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Dedication

To those who are not matched by anyone in the universe, to those who have made a great deal, and have given what cannot be returned, to my dear ***parents****.*
To my adorable brother ***Tayeb*** *and my adorable sisters* ***Farida*** *and* ***Chaimaa****.*
To my darling nephews ***Sohaib****,* ***Abderrahmen****.*
To my best friend ***SOLTANI Soumia*** *and* ***AICI Soumia****.*

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Abstract

This thesis focuses on the study of linear or nonlinear dynamic systems either of order fractional or on time scales or both. The aim of study on the one hand, Proof the existence and uniqueness of initial value problem of Riemann-Liouville fractional order on time scales using fixed point theorems. Then, presentation of the exact solution to a general Norton Massagué Model on time scales with examples. On the other hand, we study the stability of SAIQH Models on time scales and we prove that the system is permanent. Finally, we introduce a fractional order SAIRS model and we prove the existence and the positivity of solution, then we discuss the loacal and global stability of the system.

Key words : Fractional order model, existence of solution, dynamic equations on time scales, numerical simulations, stability .

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Acronyms

Acronyms	*Definition*
IVP	Initial Value Problem.
SAIQH	Susceptible, Asymptotic, Infectious, Quarantined, Hospitalized.
SAIRS	Susceptible, Asymptotic, Infected, Recovered, Susceptible.
$COVID-19$	CORONAVIRUS DISEASE 2019.

Notations

$\mathbb{T}$	Time scales.
$\mathbb{R}$	Real numbers.
$\mathbb{Z}$	Integers.
$\mathbb{N}$	Natural numbers.
$\mathbb{N}_0$	Nonnegative integers.
$h\mathbb{Z}$	$hz\,; z \in \mathbb{Z},\ where\, h\, is\, a\, fixed\, real\, number.$
$\mathbb{P}_{a,b}$	$\bigcup_{K=0}^{\infty}[K(a+b), K(a+b)+a]$.
$\mathbb{Q}$	Rational numbers.
$\mathbb{R}\setminus\mathbb{Q}$	Irrational numbers.
$\mathbb{C}$	Complex numbers.
$\sigma(\cdot)$	Forward jump operator.
$\rho(\cdot)$	Backward jump operator.
$\mu(\cdot)$	Graininess function.
$f^{\Delta}(\cdot)$	Hilger (or delta) derivative of $f at\, t\, on\, \mathbb{T}$.
Δ	Usual forward difference operator.
$C_{rd} = c_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T},\mathbb{R})$	Set of rd-continuous function.
$C^1_{rd} = C^1_{rd}(\mathbb{T}) = C^1_{rd}(\mathbb{T},\mathbb{R})$	Set of differentiable functions whose derivative is rd-continuous.
$\mathbb{C}_h$	Hilger complex numbers.
$z \oplus w$	"Circle plus" addition $\oplus$ on $\mathbb{C}_h$.
$\ominus z$	Additive inverse of z under operation $\oplus$ on $\mathbb{C}_h$.
$z \ominus w$	"Circle minus" subtraction $\ominus$ on $\mathbb{C}_h$.
$\mathbb{Z}_h$	$\{z \in \mathbb{C} :\ \frac{-\pi}{h} < \Im(z) \le \frac{\pi}{h}\}$.
$\xi_h(\cdot)$	Cylinder transformation .
$\mathcal{R} = \mathcal{R}(\mathbb{R}) = \mathcal{R}(\mathbb{R},\mathbb{T})$	Set of all regressive and rd-continuous functions.
$\mathcal{R}^+ = \mathcal{R}^+(\mathbb{R}) = \mathcal{R}^+(\mathbb{R},\mathbb{T})$	Set of regressive and rd-continuous functions such that $1+\mu(t)f(t) > 0,$ for all $t \in \mathbb{T}$.

$(p \oplus q)(\cdot)$	Addition on $\mathcal{R}$.
$(p \ominus q)(\cdot)$	Subtraction on $\mathcal{R}$.
$(\alpha \odot p)(\cdot)$	Product on $\mathcal{R}$.
$e_p(\cdot, s)$	Exponential function on time scales.
$\Gamma(\cdot)$	Gamma function.
$C(A)$	Space of continuous function on A.
I_t^α	Fractional integral operator of Riemann-Liouville of order $\alpha > 0$.
${}^{RL}D_t^\alpha$	Fractional derivative operator of Riemann-Liouville of order $\alpha > 0$.
${}^{C}D_t^\alpha$	Caputo fractional derivative operator of order $\alpha > 0$.
$\mathcal{R}_0$	Basic reproduction number.
${}_a^{\mathbb{T}}I_t^\alpha$	Fractional integral of Riemann-Liouville on time scales.
${}_a^{\mathbb{T}}D_t^\alpha$	Fractional derivative of Riemann-Liouville on time scales.

Introduction

Time scales calculus, which has become an essential tool in the study of different fields , was initiated in 1988 by S. Hilger in his Ph.D thesis [26] under the supervision of professor B. Aulbach to unify and generalize discrete and continuous analysis [14, 15].

The key idea behind calculus on time scales is to consider a dynamic equation where the domain of unknown function is a time scales, which is an arbitrary non empty closed subset of the real numbers. By choosing the time scales to be the set of real numbers, the general results obtained from time scales calculus can be applied to ordinary differential equations. On the other hand, by choosing the time scales to be the set of integers, the same general results can be applied to difference equations. This means that time scales calculus provides a unified framework for studying both continuous and discrete dynamical systems. By incorporating these different time scales into the calculus, we can obtain even more general results.

It has a tremendous potential for applications [4, 11, 13, 16, 27, 57]. For example [11], Let $N(t)$ be the number of plants of one particular species at time t in a certain area. By experiments we know that N grows exponentially according to $N' = N$ during the months of April until September. At the beginning of October, all the plants die, but the seeds remain in the ground and start growing again at the beginning of April, with N now being doubled. So we have the following model:

$$N'(t) = N(t) \text{ for all } \ t \in [2k, 2k+1],$$

and

$$N(2k+2) - N(2k+1) = N(2k+1) \text{ for all } \ k = 0, 1, 2, \ldots .$$

The domain of this model is different from $\mathbb{R}$, which it is $\mathbb{W} = \cup_{k=0}^{\infty}[2k, 2k+1]$. It is a closed subset of $\mathbb{R}$. This remark demonstrates the premise for the time scales calculus.

Fractional calculus is a theory of integrals and derivatives of arbitrary real numbers. It is a generalization of classical calculus and therefore retains many basic properties.

The concept of fractional calculus is generally believed to arise from a question posed in 1695 by G. A. l'Hospital to G. W. Leibniz wondering about the meaning of $\frac{d^n y}{dx^n}$ when $n = \frac{1}{2}$? In his response dated September 30, 1695, G. W. Leibniz wrote to l'Hospital as follows:"... it is an apparent paradox, from which one day, useful consequences will be drawn ..." [35]. The first serious attempt to give a logical definition for the fractional derivative is due to Liouville who published nine documents on this subject between (1832) and (1837). Independently, Riemann proposed an approach which was essentially that of Liouville, and it has since been called the "Riemann-Liouville approach". Later, other theories appeared such as that of Grunwald-Leitnikov, Weyl and Caputo.

In recent decades, this theory has begun to affect a significant number of mathematicians and other fields [39, 48]. It can provide effective results in modelling, identifying and controlling systems. Modelling consists of finding a parameterized model whose dynamic behavior approaches that of the system. This representation is used for the simulation of systems with the aim of designing and controlling systems. Recently, fractional derivatives have been used to generalize models.

D. F. M. Torres [58] allows a new concept fractional operators of Riemann -Liouville on time scales, introducing the forward jump operator of time scales in their definition. Using backward jump operator, we found a new definition [63] where we study the existence and uniqueness of solution to the following initial value problem:

$$({}_{t_0}^{\alpha}Dy)(t) = f(t, y(t)), \quad t \in [t_0, t_0 + d] = \mathcal{J} \subseteq \mathbb{T},$$

$$({}_{t_0}^{1-\alpha}Iy)(t_0) = 0,$$

where $\mathbb{T}$ is a given time scales, $0 < \alpha < 1$, $d > 0$, ${}_{t_0}^{\alpha}D$ is the proper (left) Riemann–Liouville fractional derivative operator or order α defined on $\mathbb{T}$ with ρ, ${}_{t_0}^{1-\alpha}I$ is the proper (left) Riemann–Liouville fractional integral operator of order $1-\alpha$ defined on $\mathbb{T}$ with ρ, and function $f : \mathcal{J} \times \mathbb{T} \to \mathbb{R}$ is a right dense continuous function.

The study of dynamical systems on time scales is today an active field of research [17, 12, 31, 50, 53, 54, 57, 62]. In 2006, L. Norton and J. Massagué [45] introduced the general model:

$$\frac{dV(t)}{dt} = aV^{\alpha}(t) - bV(t),$$

where $0 < \alpha < 1$ and a, b are constants of anabolism (growth) and catabolism (death) respectively. Recently, M. Bohner et al [13] studied Solow Models on Time Scales. Motivated by the work mentioned above, we focus on solving the general Norton-Simon-Massagué model on arbitrary time scales $\mathbb{T}$:

$$V^{\Delta}(t) = a(t)V^{\alpha}(t) - b(t)V(t),$$

where $0 < \alpha < 1$ and $a(t) > 0, b(t) > 0$.

M. Khuddush and K. R. Prasad [31] studied the n-species Lotka–Volterra system on time scales and derived sufficient conditions for the existence and uniform asymptotic stability of unique positive almost periodic solution of system. Motivated by aforementioned works, we prove the permanence and positive almost periodic solution of the following SAIQH type model on time scales:

$$\begin{cases} S^{\Delta}(t) = \Lambda + \omega n Q(t) - [\lambda(t)(1-p) + \phi p + \gamma]S^{\sigma}(t), \\ A^{\Delta}(t) = \lambda(t)(1-p)S(t) - [q\nu + \gamma]A^{\sigma}(t), \\ I^{\Delta}(t) = q\nu A(t) - [\delta_1 + \gamma]I^{\sigma}(t), \\ Q^{\Delta}(t) = \phi p S(t) + \delta_1 f_1 I(t) + \delta_2(1 - f_2 - f_3)H(t) - [\omega m + \gamma]Q^{\sigma}(t), \\ H^{\Delta}(t) = \delta_1(1-f_1)I(t) + \eta(1-k)H_{IC}(t) - [\delta_2(1-f_2-f_3) + \delta_2 f_2 + \alpha_1 f_3 + \gamma]\,H^{\sigma}(t), \\ H_{IC}^{\Delta}(t) = \delta_2 f_2 H(t) - [\eta(1-k) + \alpha_2 k + \gamma]H_{IC}^{\sigma}(t), \end{cases}$$

where, for all time $t \geq 0$,

$$\lambda(t) = \frac{\beta\,(l_A A(t) + I(t) + l_H H(t))}{N(t)}$$

is a bounded positive function with

$$N(t) = S(t) + A(t) + I(t) + Q(t) + H(t) + H_{IC}(t),$$

$\beta, l_A, l_H > 0$, and all the other parameters in the model are non negative. In addition,

$$p, 1-p, k, 1-k, q, f_1, 1-f_1, f_2, f_3, 1-f_2-f_3 \in [0,1].$$

For more details on the mathematical model () we refer the reader to [36]. Then, we conclude our works by given an example with numerical simulations to illustrate our theorical results.

Fractional derivatives are one of the new technique that have lately gained attention in scientific research because they effectively embed memory effects in dynamical systems. This idea has been successfully used in mathematical physics, it has also used in other fields more recently [8, 39, 48]. A. Nabti and B. Ghanbari [42] studied a fractional SVEIR epidemic model of Caputo type. Inspired by previous work, we prove the existence and uniqueness of solutions of the following fractional SAIRS epidemic model:

$$\begin{cases} {}^{C}D_t^{\alpha}S(t) = \mu - [\beta_A A(t) + \beta_I I(t)]S(t) - (\mu + \nu + \gamma)S(t) + \gamma(1 - A(t) - I(t)), \\ {}^{C}D_t^{\alpha}A(t) = [\beta_A A(t) + \beta_I I(t)]S(t) - (\eta + \delta_A + \mu)A(t), \\ {}^{C}D_t^{\alpha}I(t) = \eta A(t) - (\delta_I + \mu)I(t), \end{cases}$$

subject to the initial condition

$$S(0) = S_0 \geq 0,\ A(0) = A_0 \geq 0,\ I(0) = I_0 \geq 0,$$

where ${}^{C}D_t^{\alpha}$ is the fractional Caputo derivative having order $0 < \alpha \leq 1$ in order to describe the memory effects in the proposed epidemic model. We assume that the functions $S(t)$, $A(t)$, $I(t)$ and their Caputo fractional derivatives of order $0 < \alpha \leq 1$ are continuous functions. The parameters $\mu, \eta, \beta_A, \beta_I, \nu, \gamma, \delta_A, \delta_I$ in the fractional order SAIRS epidemic model (1) are considered to be positive values. Then, we study the local and global stability using Lyapunov method and we give some remarks with numerical simulations to illustrate our theoretical results.

This thesis is organized as follows:

In chapter one, we will explore the different basic concept related to calculus on time scales such as the differentiation, integral, and the exponential function with its properties. We give examples about the exact solution of the epidemic model and the differential equation. Then, we also present the fixed point theory like the theorem of Banach and Schauder which allow us to show the existence and uniqueness of problems associated with an ordinary differential equation.

In chapter two, we examine several properties of novel Riemann-Liouville fractional operators on time scales. We then establish sufficient conditions for both the existence of a solution and the uniqueness of the solution for a nonlinear Riemann-Liouville fractional initial value problem on time scales by using the fixed point theory.

In chapter three, we give a general nonlinear first-order Norton Massagué model on time scales. Then, we define the Cobb-Douglas production function on time scales and use it to give the solution for the equation that describes the model, the concrete examples are given.

In chapter four, we explore a SAIQH comportmental model on time scales and define the Lyapunov function on time scales which allow a stability asymptotic of the solution. Then, we prove the system's permanence, the existence of solution, and sufficient conditions indicating the existence of a unique almost periodic uniformly asymptotically stable solution for the dynamic system.

In chapter five, we examine a fractional order SAIRS model with vaccination. Then, we show the locally asymptotically stable of the disease-free (resp. endemic) equilibrium if $\mathcal{R}_0 < 1$ (resp. $\mathcal{R}_0 > 1$). Furthermore, if $\mathcal{R}_0$ is less than another threshaold $\mathcal{R}_1$(resp.$\mathcal{R}_0 > 1$ when $\gamma = 0$), we prove that the disease-free(resp. endemic) equilibrium is globally stable. Finally, we give some remarks with numerical simulations to illustrate our threshold results.

Contributions

The following article is extented from this thesis:

N. Zine, Z. Belarbi and B. Bayour, *Exact solution to a general tumor growth model on time scales*, Palestine Journal of Math., Vol 13 (1), 361-370, 2024.

N. Zine, B. Bayour and D. F. M. Torres, *Existence and uniqueness of solutions to proper fractional Riemiann-Liouville value problems on time scales,* Advanced Mathematical Analysis and its Applications. Chapman and Hall/CRC, 2023. 225-235.(Chapter book)

N. Zine, B. Bayour and D. F. M. Torres, *Permanence and Stability of SAIQH Models for COVID-19 on time scales*. (Submitted)

N. Zine, B. Bayour, N. Helal, and M. Helal, *Local and global Stability of fractional SAIRS Models*. (Submitted)

Chapter 1

Preliminaries

In this chapter, we present a basic notions about time scales, fixed point theorems, fractional operators, and stability. The definitions and results presented in this chapter can be found in [4, 14, 15, 17, 19, 21, 20, 24, 30, 33, 41].

1.1 Time Scales

1.1.1 Derivation

Definition 1.1.1. *A time scales $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$.*

Example 1.1.2. *The following sets are time scales:*

1. $\mathbb{R} = \{\text{real numbers}\}$,
2. $\mathbb{Z} = \{\text{integers}\}$,
3. $\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$,
4. $\mathbb{N}_0 = \{0, 1, 2, 3, 4, 5, \ldots\}$,
5. $h\mathbb{Z} = \{hz\,; z \in \mathbb{Z}\}$, *where h is a fixed real number,*
6. $\mathbb{P}_{a,b} = \bigcup_{K=0}^{\infty}[K(a+b), K(a+b)+a]$,
7. $[0,1] \cup [2,3], [0,1] \cup \mathbb{N}$,
8. *The Cantor set.*

Example 1.1.3. *The following sets aren't time scales:*

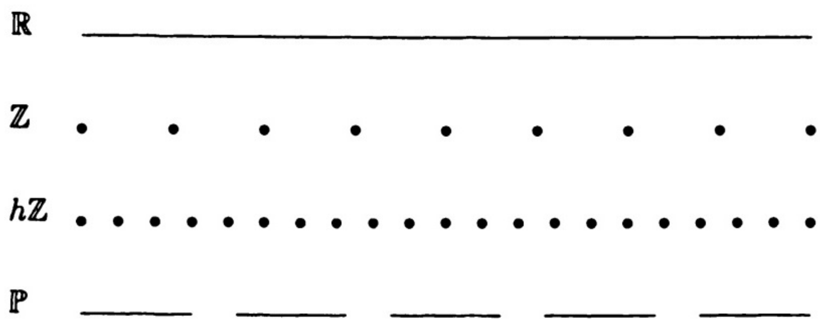


Figure 1.1: Some time scales

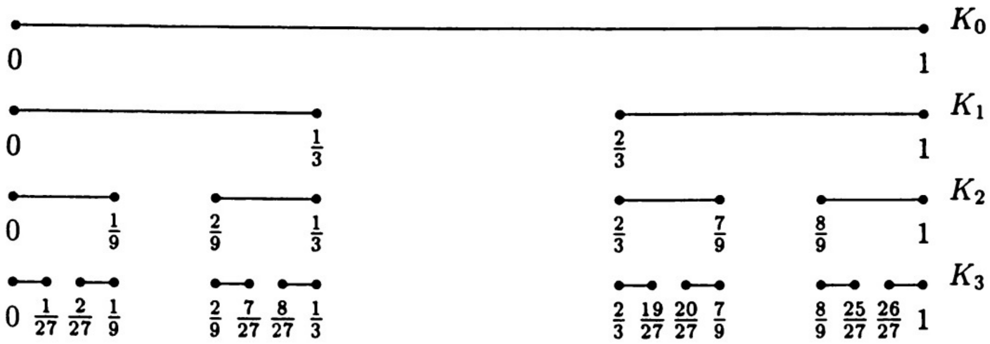


Figure 1.2: The Cantor set

1. $\mathbb{Q} = \{\textit{rational numbers}\}$,
2. $\mathbb{R} \setminus \mathbb{Q} = \{\textit{irrational numbers}\}$,
3. $\mathbb{C} = \{\textit{complex numbers}\}$,
4. $]0, 1[= \{\textit{ open interval between } 0 \textit{ and } 1\}$.

We assume throughout that a time scales $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology.

Definition 1.1.4. *Let $\mathbb{T}$ be a time scales. For $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by*

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ *by*

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

Then, one defines the graininess function $\mu : \mathbb{T} \to [0, +\infty[$ *by*

$$\mu(t) = \sigma(t) - t.$$

In this definition, we put $\inf \emptyset = \sup \mathbb{T}$ *(i.e.* $\sigma(M) = M$ *if* $\mathbb{T}$ *has a maximum* M*) and* $\sup \emptyset = \inf \mathbb{T}$ *(i.e.* $\rho(m) = m$ *if* $\mathbb{T}$ *has a minimum* m*), where* $\emptyset$ *denoted the empty set.*

Example 1.1.5. *Let us consider different time scales* $\mathbb{T}$.

- ◇ *If* $\mathbb{T} = \mathbb{R}$*, we have* $\sigma(t) = t = \rho(t)$*, and* $\mu(t) = 0$.
- ◇ *If* $\mathbb{T} = \mathbb{Z}$*, we have* $\sigma(t) = t + 1$*,* $\rho(t) = t - 1$*, and* $\mu(t) = 1$.
- ◇ *If* $\mathbb{T} = h\mathbb{Z}$*,* $h > 0$*, we have* $\sigma(t) = t + h$*,* $\rho(t) = t - h$*, and* $\mu(t) = h$.
- ◇ *If* $\mathbb{T} = q^{\mathbb{N}}$*,* $q > 1$*, we have* $\sigma(t) = qt$*,* $\rho(t) = \dfrac{t}{q}$*, and* $\mu(t) = (q - 1)t$.
- ◇ *If* $\mathbb{T} = \mathbb{N}_0^2 = \{n^2, n \in \mathbb{N}_0\}$*, we have* $\sigma(t) = (\sqrt{t} + 1)^2$*,* $\rho(t) = (\sqrt{t} - 1)^2$*, and* $\mu(t) = 2\sqrt{t} + 1$.
- ◇ *If* $\mathbb{T} = \{\sqrt{n}, n \in \mathbb{N}_0\}$*, we obtain* $\sigma(t) = \sqrt{t^2 + 1}$*,* $\rho(t) = \sqrt{t^2 - 1}$*, and* $\mu(t) = \sqrt{t^2 + 1} - t$.

Definition 1.1.6. *The operators* σ *and* ρ *allow the following classification of points on time scales* $\mathbb{T}$*:*

- *If* $\sigma(t) > t$*, then we say that* t *is right-scattered, denoted by* ***rs****.*
- *If* $\rho(t) < t$*, then* t *is left-scattered, denoted by* ***ls****.*
- *Points that are right-scattered and left-scattered at the same time are called isolated.*
- *If* $t < \sup \mathbb{T}$ *and* $\sigma(t) = t$*, then* t *is called right-dense, denoted by* ***rd****.*
- *If* $t > \inf \mathbb{T}$ *and* $\rho(t) = t$*, then* t *is called left-dense, denoted by* ***ld****.*
- *Points that are right-dense and left-dense at the same time are called dense.*

The following table illustrates the point classifications:

Nature of the point t	if ...
t is right-scattered	$\sigma(t) > t$
t is left-scattered	$\rho(t) < t$
t is isolated	$\rho(t) < t < \sigma(t)$
t is right-dense	$\sigma(t) = t$
t is left-dense	$\rho(t) = t$
t is dense	$\rho(t) = t = \sigma(t)$

Table Classification of points

Definition 1.1.7. *Let $\mathbb{T}$ be a time scales. We define the set $\mathbb{T}^\kappa$by $\mathbb{T}^\kappa = \mathbb{T} - \{M\}$, if $\mathbb{T}$ has a left-scattered maximum M. Otherwise, $\mathbb{T}^k = \mathbb{T}$.*
In summary,

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T} -]\rho(\sup \mathbb{T}), \sup \mathbb{T}] & \text{if } \sup \mathbb{T} < +\infty, \\ \mathbb{T} & \text{if } \sup \mathbb{T} = +\infty. \end{cases}$$

Example 1.1.8. *Let $\mathbb{T} = [a, b]$.*

- *If $\rho(b) = b$, then $[a, b]^\kappa = [a, b]$.*
- *If $\rho(b) < b$, then $[a, b]^\kappa = [a, b[= [a, \rho(b)]$.*

Example 1.1.9.

1. *For $\mathbb{T} = \left\{\frac{1}{n}, n \in \mathbb{N}\right\} \cup \{0\}$, we have $\sup \mathbb{T} = 1$, and $\rho(\sup \mathbb{T}) = \rho(1) = \dfrac{1}{2}$. Then*

$$\mathbb{T}^\kappa = \mathbb{T} - \left]\frac{1}{2}, 1\right] = \mathbb{T} - \{1\},$$

$$i.e. \quad \mathbb{T}^\kappa = \left\{\frac{1}{n}, n \in \mathbb{N} - \{1\}\right\} \cup \{0\}.$$

2. *For $\mathbb{T} = \{2n, n \in \mathbb{N}\}$, we have $\sup \mathbb{T} = +\infty$, then $\mathbb{T}^\kappa = \mathbb{T}$.*

Definition 1.1.10. *If $f : \mathbb{T} \to \mathbb{R}$ is a function, then we define the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ by*

$$f^\sigma(t) = f(\sigma(t)) \ \text{for all} \ \ t \in \mathbb{T},$$

i.e., $f^\sigma = f \circ \sigma$.

Definition 1.1.11. *Let $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$, then we define $f^\Delta(t)$ to be the number, provided it exists, with the property that given any $\varepsilon > 0$ there is a neighborhood U of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that*

$$\left|[f^\sigma(t) - f(s)] - f^\Delta(t)[\sigma(t) - s]\right| \leq \varepsilon|\sigma(t) - s| \ \text{for all} \ \ s \in U.$$

We call $f^{\Delta}(t)$ the delta (or the Hilger) derivative of f at t.
Moreover, we say that f is delta (or Hilger) differentiable on $\mathbb{T}^{\kappa}$ provided $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^{\kappa}$.

Theorem 1.1.12. *Assume that $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^{k}$.*
If f is Δ-differentiable at t, then

(i) f *is continuous at* t,

(ii) $f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t)$.

Proof. (i) Assume that f is Δ-differentiable at t, and for $s \in \mathbb{T}$, we have

$$\sigma(t) - s = (\sigma(t) - t) + (t - s) = \mu(t) + (t - s). \tag{1.1}$$

Let $0 < \varepsilon < 1$,

$$\varepsilon' = \frac{\varepsilon}{1 + |f^{\Delta}(t)| + 2\mu(t)}.$$

Then $0 < \varepsilon' < 1$.
By definition 1.1.11, there exists a neighborhood U of t such that

$$|f(\sigma(t)) - f(s) - (\sigma(t) - s) f^{\Delta}(t)| < \varepsilon' |\sigma(t) - s| \text{ for all } s \in U.$$

We have

$$|f(t) - f(s)| = |f(t) - f(s) + f(\sigma(t)) - f(\sigma(t)) - f^{\Delta}(t)(\sigma(t) - s) + f^{\Delta}(t)(\sigma(t) - s)|.$$

By using the condition (1.1), we obtain

$$\begin{aligned} |f(t) - f(s)| &= |f(t) - f(s) + f(\sigma(t)) - f(\sigma(t)) - f^{\Delta}(t)(\sigma(t) - s) + f^{\Delta}(t)(\sigma(t) - t + t - s)| \\ &= |f(t) - f(s) + f(\sigma(t)) - f(\sigma(t)) - f^{\Delta}(t)(\sigma(t) - s) + f^{\Delta}(t)\mu(t) + f^{\Delta}(t)(t - s)| \\ &= |[f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)] - [f(\sigma(t)) - f(t) - \mu(t) f^{\Delta}(t)] + (t - s) f^{\Delta}(t)|. \end{aligned}$$

Then, we have for all $s \in U \cap]t - \varepsilon', t + \varepsilon'[$

$$\begin{aligned} |f(t) - f(s)| &\le \varepsilon' |\sigma(t) - s| + \varepsilon' \mu(t) + |t - s||f^{\Delta}(t)| \\ &= \varepsilon' |\mu(t) + (t - s)| + \varepsilon' \mu(t) + |t - s||f^{\Delta}(t)| \\ &\le \varepsilon' \mu(t) + \varepsilon' |t - s| + \varepsilon' \mu(t) + |t - s||f^{\Delta}(t)| \end{aligned}$$

$$\begin{aligned}
&\leq \varepsilon'[\mu(t) + |t-s| + \mu(t) + |f^{\Delta}(t)|] \\
&\leq \varepsilon'[1 + |f^{\Delta}(t)| + 2\mu(t)] \\
&= \varepsilon.
\end{aligned}$$

Hence, f is continuous at t.

(ii)
- If $\sigma(t) = t$, then $\mu(t) = 0$ and we have

$$f(\sigma(t)) = f(t) = f(t) + \mu(t)f^{\Delta}(t).$$

- If $\sigma(t) > t$, and f continuous at t, and

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

Hense,

$$f(\sigma(t)) - f(t) = \mu(t)f^{\Delta}(t).$$

Then,

$$f(\sigma(t)) = f(t) + \mu(t)f^{\Delta}(t).$$

The proof is complete.

