(\begin{bmatrix} B+C & D-A \\ D-A & -(B+C) \end{bmatrix}\right) \\
&\leq \frac{1}{2}(w_p^p((C-B)-i(A+D)) + w_p^p((C-B)+i(A+D)))^{\frac{1}{p}} \\
&+ \frac{1}{2}w_p\left(\begin{bmatrix} B+C & 0 \\ 0 & -(B+C) \end{bmatrix}\right) + \frac{1}{2}w_p\left(\begin{bmatrix} 0 & D-A \\ D-A & 0 \end{bmatrix}\right) \\
&\text{(by Theorem 4.2.10)} \\
&\leq \frac{1}{2}\Big(w_p^p\big((C-B)-i(A+D)\big) + w_p^p\big((C-B)+i(A+D)\big)\Big)^{\frac{1}{p}} \\
&+ 2^{\frac{1}{p}-1}\big(w_p(B+C) + w_p(D-A)\big) \\
&\text{(by Lemma 4.2.8 and Lemma 4.1.11).}
\end{aligned}$$

■

Corollary 4.2.14. *Let $T, S \in \mathbb{B}_p(\mathcal{H})$ be with the Cartesian decompositions $T = C + iA$ and $S = -B + iD$, and let $1 \leq p \leq \infty$. Then*

$$w_p\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leq 2^{\frac{1}{p}-1}\big(w_p(T+S) + \|Re(T-S)\|_p + \|Im(S-T)\|_p\big).$$

The next lemma can be found in [1].

Lemma 4.2.15. *Let $T, S \in \mathbb{B}_p(\mathcal{H})$. Then*

(a) $w_p\left(\begin{bmatrix} T & S \\ S & T \end{bmatrix}\right) \leq \big(w_p^p(T+S) + w_p^p(T-S)\big)^{\frac{1}{p}}$ *for* $1 \leq p \leq 2$.

(b) $w_p\left(\begin{bmatrix} T & S \\ S & T \end{bmatrix}\right) \leq 2^{1-\frac{2}{p}}\big(w_p^p(T+S) + w_p^p(T-S)\big)^{\frac{1}{p}}$ *for* $2 \leq p \leq \infty$.

Theorem 4.2.16. *Let $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where $A, B, C, D \in \mathbb{B}_p(\mathcal{H})$. Then*

(a) $w_p(T) \leq \big(w_p^p(A) + w_p^p(D)\big)^{\frac{1}{p}} + 2^{\frac{1}{p}-1}\big(\|B\|_p + \|C\|_p\big)$ *for* $1 \leq p \leq 2$.

(b) $w_p(T) \le 2^{1-\frac{2}{p}}\left(w_p^p(A) + w_p^p(D)\right)^{\frac{1}{p}} + 2^{\frac{1}{p}-1}\left(\|B\|_p + \|C\|_p\right)$ *for* $2 \le p \le \infty$.

Proof. From the proof of Theorem 4.2.13, we have

$$\begin{aligned} w_p(T) &= \frac{1}{2} w_p\left(\begin{bmatrix} A+B+C+D & -A+B-C+D \\ -A-B+C+D & A-B-C+D \end{bmatrix}\right) \\ &\le \frac{1}{2} w_p\left(\begin{bmatrix} A+D & D-A \\ D-A & A+D \end{bmatrix}\right) + \frac{1}{2} w_p\left(\begin{bmatrix} B & B \\ -B & -B \end{bmatrix}\right) \\ &+ \frac{1}{2} w_p\left(\begin{bmatrix} C & -C \\ C & -C \end{bmatrix}\right) \\ &\le \frac{1}{2} w_p\left(\begin{bmatrix} A+D & D-A \\ D-A & A+D \end{bmatrix}\right) + \frac{1}{2} w_p\left(\begin{bmatrix} B & B \\ -B & -B \end{bmatrix}\right) \\ &+ \frac{1}{2} w_p\left(\begin{bmatrix} C^* & C^* \\ -C^* & -C^* \end{bmatrix}\right). \end{aligned}$$

By Lemma 4.2.15 (*a*) and Lemma 4.2.4, we obtain

$$w_p(T) \le \left(w_p^p(A) + w_p^p(D)\right)^{\frac{1}{p}} + 2^{\frac{1}{p}-1}\left(\|B\|_p + \|C\|_p\right).$$

