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Dedication

I would love to dedicate this modest work to my beloved parents.

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Abstract

Our main goal in this research is to refine some well-known numerical radius inequalities of operators on a Hilbert space or to discover new bounds for the numerical radius. In this thesis, after expressing concepts and prerequisites, we give some new upper bounds for the numerical radius of operators as well as for the numerical radii of 2×2 operator matrices. Also, we improve the triangle inequality of the operator norm. We refine some earlier existing bounds of the numerical radius. Furthermore, we derive some new Hilbert-Schmidt numerical radius inequalities for operators as well as for 2×2 operator matrices. Some of these inequalities refine some existing ones. Then we define a new norm and we study the basic properties of this norm. Finally, we provide new upper and lower bounds for the p-numerical radius of operators as well as for 2×2 operator matrices.

Key words: Numerical radius, Hilbert-Schmidt numerical radius, p-numerical radius, inequality.

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Notations

$\mathbb{N}$	The set of natural numbers.
$\mathbb{R}$	The set of real numbers.
$\mathbb{C}$	The set of complex numbers.
$\mathbb{K}$	A field.
E	A vector space.
$\mathcal{H}$	Hilbert space.
$\mathcal{B}(\mathcal{H})$	The set of all bounded linear operators on $\mathcal{H}$.
$\mathcal{B}_p(\mathcal{H})$	The Schatten p-class in $\mathcal{B}(\mathcal{H})$.
$\mathcal{B}_2(\mathcal{H})$	The Hilbert-Schmidt class in $\mathcal{B}(\mathcal{H})$.
$\mathcal{K}(\mathcal{H})$	The class of compact operators in $\mathcal{B}(\mathcal{H})$.
$\mathcal{M}_n(\mathbb{C})$	$n \times n$ matrices in $\mathbb{C}$.
$\langle x, y\rangle$	The inner product of x and y.
$\\| \cdot \\|$	Norm.
I	Identity operator.
U	Unitary operator.
$\mathcal{T}$	Block of operators in $\mathcal{B}(\mathcal{H})$.
T^*	The adjoin of T.
$\|T\|$	The absolute value of T.
$\mathcal{R}e(T)$	The real part of T.
$\mathcal{I}m(T)$	The imaginary part of T.
$W(T)$	The numerical range of T.
$w(T)$	The numerical radius of T.
$w_N(T)$	The generalized numerical radius of T.

$r(T)$	The spectral radius of T.
$m(T)$	The Crawford number of T.
$\lambda(T)$	The eigenvalue of T.
$\sigma(T)$	The spectrum of T.
$\mathcal{C}(p)$	Companion matrix of polynomial p.
$tr(T)$	Trace of T.
$\oplus$	Direct sum.
$Ker(T)$	The kernel of T.

Introduction

The study of operator and matrix theories have become an interesting topics and more popular. Mathematicians are attracted to these branches because of there relations with other scientific domains. There are many notions in operator and matrix theories, among them the eigenvalues, which have several uses in different fields such as Physics, Engineering, Economics, ...etc.

The problem of calculating the eigenvalues is delicate. However, in many scientific fields, it is sufficient to know the localization of the eigenvalues. Researchers introduced the notion of the numerical range which is related to the eigenvalues. It is defines as

$$W(T) = \{\langle Tx, x\rangle, x \in \mathcal{H}, \|x\| = 1\},$$

where T is a bounded linear operator on a complex Hilbert space, for more details see [26, 28, 35].

It is well-known that the spectrum of an operator is contained in the closure of its numerical range. The most important object related to the numerical range is the numerical radius. Several inequalities involving the numerical radius of one operator and the numerical radii of operator matrices have been established by many researchers like Kittaneh, Abu-Omar, Hirzallah, Yamzaki, and others, see [2, 30, 36, 46].

Related subjects to the numerical radius have also been introduced, in particular, the Hilbert-Schmidt numerical radius [5, 11, 44] and the p-numerical radius [10, 14]. Many papers have appeared in this domains, which highlighted and developed these topics.

For instance, inequalities among numerical radius and operator norm produce a lot of upper and lower bounds for numerical radius of a bounded linear operator on complex Hilbert space and operator matrices. In this thesis, we give a recent results for numerical radius, Hilbert-Schmidt and p-numerical radius.

This thesis is divided into four chapters.

In the first chapter, we present basic mathematical materials, that will be used later. In particular, the inner product, the norm and some other concepts will be discussed.

In the second chapter, we provide new inequalities for the classical numerical radius. Especially, we give some inequalities for the numerical radius of the sum of two operators. Also, we provide new upper bounds for the numerical radii of 2×2 operator matrices. As an application, an estimation for the zeros polynomial are given too.

In the third chapter, we investigate the Hilbert-Schmidt numerical radius. We derive some new upper and lower bounds for the Hilbert-Schmidt numerical radius for a single operator as well as for 2×2 operator matrices. Also, we define a new norm, which helps us to deduce new results concerning the Hilbert-Schmidt numerical radius.

In the fourth chapter, we study the p-numerical radius for one operator and for product of two operators as well as for 2×2 operator matrices. We give new p-numerical radius inequalities. For the particular cases $p = \infty$ and $p = 2$, we refine and rebtain earlier existing results.

Contributions

We have succeeded to published the following articles:

S. Aici, A. Frakis and F. Kittaneh, *Refinements of some numerical radius inequalities for operators*, Rendiconti del Circolo Matematico di Palermo Series 2, 72 (2023), 3815–3828.

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S. Aici and A. Frakis, *Some sharp bounds of the Hilbert-Schmidt numerical radius for operator matrices*, preprint.

S. Aici, A. Frakis, F. Kittaneh, *Further bounds for the Euclidean operator radius of a pair of operators and their applications.* Afr. Mat. 35, 48 (2024). https://doi.org/10.1007/s13370-

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Chapter 1

Preliminaries

This chapter contains the basic concepts and notions which are necessary in this thesis. In this chapter, we provide some well-known results. The material in this chapter can be found in almost every book on operator theory and matrix analysis. For details the reader can consult [15, 22, 33, 40].

1.1 Hilbert space

Definition 1.1.1. *Let E be a vector space over a field $\mathbb{K}$ ($\mathbb{C}$ or $\mathbb{R}$). A map $\|\cdot\| : E \to \mathbb{R}_+$ is called a norm on E if*

1. $\|x\| \geq 0$ *for all* $x \in E$ *and* $\|x\| = 0$ *if and only if* $x = 0$,
2. $\|\alpha x\| = |\alpha|\|x\|$ *for all* $x \in E$ *and* $\alpha \in \mathbb{K}$,
3. $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in E$ *(triangle inequality).*

We call $(E, \|\cdot\|)$ *a normed space.*

Definition 1.1.2. *Let E be a vector space over a field $\mathbb{K}$($\mathbb{C}$ or $\mathbb{R}$). A function $\langle \cdot, \cdot \rangle : E \times E \to \mathbb{C}$ is called an inner product if*

1. $\langle x, x \rangle \geq 0$ *for all* $x \in E$ *and* $\langle x, x \rangle = 0$ *if and only if* $x = 0$.
2. $\langle x, y \rangle = \overline{\langle y, x \rangle}$ *for* $x, y \in E$.
3. $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$ *for all* $x, y, z \in E$ *and* $\alpha, \beta \in \mathbb{K}$.

Theorem 1.1.1. *Let $(E, \langle\cdot,\cdot\rangle)$ be an inner product space and $\|x\| = \sqrt{\langle x, x\rangle}$. Then $\|\cdot\|$ is a norm on E.*

Definition 1.1.3. *A Hilbert space $\mathcal{H}$ is a vector space with an inner product that is complete with respect to the induced norm.*

1.2 Cauchy-Schwartz inequality

Let $x, y \in \mathcal{H}$. Then

$$|\langle x, y\rangle| \leq \|x\|\|y\|,$$

with equality if and only if x and y are linearly dependent.

1.3 Parallelogram identity

Let $x, y \in \mathcal{H}$. Then

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right).$$

Let $\mathcal{B}(\mathcal{H})$ be the set of all bounded linear operators on a complex Hilbert space.

1.4 Generalized polarization identity

Let $T \in \mathcal{B}(\mathcal{H})$ and $x, y \in \mathcal{H}$. Then

$$\langle Tx, y\rangle = \frac{1}{4}\Big(\langle T(x+y), (x+y)\rangle - \langle T(x-y), (x-y)\rangle + i\langle T(x+iy), (x+iy)\rangle - i\langle T(x-iy), (x-iy)\rangle\Big).$$

Definition 1.4.1. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

- *T is self-adjoint if and only if $T = T^*$.*
- *T is normal if and only if $T^*T = TT^*$.*
- *T is unitary if and only if $TT^* = T^*T = I$.*
- *T is definite positive if T is self adjoint and $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{H}$.*
- *T is partial isometry if $\|Tx\| = \|x\|$ for all $x \in (Ker(T))^{\perp}$.*

1.5 Operator norm

Let $T \in \mathcal{B}(\mathcal{H})$. Then

$$\|T\| = \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\|=\|y\|=1} |\langle Tx, y\rangle|.$$

1.6 Polar decomposition

Theorem 1.6.1. *Let $T \in \mathcal{B}(\mathcal{H})$. Then there exists a partial isometry operator $U \in \mathcal{B}(\mathcal{H})$ such that*

$$T = U|T|.$$

where $|T| = (T^*T)^{\frac{1}{2}}$.

1.7 Cartesian decomposition

Theorem 1.7.1. *Let $T \in \mathcal{B}(\mathcal{H})$. Then there exist self-adjoint operators A and B such that*

$$T = A + iB,$$

where $A = \mathcal{R}e(T) = \frac{1}{2}(T + T^*)$ and $B = \mathcal{I}m(T) = \frac{1}{2i}(T - T^*)$.

Theorem 1.7.2. *[15] Let $T \in \mathcal{B}(\mathcal{H})$. Then*

- $\| \, |T| \, \| = \|T\| = \|T^*\|$.
- $\|TT^*\| = \|T^*T\| = \|T\|^2$.
- $\||T||T^*|\| = \|T^2\|$.

1.8 Spectral radius

Definition 1.8.1. *Let $T \in \mathcal{M}_n(\mathbb{C})$, where $\mathcal{M}_n(\mathbb{C})$ is the set of all $n \times n$ complex matrices. The complex number λ is called an eigenvalue of T if there exists nonzero vector $x \in \mathcal{H}$ such that*

$$Tx = \lambda x.$$

The vector x is called an eigenvector of T corresponding to λ.

If $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of T, then $tr(T) = \sum_{i=1}^{n} \lambda_i$.
The set of all $\lambda \in \mathbb{C}$ that are eigenvalues of T is called the spectrum of T and it is denoted by $\sigma(T)$.

Remark 1.8.1. *It well known that $r(T) \leq \|T\|$. Moreover, if T is a normal operator, then $r(T) = \|T\|$.*

Definition 1.8.2. *Let $T \in \mathcal{B}_n(\mathcal{H})$. The spectral radius of T is defined by*

$$r(T) = \sup\{|\lambda|, \lambda \text{ the eigenvalue of } T\}.$$

Theorem 1.8.2. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$r(T) = \lim_{n \longrightarrow \infty} \|T^n\|^{\frac{1}{n}}.$$

Properties 1.8.1. *Let $T, S \in \mathcal{B}(\mathcal{H})$ and $\alpha \in \mathbb{C}$. Then*

1. $r(\alpha T) = |\alpha| r(T)$.
2. $r(T^n) = r^n(T)$.
3. $r(T^*) = r(T)$.
4. $r(TS) = r(ST)$.

1.9 Numerical range

Definition 1.9.1. *Let $T \in \mathcal{B}(\mathcal{H})$. The numerical range of T is the subset of the complex numbers given by*

$$W(T) = \{\langle Tx, x\rangle : x \in \mathcal{H}, \|x\| = 1\}.$$

Properties 1.9.1. *Let $T, S \in \mathcal{B}(\mathcal{H})$ and $\alpha, \beta \in \mathbb{C}$. Then*

- $W(\alpha I + \beta T) = \alpha + \beta W(T)$.
- $W(T + S) \subseteq W(T) + W(S)$.
- $W(T^*) = \{\overline{\lambda}, \lambda \in W(T)\}$.

- $W(U^*TU) = W(T)$ *for any unitary operator* $U \in \mathcal{B}(\mathcal{H})$.

Proof. See [26]. □

Example 1.1. *Let* $T \in \mathcal{M}_2(\mathbb{C})$*, such that* $T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ *and let* $x^t = (x_1, x_2)$ *be a unit vector in* $\mathbb{C}^2$*. Then*

$$\begin{aligned} |\langle Tx, x\rangle| &= \left| \left\langle \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle \right| \\ &= |x_2 \overline{x_1}| \\ &\leq \frac{1}{2}(|x_1|^2 + |x_2|^2) \\ &= \frac{1}{2}. \end{aligned}$$

Then

$$W(T) \subseteq \left\{ z, |z| \leq \frac{1}{2} \right\}.$$

Let $z = re^{i\theta}$, $0 \leq r \leq \frac{1}{2}$ *, if we choose* $x = \big(\cos\psi, e^{i\theta}\sin\psi\big)$*, where* $\sin 2\psi = 2r \leq 1$ *and* $0 \leq \psi \leq \frac{\pi}{4}$*. Then*

$$\langle Tx, x\rangle = e^{i\theta}\cos\psi\sin\psi = re^{i\theta}.$$

Thus

$$W(T) = \left\{ z, |z| \leq \frac{1}{2} \right\}.$$

Theorem 1.9.1. *Let* $T \in \mathcal{B}(\mathcal{H})$*. Then*

$$\sigma(T) \subseteq \overline{W(T)}.$$

Proof. See [33]. □

1.10 Numerical radius

Definition 1.10.1. *Let* $T \in \mathcal{B}(\mathcal{H})$*. The numerical radius of* T *is defined*

$$w(T) = \sup_{\|x\|=1} |\langle Tx, x\rangle| \ \ or \ w(T) = \sup_{\lambda \in W(T)} |\lambda|.$$

Notice that, for any vector $x \in \mathcal{H}$, we have

$$|\langle Tx, x\rangle| \leq w(T)\|x\|^2.$$

The following theorem is a characterization of the numerical radius.

Theorem 1.10.1. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$w(T) = \sup_{\theta\in\mathbb{R}} \|\mathcal{R}e(e^{i\theta}T)\|.$$

Also, the numerical radius defined as $w(T) = \sup_{\theta\in\mathbb{R}} \|\mathcal{I}m(e^{i\theta}T)\|$.

Proof. See [46]. □

Properties 1.10.1. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

- $w(T) = w(T^*)$.
- $w(UTU^*) = w(T)$ *for any unitary operator $U \in \mathcal{B}(\mathcal{H})$ ($w(\cdot)$ is weakly unitarily invariant norm).*
- $w(T^n) \leq w^n(T) \quad$ *for $n \in \mathbb{N}$.*

Proof. See [28]. □

Remark 1.10.2. *It is obviously that $w(\cdot)$ is a norm on $\mathcal{B}(\mathcal{H})$, but it is not a sub multiplicative norm, to see this, let $T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $S = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.*

$$w\,(TS) = w\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = 1.$$

On the other hand, we have

$$w\,(T) = \sup_{\theta\in\mathbb{R}} \left\|\mathcal{R}e\left(e^{i\theta}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)\right\| = \frac{1}{2}\sup_{\theta\in\mathbb{R}} \left\|\begin{bmatrix} 0 & e^{i\theta} \\ e^{-i\theta} & 0 \end{bmatrix}\right\| = \frac{1}{2}.$$

$w(S) = w(T^*) = \frac{1}{2}$. Hence, we have $1 = w(TS) \geq w(T)w(S) = \frac{1}{4}$.

The numerical radius is not sub multiplicative, but for $T, S \in \mathcal{B}(\mathcal{H})$, we have

- $w(TS) \leq 4w(T)w(S)$.
- *In particular, if T and S are commute, then $w(TS) \leq 2w(T)w(S)$.*
- *If T and S are normal, then $w(TS) \leq w(T)w(S)$.*

See [32].

The following theorem shows the equivalence between the operator norm and the numerical radius.

Theorem 1.10.3. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$\frac{1}{2}\|T\| \leq w(T) \leq \|T\|. \tag{1.1}$$

Proof. Let $x \in \mathcal{H}$ be any unit vector. By applying Cauchy-Schwartz inequality, we get

$$|\langle Tx, x\rangle| \leq \|Tx\|.$$

By taking the supremum in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$, we get

$$w(T) \leq \|T\|.$$

Using the generalized polarization identity, we have

$$\begin{aligned} 4|\langle Tx, y\rangle| &\leq |\langle T(x+y), (x+y)\rangle| + |\langle T(x-y), (x-y)\rangle| + |\langle T(x+iy), (x+iy)\rangle| \\ &\quad + |\langle T(x-iy), (x-iy)\rangle| \\ &\leq w(T)\Big(\|x+y\|^2 + \|x-y\|^2 + \|x+iy\|^2 + \|x-iy\|^2\Big) \\ &= 4w(T)\Big(\|x\|^2 + \|y\|^2\Big). \end{aligned}$$

By taking the supremum in the above inequality over $\|x\| = \|y\| = 1$ on both sides, we obtain the first inequality. □

Theorem 1.10.4. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$r(T) \leq w(T) \leq \|T\|.$$

Remark 1.10.5. *If T is normal, then*

$$r(T) = w(T) = \|T\|.$$

1.11 The generalized numerical radius

In [5], Abu-Omar and Kittaneh define a new norm, which generalizes the classical numerical radius.

Definition 1.11.1. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$w_N(T) = \sup_{\theta \in \mathbb{R}} N(\mathcal{R}e(e^{i\theta}T)) \text{ or } w_N(T) = \sup_{\theta \in \mathbb{R}} N(\mathcal{I}m(e^{i\theta}T)),$$

where $N(\cdot)$ is a norm on $\mathcal{B}(\mathcal{H})$.

Theorem 1.11.1. *$w_N(\cdot)$ is is a norm on $\mathcal{B}(\mathcal{H})$.*

Theorem 1.11.2. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$\frac{1}{2}N(T) \leq w_N(T).$$

Moreover, if $N(\cdot)$ is self-adjoint, then

$$w_N(T) \leq N(T).$$

Proof. Since $w_N(T) = \sup_{\theta \in \mathbb{R}} N(\mathcal{R}e(e^{i\theta}T)) \geq N(\mathcal{R}e(e^{i\theta}T)) \quad \text{for all } \theta \in \mathbb{R}.$
By taking $\theta = 0$ and $\theta = \pi/2$, we have

$$\begin{aligned} 2w_N(T) &\geq N(\mathcal{R}e(T)) + N(\mathcal{I}m(T)) \\ &\geq N(\mathcal{R}e(T) + i\mathcal{I}m(T)) \\ &= N(T). \end{aligned}$$

On the other hand, if $N(\cdot)$ is self-adjoint, then

$$\begin{aligned} w_N(T) &= \sup_{\theta\in\mathbb{R}} N(\mathcal{R}e(e^{i\theta}T)) \\ &= \frac{1}{2}\sup_{\theta\in\mathbb{R}} N(e^{i\theta}T + e^{-i\theta}T^*) \\ &\leq \frac{1}{2}\sup_{\theta\in\mathbb{R}} \big(N(e^{i\theta}T) + N(e^{-i\theta}T^*)\big) \\ &= N(T). \end{aligned}$$

□

1.12 Inequalities

1.12.1 Hölder inequality

Let $p, q \in]1, +\infty)$ such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Then for all $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n) \in \mathcal{H}$

$$\left|\sum_{i=1}^{n} x_i y_i\right| \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |y_i|^q\right)^{\frac{1}{q}}.$$

1.12.2 Arithmetic-geometric mean inequality

Let $a, b \geq 0$. Then

$$ab \leq \frac{a^2 + b^2}{2}.$$

Lemma 1.12.1. *Let $a, b \geq 0$ and $0 \leq t \leq 1$. Then*

a) $(ta^s + (1-t)b^s)^{1/s} \leq (ta^r + (1-t)b^r)^{1/r}$ *for* $0 < s \leq r$.

b) $(a^r + b^r)^{1/r} \leq (a^s + b^s)^{1/s}$ *for* $0 < s \leq r$.

Chapter 2

The numerical radius

In this chapter, we give some new upper bounds for the classical numerical radius of operators. In addition, we refine some well-known existing results. Also, we improve the triangle inequality of the operator norm. We give new inequalities for the classical numerical radii of 2×2 operator matrices. As an application, we apply one of our results to the companion matrix.

In the following theorem, Kittaneh [39] improved the inequalities (1.1) as follows.

Theorem 2.0.2. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$\frac{1}{4}\| \, |T|^2 + |T^*|^2 \| \leq w^2(T) \leq \frac{1}{2}\| \, |T|^2 + |T^*|^2 \|. \tag{2.1}$$

Proof. Let $T = A + iB$, be the Cartesian decomposition of T such that A and B are the real and the imaginary part of T, respectively. Let $x \in \mathcal{H}$ be any unit vector. By using the convexity of the function $f(t) = t^2$ on $[0, \infty)$, we get

$$\begin{aligned} |\langle Tx, x\rangle|^2 &= \langle Ax, x\rangle^2 + \langle Bx, x\rangle^2 \\ &\geq \frac{1}{2}(|\langle Ax, x\rangle| + |\langle Bx, x\rangle|)^2 \\ &\geq \frac{1}{2}|\langle (A \pm B)x, x\rangle|^2 \ . \end{aligned}$$

Now, taking the supremum on both sides in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$, we get

$$w^2(T) \geq \frac{1}{2}\|A \pm B\|^2.$$

Hence

$$\begin{aligned}2w^2(T) &\geq \frac{1}{2}\|(A+B)^2\| + \frac{1}{2}\|(A-B)^2\| \\ &\geq \|A^2+B^2\|.\end{aligned}$$

Therefore, we get the first inequality. We have

$$\begin{aligned}|\langle Tx, x\rangle|^2 &= \langle Ax, x\rangle^2 + \langle Bx, x\rangle^2 \\ &\leq \|Ax\|^2 + \|Bx\|^2 \\ &= \langle A^2x, x\rangle + \langle B^2x, x\rangle \\ &= \langle (A^2+B^2)x, x\rangle.\end{aligned}$$

By taking the supremum on both sides in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$, we get the second inequality. □

2.1 Numerical radius inequalities for the sum of two operators

In order to improve Theorem 2.0.2, we need the following proposition which can be found in [41].

Proposition 2.1.1. *Let $T, S \in \mathcal{B}(\mathcal{H})$ be self-adjoint. Then*

$$w^2(T+iS) \leq \|T^2+S^2\|.$$

It should be mentioned here that Proposition 2.1.1 can be obtained also from the inequality (2.1) by considering the Cartesian decomposition. The following lemma can be found in [28, p. 75-76]

Lemma 2.1.2. *Let $T \in \mathcal{B}(\mathcal{H})$ and $x, y \in \mathcal{H}$. Then*

$$|\langle Tx, y\rangle| \leq \langle |T|x, x\rangle^{\frac{1}{2}} \langle |T^*|y, y\rangle^{\frac{1}{2}}.$$

Recently, Moradi and Sababheh [41] have obtained the following theorem, which improve the second inequality in (2.1).

Theorem 2.1.3. *Let $T, S \in \mathcal{B}(\mathcal{H})$. Then*

$$w(T+S) \leq \frac{1}{\sqrt{2}} w(|T| + |S| + i(|T^*| + |S^*|)).$$

Proof. Let $x \in \mathcal{H}$, using Lemma 2.1.2, we get

$$\begin{aligned} |\langle (T+S)x, x\rangle| &\leq |\langle Tx, x\rangle| + |\langle Sx, x\rangle| \\ &\leq \left| \langle |T|x, x\rangle^{\frac{1}{2}} \langle |T^*|x, x\rangle^{\frac{1}{2}} + \langle |S|x, x\rangle^{\frac{1}{2}} \langle |S^*|x, x\rangle^{\frac{1}{2}} \right| \\ &\leq \frac{1}{2} \left| \langle |T|x, x\rangle + \langle |T^*|x, x\rangle + \langle |S^*|x, x\rangle + \langle |S|x, x\rangle \right| \\ &\qquad \text{(by Arithmetic-geometric mean inequality)} \\ &\leq \frac{1}{2} \left| \langle (|T| + |S|)x, x\rangle + \langle (|T^*| + |S^*|)x, x\rangle \right| \\ &\leq \frac{1}{\sqrt{2}} \left| \langle (|T| + |S| + i(|T^*| + |S^*|))x, x\rangle \right| \\ &\text{(by the scalar inequality } |a+b| \leq \sqrt{2}|a+ib| \text{ where } a, b \in \mathbb{R}). \end{aligned}$$

By taking the supremum on both sides in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$, we get the desired result. □

Taking $S = 0$ in Theorem 2.1.3, we get the following corollary.