■

Example 1.1.13.

1. *If* $\mathbb{T} = \mathbb{R}$, *we have* $\sigma(t) = t$ *then*

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} = f'(t).$$

2. *If* $\mathbb{T} = \mathbb{Z}$, *we have* $\sigma(t) = t + 1$ *then*

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} = \frac{f(t+1) - f(t)}{t + 1 - t} = f(t+1) - f(t) = \Delta f(t),$$

where Δ *is the usual forward difference operator.*

3. *If* $\mathbb{T} = h\mathbb{Z}$, *we have* $\sigma(t) = t + h$ *then*

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} = \frac{f(t+h) - f(t)}{t + h - t} = \frac{f(t+h) - f(t)}{h}.$$

Example 1.1.14. *1. Let $f : \mathbb{T} \longrightarrow \mathbb{R}$ be a function defined by $f(t) = \alpha$ for all $t \in \mathbb{T}$ where $\alpha \in \mathbb{R}$, then*

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} = \lim_{s \to t} \frac{\alpha - \alpha}{\sigma(t) - s} = 0.$$

2. Let $f : \mathbb{T} \longrightarrow \mathbb{R}$ be a function defined by $f(t) = t$, then

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} = \lim_{s \to t} \frac{\sigma(t) - s}{\sigma(t) - s} = 1.$$

3. Let $f : \mathbb{T} \longrightarrow \mathbb{R}$ be a function defined by $f(t) = \sqrt{t}$, then $\forall t \in \mathbb{T}$ we have

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} = \lim_{s \to t} \frac{\sqrt{\sigma(t)} - \sqrt{s}}{\sigma(t) - s} = \lim_{s \to t} \frac{1}{\sqrt{\sigma(t)} + \sqrt{s}} = \frac{1}{\sqrt{\sigma(t)} + \sqrt{t}}.$$

4. Let $\mathbb{T} = \{\sqrt{n}; n \in \mathbb{N}_0\}$ and let $f : \mathbb{T} \longrightarrow \mathbb{R}$ be a function defined by $f(t) = t^2$, then

$$\begin{aligned} f^{\Delta}(t) &= \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} = \lim_{s \to t} \frac{\sigma(t)^2 - s^2}{\sigma(t) - s} \\ &= \frac{(\sigma(\sqrt{n}))^2 - (\sqrt{n})^2}{(\sigma(\sqrt{n})) - \sqrt{n}} \\ &= \frac{(\sqrt{n+1})^2 - n}{\sqrt{n+1} - \sqrt{n}} = \frac{n+1-n}{\sqrt{n+1} - \sqrt{n}} \\ &= \frac{1}{\sqrt{n+1} - \sqrt{n}} = \frac{1}{\sqrt{t^2+1} - t} = \sqrt{t^2+1} + t. \end{aligned}$$

Thus, $f^{\Delta}(t) = \sqrt{t^2+1} + t$.

Theorem 1.1.15. *Assume that $f, g : \mathbb{T} \to \mathbb{R}$ are Δ-differentiable at $t \in \mathbb{T}^k$. Then:*

(i) The sum $f + g : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable at t with

$$(f+g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t).$$

(ii) For any constant α, $\alpha f : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable at t with

$$(\alpha f)^{\Delta}(t) = \alpha f^{\Delta}(t).$$

(iii) The product $fg : \mathbb{T} \to \mathbb{R}$ *is* Δ*-differentiable at* t *with*

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)).$$

(iv) If $f(t)f(\sigma(t)) \neq 0$*, then* $\dfrac{1}{f} : \mathbb{T} \to \mathbb{R}$ *is* Δ*-differentiable at* t *with*

$$\left(\frac{1}{f}\right)^{\Delta} = -\frac{f^{\Delta}(t)}{f(t)f(\sigma(t))}.$$

(v) If $g(t)g(\sigma(t)) \neq 0$*, then* $\dfrac{f}{g} : \mathbb{T} \to \mathbb{R}$ *is* Δ*-differentiable at* t *with*

$$\left(\frac{f}{g}\right)^{\Delta} = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}.$$

Example 1.1.16. *1. The derivative of* t^2 *is*

$$t + \sigma(t),$$

and the derivative of $\dfrac{1}{t}$ *is*

$$-\frac{1}{t\sigma(t)}.$$

2. Let $f : \mathbb{T} \longrightarrow \mathbb{R}$ *be a function, and for all* $t \in \mathbb{T}^{\kappa}$ *we have*

$$\begin{aligned}(f^2)^{\Delta}(t) &= (f.f)^{\Delta}(t)\\ &= f^{\Delta}(t)f(t) + f^{\sigma}(t)f^{\Delta}\\ &= f^{\Delta}(t)(f(t) + f^{\sigma}(t)).\end{aligned}$$

3. Let f*,* g*, and* h *be three functions defined in* $\mathbb{T}$*, then for all* $t \in \mathbb{T}^{\kappa}$ *we have*

$$\begin{aligned}(fgh)^{\Delta}(t) &= (fg)^{\Delta}(t)h(t) + (fg)^{\sigma}(t)h^{\Delta}(t)\\ &= f^{\Delta}(t)g(t)h(t) + f^{\sigma}(t)g^{\Delta}(t)h(t) + f^{\sigma}(t)g^{\sigma}(t)h^{\Delta}(t).\end{aligned}$$

Remark 1.

Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ *be two differentiable functions, then the derivative of* $(f \circ g)$ *is given by*

$$(f \circ g)'(t) = g'(t)f'(g(t)).$$

This rule is not valid for all time scales. Indeed :
We put $\mathbb{T} = \mathbb{Z}$ and let the function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$, $g\colon \mathbb{T} \longrightarrow \mathbb{R}$ defined by

$$f(t) = t^2 \quad et \quad g(t) = 2t.$$

We have

$$f^{\Delta}(t) = 2t \quad et \quad g^{\Delta}(t) = 2.$$

We obtain $(f \circ g)(t) = 4t^2$ and

$$\begin{aligned}(f \circ g)^{\Delta}(t) &= 4(t+1)^2 - 4t^2\\ &= 8t + 4.\end{aligned}$$

On the other hand,

$$\begin{aligned}g^{\Delta}(t) f^{\Delta}(g(t)) &= 2(2(2t))\\ &= 8t.\end{aligned}$$

We conclude that

$$(f \circ g)^{\Delta}(t) \neq g^{\Delta}(t) f^{\Delta}(g(t)), \quad \textit{for all } t \in \mathbb{T}.$$

Theorem 1.1.17.

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ continuously differentiable, and assume that $g : \mathbb{T} \longrightarrow \mathbb{R}$ is Δ-differentiable. Then $f \circ g : \mathbb{T} \longrightarrow \mathbb{R}$ is Δ-differentiable and we have the formula

$$(f \circ g)^{\Delta}(t) = g^{\Delta}(t) \int_0^1 f'[g(t) + h\mu(t) g^{\Delta}(t)]dh$$

holds.

Example 1.1.18. *Let $g : \mathbb{Z} \longrightarrow \mathbb{R}$ and $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by*

$$g(t) = t^2 \quad and \quad f(x) = e^x$$

then

$$g^{\Delta}(t) = \frac{g(\sigma(t)) - g(s)}{\sigma(t) - s} = g(t+1) - g(t) = (t+1)^2 - t^2 = 2t + 1$$

and

$$f'(x) = e^x.$$

Hence, we have by theorem 1.1.17

$$\begin{aligned}
(f \circ g)(t) &= g^{\Delta}(t) \int_0^1 f'(hg(\sigma(t)) + (1-h)g(t))dh \\
&= (2t+1) \int_0^1 e^{t^2 + h(2t+1)} dh \\
&= (2t+1)e^{t^2}. \int_0^1 e^{h(2t+1)} dh \\
&= (2t+1)e^{t^2} \left[\frac{1}{2t+1}.e^{h(2t+1)} \right]_0^1 \\
&= (2t+1)e^{t^2} \left[\frac{1}{2t+1}.e^{2t+1} - \frac{1}{2t+1} \right]_0^1 \\
&= e^{t^2}(e^{2t+1} - 1),
\end{aligned}$$

and the similar if we calculate $(f \circ g)^{\Delta}(t)$ *, we have*

$$\begin{aligned}
(f \circ g)^{\Delta}(t) &= \Delta f[g(t)] = f[g(t+1)] - f[g(t)] \\
&= e^{(t+1)^2} - e^{t^2} \\
&= e^{t^2+2t+1} - e^{2t^2} \\
&= e^{t^2}(e^{2t+1} - 1).
\end{aligned}$$

Corollary 1.1.19. *Let f be a continuous function on $[a,b]$ and it is Δ-differentiable on $[a,b[$. If $f^{\Delta}(t) = 0$ for all $t \in [a,b[$, then f is a constant function on $[a,b]$.*

Corollary 1.1.20. *Let f be a continuous function on $[a,b]$ that is Δ-differentiable on $[a,b[$. Then, f is increasing, decreasing, nondecreasing, and nonincreasing on $[a,b]$ if $f^{\Delta}(t) > 0$, $f^{\Delta}(t) < 0$, $f^{\Delta}(t) \geq 0$, and $f^{\Delta}(t) \leq 0$ for all $t \in [a,b[$, respectively.*

1.1.2 Integration

Definition 1.1.21. *A function $f : \mathbb{T} \to \mathbb{R}$ is called **regulated** provided its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exists (finite) at all left- dense points in $\mathbb{T}$.*

Example 1.1.22. *Let $\mathbb{T} = \mathbb{R}$ and*

$$f(t) = \begin{cases} \frac{1}{t} & \text{for} \quad t \in \mathbb{R}\backslash\{0\}, \\ 0 & \text{for} \quad t = 0. \end{cases}$$

All points of $\mathbb{T}$ are denses and $\lim\limits_{t\to 0^-} f(t) = -\infty$, $\lim\limits_{t\to 0^+} f(t) = +\infty$. Therefore, the function f isn't regulated on $\mathbb{R}$.

Example 1.1.23. *Let $\mathbb{T} = \mathbb{N} \cup [0, 1]$ and*

$$f(t) = \frac{t^2}{t-1}, \quad g(t) = \frac{t}{t+1}, \quad \text{for } t \in \mathbb{T}.$$

We have 1 is left dense, and $\lim\limits_{t\to 1} f(t) = \infty$ then the function f isn't regulated.
On the other hand, we have $\lim\limits_{t\to 1} g(t) = \dfrac{1}{2}$ (exists and finite) then the function g is regulated.

Definition 1.1.24. *A function $f : \mathbb{T} \to \mathbb{R}$ is called **rd-continuous** provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$.*
The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$.

The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are Δ-differentiable and whose derivative is rd-continuous is denoted by $C^1_{rd} = C^1_{rd}(\mathbb{T}) = C^1_{rd}(\mathbb{T}, \mathbb{R})$.

Example 1.1.25. *Let*

$$\mathbb{T} = \{0\} \cup \left\{\frac{1}{n}; n \in \mathbb{N}\right\} \cup \{2\} \cup \left\{2 - \frac{1}{n}; n \in \mathbb{N}\right\}.$$

We define

$$f : \mathbb{T} \longrightarrow [0.2],$$
$$t \longmapsto f(t) = \begin{cases} t & \text{if} \quad t \neq 2, \\ 0 & \text{if} \quad t = 2. \end{cases}$$

We have the point 0 is rd, and the point 2 is ld.
Then the right limit of f at 0 exists and equal to $f(0)$, thus f is continuous at 0.
On the other hand, f is discontinuous at 2 since $\lim\limits_{t\to 2} f(t) \neq f(2)$ but the left limit of f exists in 2. Therefore, f is not continuous, but it is rd-continuous.

Theorem 1.1.26. *Assume that $f : \mathbb{T} \to \mathbb{R}$. Then,*

(i) If f is continuous, then f is rd-continuous.

(ii) If f is rd-continuous, then f is regulated.

(iii) Assume that f is continuous. If $g : \mathbb{T} \to \mathbb{R}$ is regulated or rd-continuous, then $f \circ g$ has that property too.

Definition 1.1.27. *A continuous function $f : \mathbb{T} \to \mathbb{R}$ is called pre-differentiable with (region of differentiation) D, provided $D \subset \mathbb{T}^\kappa$, $\mathbb{T}^\kappa \backslash D$ is countable and contains no right- scattered elements of $\mathbb{T}$, and f is differentiable at each $t \in D$.*

Theorem 1.1.28. *(Existence of Pre-Antiderivatives) Let f be regulated. Then there exists a function F which is pre-differentiable with region of differentiation D such that*

$$F^\Delta(t) = f(t) \text{ holds for all } t \in D.$$

Definition 1.1.29. *Assume that $f : \mathbb{T} \to \mathbb{R}$ is a regulated function. Any function F as in Theorem 1.1.28 is called a pre-antiderivative of f. We define the indefinite integral of a regulated function f by*

$$\int f(t)\Delta t = F(t) + C,$$

where C is an arbitrary constant and F is pre-antiderivative of f. We define the Cauchy integral by

$$\int_r^s f(t)\Delta t = F(s) - F(r) \quad \text{for all } \; r, s \in \mathbb{T}.$$

A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided

$$F^\Delta(t) = f(t) \quad \text{holds for all } t \in \mathbb{T}^\kappa.$$

Example 1.1.30. *Let $\mathbb{T} = \mathbb{Z}$, evaluate the indefinite integral*

$$\int (3t^2 + 5t + 2)\Delta t.$$

We put $g(t) = t^3 + t^2$. We have $\sigma(t) = t + 1$, then

$$\begin{aligned} g^\Delta(t) &= (t^3)^\Delta + (t^2)^\Delta \\ &= [(\sigma(t))^2 + t\sigma(t) + t^2] + [\sigma(t) + t] \\ &= [(t+1)^2 + t(t+1) + t^2] + [t + 1 + t] \end{aligned}$$

$$\begin{aligned} &= (t^2 + 2t + 1 + t^2 + t + t^2) + 2t + 1 \\ &= 3t^2 + 5t + 2. \end{aligned}$$

Thus,

$$\int (3t^2 + 5t + 2)\Delta t = t^3 + t^2 + c,$$

where c is an arbitrary constant.

Theorem 1.1.31. *(Existence of Antiderivatives) Every rd-continuous function has an antiderivative. In particular, if $t_0 \in \mathbb{T}$, then F defined by*

$$F(t) := \int_{t_0}^{t} f(\tau)\Delta\tau \quad \text{for } t \in \mathbb{T}$$

is an antiderivative of f.

Proposition 1.1.32. *[4] Suppose that $\mathbb{T}$ is a time scales and f is an increasing continuous function on the time scales interval $[a,b]$ (i.e., $\mathbb{T} \subseteq [a,b]$). If F is the extension of f to the real interval $[a,b]$ given by*

$$F(s) := \begin{cases} f(s) & \text{if } s \in \mathbb{T}, \\ f(t) & \text{if } s \in (t,\sigma(t)) \notin \mathbb{T}. \end{cases}$$

Then,

$$\int_a^b f(t)\Delta t \leq \int_a^b F(t)dt.$$

Theorem 1.1.33. *If $f \in C_{rd}$ and $t \in \mathbb{T}^{\kappa}$, then*

$$\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t)f(t).$$

Theorem 1.1.34.

If $a,b,c \in \mathbb{T}$, $\alpha \in \mathbb{R}$ and $f,g \in C_{rd}$, then

1. $\int_a^b [f(t)+g(t)]\Delta(t) = \int_a^b f(t)\Delta(t) + \int_a^b g(t)\Delta(t)$.
2. $\int_a^b \alpha f(t)\Delta(t) = \alpha \int_a^b f(t)\Delta(t)$.
3. $\int_a^b f(t)\Delta(t) = -\int_b^a f(t)\Delta(t)$.
4. $\int_a^b f(t)\Delta(t) = \int_a^c f(t)\Delta(t) + \int_c^b f(t)\Delta(t)$.
5. $\int_a^b f(\sigma(t))g^{\Delta}(t)\Delta(t) = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(t)\Delta(t)$.
6. $\int_a^b f(t)g^{\Delta}(t)\Delta(t) = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(\sigma(t))\Delta(t)$.
7. $\int_a^a f(t)\Delta(t) = 0$.
8. *If* $|f(t)| \leq g(t)$ *on* $[a,b]$, *then* $|\int_a^b f(t)\Delta(t)| \leq \int_a^b g(t)\Delta(t)$.
9. *If* $f(t) \geq 0$ *for all* $a \leq t < b$, *then* $\int_a^b f(t)\Delta(t) \geq 0$.

Theorem 1.1.35.
Let $a, b \in \mathbb{T}$ and $f \in C_{rd}$.

(i) If $\mathbb{T} = \mathbb{R}$, the function f is Δ-integrable on $[a, b]$ and

$$\int_a^b f(t)\Delta(t) = \int_a^b f(t)dt.$$

(ii) If $[a, b]$ consists of only isolated points, then

$$\int_a^b f(t)\Delta(t) = \begin{cases} \sum_{t\in[a,b[} \mu(t)f(t) & \text{if} \quad a < b \\ 0 & \text{if} \quad a{=}b \\ -\sum_{t\in[b,a[} \mu(t)f(t) & \text{if} \quad a > b. \end{cases}$$

(iii) If $\mathbb{T} = \mathbb{Z}$, then

$$\int_a^b f(t)\Delta(t) = \begin{cases} \sum_{k=a}^{b-1} \mu(t)f(t) & \text{if} \quad a < b \\ 0 & \text{if} \quad a = b \\ -\sum_{k=b}^{a-1} \mu(t)f(t) & \text{if} \quad a > b. \end{cases}$$

(iv) If $\mathbb{T} = h\mathbb{Z}$, $h > 0$, then

$$\int_a^b f(t)\Delta(t) = \begin{cases} \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} \mu(t)f(t) & \text{if} \quad a < b \\ 0 & \text{if} \quad a{=}b \\ -\sum_{k=\frac{b}{h}}^{\frac{a}{h}-1} \mu(t)f(t) & \text{if} \quad a > b. \end{cases}$$

The proof of Theorem 1.1.34 and Theorem 1.1.35 can be directly followed from [14, 15].

Example 1.1.36. *For $\mathbb{T} = h\mathbb{Z}$, $h > 0$. Let's calculate $\int_0^t s\Delta s$ on $\mathbb{T}$, we have*

$$\int_0^t s\,\Delta s = \sum_{k=0}^{\frac{t}{h}-1}(kh)h = h^2\sum_{k=0}^{\frac{t}{h}-1} k \quad = h^2\Big[\frac{t}{2h}(\frac{t}{h} - 1)\Big]$$
$$\Rightarrow \int_0^t s\,\Delta s = \frac{t(t-h)}{2}.$$

1.1.3 The exponential function

Definition 1.1.37. *For $h > 0$, we define the Hilger complex numbers, the Hilger real axis, the Hilger alternating axis, and the Hilger imaginary circle as*

$$\mathbb{C}_h := \{z \in \mathbb{C};\ z \neq -\frac{1}{h}\},$$

$$\mathbb{R}_h := \{z \in \mathbb{C}_h;\ z \in \mathbb{R} \text{ and } z > -\frac{1}{h}\},$$

$$\mathbb{A}_h := \{z \in \mathbb{C}_h;\ z \in \mathbb{R} \text{ and } z < -\frac{1}{h}\},$$

$$\mathbb{I}_h := \{z \in \mathbb{C}_h;\ \left|z + \frac{1}{h}\right| = \frac{1}{h}\},$$

respectively. For $h = 0$, let $\mathbb{C}_0 := \mathbb{C}$, $\mathbb{R}_0 := \mathbb{R}$, $\mathbb{I}_0 := i\mathbb{R}$, and $\mathbb{A}_0 := \emptyset$.

Definition 1.1.38. *Let $h > 0$ and $z \in \mathbb{C}_h$. We define the Hilger real part of x by*

$$\mathfrak{R}_h(z) := \frac{|zh + 1| - 1}{h}$$

and the Hilger imaginary part of z by

$$\mathfrak{I}_h(z) := \frac{Arg(zh + 1)}{h},$$

where $Arg(z)$ denotes the principal argument of z (i.e.,$-\pi < Arg(z) \leq \pi$).

Note that $\mathfrak{R}_h(z)$ and $\mathfrak{I}_h(z)$ satisfy

$$-\frac{1}{h} < \mathfrak{R}_h(z) < \infty \quad \text{and} \quad -\frac{\pi}{h} < \mathfrak{I}_h(z) \leq \frac{\pi}{h},$$

respectively. In particular, $\mathfrak{R}_h(z) \in \mathbb{R}_h$.

Theorem 1.1.39. *If we define "the cicle plus" addition $\oplus$ on $\mathbb{C}_h$ by*

$$z \oplus w := z + w + zwh,$$

then $(\mathbb{C}_h, \oplus)$ is an Abelian group.

Corollary 1.1.40. *If $z \in \mathbb{C}_h$, then the additive inverse of z under the operation $\oplus$ is*

$$\ominus z := -\frac{z}{1 + zh}.$$

Definition 1.1.41. *We define "the circle minus" subtraction $\ominus$ on $\mathbb{C}_h$ by*

$$z \ominus w := z \oplus (\ominus w).$$

Remark 2. *If $z, w \in \mathbb{C}_h$, for $h > 0$, we have*

$$z \ominus w = \frac{z - w}{1 + wh}.$$

If $h = 0$, then $z \ominus w = z - w$.

Definition 1.1.42. *A function $f : \mathbb{T} \longrightarrow \mathbb{R}$ is called regressive provided*

$$1 + \mu(t) f(t) \neq 0 \;\; \text{for all} \;\; t \in \mathbb{T}^k$$

holds. The set of all regressive and rd-continuous functions $f : \mathbb{T} \longrightarrow \mathbb{R}$ will be denoted by

$$\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R}).$$

We define the set

$$\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}) = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{f \in \mathcal{R} : 1 + \mu(t) f(t) > 0, \;\; \text{for all } t \in \mathbb{T}\}.$$

Definition 1.1.43. *For $h > 0$, let $\mathbb{Z}_h$ be the strip*

$$\mathbb{Z}_h := \{z \in \mathbb{C} : \frac{-\pi}{h} < \Im(z) \leq \frac{\pi}{h}\}.$$

For $h = 0$, let $\mathbb{Z}_0 := \mathbb{C}$.

Definition 1.1.44. *For $h > 0$, we define the cylinder transformation $\xi_h : \mathbb{C}_h \to \mathbb{Z}_h$ by*

$$\xi_h(z) = \frac{1}{h} \log(1 + zh),$$

where $\log$ is the principal logarithm function. For $h = 0$, we define $\xi_0(z) := z$ for all $z \in \mathbb{C}$.

Definition 1.1.45. *Let $p, q \in \mathcal{R}$, we define the circle plus addition $\oplus$ on $\mathcal{R}$ by*

$$(p \oplus q)(t) := p(t) + q(t) + \mu(t) p(t) q(t) \quad \text{for all} \quad t \in \mathbb{T},$$

and the circle minus subtraction $\ominus$ on $\mathcal{R}$ by

$$(p \ominus q)(t) := \frac{p(t) - q(t)}{1 + \mu(t)q(t)} \quad \text{for all} \quad t \in \mathbb{T}.$$

Let introduce the notation

$$\mathcal{R}(\alpha) = \begin{cases} \mathcal{R} & \text{if } \alpha \in \mathbb{N} \\ \mathcal{R}^{+} & \text{if } \alpha \in \mathbb{R} \setminus \mathbb{N} \,. \end{cases}$$

Note that $p \in \mathcal{R}^{+}$ implies that

$$1 + \mu(t)p(t)\tau > 0 \;\; \text{for all } t \in \mathbb{T} \;\; \text{and all } \tau \in [0, 1].$$

Definition 1.1.46. *For $\alpha \in \mathbb{R}$ and $p \in \mathcal{R}(\alpha)$, we define*

$$(\alpha \odot p)(t) := \alpha p(t) \int_0^1 (1 + \mu(t)p(t)\tau)^{\alpha - 1} d\tau.$$

Theorem 1.1.47. *Suppose that $p \in \mathcal{R}$ and fix $t_0 \in \mathbb{T}$. Then the initial value problem*

$$y^{\Delta} = p(t)y, \quad y(t_0) = 1 \quad \text{on} \quad \mathbb{T}$$

has a unique solution on $\mathbb{T}$.

Definition 1.1.48. *If $p \in \mathcal{R}$ and fix $t_0 \in \mathbb{T}$, then one unique solution of the initial value problem (1.1.47) is denoted by $e_p(\cdot, s)$ called the exponential function define by*

$$e_p(t, s) := \exp \int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau \;\; \text{for} \;\; s, t \in \mathbb{T}.$$

Proposition 1.1.49. *If $p, q \in \mathcal{R}(\mathbb{T})$ and $t, s, r \in \mathbb{T}$, then*

- $e_0(t, s) = 1 \quad$ *and* $e_p(t, t) = 1$,
- $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$,
- $e_{\ominus p}(t, s) = \dfrac{1}{e_p(t, s)}$,
- $e_p(t, s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s, t)$,
- $e_p(t, s).e_q(t, s) = e_{p \oplus q}(t, s)$,
- $e_p(t, s).e_p(s, r) = e_p(t, r)$,

▹ $\dfrac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t,s)$,

▹ $\left(\dfrac{1}{e_p(\cdot,s)}\right)^{\Delta} = -\dfrac{p(t)}{e_p^{\sigma}(\cdot,s)}$.

Theorem 1.1.50. *If $\alpha \in \mathbb{R}$ and $p \in \mathcal{R}(\alpha)$, then*

$$e_{\alpha\odot p} = e_p^{\alpha}.$$

Definition 1.1.51. *Let $\alpha \in \mathbb{R} \setminus \{0\}$, $f \in C_{rd}$*

$$x^{\Delta} = [p \ominus (\frac{1}{\alpha - 1} \odot (f x^{\alpha}))]x, \tag{1.2}$$

is a Bernoulli equation on time scales.

Theorem 1.1.52. *Suppose that $\alpha \in \mathbb{R} \setminus \{0\}$, $p \in \mathcal{R}(\alpha)$, and $f \in C_{rd}$. Let $x_0 \neq 0$. If*

$$\frac{1}{x_0^{\alpha}} + \int_{t_0}^{t} e_p^{\alpha}(\tau, t_0) f(\tau) \Delta\tau > 0 \quad \text{for all} \quad t \in \mathbb{T}.$$

Then,

$$x(t) = \frac{e_p(t, t_0)}{\left[\frac{1}{x_0^{\alpha}} + \int_{t_0}^{t} e_p^{\alpha}(\tau, t_0) f(\tau)\Delta\tau\right]^{\frac{1}{\alpha}}} \tag{1.3}$$

solves the Bernoulli equation (1.2).

1.2 Point fixed theorems

Let X and Y be two Banach spaces, $\mathcal{S}$ a family of functions from X to Y, and $A \subset X$.

Definition 1.2.1. *(Uniformly bounded) we call $\mathcal{S}$ **uniformly bounded** if there exists $M > 0$ such that*

$$\|T\| = \sup_{x\in A} |T(x)| \leq M \text{ on } X \text{ for } T \in S.$$

Definition 1.2.2. *(Equicontinuous) The family $\mathcal{S}$ is **equicontinuous** on A if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for every pair of elements $x, y \in A$ and every $T \in S$ we have*

$$\|y - x\|_X < \delta \Rightarrow \|T(y) - T(x)\|_Y < \varepsilon.$$

Theorem 1.2.3. *(Ascoli-Arzela theorem) Assume that A is a compact set in X. Then a set $S \subset C(A)$ is relatively compact in $C(A)$ if and only if the functions in S are uniformly bounded and equicontinuous on A.*

Theorem 1.2.4. *(Schauder theorem) Let A be a closed convex set in Banach space X and assume that $T : A \to A$ is a continuous mapping such that $T(A)$ is relatively compact subset of A. Then T has a fixed point.*

Theorem 1.2.5. *(Banach theorem) Let T be a contraction on a Banach space X. Then T has a unique fixed point.*

1.3 Fractional operators

We now recall the celebrated gamma function.

