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Presented by: DJELLALI Noura

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Geometric properties of hypersurfaces in the geometry of Thurston Nil^4

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The jury :

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Spécialité : Géométrie différentielle

Présenté par :DJELLALI Noura

Intitulé :

Propriétés géométriques des hypersurfaces dans la géométrie de Thurston Nil^4

LE 12\05\2024

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ABSTRACT

In this research we give some geometric properties of hypersurfaces (M^3, g) in the nilpotent Lie group $(Nil^4, \widetilde{g})$. First, we give a left invariant metric, the Levi-Civita connection, Riemannian curvature, and the Ricci tensor in an orthonormal basis of vector field in Nil^4, beside, we note a classification of Codazzi hypersurfaces in a Lie group $(Nil^4, \widetilde{g})$. We also give a characterization of a class of minimal hypersurfaces in $(Nil^4, \widetilde{g})$ with an example of a minimal surface in this class.
Key words: Codazzi hypersurfaces, minimal hypersurfaces.

RÉSUMÉ

Dans cette recherche, on donnes quelques propriétés géométriques des hypersurfaces (M^3, g) dans le groupe de Lie nilpotent $(Nil^4, \widetilde{g})$. Tout d´abord, on donnes une métrique invariante á gauche, la connexion de Levi-Civita, la courbure Riemannienne et le tenseur de Ricci dans une base orthonormée de champ de vecteurs dans Nil^4. De plus, on notes une classification des hypersurfaces de Codazzi dans un groupe de Lie $(Nil^4, \widetilde{g})$. On donnes également une caractérisation d´une classe d´hypersurfaces minimales dans $(Nil^4, \widetilde{g})$ avec un exemple de surface minimale dans cette classe.
Mots clés: Hypersurfaces de Codazzi, hypersurfaces minimales.

ملخص

في هذا البحث تم إعطاء بعض الخصائص الهندسية للأسطح الفائقة (M^3, g) في زمرة لي $(Nil^4, \widetilde{g})$. في البداية نعطي الموتر المتري الريماني لا المتغير اليساري، وصلة ريمان، الانحناء ريماني، وموتر ريتشي على أساس متعامد طبيعي لحقل متجه في Nil^4، بالإضافة إلى ذلك، نحدد تصنيفا لأسطح كودازي الفائقة في زمرة لي $(Nil^4, \widetilde{g})$. نقدم أيضا تصنيفا لميزات الحد الأدنى من الأسطح الفائقة في $(Nil^4, \widetilde{g})$ مع مثال على الحد الأدنى من السطح في هذه الفئة.

الكلمات المفتاحية: السطوح الفائقة كودازي، السطوح الفائقة الحد الأدنى.

CONTENTS

INTRODUCTION

Three-dimensional Thurston model geometries are classified by W. Thurston, this classification has eight geometries. In 1983 R. Filipkiewicz[14] gave a classification to Thurston geometry from the fourth dimension, and he classified them into symmetric and non-symmetric spaces, in addition to this he considered Nil^4 as a non-symmetric space.
C. T. C. Wall[33] has studied the complex structures on 4-dimensional Thurston geometries.
S. Maier, in 1998 studied the conformal flatness of 4-dimensional Thurston geometries.
Then, in 2014 Professor BELKHELFA Mohamed and Dr. Hasni gave geometric properties for some groups in their research [17]. Other studies presented the classifications of subgroups, such as " submanifolds", especially hypersurfaces that are considered as submanifold from the third dimension.
This research focuses on Nil^4 Lie group, presented as following:

$$Nil^4 = \mathbb{R}^3 \ltimes_U \mathbb{R}$$

where $U(t) = \exp(tL)$, with

$$L = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Let (M^3, g) be a Riemannian hypersurfaces in $(Nil^4, \widetilde{g})$, where $\widetilde{g}$ is left invariant metric.
The second fundamental form B of (M^3, g) in $(Nil^4, \widetilde{g})$ is given by:

$$B(X,Y) = h(X,Y)\xi = \widetilde{\nabla}_X Y - \nabla_X Y,$$

where $X, Y \in \mathfrak{X}(M^3)$, ξ the unit normal vector field in M^3, and $\widetilde{\nabla}$(resp ∇)is the Levi-Civita connection of $(Nil^4, \widetilde{g})$ (resp (M^3, g)).

- (M^3, g) is totally geodesic $\quad if \quad h = 0.$

- (M^3, g) is parallel $\quad if \quad \nabla h = 0.$
- (M^3, g) is Codazzi $\quad if \quad (\nabla_Y h)(X, Z) = (\nabla_X h)(Y, Z).$
- (M^3, g) is minimal $\quad if \quad trace_g(h) = 0.$

This thesis aims to prove that the hypersurface (M^3, g) in $(Nil^4, \widetilde{g})$, every Codazzi is parallel and a minimal, however not every minimal is a Codazzi.

In order to get the results we used the definitions and the information distributed in the chapters as following:

The First Chapter: gives some important definitions of differentiable manifolds.

The Second Chapter: presents the basic definitions and the properties of Riemannian manifolds.

The Third Chapter: speaks about the geometries of submanifolds in a Riemannian manifold, citing the formulas of Gauss and Weingarten, the equations of Gauss and Coddazzi, especially hypersurfaces.

The Fourth Chapter: defines the geometry of Thurston Nil^4, gives its metric and the geometric properties.

The Last Chapter: exposes the results of hypersurface (M^3, g) in $(Nil^4, \widetilde{g})$, giving the conditions of the unit normal vector ξ in M^3, where (M^3, g) is Codazzi and minimal.

CHAPTER 1

SUBMANIFOLD DIFFERENTIABLE MANIFOLDS

In this chapter, we give definitions of differentiable manifolds, tangent spaces, tangent bundle, Tensor fields, Lie bracket, and diferential forms. The references used are:[1], [2], [3],[5], [8], [14], [15], [18], [20], [21], [22], [23], [24], [27], [33]. Differentiable always signifies of class C^∞ .

1.1 Differentiable manifolds

Definition 1.1.1. *A topology on a set* $\widetilde{M}$ *is any part* $\mathcal{T}$ *of* $\mathcal{P}(\widetilde{M})$ *verifying the following properties:*

1. $\emptyset, X \in \mathcal{T}$.
2. *if* $U, V \in \mathcal{T}$ *then* $U \cap V \in \mathcal{T}$.
3. *if* $\{U_i\}_{i\in I} \in \mathcal{T}$ *then* $\bigcup_{i\in I} U_i \in \mathcal{T}$.

A topological space $(\widetilde{M}, \mathcal{T})$ is called a separate space (or a Hausdorff space) for any point $x, y \in \widetilde{M}$ where $x \neq y$ there exists two open sets $U, V \in \mathcal{T}$ with

$$x \in U, \quad y \in V \quad \textit{and} \quad U \cap V = \emptyset.$$

Definition 1.1.2. *A topological space* $\widetilde{M}$ *is locally Euclidean of dimension* m *if every point* p *in* $\widetilde{M}$ *has a neighborhood* U *such that there is a homeomorphism* φ *from* U *onto an open subset of* $\mathbb{R}^m$. *We call the pair* (U, φ) *a chart.*

Definition 1.1.3. *A topological manifold $\widetilde{M}$ of dimension m is a Hausdorff, locally Euclidean space of dimension m and has a countable basis of open sets.*

Definition 1.1.4. *Two charts $(U, \varphi : U \to \mathbb{R}^m)$, $(V, \psi : V \to \mathbb{R}^m)$ of a topological are differentiable compatible if the two maps*
$\varphi \circ \psi^{-1} : \psi(U \cap V) \longrightarrow \varphi(U \cap V)$, $\psi \circ \varphi^{-1} : \varphi(U \cap V) \longrightarrow \psi(U \cap V)$ are differentiable these two maps are called the transition function between the charts with $U \cap V \neq \emptyset$.

Definition 1.1.5. *A differentiable atlas on a locally Euclidean space $\widetilde{M}$ is a collection $\mathcal{A} = (U_I, \varphi_I)$ of pairwise differentiable compatible charts that cover $\widetilde{M}$, i.e., such that $\widetilde{M} = \bigcup_I U_I$*

Definition 1.1.6. *A differentiable manifold is a topological manifold together with a differentiable atlas.*

Example 1.1.1. *1⧫ $\mathbb{R}^n$ is a differentiable manifold of dimension n of class C^∞, where $\{(\mathbb{R}^n, Id_{\mathbb{R}^n})\}$ is an atlas.*

2⧫ The standard sphere $\mathbb{S}^n = \{u \in \mathbb{R}^{n+1} \mid \|u\| = 1\}$ is a differentiable manifold of dimension n. $\mathbb{S}^n$ is a topological space, where $\mathcal{T}_{\mathbb{S}^n}$ is the topology induced by that of $\mathbb{R}^{n+1}$ (its the topology whose openings are of the form $U = \Omega \cap \mathbb{S}^n$ where Ω is an open from $\mathbb{R}^{n+1}$). Let the projections stereographic

$$\begin{array}{rcl} \varphi_N : U_N = \mathbb{S}^n - \{N\} & \longrightarrow & \mathbb{R}^n \\ (u_1, ..., u_{n+1}) & \longmapsto & \left(\dfrac{u_1}{1 - u_{n+1}}, ..., \dfrac{u_n}{1 - u_{n+1}} \right). \end{array}$$

$$\begin{array}{rcl} \varphi_S : U_S = \mathbb{S}^n - \{S\} & \longrightarrow & \mathbb{R}^n \\ (u_1, ..., u_{n+1}) & \longmapsto & \left(\dfrac{u_1}{1 + u_{n+1}}, ..., \dfrac{u_n}{1 + u_{n+1}} \right). \end{array}$$

The applications $\varphi_N : U_N \longrightarrow \mathbb{R}^n$ and $\varphi_S : U_S \longrightarrow \mathbb{R}^n$ are homeomorphism.
Using

$$1 - u_{n+1}^2 = u_1^2 + ... + u_n^2,$$

we find that

$$\begin{array}{rcl} \varphi_N^{-1} : \mathbb{R}^n & \longrightarrow & U_N \\ (x_1, ..., x_n) & \longmapsto & \left(\dfrac{2x_1}{\|x\|^2 + 1}, ..., \dfrac{2x_n}{\|x\|^2 + 1}, \dfrac{\|x\|^2 - 1}{\|x\|^2 + 1} \right). \end{array}$$

$$\begin{array}{rcl} \varphi_S^{-1} : \mathbb{R}^n & \longrightarrow & U_S \\ (y_1, ..., y_n) & \longmapsto & \left(\dfrac{2y_1}{\|y\|^2+1}, ..., \dfrac{2y_n}{\|y\|^2+1}, -\dfrac{\|y\|^2-1}{\|y\|^2+1} \right). \end{array}$$

the mapping of charts transitions are given by

$$\varphi_S \circ \varphi_N^{-1} = \frac{x}{\|x\|^2}, \quad \varphi_N \circ \varphi_S^{-1} = \frac{y}{\|y\|^2}, \quad \forall x, y \in \mathbb{R}^n - \{0\},$$

which are diffeomorphisms of differentiable. Therefore $\mathcal{A}_{\mathbb{S}^n} = \{(U_N, \varphi_N), (U_S, \varphi_S)\}$ *form a differentiable atlas.*

Definition 1.1.7. *An atlas for a differentiable manifold* $\widetilde{M}$ *is called oriented if all* $\mathcal{A} = \{(U_i, \varphi_i)\}_{i\in I}$ *such that the charts changes mapping* $\psi_{ij} = \varphi_i \circ \varphi_j^{-1}$ *has a positive Jacobian, i.e*

$$J(\psi_{ij})_p = det(d_{\varphi_j(p)}\psi_{ij}) > 0.$$

Definition 1.1.8. *A differentiable manifold is called oriented if it possesses an oriented atlas.*

Remark 1.1.1. *If* φ *be a diffeomorphism of* $\mathbb{R}^n$*, its Jacobian is defined by:*

$$J(\varphi)_p = det(d_p\varphi).$$

Example 1.1.2. *1)* $\mathbb{R}^n$ *is an orientable manifold because* $\mathcal{A} = (\mathbb{R}^n, Id_{\mathbb{R}^n})$.

2) A surface S *in* $\mathbb{R}^3$ *is orientable if it has two sides. Then, one can orient the surface by choosing one side to be the positive side. Some unusual surfaces however are not orientable because they have only one side. One classical examples is called the Möbius strip. So that a Möbius strip is not orientable because the normal vector field an this surface* S *is not orientable.*

Definition 1.1.9. *Let* M *and* $\widetilde{M}$ *two differentiable manifolds, a mapping* $f : M \longrightarrow \widetilde{M}$ *is said to be differentiable, if for every chart* (U_i, φ_i) *of* M *and every chart* (V_j, ψ_j) *of* $\widetilde{M}$ *such that* $f(U_i) \subset V_j$*, the mapping* $\psi_j \circ f \circ \varphi_i^{-1} : \varphi_i(U_i) \longrightarrow \psi_i(V_j)$ *is differentiable.*

Definition 1.1.10. F *is a smooth mapping of* M^n *into* $\widetilde{M}^m$ *if for every* $p \in M$ *there exist a coordinated neighborhood* (U, φ) *of* p *and* (V, ψ) *of* $F(p)$ *with* $F(U) \subset V$ *such that* $\psi \circ F \circ \varphi^{-1} : \varphi(U) \longrightarrow \psi(V)$ *is differentiable.*

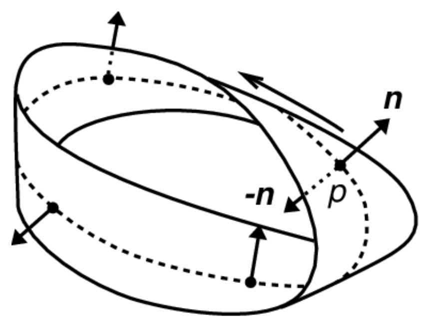

Figure 1.1: The non-oriented Möbius strip

1.2 Tangent Spaces

1.2.1 Tangent vectors

Definition 1.2.1. *Let $\widetilde{M}$ be a differentiable manifolds and D an open set of $\widetilde{M}$. A function $f : D \longrightarrow \mathbb{R}$ is called differentiable at $p \in D$, if there is chart (U, φ) of $\widetilde{M}$ with $p \in U$ and $U \subset D$ such that $f \circ \varphi^{-1} : \varphi(U) \longrightarrow \mathbb{R}$ is differentiable.*
$C^\infty(\widetilde{M})$ a set of differentiable functions on $\widetilde{M}$ at p.

Definition 1.2.2. *Let $\widetilde{M}$ be a differentiable manifold of dimension m and $p \in \widetilde{M}$. The tangent vector at p of $\widetilde{M}$ is a map*

$$\begin{array}{rcl} B_p : C^\infty(\widetilde{M}) & \longrightarrow & \mathbb{R} \\ f & \longmapsto & B_p(f), \end{array}$$

such that:

1 ∗ B_p *is a linear mapping of* $C^\infty(\widetilde{M})$ *into* $\mathbb{R}$,

2 ∗ $B_p(fg) = (B_p f)g(p) + f(p)(B_p g)$, *for all* $f, g \in C^\infty(\widetilde{M})$,

3 ∗ *If f is a constant in the neighborhood of p then $B_p(f) = 0$.*

Remark 1.2.1. *1 Let $\widetilde{M}$ be a differentiable manifold of dimension m, $p \in \widetilde{M}$, and $\gamma : I \subset \mathbb{R} \longrightarrow \widetilde{M}$ a differentiable curve with $\gamma(0) = p$ and f differentiable function on $\widetilde{M}$ at p. The tangent vector at p is the function $\gamma'(0) : C^\infty(\widetilde{M}) \longrightarrow \mathbb{R}$ given by*

$$\gamma'(0)(f) = \frac{d(f \circ \gamma)}{dt}|_{t=0}.$$

2 Let $\widetilde{M}$ be an m-dimensional differentiable manifold, (U,φ) be a chart of $\widetilde{M}$, $p \in U$, and $\varphi(p) = (x^1, ..., x^m)$ we define the map:

$$\begin{aligned} \frac{\partial}{\partial x_i}\Big|_p : C^\infty(\widetilde{M}) \quad &\longrightarrow \quad \mathbb{R} \\ f \quad &\longmapsto \quad \frac{\partial}{\partial x_i}\Big|_p(f) = \frac{\partial(f\circ\varphi^{-1})}{\partial x_i}\Big|_{\varphi(f)}. \end{aligned}$$

$\frac{\partial}{\partial x_i}|_p$ is said derivative associated to the chart (U,φ) and $\{\frac{\partial}{\partial x_1}|_p, ..., \frac{\partial}{\partial x_m}|_p\}$ be a frame for the tangent space $T_p\widetilde{M}$, for all $p \in U$.