Now, by Lemma 4.2.15 (*b*) and Lemma 4.2.4, we obtain

$$w_p(T) \le 2^{1-\frac{2}{p}}\left(w_p^p(A) + w_p^p(D)\right)^{\frac{1}{p}} + 2^{\frac{1}{p}-1}\left(\|B\|_p + \|C\|_p\right).$$

■

Remark 4.2.6. *If $A = D = 0$ in Theorem 4.2.16, we get*

$$w_p\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \le 2^{\frac{1}{p}-1}\left(\|B\|_p + \|C\|_p\right) \quad \text{for} \quad 1 \le p \le \infty. \tag{4.19}$$

This result is closely related to one given in [9]. Now, if we take $p = 2$ and $p = \infty$ in the inequality (4.19), then we obtain the following results:

$$w_2\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \le \frac{1}{\sqrt{2}}\left(\|B\|_2 + \|C\|_2\right)$$

and

$$w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \le \frac{1}{2}\left(\|B\| + \|C\|\right),$$

which have been given in [6] and [58], respectively.

Theorem 4.2.17. *Let* $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$*, where* $A, B, C, D \in \mathbb{B}_p(\mathcal{H})$ *and* $2 \le p \le \infty$*. Then*

$$w_p(T) \le \min\{\alpha, \beta\},$$

where

$$\begin{aligned}
\alpha &= \sqrt{w_p^2(A) + 2^{\frac{2}{p}-1} \sup_{\theta\in\mathbb{R}} \|Re(e^{i\theta}A)B\|_{p/2} + 2^{\frac{2}{p}-2}\|B\|_p^2} \\
&+ \sqrt{w_p^2(D) + 2^{\frac{2}{p}-1} \sup_{\theta\in\mathbb{R}} \|Re(e^{i\theta}D)C\|_{p/2} + 2^{\frac{2}{p}-2}\|C\|_p^2}. \\
\beta &= \sqrt{w_p^2(A) + 2^{\frac{2}{p}-1} \sup_{\theta\in\mathbb{R}} \|Re(e^{i\theta}A^*)C^*\|_{p/2} + 2^{\frac{2}{p}-2}\|C\|_p^2} \\
&+ \sqrt{w_p^2(D) + 2^{\frac{2}{p}-1} \sup_{\theta\in\mathbb{R}} \|Re(e^{i\theta}D^*)B^*\|_{p/2} + 2^{\frac{2}{p}-2}\|B\|_p^2}
\end{aligned}$$

Proof. Let $U = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$. Then U is unitary, and so

$$\begin{aligned}
w_p(T) &\le w_p\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right) + w_p\left(\begin{bmatrix} 0 & 0 \\ C & D \end{bmatrix}\right) \\
&= w_p\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right) + w_p\left(U^*\begin{bmatrix} D & C \\ 0 & 0 \end{bmatrix}U\right) \\
&= \sqrt{w_p^2(A) + 2^{\frac{2}{p}-1} \sup_{\theta\in\mathbb{R}} \|Re(e^{i\theta}A)B\|_{p/2} + 2^{\frac{2}{p}-2}\|B\|_p^2} \\
&+ \sqrt{w_p^2(D) + 2^{\frac{2}{p}-1} \sup_{\theta\in\mathbb{R}} \|Re(e^{i\theta}D)C\|_{p/2} + 2^{\frac{2}{p}-2}\|C\|_p^2} \\
&\quad \text{(by Theorem 4.2.12)} \\
&= \alpha.
\end{aligned}$$

Similarly,

$$\begin{aligned} w_p\left(T^*\right) &\leq w_p\left(\begin{bmatrix} A^* & C^* \\ 0 & 0 \end{bmatrix}\right) + w_p\left(\begin{bmatrix} 0 & 0 \\ B^* & D^* \end{bmatrix}\right) \\ &= w_p\left(\begin{bmatrix} A^* & C^* \\ 0 & 0 \end{bmatrix}\right) + w_p\left(U^*\begin{bmatrix} D^* & B^* \\ 0 & 0 \end{bmatrix}U\right) \\ &= \sqrt{w_p^2(A) + 2^{\frac{2}{p}-1}\sup_{\theta\in\mathbb{R}}\|Re(e^{i\theta}A^*)C^*\|_{p/2} + 2^{\frac{2}{p}-2}\|C\|_p^2} \\ &+ \sqrt{w_p^2(D) + 2^{\frac{2}{p}-1}\sup_{\theta\in\mathbb{R}}\|Re(e^{i\theta}D^*)B^*\|_{p/2} + 2^{\frac{2}{p}-2}\|B\|_p^2} \\ &= \beta. \end{aligned}$$