Definition 1.3.1 (Gamma function). *For complex numbers with a positive real part, the gamma function $\Gamma(t)$ is defined by the following convergent improper integral:*

$$\Gamma(t) := \int_0^\infty s^{t-1} e^{-s} ds.$$

Remark 3. *The gamma function satisfies the following useful property:*

$$\Gamma(t+1) = t\Gamma(t).$$

Definition 1.3.2. *For a given integrable function $h(t)$, **the fractional integral operator of Riemann Liouville** I_t^α of order $\alpha > 0$ is given by*

$$I_t^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds,$$

with

$$I_t^0 h(t) = h(t).$$

Definition 1.3.3. *For a given function $h(t)$ in $C[0,T]$, **the fractional derivative operator of Riemann Liouville sense** ${}^{RL}D_t^\alpha$ of order $\alpha > 0$ is given by*

$${}^{RL}D_t^\alpha h(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{h(s)}{(t-s)^{\alpha-n+1}} ds, \ n-1 < \alpha < n, \ n \in \mathbb{N}, \\ \frac{d^n}{dt^n} h(t), \ \alpha = n, \ n \in \mathbb{N}. \end{cases}$$

Definition 1.3.4. *Let $T > 0$ and $h(t) \in C^n[0, T]$. **The Caputo fractional derivative operator** ${}^C D_t^\alpha$ of order $\alpha > 0$ is defined by*

$$ {}^C D_t^\alpha h(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{h^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds, \; n-1 < \alpha < n, \; n \in \mathbb{N}, \\ \frac{d^n}{dt^n} h(t), \; \alpha = n, \; n \in \mathbb{N}. \end{cases} $$

The following two propositions are met in the case of Caputo fractional derivative

$$ {}^C D_t^\alpha (I_t^\alpha h)(t) = h(t), \; I_t^\alpha ({}^C D_t^\alpha h)(t) = h(t) - \sum_{k=0}^{n-1} \frac{h^{(k)}(0)}{k!} t^k, \; t > 0. $$

Definitions (1.3.3) and (1.3.4) are different from each other, and the relation between the two types of fractional derivatives is as follows

$$ {}^C D_t^\alpha h(t) = {}^{RL} D_t^\alpha h(t) - \sum_{k=0}^{n-1} r_k^\alpha(t) h^{(k)}(0), \; r_k^\alpha(t) = \frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)}. $$

The Caputo derivative has the main advantage that corresponding problem's initial condition has the same value as the ordinary differential equation. Moreover, for a constant valued function, the Caputo derivative is zero.

1.4 Stability

1.4.1 Stability of equilibrium

So it is essential to define the notion of equilibrium points. Equilibrium points play a crucial role in the study of dynamic systems Henri Poincre(1854-1912) showed that to characterize a dynamic system with several variables, there is no need to calculate detailed solutions; it is enough to know the equilibrium points and their stability. This very important result considerably simplifies the study of non-linear systems in the vicinity of these points. So to determine the stability of an equilibrium point, we must study the behavior of the solutions in a small neibohood of it. Let $f : D \to \mathbb{R}^n$, $D \subset \mathbb{R}^n$ then, we study the qualitative behavior of the solutions of the differential order system:

$$ \begin{cases} x'(t) = f(x(t)), \\ x(0) = x_0. \end{cases} \tag{1.4} $$

Definition 1.4.1. *We say that e is an equilibrium point for (1.4) if and only if $f(e) = 0$.*

Definition 1.4.2. *The equilibrium point $x = e$ of (1.4) is*

- *stable if for each $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$ such that*

$$\|x(0) - e\| < \delta \Rightarrow \|x(t) - e\| < \varepsilon, \quad \text{for all } t \geq 0,$$

- *unstable if it is not stable,*
- *asymptotically stable if it is stable and δ can be chosen such that*

$$\|x(0) - e\| < \delta \Rightarrow \lim_{t \to \infty} x(t) = e.$$

Definition 1.4.3. *A set M is said to be an invariant set with respect to the system (1.4) if*

$$x(0) \in M, \; x(t) \in M, \; \text{for all } t.$$

We also say that $x(\cdot)$ approaches a set M as t approaches infinity, if for each $\varepsilon \geq 0$ there is $T > 0$ such that

$$d(x(t), M) \leq \varepsilon \text{ for all } t > T.$$

Definition 1.4.4 (Lyapunov function). *We consider the system (1.4). Let e be an equilibrium point for (1.4), D be a domain containing e, and $V : D \to \mathbb{R}$ a continuous and differentiable function on D.*

1. *We say that V is a Lyapunov function in the broad sense at e, if it satisfies the following two properties:*

   (i) *V is positive definite.*

   (ii) *$V'(x(t)) \leq 0$ for all $x \in D$.*

2. *We say that V is a Lyapunov function in e, if satisfies the following two properties:*

   (i) *V is positive definite.*

   (ii) *$V'(x(t)) < 0$ for all $x \in D \setminus \{e\}$.*

Theorem 1.4.5 (The principle of fractional invariance Lasalle). *Let $\Omega \subset D$ be a compact set that is positively invariant with respect to (1.4). Let $V : D \to \mathbb{R}$ be a continuously differentiable function such that $V(x(t)) > 0$ and $V'(x(t)) \leq 0$ in Ω for $x(t)$ solution of system (1.4). Let E be the set of all points in Ω where $V'(x(t)) = 0$. Let M be the largest invariant set in E. Then every solution in Ω approaches M as $t \to \infty$.*

Remark 4. *Let $\alpha \in [0,1]$, the fractional system ${}^{C}D_t^{\alpha}x(t) = f(x(t))$ has the same equilibrium points as the system $x'(t) = f(x(t))$.*

1.4.2 The basic reproduction number

The basic reproduction number, denoted $\mathcal{R}_0$, is topically defined [20] as: the average number of secondary cases produced by a "typical" infected (assumed infectious) individual during his/ her entire life as infectious (infectious period) when introduced in a population of susceptibles.

$\mathcal{R}_0$ is often found through the study and computation of the eigenvalues of the Jacobian at the disease-free equilibrium. Diekmann et al [21] follow a different approach: the next generation operator approach. They define $\mathcal{R}_0$ as the spectral radius of the next generation operator.

We consider the epidemiological models that can be written in the form:

$$\begin{cases} \frac{dx}{dt} = f(x, \mathbf{E}, \mathbf{I}) \\ \frac{d\mathbf{E}}{dt} = g(x, \mathbf{E}, \mathbf{I}) \\ \frac{d\mathbf{I}}{dt} = h(x, \mathbf{E}, \mathbf{I}) \end{cases}$$

where $x \in \mathbb{R}^r$, $\mathbf{E} \in \mathbb{R}^s$, $\mathbf{I} \in \mathbb{R}^n$, r, s, $n \geq 0$, and $h(x, 0, 0) = 0$. The components of x denote the number of susceptibles, recovered, and other classes of non-infected individuals. The components of $\mathbf{E}$ represent the number of infected individuals who do not transmit the disease. The components of $\mathbf{I}$ represent the number of infected individuals capable of transmitting the disease (e.g. infectious and non-quarentined individuals).

Let $U_0 = (x^*, 0, 0) \in \mathbb{R}^{r+s+n}$ denote the disease-free equilibrium, that is, at $U_0 = (x^*, 0, 0)$, $f(x^*, 0, 0) = g(x^*, 0, 0) = h(x^*, 0, 0) = 0$. Assume that the equation $g(x^*, \mathbf{E}, \mathbf{I}) = 0$ implicitly determines a function $\mathbf{E} = \tilde{g}(x^*, \mathbf{I})$. Let $A = D_{\mathbf{I}}h(x^*, \tilde{g}(x^*, 0), 0)$ and further assume that A can be written in the form $A = M - D$, with $M \geq 0$ (That is, $m_{ij} \geq 0$) and $D > 0$, a diagonal matrix.

The spectral bound of matrix B is denoted by $m(B) = \sup\{\Re\lambda;\ \lambda \in \sigma(B)\}$, where $\Re\lambda$ means the real part of λ, while $\rho(B) = \lim_{n\to\infty} ||B^n||^{\frac{1}{n}}$ denote the spectral radius of B. The proof of the following theorem involving matrix A is found in Diekmann et al [21]:

Either

$$m(A) < 0 \Leftrightarrow \rho(MD^{-1}) < 1$$

or

$$m(A) > 0 \Leftrightarrow \rho(MD^{-1}) > 1.$$

The basic reproductive number is defined as the spectral radius (dominant eigen value) of the matrix MD^{-1}, that is,

$$\mathcal{R}_0 = \rho(MD^{-1}).$$

Example 1.4.6. *[19] Many communicable diseases can be modelled using models that include compartments for the susceptible, exposed, infected and recovered epidemiological classes. An SEIR model for a homogeneously mixing population is given by the following set of equations:*

$$\begin{cases} \frac{dS}{dt} = \Lambda - \beta S \frac{I}{N} - \mu S, \\ \frac{dE}{dt} = \beta S \frac{I}{N} - (\mu + k)E, \\ \frac{dI}{dt} = kE - (\gamma + \mu)I, \\ \frac{dR}{dt} = \gamma I - \mu R, \end{cases}$$

where E is the number of latent individuals and k is the rate at which a latent individual becomes infectious. Letting $x = (S, R)$, $\mathbf{E} = E$, $\mathbf{I} = I$, $U_0 = (\frac{\Lambda}{\mu}, 0, 0, 0)$ *and* $\tilde{g}(x^*, I) = \frac{\beta I}{\beta + k}$ *gives* $M = \frac{k\beta}{\mu + k}$ *and* $D = \gamma + \mu$. *Hence,*

$$\mathcal{R}_0 = MD^{-1} = \frac{k\beta}{(\mu + k)(\mu + \gamma)}.$$

1.4.3 Routh Hurwitz criterion

Let consider the following system:

$$\begin{cases} x_1' = a_{11}x_1 + \ldots + a_{1n}x_n, \\ \vdots \\ x_n' = a_{n1}x_1 + \ldots + a_{nn}x_n, \end{cases} \tag{1.5}$$

where all coefficients a_{ij}, $1 < i < n$, $1 < j < n$ are constants. The characteristic determinant of the system (1.5) is :

$$\det \begin{bmatrix} a_{11} - \lambda & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} - \lambda & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} - \lambda \end{bmatrix} = 0. \tag{1.6}$$

By developing the characteristic determinant given in (1.6), we obtain a polynomial of *nth* degree in λ,

$$a_0\lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n = 0, \tag{1.7}$$

where, we can assume that $a_0 > 0$. The Hurwitz's criterion, which is an algebraic condition, consists of examining the coefficients of the polynomial characteristic of classical linear system and constructing a matrix called the Hurwitz matrix which can be written in the following form:

To apply this criterion, you must first construct a square matrix of dimension n. This matrix is constructed in the following manner: Beginning with a_1, the first row is a sequential array of the coefficients with odd indices in equation (1.7). The elements of each subsequent row are formed such that for $0 < 2j - i \leq n$, the general element $a_{ij} = a_{2j-1}$, otherwise $a_{ij} = 0$. As the result of such a construction, the coefficients $a_1, \ldots, a_n$ will be on the principal diagonal of the matrix, and all elements of the last column will be equal to zero, expect the last element. The following matrix is called Hurwitz Matrix given by [41]:

$$H = \begin{bmatrix} a_1 & a_3 & a_5 & \cdots & 0 \\ a_0 & a_2 & a_4 & \cdots & 0 \\ 0 & a_1 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{bmatrix}. \tag{1.8}$$

We consider the principal diagonal minors of the matrix (1.8):

$$\Delta_1 = a_1,\ \Delta_2 = \begin{vmatrix} a_1 & a_3 \\ a_2 & a_4 \end{vmatrix},\ \ldots,\ \Delta_n = a_n\Delta_{n-1}. \tag{1.9}$$

The last expression becomes self-evident if we note that in the last column of the matrix (1.8) all elements expect a_n are equal to zero.

Theorem 1.4.7. *(Hurwitz criterion) In the characteristic polynomial (1.7), with real coefficients and a positive coefficient for the leading term, in order for all the roots to have negative real parts, necessary and sufficient condition is that all principal diagonal minors in (1.9) be positive:*

$$\Delta_1 > 0,\ \Delta_2 > 0,\ \ldots,\ \Delta_{n-1} > 0,\ \Delta_n > 0. \tag{1.10}$$

Corollary 1.4.8. *When $a_0 > 0$,*

1. *a necessary condition for all roots of equation (1.7) to have negative real part is that all coefficients $a_1, \dots, a_n$ must be positive:*

$$a_1 > 0,\ a_2 > 0,\ \dots,\ a_n > 0. \tag{1.11}$$

2. *even if one the cofficients $a_1, \dots, a_n$ is negative, then some of the roots $\lambda_1, \dots, \lambda_n$ of equation (1.7) will have positive real parts.*

Now we will consider some particular cases:

1. A first-order system ($n = 1$): The characteristic polynomial has the form:

$$a_0\lambda + a_1 = 0.$$

   For $a_0 > 0$, the asymptotic stability condition is $a_1 > 0$.

2. A second-order system ($n = 2$):

$$a_0\lambda^2 + a_1\lambda + a_2 = 0.$$

   The matrix (1.8) and Hurwitz's condition are:

$$\begin{bmatrix} a_1 & 0 \\ a_0 & a_2 \end{bmatrix},\ \Delta_1 = a_1 > 0,\ \Delta_2 = a_1a_2 > 0.$$

   For $a_0 > 0$, then the asymptotic stability conditions for a second-order system become $a_1 > 0$, $a_2 > 0$.

3. A third-order system ($n = 3$):

$$a_0\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0.$$

   We construct the corresponding matrix (1.8) and Hurwitz's condition as

$$\begin{bmatrix} a_1 & a_3 & 0 \\ a_0 & a_2 & 0 \\ 0 & a_1 & a_3 \end{bmatrix},\ \Delta_1 = a_1 > 0,\ \Delta_2 = a_1a_2 - a_0a_3 > 0,\ \Delta_3 = a_3\Delta_2 > 0.$$

   Using inequalities (1.11), we directly obtain the conditions for asymptotic stability of a third-order system, with $a_0 > 0$, as

$$a_1 > 0,\ a_2 > 0,\ a_3 > 0,\ \Delta_2 = a_1 a_2 - a_0 a_3 > 0.$$

Chapter 2

Existence and uniqueness of solutions to proper fractional Riemann-Liouville value problems on time scales

In this chapter, several properties of new Riemann–Liouville fractional operators on time scales are studied. Next, we demonstrate sufficient conditions for a nonlinear Riemann–Liouville fractional initial value problem on an arbitrary time scales to have a solution, as well as sufficient conditions for the uniqueness of solution using the fixed point theorems of Schauder and Banach.

2.1 Introduction

Let $\mathbb{T}$ be a time scales, that is, a nonempty closed subset of $\mathbb{R}$. In [9], Benkhettou, Hammoudi and Torres introduced a concept of fractional integral,

$$ {}_a^{\mathbb{T}}I_t^{\alpha}h(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}h(s)\Delta s, \tag{2.1}$$

and the concept of fractional derivative

$$ {}_a^{\mathbb{T}}D_t^{\alpha}h(t) = \frac{1}{\Gamma(1-\alpha)}\left(\int_a^t (t-s)^{-\alpha}h(s)\Delta s\right)^{\Delta} \tag{2.2}$$

of Riemann–Liouville on time scales. In [58], Torres gives more suitable definitions of fractional integral (2.1) and fractional derivative (2.2) of Riemann–Liouville on time scales, introducing the forward jump σ operator of time scales in their definition:

$$ {}_{a}^{\mathbb{T}}I_{t}^{\alpha}h(t) = \frac{1}{\mu(\alpha)} \int_{a}^{t} (t-\sigma(s))^{\alpha-1} h(s) \Delta s \tag{2.3}$$

and

$$ {}_{a}^{\mathbb{T}}D_{t}^{\alpha}h(t) = \frac{1}{\mu(1-\alpha)} \left(\int_{a}^{t} (t-\sigma(s))^{-\alpha} h(s) \Delta s \right)^{\Delta}. \tag{2.4}$$

Here we focus on definitions (2.3) and (2.4), but changing the operator σ into the backward jump operator ρ. As we shall prove, the new definitions with ρ provide proper notions with respect to existence and uniqueness of solution to the following initial value problem (IVP):

$$ ({}_{t_0}^{\alpha}Dy)(t) = f(t, y(t)), \quad t \in [t_0, t_0 + d] = \mathcal{J} \subseteq \mathbb{T}, \tag{2.5}$$

$$ ({}_{t_0}^{1-\alpha}Iy)(t_0) = 0, \tag{2.6}$$

where $\mathbb{T}$ is a given time scales, $0 < \alpha < 1$, $d > 0$, ${}_{t_0}^{\alpha}D$ is the proper (left) Riemann–Liouville fractional derivative operator or order α defined on $\mathbb{T}$ with ρ, ${}_{t_0}^{1-\alpha}I$ is the proper (left) Riemann–Liouville fractional integral operator of order $1-\alpha$ defined on $\mathbb{T}$ with ρ, and function $f : \mathcal{J} \times \mathbb{T} \to \mathbb{R}$ is a right dense continuous function. Our main results give sufficient conditions for the existence (Theorem 2.4.3) and uniqueness (Theorem 2.5.1) of solution to problem (2.5)–(2.6).

2.2 Fractional operators on time scales

Now we introduce new notions of fractional operators, analogous to the Riemann–Liouville fractional operators on time scales proposed in [58].

Definition 2.2.1 (Fractional integral on time scales)**.** *Suppose $\mathbb{T}$ is a time scales, $[a, b]$ is an interval of $\mathbb{T}$, and f is an integrable function on $[a, b]$. Let $0 < \alpha < 1$ and $t \in [a, b]$. Then the (left) Riemann–Liouville fractional integral of order α of f is defined by*

$$ ({}_{a}^{\alpha}If)(t) := \int_{a}^{t} \frac{(t-\rho(s))^{\alpha-1}}{\Gamma(\alpha)} f(s) \Delta s, \tag{2.7}$$

where Γ is the gamma function.

Definition 2.2.2 (Fractional derivative on time scales)**.** *Suppose $\mathbb{T}$ is a time scales, $[a, b]$ is an interval of $\mathbb{T}$, and f is an integrable function on $[a, b]$. Let $0 < \alpha < 1$, $t \in [a, b]$. The (left) Riemann–Liouville fractional derivative of order α of f is* *defined by*

$$({}_a^\alpha D f)(t) := \frac{1}{\Gamma(1-\alpha)}\left(\int_a^t (t-\rho(s))^{-\alpha} f(s)\Delta s\right)^{\Delta}. \tag{2.8}$$

Fractional operators of negative order are defined as follows.

Definition 2.2.3. *If $-1 < \alpha < 0$, then the (Riemann–Liouville) fractional derivative of order α is defined as the fractional integral of order $-\alpha$. Moreover, the fractional integral of order α is defined as the (Riemann–Liouville) fractional derivative of order $-\alpha$:*

$$({}_a^\alpha D f)(t) := ({}_a^{-\alpha} I f)(t), \qquad ({}_a^\alpha I f)(t) := ({}_a^{-\alpha} D f)(t).$$

Remark 5. *Along the work, we consider the order α of the fractional derivatives in the real interval $(0,1)$. We can, however, easily generalize our definitions to any positive real α. Indeed, let $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$. Then there exists $\beta \in (0,1)$ such that $\alpha = [\alpha] + \beta$, where $[\alpha]$ is the integer part of α, and we can set*

$$\left({}_a^\alpha D f\right)(t) := {}_a^\beta D\left(f^{\Delta^{[\alpha]}}\right)(t).$$

2.3 Properties of the time scales fractional operators

We begin by proving some fundamental properties of the fractional operators on time scales. After that, we prove existence of a solution to the fractional order initial value problem (2.5)–(2.6) defined on a time scales $\mathbb{T}$.

Proposition 2.3.1. *Let $\mathbb{T}$ be a time scales with derivative Δ; $0 < \alpha < 1$. Then,*

$$\left({}_a^\alpha D g\right)(t) = \left(\Delta \circ {}_a^{1-\alpha} I g\right)(t).$$

Proof. Let $g : \mathbb{T} \to \mathbb{R}$. From (2.8) we have

$$\begin{aligned}({}_a^\alpha D g)(t) &= \frac{1}{\Gamma(1-\alpha)}\left(\int_a^t (t-\rho(s))^{-\alpha} g(s)\Delta s\right)^{\Delta} \\ &= \left({}_a^{1-\alpha} I g)(t)\right)^{\Delta} \\ &= (\Delta \circ ({}_a^{1-\alpha} I g))(t).\end{aligned}$$

The proof is complete. ■

Proposition 2.3.2. *For any integrable function g on $[a,b]\cap\mathbb{T}$, the Riemann–Liouville Δ–fractional integral satisfies*

$$ {}_a^{\alpha}I \circ {}_a^{\beta}I(g) = {}_a^{\alpha+\beta}I(g), \quad \text{for } \alpha > 0 \text{ and } \beta > 0. $$

Proof. Similar to the proof of Proposition 16 of [9]. ■

Proposition 2.3.3. *For any integrable function g on $[a,b]\cap\mathbb{T}$ one has*

$$ {}_a^{\alpha}D \circ {}_a^{\alpha}Ig = g, \qquad 0 < \alpha < 1. $$

Proof. By Propositions 2.3.1 and 2.3.2, we have

$$ {}_a^{\alpha}D \circ {}_a^{\alpha}Ig = \left[\left({}_a^{1-\alpha}I\right)\left({}_a^{\alpha}Ig\right)(t)\right]^{\Delta} = \left[\left({}_a^{1}Ig\right)(t)\right]^{\Delta} = g(t). $$

The proof is complete. ■

Corollary 2.3.4. *For $0 < \alpha < 1$, we have*

$$ ({}_a^{\alpha}D) \circ ({}_a^{-\alpha}D) = Id $$

and

$$ {}_a^{-\alpha}I \circ {}_a^{\alpha}I = Id, $$

where Id denotes the identity operator.

Proof. From Definition 2.2.3 and Proposition 2.3.3, we have that

$$ {}_a^{\alpha}D \circ {}_a^{-\alpha}D = {}_a^{\alpha}D \circ {}_a^{\alpha}I = Id $$

and

$$ {}_a^{-\alpha}I \circ {}_a^{\alpha}I = {}_a^{\alpha}D \circ {}_a^{\alpha}I = Id. $$

The proof is complete. ■

Definition 2.3.5. *For $\alpha > 0$, we denote by ${}_a^{\alpha}I([a,b])$ the space of functions that can be represented by the Riemann–Liouville Δ-integral of order α of some $C_{rd}([a,b])$ function.*

Theorem 2.3.6. *Let $f \in C_{rd}([a,b])$ and $\alpha > 0$. In order that $f \in {}_a^{\alpha}I([a,b])$, it is necessary and sufficient that*

$$ ({}_a^{1-\alpha}If) \in C_{rd}^1([a,b]) \tag{2.9} $$

and

$$\left(\left({}_a^{1-\alpha}If\right)(t)\right)\big|_{t=a} = 0. \tag{2.10}$$

Proof. Assume that $f \in {}_a^{\alpha}I([a,b])$, $f(t) = ({}_a^{\alpha}Ih)(t)$ for some $h \in C_{rd}([a,b])$, and

$$({}_a^{1-\alpha}If)(t) = ({}_a^{1-\alpha}I)({}_a^{\alpha}Ih)(t).$$

From Proposition 2.3.2, we have

$$({}_a^{1-\alpha}If)(t) = ({}_a^{1}Ih)(t) = \int_a^t h(s)\Delta s.$$

Therefore,

$$({}_a^{1-\alpha}If) \in C_{rd}^1([a,b])$$

and

$$({}_a^{1-\alpha}If)(t))|_{t=a} = \int_a^a h(s)\Delta s = 0.$$

Conversely, assume that $f \in C_{rd}([a,b])$ satisfies (2.9) and (2.10). From Taylor's formula applied to function $I_a^{1-\alpha}f$, one has

$$({}_a^{1-\alpha}If)(t) = \int_a^t \frac{\Delta}{\Delta s}({}_a^{1-\alpha}If)(s)\Delta s, \quad \text{for all } t \in [a,b].$$

Let $\varphi(t) := \frac{\Delta}{\Delta t}({}_a^{1-\alpha}If)(t)$. Note that, by (2.9), $\varphi \in C_{rd}([a,b])$. From Proposition 2.3.2, we have

$$({}_a^{1-\alpha}If)(t) = ({}_a^{1}I\varphi)(t) = ({}_a^{1-\alpha}I)({}_a^{\alpha}I\varphi)(t)$$

and thus

$$({}_a^{1-\alpha}If)(t) - ({}_a^{1-\alpha}I)({}_a^{\alpha}I\varphi)(t) \equiv 0.$$

Then,

$$[{}_a^{1-\alpha}I(f - ({}_a^{\alpha}I\varphi)](t) \equiv 0.$$

This implies that

$$f - ({}_a^{\alpha}I\varphi) \equiv 0.$$

We conclude that $f = {}_a^{\alpha}I\varphi$ and $f \in {}_a^{\alpha}I([a,b])$. ■

Corollary 2.3.7. *Let $0 < \alpha < 1$ and $f \in C_{rd}([a,b])$ satisfy the condition in Theorem 2.3.6. Then,*

$$\left({}_a^{\alpha}I \circ {}_a^{\alpha}D\right)(f) = f.$$

2.4 Existence of solutions to fractional IVPs on time scales

Let $\mathbb{T}$ be a time scales and $\mathcal{J} = [t_0, t_0 + d] \subset \mathbb{T}$. A function $y \in C_{rd}(\mathcal{J}, \mathbb{R})$ is a solution to problem (2.5)–(2.6) if

$$\begin{gathered}({}_{t_0}^{\alpha}Dy)(t) = f(t, y) \text{ on } \mathcal{J}, \quad 0 < \alpha < 1,\\ \left({}_{t_0}^{1-\alpha}Iy\right)(t_0) = 0.\end{gathered}$$

To establish the existence of such solution, first we recall the definition of compact map [24].

Definition 2.4.1. *Let X and Y be topological spaces. A map $f : X \to Y$ is called compact if $f(X)$ is contained in a compact subset of Y.*

Let us define the operator

$$T : C_{rd}(\mathcal{J}, \mathbb{R}) \to C_{rd}(\mathcal{J}, \mathbb{R})$$

by

$$T(y)(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - \rho(s))^{\alpha-1} f(s, y(s))\Delta s.$$

Lemma 2.4.2. *Let $0 < \alpha < 1$, $\mathcal{J} \subseteq \mathbb{T}$, and $f : \mathcal{J} \times \mathbb{R} \to \mathbb{R}$. A function y is a solution to problem (2.5)–(2.6) if, and only if, this function is a solution to the integral equation*

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - \rho(s))^{\alpha-1} f(s, y(s))\Delta s,$$

that is, y is a fixed point of operator T: $T(y) = y$.

Proof. By Corollary 2.3.7, $({}_{t_0}^{\alpha}I) \circ ({}_{t_0}^{\alpha}Dy)(t) = y(t)$. From (2.8) we have

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - \rho(s))^{\alpha-1} f(s, y(s))\Delta s$$

and the proof is complete. ■

Theorem 2.4.3 (Existence of solution)**.** *Suppose $f : \mathcal{J} \times \mathbb{R} \to \mathbb{R}$ is a rd-continuous bounded function such that there exists $M > 0$ with $\left|f(t, y(t))\right| < M$ for all $t \in \mathcal{J}$, $y(t) \in \mathbb{R}$. Then problem (2.5)–(2.6) has a solution on $\mathcal{J}$.*

Proof. The proof is given in three steps.