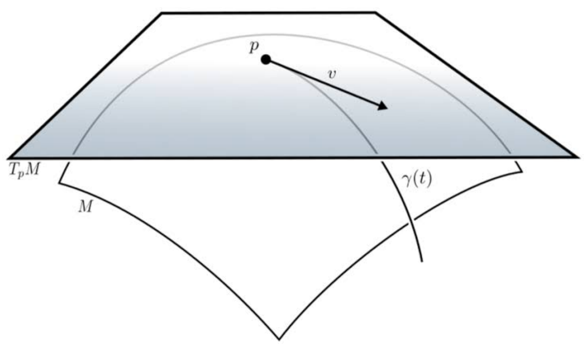

Figure 1.2: Tangent vector

Definition 1.2.3. *Let M and $\widetilde{M}$ two differentiable manifolds, and f a C^1 map $M \longrightarrow \widetilde{M}$. Let $p \in M$, $q = f(p)$, and let T_pM, $T_q\widetilde{M}$ be the tangent spaces at p, q respectively. To each $W \in T_pM$ there corresponds a tangent vector $V \in T_q\widetilde{M}$ as follows. For $g \in C^\infty(\widetilde{M})$ the function $g \circ f = g^*$, say, is in $C^\infty(M)$. Then define $V(g) = W(g^*)$. This defines a map $T_pM \longrightarrow T_q\widetilde{M}$ which is easily seen to be linear and which is called the differential of f denoted df. We can also express this definition by the formula*

$$((df)(W))(g) = W(g\circ f), \quad W \in T_pM, \quad g \in C^\infty(\widetilde{M}).$$

1.2.2 Tangent bundle

Definition 1.2.4. *Let $\widetilde{M}$ be a differentiable manifold. We define the tangent bundle of $\widetilde{M}$, denoted by*

$$T\widetilde{M} = \bigcup_{p\in\widetilde{M}} T_p\widetilde{M}.$$

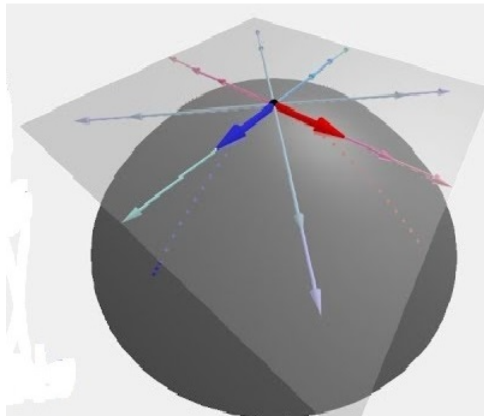

Figure 1.3: Tangent space

Remark 1.2.2. *$T_p^*\widetilde{M}$ is the dual space (cotangent space) of the tangent space $T_p\widetilde{M}$ of $\widetilde{M}$ at p. Denoted by $(dx^i|_p)$ form a basis of $T_p^*\widetilde{M}$, we have $\langle dx^i, \frac{\partial}{\partial x^j}\rangle_p = \delta_{ij}$ with:*

$$\delta_{ij} = \begin{cases} 1 & if \quad i = j \\ 0 & if \quad i \neq j \end{cases}$$

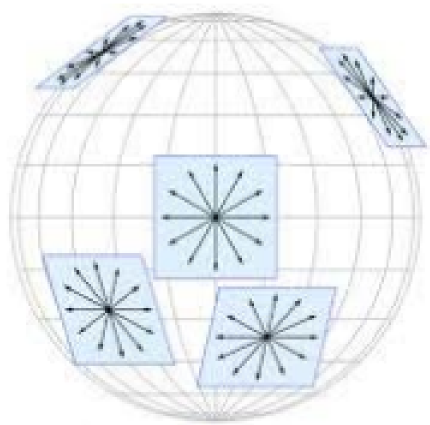

Figure 1.4: Tangent bundle on a 2-sphere

1.2.3 Vector fields

Definition 1.2.5. *Let $T_p\widetilde{M}$ be the tangent space to a differentiable manifold $\widetilde{M}$ at p. A vector field on a differentiable manifold $\widetilde{M}$ is a section of the tangent fibre $T\widetilde{M}$ of this manifold, i.e,* *the differentiable application*

$$\begin{aligned} X : \widetilde{M} &\rightarrow T\widetilde{M} \\ p &\mapsto X(p). \end{aligned}$$

The set of vector fields on $\widetilde{M}$ *is denoted by* $\mathfrak{X}(\widetilde{M})$.

1.2.4 Tensor fields

Definition 1.2.6. *For any point* p *in* $\widetilde{M}$ *we define the vector space*

$$T_p^{(r,d)}\widetilde{M} = \underbrace{T_p\widetilde{M} \otimes ... \otimes T_p\widetilde{M}}_{r-\text{once}} \otimes \underbrace{T_p^*\widetilde{M} \otimes ... \otimes T_p^*\widetilde{M}}_{d-\text{once}}.$$

Let $T^{(r,d)}\widetilde{M} = \bigcup_{p\in\widetilde{M}} T_p^{(r,d)}\widetilde{M}$. *A element* $T \in T_p^{(r,d)}\widetilde{M}$ *is a tensor of type* (r,d) *above* p. *A tensor field of type* (r,d) *on a manifold* $\widetilde{M}$ *is an assignment section of* $T^{(r,d)}\widetilde{M}$ *i.e. a tensor is a map:*

$$\begin{aligned} T : \widetilde{M} &\longrightarrow T^{(r,d)}\widetilde{M} \\ p &\longmapsto T(p) \in T_p^{(r,d)}\widetilde{M}. \end{aligned}$$

Remark 1.2.3.
1. *A function on a manifold* $\widetilde{M}$ *is a tensor of type* $(0,0)$*:* $T^{(0,0)} = C^\infty(\widetilde{M})$.
2. *A vector field* X *is a tensor of type* $(1,0)$*:* $T^{(1,0)} = \mathfrak{X}(\widetilde{M})$.
3. *A differential* $1-$ *form is a tensor of type* $(0,1)$*:* $T^{(0,1)} = \Omega^1(\widetilde{M})$.

1.2.5 Immersions- Embeddings

Definition 1.2.7. *Let M^m and $\widetilde{M}^n$ be differentiable manifolds. A differentiable mapping $f : M \longrightarrow \widetilde{M}$ is said to be an immersion if df_p: $T_pM \longrightarrow T_{f(p)}\widetilde{M}$ injective for all $p \in M$. If, in addition, f is a homeomorphism onto $f(M) \subset \widetilde{M}$, where $f(M)$ has the subspace topology induced from $\widetilde{M}$, we say that f embedding. If $M \subset \widetilde{M}$ and the inclusion $i : M \subset \widetilde{M}$ is an embedding, we say that M is a submanifold of $\widetilde{M}$.*

Remark 1.2.4. *It can be seen that if $f : M^n \longrightarrow \widetilde{M}^m$ is an immersion, then $n \leqslant m$, the difference $m - n$ is called the codimension of the immersion f.*

Example 1.2.1. • *The curve $\gamma(t) = (t^3, t^2)$ is a differentiable mapping but is not an immersion. Indeed, the condition for the map to be an immersion in this case is equivalent to the fact that $\gamma'(t) \neq 0$, which does not occur for $t = 0$ (Fig (1.5)).*
• *The curve $\gamma(t) = (t^3 - 4t, t^2 - 4)$ (Fig (1.6)) is an immersion, $\gamma : \mathbb{R} \longrightarrow \mathbb{R}^2$ which has a self-intersection for $t = 2, \quad t = -2$.*
Therefore, γ is not an embedding.

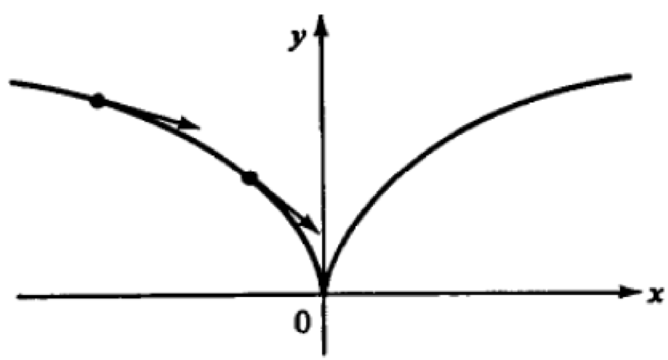

Figure 1.5: The curve is not an immersion

Proposition 1.2.1. *[8] Let $f : M^n \longrightarrow \widetilde{M}^m$, $n \leq m$, be an immersion of the differentiable manifold M into the differentiable manifold $\widetilde{M}$. For every point $p \in M$, there exists a neighborhood $U \subset M$ of p such that the restriction $f|_U \to \widetilde{M}$ is an embedding.*

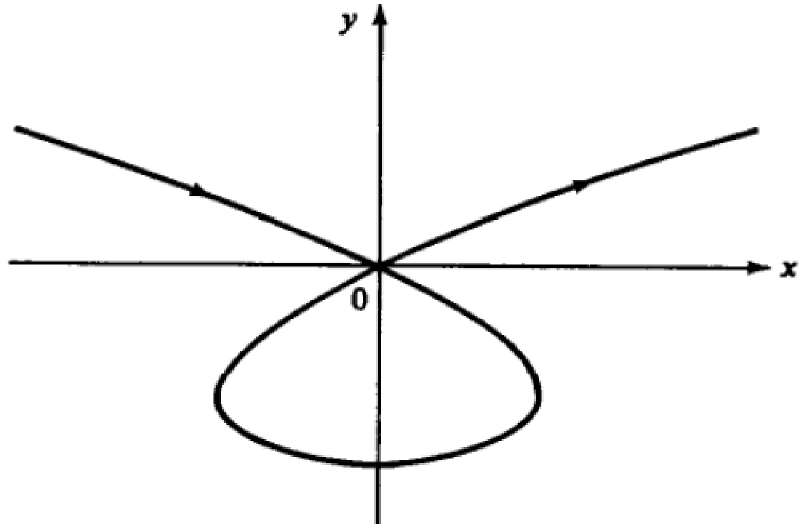

Figure 1.6: The curve is not an embedding

1.2.6 Submanifolds

Definition 1.2.8. *A subset M^n of a differentiable manifold $\widetilde{M}^m$ is said to have the n-submanifold property if each $x \in M$ has a coordinate neighborhood (V, ψ), on $\widetilde{M}$ with local coordinates $x^1, \ldots, x^m$ such that*

1) $\psi(x) = (0, \ldots, 0)$,

2) $\psi(V) = C^m_\epsilon(0)$, and

3) $\psi(V \cap M) = \{p \in C^m_\epsilon(0)/x^{n+1} = \ldots = x^m = 0\}$. If M has this property, coordinate neighborhoods of this type are called preferred coordinates (relative to M).

Where $C^m_\epsilon(0) = \{x \in \mathbb{R}^m/|x^i| < \epsilon, i = 1, \ldots, m\}$.

Definition 1.2.9. *A regular submanifold of a differentiable manifold M is any sub-space $\widetilde{M}$ with submanifold property and with a differentiable structure that the corresponding preferred coordinate neighborhoods determine on it.*

Example 1.2.2. *A regular surface $S \subset \mathbb{R}^3$ has a differentiable structure given by its parametrizations $x_\alpha : U_\alpha \longrightarrow S$. With such a structure, the mappings x_α are differentiable and, indeed, are embeddings of U_α into S, that is an immediate consequence of condition*

a) x_α are differentiables homeomorphisms, *b) The differential* $(dx_\alpha)|_p : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ *are injective for all* $p \in U_\alpha$.

We are going to show that the inclusion $i : S \subset \mathbb{R}^3$ *is an embedding, that is,* S *is a submanifold of* $\mathbb{R}^3$.

1.2.7 Lie bracket

Definition 1.2.10. *Let* $\widetilde{M}$ *be a differentiable manifold and the application* $[,]$ *is called a Lie bracket defined by:*

$$[X, Y] = XY - YX$$

for all X, $Y \in \mathfrak{X}(\widetilde{M})$.

The Lie bracket has the the following properties

Proposition 1.2.2. *[8] If* X, Y *and* Z *are differentiable vector fields on* $\widetilde{M}$, a, b *are real numbers, and* f, g *are differentiable function, then:*

1) $[X, Y] = -[Y, X]$ *(anticommutativity),*
2) $[aX + bY, Z] = a[X, Z] + b[Y, Z]$ *(linearity),*
3) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$*(Jacobi identity),*
4) $[fX, gY] = fg[Y, X] + fX(g)Y - gY(f)X$.

1.3 Lie groups and Lie algebra of a Lie group

The space $\mathbb{R}^n$ is a differentiable manifold and at the same time an abelian group with group operation given by componentwise addition. Moreover the algebraic and differentiable structures are related: $(a, b) \longmapsto a + b$ is a differentiable mapping of the product manifold $\mathbb{R}^n \times \mathbb{R}^n$ onto $\mathbb{R}^n$, that is, the group operation is differentiable. We also see that the mapping of $\mathbb{R}^n$ onto $\mathbb{R}^n$ given by taking each element a to its inverse a^{-1} is differentiable. Now, let G be a group which is at the same time a differentiable manifold. For $a, b \in G$, let ab denote their product, and a^{-1} the inverse of a.

1.3.1 Lie groups

Definition 1.3.1. *A Lie group (named Sophus Lie on* 17 *December* 1842*) is a finite dimensional smooth manifold* G *together with a group structure on* G*, such that the multiplication* $G \times G \to G$ *defined by* $(x, y) \longmapsto xy$ *and the mapping of an inverse* $G \to G$ *defined by* $x \longmapsto x^{-1}$ *are differentiable mappings.*

Example 1.3.1. *Let* $\mathbb{C}^*$ *be the nonzero complex numbers. Then* $\mathbb{C}^*$ *is a group with respect to multiplication of complex numbers, the inverse being* $Z^{-1} = \frac{1}{Z}$*. Moreover,* $\mathbb{C}^*$ *is a one-dimensional differentiable manifold covered by a single coordinate neighborhood* $U = \mathbb{C}^*$ *with coordinate map* $Z \longmapsto \varphi(Z)$ *given by* $\varphi(Z) = \varphi(x + iy) = (x, y)$ *for* $Z = x + iy$*. Using these coordinates, the product* $W = ZZ'$*,* $Z = x + iy$*, and* $Z' = x' + iy'$ *is given by*

$$(Z, Z') = ((x, y), (x', y')) \longmapsto (xx' - yy', xy' + yx') = W$$

and the mapping $Z \longmapsto Z^{-1}$ *by*

$$(x, y) \longmapsto (\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2})$$

$\mathbb{C}^*$ *is a Lie group.*

Theorem 1.3.1. *[3] If* G *and* G' *are Lie groups, then the direct product* $G \times G'$ *of these groups with the differentiable structure of the Cartesian product of manifolds is a Lie group.*

Definition 1.3.2. *Let* G *be a Lie group. We denote by* L_p *the left translation in* G *by an element* $p \in G$*:*

$$L_p(q) = pq,$$

for every $q \in G$*.*

Definition 1.3.3. *Let* G *be a Lie group. A vector field* X *on* G *is called left-invariant if it is invariant by all* L_p*, i.e.,*

$$(L_p)_*(X) = X,$$

for all $p \in G$*, is equivalent to:*

$$dL_p(X_q) = X_{pq}.$$

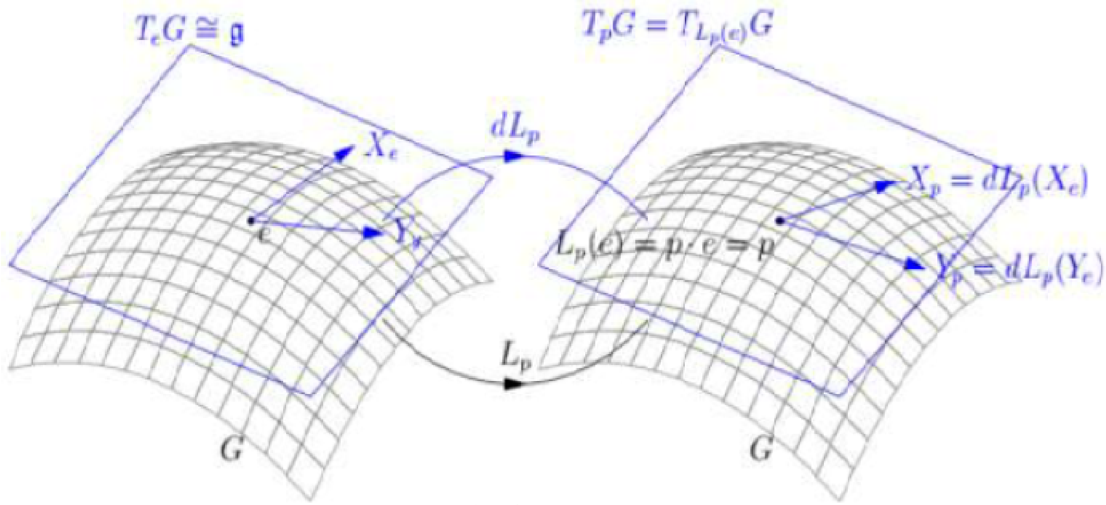

Figure 1.7: The left translation

Definition 1.3.4. *Let G be a Lie group. We denote by R_p the right translation G by an element $p \in G$:*

$$R_p(q) = qp,$$

for every $q \in G$.

Notation : We denote by $\mathfrak{X}^L(G) = \{X \in \mathfrak{X}(G) | (L_p)_*(X) = X, \forall p \in G\}$ the space of left-invariant vector fields.