Corollary 2.1.4. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$w(T) \leq \frac{1}{\sqrt{2}} w(|T| + i|T^*|). \tag{2.2}$$

Remark 2.1.5. *By using Proposition 2.1.1, we obtain*

$$w(T) \leq \frac{1}{\sqrt{2}} w(|T| + i|T^*|) \leq \frac{1}{\sqrt{2}} \| |T|^2 + |T^*|^2 \|.$$

Which refine the second inequality (2.1).

Lemma 2.1.6. *Let $T, S \in \mathcal{B}(\mathcal{H})$ be normal operators. Then*

$$w(T+S) \leq \sqrt{2} w(|T| + i|S|). \tag{2.3}$$

For more details see [41].

Theorem 2.1.7. *Let $T, S \in \mathcal{B}(\mathcal{H})$. Then*

$$w^2(T+S) \le \frac{1}{2}\left[w^2(|T|+i|S^*|)+w^2(|S|+i|T^*|)\right]+\frac{1}{4}w^2(|T|+|S^*|+i(|S|+|T^*|)).$$

Proof. Let $x \in \mathcal{H}$ be any unit vector, we have

$$\begin{aligned}
|\langle (T+S)x,x\rangle|^2 &\le |\langle Tx,x\rangle|^2+|\langle Sx,x\rangle|^2+2|\langle Tx,x\rangle||\langle Sx,x\rangle| \\
&\le \langle |T|x,x\rangle\langle |T^*|x,x\rangle+\langle |S|x,x\rangle\langle |S^*|x,x\rangle+2|\langle Tx,x\rangle||\langle Sx,x\rangle| \\
&\quad \text{(by Lemma 2.1.2)} \\
&\le \frac{1}{2}\left[\langle |T|x,x\rangle^2+\langle |T^*|x,x\rangle^2+\langle |S|x,x\rangle^2+\langle |T^*|x,x\rangle^2\right] \\
&\quad +2|\langle Tx,x\rangle\langle Sx,x\rangle| \\
&\le \frac{1}{2}\left[\langle |T|x,x\rangle^2+\langle |T^*|x,x\rangle^2+\langle |S|x,x\rangle^2+\langle |S^*|x,x\rangle^2\right] \\
&\quad +\frac{1}{2}(|\langle Tx,x\rangle|+|\langle Sx,x\rangle|)^2 \\
&\quad \text{(by the arithmetic-geometric mean inequality)} \\
&\le \frac{1}{2}\left[|\langle (|T|+i|S^*|)x,x\rangle|^2+|\langle (|S|+i|T^*|)x,x\rangle|^2\right] \\
&\quad +\frac{1}{2}\left\{\langle |T|x,x\rangle^{\frac{1}{2}}\langle |T^*|x,x\rangle^{\frac{1}{2}}+\langle |S|x,x\rangle^{\frac{1}{2}}\langle |S^*|x,x\rangle^{\frac{1}{2}}\right\}^2 \\
&\quad \text{(by Lemma 2.1.2)} \\
&\le \frac{1}{2}\left[|\langle (|T|+i|S^*|)x,x\rangle|^2+|\langle (|S|+i|T^*|)x,x\rangle|^2\right] \\
&\quad +\frac{1}{8}\left\{\langle |T|x,x\rangle+\langle |T^*|x,x\rangle+\langle |S|x,x\rangle+\langle |S^*|x,x\rangle\right\}^2 \\
&\quad \text{(by the arithmetic-geometric mean inequality)} \\
&\le \frac{1}{2}\left[|\langle (|T|+i|S^*|)x,x\rangle|^2+|\langle (|S|+i|T^*|)x,x\rangle|^2\right] \\
&\quad +\frac{1}{4}|\langle (|T|+|S^*|+i(|S|+|T^*|))x,x\rangle|^2 \\
&\quad \text{(by the scalar inequality } |a+b|\le\sqrt{2}|a+ib|\text{, where } a,b\in\mathbb{R}).
\end{aligned}$$

By taking the supremum on both sides in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$, we obtain the desired assertion. □

Corollary 2.1.8. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$w^2(T) \leq \frac{1}{4} w^2(|T| + i|T^*|) + \frac{1}{8} w^2(|T| + |T^*|). \tag{2.4}$$

Proof. By taking $S = T$ in Theorem 2.1.7, we obtain the desired result. □

Remark 2.1.9. *Using the inequality* (2.3)*, we have*

$$\begin{aligned} \frac{1}{4} w^2(|T| + i|T^*|) + \frac{1}{8} w^2(|T| + |T^*|) &\leq \frac{1}{4} w^2(|T| + i|T^*|) + \frac{1}{4} w^2(|T| + i|T^*|) \\ &= \frac{1}{2} w^2(|T| + i|T^*|). \end{aligned}$$

This means that the inequality (2.4) is better than the inequality (2.2).

The following lemma can be found in [41, 47].

Lemma 2.1.10. *Let $T, S \in \mathcal{B}(\mathcal{H})$ be normal operators. Then*

$$\|T + S\| \leq \sqrt{2} w(|T| + i|S|).$$

Using an argument similar to that used in the proof of Theorem 2.1.7, we have the following result.

Theorem 2.1.11. *Let $T, S \in \mathcal{B}(\mathcal{H})$. Then*

$$w^2(T + S) \leq \frac{1}{2} \left[w^2(|T| + i|S^*|) + w^2(|T^*| + i|S|) \right] + \frac{1}{4} w^2(|T| + |S^*| + i(|S| + |T^*|)).$$

Corollary 2.1.12. *Let $T, S \in \mathcal{B}(\mathcal{H})$ be two normal operators. Then*

$$w^2(T + S) \leq w^2(|T| + i|S|) + \frac{1}{2} w^2(|T| + |S|). \tag{2.5}$$

Proof. Since T, S are normal operators, then $|T| = |T^*|, |S| = |S^*|$. Now, the result follows immediately from Theorem 2.1.11. □

Remark 2.1.13. *We have*

$$\begin{aligned} w^2(T+S) &\le w^2(|T|+i|S|) + \frac{1}{2}w^2(|T|+|S|) \\ &= w^2(|T|+i|S|) + \frac{1}{2}\||T|+|S|\|^2 \\ &\le 2w^2(|T|+i|S|) \quad (\text{ by Lemma 2.1.10}). \end{aligned}$$

Thus, the inequality (2.5) is a refinement of the inequality (2.3).

Theorem 2.1.14. *Let $T, S \in \mathcal{B}(\mathcal{H})$. Then*

$$w^2(T+S) \le \frac{1}{2}\left[w^2(|T|+i|S^*|) + w^2(|S|+i|T^*|)\right] + w(TS) + \frac{1}{2}\left\||T^*|^2+|S|^2\right\|.$$

Proof. Let $x \in \mathcal{H}$ be any unit vector. Using the identity $\sup_{\theta\in\mathbb{R}}\left|e^{i\theta}a + e^{-i\theta}\overline{b}\right| = |a|+|b|$, where $a, b \in \mathbb{C}$, it follows that

$$\begin{aligned} |\langle (T+S)x, x\rangle|^2 &\le |\langle Tx, x\rangle|^2 + |\langle Sx, x\rangle|^2 + 2|\langle Tx, x\rangle||\langle Sx, x\rangle| \\ &\le \frac{1}{2}\left[\langle |T|x, x\rangle^2 + \langle |T^*|x, x\rangle^2 + \langle |S|x, x\rangle^2 + \langle |S^*|x, x\rangle^2\right] \\ &\quad + \frac{1}{2}(|\langle Tx, x\rangle| + |\langle Sx, x\rangle|)^2 \\ &= \frac{1}{2}\left[|\langle (|T|+i|S^*|)x, x\rangle|^2 + |\langle (|S|+i|T^*|)x, x\rangle|^2\right] \\ &\quad + \frac{1}{2}\sup_{\theta\in\mathbb{R}}\left|e^{i\theta}\langle Tx, x\rangle + e^{-i\theta}\langle S^*x, x\rangle\right|^2. \end{aligned}$$

By taking the supremum on both sides in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$, we get

$$\begin{aligned} w^2(T+S) &\le \frac{1}{2}\left[w^2(|T|+i|S^*|) + w^2(|S|+i|T^*|)\right] \\ &\quad + \frac{1}{2}\sup_{\theta\in\mathbb{R}}\left[\left\|2Re\left(e^{i\theta}TS\right)\right\| + \left\||T^*|^2+|S|^2\right\|\right] \\ &\le \frac{1}{2}\left[w^2(|T|+i|S^*|) + w^2(|S|+i|T^*|)\right] + w(TS) + \frac{1}{2}\left\||T^*|^2+|S|^2\right\|. \end{aligned}$$

□

Corollary 2.1.15. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$w^2(T) \leq \frac{1}{4}w^2(|T| + i|T^*|) + \frac{1}{4}w(T^2) + \frac{1}{8}\left\||T^*|^2 + |T|^2\right\|. \tag{2.6}$$

Proof. By taking $S = T$ in Theorem 2.1.14, the result follows directly. □

Remark 2.1.16. *Using the power inequality $w(T^2) \leq w^2(T)$, it follows that*

$$\begin{aligned} w^2(T) &\leq \frac{1}{3}w^2(|T| + i|T^*|) + \frac{1}{6}\left\||T^*|^2 + |T|^2\right\| \\ &\leq \frac{1}{2}\left\||T|^2 + |T^*|^2\right\| \quad (\textit{by Proposition } 2.1.1). \end{aligned}$$

Hence, the inequality (2.6) is a refinement of the inequality (2.1). The following lemma can be found in [36], which is called Hölder-McCarthy inequality.

Lemma 2.1.17. *Let $T \in \mathcal{B}(\mathcal{H})$ be a positive operator and let $x \in \mathcal{H}$ be any unit vector. Then*

a) $\langle Tx, x\rangle^r \leq \langle T^r x, x\rangle$ *for* $r \geq 1$.

b) $\langle T^r x, x\rangle \leq \langle Tx, x\rangle^r$ *for* $0 < r \leq 1$.

The following inequality was introduced by T. Kato [35], it is called the mixed Schwarz inequality. Generalization of this inequality have been given in [36].

Lemma 2.1.18. *Let $T \in \mathcal{B}(\mathcal{H})$ and $0 < \alpha < 1$. Then*

$$|\langle Tx, y\rangle|^2 \leq \langle |T|^{2\alpha}x, x\rangle\langle |T^*|^{2(1-\alpha)}y, y\rangle \qquad \textit{for all } x, y \in \mathcal{H}.$$

The following lemma is a direct consequence of Jensen's inequality for convex (concave) functions.

Lemma 2.1.19. *Let $a, b > 0$, $0 < \alpha < 1$. Then*

$$a^\alpha b^{1-\alpha} \leq \alpha a + (1-\alpha)b \leq [\alpha a^r + (1-\alpha)b^r]^{\frac{1}{r}} \qquad \textit{for } r \geq 1.$$

Theorem 2.1.20. *[24] Let $T \in \mathcal{B}(\mathcal{H})$, $0 < \alpha < 1$, and $r \geq 1$. Then*

$$w^r(T) \leq \frac{1}{2}\left\||T|^{2\alpha r} + |T^*|^{2(1-\alpha)r}\right\|. \tag{2.7}$$

Proof. For any unit vector $x \in \mathcal{H}$, we have

$$\begin{aligned}
|\langle Tx, x\rangle| &\leq \langle |T|^{2\alpha}x, x\rangle^{1/2}\langle |T^*|^{2(1-\alpha)}x, x\rangle^{1/2} \qquad \text{(by Lemma 2.1.18)}\\
&\leq 2^{-\frac{1}{r}}\left(\langle |T|^{2\alpha}x, x\rangle^r + \langle |T^*|^{2(1-\alpha)}x, x\rangle^r\right)^{1/r} \qquad \text{(by Lemma 2.1.19)}\\
&\leq 2^{-\frac{1}{r}}\left(\langle |T|^{2\alpha r}x, x\rangle + \langle |T^*|^{2(1-\alpha)r}x, x\rangle\right)^{1/r}. \qquad \text{(by Lemma 2.1.17(a))}
\end{aligned}$$

Thus,

$$|\langle Tx, x\rangle|^r \leq \frac{1}{2}\left(\langle |T|^{2\alpha r}x, x\rangle + \langle |T^*|^{2(1-\alpha)r}x, x\rangle\right).$$

By taking the supremum on both sides in the above inequality over $x \in \mathcal{H}$, we get the result directly. □

Theorem 2.1.21. *Let $T, S \in \mathcal{B}(\mathcal{H})$ and $r \geq 1$. Then*

$$\begin{aligned}
w^{2r}(T+S) \leq\ & 2^{2r-3}\left\{w^2(|T|^r + i|S^*|^r) + w^2(|S|^r + i|S^*|^r)\right\}\\
&+ 2^{2r-4}w^2(|T|^r + |S^*|^r + i(|S|^r + |T^*|^r)).
\end{aligned}$$

Proof. Let $x \in \mathcal{H}$ be any unit vector and let $r \geq 1$. Then

$$\begin{aligned}
(|\langle Tx, x\rangle| + |\langle Sx, x\rangle|)^r &\leq 2^{r-1}\left(|\langle Tx, x\rangle|^r + |\langle Sx, x\rangle|^r\right)\\
&\qquad \text{(by the convexity of the function} f(t) = t^r \text{ on } [0,\infty))\\
&\leq 2^{r-1}(\langle |T|x, x\rangle^{\frac{r}{2}}\langle |T^*|x, x\rangle^{\frac{r}{2}} + \langle |S|x, x\rangle^{\frac{r}{2}}\langle |S^*|x, x\rangle^{\frac{r}{2}})\\
&\qquad \text{(by Lemma 2.1.2)}\\
&\leq 2^{r-2}\left(\langle |T|x, x\rangle^r + \langle |T^*|x, x\rangle^r + \langle |S|x, x\rangle^r + \langle |S^*|x, x\rangle^r\right)\\
&\qquad \text{(by the arithmetic-geometric mean inequality)}\\
&\leq 2^{r-2}\left(\langle |T|^r x, x\rangle + \langle |T^*|^r x, x\rangle + \langle |S|^r x, x\rangle + \langle |S^*|^r x, x\rangle\right)\\
&\qquad \text{(by Lemma 2.1.17 (a))}\\
&\leq 2^{r-\frac{3}{2}}|\langle (|T|^r + |S^*|^r + i(|S|^r + |T^*|^r))x, x\rangle|\\
&\qquad \text{(by the scalar inequality } |a+b| \leq \sqrt{2}|a+ib|, \text{ where } a, b \in \mathbb{R}).
\end{aligned}$$

Hence, we obtain the inequality

$$(|\langle Tx, x\rangle| + |\langle Sx, x\rangle|)^r \leq 2^{r-\frac{3}{2}}\left[|\langle (|T|^r + |S^*|^r + i(|S|^r + |T^*|^r))x, x\rangle|\right]. \tag{2.8}$$

Therefore,

$$\begin{aligned}
|\langle (T+S)x,x\rangle|^{2r} &\le (|\langle Tx,x\rangle| + |\langle Sx,x\rangle|)^{2r} \\
&\le 2^{2r-2}\left(|\langle Tx,x\rangle|^r + |\langle Sx,x\rangle|^r\right)^2 \\
&\quad \text{(by the convexity of the function} f(t)=t^r \text{ on } [0,\infty))
\end{aligned}$$

$$\begin{aligned}
&= 2^{2r-2}\left(|\langle Tx,x\rangle|^{2r} + |\langle Sx,x\rangle|^{2r} + 2|\langle Tx,x\rangle|^r|\langle Sx,x\rangle|^r\right) \\
&\le 2^{2r-3}\left[\langle |T|^r x,x\rangle^2 + \langle |T^*|^r x,x\rangle^2 + \langle |S|^r x,x\rangle^2 + \langle |S^*|^r x,x\rangle^2\right] \\
&\quad + \frac{1}{2}(|\langle Tx,x\rangle| + |\langle Sx,x\rangle|)^{2r} \\
&\le 2^{2r-3}\left[|\langle (|T|^r + i|S^*|^r)x,x\rangle|^2 + |\langle (|S|^r + i|T^*|^r)x,x\rangle|^2\right] \\
&\quad + 2^{2r-4}|\langle (|T|^r + |S^*|^r + i(|S|^r + |T^*|^r))x,x\rangle|^2 \\
&\quad \text{(by the inequality (2.8)).}
\end{aligned}$$

By taking the supremum on both sides in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$, we get the desired statement. □

Corollary 2.1.22. *Let $T \in \mathcal{B}(\mathcal{H})$ and $r \ge 1$. Then*

$$w^{2r}(T) \le \frac{1}{4}w^2(|T|^r + i|T^*|^r) + \frac{1}{8}w^2(|T|^r + |T^*|^r). \tag{2.9}$$

Proof. By taking $S = T$ in Theorem 2.1.21, the inequality follows immediately. □

Remark 2.1.23. *By using the inequality* (2.3)*, we have*

$$\begin{aligned}
\frac{1}{4}w^2(|T|^r + i|T^*|^r) + \frac{1}{8}w^2(|T|^r + |T^*|^r) &\le \frac{1}{4}w^2(|T|^r + i|T^*|^r) + \frac{1}{4}w^2(|T|^r + i|T^*|^r) \\
&= \frac{1}{2}w^2(|T|^r + i|T^*|^r) \\
&\le \frac{1}{2}\left\||T|^{2r} + |T^*|^{2r}\right\| \ \textit{(by Proposition 2.1.1)}.
\end{aligned}$$

Hence, the inequality (2.9) is an improvement of the inequality (2.7) for $r \ge 2$ and $\alpha = \frac{1}{2}$.

Theorem 2.1.24. *Let $T \in \mathcal{B}(\mathcal{H})$, $0 < \alpha < 1$ and $r \geq 2$. Then*

$$w^r(T) \leq \frac{1}{2} w^2 \left(|T|^{\alpha r} + i |T^*|^{(1-\alpha) r} \right). \tag{2.10}$$

Proof. Let $x \in \mathcal{H}$ be any unit vector. By Lemma 2.1.17, we have

$$\begin{aligned}
|\langle Tx, x \rangle| &\leq \langle |T|^{2\alpha} x, x \rangle^{\frac{1}{2}} \langle |T^*|^{2(1-\alpha)} x, x \rangle^{\frac{1}{2}} \\
&\leq \frac{1}{2} \left(\langle |T|^{2\alpha} x, x \rangle + \langle |T^*|^{2(1-\alpha)} x, x \rangle \right) \\
&\quad \text{(by the arithmetic-geometric mean inequality).}
\end{aligned}$$

Applying Lemma 2.1.19, it follows that for $r \geq 2$, we have

$$\begin{aligned}
|\langle Tx, x \rangle|^r &\leq \left(\frac{\langle |T|^{2\alpha} x, x \rangle^{r/2} + \langle |T^*|^{2(1-\alpha)} x, x \rangle^{r/2}}{2} \right)^2 \\
&\leq \left(\frac{\langle |T|^{\alpha r} x, x \rangle + \langle |T^*|^{(1-\alpha) r} x, x \rangle}{2} \right)^2 \text{ (by Lemma 2.1.17 (a))} \\
&\leq \frac{1}{2} \left(\langle |T|^{\alpha r} x, x \rangle^2 + \langle |T^*|^{(1-\alpha) r} x, x \rangle^2 \right) \\
&\quad \text{(by the convexity of the function } f(t) = t^2 \text{ on } \mathbb{R}) \\
&= \frac{1}{2} \left| \langle (|T|^{\alpha r} + i |T^*|^{(1-\alpha) r}) x, x \rangle \right|^2.
\end{aligned}$$

By taking the supremum on both sides in the above inequality in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$, we get the desired result. □

Remark 2.1.25. *Using Proposition* (2.1.1)*, it follows that the inequality* (2.10) *is sharper than the inequality (2.7).*

The following theorem is a generalization of Theorem 2.1.20, which can be found in [24].

Theorem 2.1.26. *Let $T \in \mathcal{B}(\mathcal{H})$, $0 < \alpha < 1$, and $r \geq 1$. Then*

$$w^{2r}(T) \leq \left\| \alpha |T|^{2r} + (1-\alpha) |T^*|^{2r} \right\|. \tag{2.11}$$

Theorem 2.1.27. *Let $T \in \mathcal{B}(\mathcal{H})$, $0 < \alpha < 1$ and $r \geq 2$. Then*

$$w^{2r}(T) \leq w^2 \left(\sqrt{\alpha} |T|^r + i \sqrt{1-\alpha} |T^*|^r \right). \tag{2.12}$$

Proof. Let $x \in \mathcal{H}$ be any unit vector. By Lemma 2.1.18, we have

$$\begin{aligned}
|\langle Tx, x\rangle|^2 &\le \langle |T|^{2\alpha}x, x\rangle\langle |T^*|^{2(1-\alpha)}x, x\rangle \\
&\le \langle |T|^2x, x\rangle^{\alpha}\langle |T^*|^2x, x\rangle^{(1-\alpha)} \quad \text{(by Lemma 2.1.17 (b))} \\
&\le \left(\alpha\langle |T|^2x, x\rangle^{r/2} + (1-\alpha)\langle |T^*|^2x, x\rangle^{r/2}\right)^{2/r} \quad \text{(by Lemma 2.1.19).}
\end{aligned}$$

Therefore,

$$\begin{aligned}
|\langle Tx, x\rangle|^{2r} &\le \left(\alpha\langle |T|^2x, x\rangle^{r/2} + (1-\alpha)\langle |T^*|^2x, x\rangle^{r/2}\right)^{2} \\
&\le (\alpha\langle |T|^rx, x\rangle + (1-\alpha)\langle |T^*|^rx, x\rangle)^2 \quad \text{(by Lemma 2.1.17 (a))} \\
&\le \alpha\langle |T|^rx, x\rangle^2 + (1-\alpha)\langle |T^*|^rx, x\rangle^2 \\
&\quad \text{(by the convexity of the function } f(t) = t^2 \text{ on } \mathbb{R}) \\
&= \left|\langle(\sqrt{\alpha}|T|^r + i\sqrt{1-\alpha}|T^*|^r)x, x\rangle\right|^2.
\end{aligned}$$

By taking the supremum on both sides in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$, we obtain the required inequality. □

Remark 2.1.28. *Using Proposition 2.1.1, it follows that the inequality* (2.12) *is an improvement of the inequality* (2.11).

In [39], Kittaneh improved the inequality (2.13) as follows.

Theorem 2.1.29. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$w^2(T) \le \frac{1}{4}\||T|^2 + |T^*|^2\| + \frac{1}{2}w(T^2). \tag{2.13}$$

Proof. We have

$$\begin{aligned} w^2(T) &= \sup_{\theta\in\mathbb{R}} \|Re(e^{i\theta}T)\|^2 \\ &= \frac{1}{4}\sup_{\theta\in\mathbb{R}} \|(e^{i\theta}T + e^{-i\theta}T^*)^2\| \\ &= \frac{1}{4}\sup_{\theta\in\mathbb{R}} \|T^*T + TT^* + 2Re(e^{2i\theta}T^2)\| \\ &\le \frac{1}{4}\|T^*T + TT^*\| + \frac{1}{2}\sup_{\theta\in\mathbb{R}} \|Re(e^{2i\theta}T^2)\| \\ &= \frac{1}{4}\|T^*T + TT^*\| + \frac{1}{2}w(T^2), \end{aligned}$$

as required. □

Theorem 2.1.30. *Let $T, S \in \mathcal{B}(\mathcal{H})$. Then*

$$w(T+S) \le w^2(T) + w^2(S) + \frac{1}{2}\||T|^2 + |S^*|^2\| + w(ST). \tag{2.14}$$

Proof. See [42]. □

Theorem 2.1.31. *Let $T, S \in \mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned} w^2(T+S) \le \min&\left\{\frac{1}{2}\left\||T^*|^2 + |S|^2\right\| + w(TS), \frac{1}{2}\left\||T|^2 + |S^*|^2\right\| + w(ST)\right\} \\ &+ w^2(T) + w^2(S). \end{aligned} \tag{2.15}$$

Proof. Let $x \in \mathcal{H}$ be any unit vector. Then

$$\begin{aligned} |\langle (T+S)x, x\rangle|^2 &\le |\langle Tx, x\rangle|^2 + |\langle Sx, x\rangle|^2 + 2|\langle Tx, x\rangle||\langle Sx, x\rangle| \\ &\le |\langle Tx, x\rangle|^2 + |\langle Sx, x\rangle|^2 + \frac{1}{2}(|\langle Tx, x\rangle| + |\langle Sx, x\rangle|)^2 \\ &\quad \text{(by the arithmetic-geometric mean inequality)} \\ &= |\langle Tx, x\rangle|^2 + |\langle Sx, x\rangle|^2 + \frac{1}{2}\sup_{\theta\in\mathbb{R}} \left|e^{i\theta}\langle Tx, x\rangle + e^{-i\theta}\langle S^*x, x\rangle\right|^2. \end{aligned}$$

By taking the supremum on both sides in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$, we get

$$w^2(T+S) \le w^2(T) + w^2(B) + \frac{1}{2}\sup_{\theta\in\mathbb{R}} \left\|e^{i\theta}T + e^{-i\theta}S^*\right\|^2.$$

Using the fact that $\|TT^*\| = \|T^*T\| = \|T\|^2$, the desired result is obtained. $\square$

Remark 2.1.32. *The inequality* (2.15) *is a refinement of the inequality* (2.14).