Step 1: T is continuous. Let y_n be a sequence such that $y_n \to y$ in $C(\mathcal{J}, \mathbb{R})$. Then, for each $t \in \mathcal{J}$,

$$\begin{aligned}
&|T(y_n)(t) - T(y)(t)| \\
&\quad \leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\rho(s))^{\alpha-1} \left| f(s, y_n(s)) - f(s, y(s)) \right| \Delta s \\
&\quad \leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\rho(s))^{\alpha-1} \sup_{s\in\mathcal{J}} \left| f(s, y_n(s)) - f(s, y(s)) \right| \Delta s \\
&\quad \leq \frac{\left\| f(\cdot, y_n(\cdot)) - f(\cdot, y(\cdot)) \right\|_\infty}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\rho(s))^{\alpha-1} \Delta s \\
&\quad \leq \frac{\left\| f(\cdot, y_n(\cdot)) - f(\cdot, y(\cdot)) \right\|_\infty}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\rho(s))^{\alpha-1} ds.
\end{aligned} \tag{2.11}$$

For $0 < \alpha < 1$ we have

$$(t-\rho(s))^{\alpha-1} < (t-s)^{\alpha-1},$$

and from inequality (2.11) it follows that

$$\begin{aligned}
\left| T(y_n)(t) - T(y)(t) \right| &\leq \frac{\left\| f(\cdot, y_n(\cdot)) - f(\cdot, y(\cdot)) \right\|_\infty}{\Gamma(\alpha)} \frac{a^\alpha}{\alpha} \\
&\leq \frac{a^\alpha \left\| f(\cdot, y_n(\cdot)) - f(\cdot, y(\cdot)) \right\|_\infty}{\Gamma(\alpha+1)}.
\end{aligned}$$

Since f is a continuous function, one has

$$\left| T(y_n)(t) - T(y)(t) \right|_\infty \leq \frac{a^\alpha}{\Gamma(\alpha+1)} \left\| f(\cdot, y_n(\cdot)) - f(\cdot, y(\cdot)) \right\|_\infty \to 0$$

as $n \to \infty$.

Step 2: For the second part of the proof, we have to show that the set $T(C(\mathcal{J}, \mathbb{R}))$ is relatively compact. Let $T(y) \in T(C(\mathcal{J}, \mathbb{R}))$. Then, $\left\| T(y) \right\|_\infty \leq l$. By hypothesis, for each $t \in \mathcal{J}$ we have

$$\begin{aligned}\left|T(y)(t)\right| &\leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\rho(s))^{\alpha-1} \left|f(s, y(s))\right| \Delta s \\ &\leq \frac{M}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\rho(s))^{\alpha-1} \Delta s \\ &\leq \frac{M}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\rho(s))^{\alpha-1} ds. \qquad (2.12)\end{aligned}$$

For $0 < \alpha < 1$, we know that

$$(t-\rho(s))^{\alpha-1} \leq (t-s)^{\alpha-1},$$

and from inequality (2.12) and Proposition 1.1.32 we can write that

$$\left|T(y)(t)\right| \leq \frac{Ma^{\alpha}}{\alpha\Gamma(\alpha)} = \frac{Ma^{\alpha}}{\Gamma(\alpha+1)} = l.$$

Therefore, $T(C(\mathcal{J}, \mathbb{R}))$ is uniformly bounded. This set is also equicontinuous since for every $t_1,\ t_2 \in \mathcal{J}$, $t_1 < t_2$. Let $A = \left|T(y)(t_1) - T(y)(t_2)\right|$. Then we can write that

$$\begin{aligned}A &\leq \frac{1}{\Gamma(\alpha)} \left| \int_{t_0}^{t_1} (t_1-\rho(s))^{\alpha-1} f(s, y(s)) \Delta s - \int_{t_0}^{t_2} (t_2-\rho(s))^{\alpha-1} f(s, y(s)) \Delta s \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \left| \int_{t_0}^{t_1} (t_1-\rho(s))^{\alpha-1} f(s, y(s)) \Delta s \right| + \left| \int_{t_0}^{t_2} (t_2-\rho(s))^{\alpha-1} f(s, y(s)) \Delta s \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} (t_1-\rho(s))^{\alpha-1} \left|f(s, y(s))\right| \Delta s + \int_{t_0}^{t_2} (t_2-\rho(s))^{\alpha-1} \left|f(s, y(s))\right| \Delta s,\end{aligned}$$

that is,

$$A \leq \frac{M}{\Gamma(\alpha)} \left(\int_{t_0}^{t_1} ((t_1-\rho(s))^{\alpha-1} - (t_2-\rho(s))^{\alpha-1}) \Delta s + \int_{t_1}^{t_2} (t_2-\rho(s))^{\alpha-1} \Delta s \right). \qquad (2.13)$$

For $0 < \alpha < 1$,

$$(t-\rho(s))^{\alpha-1} < (t-s)^{\alpha-1}$$

and it follows that

$$
\begin{aligned}
&|T(y)(t_1) - T(y)(t_2)| \\
&\leq \frac{M}{\Gamma(\alpha)}\left(\int_{t_0}^{t_1}((t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1})ds + \int_{t_1}^{t_2}(t_2 - s)^{\alpha-1}ds\right) \\
&\leq \frac{M}{\alpha\Gamma(\alpha+1)}[(t_2 - t_1)^{\alpha} + (t_1 - t_0)^{\alpha} - (t_2 - t_0)^{\alpha} + (t_2 - t_1)^{\alpha}] \\
&= \frac{2M}{\alpha\Gamma(\alpha+1)}(t_2 - t_1)^{\alpha} + \frac{M}{\alpha\Gamma(\alpha+1)}[(t_1 - t_0)^{\alpha} - (t_2 - t_0)^{\alpha}].
\end{aligned}
$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. From the Arzela–Ascoli theorem, adapted to our context, it follows that $T(C(\mathcal{J}, \mathbb{R}))$ is relatively compact.

Step 3: conclusion. As a consequence of Schauder's fixed point theorem, we conclude that T has a fixed point, which is solution of problem (2.5)–(2.6). ■

2.5 Uniqueness of solutions to fractional IVPs on time scales

Theorem 2.5.1 (Existence and uniqueness of solution). *Let $\mathcal{J} = [t_0, t_0 + d] \subseteq \mathbb{T}$. The initial value problem (2.5)–(2.6) has a unique solution on $\mathcal{J}$ if function $f(t, y(t))$ is a right-dense continuous bounded function such that there exists $M > 0$ for which $\left|f(t, y(t))\right| < M$ on $\mathcal{J}$ and the Lipshitz condition*

$$
\left|f(t, x(t)) - f(t, y(t))\right| \leq L\left\|x - y\right\|_{\infty}
$$

holds for some $L > 0$, for all $t \in \mathcal{J}$ and all $x(t)$, $y(t) \in \mathbb{R}$.

Proof. Let $\mathcal{S}$ be the set of rd-continuous functions on $\mathcal{J} \subseteq \mathbb{T}$. For $y \in \mathcal{S}$, define $\left\|y\right\| = \sup_{t\in\mathcal{J}}\left\|y(t)\right\|$. It is easy to see that S is a Banach space with this norm. The subset of $\mathcal{S}(R)$ and the operator T are defined by

$$
\mathcal{S}(R) = \{X \in \mathcal{S} : \|X_s\| \leq R\}
$$

and

$$
T(y) = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t - \rho(s))^{\alpha-1}f(s, y(s))\Delta s.
$$

Then,

$$\left|T(y(t))\right| \leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\rho(s))^{\alpha-1} \left|f(s,y(s))\right| \Delta s$$
$$\leq \frac{M}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\rho(s))^{\alpha-1} \Delta s.$$

Since $(t-\rho(s))^{\alpha-1}$ is an increasing monotone function, by using Proposition 1.1.32 we can write that

$$\int_{t_0}^{t} (t-\rho(s))^{\alpha-1} \Delta s \leq \int_{t_0}^{t} (t-\rho(s))^{\alpha-1} ds.$$

Consequently,

$$\left|T(y(t))\right| \leq \frac{M}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\rho(s))^{\alpha-1} ds. \tag{2.14}$$

For $0<\alpha<1$ we have

$$(t-\rho(s))^{\alpha-1} < (t-s)^{\alpha-1}$$

and from equation (2.14) it follows that

$$\left|T(y(t))\right| \leq \frac{M}{\Gamma(\alpha)} \frac{a^\alpha}{\alpha} =: \bar{R}.$$

With $\bar{R} = \frac{Ma^\alpha}{\Gamma(\alpha+1)}$, we conclude that T is an operator from $\mathcal{S}(R)$ to $\mathcal{S}(\bar{R})$. Moreover,

$$\begin{aligned}
\left\|T(x)-T(y)\right\| &\leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\rho(s))^{\alpha-1} \mid f(s,x(s)) - f(s,y(s)) \mid \Delta s \\
&\leq \frac{L\left\|x-y\right\|_\infty}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\rho(s))^{\alpha-1} \Delta s \\
&\leq \frac{L\left\|x-y\right\|_\infty}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\rho(s))^{\alpha-1} ds.
\end{aligned} \tag{2.15}$$

It follows from equation (2.15) that

$$\begin{aligned}
\left\|T(x)-T(y)\right\| &\leq \frac{L\left\|x-y\right\|_\infty}{\Gamma(\alpha)} \frac{a^\alpha}{\alpha} \\
&= \frac{La^\alpha}{\Gamma(\alpha+1)} \left\|x-y\right\|_\infty
\end{aligned}$$

for $x,\, y \in \mathcal{S}(R)$. If $\dfrac{La^\alpha}{\Gamma(\alpha+1)} \leq 1$, then one has a contraction map. This implies the uniqueness of solution to problem (2.5)–(2.6). ■

Chapter 3

Exact solution to a general Norton Massagué Model on time scales

In this chapter, we give a general nonlinear first-order Norton-Simon-Massagué model on time scales. Then we define the Cobb-Douglas production function on time scales and use it to give the solution for the equation that describes the model. Concrete examples are given.

3.1 Introduction

Mathematical models are powerful tools that are often used to describe real-world problems, illuminating different scientific and technical disciplines [1, 18, 51, 60]. In the literature, we find several mathematical models of tumor growth that have been proposed, each with its own details and parameters [5]. We mention, among them, a plethora of macroscopic tumor growth models [10]. In 1960, Bertalanffy derived the equation that can be used to describe a tumor growth process

$$\frac{dV}{dt} = aV^{\frac{2}{3}}(t) - bV(t) \tag{3.1}$$

where a and b are proportionality constants. The solution to the equation (3.1) is the following form:

$$V(t) = \left[\frac{a}{b} - (\frac{a}{b} - c)exp(\frac{-bt}{3})\right]^3, \text{ where } c \in \mathbb{R}.$$

In 2005, L. Norton [44] introduced the model.

$$\frac{dV}{dt} = aV^{\frac{d}{3}}(t) - bV(t) \tag{3.2}$$

$$V(t_0) = V_0. \tag{3.3}$$

Here $d > 0$. We point out that the Norton-Massagué equation (3.2)-(3.3) may be solved in closed form, namely,

$$V(t) = V(0)\left[\frac{a}{b}V(0)^{\frac{d}{3}-1} + e^{(b(\frac{d}{3}-1)t)}(1 - \frac{a}{b}V(0)^{\frac{b}{3}-1)})\right]^{\frac{3}{d-3}}.$$

In 2006, L. Norton and J. Massagué [45] introduced the general model

$$\frac{dV(t)}{dt} = aV^{\alpha}(t) - bV(t) \tag{3.4}$$

where $0 < \alpha < 1$ and a, b are constants of anabolism (growth) and catabolism (death), respectively. In this chapter, we will focus on solving the general Norton-Massagué model on arbitrary time scales $\mathbb{T}$

$$V^{\Delta}(t) = a(t)V^{\alpha}(t) - b(t)V(t) \tag{3.5}$$

where $0 < \alpha < 1$ and $a(t) > 0, b(t) > 0$. To our knowledge, this is the first study that considers the resolution of the so-called general Norton-Massagué tumor growth model on time scales. The study of dynamical systems on time scales is today an active field of research. Recently, Martin Bohner et al [13] studied solow models on time scales. Motivated by the work as mentioned above, we study the general tumor growth model on time scales.

3.2 General tumor growth model on time scales

In this section, we consider the general Norton-Massagué tumor growth model on time scales

$$V^{\Delta}(t) = a(t)V^{\alpha}(t) - b(t)V(t) \tag{3.6}$$

$$V(t_0) = V_0 \tag{3.7}$$

where $0 < \alpha < 1$ and $a, b \in C_{rd}$ or $a(t) > 0, b(t) > 0$ for all $t \in \mathbb{T}$.

Let $f(x) = x^\alpha, w(t) = (\frac{1}{\alpha-1} \odot \frac{bg}{a})(t)$ and $g(t) = (1-\alpha)a(t)$.
We recall the following definition of generalized Cobb-Douglas production function on time scales [13].

Definition 3.2.1. *We define the generalized Cobb-Douglas production function on time scales by $f(x) = x\tilde{f}(x)$. Provided that*

$$\tilde{f}(x) := \frac{b(t) + (w \ominus (\frac{1}{\alpha-1} \odot (gx^{\alpha-1}))(t)}{a(t)} \tag{3.8}$$

is independent of $t \in \mathbb{T}$.

Lemma 3.2.2. *If $\mu(t) = 0$ at t, then*

$$\frac{b(t) + (w \ominus (\frac{1}{\alpha-1} \odot (gx^{\alpha-1})))(t)}{a(t)} = x^{\alpha-1}. \tag{3.9}$$

Proof. Suppose $\mu(t) = 0$ at t, then we have

$$\begin{aligned} \frac{b(t) + \left(w \ominus (\frac{1}{\alpha-1} \odot (gx^{\alpha-1}))\right)(t)}{a(t)} &= \frac{b(t) + w(t) - \frac{g(t)x^{\alpha-1}}{\alpha-1}}{a(t)} \\ &= \frac{b(t) + \frac{b(t)g(t)}{(\alpha-1)a(t)} - \frac{g(t)x^{\alpha-1}}{\alpha-1}}{a(t)} \\ &= \frac{b(t) - b(t) + a(t)x^{\alpha-1}}{a(t)} \\ &= x^{\alpha-1}. \end{aligned}$$

The proof is complete. ∎

Example 3.2.3. *If $\mathbb{T} = \mathbb{R}$, then $\tilde{f}(x) = x^{\alpha-1}$, and thus (3.8) holds. Hence the Cobb-Douglas production function is defined and equals*

$$f(x) = x\tilde{f}(x) = xx^{\alpha-1} = x^\alpha.$$

Lemma 3.2.4. *Let $t \in \mathbb{T}$. If $\mu(t) > 0$, suppose $A = b(t) + (w \ominus (\frac{1}{\alpha-1} \odot (gx^{\alpha-1})))(t)$ then*

$$A = \frac{1}{\mu(t)} \left\{ b(t)\mu(t) - 1 + \left(\frac{1 + (1-\alpha)\mu(t)a(t)x^{\alpha-1}}{1 + (1-\alpha)\mu(t)b(t)} \right)^{\frac{1}{1-\alpha}} \right\}. \tag{3.10}$$

Proof. Let $\mu(t) > 0$ at $t \in \mathbb{T}$, then we have

$$\begin{aligned}
\frac{1}{\alpha-1} \odot (gx^{\alpha-1})(t) &= \frac{1}{\alpha-1}(gx^{\alpha-1})(t) \int_0^1 (1+\mu(t)g(t)x^{\alpha-1}\tau)^{\frac{1}{\alpha-1}-1} d\tau \\
&= \frac{1}{\alpha-1} gx^{\alpha-1} \frac{\left(1+\mu(t)(gx^{\alpha-1})(t)\tau\right)^{\frac{1}{\alpha-1}}]_0^1}{\frac{1}{\alpha-1}\mu(t)gx^{\alpha-1}} \\
&= \frac{1}{\mu(t)}\left(\left(1+\mu(t)gx^{\alpha-1}\right)^{\frac{1}{\alpha-1}} - 1\right).
\end{aligned}$$

Again, we have

$$\begin{aligned}
w(t) &= \frac{1}{\alpha-1} \odot \frac{bg(t)}{a(t)} \\
&= \frac{1}{\alpha-1} \odot (b(t)(1-\alpha)) \\
&= \frac{1}{\alpha-1} b(t)(1-\alpha) \int_0^1 (1+\mu(t)b(t)(1-\alpha)\tau)^{\frac{1}{\alpha-1}-1} d\tau \\
&= b(t)(1-\alpha)\left[\frac{(1+\mu(t)b(t)(1-\alpha)\tau)^{\frac{1}{\alpha-1}}}{\mu(t)b(t)(1-\alpha)}\right]_0^1 \\
&= \frac{1}{\mu(t)}\left[(1+\mu(t)b(t)(1-\alpha))^{\frac{1}{\alpha-1}} - 1\right];
\end{aligned}$$

hence

$$\begin{aligned}
w \ominus \left(\frac{1}{\alpha-1} \odot (gx^{\alpha-1})\right) &= \frac{w - \frac{(1+\mu((gx)^{\alpha-1})^{\frac{1}{\alpha-1}-1}-1}{\mu}}{1+\mu\frac{(1+\mu gx^{\alpha-1})^{\frac{1}{\alpha-1}}-1}{\mu}} \\
&= \frac{w-(1+\mu gx^{\alpha-1})^{\frac{1}{\alpha-1}} - 1}{(1+\mu gx^{\alpha-1})^{\frac{1}{\alpha-1}}} \\
&= \frac{\frac{(1+\mu b(1-\alpha))^{\frac{1}{\alpha-1}}-1}{\mu} - \frac{(1+\mu gx^{\alpha-1})^{\frac{1}{\alpha-1}}-1}{\mu}}{(1+\mu gx^{\alpha-1})^{\frac{1}{\alpha-1}}} \\
&= \frac{1}{\mu}\left\{-1+\left(\frac{1+\mu b(1-\alpha)}{1+\mu(1-\alpha)ax^{\alpha-1}}\right)^{\frac{1}{\alpha-1}}\right\} \\
&= \frac{1}{\mu}\left\{-1+\left(\frac{1+\mu(1-\alpha)ax^{\alpha-1}}{1+\mu b(1-\alpha)}\right)^{\frac{1}{1-\alpha}}\right\}.
\end{aligned}$$

Which ends the proof. ■

We assume the following additional hypothesis:

$H(1)$: Let $a(t) > 0$ and $b(t) > 0$, for all $t \in \mathbb{T}$ and suppose $\tilde{a} = a(t)\mu(t)$, $\tilde{b} = b(t)\mu(t)$ are independents of $t \in \mathbb{T}$.

Theorem 3.2.5. *Let $\mu(t) > 0$, assume that $H(1)$ holds, then (3.8) holds and the Cobb-Douglas production is defined by*

$$f(x) = \frac{x}{\tilde{a}} \left\{ \tilde{b} - 1 + \left(\frac{1 + (1-\alpha)\tilde{a}x^{\alpha-1}}{1 + (1-\alpha)\tilde{b}} \right)^{\frac{1}{1-\alpha}} \right\}.$$

Proof. In view of lemma 3.2.4, we obtain

$$\begin{aligned} \frac{b(t) + \left(w \ominus (\frac{1}{\alpha-1} \odot (gx^{\alpha-1}))\right)(t)}{a(t)} &= \frac{1}{\mu(t)a(t)} \Big\{ b(t)\mu(t) - 1 \\ &\quad + \left(\frac{(1 + (1-\alpha)\mu(t)a(t)x^{\alpha-1})}{1 + (1-\alpha)\mu(t)b(t)} \right)^{\frac{1}{1-\alpha}} \Big\} \\ &= \frac{1}{\tilde{a}} \left\{ \tilde{b} - 1 + \left(\frac{1 + (1-\alpha)\tilde{a}x^{\alpha-1}}{1 + (1-\alpha)\tilde{b}} \right)^{\frac{1}{1-\alpha}} \right\} \end{aligned}$$

is independent of t and therefore equals $\tilde{f}(x)$ hence, $f(x) = x\tilde{f}(x)$. ■

Example 3.2.6. *Assuming $\mathbb{T} = \mathbb{Z}$ and that a and b are constants then, the cobb-Douglas production function is defined and equals*

$$f(x) = \frac{x}{a} \Big\{ b - 1 + \Big(\frac{1 + (1-\alpha)ax^{\alpha-1}}{1 + (1-\alpha)b} \Big)^{\frac{1}{1-\alpha}} \Big\}.$$

Example 3.2.7. *Assuming $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$ and If $\tilde{a}(t) := (q-1)ta(t)$ and $\tilde{b} := (q-1)tb(t)$ are independent of $t \in \mathbb{T}$ then, the cobb-Douglas production function is defined and equals*

$$f(x) = \frac{x}{\tilde{a}} \Big\{ \tilde{b} - 1 + \Big(\frac{1 + (1-\alpha)\tilde{a}x^{\alpha-1}}{1 + (1-\alpha)\tilde{b}} \Big)^{\frac{1}{1-\alpha}} \Big\}.$$

Theorem 3.2.8. *Suppose $H(1)$ and (3.8) holds. Let f be defined by $f(x) = x\tilde{f}(x)$, then (3.6) holds if and only if*

$$V^{\Delta}(t) = \{w \ominus (\frac{1}{\alpha-1} \odot (gV^{\alpha-1}))\}(t)V(t) \tag{3.11}$$

where $g = (1-\alpha)a$ and $w = \frac{1}{\alpha-1} \odot \frac{bg}{a}$.

Remark 6. *The general Norton-Massagué equation is a special case of a Bernoulli equation on time scales.*

Corollary 3.2.9. *Suppose* $(\alpha - 1) \in \mathbb{R} - \{0\}$, $w = \frac{1}{\alpha-1} \odot \frac{bg}{a} \in \mathcal{R}(\alpha - 1)$ *and* $g = (1-\alpha)a \in C_{rd}$*. Let* $V_0 \neq 0$*, if*

$$\frac{1}{V_0^{\alpha-1}} + \int_0^t e_w^{\alpha-1}(\tau, 0)g(\tau)\Delta\tau > 0 \quad \text{for all} \quad t \in \mathbb{T},$$

then

$$V(t) = \frac{e_w(t,0)}{\left[\frac{1}{V_0^{\alpha-1}} + \int_0^t e_w^{\alpha-1}(\tau,0)g(\tau)\Delta\tau\right]^{\frac{1}{\alpha-1}}} \tag{3.12}$$

solves the general Norton-Massagué equation (3.6)-(3.7).

Theorem 3.2.10. *Assume that* $a(t) > 0, b(t) > 0$ *for all* $t \in \mathbb{T}$ *and* $\lambda := \dfrac{a(t)}{b(t)}$ *is independent of* $t \in \mathbb{T}$*. If we define* $p \in \mathcal{R}$ *by*

$$p(t) = (1-\alpha)b(t) \, \text{for all} \ t \in \mathbb{T}, \tag{3.13}$$

then the solution of (3.6)-(3.7) is given by

$$V(t) = \left\{\lambda + \frac{V_0^{1-\alpha} - \lambda}{e_p(t,t_0)}\right\}^{\dfrac{1}{1-\alpha}} \quad \text{for all } t \in \mathbb{T} \tag{3.14}$$

provided that $\lambda + \dfrac{V_0^{1-\alpha} - \lambda}{e_p(t,t_0)} > 0.$

Proof. Suppose V solves (3.6) such that $V(t_0) = V_0$. Let $\tilde{x} := V^{\alpha-1}$ by [[15], theorem 2.37], we have:

$$\begin{aligned}
\frac{\tilde{x}^\Delta}{\tilde{x}} &= (\alpha-1) \odot \frac{V^\Delta}{V} \\
&= (\alpha-1) \odot \{\omega \ominus [\frac{1}{\alpha-1} \odot (gV^{\alpha-1})]\} \\
&= [(\alpha-1) \odot \omega] \ominus (gV^{\alpha-1}) \\
&= (b(1-\alpha) \ominus (gV^{\alpha-1}));
\end{aligned}$$

so

$$\tilde{x}^\Delta = (p \ominus (g\tilde{x}))\tilde{x}.$$

Let $z := \dfrac{1}{\tilde{x}}$ we get

$$\begin{aligned} z^{\Delta} &= (\frac{1}{\tilde{x}})^{\Delta} \\ &= -\frac{(\tilde{x})^{\Delta}}{\tilde{x}\tilde{x}^{\sigma}} \\ &= -(p \ominus (g\tilde{x}))z^{\sigma} \\ &= \frac{g\tilde{x} - p}{1 + \mu g\tilde{x}} z^{\sigma} \end{aligned}$$

hence

$$(1 + \mu g\tilde{x})z^{\Delta} = g\tilde{x}z^{\sigma} - pz^{\sigma};$$

it means

$$z^{\Delta} + \mu z^{\Delta} g\tilde{x} = g\tilde{x}z^{\sigma} - pz^{\sigma}.$$

Using the simple useful formula

$$z^{\Delta} + g\tilde{x}(z^{\sigma} - z) = g\tilde{x}z^{\sigma} - pz^{\sigma}.$$

Seen that

$$\begin{aligned} g &= (1 - \alpha)a \\ &= (1 - \alpha)b\lambda \\ &= \lambda p \end{aligned}$$

we get

$$z^{\Delta} = -pz^{\sigma} + g \tag{3.15}$$

the variation of constants in the formula [[15] Theorem 2.74]. The solution to 3.15 is given by

$$\begin{aligned} z(t) &= z_0 e_{\ominus p}(t, t_0) + \int_{t_0}^{t} g(\tau) e_{\ominus p}(t, \tau) \Delta\tau \\ &= z_0 e_{\ominus p}(t, t_0) + \int_{t_0}^{t} \lambda p(\tau) e_p(\tau, t) \Delta\tau \\ &= z_0 e_{\ominus p}(t, t_0) + \lambda \int_{t_0}^{t} p(\tau) e_p(\tau, t) \Delta\tau \\ &= z_0 e_{\ominus p}(t, t_0) + \lambda e_p(\tau, t) \mid_{t_0}^{t} \\ &= z_0 e_{\ominus p}(t, t_0) + \lambda(1 - e_{\ominus p}(t, t_0)). \end{aligned}$$

We have $z_0 = V_0^{1-\alpha}$ as well as $V(t) = \frac{1}{z(t)^{\frac{1}{\alpha-1}}}$ which shows (3.14). Conversely, V given by (3.14) is easily seen to be a solution to (3.6)-(3.7). ■