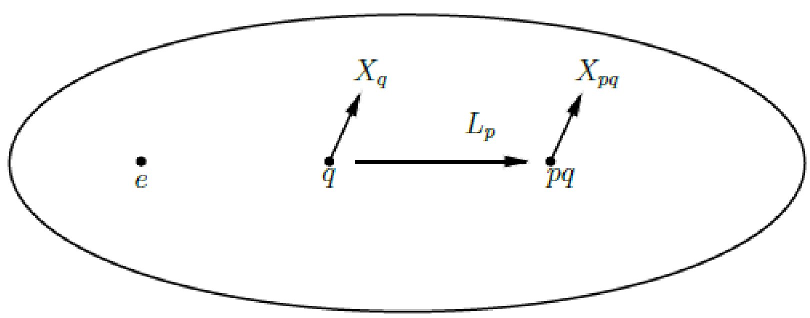

Figure 1.8: left-invariant vector field

Example 1.3.2. *The real three dimensional Heisenberg group is a Lie group in* $GL(3, \mathbb{R}$*, defined in the following way*

$$\mathbb{H}_3 = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \forall x, y, z \in \mathbb{R} \right\}.$$

The group operation is the standard matrix multiplication, which gives the multiplication rule:

$$(a, b, c).(a', b', c' = (a + a', b + b', c + c' + ab').$$

where $e = (0, 0, 0)$ *is the identity element, and for any matrix* $(x, y, z) \in \mathbb{H}_3$*, the inverse is given by*

$$(x, y, z)^{-1} = (-x, -y, xy - z).$$

In this example we will show that the three vector fields:

$$X = \frac{\partial}{\partial x}, \quad Y = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \quad Z = \frac{\partial}{\partial z}.$$

are left-invariant vector fields on $\mathbb{H}_3$*.*

In deed, the left translation in this group is given by

$$\forall (a, b, c) \in \mathbb{H}_3, \quad L(x, y, z) = (x + a, y + b, z + c + xb).$$

We obtain the following results:

$$dL(x, y, z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x & 1 \end{pmatrix}.$$

So, X *is left-invariant vector field, if*

$$d_eL(x, y, z)(X_e) = X_{(x,y,z)}.$$

Which means that:

$$d_eL(x, y, z)(\frac{\partial}{\partial x}|_e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{\partial}{\partial x}|_{(x,y,z)},$$

$$X = \frac{\partial}{\partial x},$$

for the fields Y, Z *we have*

$$d_eL(x, y, z)((\frac{\partial}{\partial y} + x\frac{\partial}{\partial z})|_e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ x \end{pmatrix} = (\frac{\partial}{\partial y} + x\frac{\partial}{\partial z})|_{(x,y,z)},$$

$$Y = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z},$$

and

$$d_eL(x,y,z)(\frac{\partial}{\partial z}|_e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{\partial}{\partial z}|_{(x,y,z)},$$

$$Z = \frac{\partial}{\partial z}.$$

Theorem 1.3.2. *[3] Let G be a Lie group and H a subgroup which is also a regular submanifold. Then with its differentiable structure as a submanifold H is a Lie group.*

Example 1.3.3. $O(n) = \{A \in GL(n,\mathbb{R}) | A^tA = I\}$*, the subgroup of orthogonal $n \times n$ matrices is a regular submanifold and thus a Lie group.*

Definition 1.3.5. *Let G be a Lie group and X a differentiable manifold. Then G is said to action on X(on the left) if there is a differentiable mapping $\phi : G \times X \to X$ satisfying two conditions:*

i) If e is the identity element of G, then $\phi(e,x) = x$ for all $x \in X$;

ii) If $g, g' \in G$, then $\phi(g, \phi(g', x)) = \phi(g.g', x)$ for all $x \in X$.

Definition 1.3.6. *An action $G \times X \to X$ of a Lie group G on a differentiable manifold X is called transitive if it has a single orbit, i.e. for any two elements $x, y \in X$, there exist $g \in G$ such that*

$$y = \phi(g,x) = g.x.$$

Definition 1.3.7. *Given an action $G \times X \to X$ of a Lie group G on a differentiable manifold X, for every element $x \in X$, the stabilizer subgroup of x (also called the isotropy group of x) is the set of all elements in G that leave x fixed*

$$stab(x) = \{g \in G \mid \phi(g,x) = x\}.$$

Definition 1.3.8. *Let G and G' be groups and let $Aut(G')$ the automorphism group of G' for the law "$\circ$". The direct product $G' \times G$ of G' and G is the group whose underlying set is the* *product set $G' \times G$, with the law $(x, y)(x', y') = (xx', yy')$ for all $x, x' \in G'$ and $y, y' \in G$. The semi-direct product is a generalization of this notion. Let $\phi : G \to Aut(G')$ a group morphism which in particular defines an action $y.x = \phi(y, x)$ of G' on G.*

Proposition 1.3.1. *[3] We define a group law on the product set $G' \times G$ in posing:*

$$(x.y).(x'.y') = (x\phi(y, x'), yy').$$

This group is called the semi-direct product of G' by G relative to the action ϕ, it is denoted $G \ltimes_\phi G'$ (or simply $G \ltimes G'$).

1.3.2 Lie algebra of a Lie group

Definition 1.3.9. *The Lie algebra of a Lie group G is the vector space $\mathcal{G} = T_eG$, equipped with the Lie bracket $[.,.]$.*

Proposition 1.3.2. *[3] If G is a Lie group (with neutral element denoted e), the map $X \longmapsto X_e$ is an isomorphism between the vector space of invariant vector fields at left on G and the tangent space T_eG.*

Definition 1.3.10. *Let G be a Lie group and $\mathcal{G}$ a Lie algebra of G. We associate to it the set*

$$\mathcal{G} = Lie(G) = \{A \in \mathsf{M}(n, \mathbb{R}) | \forall t \in \mathbb{R}, \exp(tA) \in G\},$$

where the exponential of a matrix $A \in \mathsf{M}(n, \mathbb{R})$ is defined by

$$\exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

1.3.3 Nilpotent Lie groups

Definition 1.3.11. *Let G be a group of neutral element denoted by e. The commutator of two elements A and B of G is defined by*

$$[A, B] = A^{-1}B^{-1}AB.$$

If H and K are two subgroups of G, we denote $[H, K]$ the subgroup which is generated by the commutators of the form $[A, B]$ for $A \in H$ and $B \in K$.

We define by recurrence a sequence (the descending central sequence of G) of subgroups by G, by

$$C^1(G) = G, \quad C^n(G) = [C^{n-1}(G), G].$$

Definition 1.3.12. *The group G is said to be nilpotent if there exists an integer n such that $C^n(G) = \{e\}$.*

Definition 1.3.13. *The algebra of Lie $\mathcal{G}$ is said to be nilpotent if there exists an integer $n \geqslant 1$ such that $C^n(\mathcal{G}) = \{0\}$.*

Proposition 1.3.3. *[3]*

1 *If G is nilpotent then the Lie algebra $\mathcal{G}$ is nilpotent.*

2 *If G is connected and Lie algebra $\mathcal{G}$ is nilpotent then G is nilpotent.*

Example 1.3.4. *(Heisenberg group). Let*

$$\mathbb{H}_3 = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, x, y, z \in \mathbb{R} \right\}.$$

The Lie algebra of $\mathbb{H}_3$ is given by

$$\mathcal{G} = T_{Id}\mathbb{H}_3 = \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}, x, y, z \in \mathbb{R} \right\}$$

So that, for all $A \in \mathcal{G}$, we have

$$\begin{aligned} A^3 = \left(\begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \right)^3 &= \begin{pmatrix} 0 & 0 & xy \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

This Lie algebra has a basis

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore $\mathcal{G}$ is a 3-step nilpotent Lie algebra and $\mathbb{H}_3$ nilpotent Lie group.

CHAPTER 2

RIEMANNIAN MANIFOLDS

In this chapter we recall some properties and definitions of Riemannian manifolds. The references used are: [8], [11], [14], [17], [22], [24], [25], [27], [30], [35].

2.1 Riemannian manifold

2.1.1 Riemannian metrics

Definition 2.1.1. *A Riemannian metric (or Riemannian structure) on a differentiable manifold $\widetilde{M}$ is a correspondence which associates to each point $p \in \widetilde{M}$ an inner product $\langle , \rangle_p$ (that is, a bilinear, symmetric, positive definite form) on the tangent space $T_p\widetilde{M}$, which varies differentiably in the following sense.*
If $\varphi : U \subset \mathbb{R}^m \to \widetilde{M}$ is a system of coordinates around p, with

$$q = \varphi(x_1, x_2, ..., x_m) \in \varphi(U),$$

and

$$\frac{\partial}{\partial x_i}(q) = d\varphi_q(0, ..., \underbrace{1}_{i^{\text{th}}}, ..., 0),$$

then $\langle \frac{\partial}{\partial x_i}(q), \frac{\partial}{\partial x_i}(q)\rangle_q = \widetilde{g}_{ij}(x_1, ..., x_m)$ is a differentiable function on U.

Definition 2.1.2. *The function $\widetilde{g}_{ij}(= \widetilde{g}_{ji})$ is called the local representation of the Riemannian metric in the coordinate system $\varphi : U \subset \mathbb{R}^m \to \widetilde{M}$. A differentiable manifold with a give Riemannian metric will be called a Riemannian manifold, denoted by $(\widetilde{M}, \widetilde{g})$.*

Definition 2.1.3. *Let M and $\widetilde{M}$ be Riemannian manifolds. A diffeomorphism $f : M \to \widetilde{M}$ (that is, f is a differentiable bijection with a differentiable inverse) is called an isometry if*

$$\langle X, Y\rangle_p = \langle df_p(X), df_p(Y)\rangle_{f(p)}, \tag{2.1}$$

for all $p \in M$, $X, Y \in T_pM$.

Definition 2.1.4. *Let M and $\widetilde{M}$ be Riemannian manifolds. A differentiable mapping $f : M \to \widetilde{M}$ is a local isometry at $p \in M$ if there is a neighborhood $U \subset M$ of p such that $f : U \to f(U)$ is a diffeomorphism satisfying* (2.1)*.*

Example 2.1.1. (1) *$M = \mathbb{R}^n$ with $\frac{\partial}{\partial x_i}$ identified with $e_i = (0, ..., 1, ..., 0)$. The Riemannian metric is given by $\langle e_i, e_j\rangle = \delta_{ij}$. $\mathbb{R}^n$ is called Euclidean space of dimension n.*
(2) *The antipodal mapping $A : \mathbb{S}^n \longrightarrow \mathbb{S}^n$ given by $A(p) = -p$ is an isometry of $\mathbb{S}^n$.*
(3) *Let $\mathbb{S}^2$ the unit sphere defined by*

$$\mathbb{S}^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}.$$

We consider the parametrization

$$\begin{cases} x_1 = \cos\theta\cos\alpha \\ x_2 = \sin\theta\cos\alpha \\ x_3 = \sin\alpha \end{cases}$$

Where $0 < \alpha < \frac{\pi}{2}$ and $0 < \theta < 2\pi$.
We compute the induce Riemannian metric on $\mathbb{S}^2$ by the (standard) metric $\langle,\rangle$ of $\mathbb{R}^3$.
The basic fields on $\mathbb{S}^2$ are given by

$$\begin{aligned} \frac{\partial}{\partial\theta} &= (\frac{\partial x_1}{\partial\theta}, \frac{\partial x_2}{\partial\theta}, \frac{\partial x_3}{\partial\theta}) \\ &= (-\sin\theta\cos\alpha, \cos\theta\cos\alpha, 0), \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial\alpha} &= (\frac{\partial x_1}{\partial\alpha}, \frac{\partial x_2}{\partial\alpha}, \frac{\partial x_3}{\partial\alpha}) \\ &= (-\cos\theta\sin\alpha, -\sin\theta\sin\alpha, \cos\theta). \end{aligned}$$

So that, the components of the Riemannian metric g are given by

$$\begin{aligned} g_{11} &= \langle\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}\rangle_{\mathbb{R}^3} \\ &= \sin^2\theta\cos^2\alpha + \cos^2\theta\cos^2\alpha \\ &= \cos^2\alpha, \end{aligned}$$

$$
\begin{aligned}
g_{12} &= g_{21} \\
&= \langle \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \alpha} \rangle_{\mathbb{R}^3} \\
&= \sin\theta\cos\alpha\cos\theta\sin\alpha - \cos\theta\cos\alpha\sin\theta\sin\alpha \\
&= 0,
\end{aligned}
$$

$$
\begin{aligned}
g_{22} &= \langle \frac{\partial}{\partial \alpha}, \frac{\partial}{\partial \alpha} \rangle_{\mathbb{R}^3} \\
&= \cos^2\theta\sin^2\alpha + \sin^2\theta\sin^2\alpha + \cos^2\alpha \\
&= 1.
\end{aligned}
$$

Thus

$$
\begin{aligned}
g &= g_{11} d\theta \otimes d\theta + g_{12} d\theta \otimes d\alpha + g_{21} d\alpha \otimes d\theta + g_{22} d\alpha \otimes d\alpha \\
&= \cos^2\alpha d\theta^2 + d\alpha^2.
\end{aligned}
$$

2.2 Linear connection

Definition 2.2.1. *A linear connection $\widetilde{\nabla}$ on a differentiable manifold $\widetilde{M}$ is a mapping*

$$
\begin{aligned}
\widetilde{\nabla} : \mathfrak{X}(\widetilde{M}) \times \mathfrak{X}(\widetilde{M}) &\longrightarrow \mathfrak{X}(\widetilde{M}), \\
(X, Y) &\longmapsto \widetilde{\nabla}_X Y
\end{aligned}
$$

which satisfies the following properties

i) $\widetilde{\nabla}_X (Y + Z) = \widetilde{\nabla}_X Y + \widetilde{\nabla}_X Z$,

ii) $\widetilde{\nabla}_X (fY) = X(f) Y + f \widetilde{\nabla}_X Y$,

iii) $\widetilde{\nabla}_{fX+gY} Z = f \widetilde{\nabla}_X Z + g \widetilde{\nabla}_Y Z$,

for all $X, Y, Z \in \mathfrak{X}(\widetilde{M})$ and $f, g \in C^\infty(\widetilde{M})$.

2.2.1 Torsion Tensor

Definition 2.2.2. *Let $\widetilde{M}$ be a differentiable manifold, and $\widetilde{\nabla}$ be a linear connection on $\widetilde{M}$, then the torsion T of connection $\widetilde{\nabla}$ is a tensor field of type $(1, 2)$ defined by*

$$
\begin{aligned}
T : \mathfrak{X}(\widetilde{M}) \times \mathfrak{X}(\widetilde{M}) &\longrightarrow \mathfrak{X}(\widetilde{M}). \\
(X, Y) &\longmapsto \widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X - [X, Y]
\end{aligned}
$$

The connection $\widetilde{\nabla}$ on the tangent bundle $T\widetilde{M}$ is said to be torsion-free if the corresponding torsion T vanishes i.e.

$$[X,Y] = \widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X, \quad \forall X, Y \in \mathfrak{X}(\widetilde{M}).$$

Remark 2.2.1. $T(X,Y) = -T(Y,X)$ *(T is antisymmetric),*$\forall X, Y \in \mathfrak{X}(\widetilde{M})$.

2.2.2 Levi-Civita connection

Definition 2.2.3. *A linear connection $\widetilde{\nabla}$ on a Riemannian manifold $(\widetilde{M}, \widetilde{g})$ is compatible with $\widetilde{g}$ if*

$$Z(\widetilde{g}(X,Y)) = \widetilde{g}(\widetilde{\nabla}_Z X, Y) + \widetilde{g}(X, \widetilde{\nabla}_Z Y),$$

for all $X, Y, Z \in \mathfrak{X}(\widetilde{M})$.

Proposition 2.2.1. *[8] Let $(\widetilde{M}, \widetilde{g})$ be a Riemannian manifold. Then $\widetilde{\nabla}$ defined by the Koszul formula*

$$\begin{aligned} 2\widetilde{g}(\widetilde{\nabla}_X Y, Z) &= X(\widetilde{g}(Y,Z)) + Y(\widetilde{g}(Z,X)) - Z(\widetilde{g}(X,Y)) \\ &\quad + \widetilde{g}(Z,[X,Y]) + \widetilde{g}(Y,[Z,X]) - \widetilde{g}(X,[Y,Z]), \end{aligned} \tag{2.2}$$

for all $X, Y, Z \in \mathfrak{X}(\widetilde{M})$, is linear connection $\widetilde{\nabla}$ on $\widetilde{M}$: called the Levi-Cevita connection of $\widetilde{M}$.

Theorem 2.2.1. *[8] Given a Riemannian manifold $(\widetilde{M}, \widetilde{g})$. The Levi-Civita connection $\widetilde{\nabla}$ on $(\widetilde{M}, \widetilde{g})$ satisfying the conditions*

a) $\widetilde{\nabla}$ is torsion-free,

b) $\widetilde{\nabla}$ is compatible with the Riemannian metric $\widetilde{g}$.

Remark 2.2.2. *In a coordinate system (x_i) on $\widetilde{M}$, $\widetilde{\nabla}$ is completely defined by the Christoffel symbols $\widetilde{\Gamma}_{ij}^k$ defined by:*

$$\widetilde{\nabla}_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^{m} \widetilde{\Gamma}_{ij}^k \frac{\partial}{\partial x_k}.$$

Let $X = \sum_{i=1}^m X^i \frac{\partial}{\partial x_i}$, and $Y = \sum_{j=1}^m Y^j \frac{\partial}{\partial x_j}$, then

$$\widetilde{\nabla}_X Y = \sum_{i,k=1}^{m} X^i \left(\frac{\partial Y^k}{\partial x_i} + \sum_{j=1}^{m} \widetilde{\Gamma}_{ij}^k Y^j \right) \frac{\partial}{\partial x_k}.$$

Proposition 2.2.2. *[8] Let $(\widetilde{M}, \widetilde{g})$ a Riemannian manifold with Levi-Civita connection $\widetilde{\nabla}$. Further let (U, φ) be a local coordinate on $\widetilde{M}$ and put $\partial_i = \frac{\partial}{\partial x_i} \in \mathfrak{X}(U)$. The local frame of $T\widetilde{M}$ on U. Then the Christoffel symbols $\widetilde{\Gamma}_{ij}^k : U \to \mathbb{R}$ of the Levi-Civita connection $\widetilde{\nabla}$ with respect to (U, φ) are given by*

$$\widetilde{\Gamma}_{ij}^k = \frac{1}{2} \sum_{l=1}^{m} \widetilde{g}^{kl} \{ \frac{\partial \widetilde{g}_{jl}}{\partial x_i} + \frac{\partial \widetilde{g}_{il}}{\partial x_j} - \frac{\partial \widetilde{g}_{ij}}{\partial x_l} \},$$

where $\widetilde{g}_{ij} = \widetilde{g}(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ are the components of $\widetilde{g}$, and $\widetilde{g}^{ij} = (\widetilde{g}_{ij})^{-1}$ is the inverse matrix.