Remark 2.1.33. *By taking $S = T$ in Theorem 2.1.31, we reobtain* (2.13).

The next lemma can be found in [21], which is called the Buzano extension of Schwarz's inequality.

Lemma 2.1.34. *Let $x, y, e \in \mathcal{H}$ with $\|e\| = 1$. Then*

$$|\langle x, e\rangle\langle e, y\rangle| \leq \frac{1}{2}(\|x\|\|y\| + |\langle x, y\rangle|).$$

Theorem 2.1.35. *Let $T, S \in \mathcal{B}(\mathcal{H})$. Then*

$$\|T + S\|^2 \leq \frac{1}{2}\left\{w^2(|T| + i|S|) + w^2(|T^*| + i|S^*|)\right\} + w(S^*T) + \|T\|\|S\|. \tag{2.16}$$

Proof. Let $x, y \in \mathcal{H}$ be any unit vectors. Then

$$\begin{aligned}
|\langle (T+S)x, y\rangle|^2 &\leq |\langle Tx, y\rangle|^2 + |\langle Sx, y\rangle|^2 + 2|\langle Tx, y\rangle||\langle S, y\rangle| \\
&\leq \langle |T|x, x\rangle\langle |T^*|y, y\rangle + \langle |T|x, x\rangle\langle |S^*|y, y\rangle + 2|\langle Tx, y\rangle||\langle S, y\rangle| \\
&\text{(by Lemma 2.1.2)} \\
&\leq \frac{1}{2}\left[\langle |T|x, x\rangle^2 + \langle |T^*|y, y\rangle^2 + \langle |S|x, x\rangle^2 + \langle |S^*|y, y\rangle^2\right] \\
&+\ 2|\langle Tx, y\rangle\langle y, Sx\rangle| \\
&\text{(by the arithmetic-geometric mean inequality)} \\
&= \frac{1}{2}\left[\langle |T|x, x\rangle^2 + \langle |S|x, x\rangle^2 + \langle |T^*|y, y\rangle^2 + \langle |S^*|y, y\rangle^2\right] \\
&+\ 2|\langle Tx, y\rangle\langle y, Sx\rangle| \\
&\leq \frac{1}{2}\left[|\langle (|T| + i|S|)x, x\rangle|^2 + |\langle (|T^*| + i|S^*|)y, y\rangle|^2\right] \\
&+\ |\langle Tx, Sx\rangle| + \|Tx\|\|Sx\| \quad \text{(by Lemma 2.1.34)}.
\end{aligned}$$

By taking the supremum on both sides in the above inequality over $x, y \in \mathcal{H}$ with $\|x\| = \|y\| = 1$, we get

$$\|T + S\|^2 \leq \frac{1}{2}\left\{w^2(|T| + i|S|) + w^2(|T^*| + i|S^*|)\right\} + w(S^*T) + \|T\|\|S\|,$$

as required. $\square$

Remark 2.1.36. *Using Proposition 2.1.1, we get*

$$\begin{aligned}
\|T+S\|^2 &\leq \frac{1}{2}\left\{w^2\left(|T|+i|S|\right)+w^2(|T^*|+i|S^*|)\right\}+w(S^*T)+\|T\|\|S\| \\
&\leq \frac{1}{2}\left\{\left\||T|^2+|S|^2\right\|+\left\||T^*|^2+|S^*|^2\right\|\right\}+2\|T\|\|S\| \\
&\leq \|T\|^2+\|S\|^2+2\|T\|\|S\| \\
&= (\|T\|+\|S\|)^2.
\end{aligned}$$

Hence, the inequality (2.16) is an improvement of the triangle inequality.

2.2 Numerical radius inequalities for 2×2 operator matrices

In the rest of this work in this chapter, we need the following equalities of the numerical radii for the diagonal and off-diagonal parts of 2×2 operator matrices.

Theorem 2.2.1. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then*

$$w\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = \max\{w(A), w(B)\}.$$

Proof. We have

$$\begin{aligned}
w\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) &= \sup_{\theta\in\mathbb{R}}\left\|\mathcal{R}e\left(e^{i\theta}\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right)\right\| \\
&= \sup_{\theta\in\mathbb{R}}\left\|\begin{bmatrix} \mathcal{R}e(e^{i\theta}A) & 0 \\ 0 & \mathcal{R}e(e^{i\theta}B) \end{bmatrix}\right\| \\
&= \sup_{\theta\in\mathbb{R}}\max\{\|\mathcal{R}e(e^{i\theta}A)\|, \|\mathcal{R}e(e^{i\theta}B)\|\} \\
&= \max\{w(A), w(B)\},
\end{aligned}$$

as required. $\square$

Theorem 2.2.2. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then*

$$w\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) = \frac{1}{2} \sup_{\theta \in \mathbb{R}} \left\| e^{i\theta} A + e^{-i\theta} B^* \right\|. \tag{2.17}$$

Proof. We have

$$\begin{aligned} w\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) &= \sup_{\theta \in \mathbb{R}} \left\| \mathcal{R}e \left(e^{i\theta} \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) \right\| \\ &= \frac{1}{2} \sup_{\theta \in \mathbb{R}} \left\| \begin{bmatrix} 0 & e^{i\theta} A + e^{-i\theta} B^* \\ e^{-i\theta} A^* + e^{i\theta} B & 0 \end{bmatrix} \right\| \\ &= \frac{1}{2} \sup_{\theta \in \mathbb{R}} \left\| e^{i\theta} A + e^{-i\theta} B^* \right\|. \end{aligned}$$

□

Remark 2.2.3. *For $B = A$ in Theorem 2.2.2, we find*

$$w\left(\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}\right) = w(A).$$

The direct sum of two copies of $\mathcal{H}$ denoted by $\mathcal{H} \oplus \mathcal{H}$. If $A, B, C, D \in \mathcal{B}(\mathcal{H})$, then the operator matrix $\mathcal{T} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ can be considered as an operator in $\mathcal{B}(\mathcal{H} \oplus \mathcal{H})$, which is defined by $\mathcal{T}x = \begin{bmatrix} Ax_1 + Bx_2 \\ Cx_1 + Dx_2 \end{bmatrix}$ for every vector $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathcal{H} \oplus \mathcal{H}$.

The next lemma was given by Hou and Du [34].

Lemma 2.2.4. *Let $A, B, C, D \in \mathcal{B}(\mathcal{H})$. Then*

$$\left\| \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} \|A\| & \|B\| \\ \|C\| & \|D\| \end{bmatrix} \right\|.$$

Theorem 2.2.5. *Let $\mathcal{T}$ be as described above. Then*

$$w(\mathcal{T}) \leq \max\{w(A), w(D)\} + \frac{1}{2} \left(\|B\| + \|C\| \right). \tag{2.18}$$

Proof. Using Theorem 2.2.1 and Theorem 2.2.2, we get

$$\begin{aligned} w(\mathcal{T}) &\le w\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right) + w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \\ &= \max\{w(A), w(D)\} + \frac{1}{2}\sup_{\theta\in\mathbb{R}} \left\| e^{i\theta}B + e^{-i\theta}C^* \right\| \\ &\le \max\{w(A), w(D)\} + \frac{1}{2}\left(\|B\| + \|C\|\right). \end{aligned}$$

□

Theorem 2.2.6. *Let* $\mathcal{T}$ *be as described above and* $0 < \alpha < 1$. *Then*

$$w\left(\mathcal{T}\right) \le \frac{1}{2}\left\{w(A) + 2w(D) + \sqrt{\alpha^2 w^2(A) + \|B\|^2} + \sqrt{(1-\alpha)^2 w^2(A) + \|C\|^2}\right\}. \tag{2.19}$$

Proof. For any $\theta \in \mathbb{R}$, we have

$$\begin{aligned} 2\|\mathcal{R}e(e^{i\theta}\mathcal{T})\| &= 2w(\mathcal{R}e(e^{i\theta}\mathcal{T})) \\ &\le w\left(\begin{bmatrix} 2\alpha(\mathcal{R}e(e^{i\theta}A)) & e^{i\theta}B \\ e^{-i\theta}B^* & 0 \end{bmatrix}\right) + w\left(\begin{bmatrix} 2(1-\alpha)(\mathcal{R}e(e^{i\theta}A)) & e^{-i\theta}C^* \\ e^{i\theta}C & 0 \end{bmatrix}\right) + 2w(D) \\ &\le r\left(\begin{bmatrix} 2\alpha\|\mathcal{R}e(e^{i\theta}A)\| & \|B\| \\ \|B\| & 0 \end{bmatrix}\right) + r\left(\begin{bmatrix} 2(1-\alpha)\|\mathcal{R}e(e^{i\theta}A)\| & \|C\| \\ \|C\| & 0 \end{bmatrix}\right) + 2w(D) \\ &\le w(A) + \sqrt{\alpha^2 w^2(A) + \|B\|^2} + \sqrt{(1-\alpha)^2 w^2(A) + \|C\|^2} + 2w(D). \end{aligned}$$

By taking the supremum in the above inequality over $\theta \in \mathbb{R}$, we get the desired result.

□

Theorem 2.2.7. *Let* $\mathcal{T}$ *be as described above and* $0 < \alpha < 1$. *Then*

$$w(\mathcal{T}) \le \frac{1}{2}\left[w(A) + w(D) + \sqrt{\alpha^2(w(A) - w(D))^2 + \|B\|^2} + \sqrt{(1-\alpha)^2(w(A) - w(D))^2 + \|C\|^2}\right]. \tag{2.20}$$

Proof. We have

$$\begin{aligned}
w(\mathcal{T}) &= \frac{1}{2}\sup_{\theta\in\mathbb{R}}\left\|\begin{bmatrix} 2\mathcal{R}e(e^{i\theta}A) & e^{i\theta}B + e^{-i\theta}C^* \\ e^{i\theta}C + e^{-i\theta}B^* & 2\mathcal{R}e(e^{i\theta}D)\end{bmatrix}\right\| \\
&\le \frac{1}{2}\sup_{\theta\in\mathbb{R}}\left\|\begin{bmatrix} 2\alpha\mathcal{R}e(e^{i\theta}A) & e^{i\theta}B \\ e^{-i\theta}B^* & 2\alpha\mathcal{R}e(e^{i\theta}D)\end{bmatrix}\right\| \\
&\quad + \frac{1}{2}\sup_{\theta\in\mathbb{R}}\left\|\begin{bmatrix} 2(1-\alpha)\mathcal{R}e(e^{i\theta}A) & e^{-i\theta}C^* \\ e^{i\theta}C & 2(1-\alpha)\mathcal{R}e(e^{i\theta}D)\end{bmatrix}\right\| \\
&\le \frac{1}{2}\left(\left\|\begin{bmatrix} 2\alpha w(A) & \|B\| \\ \|B\| & 2\alpha w(D)\end{bmatrix}\right\| + \left\|\begin{bmatrix} 2(1-\alpha)w(A) & \|C\| \\ \|C\| & 2(1-\alpha)w(D)\end{bmatrix}\right\|\right) \\
&\quad \text{(by Lemma 2.1.2)} \\
&= \frac{1}{2}\left[w(A) + w(D) + \sqrt{\alpha^2(w(A)-w(D))^2 + \|B\|^2} + \sqrt{(1-\alpha)^2(w(A)-w(D))^2 + \|C\|^2}\right].
\end{aligned}$$

□

We give an example in which we can see that the inequality (2.20) gives a better estimate than the inequality (2.19).

Example 2.1. *Let* $\mathcal{T} = \begin{bmatrix} A & B \\ C & D\end{bmatrix}$, *where* $A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0\end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1\end{bmatrix}$ *and* $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{bmatrix}$. *Then the upper bounds for* $w(\mathcal{T})$ *using the inequalities (2.20) and (2.19) are as follows:*
For the inequality (2.20), $w(\mathcal{T}) \le \frac{1}{2}\left(1.7 + \sqrt{\alpha^2 0.08 + 1} + \sqrt{(1-\alpha)^2 0.08 + 1}\right)$.
For the inequality (2.19), $w(\mathcal{T}) \le \frac{1}{2}\left(2.7 + \sqrt{\alpha^2 0.5 + 1} + \sqrt{(1-\alpha)^2 0.5 + 1}\right)$.

Corollary 2.2.8. *Let* $\mathcal{T}$ *be as described above. Then*

$$w(\mathcal{T}) \le \frac{1}{2}\left[w(A) + w(D) + \sqrt{\frac{1}{4}(w(A)-w(D))^2 + \|B\|^2} + \sqrt{\frac{1}{4}(w(A)-w(D))^2 + \|C\|^2}\right]. \tag{2.21}$$

Proof. The result is obtained by taking $\alpha = \frac{1}{2}$ in Theorem 2.2.7. □

Remark 2.2.9. *It is easy to see that the inequality (2.21) is better than the inequality (2.18).*

Theorem 2.2.10. *Let $\mathcal{T}$ be as described above. Then*

$$\begin{aligned} w^2(\mathcal{T}) &\leq \max\{w^2(A), w^2(D)\} + w^2\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) + w\left(\begin{bmatrix} 0 & BD \\ CA & 0 \end{bmatrix}\right) \\ &\quad + \frac{1}{2}\max\left\{\||A|^2+|B^*|^2\|, \||C^*|^2+|D|^2\|\right\}. \end{aligned} \tag{2.22}$$

Proof. See [13]. □

Theorem 2.2.11. *Let $\mathcal{T}$ be as described above. Then*

$$w^2(\mathcal{T}) \leq \min\{\alpha, \beta\}, \tag{2.23}$$

where

$$\begin{aligned} \alpha = \max\{w^2(A), w^2(D)\} + w^2\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) + \frac{1}{4}\left[\left\||A^*|^2+|C|^2\right\| + \left\||B|^2+|D^*|^2\right\|\right] \\ + \frac{1}{4}\sqrt{\left(\||A^*|^2+|C|^2\| - \||B|^2+|D^*|^2\|\right)^2 + 16\, w^2\left(\begin{bmatrix} 0 & AB \\ DC & 0 \end{bmatrix}\right)} \end{aligned}$$

and

$$\begin{aligned} \beta = \max\{w^2(A), w^2(D)\} + w^2\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) + \frac{1}{4}\left[\left\||A|^2+|B^*|^2\right\| + \left\||C^*|^2+|D|^2\right\|\right] \\ + \frac{1}{4}\sqrt{\left(\||A|^2+|B^*|^2\| - \||C^*|^2+|D|^2\|\right)^2 + 16\, w^2\left(\begin{bmatrix} 0 & BD \\ CA & 0 \end{bmatrix}\right)}. \end{aligned}$$

Proof. Let $x \in \mathcal{H} \oplus \mathcal{H}$ be any unit vector. Than

$$\begin{aligned} |\langle Tx, x\rangle|^2 &\leq \left|\left\langle \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} x, x\right\rangle\right|^2 + \left|\left\langle \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} x, x\right\rangle\right|^2 + 2\left|\left\langle \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} x, x\right\rangle\right|\left|\left\langle \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} x, x\right\rangle\right| \\ &\leq \left|\left\langle \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} x, x\right\rangle\right|^2 + \left|\left\langle \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} x, x\right\rangle\right|^2 + \frac{1}{2}\left(\left|\left\langle \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} x, x\right\rangle\right| + \left|\left\langle \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} x, x\right\rangle\right|\right)^2 \end{aligned}$$

$$
\begin{aligned}
&\leq \left|\left\langle \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} x, x \right\rangle\right|^2 + \left|\left\langle \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} x, x \right\rangle\right|^2 \\
&\quad + \frac{1}{2} \sup_{\theta \in \mathbb{R}} \left|\left\langle \left(e^{i\theta} \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} + e^{-i\theta} \begin{bmatrix} 0 & C^* \\ B^* & 0 \end{bmatrix} \right) x, x \right\rangle\right|^2
\end{aligned}
$$

By taking the supremum in the above inequality over $x \in \mathcal{H} \oplus \mathcal{H}$ with $\|x\| = 1$, we get

$$
\begin{aligned}
w^2(T) &\leq \max\{w^2(A), w^2(D)\} + w^2\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right) + \frac{1}{2} \sup_{\theta \in \mathbb{R}} \left\| \begin{bmatrix} e^{i\theta} A & e^{-i\theta} C^* \\ e^{-i\theta} B^* & e^{i\theta} D \end{bmatrix} \right\|^2 \\
&\leq \max\{w^2(A), w^2(D)\} + w^2\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right) \\
&\quad + \frac{1}{2} \sup_{\theta \in \mathbb{R}} \left\| \begin{bmatrix} |A^*|^2 + |C|^2 & e^{2i\theta} AB + e^{-2i\theta} C^* D^* \\ e^{-2i\theta} B^* A^* + e^{2i\theta} DC & |B|^2 + |D^*|^2 \end{bmatrix} \right\| \\
&\leq \max\{w^2(A), w^2(D)\} + w^2\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right) \\
&\quad + \frac{1}{2} \sup_{\theta \in \mathbb{R}} \left\| \begin{bmatrix} \||A^*|^2 + |C|^2\| & \|e^{2i\theta} AB + e^{-2i\theta} C^* D^*\| \\ \|e^{2i\theta} AB + e^{-2i\theta} C^* D^*\| & \||B|^2 + |D^*|^2\| \end{bmatrix} \right\| \\
&\quad \text{(by Lemma 2.2.4)} \\
&\leq \max\{w^2(A), w^2(D)\} + w^2\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right) + \frac{1}{4} \left[\||A^*|^2 + |C|^2\| + \||B|^2 + |D^*|^2\| \right] \\
&\quad + \frac{1}{4} \sqrt{ \left(\||A^*|^2 + |C|^2\| - \||B|^2 + |D^*|^2\| \right)^2 + 16\, w^2\left(\begin{bmatrix} 0 & AB \\ DC & 0 \end{bmatrix} \right) } \\
&\quad \text{(by Theorem (2.2.2))} \\
&= \alpha.
\end{aligned}
$$

Applying the same argument to T^*, and observing that $w(\mathcal{T}) = w(\mathcal{T}^*)$, we get

$$\begin{aligned} w^2(\mathcal{T}) &\leq \max\{w^2(A), w^2(D)\} + w^2\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) + \frac{1}{4}\left[\||A|^2 + |B^*|^2\| + \||C^*|^2 + |D|^2\|\right] \\ &\quad + \frac{1}{4}\sqrt{\left(\||A|^2 + |B^*|^2\| - \||C^*|^2 + |D|^2\|\right)^2 + 16\, w^2\left(\begin{bmatrix} 0 & BD \\ CA & 0 \end{bmatrix}\right)} \\ &= \beta. \end{aligned}$$

□

Remark 2.2.12. *If we put*

$$\begin{aligned} &\frac{1}{4}\left[\||A|^2 + |B^*|^2\| + \||C^*|^2 + |D|^2\|\right] \\ &\quad + \frac{1}{4}\sqrt{\left(\||A|^2 + |B^*|^2\| - \||C^*|^2 + |D|^2\|\right)^2 + 16w^2\left(\begin{bmatrix} 0 & BD \\ CA & 0 \end{bmatrix}\right)} = d, \end{aligned}$$

then

$$d \leq \frac{1}{2}\max\{\||A|^2 + |B^*|^2\|, \||C^*|^2 + |D|^2\|\} + w\left(\begin{bmatrix} 0 & BD \\ CA & 0 \end{bmatrix}\right).$$

This means that the inequality (2.23) is sharper than the inequality (2.22).

Theorem 2.2.13. *Let $\mathcal{T}$ be as described above. Then*

$$\begin{aligned} \|\mathcal{T}\|^2 \quad &\leq \quad \max\{\|A\|^2, \|D\|^2\} + \max\{\|B\|^2, \|C\|^2\} + w\left(\begin{bmatrix} 0 & C^*D \\ B^*A & 0 \end{bmatrix}\right) \\ &\quad + \max\{\|A\|, \|D\|\}\max\{\|B\|, \|C\|\}. \end{aligned} \tag{2.24}$$

Proof. See [13]. □

Corollary 2.2.14. *[13] Let $A, B \in \mathcal{B}(\mathcal{H})$. Then*

$$\left\|\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right\|^2 \leq \|A\|^2 + \|B\|^2 + \|A\|\|B\| + w(B^*A). \tag{2.25}$$

Theorem 2.2.15. *Let $\mathcal{T}$ be as described above. Then*

$$\begin{aligned}
\|\mathcal{T}\|^2 \leq & \frac{1}{2}\left(\max\left\{\left\||A|^2+|C|^2\right\|,\left\||D|^2+|B|^2\right\|\right\}+\max\left\{\left\||A^*|^2+|B^*|^2\right\|,\left\||D^*|^2+|C^*|^2\right\|\right\}\right) \\
& + \sqrt{\frac{1}{2}\left[\max\{\|A\|^2,\|D\|^2\}\max\{\|B\|^2,\|C\|^2\}+\max\left\{w\left(|C|^2|A|^2\right),w\left(|B|^2|D|^2\right)\right\}\right]} \\
& + w\left(\begin{bmatrix} 0 & C^*D \\ B^*A & 0 \end{bmatrix}\right). \qquad (2.26)
\end{aligned}$$

Proof. Let $x, y \in \mathcal{H} \oplus \mathcal{H}$ be any vectors with $\|x\| = \|y\| = 1$. Then

$$\begin{aligned}
|\langle Tx, y\rangle|^2 \leq & \left|\left\langle \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} x, y\right\rangle\right|^2 + \left|\left\langle \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} x, y\right\rangle\right|^2 + 2\left|\left\langle \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} x, y\right\rangle\right|\left|\left\langle \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} x, y\right\rangle\right| \\
\leq & \frac{1}{2}\left\langle \begin{bmatrix} |A|^2+|C|^2 & 0 \\ 0 & |D|^2+|B|^2 \end{bmatrix} x, x\right\rangle + \frac{1}{2}\left\langle \begin{bmatrix} |A^*|^2+|B^*|^2 & 0 \\ 0 & |D^*|^2+|C^*|^2 \end{bmatrix} y, y\right\rangle \\
& + \left|\left\langle \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} x, \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} x\right\rangle\right| + \left\|\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} x\right\|\left\|\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} x\right\| \text{ (by Lemma 2.1.34)} \\
= & \frac{1}{2}\left\langle \begin{bmatrix} |A|^2+|C|^2 & 0 \\ 0 & |D|^2+|B|^2 \end{bmatrix} x, x\right\rangle + \frac{1}{2}\left\langle \begin{bmatrix} |A^*|^2+|B^*|^2 & 0 \\ 0 & |D^*|^2+|C^*|^2 \end{bmatrix} y, y\right\rangle \\
& + \left|\left\langle \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} x, \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} x\right\rangle\right| + \sqrt{\left\langle \left|\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right|^2 x, x\right\rangle \left\langle x, \left|\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right|^2 x\right\rangle} \\
\leq & \frac{1}{2}\left\langle \begin{bmatrix} |A|^2+|C|^2 & 0 \\ 0 & |D|^2+|B|^2 \end{bmatrix} x, x\right\rangle + \frac{1}{2}\left\langle \begin{bmatrix} |A^*|^2+|B^*|^2 & 0 \\ 0 & |D^*|^2+|C^*|^2 \end{bmatrix} y, y\right\rangle \\
& + \sqrt{\frac{1}{2}\left[\left\|\left|\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right|^2 x\right\|\left\|\left|\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right|^2 x\right\| + \left|\left\langle \begin{bmatrix} |C|^2|A|^2 & 0 \\ 0 & |B|^2|D|^2 \end{bmatrix} x, x\right\rangle\right|\right]} \\
& + \left|\left\langle \begin{bmatrix} 0 & C^*D \\ B^*A & 0 \end{bmatrix} x, x\right\rangle\right| \text{ (by Lemma 2.1.34).}
\end{aligned}$$

By taking the supremum on both sides in the above inequality over $x, y \in \mathcal{H} \oplus \mathcal{H}$ with

$\|x\| = \|y\| = 1$, we obtain

$$\begin{aligned}\|\mathcal{T}\|^2 &\le \frac{1}{2}\left(\max\{\||A|^2 + |C|^2\|, \||D|^2 + |B|^2\|\} + \max\{\||A^*|^2 + |B^*|^2\|, \||D^*|^2 + |C^*|^2\|\}\right)\\ &+ \sqrt{\frac{1}{2}\left[\max\{\|A\|^2, \|D\|^2\} \max\{\|B\|^2, \|C\|^2\} + \max\{w(|C|^2|A|^2), w(|B|^2|D|^2)\}\right]}\\ &+ w\left(\begin{bmatrix} 0 & C^*D \\ B^*A & 0 \end{bmatrix}\right),\end{aligned}$$

as required. □

Remark 2.2.16. *We set*

$$\begin{aligned}&\frac{1}{2}\left(\max\{\||A|^2 + |C|^2\|, \||D|^2 + |B|^2\|\} + \max\{\||A^*|^2 + |B^*|^2\|, \||D^*|^2 + |C^*|^2\|\}\right)\\ &+ \sqrt{\frac{1}{2}\left[\max\{\|A\|^2, \|D\|^2\} \max\{\|B\|^2, \|C\|^2\} + \max\{w(|C|^2|A|^2), w(|B|^2|D|^2)\}\right]} = c\end{aligned}$$

Then

$$\begin{aligned}c &\le \frac{1}{2}\left(\max\{\|A\|^2 + \|C\|^2, \|D\|^2 + \|B\|^2\} + \max\{\|A\|^2 + \|B\|^2, \|D\|^2 + \|C\|^2\}\right)\\ &+ \frac{1}{\sqrt{2}}\sqrt{\max\{\|A\|^2, \|D\|^2\} \max\{\|B\|^2, \|C\|^2\} + \max\{\||C|^2|A|^2\|, \||B|^2|D|^2\|\}}\\ &\le \max\{\|A\|^2, \|D\|^2\} + \max\{\|B\|^2, \|C\|^2\} + \max\{\|A\|, \|D\|\} \max\{\|B\|, \|C\|\}.\end{aligned}$$

This proves that the inequality (2.26) is a refinement of the inequality (2.24).