By observing that $w_p(T) = w_p(T^*)$, we get the desired result. ■

In the following theorems, we give some new lower bounds for the numerical radii of the off-diagonal parts of 2×2 operator matrices.

Theorem 4.2.18. *Let $B, C \in \mathbb{B}_p(\mathcal{H})$ and $1 \leq p \leq \infty$. Then*

$$2^{\frac{1}{p}-1}\max\left\{\|B\|_p, \|C\|_p\right\} + 2^{\frac{1}{p}-2}\left|\|B + C^*\|_p - \|B - C^*\|_p\right| \leq w_p\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right).$$

In particular, if $B = C$, then we have

$$\frac{1}{2}\left(\|B\|_p + \left|\|Re(B)\|_p - \|Im(B)\|_p\right|\right) \leq w_p(B).$$

Proof. We have

$$\left\|Re\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)\right\|_p \leq w_p\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \text{ and } \left\|Im\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)\right\|_p \leq w_p\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right).$$

Then,

$$\begin{aligned} w_p\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) &\geq 2^{\frac{1}{p}-1}\max\left\{\|B + C^*\|_p, \|B - C^*\|_p\right\} \\ &= 2^{\frac{1}{p}-2}\left(\|B + C^*\|_p + \|B - C^*\|_p + \left|\|B + C^*\|_p - \|B - C^*\|_p\right|\right) \\ &\geq 2^{\frac{1}{p}-2}\left(\|(B + C^*) \pm (B - C^*)\|_p + \left|\|B + C^*\|_p - \|B - C^*\|_p\right|\right). \end{aligned}$$

Hence,

$$w_p\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \geq 2^{\frac{1}{p}-1} \max\left\{\|B\|_p, \|C\|_p\right\} + 2^{\frac{1}{p}-2}\left|\|B + C^*\|_p - \|B - C^*\|_p\right|.$$

■

Remark 4.2.7. *If we put $p = \infty$ and $p = 2$ in Theorem 4.2.18, then we get the results:*

$$\frac{1}{2}\max\left\{\|B\|, \|C\|\right\} + \frac{1}{4}\left|\|B + C^*\| - \|B - C^*\|\right| \leq w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)$$

and

$$\frac{1}{\sqrt{2}}\max\left\{\|B\|_2, \|C\|_2\right\} + \frac{1}{2\sqrt{2}}\left|\|B + C^*\|_2 - \|B - C^*\|_2\right| \leq w_2\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right),$$

which have been given in [20] and [10], respectively.

Theorem 4.2.19. *Let $B, C \in \mathbb{B}_p(\mathcal{H})$ and $1 \leq p \leq \infty$. Then*

$$2^{\frac{1}{p}} \max\left\{w_p(B), w_p(C)\right\} \leq w_p\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) + 2^{\frac{1}{p}-1} \min\left\{\|B + C\|_p, \|B - C\|_p\right\}.$$

Proof. Let $U = \frac{1}{\sqrt{2}}\begin{bmatrix} I & -I \\ I & I \end{bmatrix}$. Then U is unitary, and so

$$\begin{aligned} w_p\left(\begin{bmatrix} 0 & -C \\ C & 0 \end{bmatrix}\right) &= w_p\left(U^*\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}U - \frac{1}{2}\begin{bmatrix} B + C & B + C \\ -(B + C) & -(B + C) \end{bmatrix}\right) \\ &\leq w_p\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) + \frac{1}{2}w_p\left(\begin{bmatrix} B + C & B + C \\ -(B + C) & -(B + C) \end{bmatrix}\right). \end{aligned}$$