Proof 2.2.1. *Since $[\partial_i, \partial_j] = 0$, $\forall i, j = 1, ..., m$, we have*

$$\begin{aligned} 2\widetilde{g}(\widetilde{\nabla}_{\partial_i} \partial_j, \partial_l) &= 2 \sum_{s=1}^{m} \widetilde{g}(\widetilde{\Gamma}_{ij}^s \partial_j, \partial_l) \\ &= 2 \sum_{s=1}^{m} \widetilde{\Gamma}_{ij}^s \widetilde{g}_{sl}, \end{aligned}$$

and according the Koszul's formula

$$2\widetilde{g}(\widetilde{\nabla}_{\partial_i} \partial_j, \partial_l) = \partial_i(\widetilde{g}(\partial_j, \partial_l)) + \partial_j(\widetilde{g}(\partial_l, \partial_i)) - \partial_l(\widetilde{g}(\partial_i, \partial_j)),$$

we find that

$$\sum_{s=1}^{m} \widetilde{\Gamma}_{ij}^s \widetilde{g}_{sl} = \frac{1}{2} \left\{ \frac{\partial \widetilde{g}_{jl}}{\partial x_i} + \frac{\partial \widetilde{g}_{il}}{\partial x_j} - \frac{\partial \widetilde{g}_{ij}}{\partial x_l} \right\},$$

so that

$$\sum_{s=1}^{m} \widetilde{\Gamma}_{ij}^s \widetilde{g}_{sl} \widetilde{g}^{lk} = \frac{1}{2} \widetilde{g}^{lk} \left\{ \frac{\partial \widetilde{g}_{jl}}{\partial x_i} + \frac{\partial \widetilde{g}_{il}}{\partial x_j} - \frac{\partial \widetilde{g}_{ij}}{\partial x_l} \right\}.$$

As

$$\sum_{l=1}^{m} \widetilde{g}_{sl} \widetilde{g}^{lk} = \delta_{ks},$$

where δ_{ks} is the Kronecker symbol.
We conclude that

$$\sum_{s,l=1}^{m} \widetilde{\Gamma}_{ij}^s \widetilde{g}_{sl} \widetilde{g}^{lk} = \frac{1}{2} \sum_{l=1}^{m} \left\{ \frac{\partial \widetilde{g}_{jl}}{\partial x_i} + \frac{\partial \widetilde{g}_{il}}{\partial x_j} - \frac{\partial \widetilde{g}_{ij}}{\partial x_l} \right\},$$

we get

$$\widetilde{\Gamma}_{ij}^k = \frac{1}{2} \sum_{l=1}^{m} \widetilde{g}^{kl} \left\{ \frac{\partial \widetilde{g}_{jl}}{\partial x_i} + \frac{\partial \widetilde{g}_{il}}{\partial x_j} - \frac{\partial \widetilde{g}_{ij}}{\partial x_l} \right\}.$$

2.2.3 Curvatures on manifolds

Definition 2.2.4. *Let* $(\widetilde{M}, \widetilde{g})$ *be a Riemannian manifold of dimension* m*, and* $\widetilde{\nabla}$ *Levi-Civita connection of* $(\widetilde{M}, \widetilde{g})$*. The tensor of type* $(1,3)$ *on* $\widetilde{M}$

$$\widetilde{R} : \mathfrak{X}(\widetilde{M}) \times \mathfrak{X}(\widetilde{M}) \times \mathfrak{X}(\widetilde{M}) \longrightarrow \mathfrak{X}(\widetilde{M}) \ defined \ by$$

$$\widetilde{R}(X,Y)Z = \widetilde{\nabla}_X \widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y \widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]} Z \quad \forall X, Y, Z \in \mathfrak{X}(\widetilde{M}),$$

called a curvature tensor.
The curvature tensor type $(1,4)$ *is given by*

$$\widetilde{R}(X,Y,Z,W) = \widetilde{g}(\widetilde{R}(X,Y)Z, W).$$

Remark 2.2.3. *The curvature tensor* $\widetilde{R}$ *is expressed as a function of the Christoffel symbols*

$$\widetilde{R}(\partial_i, \partial_j)\partial_k = \sum_{s=1}^{m} \widetilde{R}^s_{ijk} \partial_s,$$

where $\{\partial_i\}$ *is a local basis of the vector fields on* $\widetilde{M}$*.*
Since $[\partial_i, \partial_j] = 0$ *we have*

$$\begin{aligned}
\widetilde{R}(\partial_i, \partial_j)\partial_k &= \widetilde{\nabla}_{\partial_i} \widetilde{\nabla}_{\partial_j} \partial_k - \nabla_{\partial_j} \widetilde{\nabla}_{\partial_i} \partial_k \\
&= \sum_l (\widetilde{\nabla}_{\partial_i}(\widetilde{\Gamma}^l_{jk} \partial_l) - \widetilde{\nabla}_{\partial_j}(\widetilde{\Gamma}^l_{ik} \partial_l)) \\
&= \sum_l (\frac{\partial \widetilde{\Gamma}^l_{jk}}{\partial x_i} \partial_l + \widetilde{\Gamma}^l_{jk} \widetilde{\nabla}_{\partial_i} \partial_l - \frac{\partial \widetilde{\Gamma}^l_{ik}}{\partial x_j} \partial_l + \widetilde{\Gamma}^l_{ik} \nabla_{\partial_j} \partial_l) \\
&= \sum_l (\frac{\partial \widetilde{\Gamma}^l_{jk}}{\partial x_i} \partial_l + \sum_s \widetilde{\Gamma}^l_{jk} \widetilde{\Gamma}^s_{il} \partial_s - \frac{\partial \widetilde{\Gamma}^l_{ik}}{\partial x_j} \partial_l + \sum_s \widetilde{\Gamma}^l_{ik} \widetilde{\Gamma}^s_{jl} \partial_s) \\
&= \sum_s \left\{ \frac{\partial \widetilde{\Gamma}^s_{jk}}{\partial x_i} - \frac{\partial \widetilde{\Gamma}^s_{ik}}{\partial x_j} + \sum_l (\widetilde{\Gamma}^l_{jk} \widetilde{\Gamma}^s_{il} - \widetilde{\Gamma}^l_{ik} \widetilde{\Gamma}^s_{jl}) \right\} \partial_s.
\end{aligned}$$

Therefore, the components of the curvature tensor $\widetilde{R}$ *is given by*

$$\widetilde{R}^s_{ijk} = \sum_l (\widetilde{\Gamma}^l_{jk} \widetilde{\Gamma}^s_{il} - \widetilde{\Gamma}^l_{ik} \widetilde{\Gamma}^s_{jl}) + \frac{\partial \widetilde{\Gamma}^s_{jk}}{\partial x_i} - \frac{\partial \widetilde{\Gamma}^s_{ik}}{\partial x_j}.$$

Proposition 2.2.3. *[8] Let* $(\widetilde{M}, \widetilde{g})$ *be a Riemannian manifold. We have*

1. $\widetilde{R}(X,Y)Z = -\widetilde{R}(Y,X)Z$ *(antisymmetric),*

2. $\widetilde{g}(\widetilde{R}(X,Y)Z,W) = -\widetilde{g}(\widetilde{R}(X,Y)W,Z)$,
3. $\widetilde{g}(\widetilde{R}(X,Y)Z,W) = \widetilde{g}(\widetilde{R}(Z,W)X,Y)$,
4. $\widetilde{R}$ *verified Bianchi's identity algebraic*

$$\widetilde{R}(X,Y)Z + \widetilde{R}(Y,Z)X + \widetilde{R}(Z,X)Y = 0,$$

5. $\widetilde{R}$ *verified Bianchi's identity differential*

$$(\widetilde{\nabla}_X\widetilde{R})(Y,Z) + (\widetilde{\nabla}_Y\widetilde{R})(Z,X) + (\widetilde{\nabla}_Z\widetilde{R})(X,Y) = 0.$$

for all $X,Y,Z,W \in \mathfrak{X}(\widetilde{M})$.

Definition 2.2.5. *For a point* $p \in \widetilde{M}$ *the function*

$$\begin{array}{rcl} K_p : T_p(\widetilde{M}) \times T_p(\widetilde{M}) & \to & \mathbb{R}, \\ (X,Y) & \mapsto & \dfrac{\widetilde{g}(\widetilde{R}(X,Y)Y,X)}{\widetilde{g}(X,X)\widetilde{g}(Y,Y) - \widetilde{g}(X,Y)^2} \end{array}$$

is called the sectional curvature at p *on* $(\widetilde{M},\widetilde{g})$.
The Riemannian manifold $(\widetilde{M},\widetilde{g})$ *is said to be of constant curvature if there exists* $k \in \mathbb{R}$ *such that* $K(X,Y) = k$, *for all* $X, Y, \in T_p(\widetilde{M})$.

Definition 2.2.6. *Let* $(\widetilde{M},\widetilde{g})$ *be a Riemannian manifold. We define the tensor field* k_1 : $\mathfrak{X}(\widetilde{M}) \times \mathfrak{X}(\widetilde{M}) \times \mathfrak{X}(\widetilde{M}) \to \mathfrak{X}(\widetilde{M})$ *of type* $(3,\ 1)$ *by*

$$k_1(X,Y)Z = \widetilde{g}(Y,Z)X - \widetilde{g}(X,Z)Y, \tag{2.3}$$

for all $X,Y,Z \in \mathfrak{X}(\widetilde{M})$.

Corollary 2.2.1. *Let* $(\widetilde{M},\widetilde{g})$ *of dimension* m, *with* $(m \geq 2)$ *be a Riemannian manifold of constant curvature* k. *Then the curvature tensor* $\widetilde{R}$ *is given by*

$$\widetilde{R}(X,Y)Z = k[k_1(X,Y)Z],$$

for all $X,Y,Z \in \mathfrak{X}(\widetilde{M})$.

Definition 2.2.7. *The Ricci curvature of a Riemannian manifold* $(\widetilde{M}, \widetilde{g})$ *of dimension* m *is a tensor of type* $(0,2)$ *defined by*

$$\begin{aligned} \mathrm{Ric}(X,Y) &= \mathrm{trace}_{\widetilde{g}}(Z \longmapsto \widetilde{R}(Z,X)Y) \\ &= \sum_{i=1}^{m} \widetilde{g}(\widetilde{R}(e_i,X)Y,e_i), \end{aligned}$$

for all $X, Y \in \mathfrak{X}(\widetilde{M})$, *where* $\{e_i\}$ *is an orthonormal frame on* $(\widetilde{M}, \widetilde{g})$.

Proposition 2.2.4. *[8] The Ricci curvature is symmetric. Indeed*

$$\begin{aligned} \mathrm{Ric}(X,Y) &= \sum_{i=1}^{m} \widetilde{g}(\widetilde{R}(e_i,X)Y,e_i) \\ &= \sum_{i=1}^{m} \widetilde{g}(\widetilde{R}(Y,e_i)e_i,X) \\ &= \sum_{i=1}^{m} \widetilde{g}(\widetilde{R}(e_i,Y)X,e_i) \\ &= \mathrm{Ric}(Y,X), \end{aligned}$$

$\forall X, Y \in \mathfrak{X}(\widetilde{M})$.

Definition 2.2.8. *The Ricci tensor of a Riemannian manifold* $(\widetilde{M}, \widetilde{g})$ *of dimension* m *is a tensor of type* $(1,1)$ *defined by*

$$\mathrm{Ricci}(X) = \sum_{i=1}^{m} \widetilde{R}(X,e_i)e_i, \quad \forall X \in \mathfrak{X}(\widetilde{M}).$$

Remark 2.2.4. *For all* $X, Y \in \mathfrak{X}(\widetilde{M})$ *we have*

$$\mathrm{Ric}(X,Y) = \widetilde{g}(\mathrm{Ricci}(X),Y).$$

Definition 2.2.9. *We call scalar curvature of a Riemannian manifold* $(\widetilde{M}, \widetilde{g})$ *of dimension* m, *the function defined on* $\widetilde{M}$ *by*

$$\begin{aligned} S &= \mathrm{trace}_{\widetilde{g}}\mathrm{Ric} \\ &= \sum_{i,j=1}^{m} \widetilde{g}(\widetilde{R}(e_i,e_j)e_j,e_i), \end{aligned}$$

where $\{e_i\}$ *is an orthonormal frame on* $(\widetilde{M}, \widetilde{g})$.

Corollary 2.2.2. *Let $(\widetilde{M}, \widetilde{g})$ be a Riemannian manifold of constant curvature k, then*

1. $\mathrm{Ricci}(X) = (m-1)kX$,
2. $\mathrm{Ric}(X,Y) = (m-1)k\widetilde{g}(X,Y)$,
3. $S = m(m-1)k$.

where m is the dimension of $(\widetilde{M}, \widetilde{g})$.

Proof 2.2.2. *By using the formula (2.3), we have*

1.

$$\begin{aligned} \mathrm{Ricci}(X) &= \widetilde{R}(X, e_i)e_i \\ &= k(\widetilde{g}(e_i, e_i)X, e_i)X - \widetilde{g}(e_i, X)e_i) \\ &= kmX - kX \\ &= (m-1)kX; \end{aligned}$$

2.

$$\begin{aligned} \mathrm{Ric}(X,Y) &= \widetilde{g}(\mathrm{Ricci}(X), Y) \\ &= (m-1)k\widetilde{g}(X,Y); \end{aligned}$$

3.

$$\begin{aligned} S &= \widetilde{g}(\widetilde{R}(e_i, e_j)e_j, e_i) \\ &= k\widetilde{g}(\widetilde{g}(e_j, e_j)e_i - \widetilde{g}(e_j, e_i)e_j, e_i) \\ &= k\widetilde{g}((\delta_{jj}e_i - \delta_{ji}e_j), e_i) \\ &= k\widetilde{g}(me_i - \delta_{ji}e_j, e_i) \\ &= k\widetilde{g}(me_i - e_i, e_i) \\ &= k(m-1)\widetilde{g}(e_i, e_i) \\ &= k(m-1)\delta_{ii} \\ &= k(m-1)m. \end{aligned}$$

Example 2.2.1. *1. Let $\mathbb{H}^m$ be the upper halfspace model of real hyperbolic $m-$space*

$$\mathbb{H}^m = \{(x_1, x_2, ..., x_m) \in \mathbb{R}^m,\ x_m > 0\}.$$

2. *The hyperbolic metric* $\widetilde{g}$ *on* $\mathbb{H}^m$ *is given by*

$$\widetilde{g}_{ii} = \frac{1}{x_m^2} \quad , \quad \widetilde{g}_{ij} = 0 \quad for \quad i \neq j,$$

3.

$$K(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = -1,$$

4. $\mathrm{Ric}(x_i, x_j) = -(m-1)\widetilde{g}(x_i, x_j)$,

5. $S = -m(m-1)$, *for all pairs* (i, j) *at every point of* $\mathbb{H}^m$.

2.3 Geodesics and Parallel transport

Let $(\widetilde{M}, \widetilde{g})$ be a Riemannian manifold with the Levi-Civita connection $\widetilde{\nabla}$ and let $\alpha : I \subset \mathbb{R} \longrightarrow (\widetilde{M}, \widetilde{g})$ a curve on $\widetilde{M}$

Definition 2.3.1. *A vector field* $X(t)$ *along a curve* α *is said to be parallel along* α*, if*

$$(\widetilde{\nabla}_{\dot{\alpha}} X)|_t = 0,$$

where $\dot{\alpha}(t) = d\alpha(\frac{d}{dt})|_t$*, for all* $t \in I$.

Proposition 2.3.1. *[27] Let* $\alpha : I \subset \mathbb{R} \longrightarrow (\widetilde{M}, \widetilde{g})$ *be a curve,* $t_0 \in I$*, and* $v \in T_{\alpha(t_0)}\widetilde{M}$*, Then, there is a unique vector field* X_v *parallel along* α *such that* $X_v(t_0) = v$.

Definition 2.3.2. *Let* $\alpha : I \subset \mathbb{R} \longrightarrow (\widetilde{M}, \widetilde{g})$ *be a curve.* α *is a geodesic if*

$$\widetilde{\nabla}_{\dot{\alpha}}(t)\dot{\alpha}(t) = 0 \quad , \quad \forall t \in I. \tag{2.4}$$

Proposition 2.3.2. *[27] Let* $(\widetilde{M}, \widetilde{g})$ *be a Riemannian manifold with the Levi-Civita connection* $\widetilde{\nabla}$*. For any point* $p \in \widetilde{M}$ *and for any vector* $V \in T\widetilde{M}$ *(the tangent space to* $\widetilde{M}$ *at* p*) there exists a unique geodesic* $\alpha : I \subset \mathbb{R} \longrightarrow (\widetilde{M}, \widetilde{g})$ *such that* $\alpha(0) = p$ *and* $\dot{\alpha}(0) = V$.