Corollary 2.2.17. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then*

$$\left\|\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right\|^2 \le \frac{1}{2}\left(\left\||A|^2 + |B|^2\right\| + \left\||A^*|^2 + |B^*|^2\right\|\right) + w(B^*A) + \frac{1}{\sqrt{2}}\sqrt{\|A\|^2\|B\|^2 + w(|B|^2|A|^2)}. \tag{2.27}$$

Proof. The result follows by taking $C = B$ and $D = A$ in Theorem 2.2.15. □

Remark 2.2.18. *It easy to check that the inequality (2.27) is an improvement of the inequality (2.25).*

2.3 Estimation for the zeros of polynomials

Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a monic polynomial of degree $n \geq 2$ with complex coefficients $a_0, a_1, \cdots, a_{n-1}$. Let $\mathcal{C}(p)$ be the Frobenius companion matrix of $p(z)$.

$$\mathcal{C}(p) = \begin{bmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ \vdots & 0 & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

It is well known that the eigenvalues of $\mathcal{C}(p)$ are exactly the zeros of the polynomial $p(z)$. Let λ be any zero of the polynomial $p(z)$. Then

$$|\lambda| \leq w\big(\mathcal{C}(p)\big) \quad \text{as} \quad \sigma\big(\mathcal{C}(p)\big) \subseteq W\big(\mathcal{C}(p)\big).$$

Several mathematicians have provided lot of estimations of $|\lambda|$ using different techniques over the years. We cite some of them

1. Cauchy [33] gave the following estimate

$$|\lambda| \leq 1 + \max\{|a_i| : \ i = 0, \ldots, n-1\}.$$

2. Montel [33] provided the following estimate

$$|\lambda| \leq \max\Big\{1, \sum_{i=0}^{n-1} |a_i|\Big\}.$$

3. Carmichael and Mason [33] presented the following estimate

$$|\lambda| \leq \sqrt{1 + \sum_{i=0}^{n-1} |a_i|^2}.$$

4. Paul and Bag [43] provided the following estimate

$$|\lambda| \le \frac{1}{2}\left(w(A) + \cos\left(\frac{\pi}{n-2}\right) + \sqrt{\left(w(A) - \cos\left(\frac{\pi}{n-2}\right)\right)^2 + \left(1 + \sqrt{\sum_{i=3}^{n} |a_{n-i}|^2}\right)^2}\right)$$

with $A = \begin{bmatrix} -a_{n-1} & -a_{n-2} \\ 1 & 0 \end{bmatrix}$.

5. Fujii and Kubo [25] gave the following estimate

$$|\lambda| \le \cos\left(\frac{\pi}{n+1}\right) + \frac{1}{2}\left(|a_{n-1}| + \sqrt{\sum_{i=0}^{n-1} |a_i|^2}\right).$$

6. Abu-Omar and Kittaneh [2] provided the following estimate

$$|\lambda| \le \frac{1}{2}\left(\frac{1}{2}(\beta + |a_{n-1}|) + \cos\left(\frac{\pi}{n}\right) + \sqrt{\left(\frac{\beta + |a_{n-1}|}{2} - \cos\left(\frac{\pi}{n}\right)\right)^2 + 4\gamma}\right),$$

with $\beta = \sqrt{\sum_{i=0}^{n-1} |a_i|^2}$ and $\gamma = \sqrt{\sum_{i=0}^{n-2} |a_i|^2}$.

7. Al-Dolat, Jaradat and Al Husban [9] provided the following estimate

$$|\lambda| \le \frac{1}{2}\left(|a_{n-1}| + 2\cos\left(\frac{\pi}{n-1}\right) + \sqrt{\alpha^2 |a_{n-1}|^2 + \sum_{i=0}^{n-2} |a_i|^2 + \sqrt{1 + (1-\alpha)^2 |a_{n-1}|^2}}\right),$$

where $\alpha \in [0, 1]$.

Theorem 2.3.1. *Let λ be any zero of $p(z)$ and let $\alpha \in [0, 1]$. Then*

$$|\lambda| \le \frac{1}{2}\left(|a_{n-1}| + \cos\left(\frac{\pi}{n}\right) + \sqrt{\alpha^2 \left(|a_{n-1}| - \cos\left(\frac{\pi}{n}\right)\right)^2 + \sum_{i=0}^{n-2} |a_i|^2 + \sqrt{(1-\alpha)^2 \left(|a_{n-1}| - \cos\left(\frac{\pi}{n}\right)\right)^2 + 1}}\right).$$

Proof. The result follows by taking A, B, C and D in Theorem 2.2.7 as follows $A = [-a_{n-1}]$, $B = [-a_{n-2}, \cdots, -a_0]$, $C^t = [1, 0, \cdots, 0]$ and

$$D = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & \ddots & \cdots & 0 \\ 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}$$ and using the fact that $w(D) = \cos(\frac{\pi}{n})$, see [26]. □

Example 2.2. *Let* $p(z) = z^5 - 4z^2 + 2$. *Then the upper bounds for the zeros of the polynomial* $p(z)$ *estimated by the mathematicians above are as presented in the following table.:*

Cauchy	*5*
Carmichael and Mason	*4.582*
Montel	*6*
Paul and Bag	*3.497*
Fujii and Kubo	*3.236*
Abu-Omar and Kittaneh	*3.821*
Al-Dolat, Jaradat and Al-Husban	*3.735*

If λ *is a zero of the polynomial* $p(z)$, *then for* $\alpha = 0.8$ *in Theorem 2.3.1, we obtain* $|\lambda| \leq 3.112$, *which is better than all of the above estimations.*

Example 2.3. *Let* $p(z) = z^4 - 2z + 3$. *Then the upper bounds for the zeros of the polynomial* $p(z)$ *estimated by the mathematicians above are as presented in the following table.:*

Cauchy	*4*
Carmichael and Mason	*3.741*
Montel	*5*
Paul and Bag	*0.809*
Fujii and Kubo	*7.49*
Abu-Omar and Kittaneh	*3.491*
Al-Dolat, Jaradat and Al-Husban	*3.436*

If λ *is a zero of the polynomial* $p(z)$, *then for* $\alpha = 0.795$ *in Theorem 2.3.1, we obtain* $|\lambda| \leq 2.816$, *which is better than all of the above estimations except the estimation of Paul and Bag.*

Chapter 3

Hilbert-Schmidt numerical radius

In this chapter, we give new upper and lower bounds for the Hilbert-Schmidt numerical radius. We refine some existing inequalities. Also, we introduce a new norm on $\mathcal{B}_2(\mathcal{H}) \times \mathcal{B}_2(\mathcal{H})$, where $\mathcal{B}_2(\mathcal{H})$ is the Hilbert-Schmidt class. We study basic properties of this norm and prove inequalities involving it. As an application, we deduce a chain of new bounds for the Hilbert-Schmidt numerical radii of 2×2 operator matrices. Connection with the classical Hilbert-Schmidt numerical radius of a single operator are also provided.

3.1 The Hilbert-Schmidt norm

One says that T belongs to Hilbert-Schmidt class $\mathcal{B}_2(\mathcal{H})$, if $\|T\|_2 = (trT^*T)^{\frac{1}{2}} < \infty$, where $\|\cdot\|_2$ is called the Hilbert-Schmidt norm, which is unitarily invariant norm, that is, $\|UTV\|_2 = \|T\|_2$ for any unitary operators $U, V \in \mathcal{B}(\mathcal{H})$.

The next lemma can be found in [15, p. 96], which is called the Cauchy-Schwarz inequality.

Lemma 3.1.1. *Let $T, S \in \mathcal{B}_2(\mathcal{H})$. Then*

$$|tr(TS)| \leq \|T\|_2 \|S\|_2. \tag{3.1}$$

3.2 Hilbert-Schmidt numerical radius inequalities

As we have seen in the first chapter, if $N(\cdot) = \|\cdot\|_2$, then we get the Hilbert-Schmidt numerical radius $w_2(\cdot)$. For $T \in \mathcal{B}_2(\mathcal{H})$, the Hilbert-Schmidt numerical radius is defined as

$$w_2(T) = \sup_{\theta\in\mathbb{R}} \|\mathcal{R}e(e^{i\theta}T)\|_2 \text{ or } w_2(T) = \sup_{\theta\in\mathbb{R}} \|\mathcal{I}m(e^{i\theta}T)\|_2.$$

The following theorem is a characterization of the Hilbert-Schmidt numerical radius, given by Abu-Omar and Kittaneh [5].

Theorem 3.2.1. *Let $T \in \mathcal{B}_2(\mathcal{H})$. Then*

$$w_2^2(T) = \frac{1}{2}\|T\|_2^2 + \frac{1}{2}|trT^2|.$$

Proof. We have

$$\begin{aligned} w_2^2(T) &= \sup_{\theta\in\mathbb{R}} \|\mathcal{R}e(e^{i\theta}T)\|_2^2 \\ &= \sup_{\theta\in\mathbb{R}} tr(\mathcal{R}e(e^{i\theta}T))^2 \\ &= \frac{1}{2}\sup_{\theta\in\mathbb{R}} \left(\|T\|_2^2 + \mathcal{R}e(e^{2i\theta}trT^2)\right) \\ &= \frac{1}{2}\left(\|T\|_2^2 + |trT^2|\right). \end{aligned}$$

□

The following theorem gives an equivalence between the Hilbert-Schmidt numerical radius and the Hilbert-Schmidt norm.

Theorem 3.2.2. *Let $T \in \mathcal{B}_2(\mathcal{H})$. Then*

$$\frac{1}{\sqrt{2}}\|T\|_2 \le w_2(T) \le \|T\|_2. \tag{3.2}$$

Proof. See [5]. □

In the following theorem, we introduce a formula for the Hilbert-Schmidt numerical radius by using the Cartesian decomposition.

Theorem 3.2.3. *Let $T \in \mathcal{B}_2(\mathcal{H})$ have the Cartesian decomposition $T = A + iB$. Then*

$$w_2^2(T) = \frac{1}{2}\left[\|A\|_2^2 + \|B\|_2^2 + \sqrt{\left(\|A\|_2^2 - \|B\|_2^2\right)^2 + 4(trAB)^2}\right].$$

Proof. We have $w_2(T) = \sup_{\theta\in\mathbb{R}} \|A\cos\theta - B\sin\theta\|_2$, by putting $\alpha = \cos\theta$ and $\beta = -\sin\theta$, we obtain

$$\begin{aligned} w_2^2(T) &= \sup_{\alpha^2+\beta^2=1} \|\alpha A + \beta B\|_2^2 \\ &= \sup_{\alpha^2+\beta^2=1} [tr(\alpha A^* + \beta B^*)(\alpha A + \beta B)] \\ &= \sup_{\alpha^2+\beta^2=1} \left[\alpha^2\|A\|_2^2 + \beta^2\|B\|_2^2 + 2\alpha\beta tr(AB)\right] \\ &= \frac{1}{2}\left[\|A\|_2^2 + \|B\|_2^2 + \sqrt{\left(\|A\|_2^2 - \|B\|_2^2\right)^2 + 4(tr(AB))^2}\right]. \end{aligned}$$

□

Corollary 3.2.4. *Let $T \in \mathcal{B}_2(\mathcal{H})$. Then*

$$w_2(T) \le \sqrt{\|\mathcal{R}e(T)\|_2^2 + \|\mathcal{I}m(A)\|_2^2}.$$

Proof. Using the inequality (3.1) and Theorem 3.2.3, we get the desired result. □

Theorem 3.2.5. *Let $T \in \mathcal{B}_2(\mathcal{H})$ have the Cartesian decomposition $T = A + iB$. Then*

$$\frac{1}{4}\|TT^* + T^*T\|_2 + \frac{1}{2}\sqrt{\left(\|A\|_2^2 - \|B\|_2^2\right)^2 + 4(trAB)^2} \le w_2^2(T).$$

Proof. We have

$$\begin{aligned} w_2^2(T) &\ge \frac{1}{2}\left[\|A^2\|_2 + \|B^2\|_2 + \sqrt{\left(\|A\|_2^2 - \|B\|_2^2\right)^2 + 4(trAB)^2}\right] \\ &\ge \frac{1}{2}\left[\|A^2 + B^2\|_2 + \sqrt{\left(\|A\|_2^2 - \|B\|_2^2\right)^2 + 4(trAB)^2}\right]. \end{aligned}$$

Then, we get the desired inequality. □

Theorem 3.2.6. *Let $T \in \mathcal{B}_2(\mathcal{H})$. Then*

$$\frac{1}{2}\|T\|_2^2 + \frac{1}{2}\left|\mathcal{R}e(tr(T^2))\right| \le w_2^2(T). \tag{3.3}$$

Proof. We have $\max\left\{\|\mathcal{R}e(T)\|_2^2, \|\mathcal{I}m(T)\|_2^2\right\} \leq w_2^2(T)$. Then

$$\begin{aligned} w_2^2(T) &\geq \frac{1}{2}\left(\|\mathcal{R}e(T)\|_2^2 + \|\mathcal{I}m(T)\|_2^2 + \left|\|\mathcal{R}e(T)\|_2^2 - \|\mathcal{I}m(T)\|_2^2\right|\right) \\ &= \frac{1}{2}\|T\|_2^2 + \frac{1}{2}\left|\mathcal{R}e(tr(A^2))\right|. \end{aligned}$$

□

Remark 3.2.7. *The inequality* (3.3) *is better than the first inequality in* (3.2)*.*

Theorem 3.2.8. *Let $T \in \mathcal{B}_2(\mathcal{H})$. Then*

$$\frac{1}{2}\|T\|_2 + \frac{1}{2}|\|\mathcal{R}eT\|_2 - \|\mathcal{I}mT\|_2| \leq w_2(T). \tag{3.4}$$

Proof. See [44]. □

3.3 Hilbert-Schmidt numerical radii inequalities for 2×2 operator matrices

In this section, we give new bounds of the Hilbert-Schmidt numerical radii for the diagonal and the off-diagonal parts of 2×2 operator matrices.

In order to give the rest of our results, we need the following lemma.

Lemma 3.3.1. *Let $A, B \in \mathcal{B}_2(\mathcal{H})$. Then*

$$w_2\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) \leq \sqrt{w_2^2(A) + w_2^2(B)}. \tag{3.5}$$

Proof. We have

$$\begin{aligned} w_2^2\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) &= \sup_{\theta\in\mathbb{R}} \left\| \mathcal{R}e\left(e^{i\theta}\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right)\right\|_2^2 \\ &= \sup_{\theta\in\mathbb{R}} \left\| \begin{bmatrix} \mathcal{R}e(e^{i\theta}A) & 0 \\ 0 & Re(e^{i\theta}B) \end{bmatrix}\right\|_2^2 \\ &= \sup_{\theta\in\mathbb{R}} \left(\|\mathcal{R}e(e^{i\theta}A)\|_2^2 + \|\mathcal{R}e(e^{i\theta}B)\|_2^2\right) \\ &\leq w_2^2(A) + w_2^2(B). \end{aligned}$$

□

Our next result yields an improvement of the inequality (3.5).

Theorem 3.3.2. *Let $A, B \in \mathcal{B}_2(\mathcal{H})$. Then*

$$w_2^2\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) \leq \frac{1}{2}\left[w_2^2(A) + w_2^2(B) + \sqrt{(w_2^2(A) - w_2^2(B))^2 + 4\sup_{\theta\in\mathbb{R}} \|\mathcal{R}e(e^{i\theta}A)\|_2^2 \|\mathcal{R}e(e^{i\theta}B)\|_2^2}\right].$$

Proof. We have

$$\begin{aligned} w_2^2\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) &= \sup_{\theta\in\mathbb{R}} \left[\|\mathcal{R}e(e^{i\theta}A)\|_2^2 + \|\mathcal{R}e(e^{i\theta}B)\|_2^2\right] \\ &= \sup_{\theta\in\mathbb{R}} \sup_{\substack{\alpha,\beta\in\mathbb{R} \\ \alpha^2+\beta^2=1}} \left(\alpha\|\mathcal{R}e(e^{i\theta}A)\|_2 + \beta\|\mathcal{R}e(e^{i\theta}B)\|_2\right)^2 \\ &= \sup_{\theta\in\mathbb{R}} \sup_{\substack{\alpha,\beta\in\mathbb{R} \\ \alpha^2+\beta^2=1}} \left(\alpha^2\|\mathcal{R}e(e^{i\theta}A)\|_2^2 + \beta^2\|\mathcal{R}e(e^{i\theta}B)\|_2^2 + 2\alpha\beta\|\mathcal{R}e(e^{i\theta}A)\|_2\|\mathcal{R}e(e^{i\theta}B)\|_2\right) \\ &\leq \sup_{\substack{\alpha,\beta\in\mathbb{R} \\ \alpha^2+\beta^2=1}} \left(\alpha^2 w_2^2(A) + \beta^2 w_2^2(B) + 2\alpha\beta \sup_{\theta\in\mathbb{R}} \|\mathcal{R}e(e^{i\theta}A)\|_2\|\mathcal{R}e(e^{i\theta}B)\|_2\right) \\ &\leq \left\|\begin{bmatrix} w_2^2(A) & \sup_{\theta\in\mathbb{R}} \|\mathcal{R}e(e^{i\theta}A)\|_2\|\mathcal{R}e(e^{i\theta}B)\|_2 \\ \sup_{\theta\in\mathbb{R}} \|\mathcal{R}e(e^{i\theta}A)\|_2\|\mathcal{R}e(e^{i\theta}B)\|_2 & w_2^2(D) \end{bmatrix}\right\| \\ &= \frac{1}{2}\left[w_2^2(A) + w_2^2(B) + \sqrt{(w_2^2(A) - w_2^2(B))^2 + 4\sup_{\theta\in\mathbb{R}} \|\mathcal{R}e(e^{i\theta}A)\|_2^2 \|\mathcal{R}e(e^{i\theta}B)\|_2^2}\right]. \end{aligned}$$

□

Corollary 3.3.3. *Let $A, B \in \mathcal{B}_2(\mathcal{H})$ and $r \geq 2$. Then*

$$w_2^r\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) \leq 2^{\frac{r}{2}-1}\left[w_2^r(A) + w_2^r(B)\right].$$

Proof. Using Lemma 1.12.1 (b), we have

$$\frac{1}{\sqrt{2}} w_2\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) \leq \sqrt{\frac{w_2^2(A) + w_2^2(B)}{2}}.$$

The required result follows by using Lemma 1.12.1 (a). □

Corollary 3.3.4. *Let $A, B \in \mathcal{B}_2(\mathcal{H})$ and $0 < r \leq 2$. Then*

$$w_2^r\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) \leq w_2^r(A) + w_2^r(B).$$

Proof. Using the inequality (3.5), we have

$$w_2\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) \leq \sqrt{w_2^2(A) + w_2^2(B)}.$$

The desired inequality follows by using Lemma 1.12.1 (b). □

Theorem 3.3.5. *Let $A, B \in \mathcal{B}_2(\mathcal{H})$. Then*

$$w_2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) = \frac{1}{\sqrt{2}} \sup_{\theta\in\mathbb{R}} \left\| e^{i\theta}A + e^{-i\theta}B^* \right\|_2.$$

Proof. We have

$$\begin{aligned} w_2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) &= \sup_{\theta\in\mathbb{R}} \left\| \mathcal{R}e\left(e^{i\theta}\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right)\right\|_2 \\ &= \frac{1}{2}\sup_{\theta\in\mathbb{R}} \left\| \begin{bmatrix} 0 & e^{i\theta}A + e^{-i\theta}B^* \\ e^{-i\theta}A^* + e^{i\theta}B & 0 \end{bmatrix} \right\|_2 \\ &= \frac{1}{\sqrt{2}} \sup_{\theta\in\mathbb{R}} \left\| e^{i\theta}A + e^{-i\theta}B^* \right\|_2. \end{aligned}$$

□

Properties 3.3.1. *Let $A, B \in \mathcal{B}_2(\mathcal{H})$. Then*

1. $w_2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) = w_2\left(\begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix}\right)$.

2. $w_2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) = w_2\left(\begin{bmatrix} 0 & A \\ e^{i\theta}B & 0 \end{bmatrix}\right)$.

3. $w_2\left(\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}\right) = \sqrt{2}w_2(A)$.

Proof. See [11]. □

Lemma 3.3.6. *Let $A, B \in \mathcal{B}_2(\mathcal{H})$. Then*

$$\frac{1}{\sqrt{2}}\max\{w_2(A+B), w_2(A-B)\} \leq w_2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leq \frac{w_2(A+B)+w_2(A-B)}{\sqrt{2}}.$$

Proof. See [11]. □

Lemma 3.3.7. *Let $A, B \in \mathcal{B}_2(\mathcal{H})$. Then*

$$w_2(AB) + \mathcal{R}e(trA^*B) \leq w_2^2(A+B).$$

Proof. Using the inequality (3.2), it follows that

$$\begin{aligned} w_2^2(A+B) &\geq \frac{1}{2}\|A+B\|_2^2 \\ &= \frac{1}{2}[\|A\|_2^2 + \|B\|_2^2 + 2\mathcal{R}e(tr(A^*B))] \\ &\geq \|AB\|_2 + \mathcal{R}e(trA^*B) \\ &\geq w_2(AB) + \mathcal{R}e(trA^*B). \end{aligned}$$

□

Corollary 3.3.8. *Let $A, B \in \mathcal{B}_2(\mathcal{H})$. Then*

$$\max\{w_2(AB), w_2(BA)\} \leq w_2^2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right).$$

Proof. The result follows by applying Lemma 3.3.7 and Properties 3.3.1 (1) to the following matrices $\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ B & 0 \end{bmatrix}$. □

Theorem 3.3.9. *Let $A, B \in \mathcal{B}_2(\mathcal{H})$. Then*

$$\frac{1}{\sqrt{2}} \max\{\|A\|_2, \|B\|_2\} + \frac{1}{2\sqrt{2}} \left| \|A + B^*\|_2 - \|A - B^*\|_2 \right| \leq w_2 \left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right). \tag{3.6}$$

Proof. We have

$$\begin{aligned} w_2 \left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) &\geq \max \left\{ \left\| \mathcal{R}e \left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) \right\|_2, \left\| \mathcal{I}m \left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) \right\|_2 \right\} \\ &= \frac{1}{2} \left[\left\| \mathcal{R}e \left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) \right\|_2 + \left\| \mathcal{I}e \left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) \right\|_2 \right] \\ &+ \frac{1}{2} \left| \left\| \mathcal{R}e \left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) \right\|_2 - \left\| \mathcal{I}m \left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) \right\|_2 \right| \\ &= \frac{1}{2\sqrt{2}} \left[\|A + B^*\|_2 + \|A - B^*\|_2 + \left| \|A + B^*\|_2 - \|A - B^*\|_2 \right| \right] \\ &\geq \frac{1}{2\sqrt{2}} \left[2 \max\{\|A\|_2, \|B\|_2\} + \left| \|A + B^*\|_2 - \|A - B^*\|_2 \right| \right] \\ &= \frac{1}{\sqrt{2}} \max\{\|A\|_2, \|B\|_2\} + \frac{1}{2\sqrt{2}} \left| \|A + B^*\|_2 - \|A - B^*\|_2 \right|, \end{aligned}$$

as required. □

Remark 3.3.10. *If we take $B = A$ in the inequality (3.6), we reobtain the inequality (3.4).*

Theorem 3.3.11. *Let $A, B \in \mathcal{B}_2(\mathcal{H})$. Then*

$$\frac{1}{4} \left(\left\| |A^*|^2 + |B|^2 \right\|_2^2 + \left\| |A|^2 + |B^*|^2 \right\|_2^2 \right)^{1/2} + \frac{1}{2} \left| \mathcal{R}e(trAB) \right| \leq w_2^2 \left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right).$$

Proof. We have

$$
\begin{aligned}
w_2^2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) &\geq \max\left\{\left\|\mathcal{R}e\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right)\right\|_2^2, \left\|\mathcal{I}m\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right)\right\|_2^2\right\} \\
&= \frac{1}{2}\left[\left\|\mathcal{R}e\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right)\right\|_2^2 + \left\|\mathcal{I}m\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right)\right\|_2^2\right] \\
&+ \frac{1}{2}\left|\left\|\mathcal{R}e\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right)\right\|_2^2 - \left\|\mathcal{I}m\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right)\right\|_2^2\right|
\end{aligned}
$$

$$
\begin{aligned}
&\geq \frac{1}{2}\left[\left\|\mathcal{R}e^2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) + \mathcal{I}m^2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right)\right\|_2 + |\mathcal{R}e(trAB)|\right] \\
&= \frac{1}{8}\left[\left\|\begin{bmatrix} 2(|A^*|^2 + |B|^2) & 0 \\ 0 & 2(|A|^2 + |B^*|^2) \end{bmatrix}\right\|_2 + 4\,|\mathcal{R}e(trAB)|\right] \\
&= \frac{1}{4}\left(\||A^*|^2 + |C|^2\|_2^2 + \||A|^2 + |B^*|^2\|_2^2\right)^{1/2} + \frac{1}{2}\,|\mathcal{R}e(trAB)|\,.
\end{aligned}
$$

□

Theorem 3.3.12. *Let $A, B \in \mathcal{B}_2(\mathcal{H})$. Then*

$$
\frac{\max\left\{\|A + B^*\|_2, \|A - B^*\|_2\right\}}{\sqrt{2}} \leq w_2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leq \sqrt{\frac{\|A + B^*\|_2^2 + \|A - B^*\|_2^2}{2}}. \quad (3.7)
$$

Proof. Using Theorem 3.3.5, we get

$$
\begin{aligned}
w_2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) &= \frac{1}{\sqrt{2}}\max_{\theta \in \mathbb{R}}\|\cos\theta\,(A + B^*) + \mathrm{i}\sin\theta\,(A - B^*)\|_2 \\
&\leq \sqrt{\frac{\|A + B^*\|_2^2 + \|A - B^*\|_2^2}{2}}.
\end{aligned}
$$

To prove the first inequality, it is sufficient to take $\theta = 0$ and $\theta = \frac{\pi}{2}$. □

Remark 3.3.13. *If A, B are self-adjoint, then the inequalities in (3.7) are also refinement of the second inequalities in Lemma 3.3.6.*

Corollary 3.3.14. *Let $A, B \in \mathcal{B}_2(\mathcal{H})$. Then*

1. $w_2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leq \sqrt{\|A\|_2^2 + \|B\|_2^2}$.