3.3 Examples

Example 3.3.1. *Let $\mathbb{T} = m\mathbb{Z}$ with $m > 0$, considering the following equation*

$$V^{\Delta}(t) = \sqrt{2}V^{\frac{2}{3}}(t) - 5V(t), \quad t \in \mathbb{T} \tag{3.16}$$

$$V(0) = 1, \tag{3.17}$$

here $b(t) = 5$, $a(t) = \sqrt{2}$, $\alpha = \frac{2}{3}$, $g(t) = (1-\alpha)a(t) = (1-\frac{2}{3})\sqrt{2} = \frac{\sqrt{2}}{3}$ and $w(t) = \left(\frac{1}{\alpha-1} \odot \frac{bg}{a}\right)(t)$.
We taking

$$\begin{aligned} w(t) &= \frac{1}{\frac{2}{3}-1} \odot \frac{5\sqrt{2}}{3\sqrt{2}} \\ &= (-3) \odot \frac{5}{3} \\ &= (-3)\frac{5}{3}\int_0^1 (1+\mu(t)\frac{5}{3}\tau)^{-3-1}d\tau \\ &= \frac{1}{m}\left((1+\frac{5}{3}m)^{-3} - 1\right) \end{aligned}$$

also, $1 + \mu(t)w(t) = (1+\frac{5}{3}m)^{-3} > 0$ hence $w \in \mathcal{R}^+$. In addition

$$w \ominus (\frac{1}{\alpha-1} \odot (gx^{\alpha-1}))(t) = \frac{1}{m}\frac{((1+\frac{5}{3}m)^{-3} - 1) - \frac{\sqrt{2}m}{3}x^{-\frac{1}{3}}}{1 + m\frac{\sqrt{2}}{3}x^{-\frac{1}{3}}};$$

so

$$\begin{aligned} \tilde{f}(x) &= \frac{b(t) + (w \ominus (\frac{1}{\alpha-1} \odot (gx^{\alpha-1})))(t)}{a(t)} \\ &= \frac{1}{\sqrt{2}m}\left\{\frac{5m + (1+\frac{5}{3}m)^{-3} - \frac{\sqrt{2}}{3}mx^{-\frac{1}{3}} - 1}{1 + \frac{\sqrt{2}}{3}mx^{-\frac{1}{3}}}\right\} \end{aligned}$$

is independent of t. Let $\lambda(t) = \frac{a(t)}{b(t)} = \frac{\sqrt{2}}{5}$ and according to theorem 3.2.10 the solution of equation (3.16)-(3.17) is

$$V(t) = \left\{\frac{\sqrt{2}}{5} + \frac{1 - \frac{\sqrt{2}}{5}}{e_{\frac{5}{3}}(t,0)}\right\}^3,$$

seen that

$$\begin{aligned} e_{\frac{5}{3}}(t,o) &= exp\left(\int_0^t \zeta_{\mu(t)}(\frac{5}{3})\Delta t\right) \\ &= exp\sum_{\tau\in[0,t]_{\mathbb{T}}} \frac{1}{m}log(1+\frac{5}{3}m)\tau \\ &= (1+\frac{5}{3}m)^{\frac{t}{2m}(t+1)} \end{aligned}$$

where

$$\begin{aligned} \sum_{\tau\in I}\tau &= m(0+1+2+..+\frac{t}{m}) \\ &= \frac{m}{2}(\frac{t}{m}+1)(\frac{t}{m}) \\ &= \frac{t}{2}(\frac{t}{m}+1) \end{aligned}$$

here $I=\{0,m,2m,3m,...,t\}$, *we get*

$$V(t)=\left\{\frac{\sqrt{2}}{5}+\frac{1-\frac{\sqrt{2}}{5}}{(1+\frac{5}{3}m)^{\frac{t}{2m}(t+1)}}\right\}^3. \tag{3.18}$$

It is clear that $\tilde{b}(t)=mb(t)=5m>0$ *and* $\tilde{a}(t)=ma(t)=\sqrt{2}m>0$ *are independent of* t. *On the other hand, according to theorem 3.2.8 the system(3.16)-(3.17) equivalent to system(3.19)-(3.20)*

$$V^{\Delta}(t)=(w\ominus(\frac{1}{\alpha-1}\odot(gV^{\alpha-1})))(t)V(t), \quad t\in\mathbb{T} \tag{3.19}$$

$$V(0)=1, \tag{3.20}$$

we have

$$1+\frac{\sqrt{2}}{3}\int_0^t e_w^{\alpha-1}(\tau,0)d\tau>0 \quad for\ all \quad t\in\mathbb{T}$$

according to corollary 3.2.9, the general solution of Norton-Massagué equation (3.16)-(3.17) gives

$$\begin{aligned} V(t) &= \frac{e_w(t,0)}{\left[\frac{1}{V_0^{\alpha-1}} + \int_0^t e_w^{\alpha-1}(\tau,0)g(\tau)d\tau\right]^{\frac{1}{\alpha-1}}} \\ &= \frac{e_w(t,0)}{\left[1 + \frac{\sqrt{2}}{3}\int_0^t e_w^{-3}(\tau,0)d\tau\right]^{-3}} \end{aligned}$$

where

$$e_w(t,0) = exp\left(\int_0^t \zeta_{\mu(\tau)}(w(\tau))\Delta\tau\right), \tag{3.21}$$

since $\mathbb{T} = m\mathbb{Z}$ *with* $\mu(t) = m$

$$\begin{aligned} e_w(t,0) &= exp\left(\int_0^t \zeta_{\mu(\tau)}(w(\tau))\Delta\tau\right) \\ &= exp\left(\frac{1}{m}\frac{t}{2}(\frac{t}{m}+1)log(1+mw)\right) \end{aligned}$$

where $\zeta_{\mu(\tau)}(w(\tau)) = \frac{1}{\mu(\tau)}log(1+\mu(\tau)w(\tau))$*, and*

$$\sum_{\tau\in[0,t]_{\mathbb{T}}} \tau = \frac{t}{2}(\frac{t}{m}+1),$$

therefore

$$\begin{aligned} V(t) &= \frac{exp\left(\frac{1}{m}\frac{t}{2}(\frac{t}{m}+1)log(1+mw)\right)}{\left[1+\frac{\sqrt{2}}{3}\int_0^t exp^{\frac{-1}{3}}(\frac{1}{m}\frac{\tau}{2}(\frac{\tau}{m}+1)log(1+mw))\Delta\tau\right]^{-3}} \\ &= \frac{exp\left(\frac{1}{m}\frac{t}{2}(\frac{t}{m}+1)log(1+\frac{5}{3}m)\right)}{\left[1+\frac{\sqrt{2}}{3}\int_0^t exp^{\frac{-1}{3}}(\frac{1}{m}\frac{\tau}{2}(\frac{\tau}{m}+1)log(1+\frac{5}{3}m))\Delta\tau\right]^{-3}} \\ &= \frac{exp\left(\frac{1}{m}\frac{t}{2}(\frac{t}{m}+1)log(1+\frac{5}{3}m)\right)}{\left[1+\frac{\sqrt{2}}{3}\sum_{\tau=0}^t exp(\frac{\tau}{-6m}(\frac{\tau}{m}+1)log(1+\frac{5}{3}m))\right]^{-3}} \\ &= \frac{(1+\frac{5}{3}m)^{\frac{t}{2m}(\frac{t}{m}+1)}}{\left[1+\frac{\sqrt{2}}{3}\sum_{\rho=0}^{\frac{t}{m}}(1+\frac{5}{3}m)^{\frac{\rho(\rho+1)}{-6}}\right]^{-3}} \end{aligned}$$

it means

$$V(t) = (1+\frac{5}{3}m)^{\frac{t}{2m}(\frac{t}{m}+1)}\left[1+\frac{\sqrt{2}}{3}\sum_{\rho=0}^{\frac{t}{m}}(1+\frac{5}{3}m)^{\frac{\rho(\rho+1)}{-6}}\right]^3. \tag{3.22}$$

.

Conclusion

A new system of general nonlinear first-order Norton-Massagué tumor growth on time scales has been introduced. Using a resolved Cobb-Douglas production function on time scales to solve the proposed system and obtain the desired results generalize the continuous and discrete spaces. The results presented can be used to assess the solvability of some different classes of problems in the literature.

Chapter 4

Permanence and Stability of SAIQH Models for COVID-19 on time scales

This chapter studies a SAIQH (Susceptible-Asymptomatic-Infectious-Quarantined -Hospitalized) compartmental model on time scales where the definition of the Lyapunov function is given. Then, we prove the permanent of the system, the existence of solution, and sufficient conditions implying the dynamic system to have a unique almost periodic solution that is uniformly asymptotically stable.

4.1 Introduction

In 2019, the COVID-19 pandemic has appeared for the first time in Wuhan, China, attracting many researchers to investigate outbreaks and the spread of viruses [2, 3]. Some of the studies provided new mathematical compartmental models, illustrating well the important contributions of Mathematics to fight communicable diseases. Such models have been used to analyze the corresponding dynamics and to supply useful techniques in disease epidemiology [6, 36, 55].

Kim et al. studied a SAIQH (Susceptible, Asymptomatic, Infectious, Quarantined, Hospitalized) mathematical model to analyze the transmission dynamics of MERS and to estimate transmission rates [32]. Lemos-Paião et al. developed a SAIQH type model for COVID-19, which was important to describe and understand the pandemic in Portugal [36]. Similar to [36], here we also consider the H_{IC} class of hospitalized individuals in intensive care units.

Let the total living population under study, denoted by $N(t)$, $t \geq 0$, be divided into six classes: (i) the susceptible individuals $S(t)$; (ii) the infected individuals without (or with mild) symptoms $A(t)$ (the Asymptomatic); (iii) infected individuals $I(t)$ with visible symptoms; (iv) quarantined individuals $Q(t)$ in isolation at home; (v) hospitalized individuals $H(t)$; (vi) and hospitalized individuals $H_{IC}(t)$ in intensive care units. The mathematical model introduced and studied in [36] reads:

$$\begin{cases} \dot{S}(t) = \Lambda + \omega n Q(t) - [\lambda(t)(1-p) + \phi p + \gamma]S(t), \\ \dot{A}(t) = \lambda(t)(1-p)S(t) - [qv + \gamma]A(t), \\ \dot{I}(t) = qvA(t) - [\delta_1 + \gamma]I(t), \\ \dot{Q}(t) = \phi p S(t) + \delta_1 f_1 I(t) + \delta_2(1 - f_2 - f_3)H(t) - [\omega m + \gamma]Q(t), \\ \dot{H}(t) = \delta_1(1-f_1)I(t) + \eta(1-k)H_{IC}(t) - [\delta_2(1-f_2-f_3) + \delta_2 f_2 + \alpha_1 f_3 + \gamma]\, H(t), \\ \dot{H_{IC}}(t) = \delta_2 f_2 H(t) - [\eta(1-k) + \alpha_2 k + \gamma]H_{IC}(t), \end{cases} \tag{4.1}$$

where, for all time $t \geq 0$,

$$\lambda(t) = \frac{\beta\,(l_A A(t) + I(t) + l_H H(t))}{N(t)}$$

is a bounded positive function with

$$N(t) = S(t) + A(t) + I(t) + Q(t) + H(t) + H_{IC}(t),$$

$\beta, l_A, l_H > 0$, and all the other parameters in the model are nonnegative. In addition,

$$p, 1-p, k, 1-k, q, f_1, 1-f_1, f_2, f_3, 1-f_2-f_3 \in [0,1].$$

For more details on the mathematical model (4.1) we refer the reader to [36].

Motivated by the 2019 paper of Khuddush and Prasad on a n-species Lotka–Volterra system [31], here we extend (4.1) to a time scales $\mathbb{T}$, studying the permanence and uniform asymptotic stability of the unique positive almost periodic solution of the following SAIQH type model on time scales:

$$\begin{cases} S^{\Delta}(t) = \Lambda + \omega n Q(t) - [\lambda(t)(1-p) + \phi p + \gamma] S^{\sigma}(t), \\ A^{\Delta}(t) = \lambda(t)(1-p)S(t) - [qv + \gamma]A^{\sigma}(t), \\ I^{\Delta}(t) = qvA(t) - [\delta_1 + \gamma]I^{\sigma}(t), \\ Q^{\Delta}(t) = \phi p S(t) + \delta_1 f_1 I(t) + \delta_2(1 - f_2 - f_3)H(t) - [\omega m + \gamma]Q^{\sigma}(t), \\ H^{\Delta}(t) = \delta_1(1-f_1)I(t) + \eta(1-k)H_{IC}(t) - [\delta_2(1-f_2-f_3) + \delta_2 f_2 + \alpha_1 f_3 + \gamma]\, H^{\sigma}(t), \\ H_{IC}^{\Delta}(t) = \delta_2 f_2 H(t) - [\eta(1-k) + \alpha_2 k + \gamma]H_{IC}^{\sigma}(t), \end{cases} \tag{4.2}$$

where all parameters are real nonnegative numbers and $\lambda(t)$ is bounded positive, as mentioned before. In the particular case $\mathbb{T} = \mathbb{R}^{+}$, system (4.2) reduces to (4.1).

For convenience, in the sequel we put $x_1 = S$, $x_2 = A$, $x_3 = I$, $x_4 = Q$, $x_5 = H$, and $x_6 = H_{IC}$.

For a function $f(t)$ defined on $t \in \mathbb{T}^{+}$, we set

$$f^{L} := \inf\{f(t) : t \in \mathbb{T}^{+}\}, \quad f^{U} := \sup\{f(t) : t \in \mathbb{T}^{+}\}.$$

In the following, we give a lemma proved in [27] and some definitions [37].

Lemma 4.1.1. *Assume that $\alpha > 0$, $b > 0$, and $-\alpha \in \mathcal{R}^{+}$. If*

$$y^{\Delta}(t) \geq (\leq)\, b - \alpha y^{\sigma}(t), \quad y(t) > 0, \quad t \in [t_0, \infty)_{\mathbb{T}},$$

then

$$y(t) \geq (\leq)\, \frac{b}{\alpha}\left[1 + \left(\frac{\alpha y(t_0)}{b} - 1\right) e_{(-\alpha)}(t, t_0)\right], \quad t \in [t_0, \infty)_{\mathbb{T}},$$

where $e_{\cdot}(\cdot,\cdot)$ is the standard exponential function of the time scales calculus [15].

Definition 4.1.2. *A time scales $\mathbb{T}$ is called an almost periodic time scales if*

$$\Pi = \{\tau \in \mathbb{R} : \ t + \tau \in \mathbb{T}, \textit{ for all } t \in \mathbb{T}\} \neq \{0\}.$$

Definition 4.1.3. *Let $\mathbb{T}$ be an almost periodic time scales. A function $x \in C(\mathbb{T}, \mathbb{R}^{n})$ is called an almost periodic function if the ε-translation set of x, that is,*

$$E\{\varepsilon, x\} = \{\tau \in \Pi : \ |x(t+\tau) - x(t)| < \varepsilon \textit{ for all } t \in \mathbb{T}\},$$

is a relatively dense set in $\mathbb{T}$ for all $\varepsilon > 0$ and there exists a constant $l(\varepsilon) > 0$ such that each interval of length $l(\varepsilon)$ contains a $\tau \in E\{\varepsilon, x\}$ for which $|x(t+\tau) - x(t)| < \varepsilon$ for all $t \in \mathbb{T}$.

The value τ is known as the ε-translation number of x and $l(\varepsilon)$ is called the inclusion length of $E\{\varepsilon, x\}$.

Definition 4.1.4. *Let $\mathbb{D}$ be an open set in $\mathbb{R}^n$ and let $\mathbb{T}$ be a positive almost periodic time scales. A function $f \in C(\mathbb{T} \times \mathbb{D}, \mathbb{R}^n)$ is called an almost periodic function in $t \in \mathbb{T}$ uniformly for $x \in \mathbb{D}$ if the ε-translation set of f,*

$$E\{\varepsilon, f, \mathbb{S}\} = \{\tau \in \Pi : | f(t+\tau) - f(t) | < \varepsilon, \ \ for \ \ all \ \ (t, x) \in \mathbb{T} \times \mathbb{S}\},$$

is a relatively dense set in $\mathbb{T}$ for all $\varepsilon > 0$ and, for each compact subset $\mathbb{S}$ of $\mathbb{D}$, that is, for any given $\varepsilon > 0$ and each compact subset $\mathbb{S}$ of $\mathbb{D}$, there exists a constant $l(\varepsilon, \mathbb{S}) > 0$ such that each interval of length $l(\varepsilon, \mathbb{S})$ contains $\tau(\varepsilon, \mathbb{S}) \in E\{\varepsilon, f, \mathbb{S}\}$ for which

$$| f(t+\tau, x) - f(t, x) | < \varepsilon, \ \ for \ \ all \ \ (t, x) \in \mathbb{T} \times \mathbb{S}.$$

Consider a system

$$x^{\Delta}(t) = h(t, x), \tag{4.3}$$

where $h : \mathbb{T}^+ \times \mathbb{S}_B \longrightarrow \mathbb{R}^n$, $\mathbb{S}_B = \{x \in \mathbb{R}^n : \|x\| < B\}$ and $h(t, x)$ is almost periodic in t uniformly for $x \in \mathbb{S}_B$ and is continuous in x. In [62] the question of existence of a unique almost periodic solution $f(t, x(t)) \in \mathbb{S}$ of (4.3), which is uniformly asymptotically stable, is investigated. For our model, we obtain from [62] the following result.

Lemma 4.1.5. *Suppose that there exists a Lyapunov function $V(t, x, z)$ defined on $\mathbb{T}^+ \times \mathbb{S}_B \times \mathbb{S}_B$ satisfying the following conditions:*

(i) $a(\|x - z\|) \leq V(t, x, z) \leq b(\|x - z\|)$, *where* $a, b \in \mathbb{K}$ *with* $\mathbb{K} = \{\alpha \in C(\mathbb{R}^+, \mathbb{R}^+) : \alpha(0) = 0$ *and* α *is increasing*$\}$;

(ii) $|V(t, x, z) - V(t, x_1, z_1)| \leq L(\|x - x_1\| + \|z - z_1\|)$, *where* $L > 0$ *is a constant;*

(iii) $D^+V^{\Delta}(t, x, z) \leq -cV(t, x, z)$, *where* $c > 0$, $-c \in \mathcal{R}^+$,

and D^+V^{Δ} is the Dini derivative of V. Furthermore, if there exists a solution $x(t) \in \mathbb{S}$ of *system*

$$\begin{cases} x_1^{\Delta}(t) = \Lambda + \omega n x_4(t) - [\lambda(t)(1-p) + \phi p + \gamma] x_1^{\sigma}(t), \\ x_2^{\Delta}(t) = \lambda(t)(1-p)x_1(t) - [qv + \gamma] x_2^{\sigma}(t), \\ x_3^{\Delta}(t) = qv x_2(t) - [\delta_1 + \gamma] x_3^{\sigma}(t), \\ x_4^{\Delta}(t) = \phi p x_1(t) + \delta_1 f_1 x_3(t) + \delta_2(1 - f_2 - f_3) x_5(t) - [\omega n + \gamma] x_4^{\sigma}(t), \\ x_5^{\Delta}(t) = \delta_1(1-f_1)x_3(t) + \eta(1-k)x_6(t) - [\delta_2(1-f_2-f_3) + \delta_2 f_2 + \alpha_1 f_3 + \gamma] x_5^{\sigma}(t), \\ x_6^{\Delta}(t) = \delta_2 f_2 x_5(t) - [\eta(1-k) + \alpha_2 k + \gamma] x_6^{\sigma}(t), \end{cases} \tag{4.4}$$

for $t \in \mathbb{T}^+$, where $\mathbb{S} \cup \mathbb{S}_B$ is a compact set, then there exists a unique almost periodic solution $f(t) \in \mathbb{S}$ of system (4.4), which is uniformly asymptotically stable.

4.2 Main Results

Let $t_0 \in \mathbb{T}$ be a fixed positive initial time. Our main results are: the proof that system (4.4) is permanent (Section 4.2.1); a sufficient condition for existence of a solution to system (4.4) (Section 4.2.2); and conditions that imply the dynamic system (4.4) to have a unique almost periodic solution that is uniformly asymptotically stable (Section 4.2.3).

4.2.1 Permanence of solutions

We begin by introducing the notion of permanence of solutions.

Definition 4.2.1. *System (4.4) is said to be permanent if there exist positive constants m and M such that $m \leq \liminf_{t \longrightarrow \infty} x_i(t) \leq \limsup_{t \longrightarrow \infty} x_i(t) \leq M$, $i = 1, 2, \ldots, 6$, for any solution $(x_1(t), \ldots, x_6(t))$ of (4.4).*

Theorem 4.2.2. *System (4.4) is permanent.*

Proof. Let $Z(t) = (x_1(t), \ldots, x_6(t))$ be a positive solution of system (4.4). Then,

$$\begin{cases} x_1^{\Delta}(t) \geq \Lambda - [\lambda^U(1-p) + \phi p + \gamma] x_1^{\sigma}(t), \\ x_2^{\Delta}(t) \geq \lambda^L(1-p)m_1 - (qv + \gamma) x_2^{\sigma}(t), \\ x_3^{\Delta}(t) \geq qv m_2 - (\delta_1 + \gamma) x_3^{\sigma}(t), \\ x_4^{\Delta}(t) \geq \phi p m_1 + \delta_1 f_1 m_3 - (\omega n + \gamma) x_4^{\sigma}(t), \\ x_5^{\Delta}(t) \geq \delta_1(1-f_1)m_3 - [\delta_2(1-f_2-f_3) + \delta_2 f_2 + \alpha_1 f_3 + \gamma] x_5^{\sigma}(t), \\ x_6^{\Delta}(t) \geq \delta_2 f_2 m_5 - [\eta(1-k) + \alpha_2 k + \gamma] x_6^{\sigma}(t). \end{cases}$$

From Lemma 4.1.1, it follows that

$$x_1(t) \geq \frac{\Lambda}{\lambda^u(1-p)+\phi p+\gamma}\left[1+\left(\frac{\lambda^U(1-p)+\phi p+\gamma}{\Lambda}x_1(t_0)-1\right)e_{-(\lambda^U(1-p)+\phi p+\gamma)}(t,t_0)\right]$$

and we have $e_{-(\lambda^U(1-p)+\phi p+\gamma)}(t,t_0) \longrightarrow 0, \;\; as \;\; t \longrightarrow \infty$. Thus,

$$\begin{cases} x_1(t) \geq m_1 := \dfrac{\Lambda}{\lambda^U(1-p)+\phi p+\gamma}, & \text{for } t \geq T_1, \\ x_2(t) \geq m_2 := \dfrac{\lambda^L(1-p)m_1}{qv+\gamma}, & \text{for } t \geq T_2, \\ x_3(t) \geq m_3 := \dfrac{qvm_2}{\delta_1+\gamma}, & \text{for } t \geq T_3, \\ x_4(t) \geq m_4 := \dfrac{\phi pm_1+\delta_1 f_1 m_3}{\omega n+\gamma}, & \text{for } t \geq T_4, \\ x_5(t) \geq m_5 := \dfrac{\delta_1(1-f_1)m_3}{\delta_2(1-f_2-f_3)+\delta_2 f_2+\alpha_1 f_3+\gamma}, & \text{for } t \geq T_5, \\ x_6(t) \geq m_6 := \dfrac{\delta_2 f_2 m_5}{\eta(1-k)+\alpha_2 k+\gamma}, & \text{for } t \geq T_6. \end{cases}$$

Let $m = \min\limits_{1\leqslant i\leqslant 6} m_i$ and $T = \max\limits_{1\leqslant i\leqslant 6} T_i$. We can then write that $x_i(t) \geq m$ for all $t > T$. ■

4.2.2 Existence of solution

For system (4.4), we introduce the following assumption:

(H_1) $\lambda(t)$ is a bounded almost periodic function and satisfy $0 < \lambda^L \leq \lambda(t) \leq \lambda^U$.

To prove existence of solution, we first begin with a technical lemma.

Lemma 4.2.3. *If $(H1)$ holds, then, for any positive solution $Z(t) = (x_1(t), \ldots, x_6(t))$ of system (4.4), there exist positive constants M and T such that $x_i(t) < M$, $i = 1, \ldots, 6$, for all $t > T$.*

Proof. Let $Z(t) = (x_1(t), \ldots, x_6(t))$ be a positive solution of system (4.4). Then,

$$\begin{cases} x_1^\Delta(t) \leqslant \Lambda + \omega n\frac{\Lambda}{\gamma} - [\lambda^L(1-p)+\phi p+\gamma]x_1^\sigma(t), \\ x_2^\Delta(t) \leqslant \lambda^U(1-p)M_1 - (qv+\gamma)x_2^\sigma(t), \\ x_3^\Delta(t) \leqslant qvM_2 - (\delta_1+\gamma)x_3^\sigma(t), \\ x_4^\Delta(t) \leqslant \phi pM_1 + \delta_1 f_1 M_3 + \delta_2(1-f_2-f_3)\frac{\Lambda}{\gamma} - (\omega n+\gamma)x_4^\sigma(t), \\ x_5^\Delta(t) \leqslant \delta_1(1-f_1)M_3 + \eta(1-k)\frac{\Lambda}{\gamma} - [\delta_2(1-f_2-f_3)+\delta_2 f_2+\alpha_1 f_3+\gamma]x_5^\sigma(t), \\ x_6^\Delta(t) \leqslant \delta_2 f_2 M_5 - [\eta(1-k)+\alpha_2 k+\gamma]x_6^\sigma(t). \end{cases}$$

From Lemma 4.1.1,

$$x_1(t) \leqslant \frac{\Lambda + \omega n \frac{\Delta}{\gamma}}{\lambda^L(1-p) + \phi p + \gamma}\left[1 + \left(\frac{\lambda^L(1-p) + \phi p + \gamma}{\Lambda + \omega n \frac{\Delta}{\gamma}} x_1(t_0) - 1\right) e_{-(\lambda^L(1-p)+\phi p+\gamma)}(t, t_0)\right]$$

and we have $e_{-(\lambda^L(1-p)+\phi p+\gamma)}(t, t_0) \longrightarrow 0, \;\; as \;\; t \longrightarrow \infty$. Then,

$$x_1(t) \leqslant M_1 := \frac{\Lambda + \omega n \frac{\Delta}{\gamma}}{\lambda^L(1-p) + \phi p + \gamma},$$

$$\begin{cases} x_1(t) \leqslant M_1 := \dfrac{\Lambda + \omega n \frac{\Delta}{\gamma}}{\lambda^L(1-p) + \phi p + \gamma}, & \text{for } t \geq T_1, \\ x_2(t) \leqslant M_2 := \dfrac{\lambda^U(1-p)M_1}{q\nu + \gamma}, & \text{for } t \geq T_2, \\ x_3(t) \leqslant M_3 := \dfrac{q\nu M_2}{\delta_1 + \gamma}, & \text{for } t \geq T_3, \\ x_4(t) \leqslant M_4 := \dfrac{\phi p M_1 + \delta_1 f_1 M_3 + \delta_2(1 - f_2 - f_3)\frac{\Delta}{\gamma}}{\omega n + \gamma}, & \text{for } t \geq T_4, \\ x_5(t) \leqslant M_5 := \dfrac{\delta_1(1-f_1)M_3 + \eta(1-k)\frac{\Delta}{\gamma}}{\delta_2(1 - f_2 - f_3) + \delta_2 f_2 + \alpha_1 f_3 + \gamma}, & \text{for } t \geq T_5, \\ x_6(t) \leqslant M_6 := \dfrac{\delta_2 f_2 M_5}{\eta(1-k) + \alpha_2 k + \gamma}, & \text{for } t \geq T_6. \end{cases}$$

Let $M = \max\limits_{1 \leqslant i \leqslant 6} M_i$ and $T = \max\limits_{1 \leqslant i \leqslant 6} T_i$. Then $x_i(t) \leqslant M$ for all $t > T$. ∎

Define

$$\begin{aligned} \Omega \;\; = \;\; & \{(x_1(t), \ldots, x_6(t)) : (x_1(t), \ldots, x_6(t)) \\ & \textit{is a solution of } (4.4) \textit{ and } \; 0 < m \leq x_i \leq M, \;\; i = 1, \ldots, 6\}. \end{aligned}$$

Theorem 4.2.4. *If* $(H1)$ *holds, then the set* Ω *is nonempty.*

Proof. The almost periodicity of $\lambda(t)$ implies that there is a sequence $\{\xi_l\} \subseteq \mathbb{T}^+$ with $l \to \infty$ such that

$$\lambda(t + \xi_l) \to \lambda(t) \;\; \text{as} \;\; l \to \infty.$$

From Theorem 4.2.2 and Lemma 4.2.3, for each sufficiently small $\varepsilon > 0$, there exists a $t_1 \in \mathbb{T}^+$ such that

$$m - \varepsilon \leq x_i(t) \leq M + \varepsilon, \;\; \text{for all} \;\; t \geq t_1, \quad i = 1, \ldots, 6.$$

Set $x_{il}(t) = x_i(t + \xi_l)$ for $t \geq t_1 - \xi_l$, $l = 1, 2, \ldots$ For any positive integer k, we obtain that there exists a sequence $\{x_{il}(t) : l \geq k\}$ such that the sequence $\{x_{il}(t)\}$ has a subsequence, denoted by $\{x^*_{il}(t)\}$ ($x^*_{il}(t) = x_{il}(t + \xi^*_l)$), converging on any finite interval of $\mathbb{T}^+$ as $l \to \infty$. So we have a sequence $\{y_i(t)\}$ such that, for $t \in \mathbb{T}^+$,

$$x^*_{il}(t) \longrightarrow y_i(t), \quad \text{as } l \to \infty, \ i = 1, \ldots, 6. \tag{4.5}$$