Definition 2.3.3. *Given a curve* $\alpha : I \subset \mathbb{R} \longrightarrow (\widetilde{M}, \widetilde{g})$*, and a vector* $V_0 \in T_{\alpha(0)}\widetilde{M}$*, we define the parallel transport of* V_0 *along* $\alpha(t)$ *to be the unique solution* $V(t)$ *to the ODE*

$$\widetilde{\nabla}_{\dot{\alpha}(t)} V(t) = 0,$$

for any $t \in I$ *with the initial condition* $V(0) = V_0$.

In terms of local coordinates $f(u_1, ..., u_m)$, the parallel transport equation can be expressed as follows. Let $\alpha(t) = f(\alpha^1(t), ..., \alpha^m(t))$ and $V(t) = V^i(t)\frac{\partial}{\partial u_i}$, then

$$\dot{\alpha}(t) = \sum_{i=1}^{m} \frac{d\alpha^i}{dt}(\frac{\partial}{\partial u_i} \circ \alpha),$$

where $\alpha^i = u_i \circ \alpha$ for all $i = \overline{1, m}$. We obtain

$$\begin{aligned}
\widetilde{\nabla}_{\dot{\alpha}(t)} V(t) &= \widetilde{\nabla}_{d\alpha(\frac{d}{dt})} V^i(t)(\frac{\partial}{\partial u_i} \circ \alpha) \\
&= \nabla^{\alpha}_{\frac{d}{dt}} V^i(t)(\frac{\partial}{\partial u_i} \circ \alpha), \\
&= \frac{dV^i}{dt}(\frac{\partial}{\partial u_i} \circ \alpha) + V^i \nabla^{\alpha}_{\frac{d}{dt}}(\frac{\partial}{\partial u_i} \circ \alpha) \\
&= \frac{dV^i}{dt}(\frac{\partial}{\partial u_i} \circ \alpha) + V^i \widetilde{\nabla}_{d\alpha(\frac{d}{dt})} \frac{\partial}{\partial u_i} \\
&= \frac{dV^i}{dt}(\frac{\partial}{\partial u_i} \circ \alpha) + V^i \frac{d\alpha_j}{dt}(\widetilde{\nabla}_{\frac{\partial}{\partial u_j}} \frac{\partial}{\partial u_i}) \circ \alpha \\
&= \frac{dV^i}{dt}(\frac{\partial}{\partial u_i} \circ \alpha) + V^i \frac{d\alpha_j}{dt}(\Gamma^k_{ij} \circ \alpha)(\frac{\partial}{\partial u_k} \circ \alpha) \\
&= \sum_{k=1}^{m} \left[\frac{dV_k}{dt} + \sum_{i,j=1}^{m} V^i \frac{d\alpha_j}{dt}(\Gamma^k_{ij} \circ \alpha) \right] (\frac{\partial}{\partial u_k} \circ \alpha),
\end{aligned}$$

where ∇^{α} is the Pull-Buck connection on the inverse fibre $\alpha^{-1}T\widetilde{M}$.
Therefore, the parallel transport equation $\widetilde{\nabla}_{\dot{\alpha}(t)} V(t) = 0$ is equivalent to

$$\frac{dV^k}{dt} + \sum_{i,j=1}^{m} V^j \widetilde{\Gamma}^k_{ij} \frac{d\alpha_j}{dt} = 0,$$

for any k.
The equation for parallel transport depends only on the derivative of the curve, not on the curve itself, allowing the parallel transport of a vector along a vector field. This equation is a first-order ordinary differential equation ODE with guaranteed existence and uniqueness of solutions.

Example 2.3.1. *Diagram showing geodesics on a manifold $\widetilde{M}$. We consider a differentiable curve α in terms of local coordinates we can obtain a system of equations required for α to be a geodesic. Consider a Riemannian manifold $(\widetilde{M}, \widetilde{g})$ and some differentiable curve $\alpha : [-\epsilon, \epsilon] \longrightarrow U \subset \widetilde{M}$ such that $\alpha(0) = p$. Let $V \in T_p\widetilde{M}$ be a tangent vector to α at p and φ be a local coordinate chart such that $\varphi = (x_1, ..., x_m) : U \subset \widetilde{M} \longrightarrow \mathbb{R}^m$ with $p \in U$. We have*

$$V = \sum_{j=1}^{m} V_j \frac{\partial}{\partial x_j} \in T_p\widetilde{M}.$$

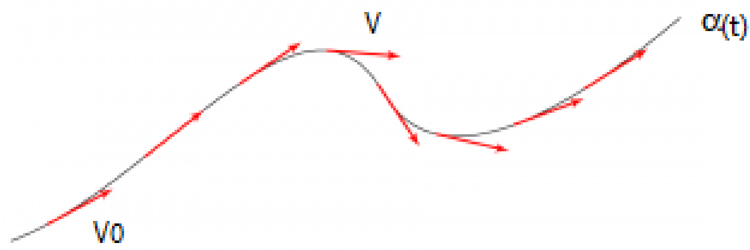

Figure 2.1: The parallel transport

Let $\alpha : [-\epsilon, \epsilon] \longrightarrow U$ *be some differentiable curve with* $\alpha(0) = p$ *and*

$$(\varphi \circ \alpha)(t) = (\alpha_1(t), ..., \alpha_m(t)).$$

Then we have

$$\dot{\alpha}(t) = \sum_{j=1}^{m} \dot{\alpha}_j(t) \frac{\partial}{\partial x_j}.$$

We can derive (see do Carmo[8]) the following system of equations that must be satisfies for α *to be a geodesic with* $\alpha(0) = p$ *and* $\dot{\alpha}(0) = V$.

1) $\ddot{\alpha}_k(t) + \sum_{i=j=1}^{m} \dot{\alpha}_i(t)\dot{\alpha}_j(t)\Gamma_{ij}^k(\alpha(t)) = 0,$

2) $\alpha_k(0) = x_k(p),$

3) $\dot{\alpha}_k(0) = V_k.$

This is a system of second order ordinary differential equations. The Theory of Ordinary Differential Equations tells us that there exists a unique solution in a neighbourhood $[-\epsilon, \epsilon]$ *of* 0. *This tells us that geodesics are unique for a particular choice of* $p \in \widetilde{M}$ *and* $V \in T_p\widetilde{M}$.

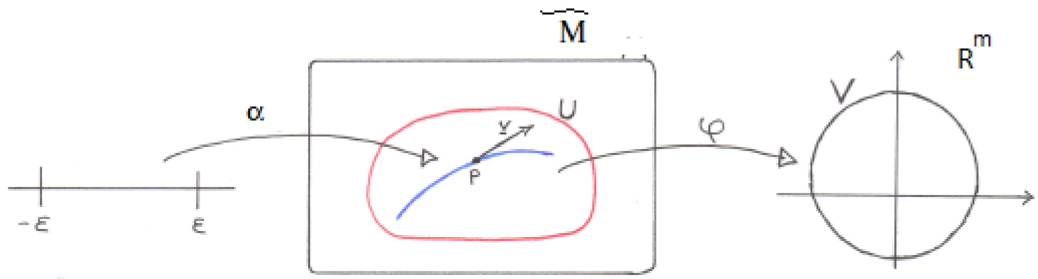

Figure 2.2: Diagram showing geodesics on a manifold $\widetilde{M}$

2.4 Left-invariant metrics on a Lie group

Definition 2.4.1. *A Riemannian metric on a Lie group G is left invariant if:*

$$\langle X, Y\rangle_b = \langle (dL_a)_b X, (dL_a)_b Y\rangle_{L_a(b)}$$

for all $a, b \in G$ and all $X, Y \in T_b G$, that is, L_a is an isometry.

Proposition 2.4.1. *Let X, Y be left invariant vector fields on G, for each $a \in G$ and for any differentiable function f on G, we have*

$$\begin{aligned} dL_a[X,Y]f &= [X,Y](f \circ L_a) \\ &= X(dL_a Y)f - Y(dL_a X)f \\ &= (XY - YX)f \\ &= [X,Y]f, \end{aligned}$$

we conclude that the bracket of any two left invariant vector fields is again a left invariant vector field. If $X_e, Y_e \in T_e G$, we put $[X_e, Y_e] = [X, Y]_e$. With this operation, $T_e G = \mathcal{G}$ the Lie algebra of G.

Remark 2.4.1. *To introduce a left invariant metric on G, take any arbitrary inner product $\langle , \rangle_e$ on $\mathcal{G}$ and define*

$$\langle X, Y\rangle_a = \langle dL_{a^{-1}}(X), dL_{a^{-1}}(Y)\rangle_e, \quad a \in G, \quad X, Y \in T_a G.$$

Since L_a depends differentiably on a, this construction actually produces a Riemannian metric, which is clearly left invariant.

Example 2.4.1. 1. *A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(t) = yt + x, \quad t, x, y \in \mathbb{R}, \quad y > 0,$ is called a proper affine function. The subset of all such functions with respect to the usual composition law forms a Lie group G. As a differentiable manifold G is simply the upper half-plane $\{(x, y) \in \mathbb{R}^2, \quad y > 0\}$ with the differentiable structure induce from $\mathbb{R}^2$. The left invariant Riemannian metric of which at the neutral element $e = (0, 1)$ coincides with the Euclidean metric*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

is given by

$$\begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix}.$$

2. *The left invariant Riemannian metric g of Heisenberg $\mathbb{H}_3$ is given by*

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & x^2 + 1 & -x \\ 0 & -x & 1 \end{pmatrix}.$$

Let $\{X, Y, Z\}$ be a left-invariant frame field , where

$$X = \frac{\partial}{\partial x},$$

$$Y = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z},$$

$$Z = \frac{\partial}{\partial z},$$

and $\theta_1 + \theta_2 + \theta_3$ is a dual conframe field

$$\theta_1 = dx,$$

$$\theta_2 = dy,$$

$$\theta_3 = -xdy + dz.$$

So then

$$g = \theta_1^2 + \theta_2^2 + \theta_3^2,$$

CHAPTER 3

GEOMETRY OF RIEMANNIAN SUBMANIFOLDS

This chapter introduces the following concepts: Riemannian submanifold , Levi-Civita connection of a submanifold, second fundamental form, mean curvature, shape operator of a submanifold normal connection of a submanifold and the formulas of Gauss and Weingarten and the equations of Gauss and Codazzi. Some geometric properties of submanifolds: paralleles and minimal. The Frobenius Theorem. The references used are: [1], [7], [8], [9], [10], [12], [25], [26], [27],[29], [35] .

3.1 Riemannian submanifolds

Definition 3.1.1. *Let M be a differentiable manifold of dimension n and $\widetilde{M}$ a differentiable manifold of dimension $(n+m)$. The differentiable mapping $f : M \to \widetilde{M}$ is an immersion if $d_pf : T_pM \to T_{f(p)}\widetilde{M}$ is injective for all $p \in M$. Then the difference between the dimensions of the two manifolds (in this case m) is defined as the codimension of the immersion f.*
If the Riemannian metric $\widetilde{g}$ on $\widetilde{M}$ induces a Riemannian metric g on M for all $X, Y \in T_pM$ such that

$$g(X,Y) = \widetilde{g}(df_p(X), df_p(Y)),$$

then f is an isometric immersion, or an embedding. If $M \subset \widetilde{M}$ is an isometric immersion then M is a submanifold of $\widetilde{M}$.

We also define the concept of tangent and normal bundles.
Remark 3.1.1. *Let M be a differentiable manifold and let the tangent bundle TM be the set of all tangent vectors for all points of the manifold such that $TM = \{(p,X) : p \in M, X \in T_pM\}$.*

Then TM is a differentiable structure of dimension $2n$.

Definition 3.1.2. *Let M be a submanifold isometrically immersed in a Riemannian manifold $(\widetilde{M}, \widetilde{g})$. Then we can compose the tangent space of the Riemannian manifold $(\widetilde{M}, \widetilde{g})$ into tangential and normal parts*

$$T_p\widetilde{M} = T_pM \oplus (T_pM)^{\perp},$$

where $(T_pM)^{\perp} = \{v \in T_p\widetilde{M} \mid \widetilde{g}(v, w) = 0, \forall X \in T_pM\}$.
We can locally decompose the tangent bundle of $\widetilde{M}$, $T\widetilde{M}$ into the tangent bundle TM and normal bundle $T^{\perp}M$.
We also can consider a vector field $\overline{X}$ in the ambient space $\widetilde{M}$ as extensions of vector field X in the submanifold M.
Proposition 3.1.1. *[6] Let X and Y be two vector fields on M and let $\overline{X}$ and $\overline{Y}$ be extensions of X and Y, respectively. Then $[\overline{X}, \overline{Y}]|_M$ is independent of the extensions, and*

$$[\overline{X}, \overline{Y}]|_M = [X, Y].$$

3.2 Covariant differentiation and the Second Fundamental Form

Let M be an n-dimensional manifold immersed in an $(n+m)$-dimensional manifold $\widetilde{M}$. Then M is a submanifold of $\widetilde{M}$. If the manifold $\widetilde{M}$ is covered by a system of coordinate neighbourhoods (U, x^i) for $i = 1, ..., n+m$ and the submanifold is covered by another system of coordinate neighbourhoods (V, y^j) for $j = 1, \dots, n$, then the submanifold M can be locally represented by

$$x^i = x^i(y^1, \dots, y^n), \forall i = 1, \dots, n+m$$

We now consider the Riemannian manifold $(\widetilde{M}, \widetilde{g})$. Then the submanifold M is also a Riemannian manifold with an induced metric g, given by

$$g(X, Y) = \widetilde{g}(X, Y).$$

We now define The Levi-Civita connection $\nabla_X Y$ for a submanifold M in terms of the manifold $\widetilde{M}$ with the Levi-Civita connection on $\widetilde{\nabla}_X Y$ in the following proposition.
Proposition 3.2.1. *[35] Let M be an n-dimensional manifold immersed in a Riemannian manifold $(\widetilde{M}, \widetilde{g})$ of dimension $(n+m)$. The Levi-Civita connection on $\widetilde{M}$ is $\widetilde{\nabla}$. Since M has codimension m, at any point we can choose m fields of normal vectors $\xi_1, ..., \xi_m$ of the normal bundle $TM^{\perp}$. We can assume that $\xi_1, ..., \xi_m$ are orthonormal for every $p \in M$. We*

define covariant differentiation at a point p on the submanifold M by separating out normal and tangential components as follows

$$(\widetilde{\nabla}_X Y)_p = (\nabla_X Y)_p + B_p(X, Y), \tag{3.1}$$

where $(\nabla_X Y)_p \in TM$ and $B_p(X, Y) \in T_pM^\perp$.

Motivated by this, we define ∇ as the induced connection on M. By checking the properties of covariant differentiation, it can be shown that $(\nabla_X Y)$ is the covariant derivative on M.
We also define the symmetric bilinear mapping $B : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)^\perp$ as the second fundamental form of M.
In terms of a particular $B_p : T_pM \times T_pM \to T_pM^\perp$ is the second fundamental form of M at p.
More generally, if M is a submanifold of codimension m (i.e. it is immersed in a manifold $\widetilde{M}$ of dimension $n+m$ then we can choose m fields of orthonormal vectors $\xi_1, ..., \xi_m$. We can then express B in terms of h in the following way

$$B(X, Y) = \sum_{i=1}^{m} h^i(X, Y)\xi_i. \tag{3.2}$$

Similarly, we can separate the covariant derivative of a normal vector field ξ on M, $\widetilde{\nabla}_X \xi$, into tangential and normal components

$$(\widetilde{\nabla}_X \xi)_p = -(A_\xi X)_p + (\nabla^\perp_X \xi)_p \tag{3.3}$$

Here, $A_\xi X$ is the tangential component of $\widetilde{\nabla}_X \xi$ and $\nabla^\perp_X \xi$ is the normal.
Definition 3.2.1. *The mapping $(X, \xi) : T_pM \times T_pM^\perp \to A_\xi(X) \in T_pM$ is bilinear and consequently $(A_\xi X)_p : T_pM \times T_pM^\perp \to T_pM$ depends only on X and ξ at p.*
The application A is called the Weingarten operator.