2. $\frac{1}{2}|\|A+B^*\|_2 - \|A-B^*\|_2| + \max\{\|A\|_2, \|B\|_2\} \leq \sqrt{2}w_2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right)$.

Proof. The first inequality follows by using the parallelogram law in the inequalities (3.7)

$$\|A+B^*\|_2^2 + \|A-B^*\|_2^2 = 2\left(\|A\|_2^2 + \|B\|_2^2\right).$$

To prove the second inequality, we have

$$\begin{aligned}
\sqrt{2}w_2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) &\geq \max\left\{\|A+B^*\|_2, \|A-B^*\|_2\right\} \\
&= \frac{1}{2}\left(\|A+B^*\|_2 + \|A-B^*\|_2 + |\|A+B^*\|_2 - \|A-B^*\|_2|\right) \\
&\geq \|A\|_2 + \frac{1}{2}|\|A+B^*\|_2 - \|A-B^*\|_2|.
\end{aligned}$$

Using the same argument yields

$$\sqrt{2}w_2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \geq \|B\|_2 + \frac{1}{2}|\|A+B^*\|_2 - \|A-B^*\|_2|.$$

Therefore, the second inequality is obtained. □

Theorem 3.3.15. *Let $T \in \mathcal{B}_2(\mathcal{H})$ be with the Cartesian decomposition $T = A + iB$. Then*

$$\frac{w_2(T)}{2} \leq \frac{1}{\sqrt{2}}w_2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leq w_2(T). \tag{3.8}$$

Proof. See [11]. □

Theorem 3.3.16. *Let $A, B \in \mathcal{B}_2(\mathcal{H})$. Then*

$$\frac{\max\left\{\|A+\mathrm{i}B^*\|_2, \|A-\mathrm{i}B^*\|_2\right\}}{\sqrt{2}} \leq w_2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leq \sqrt{\frac{\|A+\mathrm{i}B^*\|_2^2 + \|A-\mathrm{i}B^*\|_2^2}{2}}.$$

Proof. The results follows from Theorem 3.3.12 and Proprieties 3.3.1 (2). □

Corollary 3.3.17. *Let $T \in \mathcal{B}_2(\mathcal{H})$ have the Cartesian decomposition $T = A + iB$. Then*

$$\frac{1}{\sqrt{2}}\|T\|_2 \leq w_2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leq \|T\|_2. \tag{3.9}$$

Remark 3.3.18. *It easy to see that the inequalities (3.9) are refinements of the inequalities (3.8).*

Lemma 3.3.19. *Let $A, B \in \mathcal{B}_2(\mathcal{H})$. Then*

$$w_2^2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leq w_2^2(A+B) + w_2^2(A-B). \tag{3.10}$$

Theorem 3.3.20. *Let $\mathcal{T}$ be as described above. Then*

$$w_2\left(\mathcal{T}\right) \leq \sqrt{w_2^2(A) + w_2^2(D) + \|B\|_2^2 + \|C\|_2^2}. \tag{3.11}$$

Proof. We have

$$\begin{aligned} w_2^2\left(\mathcal{T}\right) &= \frac{1}{2}\left(\|\mathcal{T}\|_2^2 + \left|tr\mathcal{T}^2\right|\right) \\ &= \frac{1}{2}\left(\|A\|_2^2 + \|B\|_2^2 + \|C\|_2^2 + \|D\|_2^2 + |tr(A^2)| + |tr(D^2)| + 2|tr(BC)|\right) \\ &= w_2^2(A) + w_2^2(D) + \|B\|_2^2 + \|C\|_2^2. \end{aligned}$$

as required. □

Theorem 3.3.21. *Let $\mathcal{T}$ be as described above and let $0 \leq \alpha \leq 1$. Then*

$$w_2\left(\mathcal{T}\right) \leq \frac{1}{\sqrt{2}}\left[\sqrt{2\alpha^2(w_2^2(A) + w_2^2(D)) + \|B\|_2^2} + \sqrt{2(1-\alpha)^2(w_2^2(A) + w_2^2(D)) + \|C\|_2^2}\right].$$

Proof. We have

$$\begin{aligned}
w_2(\mathcal{T}) &= \frac{1}{2}\sup_{\theta\in\mathbb{R}} \left\| \begin{bmatrix} 2\mathcal{R}e(e^{i\theta}A) & e^{i\theta}B + e^{-i\theta}C^* \\ e^{i\theta}C + e^{-i\theta}B^* & 2\mathcal{R}e(e^{i\theta}D) \end{bmatrix} \right\|_2 \\
&\leq \frac{1}{2}\sup_{\theta\in\mathbb{R}} \left\| \begin{bmatrix} 2\alpha\mathcal{R}e(e^{i\theta}A) & e^{i\theta}B \\ e^{-i\theta}B^* & 2\alpha\mathcal{R}e(e^{i\theta}D) \end{bmatrix} \right\|_2 \\
&\quad + \frac{1}{2}\sup_{\theta\in\mathbb{R}} \left\| \begin{bmatrix} 2(1-\alpha)\mathcal{R}e(e^{i\theta}A) & e^{-i\theta}C^* \\ e^{i\theta}C & 2(1-\alpha)\mathcal{R}e(e^{i\theta}D) \end{bmatrix} \right\|_2 \\
&\leq \frac{1}{\sqrt{2}}\left[\sqrt{2\alpha^2(w_2^2(A)+w_2^2(D)) + \|B\|_2^2} + \sqrt{2(1-\alpha)^2(w_2^2(A)+w_2^2(D)) + \|C\|_2^2}\right].
\end{aligned}$$

□

Remark 3.3.22. *By taking $\alpha = \frac{1}{2}$, we get*

$$w_2(\mathcal{T}) \leq \frac{1}{2}\left[\sqrt{w_2^2(A)+w_2^2(D)+2\|B\|_2^2} + \sqrt{w_2^2(A)+w_2^2(D)+2\|C\|_2^2}\right]. \tag{3.12}$$

Using the concavity of the function $f(t) = t^{\frac{1}{2}}$ on $[0,\infty)$, it follows that the inequality (3.12) is a refinement of the inequality (3.11).

Theorem 3.3.23. *Let $\mathcal{T}$ be as described above. Then*

$$w_2(\mathcal{T}) \leq \min\{\alpha, \beta\},$$

where

$$\begin{aligned}
\alpha = \frac{1}{2}&\left[\sqrt{w_2^2(A+B) + w_2^2(A-B) + w_2^2\left(\begin{bmatrix} 0 & A+B \\ A-B & 0 \end{bmatrix}\right)}\right. \\
&\left. + \sqrt{w_2^2(D+C) + w_2^2(D-C) + w_2^2\left(\begin{bmatrix} 0 & D+C \\ D-C & 0 \end{bmatrix}\right)}\right]
\end{aligned}$$

and

$$\beta = \frac{1}{2}\left[\sqrt{w_2^2(A+C)+w_2^2(A-C)+w_2^2\left(\begin{bmatrix} 0 & A+C \\ A-C & 0\end{bmatrix}\right)}\right.$$
$$\left. + \sqrt{w_2^2(D+B)+w_2^2(D-B)+w_2^2\left(\begin{bmatrix} 0 & D+B \\ D-B & 0\end{bmatrix}\right)}\right].$$

Proof. Let $T_1 = \begin{bmatrix} A & B \\ 0 & 0\end{bmatrix}$ and $V = \frac{1}{\sqrt{2}}\begin{bmatrix} I & -I \\ I & I\end{bmatrix}$ be an unitary operator. Then

$$\begin{aligned}
w_2^2(T_1) &= \sup_{\theta\in\mathbb{R}} \left\|\mathcal{R}e\left(e^{i\theta}\begin{bmatrix} A & B \\ 0 & 0\end{bmatrix}\right)\right\|_2^2 \\
&= \frac{1}{4}\sup_{\theta\in\mathbb{R}} \left\|\begin{bmatrix} 2\mathcal{R}e(e^{i\theta}A) & e^{i\theta}B \\ e^{-i\theta}B^* & 0\end{bmatrix}\right\|_2^2 \\
&= \frac{1}{4}\sup_{\theta\in\mathbb{R}} \left\|V^*\begin{bmatrix} 2\mathcal{R}e(e^{i\theta}A) & e^{i\theta}B \\ e^{-i\theta}B^* & 0\end{bmatrix}V\right\|_2^2 \\
&= \frac{1}{16}\sup_{\theta\in\mathbb{R}} \left\|\begin{bmatrix} 2\mathcal{R}e(e^{i\theta}(A+B)) & e^{i\theta}(B-A)+e^{-i\theta}(-A^*-B^*) \\ e^{i\theta}(-A-B)+e^{-i\theta}(-A^*+B^*) & 2\mathcal{R}e(e^{i\theta}(A-B))\end{bmatrix}\right\|_2^2
\end{aligned}$$

$$= \frac{1}{16}\sup_{\theta\in\mathbb{R}}\left[\|2\mathcal{R}e(e^{i\theta}(A+B))\|_2^2 + \|\mathcal{R}e(e^{i\theta}(A-B))\|_2^2 + 2\|e^{i\theta}(A-B)+e^{-i\theta}(A^*+B^*)\|_2^2\right]$$
$$\leq \frac{1}{4}\left[w_2^2(A+B)+w_2^2(A-B)+w_2^2\left(\begin{bmatrix} 0 & A+B \\ A-B & 0\end{bmatrix}\right)\right].$$

Let $U = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ be a unitary operator. Then

$$\begin{aligned}
w_2(\mathcal{T}) &\leq w_2\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right) + w_2\left(\begin{bmatrix} 0 & 0 \\ C & D \end{bmatrix}\right) \\
&= w_2\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right) + w_2\left(U^* \begin{bmatrix} D & C \\ 0 & 0 \end{bmatrix} U\right) \\
&= w_2\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right) + w_2\left(\begin{bmatrix} D & C \\ 0 & 0 \end{bmatrix}\right) \\
&\leq \frac{1}{2}\sqrt{w_2^2(A+B) + w_2^2(A-B) + w_2^2\left(\begin{bmatrix} 0 & A+B \\ A-B & 0 \end{bmatrix}\right)} \\
&+ \frac{1}{2}\sqrt{w_2^2(D+C) + w_2^2(D-C) + w_2^2\left(\begin{bmatrix} 0 & D+C \\ D-C & 0 \end{bmatrix}\right)} \\
&= \alpha.
\end{aligned}$$

Similarly,

$$\begin{aligned}
w_2(\mathcal{T}^*) &\leq \frac{1}{2}\sqrt{w_2^2(A+C) + w_2^2(A-C) + w_2^2\left(\begin{bmatrix} 0 & A+C \\ A-C & 0 \end{bmatrix}\right)} \\
&+ \frac{1}{2}\sqrt{w_2^2(D+B) + w_2^2(D-B) + w_2^2\left(\begin{bmatrix} 0 & D+B \\ D-B & 0 \end{bmatrix}\right)} \\
&= \beta.
\end{aligned}$$

□

For $T \in \mathcal{B}_2(\mathcal{H})$, define $\gamma(T) = \sqrt{w_2^2(T) + \|T\|_2^2}$.

Corollary 3.3.24. *Let $T, T', S, S' \in \mathcal{B}_2(\mathcal{H})$ be with the Cartesian decompositions $T = A + iB$, $T' = A + iC$, $S = D + iC$ and $S' = D + iB$, respectively. Then*

$$w_2\left(\begin{bmatrix} A & iB \\ iC & D \end{bmatrix}\right) \leq \frac{1}{\sqrt{2}} \min\{\gamma(T) + \gamma(S), \gamma(T') + \gamma(S')\}.$$

Proof. By replacing B by iB and C by iC, respectively, in Theorem 3.3.23, the result follows immediately. □

Now we give lower bounds for $w_2\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right)$. The following two theorems can be found in [27].

Theorem 3.3.25. *Let $A, B \in \mathcal{B}_2(\mathcal{H})$. Then*

$$\frac{1}{\sqrt{2}} \max\{w_2(A+B), w_2(A-B)\} \leq w_2\left(\begin{bmatrix} A & B \\ -A & -B \end{bmatrix}\right). \tag{3.13}$$

Theorem 3.3.26. *Let $A, B \in \mathcal{B}_2(\mathcal{H})$. Then*

a) If A is self-adjoint, then

$$\sqrt{2} \max\{w_2(A), w_2(B)\} \leq w_2\left(\begin{bmatrix} A & -B \\ B & A \end{bmatrix}\right). \tag{3.14}$$

b) If B is self-adjoint, then

$$\frac{1}{4} \max\{w_2(A), w_2(B)\} \leq w_2\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right). \tag{3.15}$$

Theorem 3.3.27. *Let $A, B \in \mathcal{B}_2(\mathcal{H})$. Then*

$$\sqrt{2} \max\{w_2(A), w_2(B)\} \leq w_2\left(\begin{bmatrix} A & B \\ -B & -A \end{bmatrix}\right). \tag{3.16}$$

Proof. See [44]. □

Theorem 3.3.28. *Let $\mathcal{T}$ be as described above. Then*

$$\frac{1}{\sqrt{2}} \max\{w_2(A+D), w_2(A-D), w_2(B+C), w_2(B-C)\} \leq w_2(\mathcal{T}). \tag{3.17}$$

Proof. We have

$$\begin{aligned} w_2\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right) &= \sup_{\theta\in\mathbb{R}} \left\| \begin{bmatrix} Re(e^{i\theta}A) & 0 \\ 0 & Re(e^{i\theta}D) \end{bmatrix} \right\|_2 \\ &= \sup_{\theta\in\mathbb{R}} \sqrt{\|Re(e^{i\theta}A)\|_2^2 + \|Re(e^{i\theta}D)\|_2^2} \\ &\geq \frac{1}{\sqrt{2}} \sup_{\theta\in\mathbb{R}} \left(\|Re(e^{i\theta}A)\|_2 + \|Re(e^{i\theta}D)\|_2\right) \\ &\geq \frac{1}{\sqrt{2}} \sup_{\theta\in\mathbb{R}} \|Re(e^{i\theta}(A+D))\|_2 \\ &= \frac{1}{\sqrt{2}} w_2(A+D). \end{aligned}$$

So,

$$w_2\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right) \geq \frac{1}{\sqrt{2}} w_2(A+D). \tag{3.18}$$

Replacing D by $-D$ in the inequality (3.18), we get

$$w_2\left(\begin{bmatrix} A & 0 \\ 0 & -D \end{bmatrix}\right) = w_2\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right) \geq \frac{1}{\sqrt{2}} w_2(A-D).$$

Hence,

$$w_2\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right) \geq \frac{1}{\sqrt{2}} \max\{w_2(A+D), w_2(A-D)\}.$$

Since $w_2(\cdot)$ is a weakly unitarily invariant norm, we have the pinching inequality

$$w_2\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \geq \max\left\{ w_2\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right), w_2\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)\right\}.$$

Using Lemma 3.3.6, we obtain the required inequality. □

Remark 3.3.29. *1. If we take $C = -A$ and $D = -B$ in inequality (3.17), then we reobtain the inequality (3.13).*

2. Also, if we take $C = -B$ and $D = -A$ in inequality (3.17), then we reobtain the inequality (3.16).

Corollary 3.3.30. *Let $T, S \in \mathcal{B}_2(\mathcal{H})$ be with the Cartesian decompositions $T = A + iD$*

and $S = B + iC$. Then

$$\max\{w_2(T), w_2(S)\} \leq \sqrt{2} w_2 \left(\begin{bmatrix} A & B \\ iC & iD \end{bmatrix} \right).$$

Proof. The result follows by replacing C and D by iC and iD, respectively in the inequality (3.17). □

Corollary 3.3.31. *Let $T \in \mathcal{B}_2(\mathcal{H})$ be with the Cartesian decomposition $T = A + iB$. Then*

$$w_2(T) \leq \sqrt{2} w_2 \left(\begin{bmatrix} A & 0 \\ 0 & iB \end{bmatrix} \right) \leq 2 w_2(T).$$

Proof. By replacing D by iB in the inequality (3.17), we get

$$\frac{1}{\sqrt{2}} w_2(T) \leq w_2 \left(\begin{bmatrix} A & 0 \\ 0 & iB \end{bmatrix} \right).$$

On the other hand, we have

$$\begin{aligned} w_2 \left(\begin{bmatrix} A & 0 \\ 0 & iB \end{bmatrix} \right) &\leq \sqrt{\|A\|_2^2 + \|B\|_2^2} \\ &\leq \sqrt{2} w_2(T). \end{aligned}$$

□

Corollary 3.3.32. *Let $A, T \in \mathcal{B}_2(\mathcal{H})$. Then*

$$\sqrt{2} \max\{w_2(A), w_2(T)\} \leq w_2 \left(\begin{bmatrix} A & -T \\ T & A \end{bmatrix} \right). \tag{3.19}$$

Proof. The inequality (3.19) follows by taking $D = A$, $B = -T$ and $C = T$ in the inequality (3.17). □

Remark 3.3.33. *Note that the inequality (3.19) is a generalization of the inequality (3.14).*

Corollary 3.3.34. *Let $A, B \in \mathcal{B}_2(\mathcal{H})$. Then*

$$\frac{1}{\sqrt{2}} \max\{w_2(A), w_2(B)\} \leq w_2\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right). \tag{3.20}$$

Proof. Taking $C = D = 0$ in the inequality (3.17), the desired inequality is obtained. □

Remark 3.3.35. *Also, note that the inequality (3.20) is an improvement of the inequality (3.15).*

Theorem 3.3.36. *Let $\mathcal{T}$ be as described above. Then '*

$$\frac{1}{\sqrt{2}} \max\{w_2(AB + DC), w_2(AB - DC)\} \leq w_2^2(\mathcal{T}). \tag{3.21}$$

Proof. By applying Lemma 3.3.7 and the first inequality in Lemma 3.3.6, we have

$$\begin{aligned} w_2^2(\mathcal{T}) &\geq w_2\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \\ &= w_2\left(\begin{bmatrix} 0 & AB \\ DC & 0 \end{bmatrix}\right) \\ &\geq \frac{1}{\sqrt{2}} \max\{w_2(AB + DC), w_2(AB - DC)\}. \end{aligned}$$

□

Lemma 3.3.37. *Let $A, B \in \mathcal{B}_2(\mathcal{H})$. Then*

$$w_2\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right) \leq \sqrt{w_2^2(A + B) + w_2^2(A + B)}.$$

Proof. See [11]. □

Proposition 3.3.38. *Let $T = [T_{ij}]$, where $T_{ij} \in \mathcal{B}_2(\mathcal{H})$, for $i, j \in \{1, \cdots, n\}$. Then*

$$\|T\|_2^2 = \|[\|T_{ij}\|_2]\|_2^2 = \sum_{i,j=1}^{n} \|T_{ij}\|_2^2.$$

Theorem 3.3.39. *Let $T = [T_{ij}]$, where $T_{ij} \in \mathcal{B}_2(\mathcal{H})$, for $i, j \in \{1, \cdots, n\}$. Then*

$$w_2(T) \leq w_2([t_{ij}]),$$

where

$$t_{ij} = \begin{cases} w_2(T_{ii}) & if \quad i = j, \\ \dfrac{1}{\sqrt{2}} w_2\left(\begin{bmatrix} 0 & T_{ij} \\ T_{ji} & 0 \end{bmatrix}\right) & if \quad i \neq j. \end{cases}$$

Proof. We have

$$\begin{aligned} \|\mathcal{R}e(e^{i\theta}T)\|_2 &= \left\| \begin{bmatrix} \mathcal{R}e(e^{i\theta}T_{11}) & \frac{1}{2}(e^{i\theta}T_{12} + e^{-i\theta}T_{21}^*) & \dots & \frac{1}{2}(e^{i\theta}T_{1n} + e^{-i\theta}T_{n1}^*) \\ \frac{1}{2}(e^{i\theta}T_{21} + e^{-i\theta}T_{12}^*) & \mathcal{R}e(e^{i\theta}T_{22}) & \dots & \frac{1}{2}(e^{i\theta}T_{2n} + e^{-i\theta}T_{n2}^*) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}(e^{i\theta}T_{n1} + e^{-i\theta}T_{1n}^*) & \frac{1}{2}(e^{i\theta}T_{n2} + e^{-i\theta}T_{2n}^*) & \dots & \mathcal{R}e(e^{i\theta}T_{nn}) \end{bmatrix} \right\|_2 \\ &= \left\| \begin{bmatrix} \|\mathcal{R}e(e^{i\theta}T_{11})\|_2 & \frac{1}{2}\|e^{i\theta}T_{12} + e^{-i\theta}T_{21}^*\|_2 & \dots & \frac{1}{2}\|e^{i\theta}T_{1n} + e^{-i\theta}T_{n1}^*\|_2 \\ \frac{1}{2}\|e^{i\theta}T_{21} + e^{-i\theta}T_{12}^*\|_2 & \|\mathcal{R}e(e^{i\theta}T_{22})\|_2 & \dots & \frac{1}{2}\|e^{i\theta}T_{2n} + e^{-i\theta}T_{n2}^*\|_2 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}\|e^{i\theta}T_{n1} + e^{-i\theta}T_{1n}^*\|_2 & \frac{1}{2}\|e^{i\theta}T_{n2} + e^{-i\theta}T_{2n}^*\|_2 & \dots & \|\mathcal{R}e(e^{i\theta}T_{nn})\|_2 \end{bmatrix} \right\|_2 \\ &\leq \|[t_{ij}]\|_2 \end{aligned}$$

(by the norm monotonicity of matrices with nonnegative entries and by Theorem 3.3.5.)

Now, since the matrix $[t_{ij}]$ is real symmetric, then we have $\|[t_{ij}]\|_2 = w_2([t_{ij}])$.
Hence

$$w_2(T) \leq w_2([t_{ij}]).$$

□

Theorem 3.3.40. *Let $\mathcal{T}$ be as described above. Then*

$$w_2(\mathcal{T}) \leq \sqrt{w_2^2(A) + w_2^2(D) + w_2^2\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)}. \tag{3.22}$$

Proof. By Theorem 3.3.39 and Preposition 3.3.1 (1), we obtain

$$\begin{aligned} w_2(\mathcal{T}) &\le w_2\left(\begin{bmatrix} w_2(A) & \frac{1}{\sqrt{2}}w_2\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \\ \frac{1}{\sqrt{2}}w_2\left(\begin{bmatrix} 0 & C \\ B & 0 \end{bmatrix}\right) & w_2(D) \end{bmatrix}\right) \\ &= w_2(M). \end{aligned}$$

Since M is real symmetric, then $w_2(M) = \|M\|_2$. Hence we get the result directly. $\square$

Remark 3.3.41. *From the first result in Corollary 3.3.14, one can deduce that the inequality (3.22) is sharper than the inequality (3.11).*

3.4 Hilbert-Schmidt numerical radius of a pair of operators

We define the Hilbert-Schmidt numerical radius of a pair of bounded linear operators $A, B \in \mathcal{B}_2(\mathcal{H})$ as follows.