It is easy to see that the above sequence $\{\xi^*_l\} \subseteq \mathbb{T}^+$ with $\xi^*_l \to \tau$ for $l \to \infty$ is such that

$$\lambda(t + \xi^*_l) \longrightarrow \lambda(t), \quad \text{as} \quad l \to \infty,$$

which, together with (4.5) and

$$\begin{cases} x^{*\Delta}_{1l}(t) = \Lambda + \omega n x^*_{4l}(t) - [\lambda(t + \xi^*_l)(1-p) + \phi p + \gamma] x^{*\sigma}_{1l}(t), \\ x^{*\Delta}_{2l}(t) = \lambda(t + \xi^*_l))(1-p) x^*_{1l}(t) - [qv + \gamma] x^{*\sigma}_{2l}(t), \\ x^{*\Delta}_{3l}(t) = qv x^*_{2l}(t) - [\delta_1 + \gamma] x^{*\sigma}_{3l}(t), \\ x^{*\Delta}_{4l}(t) = \phi p x^*_{1l}(t) + \delta_1 f_1 x^*_{3l}(t) + \delta_2(1 - f_2 - f_3) x^*_{5l}(t) - [\omega n + \gamma] x^{*\sigma}_{4l}(t), \\ x^{*\Delta}_{5l}(t) = \delta_1(1 - f_1) x^*_{3l}(t) + \eta(1-k) x^*_{6l}(t) - [\delta_2(1 - f_2 - f_3) + \delta_2 f_2 + \alpha_1 f_3 + \gamma] x^{*\sigma}_{5l}(t), \\ x^{*\Delta}_{6l}(t) = \delta_2 f_2 x^*_{5l}(t) - [\eta(1-k) + \alpha_2 k + \gamma] x^{*\sigma}_{6l}(t), \end{cases}$$

yields

$$\begin{cases} y^{\Delta}_1(t) = \Lambda + \omega n y_4(t) - [\lambda(t)(1-p) + \phi p + \gamma] y^{\sigma}_1(t), \\ y^{\Delta}_2(t) = \lambda(t)(1-p) y_1(t) - [qv + \gamma] y^{\sigma}_2(t), \\ y^{\Delta}_3(t) = qv y_2(t) - [\delta_1 + \gamma] y^{\sigma}_3(t), \\ y^{\Delta}_4(t) = \phi p y_1(t) + \delta_1 f_1 y_3(t) + \delta_2(1 - f_2 - f_3) y_5(t) - [\omega n + \gamma] y^{\sigma}_4(t), \\ y^{\Delta}_5(t) = \delta_1(1 - f_1) y_3(t) + \eta(1-k) y_6(t) - [\delta_2(1 - f_2 - f_3) + \delta_2 f_2 + \alpha_1 f_3 + \gamma]\, y^{\sigma}_5(t), \\ y^{\Delta}_6(t) = \delta_2 f_2 y_5(t) - [\eta(1-k) + \alpha_2 k + \gamma]\, y^{\sigma}_6(t). \end{cases}$$

It is clear that $(y_1(t), \ldots, y_6(t))$ is a solution of system (4.4) and

$$m - \varepsilon \leq y_i(t) \leq M + \varepsilon, \ \text{ for all } \ t \in \mathbb{T}^+, \quad i = 1, \ldots, 6.$$

Since ε is arbitrary, it follows that

$$m \leq y_i(t) \leq M, \ \text{ for } \ t \in \mathbb{T}^+, \quad i = 1, \ldots, 6.$$

The proof is complete. ∎

4.2.3 Uniform asymptotic stability

Now, we establish sufficient conditions for the existence of a unique positive almost periodic solution to system (4.4) that is uniform asymptotically stable. We introduce some more notations. Let

$$\begin{aligned}
A_1 &:= \lambda^L(1-p)+\phi p+\gamma; \quad A_2 := qv+\gamma;\\
A_3 &:= \delta_1+\gamma; \quad A_4 := \omega n+\gamma;\\
A_5 &:= \delta_2(1-f_3)+\alpha_1 f_3+\gamma; \quad A_6 := \eta(1-k)+\alpha_2 k+\gamma;\\
B_1 &:= \lambda^U(1-p)+\phi p; \quad B_2 := qv+2\frac{\gamma\beta l_A(1-p)M}{\Lambda};\\
B_3 &:= \delta_1+2\frac{\gamma\beta(1-p)M}{\Lambda}; \quad B_4 := \omega n;\\
B_5 &:= \delta_2(1-f_3)+2\frac{\gamma\beta l_H(1-p)M}{\Lambda}; \quad B_6 := \eta(1-k).
\end{aligned}$$

Moreover, let $A := \min_{1\le i\le 6} A_i$ and $B := \max_{1\le i\le 6} B_i$.

In our next result (Theorem 4.2.5), we assume the following additional hypothesis:

(H2) $B < A$.

Theorem 4.2.5. *If (H1) and (H2) hold, then the dynamic system (4.4) has a unique almost periodic solution $Z(t) = (x_1(t),\dots,x_6(t)) \in \Omega$ that is uniformly asymptotically stable.*

Proof. According to Theorem 4.2.2, every solution $Z(t) = (x_1(t),\dots,x_6(t))$ of system (4.4) satisfies $x_i^L \le x_i \le x_i^U$. Hence, $|x_i(t)| \le K_i$, $i = 1,\dots,6$. Suppose that $Z(t) = (x_1(t),\dots,x_6(t))$ and $\widehat{Z}(t) = (\widehat{x_1}(t),\dots,\widehat{x_6}(t))$ are two positive solutions of system (4.4). We have

$$\begin{cases}
x_1^\Delta(t) = \Lambda+\omega n x_4(t)-[\lambda(t)(1-p)+\phi p+\gamma]x_1^\sigma(t),\\
x_2^\Delta(t) = \lambda(t)(1-p)x_1(t)-[qv+\gamma]x_2^\sigma(t),\\
x_3^\Delta(t) = qvx_2(t)-[\delta_1+\gamma]x_3^\sigma(t),\\
x_4^\Delta(t) = \phi p x_1(t)+\delta_1 f_1 x_3(t)+\delta_2(1-f_2-f_3)x_5(t)-[\omega n+\gamma]x_4^\sigma(t),\\
x_5^\Delta(t) = \delta_1(1-f_1)x_3(t)+\eta(1-k)x_6(t)-[\delta_2(1-f_2-f_3)+\delta_2 f_2+\alpha_1 f_3+\gamma]x_5^\sigma(t),\\
x_6^\Delta(t) = \delta_2 f_2 x_5(t)-[\eta(1-k)+\alpha_2 k+\gamma]x_6^\sigma(t),
\end{cases}$$

and

$$\begin{cases} \widehat{x_1^\Delta}(t) = \Lambda + \omega n\widehat{x_4}(t) - [\widehat{\lambda}(t)(1-p) + \phi p + \gamma]\widehat{x_1^\sigma}(t), \\ \widehat{x_2^\Delta}(t) = \widehat{\lambda}(t)(1-p)\widehat{x_1}(t) - [qv + \gamma]\widehat{x_2^\sigma}(t), \\ \widehat{x_3^\Delta}(t) = qv\widehat{x_2}(t) - [\delta_1 + \gamma]\widehat{x_3^\sigma}(t), \\ \widehat{x_4^\Delta}(t) = \phi p\widehat{x_1}(t) + \delta_1 f_1\widehat{x_3}(t) + \delta_2(1 - f_2 - f_3)\widehat{x_5}(t) - [\omega n + \gamma]\widehat{x_4^\sigma}(t), \\ \widehat{x_5^\Delta}(t) = \delta_1(1-f_1)\widehat{x_3}(t) + \eta(1-k)\widehat{x_6}(t) - [\delta_2(1-f_2-f_3) + \delta_2 f_2 + \alpha_1 f_3 + \gamma]\widehat{x_5^\sigma}(t), \\ \widehat{x_6^\Delta}(t) = \delta_2 f_2\widehat{x_5}(t) - [\eta(1-k) + \alpha_2 k + \gamma]\widehat{x_6^\sigma}(t). \end{cases}$$

Denote

$$||Z|| = ||(x_1(t), \dots, x_6(t))|| = \sup_{t\in\mathbb{T}^+} \sum_{i=1}^{6} |x_i(t)|.$$

Then $||Z|| \le K$ and $\left\|\widehat{Z}\right\| \le K$ where $K = \sum_{i=1}^{6} K_i$. Define the Lyapunov function $V(t, Z, \widehat{Z})$ on $\mathbb{T}^+ \times \Omega \times \Omega$ as

$$V(t, Z, \widehat{Z}) = \sum_{i=1}^{6} \left|x_i(t) - \widehat{x_i}(t)\right| = \sum_{i=1}^{6} V_i(t), \tag{4.6}$$

where $V_i(t) = \left|x_i(t) - \widehat{x_i}(t)\right|$. Then the two norms

$$\left\|Z - \widehat{Z}\right\| = \sum_{i=1}^{6} \left|x_i(t) - \widehat{x_i}(t)\right|$$

and

$$\left\|Z - \widehat{Z}\right\|_* = \sup_{t\in\mathbb{R}_+} \left[\sum_{i=1}^{6} (x_i(t) - \widehat{x_i}(t))^2\right]^{\frac{1}{2}}$$

are equivalent, that is, there exist two constants η_1, $\eta_2 > 0$ such that

$$\eta_1 \left\|Z - \widehat{Z}\right\|_* \le \left\|Z - \widehat{Z}\right\| \le \eta_2 \left\|Z - \widehat{Z}\right\|_*.$$

Hence,

$$\eta_1 \left\|Z - \widehat{Z}\right\|_* \le V(t, Z, \widehat{Z}) \le \eta_2 \left\|Z - \widehat{Z}\right\|_*.$$

Let a, $b \in C(\mathbb{R}^+, \mathbb{R}^+)$, $a(x) = \eta_1 x$, and $b(x) = \eta_2 x$. Then the assumption (i) of

Lemma 4.1.5 is satisfied. On the other hand, we have

$$\begin{aligned}\left|V(t,Z(t),\widehat{Z}(t))-V(t,Z^*(t),\widehat{Z}^*(t))\right| &= \left|\sum_{i=1}^{6}\left|x_i(t)-\widehat{x}_i(t)\right|-\sum_{i=1}^{6}\left|x_i^*(t)-\widehat{x}_i^*(t)\right|\right|\\ &\leq \sum_{i=1}^{6}\left|\left|x_i(t)-\widehat{x}_i(t)\right|-\left|x_i^*(t)-\widehat{x}_i^*(t)\right|\right|\\ &\leq \sum_{i=1}^{6}\left|x_i(t)-x_i^*(t)\right|+\sum_{i=1}^{6}\left|\widehat{x}_i(t)-\widehat{x}_i^*(t)\right|\\ &\leq \|Z-Z^*\|+\left\|\widehat{Z}-\widehat{Z}^*\right\|,\end{aligned}$$

where $L=1$, so condition (ii) of Lemma 4.1.5 is also satisfied.

Now, using Lemma 4.2 of [38], it follows that

$$D^+V_i^\Delta(t)\leq sign(x_i^\sigma(t)-\widehat{x}_i^\sigma(t))(x_i^\Delta(t)-\widehat{x}_i^\Delta(t)),\quad i=1,\ldots,6,$$

where $D^+V_i^\Delta$ is the Dini derivative of V_i. For $i=1$,

$$\begin{aligned}D^+V_1^\Delta(t) &\leq sign(x_1^\sigma(t)-\widehat{x}_1^\sigma(t))(x_1^\Delta(t)-\widehat{x}_1^\Delta(t))\\ &= sign(x_1^\sigma(t)-\widehat{x}_1^\sigma(t))[\omega n(x_4(t)-\widehat{x}_4(t))-\lambda(t)(1-p)x_1^\sigma(t)\\ &\quad-[\phi p+\gamma](x_1^\sigma(t)-\widehat{x}_1^\sigma(t))+\widehat{\lambda}(t)(1-p)\widehat{x}_1^\sigma(t)]\\ &= sign(x_1^\sigma(t)-\widehat{x}_1^\sigma(t))[\omega n(x_4(t)-\widehat{x}_4(t))-\lambda(t)(1-p)x_1^\sigma(t)\\ &\quad-[\phi p+\gamma](x_1^\sigma(t)-\widehat{x}_1^\sigma(t))+\widehat{\lambda}(t)(1-p)\widehat{x}_1^\sigma(t)+\lambda(t)(1-p)\widehat{x}_1^\sigma(t)-\lambda(t)(1-p)\widehat{x}_1^\sigma(t)]\\ &= sign(x_1^\sigma(t)-\widehat{x}_1^\sigma(t))[\omega n(x_4(t)-\widehat{x}_4(t))\\ &\quad-[\lambda(t)(1-p)+\phi p+\gamma](x_1^\sigma(t)-\widehat{x}_1^\sigma(t))\\ &\quad+(\widehat{\lambda}(t)-\lambda(t))(1-p)\widehat{x}_1^\sigma(t)]\\ &= sign(x_1^\sigma(t)-\widehat{x}_1^\sigma(t))[\omega n(x_4(t)-\widehat{x}_4(t))-[\lambda(t)(1-p)+\phi p+\gamma](x_1^\sigma(t)-\widehat{x}_1^\sigma(t))\\ &\quad+(\widehat{\lambda}(t)-\lambda(t))(1-p)\widehat{x}_1^\sigma(t)]\\ &\leq \omega n\left|x_4(t)-\widehat{x}_4(t)\right|-[\lambda^L(1-p)+\phi p+\gamma]\left|x_1^\sigma(t)-\widehat{x}_1^\sigma(t)\right|\\ &\quad+\frac{\gamma\beta l_A(1-p)M}{\Lambda}\left|x_2(t)-\widehat{x}_2(t)\right|+\frac{\gamma\beta(1-p)M}{\Lambda}\left|x_3(t)-\widehat{x}_3(t)\right|\\ &\quad+\frac{\gamma\beta l_H(1-p)M}{\Lambda}\left|x_5(t)-\widehat{x}_5(t)\right|;\end{aligned}$$

for $i = 2$,

$$\begin{aligned}
D^+V_2^\Delta(t) &\le sign(x_2^\sigma(t) - \widehat{x_2^\sigma}(t))(x_2^\Delta(t) - \widehat{x_2^\Delta}(t)) \\
&= sign(x_2^\sigma(t) - \widehat{x_2^\sigma}(t))[\lambda(t)(1-p)x_1(t) - [qv+\gamma](x_2^\sigma(t) - \widehat{x_2^\sigma}(t)) - \widehat{\lambda}(t)(1-p)\widehat{x_1}(t)] \\
&= sign(x_2^\sigma(t) - \widehat{x_2^\sigma}(t))[\lambda(t)(1-p)x_1(t) - [qv+\gamma](x_2^\sigma(t) - \widehat{x_2^\sigma}(t)) \\
&\quad -\widehat{\lambda}(t)(1-p)\widehat{x_1}(t) + \lambda(t)(1-p)\widehat{x_1}(t) - \lambda(t)(1-p)\widehat{x_1}(t)] \\
&= sign(x_2^\sigma(t) - \widehat{x_2^\sigma}(t))[-[qv+\gamma](x_2^\sigma(t) - \widehat{x_2^\sigma}(t)) \\
&\quad +\lambda(t)(1-p)(x_1(t) - \widehat{x_1}(t)) - (\widehat{\lambda}(t) - \lambda(t))(1-p)\widehat{x_1}(t)] \\
&\le \lambda^U(1-p)\left|x_1(t) - \widehat{x_1}(t)\right| - [qv+\gamma]\left|x_2^\sigma(t) - \widehat{x_2^\sigma}(t)\right| \\
&\quad +(1-p)M\left|\lambda(t) - \widehat{\lambda}(t)\right| \\
&\le \lambda^U(1-p)\left|x_1(t) - \widehat{x_1}(t)\right| - [qv+\gamma]\left|x_2^\sigma(t) - \widehat{x_2^\sigma}(t)\right| \\
&\quad +\frac{\gamma\beta l_A(1-p)M}{\Lambda}\left|x_2(t) - \widehat{x_2}(t)\right| \\
&\quad +\frac{\gamma\beta(1-p)M}{\Lambda}\left|x_3(t) - \widehat{x_3}(t)\right| + \frac{\gamma\beta l_H(1-p)M}{\Lambda}\left|x_5(t) - \widehat{x_5}(t)\right|;
\end{aligned}$$

for $i = 3$,

$$\begin{aligned}
D^+V_3^\Delta(t) &\le sign(x_3^\sigma(t) - \widehat{x_3^\sigma}(t))(x_3^\Delta(t) - \widehat{x_3^\Delta}(t)) \\
&= sign(x_3^\sigma(t) - \widehat{x_3^\sigma}(t))[qv(x_2(t) - \widehat{x_2}(t)) \\
&\quad -[\delta_1+\gamma](x_3^\sigma(t) - \widehat{x_3^\sigma}(t))] \\
&\le qv\left|x_2(t) - \widehat{x_2}(t)\right| - [\delta_1+\gamma]\left|x_3^\sigma(t) - \widehat{x_3^\sigma}(t)\right|;
\end{aligned}$$

for $i = 4$,

$$\begin{aligned}
D^+V_4^\Delta(t) &\le sign(x_4^\sigma(t) - \widehat{x_4^\sigma}(t))(x_4^\Delta(t) - \widehat{x_4^\Delta}(t)) \\
&= sign(x_4^\sigma(t) - \widehat{x_4^\sigma}(t))[\phi p(x_1(t) - \widehat{x_1}(t)) + \delta_1 f_1(x_3(t) - \widehat{x_3}(t)) \\
&\quad +\delta_2(1-f_2-f_3)(x_5(t) - \widehat{x_5}(t)) - [\omega n+\gamma](x_4^\sigma(t) - \widehat{x_4^\sigma}(t))] \\
&\le \phi p\left|x_1(t) - \widehat{x_1}(t)\right| + \delta_1 f_1\left|x_3(t) - \widehat{x_3}(t)\right| \\
&\quad +\delta_2(1-f_2-f_3)\left|x_5(t) - \widehat{x_5}(t)\right| - [\omega n+\gamma]\left|x_4^\sigma(t) - \widehat{x_4^\sigma}(t)\right|;
\end{aligned}$$

for $i = 5$,

$$\begin{aligned}
D^+V_5^\Delta(t) &\leq sign(x_5^\sigma(t) - \widehat{x_5^\sigma}(t))(x_5^\Delta(t) - \widehat{x_5^\Delta}(t)) \\
&= sign(x_5^\sigma(t) - \widehat{x_5^\sigma}(t))[\delta_1(1-f_1)(x_3(t) - \widehat{x_3}(t)) + \eta(1-k)(x_6(t) - \widehat{x_6}(t)) \\
&\quad -[\delta_2(1-f_2-f_3) + \delta_1 f_1 + \alpha_1 f_3 + \gamma](x_5^\sigma(t) - \widehat{x_5^\sigma}(t))] \\
&\leq \delta_1(1-f_1)\left|x_3(t) - \widehat{x_3}(t)\right| + \eta(1-k)\left|x_6(t) - \widehat{x_6}(t)\right| \\
&\quad -[\delta_2(1-f_2-f_3) + \delta_1 f_1 + \alpha_1 f_3 + \gamma]\left|x_5^\sigma(t) - \widehat{x_5^\sigma}(t)\right|;
\end{aligned}$$

and, finally, for $i = 6$,

$$\begin{aligned}
D^+V_6^\Delta(t) &\leq sign(x_6^\sigma(t) - \widehat{x_6^\sigma}(t))(x_6^\Delta(t) - \widehat{x_6^\Delta}(t)) \\
&= sign(x_6^\sigma(t) - \widehat{x_6^\sigma}(t))[\delta_2 f_2(x_5(t) - \widehat{x_5}(t)) \\
&\quad -[\eta(1-k) + \alpha_2 k + \gamma](x_6^\sigma(t) - \widehat{x_6^\sigma}(t) \\
&\leq \delta_2 f_2\left|x_5(t) - \widehat{x_5}(t)\right| - [\eta(1-k) + \alpha_2 k + \gamma]\left|x_6^\sigma(t) - \widehat{x_6^\sigma}(t)\right|.
\end{aligned}$$

It follows that

$$\begin{aligned}
D^+V^\Delta(t) &\leq \omega n\left|x_4(t) - \widehat{x_4}(t)\right| - [\lambda^L(1-p) + \phi p + \gamma]\left|x_1^\sigma(t) - \widehat{x_1^\sigma}(t)\right| \\
&\quad + \frac{\gamma\beta l_A(1-p)M}{\Lambda}\left|x_2(t) - \widehat{x_2}(t)\right| + \frac{\gamma\beta(1-p)M}{\Lambda}\left|x_3(t) - \widehat{x_3}(t)\right| \\
&\quad + \frac{\gamma\beta l_H(1-p)M}{\Lambda}\left|x_5(t) - \widehat{x_5}(t)\right| + \lambda^U(1-p)\left|x_1(t) - \widehat{x_1}(t)\right| \\
&\quad -[q\nu + \gamma]\left|x_2^\sigma(t) - \widehat{x_2^\sigma}(t)\right| + \frac{\gamma\beta l_A(1-p)M}{\Lambda}\left|x_2(t) - \widehat{x_2}(t)\right| \\
&\quad + \frac{\gamma\beta(1-p)M}{\Lambda}\left|x_3(t) - \widehat{x_3}(t)\right| + \frac{\gamma\beta l_H(1-p)M}{\Lambda}\left|x_5(t) - \widehat{x_5}(t)\right| \\
&\quad + q\nu\left|x_2(t) - \widehat{x_2}(t)\right| - [\delta_1 + \gamma]\left|x_3^\sigma(t) - \widehat{x_3^\sigma}(t)\right| \\
&\quad + \phi p\left|x_1(t) - \widehat{x_1}(t)\right| + \delta_1 f_1\left|x_3(t) - \widehat{x_3}(t)\right| \\
&\quad + \delta_2(1-f_2-f_3)\left|x_5(t) - \widehat{x_5}(t)\right| - [\omega n + \gamma]\left|x_4^\sigma(t) - \widehat{x_4^\sigma}(t)\right| \\
&\quad + \delta_1(1-f_1)\left|x_3(t) - \widehat{x_3}(t)\right| + \eta(1-k)\left|x_6(t) - \widehat{x_6}(t)\right| \\
&\quad -[\delta_2(1-f_2-f_3) + \delta_1 f_1 + \alpha_1 f_3 + \gamma]\left|x_5^\sigma(t) - \widehat{x_5^\sigma}(t)\right| \\
&\quad + \delta_2 f_2\left|x_5(t) - \widehat{x_5}(t)\right| - [\eta(1-k) + \alpha_2 k + \gamma]\left|x_6^\sigma(t) - \widehat{x_6^\sigma}(t)\right|
\end{aligned}$$

$$
\begin{aligned}
&= -[\lambda(t)(1-p)+\phi p+\gamma]\left|x_1^\sigma(t)-\widehat{x_1^\sigma}(t)\right|-[qv+\gamma]\left|x_2^\sigma(t)-\widehat{x_2^\sigma}(t)\right| \\
&\quad -[\delta_1+\gamma]\left|x_3^\sigma(t)-\widehat{x_3^\sigma}(t)\right|-[\omega n+\gamma]\left|x_4^\sigma(t)-\widehat{x_4^\sigma}(t)\right| \\
&\quad -[\delta_2(1-f_2-f_3)+\delta_1 f_1+\alpha_1 f_3+\gamma]\left|x_5^\sigma(t)-\widehat{x_5^\sigma}(t)\right| \\
&\quad -[\eta(1-k)+\alpha_2 k+\gamma]\left|x_6^\sigma(t)-\widehat{x_6^\sigma}(t)\right| \\
&\quad +\left\{\lambda^U(1-p)+\phi p\right\}\left|x_1(t)-\widehat{x_1}(t)\right| \\
&\quad +\left\{qv+\frac{\gamma\beta l_A(1-p)M}{\Lambda}+\frac{\gamma\beta l_A(1-p)M}{\Lambda}\right\}\left|x_2(t)-\widehat{x_2}(t)\right| \\
&\quad +\left\{\delta_1 f_1+\delta_1(1-f_1)+\frac{\gamma\beta(1-p)M}{\Lambda}+\frac{\gamma\beta(1-p)M}{\Lambda}\right\}\left|x_3(t)-\widehat{x_3}(t)\right| \\
&\quad +\omega n\left|x_4(t)-\widehat{x_4}(t)\right|+\left\{\delta_2(1-f_2-f_3)+\delta_2 f_2+\frac{\gamma\beta l_H(1-p)M}{\Lambda}\right. \\
&\quad \left.+\frac{\gamma\beta l_H(1-p)M}{\Lambda}\right\}\left|x_5(t)-\widehat{x_5}(t)\right|+\eta(1-k)\left|x_6(t)-\widehat{x_6}(t)\right| \\
&= -A_1\left|x_1^\sigma(t)-\widehat{x_1^\sigma}(t)\right|-A_2\left|x_2^\sigma(t)-\widehat{x_2^\sigma}(t)\right| \\
&\quad -A_3\left|x_3^\sigma(t)-\widehat{x_3^\sigma}(t)\right|-A_4\left|x_4^\sigma(t)-\widehat{x_4^\sigma}(t)\right| \\
&\quad -A_5\left|x_5^\sigma(t)-\widehat{x_5^\sigma}(t)\right|-A_6\left|x_6^\sigma(t)-\widehat{x_6^\sigma}(t)\right| \\
&\quad +B_1\left|x_1(t)-\widehat{x_1}(t)\right|+B_2\left|x_2(t)-\widehat{x_2}(t)\right| \\
&\quad +B_3\left|x_3(t)-\widehat{x_3}(t)\right|+B_4\left|x_4(t)-\widehat{x_4}(t)\right| \\
&\quad +B_5\left|x_5(t)-\widehat{x_5}(t)\right|+B_6\left|x_6(t)-\widehat{x_6}(t)\right| \\
&= -AV(\sigma(t))+BV(t) \\
&= (B-A)V(t)-A\mu(t)D^+V^\Delta(t)
\end{aligned}
$$

and $D^+V^\Delta(t) \le \frac{B-A}{1+A\mu(t)}V(t) \le -\psi(t)V(t)$ with $\psi = \frac{A-B}{1+A\mu^U}$. By $(H2)$, we have $\psi(t) = \frac{A-B}{1+A\mu^U} > 0$ and $1-\psi\mu(t) = 1+A(\mu^U-\mu(t))+\mu(t)B > 0$. Hence, $-\psi \in \mathcal{R}^+$. Thus, the assumption (iii) of Lemma 4.1.5 is satisfied and it follows from Lemma 4.1.5 that there exists a unique almost periodic solution $Z(t) = (x_1(t),\dots,x_6(t))$ of the dynamic system (4.4) that is uniformly asymptotically stable with $Z(t) \in \Omega$. ■

We illustrate our results with an example.