We demonstrate how $A_\xi(X)$ is related to the second fundamental form B in the following proposition
Proposition 3.2.2. *[35] For each normal vector field ξ on M we have*

$$g(A_\xi(X), Y) = g(B(X, Y), \xi), \forall X, Y \in \mathfrak{X}(M).$$

We also can define the operator $\nabla^{\perp}_X \xi$ as follows

Definition 3.2.2. *The mapping* $(X, \xi) : T_pM \times T_pM^{\perp} \to \nabla^{\perp}_X \xi \in T_pM^{\perp}$ *defines the covariant derivative of* $\xi \in T_pM^{\perp}$ *in the* X *direction, with* $\nabla^{\perp}$ *being the Riemannian connection for the normal space* $TM^{\perp}$*.*

The equations (3.1) *and* (3.3) *are known as the Gauss formula and Weingarten formula respectively*

$$\widetilde{\nabla}_X Y = \nabla_X Y + B(X, Y) \qquad (Gauss \quad Formula),$$

$$\widetilde{\nabla}_X \xi = -A_\xi X + \nabla^{\perp}_X \xi \qquad (Weingarten \quad Formula).$$

Example 3.2.1. *(Hypersurfaces). We now consider the Gauss and Weingarten formula for hypersurfaces. A hypersurface* M^n *in* $\widetilde{M}^{n+1}$*, and therefore has a unique normal* ξ*. Then* $g(\xi, \xi) = 1$*. We differentiate this to obtain* $g(\widetilde{\nabla}_X \xi, \xi) = 0$ *and hence* $g(\nabla^{\perp}_X \xi, \xi) = 0$*. Thus* $\nabla^{\perp}_X \xi = 0$*. Weingarten's formula then becomes* $\widetilde{\nabla}_X \xi = -A_\xi X$*.*

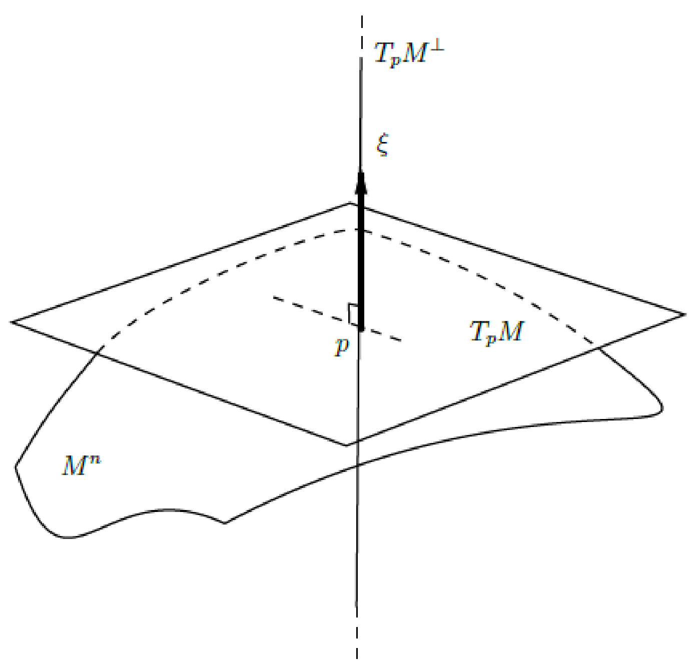

Figure 3.1: Hypersurface

3.3 The Gauss-Codazzi-Ricci equations

Let (M, g) be a Riemannian submanifold of dimension n in a Riemannian manifold $(\widetilde{M}, \widetilde{g})$ of dimension $(n+m)$. We denote by R (resp. $\widetilde{R}$) Riemannian curvature tensor of M (resp. of $\widetilde{M}$), and ∇ (resp. $\widetilde{\nabla}$) the Levi-Civita connection on M (resp. of $\widetilde{M}$).
The Riemannian curvature tensor for the Riemannian manifold $(\widetilde{M}, \widetilde{g})$

$$\widetilde{R}(X,Y)Z = \widetilde{\nabla}_X(\widetilde{\nabla}_Y Z) - \widetilde{\nabla}_Y(\widetilde{\nabla}_X Z) - \widetilde{\nabla}_{[X,Y]}Z$$

We consider $\widetilde{\nabla}_X(\widetilde{\nabla}_Y Z)$. By using the Gauss and Weingarten formulae this is equal to

$$\begin{aligned}
\widetilde{\nabla}_X(\widetilde{\nabla}_Y Z)) &= \widetilde{\nabla}_X(\nabla_Y Z + B(Y,Z)) \\
&= \widetilde{\nabla}_X(\nabla_Y Z) + \widetilde{\nabla}_X(\sum h^i(Y,Z)\xi_i) \\
&= \nabla_X(\nabla_Y Z) + \sum h^i(X, \nabla_Y Z)\xi_i + \widetilde{\nabla}_X \sum h^i(Y,Z)\xi_i \\
&= \nabla_X(\nabla_Y Z - \sum h^i(Y,Z)A_i X + \sum\{X.h^i(Y,Z) + h^i(X,\nabla_Y Z)\}\xi_i \\
&+ \sum h^i(Y,Z)\nabla^{\perp}_X \xi_i,
\end{aligned}$$

where

$$X.h^i(Y,Z) = \nabla_X h^i(Y,Z) + h^i(\nabla_X Y, Z) + h^i(Y, \nabla_X Z).$$

We immediately notice that $\widetilde{\nabla}_Y(\widetilde{\nabla}_X Z)$ yields the same equation as above by interchanging X and Y. For $\widetilde{\nabla}_{[X,Y]}Z$ we use the Gauss Formula to obtain

$$\begin{aligned}
\widetilde{\nabla}_{[X,Y]}Z &= \nabla_{[X,Y]}Z + \sum h^i([X,Y],Z)\xi_i \\
&= \nabla_{[X,Y]}Z + \sum\{h^i(\nabla_X Y, Z) - h^i(\nabla_Y X, Z)\}\xi_i.
\end{aligned}$$

We obtain an expression for $\widetilde{R}(X,Y)Z$ in terms of the induced connection ∇ and the second fundamental form

$$\begin{aligned}
\widetilde{R}(X,Y)Z &= R(X,Y)Z + \sum h^i([X,Y],Z)\xi_i \\
&= \nabla_{[X,Y]}Z + \sum\{(\nabla_X h^i)(Y,Z) - (\nabla_Y h^i)(X,Z)\}\xi_i \\
&+ \sum h^i(X,Z)A_i(Y) - \sum h^i(Y,Z)A_i(X) \\
&+ \sum h^i(Y,Z)\nabla^{\perp}_X\xi_i - \sum h^i(X,Z)\nabla^{\perp}_Y\xi_i.
\end{aligned}$$

The tangential component of $\widetilde{R}(X,Y)Z$ according to the equation above is equal to

$$R(X,Y)Z + \sum h^i(X,Z)A_i(Y) - \sum h^i(Y,Z)A_i(X).$$

Let W be a vector tangent to M at a particular point. We consider the inner product of $\widetilde{R}(X,Y)Z$ and W in order to derive the Gauss equation. We obtain

$$\begin{aligned}
\widetilde{R}(X,Y,Z,W) &= g(R(X,Y)Z,W) + \sum h^i(X,Z)g(A_i(Y),W) \\
&- \sum h^i(Y,Z)g(A_i(X),W) \\
&= g(R(X,Y)Z,W) + \sum h^i(X,Z)h^i(Y,W) \\
&- \sum h^i(Y,Z)h^i(X,W) \\
&= g(R(X,Y)Z,W) + g(B(X,Z),B(Y,W)) - g(B(Y,Z),B(X,W)).
\end{aligned}$$

Definition 3.3.1. *The Gauss Equation for a Riemannian submanifold (M,g) of dimension n in a Riemannian manifold $(\widetilde{M},\widetilde{g})$ of dimension $(n+m)$ is the tangential component of $\widetilde{R}(X,Y)Z$ and can be expressed as*

$$\widetilde{R}(X,Y,Z,W) = g(R(X,Y)Z,W) + g(B(X,Z),B(Y,W)) - g(B(Y,Z),B(X,W)), \quad (3.4)$$

where X,Y,Z,W are tangent vectors to M at a particular point p.

We now consider the normal components of Equation in order to derive the Codazzi equation. The normal component of $\widetilde{R}(X,Y)Z$ from Equation is equal to

$$\sum\{(\nabla_X h^i)(Y,Z) - (\nabla_Y h^i)(X,Z)\}\xi_i + \sum h^i(Y,Z)\nabla^{\perp}_X\xi_i - \sum h^i(X,Z)\nabla^{\perp}_Y\xi_i.$$

We can simplify this expression by defining the covariant derivative of the second Fundamental form $\widetilde{\nabla}_X B$ as

$$\widetilde{\nabla}_X B(Y,Z) = \nabla^{\perp}_X(B(Y,Z)) - B(\nabla_X Y,Z) - B(Y,\nabla_X Z), \quad (3.5)$$

we can simplify this to

$$\begin{aligned}
\widetilde{\nabla}_X B(Y,Z) &= \nabla^{\perp}_X(\sum h^i(Y,Z)\xi_i) - \sum\{h^i(\nabla_X Y,Z) + \sum h^i(Y,\nabla_X Z)\}\xi_i \\
&= \sum X.h^i(Y,Z)\xi_i + \sum h^i(Y,Z)\nabla^{\perp}_X\xi_i \\
&- \sum\{h^i(\nabla_X Y,Z) + \sum h^i(Y,\nabla_X Z)\}\xi_i \\
&= \sum(\nabla_X h^i)(Y,Z)\xi_i + \sum h^i(Y,Z)\nabla^{\perp}_X\xi_i.
\end{aligned}$$

We then see that the normal component of $\widetilde{R}(X,Y)Z$ can also be expressed by

$$\widetilde{\nabla}_X B(Y,Z) - \widetilde{\nabla}_Y B(X,Z).$$

This motivates us to define the Codazzi equation.

Definition 3.3.2. *The Codazzi equation for a Riemannian submanifold* (M, g) *of dimension* n *in a Riemannian manifold* $(\widetilde{M}, \widetilde{g})$ *of dimension* $(n+m)$ *is the normal component of* $\widetilde{R}(X, Y)Z$ *and can be expressed as*

$$\begin{aligned}\widetilde{\nabla}_X B(Y, Z) - \widetilde{\nabla}_Y B(X, Z) &= \sum\{(\nabla_X h^i)(Y, Z) - (\nabla_Y h^i)(X, Z)\}\xi_i \\ &+ \sum h^i(Y, Z)\nabla^{\perp}_X \xi_i - \sum h^i(X, Z)\nabla^{\perp}_Y \xi_i.\end{aligned}$$

We now derive the Ricci equation. Consider two particular normal vectors ξ and η. Then:

$$\widetilde{R}(X, Y)\xi = \widetilde{\nabla}_X(\widetilde{\nabla}_Y \xi) - \widetilde{\nabla}_Y(\widetilde{\nabla}_X \xi) - \widetilde{\nabla}_{[X,Y]}\xi$$

Using the Gauss and Weingarten formulae this becomes

$$\begin{aligned}\widetilde{R}(X, Y)\xi &= \widetilde{\nabla}_X(-A_\xi Y + \nabla^{\perp}_Y \xi) - \widetilde{\nabla}_Y(-A_\xi X + \nabla^{\perp}_X \xi) - \nabla^{\perp}_{[X,Y]}\xi) \\ &= \widetilde{\nabla}_X \nabla^{\perp}_Y \xi - \widetilde{\nabla}_Y \nabla^{\perp}_X \xi + \widetilde{\nabla}_Y A_\xi X - \widetilde{\nabla}_X A_\xi Y \\ &+ \textit{other terms} \\ &= \nabla^{\perp}_X \nabla^{\perp}_Y \xi - \nabla^{\perp}_Y \nabla^{\perp}_X \xi - \nabla^{\perp}_{[X,Y]}\xi + h(Y, A_\xi X) - h(X, A_\xi Y) \\ &+ \textit{other terms}\end{aligned}$$

Motivated by this, we define $R^{\perp}$ as the Riemannian curvature tensor of the normal connection $\nabla^{\perp}$ on the normal bundle $T^{\perp}M$ in the following way

Definition 3.3.3. *The Riemannian curvature tensor of the normal connection* $\nabla^{\perp}$ *is defined as*

$$R^{\perp}(X, Y)\xi = \nabla^{\perp}_X \nabla^{\perp}_Y \xi - \nabla^{\perp}_Y \nabla^{\perp}_X \xi - \nabla^{\perp}_{[X,Y]}\xi. \tag{3.6}$$

We take the inner product of this expression with the normal vector $\eta$to obtain the Ricci equation. The resulting equation is

$$\begin{aligned}\widetilde{R}(X, Y, \xi, \eta) &= R^{\perp}(X, Y, \xi, \eta) + \widetilde{g}(h(Y, A_\xi X), \eta) - \widetilde{g}(h(X, A_\xi Y), \eta) \\ &= R^{\perp}(X, Y, \xi, \eta) + g(A_\eta A_\xi X, Y) - g(A_\xi A_\eta X, Y).\end{aligned}$$

Definition 3.3.4. *The Ricci Equation for a Riemannian submanifold* (M, g) *of dimension* n *in a Riemannian manifold* $(\widetilde{M}, \widetilde{g})$ *of dimension* $(n + m)$ *is defined as*

$$\widetilde{R}(X, Y, \xi, \eta) = R^{\perp}(X, Y, \xi, \eta) + g(A_\eta A_\xi X, Y) - g(A_\xi A_\eta X, Y),$$

where

$$R^{\perp}(X, Y, \xi, \eta) = \widetilde{g}(R^{\perp}(X, Y)\xi, \eta).$$

Proposition 3.3.1. *[26] If $\widetilde{M}$ is a space of constant curvature k, then the Codazzi equation becomes*

$$(\widetilde{\nabla}_X B)(Y,Z) = (\widetilde{\nabla}_Y B)(X,Z),$$

$\widetilde{R}(X,Y)Z$ is tangent to M for any vectors $X,Y,Z \in M$. This implies that it's normal component is zero.

Example 3.3.1. *(Hypersurfaces in $\mathbb{R}^{n+1}$). If we assume that $\widetilde{M} = \mathbb{R}^{n+1}$ and M is a hypersurface, then we have*

$$(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z)$$
$$\Rightarrow (\nabla_X A)(Y) = (\nabla_Y A)(X).$$

Proposition 3.3.2. *[26] If $(\widetilde{M}, \widetilde{g})$ be a Riemannian manifold of constant curvature k. then the Ricci equation becomes*

$$R^{\perp}(X,Y,\xi,\eta) = g(A_\eta A_\xi X, Y) - g(A_\xi A_\eta X, Y),$$

$\widetilde{R}(X,Y)Z$ is tangent to (M,g) for any vectors $X,Y,Z \in M$. This implies that it's normal component is zero. Therefore the inner product of $\widetilde{R}(X,Y)\xi$ with a normal vector η is zero.

3.4 Geometric of Submanifolds

3.4.1 Minimal Submanifolds

Definition 3.4.1. *Let (M,g) of dimension n be a Riemannian manifold in $(\widetilde{M}, \widetilde{g})$ of dimension $(n+m)$. The mean curvature vector H is defined by*

$$H = \frac{1}{n} trace_g(B),$$

where

$$trace_g(B) = \sum_1^n B(e_i, e_i)$$

for a local field of orthonormal frame $\{e_1, ..., e_n\}$ in TM.

Definition 3.4.2. *A submanifold (M, g) of dimension n in a Riemannian manifold $(\widetilde{M}, \widetilde{g})$ of dimension $(n+m)$ is said to be minimal if*

$$H = 0.$$

3.4.2 Totally geodesic Submanifolds

Definition 3.4.3. *A submanifold (M, g) of dimension n in a Riemannian manifold $(\widetilde{M}, \widetilde{g})$ of dimension $(n+m)$ is called totally geodesic at $p \in M$ if every geodesic at p in the ambient space $\widetilde{M}$ tangent to the submanifold M is contained in M.*
If M is totally geodesic for all p then M is a totally geodesic submanifold.

Theorem 3.4.1. *[9] A submanifold (M, g) of dimension n in a Riemannian manifold $(\widetilde{M}, \widetilde{g})$ of dimension $(n+m)$ is totally geodesic if and only if its second fundamental form B is identically zero.*

Remark 3.4.1. *Every totally geodesic submanifold (M, g) of dimension n of a Riemannian manifold $(\widetilde{M}, \widetilde{g})$ of dimension $(n+m)$ is minimal.*

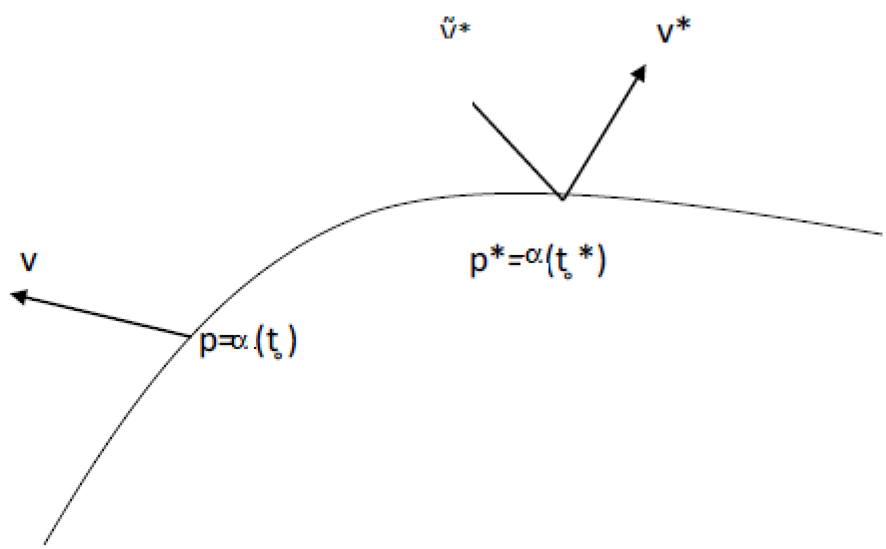

Figure 3.2: Totally geodesic Submanifold

3.4.3 Parallel Submanifolds

Let (M, g) of dimension n be a Riemannian manifold in $(\widetilde{M}, \widetilde{g})$ of dimension $(n+m)$. Given a curve α in (M, g) and two vectors $u, v \in T_pM$, with $\alpha(0) = p$, we have the vector $B(u, v)$ in the normal space of M at the point p, $T_p^{\perp}M$. At the point $\alpha(t_0) = q$, we can consider two normal vectors. First, the parallel translate of $B(u, v)$ by $\nabla^{\perp}$, which we denote by $B(u, v)^{*\perp}$, and secondly, the vector $B(u^*, v^*)$ obtained after first parallelly translating u and v by ∇, and then applying B. A submanifold (M, g) of dimension n in a Riemannian manifold $(\widetilde{M}, \widetilde{g})$ of dimension $(n+m)$ is called parallel or extrinsically symmetric when $\widetilde{\nabla} B = 0$.