Definition 3.4.1. *Let $A, B \in \mathcal{B}_2(\mathcal{H})$. The Hilbert-Schmidt numerical radius of (A, B) is defined by*

$$w_2(A, B) = \sup_{\theta\in\mathbb{R}} \sqrt{\|\mathcal{R}e(e^{i\theta}A)\|_2^2 + \|\mathcal{R}e(e^{i\theta}B)\|_2^2}.$$

Theorem 3.4.1. *$w_2(\cdot,\cdot)$ is a norm on $\mathcal{B}_2(\mathcal{H}) \times \mathcal{B}_2(\mathcal{H})$.*

Proof. Let $A, B \in \mathcal{B}_2(\mathcal{H})$. It is obvious that $w_2(A, B) \ge 0$. Assume that $w_2(A, B) = 0$. Then $A = B = 0$. Now, let $\lambda \in \mathbb{C}$. Then there exists a $\psi \in \mathbb{R}$ such that $\lambda = |\lambda|e^{i\psi}$. So

$$\begin{aligned} w_2(\lambda A, \lambda B) &= \sup_{\theta\in\mathbb{R}} \sqrt{\|\mathcal{R}e(e^{i\theta}\lambda A)\|_2^2 + \|\mathcal{R}e(e^{i\theta}\lambda B)\|_2^2} \\ &= \sup_{\theta\in\mathbb{R}} \sqrt{|\lambda|^2\|\mathcal{R}e(e^{i(\theta+\psi)}A)\|_2^2 + |\lambda|^2\|\mathcal{R}e(e^{i(\theta+\psi)}B)\|_2^2} \\ &= |\lambda| w_2(A, B). \end{aligned}$$

Let $A_1, A_2, B_1, B_2 \in \mathcal{B}_2(\mathcal{H})$. Then

$$\begin{aligned}
w_2(A_1+A_2, B_1+B_2) &= \sup_{\theta\in\mathbb{R}} \sqrt{\|\mathcal{R}e(e^{i\theta}(A_1+A_2))\|_2^2 + \|\mathcal{R}e(e^{i\theta}(B_1+B_2))\|_2^2} \\
&= \sup_{\theta\in\mathbb{R}} \sup_{\substack{\alpha,\beta\in\mathbb{R} \\ \alpha^2+\beta^2=1}} \big(\alpha\|\mathcal{R}e(e^{i\theta}(A_1+A_2))\|_2 + \beta\|\mathcal{R}e(e^{i\theta}(B_1+B_2))\|_2\big) \\
&\leq \sup_{\theta\in\mathbb{R}} \sup_{\substack{\alpha,\beta\in\mathbb{R} \\ \alpha^2+\beta^2=1}} \big(\alpha\|\mathcal{R}e(e^{i\theta}A_1)\|_2 + \beta\|\mathcal{R}e(e^{i\theta}B_1)\|_2\big) \\
&+ \sup_{\theta\in\mathbb{R}} \sup_{\substack{\alpha,\beta\in\mathbb{R} \\ \alpha^2+\beta^2=1}} \big(\alpha\|\mathcal{R}e(e^{i\theta}A_2)\|_2 + \beta\|\mathcal{R}e(e^{i\theta}B_2)\|_2\big) \\
&= \sup_{\theta\in\mathbb{R}} \sqrt{\|\mathcal{R}e(e^{i\theta}A_1)\|_2^2 + \|\mathcal{R}e(e^{i\theta}B_1)\|_2^2} \\
&+ \sup_{\theta\in\mathbb{R}} \sqrt{\|\mathcal{R}e(e^{i\theta}A_2)\|_2^2 + \|\mathcal{R}e(e^{i\theta}B_2)\|_2^2} \\
&= w_2(A_1, B_1) + w_2(A_2, B_2).
\end{aligned}$$

□

Remark 3.4.2. *Replacing A and B by iA and iB, respectively, in Definition 3.4.1, yields*

$$w_2(A,B) = \sup_{\theta\in\mathbb{R}} \sqrt{\|\mathcal{I}m(e^{i\theta}A)\|_2^2 + \|\mathcal{I}m(e^{i\theta}B)\|_2^2}.$$

It is easy to check the following inequalities, which follow from Definition 3.4.1:

$$\max\{w_2(A), w_2(B)\} \leq w_2(A,B) \leq \sqrt{w_2^2(A) + w_2^2(B)}. \tag{3.23}$$

Theorem 3.4.3. *Let $A, B \in \mathcal{B}_2(\mathcal{H})$. Then*

$$w_2^2(A,B) = \frac{1}{2} w_2^2\left(A+B, A-B\right).$$

Proof. We have

$$\begin{aligned} w_2^2(A,B) &= \sup_{\theta\in\mathbb{R}} \left(\|\mathcal{R}e(e^{i\theta}A)\|_2^2 + \|\mathcal{R}e(e^{i\theta}B)\|_2^2\right) \\ &= \frac{1}{2}\sup_{\theta\in\mathbb{R}} \left(\|\mathcal{R}e(e^{i\theta}(A+B))\|_2^2 + \|\mathcal{R}e(e^{i\theta}(A-B))\|_2^2\right) \\ &\quad \text{(by the parallelogram identity)} \\ &= \frac{1}{2} w_2^2\left(A+B, A-B\right). \end{aligned}$$

□

Using Theorem 3.4.3, the following corollary follows from the inequalities (3.23).

Corollary 3.4.4. *Let* $A, B \in \mathcal{B}_2(\mathcal{H})$*. Then*

$$\frac{1}{2}\max\left\{w_2^2(A+B), w_2^2(A-B)\right\} \le w_2^2(A,B) \le \frac{1}{2}\left(w_2^2(A+B) + w_2^2(A-B)\right).$$

In particular, if $T = A + iB$ is the Cartesian decomposition of T, then replacing B by iB in Corollary 3.4.4, gives the following corollary.

Corollary 3.4.5. *Let* $T \in \mathcal{B}_2(\mathcal{H})$ *have the Cartesian decomposition* $T = A + iB$*. Then*

$$\frac{1}{\sqrt{2}} w_2(T) \le w_2(A, iB) \le w_2(T).$$

The following theorem is a characterization of $w_2(A,B)$.

Theorem 3.4.6. *Let* $A, B \in \mathcal{B}_2(\mathcal{H})$*. Then*

$$w_2^2(A,B) = \frac{1}{2}\left(\|A\|_2^2 + \|B\|_2^2 + \left|tr(A^2+B^2)\right|\right).$$

Proof. We have

$$\begin{aligned} w_2^2(A,B) &= \sup_{\theta\in\mathbb{R}} \left(tr|Re(e^{i\theta}A)|^2 + tr|\mathcal{R}e(e^{i\theta}B)|^2\right) \\ &= \frac{1}{2}\sup_{\theta\in\mathbb{R}} \left(\|A\|_2^2 + \|B\|_2^2 + tr\mathcal{R}e(e^{2i\theta}(A^2+B^2))\right) \\ &= \frac{1}{2}\left(\|A\|_2^2 + \|B\|_2^2 + \left|tr(A^2+B^2)\right|\right). \end{aligned}$$

□

Corollary 3.4.7. *Let $A, B \in \mathcal{B}_2(\mathcal{H})$ be self-adjoint. Then*

$$w_2^2(A,B) = w_2^2(A) + w_2^2(B).$$

Proof. Since A, B are self-adjoint, then A^2, B^2 are positive. Therefore, $|tr(A^2+B^2)| = |trA^2| + |trB^2|$. Hence, the result follows from Theorem 3.4.6 and Theorem 3.2.1. □

Theorem 3.4.8. *Let $A, B \in \mathcal{B}_2(\mathcal{H})$ and $\alpha, \beta \in \mathbb{R}$. Then*

$$\sup_{\alpha^2+\beta^2=1} \max\left\{w_2^2(\alpha A \pm \beta B), w_2^2(\beta A \pm \alpha B)\right\} \leq w_2^2(A,B) \leq \inf_{\alpha^2+\beta^2=1}\left(w_2^2(\alpha A, \beta B) + w_2^2(\beta A, \alpha B)\right).$$

Proof. Assume that $\alpha^2+\beta^2=1$. Then

$$\begin{aligned}
w_2^2(A,B) &= \sup_{\theta\in\mathbb{R}}\left(\|\mathcal{R}e(e^{i\theta}A)\|_2^2 + \|\mathcal{R}e(e^{i\theta}B)\|_2^2\right)\\
&= \sup_{\theta\in\mathbb{R}}\left((\alpha^2+\beta^2)\|\mathcal{R}e(e^{i\theta}A)\|_2^2 + (\alpha^2+\beta^2)\|\mathcal{R}e(e^{i\theta}B)\|_2^2\right)\\
&= \sup_{\theta\in\mathbb{R}}\left(\|\mathcal{R}e(e^{i\theta}\alpha A)\|_2^2 + \|\mathcal{R}e(e^{i\theta}\beta B)\|_2^2 + \|\mathcal{R}e(e^{i\theta}\beta A)\|_2^2 + \|\mathcal{R}e(e^{i\theta}\alpha B)\|_2^2\right)\\
&\leq \sup_{\theta\in\mathbb{R}}\left(\|\mathcal{R}e(e^{i\theta}\alpha A)\|_2^2 + \|Re(e^{i\theta}\beta B)\|_2^2\right) + \sup_{\theta\in\mathbb{R}}\left(\|\mathcal{R}e(e^{i\theta}\beta A)\|_2^2 + \|\mathcal{R}e(e^{i\theta}\alpha B)\|_2^2\right)\\
&= w_2^2(\alpha A, \beta B) + w_2^2(\beta A, \alpha B).
\end{aligned}$$

Hence, we obtain the second inequality.

Also,

$$\begin{aligned}
w_2^2(A,B) &= \sup_{\theta\in\mathbb{R}}\left(\|\mathcal{R}e(e^{i\theta}A)\|_2^2 + \|\mathcal{R}e(e^{i\theta}B)\|_2^2\right)\\
&= \sup_{\theta\in\mathbb{R}}\left(\alpha^2+\beta^2\right)\left(\|\mathcal{R}e(e^{i\theta}A)\|_2^2 + \|\mathcal{R}e(e^{i\theta}B)\|_2^2\right)\\
&\geq \sup_{\theta\in\mathbb{R}}\max\left\{\left(|\alpha|\|\mathcal{R}e(e^{i\theta}A)\|_2 + |\beta|\|\mathcal{R}e(e^{i\theta}B)\|_2\right)^2, \left(|\beta|\|\mathcal{R}e(e^{i\theta}A)\|_2 + |\alpha|\|\mathcal{R}e(e^{i\theta}B)\|_2\right)^2\right\}\\
&\quad \text{(by the Cauchy-Schwarz inequality)}\\
&\geq \sup_{\theta\in\mathbb{R}}\max\left\{\|\alpha\mathcal{R}e(e^{i\theta}A) \pm \beta\mathcal{R}e(e^{i\theta}B)\|_2^2, \|\beta\mathcal{R}e(e^{i\theta}A) \pm \alpha\mathcal{R}e(e^{i\theta}B)\|_2^2\right\}\\
&\geq \sup_{\theta\in\mathbb{R}}\max\{\|\mathcal{R}e(e^{i\theta}(\alpha A \pm \beta B))\|_2^2, \|\mathcal{R}e(e^{i\theta}(\beta A \pm \alpha B))\|_2^2\}\\
&= \max\{w_2^2(\alpha A \pm \beta B), w_2^2(\beta A \pm \alpha B)\}.
\end{aligned}$$

Hence, we get the first inequality. □

Theorem 3.4.9. *Let $A, B \in \mathcal{B}_2(\mathcal{H})$. Then*

$$w_2^2(A,B) \leq \min\{w_2^2(A+B), w_2^2(A-B)\} + 2w_2(A)w_2(B).$$

Proof. We have

$$\|\mathcal{R}e(e^{i\theta}A)\|_2^2 + \|\mathcal{R}e(e^{i\theta}B)\|_2^2 = \|\mathcal{R}e(e^{i\theta}(A-B))\|_2^2 + 2tr\big(\mathcal{R}e(e^{i\theta}A)\mathcal{R}e(e^{i\theta}B)\big).$$

Then, by taking the supremum over $\theta \in \mathbb{R}$, in the above identity, we get

$$\begin{aligned} w_2^2(A,B) &\leq w_2^2(A-B) + \sup_{\theta\in\mathbb{R}} 2\big|tr\big(\mathcal{R}e(e^{i\theta}A)\mathcal{R}e(e^{i\theta}B)\big)\big| \\ &\leq w_2^2(A-B) + 2\sup_{\theta\in\mathbb{R}} \|\mathcal{R}e(e^{i\theta}A)\|_2\|\mathcal{R}e(e^{i\theta}B)\|_2 \\ &\text{(by the inequality (3.1))} \\ &\leq w_2^2(A-B) + 2w_2(A)w_2(B). \end{aligned}$$

Thus,

$$w_2^2(A,B) \leq w_2^2(A-B) + 2w_2(A)w_2(B).$$

By replacing B by $-B$ in the above inequality, we get

$$w_2^2(A,B) \leq w_2^2(A+B) + 2w_2(A)w_2(B).$$

Therefore, we get the required result. □

Corollary 3.4.10. *Let $A, B \in \mathcal{B}_2(\mathcal{H})$. Then*

$$w_2^2\,(A,B) \leq 2\min\left\{w_2^2(A), w_2^2(B)\right\} + w_2(A+B)w_2(A-B).$$

Proof. The result follows from Theorem 3.4.3 and Theorem 3.4.9. □

Theorem 3.4.11. *Let $A, B \in \mathcal{B}_2(\mathcal{H})$. Then*

$$w_2^2\,(A,B) \leq \frac{1}{2}\Big(w_2^2(A)+w_2^2(B)+\sqrt{\big(w_2^2(A)-w_2^2(B)\big)^2 + 4\sup_{\theta\in\mathbb{R}}\|\mathcal{R}e(e^{i\theta}A)\|_2^2\|\mathcal{R}e(e^{i\theta}B)\|_2^2}\Big). \tag{3.24}$$

Proof. Using the same argument of Theorem 3.3.2, we get the result directly. □

Remark 3.4.12. *The inequality (3.24) is an improvement of the second inequality of (3.23).*

Theorem 3.4.13. *Let $A, B \in \mathcal{B}_2(\mathcal{H})$. Then*

a) $w_2^r(A,B) \leq w_2^r(A) + w_2^r(B)$ *for* $1 \leq r \leq 2$.

b) $w_2^r(A,B) \leq 2^{\frac{r}{2}-1}\big(w_2^r(A) + w_2^r(B)\big)$ *for* $2 \leq r < \infty$.

Proof. The case $r = 1$ is obvious. Let $r, s > 1$ be such that $\frac{1}{r} + \frac{1}{s} = 1$. Then

$$\begin{aligned} w_2(A,B) &= \sup_{\theta\in\mathbb{R}} \sqrt{\|\mathcal{R}e(e^{i\theta}A)\|_2^2 + \|\mathcal{R}e(e^{i\theta}B)\|_2^2} \\ &= \sup_{\theta\in\mathbb{R}} \sup_{\substack{\alpha,\beta\in\mathbb{R} \\ \alpha^2+\beta^2=1}} \big(\alpha\|\mathcal{R}e(e^{i\theta}A)\|_2 + \beta\|\mathcal{R}e(e^{i\theta}B)\|_2\big) \\ &\leq \sup_{\substack{\alpha,\beta\in\mathbb{R} \\ \alpha^2+\beta^2=1}} \left(|\alpha|^s + |\beta|^s\right)^{1/s} \big(w_2^r(A) + w_2^r(B)\big)^{1/r} \text{(by Hölder's inequality).} \end{aligned}$$

If $1 \leq r \leq 2$, then $2 \leq s < \infty$, and $\sup\limits_{\substack{\alpha,\beta\in\mathbb{R} \\ \alpha^2+\beta^2=1}} \left(|\alpha|^s + |\beta|^s\right)^{1/s} = 1$.

If $2 \leq r < \infty$, then $1 < s \leq 2$, and $\sup\limits_{\substack{\alpha,\beta\in\mathbb{R} \\ \alpha^2+\beta^2=1}} \left(|\alpha|^s + |\beta|^s\right)^{1/s} = 2^{\frac{1}{2}-\frac{1}{r}}$.

This completes the proof. □

Theorem 3.4.14. *Let $A, B \in \mathcal{B}_2(\mathcal{H})$. Then*

$$\frac{1}{2}\max\big\{w_2^2(A-B) + m_2^2(A+B), w_2^2(A+B) + m_2^2(A-B)\big\} \leq w_2^2(A,B), \tag{3.25}$$

where $m_2(A) = \inf\limits_{\theta\in\mathbb{R}} \|Re(e^{i\theta}A)\|_2$.

Proof. We have

$$\begin{aligned} w_2^2(A,B) &= \sup_{\theta\in\mathbb{R}} \left(\|\mathcal{R}e(e^{i\theta}A)\|_2^2 + \|\mathcal{R}e(e^{i\theta}B)\|_2^2\right) \\ &= \frac{1}{2}\sup_{\theta\in\mathbb{R}} \left(\|\mathcal{R}e(e^{i\theta}(A+B))\|_2^2 + \|\mathcal{R}e(e^{i\theta}(A-B))\|_2^2\right) \\ &\geq \frac{1}{2}\sup_{\theta\in\mathbb{R}} \left(\|\mathcal{R}e(e^{i\theta}(A+B))\|_2^2 + \inf_{\theta\in\mathbb{R}}\|\mathcal{R}e(e^{i\theta}(A-B))\|_2^2\right) \\ &= \frac{1}{2}\left(w_2^2(A+B) + m_2^2(A-B)\right). \end{aligned}$$

A similar argument to the previous one yields

$$w_2^2(A,B) \geq \frac{1}{2}\left(w_2^2(A-B) + m_2^2(A+B)\right).$$

Therefore, we get the desired result. □

Remark 3.4.15. *The inequality (3.25) is a refinement of the first inequality of Corollary (3.4.4).*

Theorem 3.4.16. *Let $A, B \in \mathcal{B}_2(\mathcal{H})$. Then*

$$\frac{1}{4}\left(\|A\|_2^2 + \|B\|_2^2\right) + \frac{1}{2}\left|\mathcal{R}e\left(trAB^*\right)\right| \leq w_2^2(A,B).$$

Proof. From Corollary 3.4.4, we have

$$\begin{aligned} w_2^2(A,B) &\geq \frac{1}{2}\max\left\{w_2^2(A+B), w_2^2(A-B)\right\} \\ &\geq \frac{1}{4}\max\{\|A+B\|_2^2, \|A-B\|_2^2\} \quad \text{(by the first inequality in (3.2))} \\ &= \frac{1}{8}\left(\|A+B\|_2^2 + \|A-B\|_2^2 + \left|\|A+B\|_2^2 - \|A-B\|_2^2\right|\right) \\ &= \frac{1}{4}\left(\|A\|_2^2 + \|B\|_2^2\right) + \frac{1}{2}\left|\mathcal{R}e\left(trAB^*\right)\right|, \end{aligned}$$

as required. □

Theorem 3.4.17. *Let $A, B \in \mathcal{B}_2(\mathcal{H})$. Then*

$$\frac{1}{4}\max\{\|A^2+B^2\|_2, \|AB+BA\|_2\} + \frac{1}{4}\left|\|A\|_2^2 - \|B\|_2^2\right| \leq w_2^2(A,B).$$

Proof. From Theorem 3.4.16, one can deduce that

$$\frac{1}{4}\max\{\|A^2+B^2\|_2, \|A^2-B^2\|_2\} + \frac{1}{2}\big|\mathcal{R}e\big(trAB^*\big)\big| \leq w_2^2(A,B).$$

Now, by Theorem 3.4.3, we get

$$\begin{aligned}
w_2^2(A,B) &\geq \frac{1}{8}\max\{\|(A+B)^2+(A-B)^2\|_2, \|(A+B)^2-(A-B)^2\|_2\} \\
&+ \frac{1}{8}\big|\mathcal{R}e\big(tr(A+B)(A^*-B^*)\big)\big| \\
&= \frac{1}{4}\max\{\|A^2+B^2\|_2, \|AB+BA\|_2\} + \frac{1}{4}\big|\mathcal{R}e\big(tr(AA^*-AB^*+BA^*-BB^*)\big)\big| \\
&= \frac{1}{4}\max\{\|A^2+B^2\|_2, \|AB+BA\|_2\} + \frac{1}{4}\big|\mathcal{R}e\big(\|A\|_2^2-\|B\|_2^2+2iIm(trBA^*)\big)\big| \\
&= \frac{1}{4}\max\{\|A^2+B^2\|_2, \|AB+BA\|_2\} + \frac{1}{4}\left|\|A\|_2^2-\|B\|_2^2\right|.
\end{aligned}$$

□

Theorem 3.4.18. *Let $A, B \in \mathcal{B}_2(\mathcal{H})$. Then*

$$\frac{1}{2}|trAB| + \frac{1}{4}\Big(\max\{|tr(A^2+B^2)|, 2|trAB|\} + \big|w_2^2(A+B)-w_2^2(A-B)\big|\Big) \leq w_2^2(A,B).$$

Proof. From Corollary 3.4.4, we have

$$\begin{aligned}
w_2^2(A,B) &\geq \frac{1}{2}\max\left\{w_2^2(A+B), w_2^2(A-B)\right\} \\
&= \frac{1}{4}\left(w_2^2(A+B)+w_2^2(A-B)\right) + \frac{1}{4}\left|w_2^2(A+B)-w_2^2(A-B)\right| \\
&= \frac{1}{8}\left(\|A+B\|_2^2+\|A-B\|_2^2+|tr(A+B)^2|+|tr(A-B)^2|\right) \\
&+ \frac{1}{4}\left|w_2^2(A+B)-w_2^2(A-B)\right| \quad \text{(by Theorem 3.2.1)} \\
&\geq \frac{1}{4}\big(\|A\|_2^2+\|B\|_2^2+\max\{|tr(A^2+B^2)|, 2|trAB|\}\big) + \frac{1}{4}\left|w_2^2(A+B)-w_2^2(A-B)\right| \\
&\geq \frac{1}{2}\|A\|_2\|B\|_2 + \frac{1}{4}\Big(\max\{|tr(A^2+B^2)|, 2|trAB|\} + \big|w_2^2(A+B)-w_2^2(A-B)\big|\Big) \\
&\geq \frac{1}{2}|trAB| + \frac{1}{4}\Big(\max\{|tr(A^2+B^2)|, 2|trAB|\} + \big|w_2^2(A+B)-w_2^2(A-B)\big|\Big).
\end{aligned}$$

□

Theorem 3.4.19. *Let $A, B \in \mathcal{B}_2(\mathcal{H})$. Then*

$$\max\{w_2^2(A+B)), w_2^2(A-B))\} - 2w_2(A)w_2(B) \leq w_2^2(A,B).$$

Proof. We have

$$\|\mathcal{R}e(e^{i\theta}(A+B))\|_2^2 = \|\mathcal{R}e(e^{i\theta}A)\|_2^2 + \|\mathcal{R}e(e^{i\theta}B)\|_2^2 + 2tr\big(\mathcal{R}e(e^{i\theta}A)\mathcal{R}e(e^{i\theta}B)\big).$$

Then

$$\begin{aligned}\|\mathcal{R}e(e^{i\theta}(A+B))\|_2^2 &\leq \|\mathcal{R}e(e^{i\theta}A)\|_2^2 + \|\mathcal{R}e(e^{i\theta}B)\|_2^2 + 2|tr(\mathcal{R}e(e^{i\theta}A)(\mathcal{R}e(e^{i\theta}B))| \\ &\leq \|\mathcal{R}e(e^{i\theta}A)\|_2^2 + \|\mathcal{R}e(e^{i\theta}B)\|_2^2 + 2\|\mathcal{R}e(e^{i\theta}A)\|_2\|\mathcal{R}e(e^{i\theta}B)\|_2 \\ &\text{(by the inequality (3.1))} \\ &\leq w_2^2(A,B) + 2w_2(A)w_2(B).\end{aligned}$$

By taking the suprumum over $\theta \in \mathbb{R}$, in the above inequality, we get the following inequality

$$w_2^2(A+B) \leq w^2(A,B) + 2w_2(A)w_2(B).$$

By replacing B by $-B$ in the above inequality, we get

$$w_2^2(A-B) \leq w_2^2(A,B) + 2w_2(A)w_2(B).$$

Hence, we get the desired result. □

3.5 Relation between $w_2(\cdot,\cdot)$ and $w_2(\cdot)$

We begin this section with following relation between our new norm $w_2(\cdot,\cdot)$ and $w_2(\cdot)$.