Example 4.2.6. *Based on [36], let us consider the following system on the time scales*

$\mathbb{T} = \mathbb{Z}_0^+$:

$$\begin{cases} x_1^\Delta(t) = \Lambda + \omega n x_4(t) - [\lambda(t)(1-p) + \phi p + \gamma] x_1^\sigma(t), \\ x_2^\Delta(t) = \lambda(t)(1-p) x_1(t) - [q\nu + \gamma] x_2^\sigma(t), \\ x_3^\Delta(t) = q\nu x_2(t) - [\delta_1 + \gamma] x_3^\sigma(t), \\ x_4^\Delta(t) = \phi p x_1(t) + \delta_1 f_1 x_3(t) + \delta_2 (1 - f_2 - f_3) x_5(t) - [\omega n + \gamma] x_4^\sigma(t), \\ x_5^\Delta(t) = \delta_1 (1 - f_1) x_3(t) + \eta(1-k) x_6(t) - [\delta_2(1 - f_2 - f_3) + \delta_2 f_2 + \alpha_1 f_3 + \gamma] x_5^\sigma(t), \\ x_6^\Delta(t) = \delta_2 f_2 x_5(t) - [\eta(1-k) + \alpha_2 k + \gamma] x_6^\sigma(t), \end{cases} \tag{4.7}$$

subject to

$$x_1(0) = 10283785, \quad x_2(0) = 13, \quad x_3(0) = 2, \quad x_4(0) = 0, \quad x_5(0) = 0, \quad x_6(0) = 0,$$

where $\Lambda = \frac{22614}{53}$, $\omega = 1/31$, $n = 0.075$,

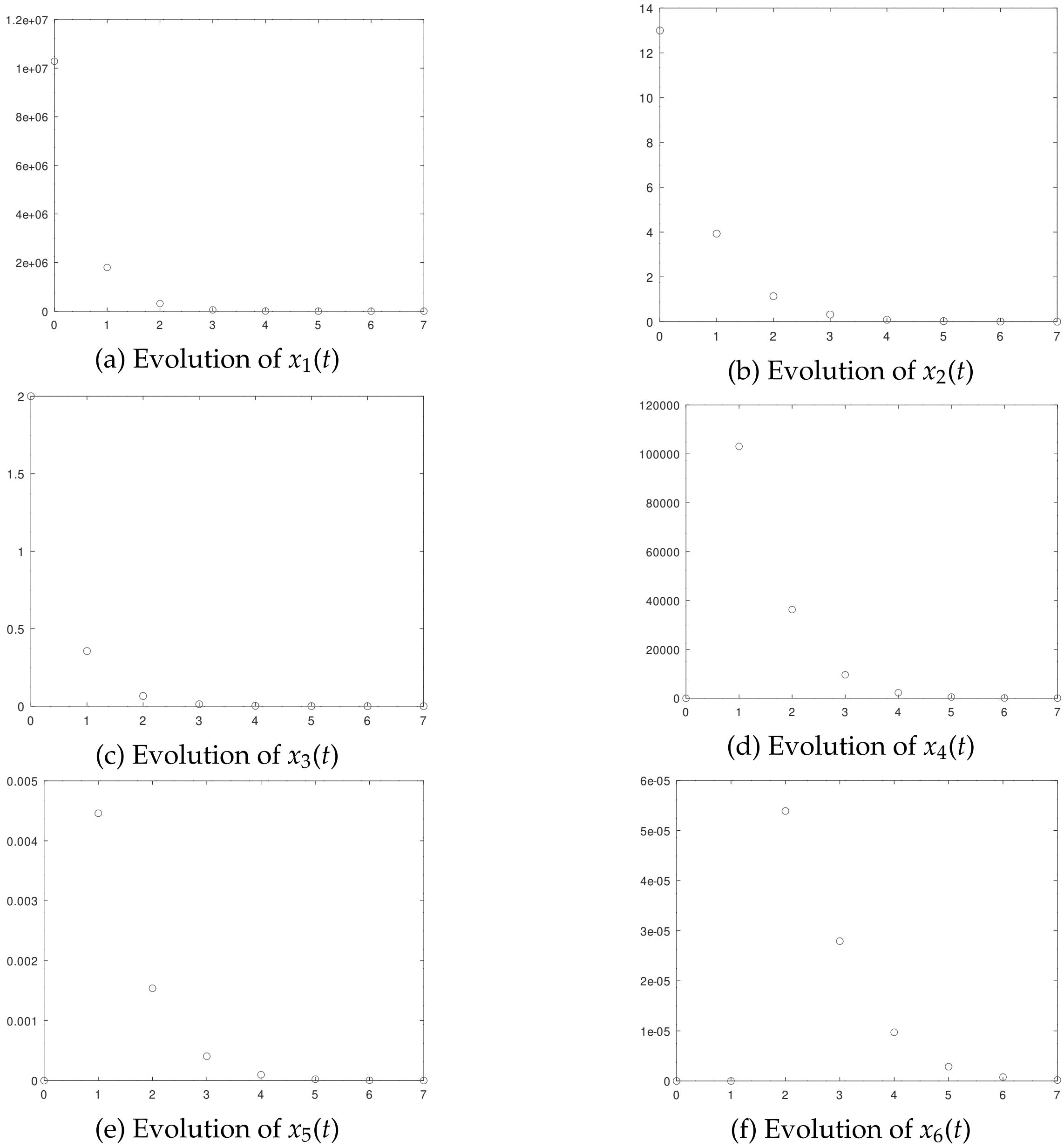
$$\lambda(t) = \frac{\beta\,(l_A x_2(t) + x_3(t) + l_H x_5(t))}{N(t)}$$

with $\beta = 1.93$, $l_A = 1$, $l_H = 0.1$, *and* $N(t) = \sum_{i=1}^{6} x_i(t)$, $p = 0.68$, $\phi = 1/12$, $\gamma = \frac{47833615}{N_0}$ *with* $N_0 = N(0)$, $q = 0.15$, $\nu = 1/15$, $\delta_1 = 1/3$, $\delta_2 = 1/3$, $f_1 = 0.96$, $f_2 = 0.21$, $f_3 = 0.03$, $\eta = 1/7$, $k = 0.03$, $\alpha_1 = 1/7$, *and* $\alpha_2 = 1/15$.

System (4.7) *is permanent with* $\lambda^L = 1.876738171 \times 10^{-7}$, $\lambda^U = 1.93$, $M_1 = 65.83271997$, $M_2 = 8.722416333$, $M_3 = 0.017498412509$, $M_4 = 4.428264471$, $M_6 = 0.02788991356$, $M = \max_{i=1,\ldots,6}(M_i) = M_1$, $m_1 = 58.16800031$, $m_2 = 7.7068897$, $m_3 = 0.0154611$, $m_4 = 0.7093454$, $m_5 = 0.0000414$, $m_6 = 6.0482208 \times 10^{-7}$, *and* $m = \min_{i=1,\ldots,6}(m_i) = m_6$. *In addition, the conditions of Theorem 4.2.5 are verified and we have*

$$4.653775371 = A > B = 4.148857053, \quad \psi = 0.0505839, \quad 1 - \psi\mu(t) = 0.94941603 > 0.$$

We conclude that system (4.7) *has a unique positive almost periodic solution, which is uniformly asymptotic stable. This is illustrated in Figure 4.1.*



(a) Evolution of $x_1(t)$

(b) Evolution of $x_2(t)$

(c) Evolution of $x_3(t)$

(d) Evolution of $x_4(t)$

(e) Evolution of $x_5(t)$

(f) Evolution of $x_6(t)$

Figure 4.1: Example 4.2.6: solution of (4.7) during 7 days.

Chapter 5

Local and global Stability of fractional SAIRS Models

In this chapter, we consider a fractional order SAIRS (Susceptible-Asymptomatic infected-symptomatic Infected-Recovered-Susceptible) model with vaccination. Which represents the interaction of four distinct compartments of people in community with an epidemic. We show that the disease free (resp. endemic) equilibrium is locally asymptotically stable if $\mathcal{R}_0 < 1$ (resp. $\mathcal{R}_0 > 1$). Moreover, we prove that the disease free (resp. endemic) equilibrium is globally asymptotically stable if $\mathcal{R}_0$ is less than another threshold $\mathcal{R}_1$ (resp. $\mathcal{R}_0 > 1$ when $\gamma = 0$). To conclude this work we give some remarks with numerical simulations to illustrate our theoretical results.

5.1 Introduction

We consider an extension of the SAIRS model presented in [50]. The system of ODEs which describes the model is given by

$$\begin{cases} \dot{S}(t) = \mu - [\beta_A A(t) + \beta_I I(t)]S(t) - (\mu + \nu)S(t) + \gamma R(t), \\ \dot{A}(t) = [\beta_A A(t) + \beta_I I(t)]S(t) - (\eta + \delta_A + \mu)A(t), \\ \dot{I}(t) = \eta A(t) - (\delta_I + \mu)I(t), \\ \dot{R}(t) = \delta_A A(t) + \delta_I I(t) + \nu S(t) - (\gamma + \mu)R(t), \end{cases} \tag{5.1}$$

with initial condition (S(0),A(0),I(0),R(0)) belonging to the set

$$\overline{\Gamma} = \{(S, A, I, R) \in \mathbb{R}^4_+ : \ S + A + I + R = 1\}.$$

where the parameters μ, η, β_A, β_I, ν, γ, δ_A, δ_I in the SAIRS epidemic model (5.1) are considered to be positive values. Hence, system (5.1) is equivalent to the following three dimensional dynamical system

$$\begin{cases} \dot{S}(t) = \mu - [\beta_A A(t) + \beta_I I(t)]S(t) - (\mu + \nu + \gamma)S(t) + \gamma(1 - A(t) - I(t)), \\ \dot{A}(t) = [\beta_A A(t) + \beta_I I(t)]S(t) - (\eta + \delta_A + \mu)A(t), \\ \dot{I}(t) = \eta A(t) - (\delta_I + \mu)I(t), \end{cases} \tag{5.2}$$

with initial condition (S(0),A(0),I(0)) belonging to the set

$$\Omega = \{(S, A, I) \in \mathbb{R}^3_+ : S + A + I \leq 1\}.$$

5.2 Definitions

Definition 5.2.1. *The Mittag Leffler function of two parameters is given by*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \; z \in \mathbb{C},$$

where α, $\beta > 0$, and $\mathbb{C}$ denote the complex plane.

Note that, when $\alpha = \beta = 1$, the Mittag Leffler function $E_{1,1}(z)$ reduces to the exponential function $\exp(z)$. Also, the Mittag Leffler function satisfies the following useful equality:

$$E_{\alpha,\beta}(z) = zE_{\alpha,\alpha+\beta}(z) + \frac{1}{\Gamma(\beta)}, \; \alpha, \beta > 0.$$

Lemma 5.2.2. *For $a \in \mathbb{R}$ and α, $\beta > 0$, we obtain [[29] Lemme 2.1]*

$$\mathcal{L}(t^{\beta-1} E_{\alpha,\beta}(at^{\alpha})) = \frac{s^{\alpha-\beta}}{s^{\alpha} - a}.$$

Also,

$$\mathcal{L}({}^{C}D_t^{\alpha} h(t)) = s^{\alpha}\widehat{h}(s) - \sum_{k=0}^{n-1} h^{(k)}(0)s^{\alpha-k-1},$$

where $\widehat{h}(s) = \mathcal{L}(h(t))$ and $\mathcal{L}$ denoted the Laplace transformation.

In the following, we give an elementary lemma proved in [?] , which describes the Volterra type Lyapunov function for the fractional order epidemic systems.

Lemma 5.2.3. *Let $0 < \alpha < 1$, and $\zeta \in C[0,T]$ be positive valued function. Then, for all $t \in [0,T)$, one has*

$$ {}^{C}D_t^{\alpha}\left(\zeta(t) - \zeta^* - \zeta^* \ln \frac{\zeta(t)}{\zeta^*}\right) \le \left(1 - \frac{\zeta^*}{\zeta(t)}\right) {}^{C}D_t^{\alpha}\zeta(t), $$

for all $\zeta^* \in \mathcal{R}_+$.

5.2.1 The fractional SAIRS model

Motivated by the classical SAIRS epidemic system (5.2), we deal with the following fractional SAIRS epidemic model

$$ \begin{cases} {}^{C}D_t^{\alpha}S(t) = \mu - [\beta_A A(t) + \beta_I I(t)]S(t) - (\mu + \nu + \gamma)S(t) + \gamma(1 - A(t) - I(t)), \\ {}^{C}D_t^{\alpha}A(t) = [\beta_A A(t) + \beta_I I(t)]S(t) - (\eta + \delta_A + \mu)A(t), \\ {}^{C}D_t^{\alpha}I(t) = \eta A(t) - (\delta_I + \mu)I(t), \end{cases} \tag{5.3} $$

subject to the initial condition

$$ S(0) = S_0 \ge 0,\ A(0) = A_0 \ge 0,\ I(0) = I_0 \ge 0, $$

where ${}^{C}D_t^{\alpha}$ is the fractional Caputo derivative having order $0 < \alpha \le 1$ in order to describe the memory effects in the proposed epidemic model. We assume that the functions $S(t)$, $A(t)$, $I(t)$ and their Caputo fractional derivatives of order $0 < \alpha \le 1$ are continuous functions. The parameters $\mu, \eta, \beta_A, \beta_I, \nu, \gamma, \delta_A, \delta_I$ in the fractional order SAIRS epidemic model (5.3) are considered to be positive values.

5.3 Main results

Proposition 5.3.1.

$$ \Omega = \{(S, A, I) \in \mathbb{R}_+^3;\ S + A + I \le 1\} $$

is a positively invariant region for system (5.3).

Proof. We have

$$ N(t) = S(t) + A(t) + I(t). $$

Consequently, adding equation yields

$$^{C}D_t^{\alpha}N(t) = \mu + \gamma - (\mu + \gamma)N(t) - \nu S(t) - \delta_A A(t) - \delta_I I(t). \tag{5.4}$$

Then

$$\begin{aligned} ^{C}D_t^{\alpha}N(t) &= \mu + \gamma - (\mu + \gamma)N(t) - \nu S(t) - \delta_A A(t) - \delta_I I(t) \\ &\le \mu + \gamma - (\mu + \gamma)N(t). \end{aligned} \tag{5.5}$$

Taking the Laplace transform in inequality (5.5) into account, we get

$$x^{\alpha}\widehat{N}(x) - x^{\alpha-1}N(0) \le \frac{\mu + \gamma}{x} - (\mu + \gamma)\widehat{N}(x).$$

Hence,

$$(x^{\alpha} + \mu + \gamma)\widehat{N}(x) = \frac{\mu + \gamma}{x} + x^{\alpha-1}N(0),$$

$$\widehat{N}(x) \le (\mu + \gamma)\frac{x^{\alpha-(1+\alpha)}}{x^{\alpha} + \mu + \gamma} + N(0)\frac{x^{\alpha-1}}{x^{\alpha} + \mu + \gamma}.$$

Accordingly, we have

$$\begin{aligned} N(t) &\le (\mu + \gamma)t^{\alpha}E_{\alpha,1+\alpha}(-(\mu + \gamma)t^{\alpha}) + N(0)E_{\alpha,1}(-(\mu + \gamma)t^{\alpha}) \\ &\le 1 - E_{\alpha,1}(-(\mu + \gamma)t^{\alpha}) + N(0)E_{\alpha,1}(-(\mu + \gamma)t^{\alpha}) \\ &\le 1. \end{aligned}$$

Since $0 \le E_{\alpha,1}(-(\mu + \gamma)t) \le 1$ holds and $N(0) \le 1$, then one obtains $N(t) \le 1$. Thus, Ω is a positively invariant set, and all initial solutions belong to Ω remain in Ω for all $t > 0$. ■

Consider the following fractional order system in Caputo sense, as

$$^{C}D_t^{\alpha}u(t) = \phi(t, u(t)), \text{ for all } t \ge 0, \; u(0) = u_0 \in \mathbb{R}^n \tag{5.6}$$

where $\phi : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ with $n \ge 1$.

Lemma 5.3.2. *Assume that $\phi(t, u(t))$ satisfies the following:*

1. *ϕ is a continuous function with respect to t for all $u(t) \in \mathbb{R}^n$,*
2. *ϕ and $\frac{\partial \phi}{\partial u}$ are continuous funtions respect to $u(t) \in \mathbb{R}^n$,*
3. *$\left\|\phi\right\| \le a_1 + a_2\left\|u\right\|$ for all $u \in \mathbb{R}^n$, and all a_1, $a_2 > 0$.*

Then, system (5.6) possesses a unique solution on $[0, +\infty)$.

The proof of Lemma 5.3.2 can be directly followed from [40], we require the following lemmas in proving the non negativity of solutions to system subject to non negative initial condition for the system (5.3).

Lemma 5.3.3. *[49] Assume that* $\varphi(t) \in C[a, b]$ *and* ${}^{C}D_t^{\alpha}\varphi \in C[a, b]$*, with* $0 < \alpha \leq 1$. *Then, one has*

$$\varphi(t) = \varphi(a) + \frac{1}{\Gamma(\alpha)}\, {}^{C}D_t^{\alpha}\varphi(\xi)(t-a)^{\alpha},\ a \leq \xi \leq t, \text{for } t \in (a, b]. \tag{5.7}$$

Lemma 5.3.4. *[49] Let* $\varphi(t) \in C[a, b]$ *and* ${}^{C}D_t^{\alpha}\varphi \in C[a, b]$*, with* $0 < \alpha \leq 1$. *If* ${}^{C}D_t^{\alpha}\varphi(t) \geq 0$*, then* $\varphi(t)$*is non decreasing function for* $t \in [a, b]$. *If* ${}^{C}D_t^{\alpha}\varphi(t) \leq 0$*, then* $\varphi(t)$*is non increasing function for* $t \in [a, b]$.

Let define the set

$$\Theta = \{(S, A, I) \in \mathbb{R}^3 : S(t) \geq 0,\ A(t) \geq 0,\ I(t) \geq 0\}.$$

Theorem 5.3.5. *System (5.3) attains a unique solution on* $[0, +\infty)$. *Further, the solution of system (5.3) remains non negative and bounded for all* $t \geq 0$. *In addition, we have*

$$\begin{aligned} S(t) &\leq \frac{\mu + \gamma}{\mu + \upsilon + \gamma} + S(0), \\ A(t) &\leq A(0), \\ I(t) &\leq I(0). \end{aligned}$$

Proof. Let us reformulate system (5.3) in the form of a Caputo fractional derivative system of order $0 < \alpha \leq 1$, as follows

$${}^{C}D_t^{\alpha}u(t) = \phi(t, u(t)),\ \text{for all}\ t \geq 0,\ u(0) = u_0 \in \mathbb{R}^3_+, \tag{5.8}$$

where $\phi : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}^3$,

$$\phi(t, u(t)) = b + Eu(t) + S(t)Fu(t) + Gu(t), \tag{5.9}$$

$u(t) = (S(t), A(t), I(t))^T$, $u_0 = (S_0, A_0, I_0)^T$, $b = (\mu + \gamma, 0, 0)^T$, and also

$$E = \begin{bmatrix} -(\mu + \nu + \gamma) & 0 & 0 \\ 0 & -(\eta + \delta_A + \mu) & 0 \\ 0 & 0 & -(\delta_I + \mu) \end{bmatrix}, F = \begin{bmatrix} 0 & -\beta_A & -\beta_I \\ 0 & \beta_A & \beta_I \\ 0 & 0 & 0 \end{bmatrix},$$

$G = \begin{bmatrix} 0 & -\gamma & -\gamma \\ 0 & 0 & 0 \\ 0 & \eta & 0 \end{bmatrix}$. The vector function $u(t)$ also satisfies the first and second conditions of Lemma 5.3.2. Now, we verify the third condition. From systems (5.8)and (5.9), one obtain

$$\left\| {}^{C}D_t^{\alpha} u(t) \right\| \leq \|b\| + (\|E\| + \|S(t)\| \, \|F\| + \|G\|) \, \|u(t)\| \, .$$

Hence, the third condition of Lemma 5.3.2 is verified. Then, system (5.3) has a unique solution on $[0, +\infty)$. Further, we show that the system possesses a non negative solution. To this end, Suppose that

$$(S_0, A_0, I_0) \in S(t)\text{-axis} = \{(S(t), 0, 0) \, : S(t) \geq 0\}$$

Taking the Laplace transform into account system along with the vector $S(t)$ axis. Let

$${}^{C}D_t^{\alpha} S(t) = \mu + \gamma - (\mu + \nu + \gamma) S(t).$$

We get

$$x^{\alpha} \widehat{S}(x) - x^{\alpha-1} S(0) = \frac{\mu + \gamma}{x} - (\mu + \nu + \gamma) \widehat{S}(x).$$

Hence,

$$\widehat{S}(x) = (\mu + \gamma) \frac{x^{\alpha-(1+\alpha)}}{x^{\alpha} + \mu + \nu + \gamma} + S(0) \frac{x^{\alpha-1}}{x^{\alpha} + \mu + \nu + \gamma}.$$

Accordingly, we have

$$S(t) = (\mu + \gamma) t^{\alpha} E_{\alpha, 1+\alpha}(-(\mu + \nu + \gamma) t^{\alpha}) + S(0) E_{\alpha,1}(-(\mu + \nu + \gamma) t^{\alpha}).$$

Seen

$$E_{(\alpha,\alpha+1)}(-(\mu + v + \gamma) t^{\alpha}) = -\frac{1}{(\mu + v + \gamma) t^{\alpha}} E_{(\alpha,1)}(-(\mu + v + \gamma) t^{\alpha}) + \frac{1}{(\mu + v + \gamma) t^{\alpha}}$$

Hence,

$$S(t) = -\frac{\mu + \gamma}{\mu + v + \gamma} E_{(\alpha,1)}(-(\mu + v + \gamma) t^{\alpha}) + \frac{\mu + \gamma}{\mu + v + \gamma} + S(0) E_{(\alpha,1)}(-(\mu + v + \gamma) t^{\alpha})$$

Since $0 \leq E_{(\alpha,1)}(-(\mu+\upsilon+\gamma)t^{\alpha}) \leq 1$ so one achieves

$$S(t) \geq 0 and S(t) \leq \frac{\mu+\gamma}{\mu+\upsilon+\gamma} + S(0),$$

also
$\left(-\frac{\mu+\gamma}{\mu+\upsilon+\gamma}E_{(\alpha,1)}(-(\mu+\upsilon+\gamma)t^{\alpha}) + \frac{\mu+\gamma}{\mu+\upsilon+\gamma} + S(0)E_{(\alpha,1)}(-(\mu+\upsilon+\gamma)t^{\alpha}), 0, 0\right) \in S(t)$-axis.
By the same argument with $A(t)$-axis, i.e

$${}^{C}D_{t}^{\alpha}A(t) = -(\eta+\delta_A+\mu)A(t).$$

Taking the Laplace transform, we get

$$x^{\alpha}\widehat{A}(x) - x^{\alpha-1}A(0) = -(\eta+\delta_A+\mu)\widehat{A}(x).$$

Hence,

$$\widehat{A}(x) = A(0)\frac{x^{\alpha-1}}{x^{\alpha}+\eta+\delta_A+\mu}.$$

Accordingly, we have

$$A(t) = A(0)E_{\alpha,1}(-(\alpha+\delta_A+\mu)t^{\alpha}).$$

s Seen $0 \leq E_{\alpha,1}(-(\alpha+\delta_A+\mu)t^{\alpha}) \leq 1$. Hence, $A(t) \leq A(0)$ and $A(t) \geq 0$ also

$$(0, A(0)E_{\alpha,1}(-(\eta+\delta_A+\mu)t^{\alpha}), 0) \in A(t)\text{-axis}.$$

By the same argument with $I(t)$-axis, let

$${}^{C}D_{t}^{\alpha}I(t) = -(\delta_I+\mu)I(t).$$

Taking the Laplace transform, we get

$$x^{\alpha}\widehat{I}(x) - x^{\alpha-1}I(0) = -(\delta_I+\mu)\widehat{I}(x).$$

Hence,

$$\widehat{I}(x) = I(0)\frac{x^{\alpha-1}}{x^{\alpha}+\delta_I+\mu}.$$

Accordingly, we have

$$I(t) = I(0)E_{\alpha,1}(-(\delta_I+\mu)t^{\alpha}).$$

Hence, $I(t) \leq I(0)$ and $I(t) \geq 0$ also

$$(0, 0, I(0)E_{\alpha,1}(-(\delta_I + \mu)t^{\alpha})) \in I(t)\text{-axis}.$$

This indicates that $S(t)$, $A(t)$, and $I(t)$ are solutions of the system and positive invariants sets. In sequel, we prove that Θ is a positive invariant set. Let $(S_0, A_0, I_0) \in \Theta$. On the contrary, we suppose there exists a solution $(S(t), A(t), I(t))$ to escape of Θ. Then, by uniqueness of the solution, $(S(t), A(t), I(t))$ do not cross the axes.

If the solution $(S(t), A(t), I(t))$ escapes by the plan $S(t) = 0$, there exists t_0 such that $S(t_0) = 0$, $A(t_0) > 0$, $I(t_0) > 0$ and for all $t > t_0$ sufficiently near t_0, we have $S(t) < 0$. Moreover, we have

$$\begin{aligned} {}^{C}D_t^{\alpha} S(t) \mid_{t=t_0} &= \mu + \gamma - \gamma(A(t) + I(t)) \\ &\geq \mu. \end{aligned}$$

Thus, $S(t) \geq 0$ for all $t > t_0$, and this is absurd, which implies that $S(t) \geq 0$ for all $t \geq 0$. With a similar manner, we prove that $A(t) \geq 0$ and $I(t) \geq 0$ for all $t \geq 0$. ∎

5.3.1 Local Stability

System (5.3) admits a unique disease free equilibrium point $X^0 = (S^0, 0, 0)$, where $S^0 = \frac{\mu+\gamma}{\mu+\nu+\gamma}$. To consider the existence and uniqueness of endemic equilibrium $X^* = (S^*, A^*, I^*)$, we firstly compute the basic reproductive number $\mathcal{R}_0$ of system (5.3).

Let $U = (A, I, S)^T$. Then, system (5.3) can be written as ${}^{C}D_t^{\alpha} U(t) = F(U) - V(U)$, where $F(U) = \begin{bmatrix} (\beta_A A(t) + \beta_I I(t))S(t) \\ 0 \\ 0 \end{bmatrix}$ and $V(U) = \begin{bmatrix} (\eta + \delta_A + \mu)A(t) \\ -\eta A(t) + (\delta_I + \mu)I(t) \\ (\beta_A A(t) + \beta_I I(t))S(t) + (\mu + \nu + \gamma)S(t) + \gamma(A(t) + I(t)) - \mu - \gamma \end{bmatrix}$. The Jacobian matrices of F and V at the disease free equilibrium X^0 are given by

$$J(F(X^0)) = \begin{bmatrix} 0 & \beta_A S^0 & \beta_I S^0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{and } J(V(X^0)) = \begin{bmatrix} 0 & \eta + \delta_A + \mu & 0 \\ 0 & -\eta & \delta_I + \mu \\ \mu + \nu + \gamma & \beta_A S^0 + \gamma & \beta_I S^0 + \gamma \end{bmatrix}.$$

Now, let us employ the next generation matrix method to compute the basic reproductive ratio $\mathcal{R}_0$ using only the two disease comprtatments, denoted by A and I from system (5.3). The next generation matrix is the product of matrices $\mathcal{F}$ and $\mathcal{V}^{-1}$, where

$$\mathcal{F} = \begin{bmatrix} \beta_A S^0 & \beta_I S^0 \\ 0 & 0 \end{bmatrix} \text{ and } \mathcal{V} = \begin{bmatrix} \eta + \delta_A + \mu & 0 \\ -\eta & \delta_I + \mu \end{bmatrix}.$$

The basic reproductive ratio $\mathcal{R}_0$, defined as the spectral radius of the matrix $\mathcal{F}\mathcal{V}^{-1}$, is obtained as

$$\mathcal{R}_0 = \rho(\mathcal{F}\mathcal{V}^{-1}) = \left(\beta_A + \frac{\eta\beta_I}{\delta_I + \mu}\right)\frac{\gamma + \mu}{(\eta + \delta_A + \mu)(\nu + \gamma + \mu)} = \left(\beta_A + \frac{\eta\beta_I}{h_2}\right)\frac{S^0}{h_1},$$

where $S^0 = \frac{\mu+\gamma}{h_0}$, $h_0 = \mu + \nu + \gamma$, $h_1 = \eta + \delta_A + \mu$ and $h_2 = \delta_I + \mu$.
In the case when ($A \neq 0$ and $I \neq 0$), system (5.3) admits $X^* = (S^*, A^*, I^*)$ as a unique endemic equilibrium point, where

$$S^* = \frac{h_1 h_2}{\beta_A h_2 + \beta_I \eta},$$

$$I^* = \frac{\eta h_0 S^* \left(\frac{S^0}{S^*} - 1\right)}{h_1 h_2 + \gamma(h_2 + \eta)} = \frac{\eta h_0 S^* (\mathcal{R}_0 - 1)}{h_1 h_2 + \gamma(h_2 + \eta)},$$

$$A^* = \frac{h_2}{\eta} I^*.$$

Clearly, it is evident that if $\mathcal{R}_0 < 1$, then system does not admit any positive endemic equilibrium (it has no biological sense to get negative values for A^* and I^*). Thus, we require $\mathcal{R}_0 > 1$, to assure the existence and positivity of the endemic equilibrium point.