Proposition 3.4.1. *[10] A submanifold (M, g) ba a submanifold of dimension n in a Riemannian manifold $(\widetilde{M}, \widetilde{g})$ of dimension $(n+m)$ is parallel if and only if the parallel transport of the second fundamental form with respect to $\nabla^{\perp}$ along any curve in M is equal to the second fundamental form acting on the parallel transport of two tangent vectors to M along the same curve, that's to say $B(u, v)^{*\perp} = B(u^*, v^*)$.*

Proof 3.4.1. *Let $p \in M$ and $\alpha : I \subset \mathbb{R} \longrightarrow$ a curve in M with $\alpha(0) = p$. Consider two vector fields $u, v \in \mathfrak{X}(\alpha)$ so that $u_p = u$ and $v_p = v$, and $\nabla_{\dot{\alpha}} u = \nabla_{\dot{\alpha}} v = 0$.*
Assume that M is parallel, i.e. $\widetilde{\nabla} B = 0$. Because the parallel transport defines a unique vector field it is sufficient to prove that $\nabla^{\perp}_{\dot{\alpha}} B(u, v) = 0$. In fact,

$$\nabla^{\perp}_{\dot{\alpha}} B(u, v) = B(\nabla_{\dot{\alpha}} u, v) + B(u, \nabla_{\dot{\alpha}} v) = 0.$$

Conversely, let us assume that $B(u, v)^{*\perp} = B(u^*, v^*)$. Then,

$$\begin{aligned} \widetilde{\nabla} B(\dot{\alpha}, u, v) \mid_p &= (\nabla^{\perp}_{\dot{\alpha}} B(u, v) - B(\nabla_{\dot{\alpha}} u, v) - B(u, \nabla_{\dot{\alpha}} v)) \mid_p \\ &= \nabla^{\perp}_{\dot{\alpha}} B(u, v) \mid_p = 0. \end{aligned}$$

Example 3.4.1. *We can obtain the geodesics of S^2 by intersecting S^2 with a plane containing the origin. These geodesics are the great circles, as shown in Figure (3.4), and can be considered to be totally geodesic submanifolds of dimension 1 of S^2.*

3.4.4 The Frobenius Theorem

We are going to study completely integrable distributions. In particular, we will state the Frobenius Theorem, which gives us the conditions to generalize the result that was given in the motivation.

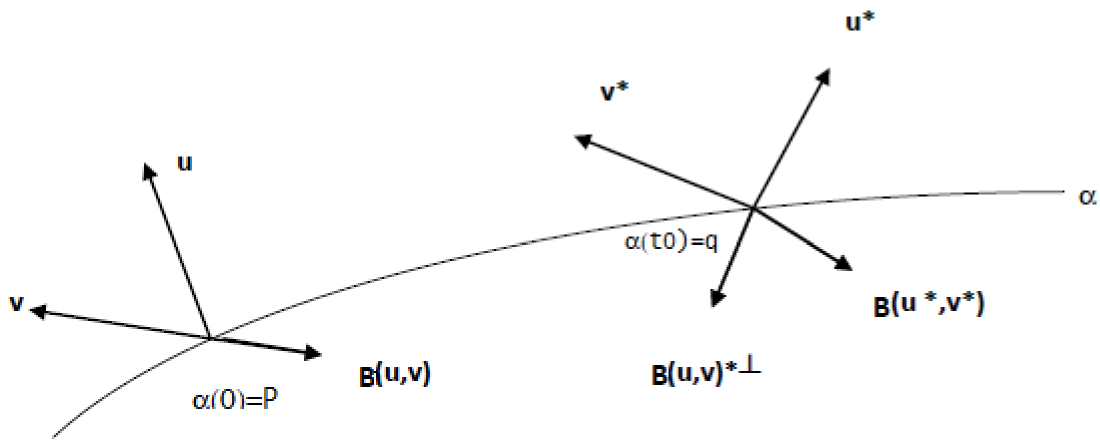

Figure 3.3: Parallel Submanifold

Definition 3.4.4. *An d-dimensional distribution D on $\widetilde{M}$ is an differentiable assigment of an d-dimensional subspace D_p of $T_p\widetilde{M}$ at each point $p \in \widetilde{M}$, such that D_p is differentiable with respect to p.*
We also say that a vector field X on $\widetilde{M}$ belongs to D if $X_p \in D_p$ for any point $p \in \widetilde{M}$.

Definition 3.4.5. *A submanifold (M, g) ba a submanifold of dimension n in a Riemannian manifold $(\widetilde{M}, \widetilde{g})$ of dimension $(n + m)$ is called an integral manifold of D, if $T_pM = D_p$ for any point $p \in M$. Moreover, if an integral manifold of D exists through each point of $\widetilde{M}$, D is said to be completely integrable.*

Theorem 3.4.2. *[29] Let D be a distribution on a Riemannian manifold $(\widetilde{M}, \widetilde{g})$. Then, D is completely integrable if and only if for any two vector fields X, Y belonging to D, the lie-bracket $[X, Y]$ also belongs to D (a distribution with this property is said to be involutive).*

Proof 3.4.2. *See p.3 of [29].*

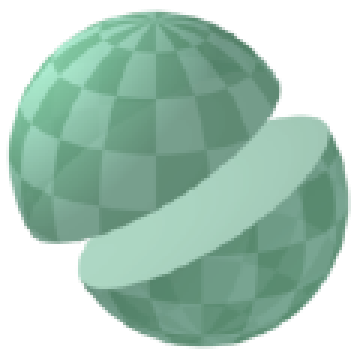

Figure 3.4: geodesics of S^2

CHAPTER 4

THURSTON'S GEOMETRY NIL^4

The objective of this chapter is to introduce a Thurston model geometry Nil^4, define the metric Nil^4, and study the geometric properties of Nil^4 space. The references used are: [13], [14], [17], [21], [32], [33], [34].

4.1 Model geometries

Definition 4.1.1. *A model geometry (G, X) is a manifold X with a Lie group G of diffeomorphisms of X, such that*

1. *X is connected and simply connected,*
2. *The action of G on X is transitive with compact stabilizer,*
3. *G is maximal in the sense that it is not contained in a group of diffeomorphisms of X with compact stabilizer,*
4. *there exists at least one compact manifold modeled (G, X)-manifold.*

4.2 Lie group Nil^4

A model geometry (Nil^4, Nil^4) is considred to be a manifold and a Lie group of diffeomorphisms of Nil^4.

The Nil^4 is a nilpotent Lie group and, we have $Nil^4 = \mathbb{R}^3 \ltimes_U \mathbb{R}$ where the $U(t) = \exp(tL)$, with

$$L = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \exp(tL) = I_3 + tL + \frac{t^2}{2}L^2 = \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.1}$$

Component of its isometry group with identity is Nil^4 itself as left translation. The semidirect product in Nil^4 is given by

$$\begin{aligned}(V,t)(V',t') &= (V + \exp(tL)V', t+t') \\ &= \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}, t+t' \right) \\ &= \left(\begin{pmatrix} x + x' + ty' + \frac{t^2}{2}z' \\ y + y' + tz' \\ z + z' \end{pmatrix}, t+t' \right),\end{aligned} \tag{4.2}$$

for all $V = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $V' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \in \mathbb{R}^3$, and $t, t' \in \mathbb{R}$. We have the parameterization

$$\begin{aligned}\phi : Nil^4 &\longrightarrow \mathbb{R}^4. \\ \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}, t \right) &\longmapsto (x, y, z, t)\end{aligned} \tag{4.3}$$

4.3 Metric left-invariant by the nilpotent Lie group Nil^4

A Riemannian metric on the nilpotent Lie group Nil^4 by definition is, a left invariant metric, if it is invariant with all left translations L_a. More precisely, we have the following definition

$$\forall a, a' \in Nil^4, \xi, \eta \in T_{a'}Nil^4 \quad \widetilde{g}_{a'}((dL_a)_{a'}(\xi), (dL_a)_{a'}(\eta)) = \widetilde{g}_{a'}(\xi, \eta).$$

According to the definition above, $\widetilde{g}$ is a left invariant metric on Nil^4 if for all $\sigma \in Nil^4$

1. $\widetilde{g}_\sigma$ is a scalar product on $T_\sigma Nil^4$,
2. $\widetilde{g}_\sigma = dL_\sigma(e)_*.\widetilde{g}_e$,
3. $\widetilde{g}_e = dL_\sigma(e)^*.\widetilde{g}_\sigma$ with $dL_\sigma(e) : T_e Nil^4 \longrightarrow T_\sigma Nil^4$.

Lemma 4.3.1. *Let Nil^4 be a nilpotent Lie group. Then, the following vectors field*

$$\begin{aligned} e_1 &= \frac{\partial}{\partial x}, \\ e_2 &= t\frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \\ e_3 &= \frac{t^2}{2}\frac{\partial}{\partial x} + t\frac{\partial}{\partial y} + \frac{\partial}{\partial z}, \\ e_4 &= \frac{\partial}{\partial t}. \end{aligned} \tag{4.4}$$

Proof. For any point $p(x, y, z, t)$ at Nil^4, the left translation L_p

$$\begin{array}{rcl} L_p : Nil^4 & \longrightarrow & Nil^4 \\ q(x', y', z', t') & \longmapsto & p.q \end{array}$$

We have

$$L_p(q) = p.q = (\begin{pmatrix} x + x' + ty' + \frac{t^2}{2}z' \\ y + y' + tz' \\ z + z' \end{pmatrix}, t + t')$$

The differential L_p is calculated as following

$$D_q L_p = \begin{pmatrix} 1 & t & \frac{t^2}{2} & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\{V_1, V_2, V_3, V_4\}$ is the frame field on $T_e Nil^4$ at the neutral element $e = (0, 0, 0, 0)$, $(\frac{\partial}{\partial x}(p), \frac{\partial}{\partial y}(p), \frac{\partial}{\partial z}(p), \frac{\partial}{\partial t}(p))$ is the basis of $T_p Nil^4$ in the point $p(x, y, s, t)$, Therefore, $\{e_1, e_2, e_3, e_4\}$ are a left-invariant

vector fields if: $D_eL_p(V_i) = e_i$, for $i \in \{1, 2, 3, 4\}$.
Then

$$\begin{aligned} D_eL_p : T_eNil^4 &\longrightarrow T_pNil^4 \\ V_i &\longmapsto e_i = D_eL_p(V_i) \end{aligned}$$

So, that

$$e_1 = D_eL_p(\begin{pmatrix}1\\0\\0\\0\end{pmatrix}) = \frac{\partial}{\partial x},$$

$$e_2 = D_eL_p(\begin{pmatrix}0\\1\\0\\0\end{pmatrix}) = t\frac{\partial}{\partial x} + \frac{\partial}{\partial y},$$

$$e_3 = D_eL_p(\begin{pmatrix}0\\0\\1\\0\end{pmatrix}) = \frac{t^2}{2}\frac{\partial}{\partial x} + t\frac{\partial}{\partial y} + \frac{\partial}{\partial z},$$

$$e_4 = D_eL_p(\begin{pmatrix}0\\0\\0\\1\end{pmatrix}) = \frac{\partial}{\partial t}.$$

□

So that, the dual coframe fields are given by

$$\begin{aligned}
\theta_1 &= dx - tdy + \frac{t^2}{2}dz, \\
\theta_2 &= dy - tdz, \\
\theta_3 &= dz, \\
\theta_4 &= dt.
\end{aligned} \tag{4.5}$$

The matrix of a Riemannian metric on Nil^4

$$\widetilde{g} = \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2$$

is given by

$$\widetilde{g} = \begin{pmatrix} 1 & -t & \frac{t^2}{2} & 0 \\ -t & 1+t^2 & -t(1+\frac{t^2}{2}) & 0 \\ \frac{t^2}{2} & -t(1+\frac{t^2}{2}) & 1+t^2+\frac{t^4}{4} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

4.4 Geometric properties of $(Nil^4, \widetilde{g})$

Let the nilpotent Lie group $(Nil^4, \widetilde{g})$, and $\widetilde{\nabla}$ the Levi-Civita of connections $(Nil^4, \widetilde{g})$. The Christoffel symbols of $\widetilde{\nabla}$ for a point of Nil^4 are $\widetilde{\Gamma}_{ij}^k$ for $i, j, k \in \{1, 2, 3, 4\}$.

Proposition 4.4.1. *By Kozsul´s formula, the non-zero of the Levi-Civita connections $\widetilde{\nabla}$ of $(Nil^4, \widetilde{g})$ are as follows:*

$$\begin{aligned}
\widetilde{\nabla}_{e_1} e_2 &= \frac{1}{2} e_4\,, & \widetilde{\nabla}_{e_1} e_4 &= -\frac{1}{2} e_2 \\
\widetilde{\nabla}_{e_2} e_1 &= \frac{1}{2} e_4\,, & \widetilde{\nabla}_{e_2} e_3 &= \frac{1}{2} e_4 \\
\widetilde{\nabla}_{e_2} e_4 &= -\frac{1}{2}(e_1 + e_3)\,, & \widetilde{\nabla}_{e_3} e_2 &= \frac{1}{2} e_4 \\
\widetilde{\nabla}_{e_3} e_4 &= -\frac{1}{2} e_2\,, & \widetilde{\nabla}_{e_4} e_1 &= -\frac{1}{2} e_2 \\
\widetilde{\nabla}_{e_4} e_2 &= \frac{1}{2}(e_1 - e_3)\,, & \widetilde{\nabla}_{e_4} e_3 &= \frac{1}{2} e_2.
\end{aligned}$$

Proof. Note that, the non-zero of Christoffel symbols $\widetilde{\Gamma}_{ij}^k$ for $i, j, k \in \{1, 2, 3, 4\}$ are given by

$$
\begin{aligned}
\widetilde{\Gamma}_{12}^4 &= \frac{1}{2}\,, & \widetilde{\Gamma}_{13}^4 = -\frac{t}{2} \\
\widetilde{\Gamma}_{14}^1 &= -\frac{t}{2}\,, & \widetilde{\Gamma}_{14}^2 = -\frac{1}{2} \\
\widetilde{\Gamma}_{22}^4 &= -t\,, & \widetilde{\Gamma}_{23}^4 = \frac{1}{2} + \frac{3t^2}{4} \\
\widetilde{\Gamma}_{24}^1 &= -\frac{1}{2} + \frac{t^2}{4}\,, & \widetilde{\Gamma}_{24}^3 = -\frac{1}{2} \\
\widetilde{\Gamma}_{33}^4 &= -t(1 + \frac{t^2}{2})\,, & \widetilde{\Gamma}_{34}^2 = -\frac{1}{2} + \frac{t^2}{4} \\
\widetilde{\Gamma}_{34}^3 &= \frac{t}{2}\,.
\end{aligned}
$$

Proposition 4.4.1 follows from (4.4). □

Corollary 4.4.1. *The non-zero Lie brackets of the basis $\{e_i\}_{1 \le i \le 4}$ are given by*

$$[e_4, e_2] = e_1 \quad , \quad [e_4, e_3] = e_2.$$

Proof. Follows directly by Proposition 4.4.1, with $[e_i, e_j] = \widetilde{\nabla}_{e_i} e_j - \widetilde{\nabla}_{e_j} e_i$ for all $i, j = 1, 2, 3, 4$. □

Proposition 4.4.2. *The only non-zero components of Riemannian curvature of $(Nil^4, \widetilde{g})$ are given by*

$$
\begin{aligned}
\widetilde{g}(\widetilde{R}(e_1, e_2)e_1, e_2) &= -\frac{1}{4}\,, & \widetilde{g}(\widetilde{R}(e_1, e_2)e_2, e_3) = \frac{1}{4} \\
\widetilde{g}(\widetilde{R}(e_1, e_4)e_1, e_4) &= -\frac{1}{4}\,, & \widetilde{g}(\widetilde{R}(e_1, e_4)e_3, e_4) = \frac{1}{4} \\
\widetilde{g}(\widetilde{R}(e_2, e_1)e_2, e_3) &= -\frac{1}{4}\,, & \widetilde{g}(\widetilde{R}(e_2, e_3)e_2, e_3) = -\frac{1}{4} \\
\widetilde{g}(\widetilde{R}(e_2, e_4)e_2, e_4) &= \frac{1}{2}\,, & \widetilde{g}(\widetilde{R}(e_3, e_4)e_3, e_4) = \frac{3}{4}.
\end{aligned}
$$

Proof. Using the definition of Riemannian curvature $\widetilde{R}(X, Y)Z = [\widetilde{\nabla}_X, \widetilde{\nabla}_Y]Z - \widetilde{\nabla}_{[X,Y]}Z$, the Proposition 4.4.1, and the Corollary 4.4.1. □

According to Proposition 4.4.2, we have the following Corollary.