Theorem 3.5.1. *Let $A, B \in \mathcal{B}_2(\mathcal{H})$. Then*

$$w_2(A,B) = w_2\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right). \tag{3.26}$$

Proof. We have

$$\begin{aligned} w_2^2(A,B) &= \sup_{\theta\in\mathbb{R}} \left(\|\mathcal{R}e(e^{i\theta}A)\|_2^2 + \|\mathcal{R}e(e^{i\theta}B)\|_2^2\right) \\ &= \sup_{\theta\in\mathbb{R}} \left\| \begin{bmatrix} \mathcal{R}e(e^{i\theta}A) & 0 \\ 0 & \mathcal{R}e(e^{i\theta}B) \end{bmatrix} \right\|_2^2 \\ &= sup_{\theta\in\mathbb{R}} \left\| \mathcal{R}e\left(e^{i\theta}\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) \right\|_2^2 \\ &= w_2^2\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right). \end{aligned}$$

□

It is clear that the inequality (3.5) can be obtained from Theorem 3.5.1 and the second inequality in (3.23).

Remark 3.5.2. *In view of Theorem 3.4.14 and Theorem 3.5.1, we see that if $A, B \in \mathcal{B}_2(\mathcal{H})$, then*

$$\frac{1}{2}\max\left\{w_2^2(A-B)+m_2^2(A+B), w_2^2(A+B)+m_2^2(A-B)\right\} \leq w_2^2\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right). \quad (3.27)$$

Thus, the inequality (3.27) is an improvement of the inequality of Lemma 3.3.6.

Corollary 3.5.3. *Let $T \in \mathcal{B}_2(\mathcal{H})$ be with the Cartesian decomposition $T = A + iB$. Then*

$$\frac{1}{2}\left(w_2^2(T) + m_2^2(T)\right) \leq w_2^2\left(\begin{bmatrix} A & 0 \\ 0 & iB \end{bmatrix}\right) \leq w_2^2(T). \quad (3.28)$$

Proof. The first inequality follows from replacing B by iB in the inequality (3.27) and the second inequality follows from the second inequality in Corollary 3.4.4. □

Remark 3.5.4. *It is clear that the inequality (3.28) is a refinement of the inequality (3.17).*

Theorem 3.5.5. *Let $\mathcal{T}$ be as described above. Then*

$$\frac{1}{2}\max\{w_2^2(A+D)+m_2^2(A-D), w_2^2(A-D)+m_2^2(A+D), w_2^2(B+C), w_2^2(B-C)\} \leq w_2^2(\mathcal{T}). \quad (3.29)$$

Proof. We have the pinching inequality

$$w_2(\mathcal{T}) \geq \max\left\{ w_2\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right), w_2\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)\right\}.$$

Now, the desired inequality follows from Theorem 3.3.5 and the inequality (3.27). □

Clearly the inequality (3.29) is better than the inequality (3.17).

Corollary 3.5.6. *Let $A, B \in \mathcal{B}_2(\mathcal{H})$. Then*

$$\frac{1}{2}\max\{w_2^2(A) + m_2^2(A), w_2^2(B)\} \leq w_2^2\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right). \tag{3.30}$$

Remark 3.5.7. *The inequality (3.30) is better than the inequality (3.20).*

Theorem 3.5.8. *Let $A, B \in \mathcal{B}_2(\mathcal{H})$. Then*

$$w_2(A, B) = \frac{1}{\sqrt{2}} w_2\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right). \tag{3.31}$$

Proof. We have

$$\begin{aligned} w_2^2(A, B) &= \sup_{\theta \in \mathbb{R}} \left(\|\mathcal{R}e(e^{i\theta} A)\|_2^2 + \|\mathcal{R}e(e^{i\theta} B)\|_2^2 \right) \\ &= \frac{1}{2} \sup_{\theta \in \mathbb{R}} \left\| \begin{bmatrix} \mathcal{R}e(e^{i\theta} A) & \mathcal{R}e(e^{i\theta} B) \\ \mathcal{R}e(e^{i\theta} B) & \mathcal{R}e(e^{i\theta} A) \end{bmatrix} \right\|_2^2 \\ &= \frac{1}{2} \sup_{\theta \in \mathbb{R}} \left\| \mathcal{R}e\left(e^{i\theta} \begin{bmatrix} A & B \\ B & A \end{bmatrix} \right) \right\|_2^2 \\ &= \frac{1}{2} w_2^2\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right). \end{aligned}$$

□

Remark 3.5.9. *If $U = \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ -I & I \end{bmatrix}$, then U is unitary and*

$$U \begin{bmatrix} A & B \\ B & A \end{bmatrix} U^* = \begin{bmatrix} A + B & 0 \\ 0 & A - B \end{bmatrix}.$$

Thus, $w_2\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right) = w_2\left(\begin{bmatrix} A+B & 0 \\ 0 & A-B \end{bmatrix}\right)$. *Using Theorem 3.5.1 and Theorem 3.5.8, we have*

$$w_2^2(A,B) = \frac{1}{2} w_2^2\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right) = \frac{1}{2} w_2^2\left(\begin{bmatrix} A+B & 0 \\ 0 & A-B \end{bmatrix}\right) = \frac{1}{2} w_2^2(A+B, A-B),$$

which gives another proof of Theorem 3.4.3.

It is easy to see that the inequality of Lemma 3.3.37 can be obtained from Theorem 3.5.8 and Corollary 3.4.4.

Remark 3.5.10. *In view of Theorem 3.4.14 and Theorem 3.5.8, we see that if* $A, B \in \mathcal{B}_2(\mathcal{H})$*, then*

$$\max\left\{w_2^2(A-B) + m_2^2(A+B), w_2^2(A+B) + m_2^2(A-B)\right\} \le w_2^2\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right).$$

From Theorem 3.5.1 and Theorem 3.5.8, we obtain the following corollary.

Corollary 3.5.11. *Let* $A, B \in \mathcal{B}_2(\mathcal{H})$*. Then*

$$w_2\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = \frac{1}{\sqrt{2}} w_2\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right).$$

Theorem 3.5.12. *Let* $\mathcal{T}$ *be as described above. Then*

$$\frac{1}{\sqrt{2}} \max\{w_2(AB+DC), w_2(AB-DC)\} + \frac{1}{2}\left|trA^2 + trD^2 + 2trBC\right| \le w_2^2(\mathcal{T}). \quad (3.32)$$

Proof. From Corollary 3.4.4, we have

$$w_2^2(A,B) \le \frac{1}{2}\left(w_2^2(A+B) + w_2^2(A-B)\right). \quad (3.33)$$

Also, by Theorem 3.4.6, we have

$$\begin{aligned} w_2^2(A,B) &= \frac{1}{2}\big(\|A\|_2^2+\|B\|_2^2+|tr(A^2+B^2)|\big) \\ &\geq \|A\|_2\|B\|_2+\frac{1}{2}|tr(A^2+B^2)| \\ &\geq w_2(AB)+\frac{1}{2}|tr(A^2+B^2)|. \end{aligned} \tag{3.34}$$

Now , by combining the inequalities (3.33) and (3.34), we obtain

$$w_2(AB)+\frac{1}{2}\big|tr(A^2+B^2)\big| \leq \frac{1}{2}\left(w_2^2(A+B)+w_2^2(A-B)\right). \tag{3.35}$$

Applying the inequality (3.35) to the operator matrices $\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$ and $\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$, gives

$$\begin{aligned} w_2^2(\mathcal{T}) &= \frac{1}{2}\left(w_2^2(\mathcal{T})+w_2^2\left(\begin{bmatrix} A & -B \\ -C & D \end{bmatrix}\right)\right) \\ &\geq w_2\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)+\frac{1}{2}\left|tr\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}^2+\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}^2\right)\right|, \\ &= w_2\left(\begin{bmatrix} 0 & AB \\ DC & 0 \end{bmatrix}\right)+\frac{1}{2}\left|tr\left(A^2+D^2+2BC\right)\right| \\ &\geq \frac{1}{\sqrt{2}}\max\{w_2(AB+DC),w_2(AB-DC)\}+\frac{1}{2}\left|trA^2+trD^2+2trBC\right| \end{aligned}$$

(by Lemma 3.3.7).

□

It is easy to see that the inequality (3.32) is an improvement of the inequality (3.21). By taking $B=A$ in the inequality (3.34), we get the following corollary, which includes a power inequality for $w_2(\cdot)$.

Corollary 3.5.13. *Let $A\in\mathcal{B}_2(\mathcal{H})$. Then*

$$w_2(A^2)\leq 2w_2^2(A)-|trA^2|.$$

Chapter 4

The p-numerical radius

In this chapter we present some new inequalities for the p-numerical radius for a single operator and product of two operators as well as for the p-numerical radii of 2×2 operator matrices. Also, we improve some existing ones.

4.1 The Schatten p-norm

Let $\mathcal{K}(\mathcal{H})$ be the class of compact operators in $\mathcal{B}(\mathcal{H})$. For a compact operator $T \in \mathcal{K}(\mathcal{H})$ the Schatten p-norm of T is defined by $\|T\|_p = \big(tr|T|^p\big)^{\frac{1}{p}}$, where $1 \leq p \leq \infty$ and $|T| = (T^*T)^{\frac{1}{2}}$. For $0 < p < 1$, $\|\cdot\|_p$ is a quasi-norm (it does not satisfy the triangle inequality). The p-Schatten class in $\mathcal{B}(\mathcal{H})$, denoted by $\mathcal{B}_p(\mathcal{H})$, is defined by

$$\mathcal{B}_p(\mathcal{H}) = \{T \in \mathcal{K}(\mathcal{H}) : \|T\|_p < \infty\}.$$

for $1 \leq p \leq q \leq \infty$, the Schatten p-norm of T satisfies the monotonicity property

$$\|T\|_\infty \leq \|T\|_q \leq \|T\|_p \leq \|T\|_1.$$

The Schatten p-norm satisfies the unitarily invariant norm condition i.e for $T \in \mathcal{B}_p(\mathcal{H})$ and U, V are unitary operators.

$$\|UTV\|_p = \|T\|_p.$$

For $T \in \mathcal{B}_p(\mathcal{H})$, where $0 < p \leq \infty$, we have the following relations :

$$\|T\|_{tp}^t = \||T|^t\|_p = \||T^*|^t\|_p \quad \text{for } t > 0. \tag{4.1}$$

Let $T \in \mathcal{B}_p(\mathcal{H})$, $S \in \mathcal{B}_q(\mathcal{H})$. If $p, q, r \in [1, \infty)$ are such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, then $TS \in \mathcal{B}_r(\mathcal{H})$ and

$$\|TS\|_r \leq \|T\|_p \|S\|_q. \tag{4.2}$$

When $p = \infty$ and $p = 2$ the Schatten p-norms are the operator norm $\|T\| = \sup\limits_{\|x\|=1} \|Tx\|$ and the Hilbert–Schmidt norm $\|T\|_2 = (trT^*T)^{1/2}$, respectively.

Lemma 4.1.1. *[1] Let $A, B \in \mathcal{B}_p(\mathcal{H})$, where $0 < p < \infty$, we have*

$$\left\| \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right\|_p = \left\| \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right\|_p = \begin{cases} (\|A\|_p^p + \|B\|_p^p)^{1/p} & \text{for } 0 < p < \infty \\ \max\{\|A\|, \|B\|\} & \text{for } p = \infty. \end{cases}$$

4.2 The p-numerical radius of operators

As we have seen in the first chapter, if $N(\cdot) = \| \cdot \|_p$, then we get the p-numerical radius $w_p(\cdot)$. For $T \in \mathcal{B}_p(\mathcal{H})$, the p-numerical radius is defined as

$$w_p(T) = \sup_{\theta \in \mathbb{R}} \|\mathcal{R}e(e^{i\theta}T)\|_p \text{ or } w_p(T) = \sup_{\theta \in \mathbb{R}} \|\mathcal{I}m(e^{i\theta}T)\|_p.$$

For $T \in \mathcal{B}_p(\mathcal{H})$, it easy to see that

$$\frac{1}{2}\|T\|_p \leq w_p(T) \leq \|T\|_p. \tag{4.3}$$

It should be mentioned here that Bottazzi and Conde [20] have proved that

$$2^{-\frac{1}{p}}\|T\|_p \leq w_p(T) \leq \|T\|_p \qquad \text{for } 1 \leq p \leq 2, \tag{4.4}$$

and

$$2^{\frac{1}{p}-1}\|T\|_p \leq w_p(T) \leq \|T\|_p \qquad \text{for } 2 \leq p < \infty. \tag{4.5}$$

Theorem 4.2.1. *Let $T \in \mathcal{B}_p(\mathcal{H})$ and $p \geq 2$. Then*

$$w_p^2(T) \leq \frac{1}{4}\|TT^* + T^*T\|_{p/2} + \frac{1}{2}w_{p/2}(T^2). \tag{4.6}$$

Proof. See [20]. □

In the following theorem, we give a lower bound for $w_p(\cdot)$.

Theorem 4.2.2. *Let $T \in \mathcal{B}_p(\mathcal{H})$ and $p \geq 1$. Then*

$$\frac{1}{4}\|T\|_p^2 + \frac{1}{2}\sup_{\psi\in\mathbb{R}}\left|\|\mathcal{R}e(e^{i\psi}T)\|_p^2 - \|\mathcal{I}m(e^{i\psi}T)\|_p^2\right| \leq w_p^2(T). \tag{4.7}$$

Proof. We have $\max\left\{\|\mathcal{R}e(e^{i\psi}T)\|_p^2, \|\mathcal{I}m(e^{i\psi}T)\|_p^2\right\} \leq w_p^2(T)$. Then

$$\begin{aligned}
w_p^2(T) &\geq \frac{1}{2}\left(\|\mathcal{R}e(e^{i\psi}T)\|_p^2 + \|\mathcal{I}m(e^{i\psi}T)\|_p^2\right) + \frac{1}{2}\left|\|\mathcal{R}e(e^{i\psi}T)\|_p^2 - \|\mathcal{I}m(e^{i\psi}T)\|_p^2\right| \\
&\geq \frac{1}{4}\|\mathcal{R}e(e^{i\psi}T) + i\mathcal{I}m(e^{i\psi}T)\|_p^2 + \frac{1}{2}\left|\|\mathcal{R}e(e^{i\psi}T)\|_p^2 - \|\mathcal{I}m(e^{i\psi}T)\|_p^2\right| \\
&= \frac{1}{4}\|T\|_p^2 + \frac{1}{2}\left|\|\mathcal{R}e(e^{i\psi}T)\|_p^2 - \|\mathcal{I}m(e^{i\psi}T)\|_p^2\right|.
\end{aligned}$$

Hence, we obtain the required result. □

Remark 4.2.3. *It is easy to see that the inequality (4.7) is also an improvement of the first inequality in (4.3).*

Theorem 4.2.4. *Let $T \in \mathcal{B}_p(\mathcal{H})$ and $p \geq 1$. Then*

$$\frac{1}{2}\|T\|_p + \frac{1}{2}\sup_{\psi\in\mathbb{R}}\left|\|\mathcal{R}e(e^{i\psi}T)\|_p - \|\mathcal{I}m(e^{i\psi}T)\|_p\right| \leq w_p\left(T\right). \tag{4.8}$$

Proof. We have $\max\left\{\|\mathcal{R}e(e^{i\psi}T)\|_p, \|\mathcal{I}m(e^{i\psi}T)\|_p\right\} \leq w_p(T)$. Then

$$\begin{aligned}
w_p\left(T\right) &\geq \frac{1}{2}\left(\|\mathcal{R}e(e^{i\psi}T)\|_p + \|Im(e^{i\psi}T)\|_p + \left|\|\mathcal{R}e(e^{i\psi}T)\|_p - \|\mathcal{I}m(e^{i\psi}T)\|_p\right|\right) \\
&\geq \frac{1}{2}\left(\|\mathcal{R}e(e^{i\psi}T) + i\mathcal{I}m(e^{i\psi}T)\|_p + \left|\|\mathcal{R}e(e^{i\psi}T)\|_p - \|\mathcal{I}m(e^{i\psi}T)\|_p\right|\right) \\
&= \frac{1}{2}\|T\|_p + \frac{1}{2}\left|\|\mathcal{R}e(e^{i\psi}T)\|_p - \|\mathcal{I}m(e^{i\psi}T)\|_p\right|.
\end{aligned}$$

Therefore, we get the desired inequality. □

Remark 4.2.5. *It is clear that the inequality (4.8) is an improvement of the first inequality in (4.3).*

In [14], Benmakhlouf, Hirzallah and Kittaneh have obtained the following theorem.

Theorem 4.2.6. *Let $T \in \mathcal{B}_p(\mathcal{H})$. Then*

$$w_p^p(T) \leq \|\mathcal{R}e(T)\|_p^p + \|\mathcal{I}m(T)\|_p^p \qquad \text{for } 1 \leq p \leq 2 \tag{4.9}$$

and

$$w_p^p(T) \leq 2^{\frac{p}{2}-1}(\|\mathcal{R}e(T)\|_p^p + \|\mathcal{I}m(T)\|_p^p) \qquad \text{for } 2 \leq p < \infty. \tag{4.10}$$

Proof. It can be shown that another form of the p-numerical radius is that

$$w_p(T) = \sup_{\substack{\alpha,\beta\in\mathbb{R}\\ \alpha^2+\beta^2=1}} \|\alpha\mathcal{R}e(T) + \beta\mathcal{I}m(T)\|_p.$$

In particular, we have

$$w_p(T) = \sup_{\theta\in\mathbb{R}} \|\cos\theta\mathcal{R}e(T) + \sin\theta\mathcal{I}m(T)\|_p.$$

So,

$$\begin{aligned} w_p(T) &\leq \sup_{\theta\in\mathbb{R}} (|\cos\theta|\|\mathcal{R}e(T)\|_p + |\sin\theta|\|\mathcal{I}m(T)\|_p) \\ &\leq \sup_{\theta\in\mathbb{R}} (|\cos\theta|^q + |\sin\theta|^q)^{1/q} \left(\|\mathcal{R}e(T)\|_p^p + \|\mathcal{I}m(T)\|_p^p\right)^{1/p} \\ &\qquad \text{(by Hölder inequality).} \end{aligned}$$

If $1 \leq p \leq 2$, then $2 \leq q \leq \infty$, which implies that $\sup_{\theta\in\mathbb{R}}(\cos^q\theta + \sin^q\theta)^{\frac{1}{q}} = 1$, and so

$$w_p^p(T) \leq \|\mathcal{R}e(T)\|_p^p + \|\mathcal{I}m(T)\|_p^p,$$

while if $2 \leq p < \infty$, then $1 < q \leq 2$, which implies that $\sup_{\theta\in\mathbb{R}}(\cos^q\theta + \sin^q\theta)^{\frac{1}{q}} = 2^{\frac{1}{q}-\frac{1}{2}} = 2^{\frac{1}{2}-\frac{1}{p}}$, and so

$$w_p^p(T) \leq 2^{\frac{1}{2}-\frac{1}{p}} \left(\|\mathcal{R}e(T)\|_p^p + \|\mathcal{I}m(T)\|_p^p\right).$$

□

4.3 The p-numerical radius inequalities for products of operators

To give our results, we need the following lemmas which we can be found in [37], [38] and [12], respectively.

Lemma 4.3.1. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be such that AB is self-adjoint and $p \geq 1$. Then*

$$\|AB\|_p \leq \|Re(BA)\|_p$$

Lemma 4.3.2. *Let $A, B \in \mathcal{B}_p(\mathcal{H})$ and $p > 0$. Then*

$$\left\| \begin{bmatrix} A & B \\ B & A \end{bmatrix} \right\|_p = \left(\|A+B\|_p^p + \|A-B\|_p^p\right)^{\frac{1}{p}}.$$

Lemma 4.3.3. *Let $A, B \in \mathcal{B}_p(\mathcal{H})$ be such that A, B are self-adjoint operators and $p \geq 1$. Then*

$$\||A+B|^r\|_p \leq 2^{r-1}\||A|^r + |B|^r\|_p \quad \textit{for } r \geq 1.$$

Lemma 4.3.4. *Let $A, B \in \mathcal{B}(\mathcal{H})$ and $p \geq 1$. Then*

$$\|AB\|_p \leq \frac{1}{2}\||A|^2 + |B^*|^2\|_p.$$

Proof. See [18]. □

Our first theorem can be stated as follows.

Theorem 4.3.5. *Let $A, B \in \mathcal{B}_p(\mathcal{H})$ and $p \geq 1$. Then*

$$w_p^r(AB) \leq \frac{1}{2}\||A|^{2r} + |B^*|^{2r}\|_{p/r} \qquad \textit{for } r \geq 1. \tag{4.11}$$

Proof. We have

$$\begin{aligned} w_p^r(AB) &\leq \|AB\|_p^r \\ &\leq 2^{-r}\||A|^2 + |B^*|^2\|_p^r \quad \text{(by Lemma 4.3.4)} \\ &= 2^{-r}\|\left||A|^2 + |B^*|^2\right|^r\|_{p/r} \\ &\leq \frac{1}{2}\||A|^{2r} + |B^*|^{2r}\|_{p/r} \quad \text{(by Lemma 4.3.3)}. \end{aligned}$$

□

Remark 4.3.6. *If we take $p = \infty$ in the inequality (4.11), then we reobtain*

$$w^r(AB) \leq \frac{1}{2}\||A|^{2r} + |B^*|^{2r}\| \qquad \text{for } r \geq 1.$$

Which is proved by Dragomir [23].

Theorem 4.3.7. *Let $A, B \in \mathcal{B}_p(\mathcal{H})$ and $p \geq 1$. Then*

$$w_p(AB) \leq 2^{\frac{1}{p}-1}w_p(BA) + 2^{\frac{1}{p}-2}\||A|^2 + |B^*|^2\|_p.$$

Proof. We have

$$\begin{aligned}
w_p(AB) &= \sup_{\theta\in\mathbb{R}} \|Re(e^{i\theta}AB)\|_p \\
&= \frac{1}{2}\sup_{\theta\in\mathbb{R}} \|e^{i\theta}AB + e^{-i\theta}B^*A^*\|_p \\
&= \frac{1}{2}\sup_{\theta\in\mathbb{R}} \left\| \begin{bmatrix} e^{i\theta}A & B^* \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B & 0 \\ e^{-i\theta}A^* & 0 \end{bmatrix} \right\|_p \\
&\leq \frac{1}{2}\sup_{\theta\in\mathbb{R}} \left\| Re\left(\begin{bmatrix} B & 0 \\ e^{-i\theta}A^* & 0 \end{bmatrix} \begin{bmatrix} e^{i\theta}A & B^* \\ 0 & 0 \end{bmatrix} \right) \right\|_p \quad \text{(by Lemma 4.3.1)} \\
&= \frac{1}{2}\sup_{\theta\in\mathbb{R}} \left\| Re\left(\begin{bmatrix} e^{i\theta}BA & |B^*|^2 \\ |A|^2 & e^{-i\theta}A^*B^* \end{bmatrix} \right) \right\|_p \\
&= \frac{1}{4}\sup_{\theta\in\mathbb{R}} \left\| \begin{bmatrix} 2Re(e^{i\theta}BA) & |A|^2+|B^*|^2 \\ |A|^2+|B^*|^2 & 2Re(e^{i\theta}BA) \end{bmatrix} \right\|_p \\
&= \frac{1}{4}\sup_{\theta\in\mathbb{R}} \left(\|2Re(e^{i\theta}BA)+|A|^2+|B^*|^2\|_p^p + \|2Re(e^{i\theta}BA)-|A|^2-|B^*|^2\|_p^p \right)^{\frac{1}{p}} \\
&\quad \text{(by Lemma 4.3.2)} \\
&\leq 2^{\frac{1}{p}-2}\sup_{\theta\in\mathbb{R}} \left(\|2Re(e^{i\theta}BA)\|_p + \||A|^2+|B^*|^2\|_p \right) \quad \text{(by the triangle inequality)} \\
&= 2^{\frac{1}{p}-2}\left(2w_p(BA) + \||A|^2+|B^*|^2\|_p \right).
\end{aligned}$$

Hence, we get the desired inequality. □

Corollary 4.3.8. *Let $A, B \in \mathcal{B}_p(\mathcal{H})$ and $p \geq 1$. Then*

$$w_p^r(AB) \leq 2^{\frac{r}{p}-1}w_p^r(BA) + 2^{\frac{r}{p}-2}\||A|^{2r}+|B^*|^{2r}\|_{p/r} \qquad \textit{for } r \geq 1. \tag{4.12}$$

Proof. We have

$$\begin{aligned}
w_p^r(AB) &\leq 2^{\frac{r}{p}-2r}\left(2w_p(BA) + \||A|^2+|B^*|^2\|_p\right)^r \\
&\leq 2^{\frac{r}{p}-r-1}\left(2^r w_p^r(BA) + \||A|^2+|B^*|^2\|_p^r\right) \\
&\quad \big(\text{by the convexity of the function } f(t)=t^r \text{on } [0,\infty)\big) \\
&= 2^{\frac{r}{p}-1}w_p^r(BA) + 2^{\frac{r}{p}-r-1}\|||A|^2+|B^*|^2|^r\|_{p/r} \\
&\leq 2^{\frac{r}{p}-1}w_p^r(BA) + 2^{\frac{r}{p}-2}\||A|^{2r}+|B^*|^{2r}\|_{p/r} \quad \text{(by Lemma 4.3.3)},
\end{aligned}$$

as required. □

Remark 4.3.9. *If we choose $p = \infty$ in the inequality (4.12), then we obtain*

$$w^r(AB) \leq \frac{1}{2} w^r(BA) + \frac{1}{4} \| |A|^{2r} + |B^*|^{2r} \| \qquad \text{for } r \geq 1.$$

Which is given in [45].