Using Lemma 4.2.3, Lemma 4.1.11 and Lemma 4.2.4, it follows that

$$2^{\frac{1}{p}}w_p(C) \leq w_p\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) + 2^{\frac{1}{p}-1}\|B + C\|_p. \tag{4.20}$$

Now, replacing C by $-C$ in the inequality (4.20), we obtain

$$2^{\frac{1}{p}}w_p(C) \leq w_p\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) + 2^{\frac{1}{p}-1}\|B - C\|_p. \tag{4.21}$$

From the inequalities (4.20) and (4.21), we deduce that

$$2^{\frac{1}{p}} w_p(C) \leq w_p\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) + 2^{\frac{1}{p}-1} \min\left\{\|B+C\|_p, \|B-C\|_p\right\}. \tag{4.22}$$

Now, interchanging B and C in the inequality (4.22), we have

$$2^{\frac{1}{p}} w_p(B) \leq w_p\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) + 2^{\frac{1}{p}-1} \min\left\{\|B+C\|_p, \|B-C\|_p\right\}. \tag{4.23}$$

Combining the inequalities (4.22) and (4.23), we deduce the required result. ■

Theorem 4.2.20. *Let $B, C \in \mathbb{B}_p(\mathcal{H})$ and $2 \leq p \leq \infty$. Then*

$$\begin{aligned} w_p^2\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) &\geq \frac{1}{4}\left(\left\||B^*|^2 + |C|^2\right\|_{p/2}^{\frac{p}{2}} + \left\||B|^2 + |C^*|^2\right\|_{p/2}^{\frac{p}{2}}\right)^{\frac{2}{p}} \\ &+ 2^{\frac{2}{p}-3}\left|\|B+C^*\|_p^2 - \|B-C^*\|_p^2\right|. \end{aligned}$$

In particular, if $B = C$, then

$$\frac{1}{4}\left\||B^*|^2 + |B|^2\right\|_{p/2} + \frac{1}{2}\left|\|Re(B)\|_p^2 - \|Im(B)\|_p^2\right| \leq w_p^2(B). \tag{4.24}$$

Proof. We have

$$
\begin{aligned}
w_p^2\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) &\geq \max\left\{\left\|Re\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)\right\|_p^2, \left\|Im\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)\right\|_p^2\right\} \\
&= \frac{1}{2}\left(\left\|Re\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)\right\|_p^2 + \left\|Im\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)\right\|_p^2\right) \\
&+ \frac{1}{2}\left|\left\|Re\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)\right\|_p^2 - \left\|Im\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)\right\|_p^2\right| \\
&\geq \frac{1}{2}\left(\left\|\left(Re\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)\right)^2\right\|_{p/2} + \left\|\left(Im\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)\right)^2\right\|_{p/2}\right) \\
&+ 2^{\frac{2}{p}-3}\left|\|B + C^*\|_p^2 - \|B - C^*\|_p^2\right| \\
&\geq \frac{1}{2}\left(\left\|\left(Re\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)\right)^2 + \left(Im\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)\right)^2\right\|_{p/2}\right) \\
&+ 2^{\frac{2}{p}-3}\left|\|B + C^*\|_p^2 - \|B - C^*\|_p^2\right| \\
&= \frac{1}{4}\left\|\begin{bmatrix} |B^*|^2 + |C|^2 & 0 \\ 0 & |B|^2 + |C^*|^2 \end{bmatrix}\right\|_{p/2} + 2^{\frac{2}{p}-3}\left|\|B + C^*\|_p^2 - \|B - C^*\|_p^2\right| \\
&= \frac{1}{4}\left(\||B^*|^2 + |C|^2\|_{p/2}^{\frac{p}{2}} + \||B|^2 + |C^*|^2\|_{p/2}^{\frac{p}{2}}\right)^{\frac{2}{p}} + 2^{\frac{2}{p}-3}\left|\|B + C^*\|_p^2 - \|B - C^*\|_p^2\right|
\end{aligned}
$$

(by Lemma 4.2.8).

■

Remark 4.2.8. *If we put $p = \infty$ in the inequality (4.24), then we obtain the inequality*

$$\frac{1}{4}\left\||B^*|^2 + |B|^2\right\| + \frac{1}{2}\left|\|Re(B)\|^2 - \|Im(B)\|^2\right| \leq w^2(B),$$

which has recently been given in [20].

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