Theorem 5.3.6. *System (5.3) always has a disease-free equilibrium $X^0 = (S^0, 0, 0)$. In addition, if $\mathcal{R}_0 > 1$, then there exists a unique endemic equilibrium point $X^* = (S^*, A^*, I^*)$.*

Next, we will discuss the stability of the equilibrium points of system (5.3). At the point $X(S, A, I)$, the Jacobian matrix of system (5.3) is given by

$$J(X) = \begin{pmatrix} -[\beta_A A + \beta_I I] - h_0 & -\beta_A S - \gamma & -\beta_I S - \gamma \\ \\ \beta_A A + \beta_I I & \beta_A S - h_1 & \beta_I S \\ \\ 0 & \eta & -h_2 \end{pmatrix}. \tag{5.10}$$

Using the Jacobian matrix (5.10) and the Matignon condition [48], [52], the local stability of the equilibrium points of the fractional-order system (5.3) is investigated. We have the following theorems.

Theorem 5.3.7. *The disease-free equilibrium $X^0 = (S^0, 0, 0)$ of the fractional-order system (5.3) is locally asymptotically stable if $\mathcal{R}_0 < 1$ and unstable if $\mathcal{R}_0 > 1$.*

Proof. The Jacobian matrix (5.10) around the disease-free equilibrium X^0 is as follows

$$J(X^0) = \begin{bmatrix} -h_0 & -\beta_A S^0 - \gamma & -\beta_I S^0 - \gamma \\ \\ 0 & \beta_A S^0 - h_1 & \beta_I S^0 \\ \\ 0 & \eta & -h_2 \end{bmatrix}.$$

The eigenvalues of the Jacobian matrix $J(X^0)$ around the disease-free equilibrium X^0 are $\lambda_1 = -h_0$ and the roots of the characteristic polynomial of the minor matrix of $J(X^0)$ given by

$$C(\lambda) = \lambda^2 + (h_1 + h_2 - \beta_A S^0)\lambda + h_1 h_2 (1 - \mathcal{R}_0) = 0. \tag{5.11}$$

When $\mathcal{R}_0 < 1$, it is evident that $h_1 + h_2 - \beta_A S^0 = h_2 + h_1 \left(1 - \beta_A \frac{S^0}{h_1}\right) > h_2 + h_1 (1 - \mathcal{R}_0) > 0$. Thus, from the Routh-Hurwitz criterion, all the roots λ_i of the characteristic equation (5.11) have negative real parts. By using Matignon's condition [48], [52], it can be observed that $|arg(\lambda_i)| > \alpha\frac{\pi}{2}$ for all $0 < \alpha < 1$. Therefore, the disease free equilibrium point X^0 is locally asymptotically stable if $\mathcal{R}_0 < 1$.
When $\mathcal{R}_0 > 1$, we have $C(0) < 0$ and $\lim_{\lambda \to +\infty} C(\lambda) = +\infty$, then there exist positive real root $\lambda^* > 0$ of the characteristic equation (5.11), from Matignon's condition [48], [52], it can be observed that $|arg(\lambda^*)| = 0 < \alpha\frac{\pi}{2}$ for all $0 < \alpha < 1$. Thus, the disease free equilibrium point X^0 is unstable. This result completes the proof. ■

It is observed that the disease free equilibrium point X^0 is locally asymptotically stable when the endemic equilibrium X^* do not exist.

Next, the stability of the endemic equilibrium X^* is discussed.

Theorem 5.3.8. *The unique endemic equilibrium X^* of the fractional-order system (5.3) is locally asymptotically stable if $\mathcal{R}_0 > 1$.*

Proof. At the endemic equilibrium X^*, The Jacobian matrix (5.10) is given by

$$J(X^*) = \begin{pmatrix} -h_1\frac{A^*}{S^*} - h_0 - \lambda & -\beta_A S^* - \gamma & -\beta_I S^* - \gamma \\ h_1\frac{A^*}{S^*} & \beta_A S^* - h_1 - \lambda & \beta_I S^* \\ 0 & \eta & -h_2 - \lambda \end{pmatrix}.$$

The eigenvalues of the Jacobian matrix $J(X^*)$ around the endemic equilibrium X^* are the roots of the characteristic equation given by

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0, \tag{5.12}$$

where a_i, $i = 0, \ldots, 2$ are given as follow,

$$\begin{aligned} a_2 &= \left(h_0 + h_1\frac{A^*}{S^*}\right) + \left(h_2 + \frac{\beta_I I^* S^*}{A^*}\right), \\ a_1 &= h_0\left(h_2 + \frac{\beta_I I^* S^*}{A^*}\right) + (h_1 + h_2 + \gamma)h_1\frac{A^*}{S^*}, \\ a_0 &= [(h_1 + \gamma)h_2 + \eta\gamma]h_1\frac{A^*}{S^*}. \end{aligned}$$

It is evident that $a_i > 0$. Moreover, we have

$$\begin{aligned} a_1a_2 - a_0 &= h_0\left(h_2 + \tfrac{\beta_I I^* S^*}{A^*}\right)\left(h_0 + h_1\tfrac{A^*}{S^*}\right) + h_0\left(h_2 + \tfrac{\beta_I I^* S^*}{A^*}\right)^2 \\ &+\left(h_0 + h_1\tfrac{A^*}{S^*}\right)(h_1 + h_2 + \gamma)h_1\tfrac{A^*}{S^*} + \left(h_2 + \tfrac{\beta_I I^* S^*}{A^*}\right)(h_1 + h_2 + \gamma)h_1\tfrac{A^*}{S^*} \\ &-[(h_1 + \gamma)h_2 + \eta\gamma]h_1\tfrac{A^*}{S^*} \end{aligned}$$

$$
\begin{aligned}
&= h_0\left(h_2+\tfrac{\beta_I I^* S^*}{A^*}\right)\left(h_0+h_1\tfrac{A^*}{S^*}\right)+h_0\left(h_2+\tfrac{\beta_I I^* S^*}{A^*}\right)^2 \\
&\quad +\left(h_0+h_1\tfrac{A^*}{S^*}\right)(h_2+\gamma)h_1\tfrac{A^*}{S^*}+\left(h_2+\tfrac{\beta_I I^* S^*}{A^*}\right)h_1h_2\tfrac{A^*}{S^*} \\
&\quad +\left(h_1^2\tfrac{A^*}{S^*}+(h_1+\gamma)\tfrac{\beta_I I^* S^*}{A^*}\right)h_1\tfrac{A^*}{S^*}+(h_0h_1-\eta\gamma)\,h_1\tfrac{A^*}{S^*}>0.
\end{aligned}
$$

Then, according to the Routh-Hurwitz criterion, all the roots of the characteristic equation (5.12) have negative real parts. By using Matignon's condition [48], [52], it can be observed that $|arg(\lambda_1)|=\pi>\alpha\frac{\pi}{2}$ for all $0<\alpha<1$. Therefore, the endemic equilibrium point X^* is locally asymptotically stable if $\mathcal{R}_0>1$. ■

5.3.2 Global stability

Let us define a function $\Psi:\mathbb{R}_+\to\mathbb{R}_+$ given by

$$
\Psi(\zeta(t))=\zeta(t)-\zeta^*-\zeta^* ln\frac{\zeta(t)}{\zeta^*},\ \text{for all } t\geq 0,\ \zeta^*>0.
$$

Note that $\Psi(\zeta)$ is a non-negative function for any $\zeta>0$ that attains a global minimum at $\zeta=\zeta^*$.

Theorem 5.3.9. *The disease-free equilibrium X^0 of system (5.3) is globally asymptotically stable on Ω, if $\mathcal{R}_0\leq\mathcal{R}_1$ where* $\mathcal{R}_1=1-\frac{\left(\frac{\eta(\beta_A-\beta_I)^2\gamma^2}{4(\mu+\delta_I)\beta_A\beta_I}\right)}{(\mu+\eta+\delta_A)\left[\left(\frac{\eta}{\mu+\delta_I}+1\right)\gamma+(\mu+\eta+\delta_A)\right]}$.

Proof. Let us consider the following function:

$$
V_0(S,A,I)=2\left(S-S^0\right)^2+c_1A+\frac{1}{(\mu+\delta_I)}\left(4\gamma S^0+\beta_I\left[S^0-\frac{\gamma}{\beta_I}+\frac{c_1}{4}\right]^2\right)I,
$$

where $c_1=4S^0\left(\frac{(\frac{\eta}{(\mu+\delta_I)}+1)\gamma+(2-\mathcal{R}_0)(\mu+\eta+\delta_A)}{(\mu+\eta+\delta_A)\mathcal{R}_0}\right)$. It is easily seen that the above function is nonnegative and also $V_0=0$ if and only if $S=S^0$, $A=0$ and $I=0$. Applying the Caputo fractional derivative on equations of system (5.3), we obtain

$$
\begin{aligned}
{}^CD_t^\alpha V_0(S,A,I) &= 4\left(S-S^0\right){}^CD_t^\alpha S+c_1{}^CD_t^\alpha A \\
&\quad +\frac{1}{(\mu+\delta_I)}\left(4\gamma S^0+\beta_I\left[S^0-\frac{\gamma}{\beta_I}+\frac{c_1}{4}\right]^2\right){}^CD_t^\alpha I
\end{aligned}
$$

$$
\begin{aligned}
&= -4(\beta_A A + \beta_I I)\left(S^2 - S^0 S\right) - 4(\mu + \nu + \gamma)\left(S - S^0\right)^2 \\
&\quad -4\gamma(A + I)\left(S - S^0\right) \\
&\quad +c_1(\beta_A A + \beta_I I)S - c_1(\mu + \eta + \delta_A)A \\
&\quad +\frac{\left(4\gamma S^0 + \beta_I\left[S^0 - \frac{\gamma}{\beta_I} + \frac{c_1}{4}\right]^2\right)}{(\mu + \delta_I)}(\eta A - (\mu + \delta_I)I) \\
&= -4(\mu + \nu + \gamma)\left(S - S^0\right)^2 \\
&\quad -\beta_A A\left[4S^2 - 4\left(S^0 - \frac{\gamma}{\beta_A} + \frac{c_1}{4}\right)S + \left(S^0 - \frac{\gamma}{\beta_A} + \frac{c_1}{4}\right)^2\right] \\
&\quad -\beta_I I\left[4S^2 - 4\left(S^0 - \frac{\gamma}{\beta_I} + \frac{c_1}{4}\right)S + \left(S^0 - \frac{\gamma}{\beta_I} + \frac{c_1}{4}\right)^2\right] \\
&\quad -H(c_1)A \\
&= -4(\mu + \nu + \gamma)\left(S - S^0\right)^2 - \beta_A A\left[2S - \left(S^0 - \frac{\gamma}{\beta_A} + \frac{c_1}{4}\right)\right]^2 \\
&\quad -\beta_I I\left[2S - \left(S^0 - \frac{\gamma}{\beta_I} + \frac{c_1}{4}\right)\right]^2 - H(c_1)A,
\end{aligned}
$$

where

$$
\begin{aligned}
H(c_1) &= c_1(\mu + \eta + \delta_A) - 4\gamma S^0 - \beta_A\left(S^0 - \frac{\gamma}{\beta_A} + \frac{c_1}{4}\right)^2 \\
&\quad -\left(\frac{\eta}{\mu + \delta_I}\right)\left(4\gamma S^0 + \beta_I\left[S^0 - \frac{\gamma}{\beta_I} + \frac{c_1}{4}\right]^2\right) \\
&= 4S^0\left[\frac{\left(\frac{\eta+\mu+\delta_I}{\mu+\delta_I}\right)\gamma + (\mu + \eta + \delta_A)}{\mathcal{R}_0}\right](\mathcal{R}_1 - \mathcal{R}_0)\,. \qquad (5.13)
\end{aligned}
$$

From (5.13) we can show that the term $H(c_1)$ is negative if and only if $\mathcal{R}_0 \le \mathcal{R}_1$. Thus, we have ${}^C D_t^\alpha V_0(S, A, I) \le 0$ for all $(S, A, I) \in \Omega$ and ${}^C D_t^\alpha V_0(S, A, I) = 0$ if and only if $(S, A, I) = (S^0, 0, 0)$. Thus, the only invariant set contained in Ω is $\{(S^0, 0, 0)\}$. Hence, by Lemma 4.6 in [28], it is proved the convergence of the solutions (S, A, I) to $(S^0, 0, 0)$. Therefore, X^0 is globally asymptotically stable in Ω if $\mathcal{R}_0 \le \mathcal{R}_1$. ■

Theorem 5.3.10. *For $\gamma = 0$, the endemic equilibrium X^* of system (5.3) is globally asymptotically stable on $\Omega/[0, 1] \times \{(0, 0)\}$ if $\mathcal{R}_0 > 1$.*

Proof. Consider the function

$$\begin{aligned} V_1(S,A,I) &= \left(S - S^* - S^* \ln \frac{S}{S^*}\right) \\ &+ \left(A - A^* - A^* \ln \frac{A}{A^*}\right) \\ &+ \left(\frac{(\mu+\eta+\delta_A) - \beta_A S^*}{\eta}\right)\left(I - I^* - I^* \ln \frac{I}{I^*}\right). \end{aligned}$$

This function is positive (since $\frac{(\mu+\eta+\delta_A)-\beta_A S^*}{\eta} = \frac{\beta_I S^* I^*}{\eta A^*}$) and $V_1(S,A,I) = 0$ if and only if $(S,A,I) = (S^*,A^*,I^*)$.

By calculating the α-order derivative of V_1 along the solution of system (5.3) and using Lemma 3.1 in [?], we obtain

$$\begin{aligned} {}^{C}D_t^{\alpha} V_1(S,A,I) &\leq \left(1 - \frac{S^*}{S}\right){}^{C}D_t^{\alpha} S + \left(1 - \frac{A^*}{A}\right){}^{C}D_t^{\alpha} A \\ &+ \left(\frac{(\mu+\eta+\delta_A) - \beta_A S^*}{\eta}\right)\left(1 - \frac{I^*}{I}\right){}^{C}D_t^{\alpha} I \\ &= \left(1 - \frac{S^*}{S}\right)((\beta_A A^* + \beta_I I^*)S^* - (\beta_A A + \beta_I I)S - (\mu+\nu)(S - S^*)) \\ &+ \left(1 - \frac{A^*}{A}\right)((\beta_A A + \beta_I I)S - (\mu+\eta+\delta_A)A) \\ &+ \left(\frac{(\mu+\eta+\delta_A) - \beta_A S^*}{\eta}\right)\left(1 - \frac{I^*}{I}\right)(\eta A - (\mu+\delta_I)I) \\ &= -\frac{1}{S}(\mu+\nu)(S-S^*)^2 + \beta_A A^* S^*\left(2 - \frac{S^*}{S} - \frac{S}{S^*}\right) \\ &+ \beta_I I^* S^*\left(2 - \frac{S^*}{S} - \frac{I}{I^*}\frac{S}{S^*}\frac{A^*}{A}\right) \\ &+ (\beta_I S^* I^* - (\mu+\eta+\delta_A)A^* + \beta_A S^* A^*)\frac{I}{I^*} \\ &+ \eta A^*\left(\frac{(\mu+\eta+\delta_A) - \beta_A S^*}{\eta}\right)\left(1 - \frac{I^*}{I}\frac{A}{A^*}\right). \end{aligned} \tag{5.14}$$

Now, replacing

$$u = \frac{S}{S^*}, \quad v = \frac{A}{A^*}, \quad w = \frac{I}{I^*}, \quad \text{and} \quad \frac{(\mu+\eta+\delta_A) - \beta_A S^*}{\eta} = \frac{\beta_I S^* I^*}{\eta A^*},$$

in inequality (5.14), we obtain

$$\begin{aligned} {}^{C}D_t^{\alpha} V_1(S,A,I) &\leq -\frac{1}{S}(\mu+\nu)(S-S^*)^2 + \beta_A A^* S^*\left(2 - \frac{1}{u} - u\right) \\ &+ \beta_I I^* S^*\left(3 - \frac{1}{u} - \frac{uw}{v} - \frac{v}{w}\right). \end{aligned} \tag{5.15}$$

By the arithmetic mean-geometric mean inequality we have $\left(2 - \frac{1}{u} - u\right) \leq 0$ and $\left(3 - \frac{1}{u} - \frac{uw}{v} - \frac{v}{w}\right) \leq 0$ for all $u \geq 0$, $v \geq 0$ and $w \geq 0$. Hence ${}^{C}D_t^{\alpha} V_1(S, A, I) \leq 0$, and ${}^{C}D_t^{\alpha} V_1(S, A, I) = 0$ if and only if $S = S^*$ and $v = w$ (*i.e.* $\frac{I}{I^*} = \frac{A}{A^*}$). Since S must remain constant at S^*, ${}^{C}D_t^{\alpha} S$ is zero. This implies that $A = A^*$ and $I = I^*$. Thus, By Lemma 4.6 in [28], it is proved that the fully endemic equilibrium X^* is globally asymptotically stable in $\Omega/[0,1] \times \{(0,0)\}$. ■

Remark 7. *If the initial conditions starts from* $[0,1] \times \{(0,0)\}$, *then the solution obviously converges to the disease free equilibrium point* X^0.

5.4 Conclusion and simulations

In this chapter, we have considered a fractional order SAIRS model. We have investigated the existence and the stability of the equilibria. This analysis is obtained according the value of $\mathcal{R}_0$ and its position with respect to some thresholds $\mathcal{R}_1$ and 1. Using the Lyapunov functionals, we show that the disease free equilibrium X^0 is globally asymptotically stable for $\mathcal{R}_0 \leq \mathcal{R}_1$ and unstable for $\mathcal{R}_0 > 1$ (see Fig. 5.1, Fig. 5.2 and Fig. 5.3). Moreover, when the disease free equilibrium X^0 is unstable i.e. $\mathcal{R}_0 > 1$, we have showed the existence of a endemic equilibrium X^* which is also globally asymptotically stable when $\gamma = 0$ (see Fig. 5.5).

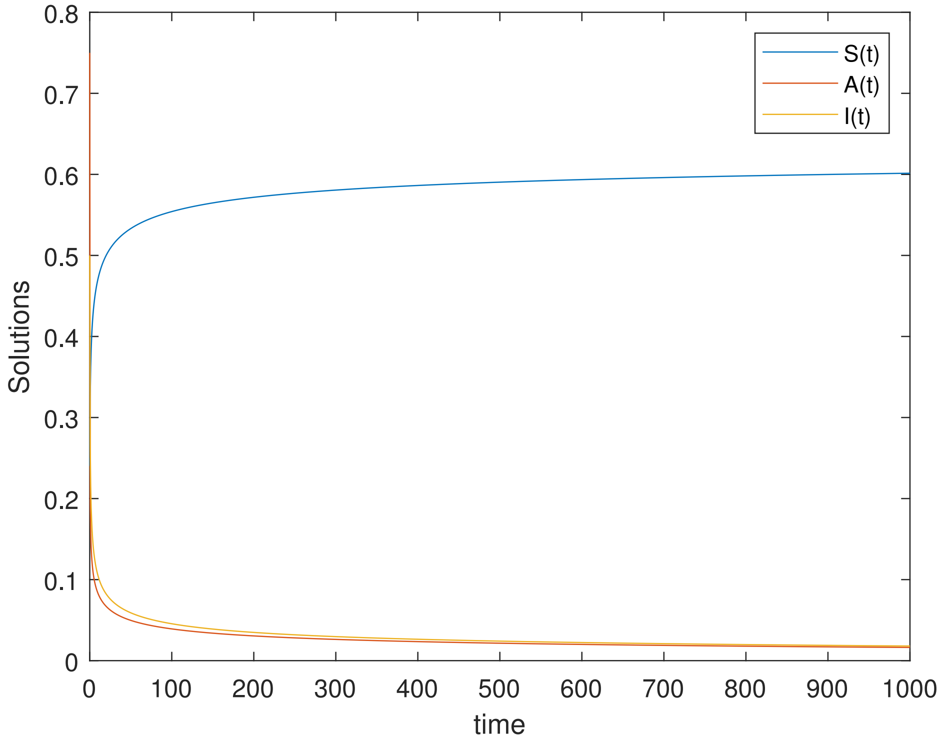


Figure 5.1: Numerical solutions of (5.3) with $\alpha = 0.5$, $\mu = 1.25$, $\beta_A = 3.5$, $\beta_I = 3.5$, $\nu = 1$, $\gamma = 0.5$, $\eta = 2.5$, $\delta_A = 1.5$, $\delta_I = 1.5$ and the initial conditions are $S_0 = 0.25$, $A_0 = 0.75$ and $I_0 = 0.5$. Note that $\mathcal{R}_0 = 0.81 < \mathcal{R}_1 = 1$, then the disease free equilibrium X^0 is globally asymptotically stable.

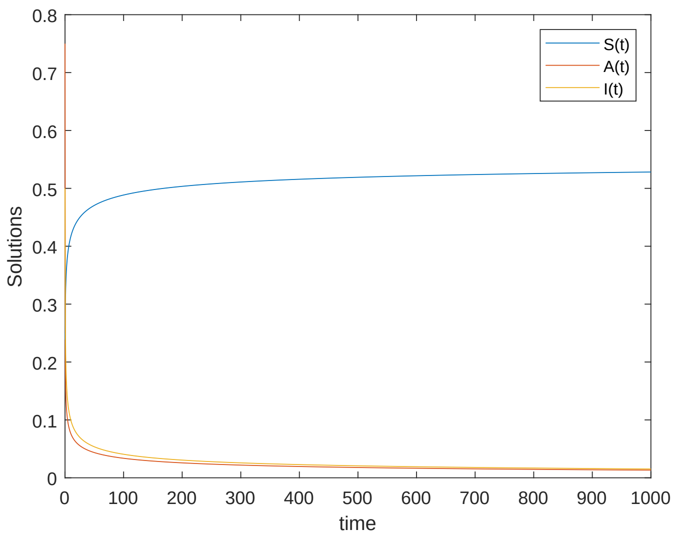


Figure 5.2: Numerical solutions of (5.3) with $\alpha = 0.5$, $\mu = 1.25$, $\beta_A = 4$, $\beta_I = 3.5$, $\nu = 1$, $\gamma = 0$, $\eta = 2.5$, $\delta_A = 1.5$, $\delta_I = 1.5$ and the initial conditions are $S_0 = 0.25$, $A_0 = 0.75$ and $I_0 = 0.5$. Note that $\mathcal{R}_0 = 0.76 < \mathcal{R}_1 = 1$, then the disease free equilibrium X^0 is globally asymptotically stable.

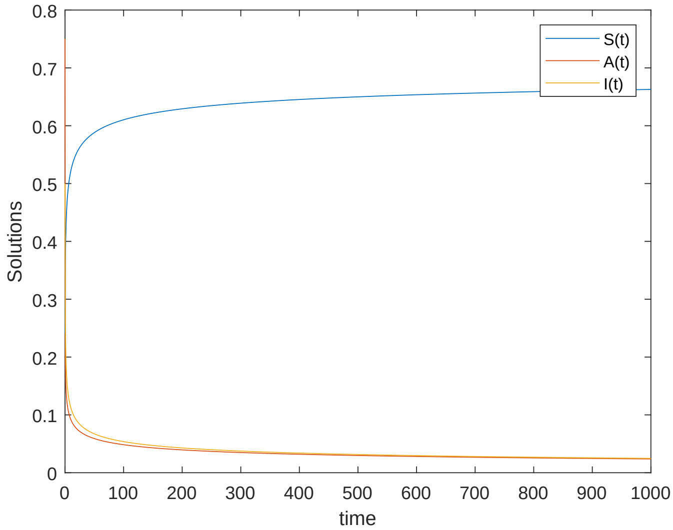


Figure 5.3: Numerical solutions of (5.3) with $\alpha = 0.5$, $\mu = 1.25$, $\beta_A = 5$, $\beta_I = 2$, $\nu = 1$, $\gamma = 1.25$, $\eta = 2.5$, $\delta_A = 1.5$, $\delta_I = 1.5$ and the initial conditions are $S_0 = 0.25$, $A_0 = 0.75$ and $I_0 = 0.5$. Note that $\mathcal{R}_0 = 0.93 < \mathcal{R}_1 = 0.99$, then the disease free equilibrium X^0 is globally asymptotically stable.

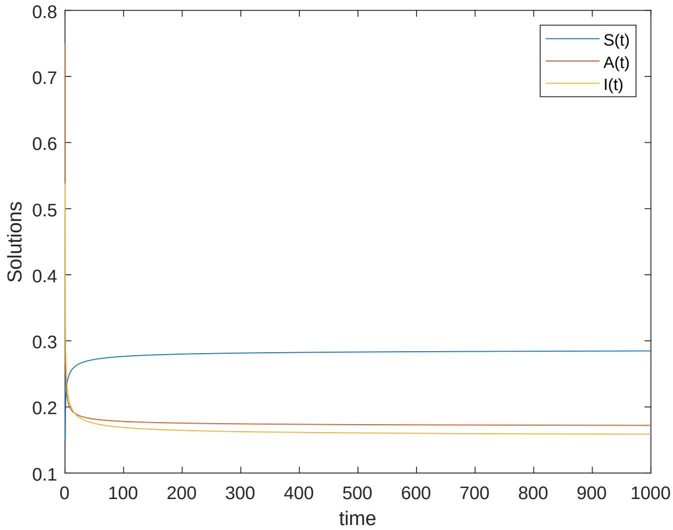


Figure 5.4: Numerical solutions of (5.3) with $\alpha = 0.5$, $\mu = 1.25$, $\beta_A = 15$, $\beta_I = 3.5$, $\nu = 0.0002$, $\gamma = 1.25$, $\eta = 2.5$, $\delta_A = 1.5$, $\delta_I = 1.5$ and the initial conditions are $S_0 = 0.25$, $A_0 = 0.75$ and $I_0 = 0.5$. Note that $\mathcal{R}_0 = 3.46 > 1$, then the endemic equilibrium X^* is locally asymptotically stable.

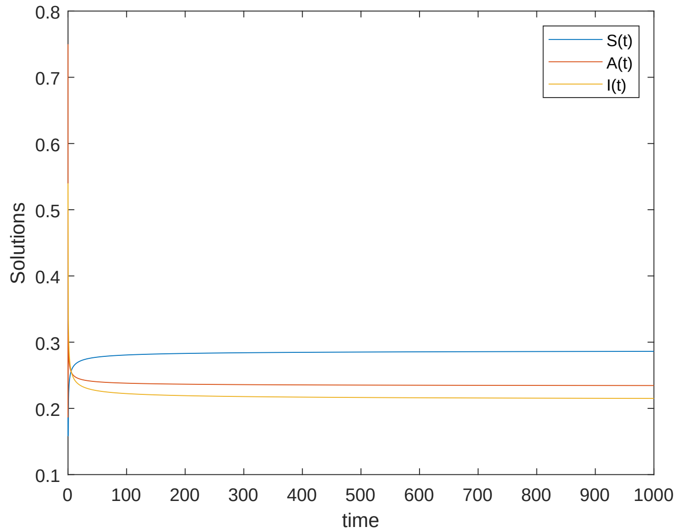


Figure 5.5: Numerical solutions of (5.3) with $\alpha = 0.5$, $\mu = 1.25$, $\beta_A = 15$, $\beta_I = 3.5$, $\nu = 0.0002$, $\gamma = 0$, $\eta = 2.5$, $\delta_A = 1.5$, $\delta_I = 1.5$ and the initial conditions are $S_0 = 0.25$, $A_0 = 0.75$ and $I_0 = 0.5$. Note that $\mathcal{R}_0 = 3.46 > 1$, then the endemic equilibrium X^* is globally asymptotically stable.

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