Theorem 4.3.10. *Let $A, B \in \mathcal{B}_p(\mathcal{H})$ and $p \geq 1$. Then*

$$w_p(AB) \leq 2^{\frac{1}{p}-1} w_p(BA) + 2^{\frac{2}{p}-2} \left(\|A\|_{2p} \|B\|_{2p} + \|AB\|_p \right).$$

Proof. By replacing A and B by $\sqrt{\frac{\|B\|_{2p}}{\|A\|_{2p}}} A$ and $\sqrt{\frac{\|A\|_{2p}}{\|B\|_{2p}}} B$, respectively, in Theorem 4.4.14, we get

$$w_p(AB) \leq 2^{\frac{1}{p}-2} \left(2 w_p(BA) + \left\| \frac{\|B\|_{2p}}{\|A\|_{2p}} |A|^2 + \frac{\|A\|_{2p}}{\|B\|_{2p}} |B^*|^2 \right\|_p \right).$$

On the other hand, we have

$$\begin{aligned} \left\| \frac{\|B\|_{2p}}{\|A\|_{2p}} |A|^2 + \frac{\|A\|_{2p}}{\|B\|_{2p}} |B^*|^2 \right\|_p &= \left\| \begin{bmatrix} \frac{\|B\|_{2p}}{\|A\|_{2p}} |A|^2 + \frac{\|A\|_{2p}}{\|B\|_{2p}} |B^*|^2 & 0 \\ 0 & 0 \end{bmatrix} \right\|_p \\ &= \left\| \begin{bmatrix} \sqrt{\frac{\|B\|_{2p}}{\|A\|_{2p}}} A^* & \frac{\|A\|_{2p}}{\|B\|_{2p}} B \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{\frac{\|B\|_{2p}}{\|A\|_{2p}}} A & 0 \\ \sqrt{\frac{\|A\|_{2p}}{\|B\|_{2p}}} B^* & 0 \end{bmatrix} \right\|_p \end{aligned}$$

$$\leq \left\| Re\left(\begin{bmatrix}\sqrt{\frac{\|B\|_{2p}}{\|A\|_{2p}}}A & 0\\ \sqrt{\frac{\|A\|_{2p}}{\|B\|_{2p}}}B^* & 0\end{bmatrix}\begin{bmatrix}\sqrt{\frac{\|B\|_{2p}}{\|A\|_{2p}}}A^* & \sqrt{\frac{\|A\|_{2p}}{\|B\|_{2p}}}B\\ 0 & 0\end{bmatrix}\right)\right\|_p$$

(by Lemma 4.3.1)

$$\leq \left\| Re\left(\begin{bmatrix}\frac{\|B\|_{2p}}{\|A\|_{2p}}|A^*|^2 & AB\\ B^*A^* & \frac{\|A\|_{2p}}{\|B\|_{2p}}|B|^2\end{bmatrix}\right)\right\|_p$$

$$= \left\| \begin{bmatrix}\frac{\|B\|_{2p}}{\|A\|_{2p}}|A^*|^2 & AB\\ B^*A^* & \frac{\|A\|_{2p}}{\|B\|_{2p}}|B|^2\end{bmatrix}\right\|_p$$

$$\leq \left\| \begin{bmatrix}\frac{\|B\|_{2p}}{\|A\|_{2p}}|A^*|^2 & 0\\ 0 & \frac{\|A\|_{2p}}{\|B\|_{2p}}|B|^2\end{bmatrix}\right\|_p + \left\| \begin{bmatrix}0 & AB\\ B^*A^* & 0\end{bmatrix}\right\|_p$$

$$= 2^{\frac{1}{p}}\left(\|A\|_{2p}\|B\|_{2p} + \|AB\|_p\right).$$

Hence, we obtain the required inequality. □

The following corollary is an immediate consequence of Theorem 4.3.10. To see this, it is sufficient to use the inequality (4.2).

Corollary 4.3.11. *Let $A, B \in \mathcal{B}_p(\mathcal{H})$ and $p \geq 1$. Then*

$$w_p(AB) \leq 2^{\frac{1}{p}-1}w_p(BA) + 2^{\frac{2}{p}-1}\|A\|_{2p}\|B\|_{2p}. \tag{4.13}$$

Remark 4.3.12. *If we put $p = \infty$ in the inequality (4.13), then we get*

$$w(AB) \leq \frac{1}{2}w(BA) + \frac{1}{2}\|A\|\|B\|,$$

which was already given in [4].

4.4 The p-numerical radii of 2×2 operator matrices

In order to give our results, we need the following lemmas, which are given in [29].

Lemma 4.4.1. *Let $A, B \in \mathcal{B}_p(\mathcal{H})$ and $p > 0$. Then*

$$w_p^p\left(\begin{bmatrix}A & 0\\ 0 & B\end{bmatrix}\right) \leq w_p^p(A) + w_p^p(B) \quad \textit{for } p > 0.$$

Proof. We have

$$\begin{aligned} w_p\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) &= \sup_{\theta\in\mathbb{R}} \left\| \mathcal{R}e\left(e^{i\theta}\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right)\right\|_p \\ &= sup_{\theta\in\mathbb{R}} \left\| \begin{bmatrix} \mathcal{R}e(e^{i\theta}A) & 0 \\ 0 & Re(e^{i\theta}B) \end{bmatrix}\right\|_p \\ &= sup_{\theta\in\mathbb{R}} \left(\|\mathcal{R}e(e^{i\theta}A)\|_p^p + \|\mathcal{R}e(e^{i\theta}B)\|_p^p\right)^{\frac{1}{p}} \ \text{(by Lemma 4.1.1)} \\ &\leq \left(w_p^p(A) + w_p^p(B)\right)^{\frac{1}{p}}, \end{aligned}$$

as required. □

Lemma 4.4.2. *Let $A, B \in \mathcal{B}_p(\mathcal{H})$ and $p > 0$. Then*

$$w_p\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) = 2^{\frac{1}{p}-1} \sup_{\theta\in\mathbb{R}} \left\|e^{i\theta}A + e^{-i\theta}B^*\right\|_p.$$

In particular,

$$w_p\left(\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}\right) = 2^{\frac{1}{p}} w_p(A).$$

Proof. We have

$$\begin{aligned} w_p\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) &= \sup_{\theta\in\mathbb{R}} \left\| \mathcal{R}e\left(e^{i\theta}\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right)\right\|_p \\ &= \frac{1}{2}\sup_{\theta\in\mathbb{R}} \left\| \begin{bmatrix} 0 & e^{i\theta}A + e^{-i\theta}B^* \\ e^{-i\theta}A^* + e^{i\theta}B & 0 \end{bmatrix}\right\|_p \\ &= 2^{\frac{1}{p}-1} \sup_{\theta\in\mathbb{R}} \left\|e^{i\theta}A + e^{-i\theta}B^*\right\|_p, \end{aligned}$$

as required. □

Lemma 4.4.3. *Let $A, B \in \mathcal{B}_p(\mathcal{H})$. Then*

$$2^{\frac{1}{p}-1}\max\{w_p(A+B), w_p(A-B)\} \leq w_p\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leq 2^{\frac{1}{p}-1}\left(w_p(A+B) + w_p(A-B)\right).$$

Lemma 4.4.4. *Let $A, B \in \mathcal{B}_p(\mathcal{H})$. Then*

$$\frac{1}{2}\left\|\begin{bmatrix} 0 & A+B \\ A^*+B^* & 0 \end{bmatrix}\right\|_p \leq w_p\left(\begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix}\right).$$

Proof. We have

$$\begin{aligned} w_p\left(\begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix}\right) &= \sup_{\theta\in\mathbb{R}}\left\|\mathcal{R}e\left(e^{i\theta}\begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix}\right)\right\|_p \\ &= \frac{1}{2}\sup_{\theta\in\mathbb{R}}\left\|\begin{bmatrix} 0 & e^{i\theta}A+e^{-i\theta}B \\ e^{i\theta}B^*+e^{-i\theta}A^* & 0 \end{bmatrix}\right\|_p \\ &\geq \frac{1}{2}\left\|\begin{bmatrix} 0 & A+B \\ A^*+B^* & 0 \end{bmatrix}\right\|_p \text{ (by letting } \theta=0), \end{aligned}$$

as required. □

Theorem 4.4.5. *Let $A, B \in \mathcal{B}_p(\mathcal{H})$. Then*

$$\|A+B\|_p \leq 2^{1-\frac{1}{p}} w_p\left(\begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix}\right). \tag{4.14}$$

Proof. By Lemma 4.4.4, we have

$$\begin{aligned} w_p\left(\begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix}\right) &\geq \frac{1}{2}\left\|\begin{bmatrix} 0 & A+B \\ A^*+B^* & 0 \end{bmatrix}\right\|_p \\ &= \frac{1}{2}\left(\|A+B\|_p^p + \|A^*+B^*\|_p\right)^{\frac{1}{p}} \text{ (by Lemma 4.1.1)} \\ &= 2^{\frac{1}{p}-1}\|A+B\|_p^p, \end{aligned}$$

which is precisely (4.14). □

Theorem 4.4.6. *[10] Let $A, B \in \mathcal{B}_p(\mathcal{H})$. Then*

$$2^{1-\frac{1}{p}} w_p\left(\begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix}\right) \leq \|A\|_p + \|B\|_p. \tag{4.15}$$

Proof. By Lemma 4.4.2, we have

$$w_p\left(\begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix}\right) = 2^{\frac{1}{p}-1} \sup_{\theta \in \mathbb{R}} \left\| e^{i\theta} A + e^{-i\theta} B^* \right\|_p$$
$$\leq 2^{\frac{1}{p}-1} \left(\|A\|_p + \|B\|_p \right) \text{ (by the triangle inequality)},$$

as required. □

Theorem 4.4.7. *Let $A, B \in \mathcal{B}_p(\mathcal{H})$. Then*

$$\|A + B\|_p \leq 2^{1-\frac{1}{p}} w_p\left(\begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix}\right) \leq \|A\|_p + \|B\|_p \tag{4.16}$$

with equality $A = B$.

Proof. The result is based on combining Theorem 4.4.5 and 4.4.6.

If $A = B$, then the boundary terms are obviously equal to $2\|A\|_p$. The middle term is

$$2^{1-\frac{1}{p}} w_p\left(\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}\right) = \left\| \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix} \right\|_p$$
$$= 2^{\frac{1}{p}} \|A\|_p \text{ (by Lemma 4.1.1)}.$$

The equality has accordingly been proven. □

The first result can be stated as follows.

Theorem 4.4.8. *Let $A, B \in \mathcal{B}_p(\mathcal{H})$ and $p \geq 2$. Then*

$$w_p\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leq 2^{\frac{1}{p}-1} \min\left\{ \sqrt{\|AA^* + B^*B\|_{p/2} + 2w_{p/2}(AB)}, \sqrt{\|A^*A + BB^*\|_{p/2} + 2w_{p/2}(BA)} \right\}. \tag{4.17}$$

Proof. Using Lemma 4.4.2, we have

$$\begin{aligned} w_p\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) &= 2^{\frac{1}{p}-1} \sup_{\theta\in\mathbb{R}} \left\| e^{i\theta}A + e^{-i\theta}B^* \right\|_p \\ &= 2^{\frac{1}{p}-1} \sup_{\theta\in\mathbb{R}} \left\| (e^{i\theta}A + e^{-i\theta}B^*)(e^{i\theta}A + e^{-i\theta}B^*)^* \right\|_{p/2}^{1/2} \text{ (by the equality (4.1))} \\ &= 2^{\frac{1}{p}-1} \sup_{\theta\in\mathbb{R}} \left\| AA^* + B^*B + 2Re(e^{i\theta}AB) \right\|_{p/2}^{1/2} \\ &\leq 2^{\frac{1}{p}-1} \sup_{\theta\in\mathbb{R}} \left(\|AA^* + B^*B\|_{p/2} + 2\|Re(e^{i\theta}AB)\|_{p/2} \right)^{1/2} \\ &= 2^{\frac{1}{p}-1} \left(\|AA^* + B^*B\|_{p/2} + 2w_{p/2}(AB) \right)^{1/2}. \end{aligned}$$

The result follows by symmetry. □

Remark 4.4.9. *For $B = A$ in the inequality (4.17), we reobtain the inequality (4.6).*

The following lemma, known as Clarkson's inequalities, which can be found in [19].

Lemma 4.4.10. *Let $T, S \in \mathcal{B}_p(\mathcal{H})$. Then*

a) $2^{p-1}(\|T\|_p^p + \|S\|_p^p) \leq \|T+S\|_p^p + \|T-S\|_p^p \leq 2(\|T\|_p^p + \|S\|_p^p)$
for $1 \leq p \leq 2$.

b) $2(\|T\|_p^p + \|S\|_p^p) \leq \|T+S\|_p^p + \|T-S\|_p^p \leq 2^{p-1}(\|T\|_p^p + \|S\|_p^p)$
for $2 \leq p < \infty$.

In the following theorem, we present further estimations for $w_p\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right)$.

Theorem 4.4.11. *Let $A, B \in \mathcal{B}_p(\mathcal{H})$. Then*

a) $2^{-\frac{1}{p}}(\|A\|_p^p + \|B\|_p^p)^{1/p} \leq w_p\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right)$ *for* $1 < p \leq 2$.

b) $2^{\frac{1}{p}-1}(\|A\|_p^p + \|B\|_p^p)^{1/p} \leq w_p\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right)$ *for* $2 \leq p < \infty$.

Proof. Using Lemma 4.4.2, we have, for $1 < p \leq 2$,

$$\begin{aligned} 2w_p^p\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) &= 2^{1-p}\big(\sup_{\theta\in\mathbb{R}} \|e^{i\theta}A + e^{i\theta}B^*\|_p^p + \sup_{\theta\in\mathbb{R}} \|e^{i\theta}A - e^{i\theta}B^*\|_p^p\big) \\ &\geq 2^{1-p}\sup_{\theta\in\mathbb{R}}(\|e^{i\theta}A + e^{i\theta}B^*\|_p^p + \|e^{i\theta}A - e^{i\theta}B^*\|_p^p) \\ &\geq \|A\|_p^p + \|B\|_p^p \qquad \text{(by Lemma 4.4.10 (a))}. \end{aligned}$$

To obtain the second lower bound for $w_p(T)$, we use an argument similar to that used in the first one. □

Theorem 4.4.12. *Let $\mathcal{T} \in \mathcal{B}_p(\mathcal{H})$ and $p \geq 1$. Then*

$$2^{\frac{1}{p}-1}\max\{w_p(A+D), w_p(A-D), w_p(B+C), w_p(B-C)\} \leq w_p\left(\mathcal{T}\right).$$

Proof. We have

$$\begin{aligned} w_p\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right) &= \sup_{\theta\in\mathbb{R}}\left\|\mathcal{R}e\left(e^{i\theta}\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right)\right\|_p \\ &= \sup_{\theta\in\mathbb{R}}\left\|\begin{bmatrix} \mathcal{R}e(e^{i\theta}A) & 0 \\ 0 & \mathcal{R}e(e^{i\theta}D) \end{bmatrix}\right\|_p \\ &= \sup_{\theta\in\mathbb{R}}\left(\|\mathcal{R}e(e^{i\theta}A)\|_p^p + \|\mathcal{R}e(e^{i\theta}D)\|_p^p\right)^{1/p} \text{(by the equality (4.1.1))} \\ &\geq 2^{\frac{1}{p}-1}\sup_{\theta\in\mathbb{R}}\left(\|\mathcal{R}e(e^{i\theta}A)\|_p + \|\mathcal{R}e(e^{i\theta}D)\|_p\right) \\ &\quad \text{(by the concavity of the function } f(t) = t^{1/p} \text{ on } [0,\infty)) \\ &\geq 2^{\frac{1}{p}-1}\sup_{\theta\in\mathbb{R}}\|\mathcal{R}e(e^{i\theta}(A+D))\|_p \\ &= 2^{\frac{1}{p}-1}w_p(A+D). \end{aligned}$$

Replacing D by $-D$ in the above argument, gives

$$w_p\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right) \geq 2^{\frac{1}{p}-1}w_p(A-D).$$

Thus,

$$2^{\frac{1}{p}-1} \max\{w_p(A+D), w_p(A-D)\} \leq w_p\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right). \tag{4.18}$$

Therefore, the required inequality follows by observing that $\max\left\{w_p\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right), w_p\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)\right\} \leq w_p(\mathcal{T})$ and using the first inequality in Lemma 4.4.3 and the inequality (4.18). □

Remark 4.4.13.

If we take $C = D = 0$ in Theorem 4.4.12, then we get

$$2^{\frac{1}{p}-1} \max\{w_p(A), w_p(B)\} \leq w_p\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right). \tag{4.19}$$

If we make $C = B$ and $D = A$ in Theorem 4.4.12, then we find that

$$2^{\frac{1}{p}} \max\{w_p(A), w_p(B)\} \leq w_p\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right). \tag{4.20}$$

For $p = \infty$ in the inequality (4.20), we get

$$\max\{w(A), w(B)\} \leq w\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right).$$

If we take $C = -A$ and $D = -B$ in Theorem 4.4.12, then we get

$$2^{\frac{1}{p}-1} \max\{w_p(A-B), w_p(A+B)\} \leq w_p\left(\begin{bmatrix} A & B \\ -A & -B \end{bmatrix}\right). \tag{4.21}$$

For $p = \infty$ in the inequality (4.21), we get

$$\frac{1}{2} \max\{w(A-B), w(A+B)\} \leq w\left(\begin{bmatrix} A & B \\ -A & -B \end{bmatrix}\right).$$

If we choose $B=-C$ and $D=A$ in Theorem 4.4.12, then we obtain

$$2^{\frac{1}{p}}\max\{w_p(A),w_p(C)\}\le w_p\left(\begin{bmatrix}A & -C\\ C & A\end{bmatrix}\right).$$

Theorem 4.4.14. *Let $A,B\in\mathcal{B}_p(\mathcal{H})$ and $p\ge 1$. Then*

$$\max\left\{w_p(A),2^{\frac{1}{p}-1}\|B\|_p\right\}\le w_p\left(\begin{bmatrix}A & B\\ 0 & 0\end{bmatrix}\right). \tag{4.22}$$

Proof. Let $T=\begin{bmatrix}A & B\\ 0 & 0\end{bmatrix}$ and $U=\begin{bmatrix}I & 0\\ 0 & -I\end{bmatrix}$. Then U is unitary, and so

$$\begin{aligned} w_p\left(T\right)&=\frac{1}{2}\big(w_p(T)+w_p(U^*TU)\big)\\ &\ge\frac{1}{2}\max\left\{w_p(T+U^*TU),w_p(T-U^*TU)\right\}\\ &=\max\left\{w_p(A),2^{\frac{1}{p}-1}\|B\|_p\right\}, \end{aligned}$$

as required. □

Remark 4.4.15. *The inequality* (4.22) *is an improvement of the inequality* (4.19)

If we take $p=2$ in Theorem 4.4.14, then we get the following corollary.

Corollary 4.4.16. *Let $A,B\in\mathcal{B}_p(\mathcal{H})$. Then*

$$\max\left\{w_2(A),\frac{1}{\sqrt{2}}\|B\|_2\right\}\le w_2\left(\begin{bmatrix}A & B\\ 0 & 0\end{bmatrix}\right). \tag{4.23}$$

Remark 4.4.17. *It is easy to check that the inequality* (4.23) *is better than the inequality* (3.20)*.*

It should be mentioned here that the inequality (4.22) has been obtained in [31] for the case $p=\infty$, i.e., for the usual numerical radius.

Theorem 4.4.18. *Let $\mathcal{T}\in\mathcal{B}_p(\mathcal{H})$ and $p\ge 1$. Then*

$$w_p\left(\mathcal{T}\right)\le\frac{1}{2}\left(w_p^p(A+D)+w_p^p\left(A-D\right)\right)^{1/p}+2^{\frac{1}{p}-1}\left(w_p(B+C)+w_p(B-C)\right).$$

Proof. Let $U = \frac{1}{\sqrt{2}}\begin{bmatrix} I & I \\ -I & I \end{bmatrix}$. Then U is unitary, and so

$$\begin{aligned} w_p\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right) &= w_p\left(U\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}U^*\right) \\ &= \frac{1}{2}w_p\left(\begin{bmatrix} A+D & -A+D \\ -A+D & A+D \end{bmatrix}\right) \\ &\le \frac{1}{2}\left(w_p^p(A+D)+w_p^p\left(A-D\right)\right)^{1/p}. \end{aligned}$$

The result follows by observing that $w_p\left(T\right) \le w_p\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right) + w_p\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)$, and using the second inequality in Lemma 4.4.3. □

Lemma 4.4.19. *Let $T=[T_{ij}]$, where $T_{ij}\in\mathcal{B}_p(\mathcal{H})$ with $i,j=1,\ldots,n$. Then*

1. $\|T\|_p^p \le \sum_{i,j=1}^n \|T_{i,j}\|_p^p$ *for* $1\le p\le 2$.
2. $\|T\|_p^p \le n^{p-2}\sum_{i,j=1}^n \|T_{i,j}\|_p^p$ *for* $2\le p<\infty$.

Proof. See [17]. □

Our last result involves $n\times n$ operator matrices.

Theorem 4.4.20. *Let $\mathcal{T}=[\mathcal{T}_{ij}]$, where $\mathcal{T}_{ij}\in\mathcal{B}_p(\mathcal{H})$ with $i,j=1,\ldots,n$. Then*

(a) $w_p^p(\mathcal{T}) \le \sum_{i,j=1}^n w_p^p([t_{ij}])$ *for* $1\le p\le 2$,

(b) $w_p^p(\mathcal{T}) \le n^{p-2}\sum_{i,j=1}^n w_p^p([t_{ij}])$ *for* $2\le p<\infty$,

where

$$t_{ij} = \begin{cases} \mathcal{T}_{ii} & if \quad i=j \\ 2^{-\frac{1}{p}}\begin{bmatrix} 0 & \mathcal{T}_{ij} \\ \mathcal{T}_{ji} & 0 \end{bmatrix} & if \quad i\ne j. \end{cases}$$

Proof. For $1\le p\le 2$, we have

$$\|\mathcal{R}e(e^{i\theta}\mathcal{T})\|_p^p = \left\|\begin{bmatrix} \mathcal{R}e(e^{i\theta}\mathcal{T}_{11}) & \frac{1}{2}(e^{i\theta}\mathcal{T}_{12}+e^{-i\theta}\mathcal{T}_{21}^*) & \dots & \frac{1}{2}(e^{i\theta}\mathcal{T}_{1n}+e^{-i\theta}\mathcal{T}_{n1}^*) \\ \frac{1}{2}(e^{i\theta}\mathcal{T}_{21}+e^{-i\theta}\mathcal{T}_{12}^*) & \mathcal{R}e(e^{i\theta}\mathcal{T}_{22}) & \dots & \frac{1}{2}(e^{i\theta}\mathcal{T}_{2n}+e^{-i\theta}\mathcal{T}_{n2}^*) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}(e^{i\theta}\mathcal{T}_{n1}+e^{-i\theta}\mathcal{T}_{1n}^*) & \frac{1}{2}(e^{i\theta}\mathcal{T}_{n2}+e^{-i\theta}\mathcal{T}_{2n}^*) & \dots & \mathcal{R}e(e^{i\theta}\mathcal{T}_{nn}) \end{bmatrix}\right\|_p^p$$

$$\leq \sum_{i=1}^{n} \|\mathcal{R}e(e^{i\theta}\mathcal{T}_{ii})\|_p^p + \frac{1}{2^p}\sum_{\substack{j=1\\ i\neq j}}^{n} \|e^{i\theta}\mathcal{T}_{ij}+e^{-i\theta}\mathcal{T}_{ji}^*\|_p^p \quad \text{(by Lemma 4.4.10).}$$

By taking the supremum over $\theta \in \mathbb{R}$ in both sides of the above inequality, and using Lemma 4.4.2, we get the first result. We use a similar argument to obtain the second inequality for $2 \leq p < \infty$. $\square$

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