Corollary 4.4.2. *The matrix of Ricci curvature of* $(Nil^4, \widetilde{g})$ *is given by*

$$(S_{ij}) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $S_{ij} = \sum_{a=1}^{n} \widetilde{g}(\widetilde{R}(e_i, e_a)e_a, e_j)$*, for all* $i, j = 1, 2, 3, 4.$
Thus, the scalar curvature of $(Nil^4, \widetilde{g})$ *is*

$$\tau = 1.$$

CHAPTER 5

GEOMETRY OF HYPERSURFACES M^3 IN THURSTON'S GEOMETRY NIL^4

This chapter presents our work on the hypersurface (M^3, g) in geometry of Thurston $(Nil^4, \widetilde{g})$. we give a classification of Codazzi hypersurfaces in a Lie group $(Nil^4, \widetilde{g})$. We also give a characterization of a class of minimal hypersurfaces in $(Nil^4, \widetilde{g})$ with an example of a minimal surface in this class. The references used are:[3], [8], [13], [14], [17], [18], [24], [25], [27], [30], [33], [34]. .

5.1 Classification of Codazzi hypersurfaces in Nil^4

Theorem 5.1.1. *A hypersurface (M^3, g) in the Lie group $(Nil^4, \widetilde{g})$ is Codazzi if and only if the unit normal vector field to (M^3, g) is $\xi = e_4$.*

Proof. Let (M^3, g) be a hypersurface in $(Nil^4, \widetilde{g})$.
We have, $\xi = ae_1 + be_2 + ce_3 + de_4$ the unit normal vector field on (M^3, g), where a, b, c, d are local functions on M^3. Thus

$$X_1 = be_1 - ae_2\,, \qquad X_2 = ce_1 - ae_3$$
$$X_3 = de_1 - ae_4\,, \qquad X_4 = ce_2 - be_3$$
$$X_5 = de_2 - be_4\,, \qquad X_6 = de_3 - ce_4$$

are tangent vectors fields to the hypersurface (M^3, g).
Now, assume that the hypersurface (M^3, g) is Codazzi, that is

$$(\nabla_X h)(Y, Z) = (\nabla_Y h)(X, Z), \quad \forall X, Y, Z \in \mathfrak{X}(M^3). \tag{5.1}$$

Then it follows from the equation of Codazzi (3.4) that

$$\widetilde{g}(\widetilde{R}(X_i, X_j)X_k, \xi) = 0, \quad \forall i, j, k \in \{1, ..., 6\}. \tag{5.2}$$

By using the curvature components given in Proposition 4.4.2, we get the following

$$\widetilde{g}(\widetilde{R}(X_1, X_2)X_3, \xi) = \frac{1}{4}abd(a - c) = 0,$$

from the which we prove that $a = 0$ or $b = 0$ or $d = 0$ or $a = c$.

• If $a = 0$, we have the following equations

$$\begin{aligned}
\widetilde{g}(\widetilde{R}(X_1, X_4)X_1, \xi) &= -\frac{1}{4}b^3c = 0, \\
\widetilde{g}(\widetilde{R}(X_2, X_4)X_4, \xi) &= \frac{1}{4}c^2b^2 + \frac{1}{4}c^2 = 0, \\
\widetilde{g}(\widetilde{R}(X_1, X_5)X_4, \xi) &= \frac{1}{2}b^3d + \frac{1}{4}bc^2d = 0.
\end{aligned}$$

Thus $c = 0$ and $bd = 0$. So that, $\xi = e_2$ or $\xi = e_4$.
Note that, in the case where $\xi = e_2$, the Lie bracket $[e_4, e_3] = e_2$ is not tangent vector field on M^3 despite e_2 and e_4 are tangent vectors fields on M^3. So, by Frobenius Theorem, this case is unacceptable. Then we have $\xi = e_4$.

• If $b = 0$, we obtain the equations

$$\begin{aligned}
\widetilde{g}(\widetilde{R}(X_1, X_2)X_1, \xi) &= \frac{1}{4}a^2(a^2 - c^2) = 0, \\
\widetilde{g}(\widetilde{R}(X_1, X_3)X_1, \xi) &= -\frac{1}{4}a^2d(3a + c) = 0.
\end{aligned}$$

For $a = 0$, we get $c = 0$. Thus $\xi = e_4$. For $a = \pm c$, we find that $ad = 0$. Hence $\xi = e_4$ or $\xi = \frac{1}{\sqrt{2}}(e_1 \pm e_3)$.
Note that, in the case where $\xi = \frac{1}{\sqrt{2}}(e_1 \pm e_3)$, the Lie bracket $[e_4, e_2]$ is tangent vector field on M^3 because e_2 and e_4 are tangent vectors fields on M^3. But $\widetilde{g}([e_4, e_2], \xi) = \widetilde{g}(e_1, \xi) = \frac{1}{\sqrt{2}} \neq 0$, we obtain a contradiction with the fact that ξ in normal to M^3. Therefore, $\xi = e_4$.

• If $d = 0$, we have the equations

$$\begin{aligned}
\widetilde{g}(\widetilde{R}(X_1, X_2)X_2, \xi) &= -\frac{1}{4}ab(a-c)^2 = 0,\\
\widetilde{g}(\widetilde{R}(X_1, X_4)X_1, \xi) &= \frac{1}{4}b(a-c)(a^2+b^2+ac) = 0,\\
\widetilde{g}(\widetilde{R}(X_2, X_4)X_4, \xi) &= -\frac{1}{4}c(a-c)(b^2+c^2+ac) = 0,\\
\widetilde{g}(\widetilde{R}(X_1, X_2)X_1, \xi) &= \frac{1}{4}a(a-c)(a^2+b^2+ac) = 0,\\
\widetilde{g}(\widetilde{R}(X_6, X_2)X_3, \xi) &= -a^2c^2 - \frac{1}{4}ac(a^2-c^2) = 0.
\end{aligned}$$

Hence $a = c = 0$. Thus, $\xi = e_2$. It is unacceptable, because in this case e_3 and e_4 are tangent vectors fields on M^3 but $[e_4, e_3] = e_2$ is not tangent vector field on M^3.

• If $a = c$, we get the following equations

$$\begin{aligned}
\widetilde{g}(\widetilde{R}(X_6, X_2)X_3, \xi) &= -c^2(c^2 + \frac{1}{2}d^2) = 0,\\
\widetilde{g}(\widetilde{R}(X_1, X_5)X_6, \xi) &= \frac{1}{2}b^2(c^2 - d^2) = 0.
\end{aligned}$$

we obtain $c = 0$ and $bd = 0$. Hence, $\xi = e_4$. Here, $\xi = e_2$ is unacceptable.

According to the previous calculations, it suffices to show that

$$\widetilde{g}(\widetilde{R}(X,Y)Z, \xi) = (\nabla_Y h)(X,Z) - (\nabla_X h)(Y,Z) = 0, \tag{5.3}$$

for all $X, Y, Z \in \mathfrak{X}(M^3)$ for $\xi = e_4$. You can easily check the equations

$$\widetilde{g}(\widetilde{R}(e_i, e_j)e_k, e_4) = 0, \quad i,j,k = 1, ..., 3.$$

□

Remark 5.1.1. *Let (M^3, g) be a Codazzi hypersurfaces in $(Nil^4, \widetilde{g})$ is given by*

$$\begin{aligned}
f : (\mathbb{R}^3, g) &\longrightarrow (Nil^4, \widetilde{g}),\\
(x,y,z) &\longmapsto (x,y,z,t_0)
\end{aligned}$$

where $t_0 \in \mathbb{R}$. Since M^3 is a hypersurface of $(Nil^4, \widetilde{g})$, then g is given by

$$g = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{pmatrix}$$

Then there exists an orthonormal frame field $\{e_1, e_2, e_3\}$ on (M^3, g), where

$$e_1 = \frac{\partial}{\partial x}, \qquad e_2 = \frac{\partial}{\partial y}$$
$$e_3 = \frac{\partial}{\partial z}.$$

So, we find that the second fundamental form h of the hypersurface (M^3, g)

$$h = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

The components h_{ij} of the second fundamental form of M^3 are related by

$$h_{ij} = \xi^t.\widetilde{g}.\widetilde{\nabla}_{\partial_{u_i}}\partial_{u_j},$$

where $\partial_{u_1} = e_1, \partial_{u_2} = e_2, \partial_{u_3} = e_3$.
In this case

- (M^3, g) *is not totally geodesic* *because* $h \neq 0$,
- (M^3, g) *is parallel* *because* $\nabla h = 0$,
- (M^3, g) *is Codazzi* *because* $\nabla h = 0$,
- (M^3, g) *is minimal* *because* $trace_g(h) = 0$.

Corollary 5.1.1. *Every hypersurface Codazzi of $(Nil^4$ is a parallel.*

The matrix of Ricci curvature of (M^3, g) in $(Nil^4, \widetilde{g})$ is described by

$$(S_{ij}) = \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix},$$

The scalar curvature of (M^3, g) in $(Nil^4, \widetilde{g})$ is given by

$$\tau = 1$$

The shape operator of (M^3, g) in $(Nil^4, \widetilde{g})$ is given by

$$A_\xi = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix}.$$

The shape operator and the second fundamental form [3] are related by $g(A_\xi X, Y) = \widetilde{g}(h(X,Y)\xi, \xi)$.
So, $A_\xi = (g^{-1})^t.h^t$.
The principal curvatures of (M^3, g) in $(Nil^4, \widetilde{g})$ are

$$-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}.$$

5.2 Characterization of a class of minimal hypersurfaces in Nil^4

Let (M^3, g) be a hypersurface in $(Nil^4, \widetilde{g})$. In this section, we search the conditions for the hypersurface (M^3, g) to be minimal in $(Nil^4, \widetilde{g})$, where the unit normal vector field on (M^3, g) is given by $\xi = ae_1 + be_2 + ce_3 + de_4$, and assume that $\{X_i\}_{1\leq i\leq 3}$ is a local orthonormal frame on (M^3, g), where $X_i = a_ie_1 + b_ie_2 + c_ie_3 + d_ie_4$ for some local functions $\{a, b, c, d, a_i, b_i, c_i, d_i\}_{1\leq i\leq 3}$ on M^3 depends only on the variable t.

Theorem 5.2.1. *The hypersurface (M^3, g) is minimal in $(Nil^4, \widetilde{g})$ if and only if*

$$\sum_{i=1}^{3} \left[a_i\left(b_i d - bd_i\right) + b_i\left(c_i d - cd_i\right) + d_i\left(aa_i' + bb_i' + cc_i' + dd_i'\right)\right] = 0.$$

Proof. The unit normal vector field on (M^3, g) is given by

$$\xi = ae_1 + be_2 + ce_3 + de_4.$$

And assume that $\{X_i\}_{1\leq i\leq 3}$ is a local orthonormal frame on (M^3, g).
Where

$$X_i = a_ie_1 + b_ie_2 + c_ie_3 + d_ie_4$$

for some local functions $\{a, b, c, d, a_i, b_i, c_i, d_i\}_{1\leq i\leq 3}$ on M^3 depends only on the variable t.
We compute

$$\begin{aligned}
\widetilde{\nabla}_{X_i} X_i &= \widetilde{\nabla}_{a_ie_1+b_ie_2+c_ie_3+d_ie_4}\left(a_ie_1 + b_ie_2 + c_ie_3 + d_ie_4\right) \\
&= a_i\left(a_i\widetilde{\nabla}_{e_1}e_1 + b_i\widetilde{\nabla}_{e_1}e_2 + c_i\widetilde{\nabla}_{e_1}e_3 + d_i\widetilde{\nabla}_{e_1}e_4\right) \\
&\quad +b_i\left(a_i\widetilde{\nabla}_{e_2}e_1 + b_i\widetilde{\nabla}_{e_2}e_2 + c_i\widetilde{\nabla}_{e_2}e_3 + d_i\widetilde{\nabla}_{e_2}e_4\right) \\
&\quad +c_i\left(a_i\widetilde{\nabla}_{e_3}e_1 + b_i\widetilde{\nabla}_{e_3}e_2 + c_i\widetilde{\nabla}_{e_3}e_3 + d_i\widetilde{\nabla}_{e_3}e_4\right) \\
&\quad +d_i\Big(a_i'e_1 + a_i\widetilde{\nabla}_{e_4}e_1 + b_i'e_2 + b_i\widetilde{\nabla}_{e_4}e_2 + c_i'e_3 + c_i\widetilde{\nabla}_{e_4}e_3 \\
&\quad +d_i'e_4 + d_i\widetilde{\nabla}_{e_4}e_4\Big).
\end{aligned} \tag{5.4}$$

From Proposition 4.4.1, and equation (5.4), we obtain

$$\begin{aligned}
\widetilde{\nabla}_{X_i} X_i &= a_i\left(\frac{b_i}{2}e_4 - \frac{d_i}{2}e_2\right) + b_i\left(\frac{a_i}{2}e_4 + \frac{c_i}{2}e_4 - \frac{d_i}{2}(e_1 + e_3)\right) \\
&\quad +c_i\left(\frac{b_i}{2}e_4 - \frac{d_i}{2}e_2\right) + d_i\Big(a_i'e_1 - \frac{a_i}{2}e_2 + b_i'e_2 + \frac{b_i}{2}(e_1 - e_3) \\
&\quad +c_i'e_3 + \frac{c_i}{2}e_2 + d_i'e_4\Big),
\end{aligned}$$

it is equivalent to the following equation

$$\widetilde{\nabla}_{X_i}X_i = a_i'd_ie_1 + d_i(b_i' - a_i)e_2 + d_i(c_i' - b_i)e_3 + [d_id_i' + b_i(a_i + c_i)]e_4. \tag{5.5}$$

From Proposition 4.4.1, and equation (5.4), we obtain By equation (5.5).
We have

$$\widetilde{g}(\widetilde{\nabla}_{X_i}X_i, \xi) = aa_i'd_i + bd_i(b_i' - a_i) + cd_i(c_i' - b_i) + d[d_id_i' + b_i(a_i + c_i)]. \tag{5.6}$$

Thus, the hypersurface (M^3, g) is minimal if

$$H = \frac{1}{3}\sum_{i=1}^{3}\widetilde{g}(\widetilde{\nabla}_{X_i}X_i, \xi) = 0. \tag{5.7}$$

The Theorem 5.2.1 follows by equations (5.6) and (5.7). □

5.3 Example of minimal hypersurfaces in Nil^4

Example 5.3.1. *Let (M^3, g) be a minimal hypersurface in $(Nil^4, \widetilde{g})$, defined by*

$$\begin{aligned} f : (M^3, g) &\longrightarrow (Nil^4, \widetilde{g}). \\ (y, z, t) &\longmapsto (2t + \frac{t^3}{3}, y, z, t) \end{aligned}$$

We consider the following vector fields

$$\begin{aligned} \xi &= \frac{2}{\sqrt{5}(2+t^2)}e_1 + \frac{2t}{\sqrt{5}(2+t^2)}e_2 + \frac{t^2}{\sqrt{5}(2+t^2)}e_3 - \frac{2}{\sqrt{5}}e_4, \\ X_1 &= -\frac{t}{\sqrt{1+t^2}}e_1 + \frac{1}{\sqrt{1+t^2}}e_2, \\ X_2 &= -\frac{t^2}{(2+t^2)\sqrt{1+t^2}}e_1 - \frac{t^3}{(2+t^2)\sqrt{1+t^2}}e_2 + \frac{2\sqrt{1+t^2}}{2+t^2}e_3, \\ X_3 &= \frac{4}{\sqrt{5}(2+t^2)}e_1 + \frac{4t}{\sqrt{5}(2+t^2)}e_2 + \frac{2t^2}{\sqrt{5}(2+t^2)}e_3 + \frac{1}{\sqrt{5}}e_4. \end{aligned}$$

We have $\{X_i\}_{1\leq i\leq 3}$ is a local orthonormal frame on (M^3, g).
Where

$$X_i = a_i e_1 + b_i e_2 + c_i e_3 + d_i e_4$$

for some local functions $\{a, b, c, d, a_i, b_i, c_i, d_i\}_{1\leq i\leq 3}$ on M^3 depends only on the variable t. The unit normal vector field ξ on (M^3, g) which is given by

$$\xi = ae_1 + be_2 + ce_3 + de_4.$$

We have

$$aa_i' d_i + bd_i(b_i' - a_i) + cd_i(c_i' - b_i) + d[d_i d_i' + b_i(a_i + c_i)] = 0,$$

So

$$\begin{cases} aa_1' d_1 + bd_1(b_1' - a_1) + cd_1(c_1' - b_1) + d[d_1 d_1' + b_1(a_1 + c_1)] = 0 \\ aa_2' d_2 + bd_2(b_2' - a_2) + cd_2(c_2' - b_2) + d[d_2 d_2' + b_2(a_2 + c_2)] = 0 \\ aa_3' d_3 + bd_3(b_3' - a_3) + cd_3(c_3' - b_3) + d[d_3 d_3' + b_3(a_3 + c_3)] = 0 \\ a^2 + b^2 + c^2 + d^2 = 1 \end{cases}$$

Assume that

$$c_1 = d_1 = d_2 = 0,$$

Thus the hypersurface (M^3, g) that is defined by these vector fields is minimal.

5.4 Conclusion

We have interested in studying the geometric properties of hypersurfaces in, one of the 4-dimensional Thurston model geometries, Nil^4. More precisely we have classified the codazzi hypersurfaces and give examples of minimal surfaces.
As Nil^4 is a Lie group, we have to work with a left invariant Riemannian metric. The next step will be, to have a classification for minimal hypersurfaces in this space with the same Riemannian metric and to see how the change of a metric, by keeping it left invariant, can affect these classifications.

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